MPO 711 - Homework #2 Spring 2020

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1) Derive the thermal wind balance $\frac{\partial u_g}{\partial z} = \frac{g}{\rho_0 f_0}\frac{\partial \rho_g}{\partial y}, \frac{\partial v_g}{\partial z} = -\frac{g}{\rho_0 f_0}\frac{\partial \rho_g}{\partial x}$ for the oceans from the hydrostatic and geostrophic balances. In your opinion, what are the main difficulties in using this relation for calculating oceanic velocities?

Geostrophic balance:

(1) $$f_0 v_g = \frac{1}{\rho_0}\frac{\partial p_g}{\partial x}$$

(2) $$f_0 u_g = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial y}$$

Hydrostatic balance:

(3) $$\frac{\partial p_g}{\partial z} = -\rho_g g$$

differentiating wrt z and usind (3)

$$\frac{\partial}{\partial z}[f_0 v_g = \frac{1}{\rho_0}\frac{\partial p_g}{\partial x}] \quad => f_0\frac{\partial v_g}{\partial z} = \frac{1}{\rho_0}\frac{\partial}{\partial x}[\frac{\partial p_g}{\partial z}] \quad => f_0\frac{\partial v_g}{\partial z} = \frac{1}{\rho_0}\frac{\partial}{\partial x}[-\rho_g g] \quad => \frac{\partial v_g}{\partial z} = -\frac{g}{\rho_0 f_0}\frac{\partial \rho_g}{\partial x}$$

$$\frac{\partial}{\partial z}[f_0 u_g = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial y}] => f_0\frac{\partial u_g}{\partial z} = -\frac{1}{\rho_0}\frac{\partial}{\partial y}[\frac{\partial p_g}{\partial z}] => f_0\frac{\partial u_g}{\partial z} = -\frac{1}{\rho_0}\frac{\partial}{\partial y}[-\rho_g g] => \frac{\partial u_g}{\partial z} = \frac{g}{\rho_0 f_0}\frac{\partial \rho_g}{\partial y}$$

2) Imagine a planet whose radius is half the radius of the Earth, and whose rotation rate is twice of the Earth. Discuss how the zonal phase speed of Rossby waves in the 1.5-layer system would be different from the one on the Earth. Discuss both very long ($k, l << \frac{1}{R_d}$) and very short ($k, l >> \frac{1}{R_d}$) waves.

Using the Rossby dispersion relation,

$$\omega = \frac{-\beta k}{k^2 + l^2 + \frac{1}{R^2}}$$

$$C_{phase} = \frac{\omega}{k} = \frac{-\beta}{k^2 + l^2 + \frac{1}{R^2}}$$

For the Earth:

$$\beta = \frac{\partial f_{EARTH}}{\partial y} = \frac{2\Omega_{EARTH} cos(\theta)}{R_{EARTH}}$$

$$C_{phase}^{EARTH} = \frac{2\Omega_{EARTH} cos(\theta)}{R_{EARTH}(k^2 + l^2 + \frac{1}{R^2})}$$

If $\Omega_{Planet} = 2\Omega_{EARTH}$ and $R_{Planet} = \frac{R_{EARTH}}{2}$,

$$\beta_{Planet} = \frac{2\Omega_{Planet} cos(\theta)}{R_{Planet}} = \frac{4\Omega_{EARTH} cos(\theta)}{\frac{R_{Planet}}{2}} = \frac{8\Omega_{EARTH} cos(\theta)}{R_{EARTH}} = 4[\frac{2\Omega_{EARTH} cos(\theta)}{R_{EARTH}}] = 4 \times \beta_{EARTH}$$

$$C_{phase}^{Planet} = \frac{8\Omega_{EARTH} cos(\theta)}{R_{EARTH}(k^2 + l^2 + \frac{1}{R^2})} = 4 \times [\frac{\beta_{EARTH}}{k^2 + l^2 + \frac{1}{R^2}}] = 4 \times C_{phase}^{EARTH}$$

The Rossby wave phase speed on that planet would be 4 times greather the Rossby wave phase speed on Earth!

Given some values: $\theta = 30^o$; $\frac{1}{k} = \frac{1}{l} = R = 50$ km; and considering $\Omega_{EARTH} = 7.2921159 \times 10^{-5}$ rad/s and

$R_{Earth} = 6356$ km

```
lat=30;
k=1/50000;
l=1/50000;
r=50000;
Omega=7.2921159e-5;
R_earth=6.356e6;

cp_earth=(2*Omega*cosd(lat))/(R_earth*(k^2+l^2+r^(-2)))
```

```
cp_earth = 0.0166
```

```
cp_planet=(8*(Omega)*cosd(lat))/(R_earth*(k^2+l^2+r^(-2)))
```

```
cp_planet = 0.0662
```

- For Long waves ($k, l << \frac{1}{R_d}$)

$$C_{phase}^{EARTH} = \frac{2\Omega_{EARTH}cos(\theta)}{R_{EARTH}(\frac{1}{R^2})} = \frac{2\Omega_{EARTH}cos(\theta)R^2}{R_{EARTH}}$$

$$C_{phase}^{Planet} = \frac{8\Omega_{EARTH}cos(\theta)}{R_{EARTH}(\frac{1}{R^2})} = \frac{8\Omega_{EARTH}cos(\theta)R^2}{R_{EARTH}}$$

```
%Considering the same values as before:
cp_earth=(2*Omega*cosd(lat))/(R_earth*(r^(-2))) %[m/s]
```

```
cp_earth = 0.0497
```

```
cp_planet=(8*(Omega)*cosd(lat))/(R_earth*(r^(-2))) %[m/s]
```

```
cp_planet = 0.1987
```

- For short waves $(k, l >> \frac{1}{R_d})$

$$C_{phase}^{EARTH} = \frac{2\Omega_{EARTH}cos(\theta)}{R_{EARTH}(\frac{1}{k^2 + l^2})}$$

$$C_{phase}^{Planet} = \frac{8\Omega_{EARTH}cos(\theta)}{R_{EARTH}(\frac{1}{k^2 + l^2})}$$

```
%Considering the same values as before:
cp_earth=(2*Omega*cosd(lat))/(R_earth*(k^2+l^2)) %[m/s]
```

```
cp_earth = 0.0248
```

```
cp_planet=(8*(Omega)*cosd(lat))/(R_earth*(k^2+l^2)) %[m/s]
```

```
cp_planet = 0.0994
```

Given the values attributed to the variables, long wave phase speed could reach approximately 5cm/s at the Earth and 20cm/s at this specific planet. On the other hand, for short waves, their phase speed were reduced to approximately 2.4 cm/s at the Earth and 10 cm/s at this specific planet.

3) The Brunt-Vaisala frequency for an arbritry fluid is defined by $N^2 = -\frac{g}{\rho_\theta}\frac{\partial \rho_\theta}{\partial z}$, where ρ_θ is the potential density. Show that for the dry ideal gas, this definition is identical to the one used in class $N^2 = -\frac{g}{\theta}\frac{\partial \theta}{\partial z}$.

For an general fluid: $N^2 = -\frac{g}{\rho_\theta}\frac{\partial \rho_\theta}{\partial z}$. (1)

Considering the ideal gas:

$P_r = \rho RT$ (2)

$\rho = \frac{P_r}{RT}$ (3)

Also, potential temperature could be defined as:

$\theta = T\left(\frac{P_r}{P}\right)^{\kappa}$ (4)

For a dry gas, $P_r = P$, therefore:

$\theta = T$ (5)

Using (5) on (3)

$\rho = \frac{P_r}{R\theta}$ (6)

Using (6) on (1) and considering P_r and R constants:

$N^2 = -\frac{g}{\rho_\theta}\frac{\partial \rho_\theta}{\partial z} = -\frac{gR\theta}{P_r}\frac{\partial\left(\frac{P_r}{R\theta}\right)}{\partial z} = -\frac{g\theta P_r R}{P_r R}\frac{\partial\left(\frac{1}{\theta}\right)}{\partial z} = -g\theta\frac{\partial}{\partial z}\left(\frac{1}{\theta}\right)$ (7)

Knowing that $\frac{\partial}{\partial z}\left(\frac{1}{\theta}\right) = \frac{1}{\theta^2}\frac{\partial\theta}{\partial z}$

We have:

$N^2 = -\frac{g\theta}{\theta^2}\frac{\partial\theta}{\partial z} = -\frac{g}{\theta}\frac{\partial\theta}{\partial z}$

4) Consider a wave-like disturbance in a stratified fluid of 4km depth. For x- and y wavelengths of 500km, calculate the time required for the phase of the barotropic and first baroclinic mode to travel westward over 6,000km at 30^o N. Use a constant value of $2\times10^{-3}s^{-1}$ for the Brunt-Vaisala frequency.

$H = 4\times10^3[m]; k, l = \frac{1}{5\times10^5}[m^{-1}]; \Delta x = 6\times10^6[m]; \theta = 30[^oN]; N^2 = 2\times10^{-3}[s^{-1}]$

(1) $c_x = \frac{\omega}{k} = \frac{1}{k}[\frac{-\beta k}{k^2+l^2+\lambda_i^{-2}}] = \frac{-\beta}{k^2+l^2+\lambda_i^{-2}}$

$\beta = \frac{\partial f}{\partial y} = \frac{2\Omega cos(\theta)}{a}$, where a is the Earth radius.

- for $i = 0$ (barotropic mode)

```
clear variables
clc
H=4e3;%depth
k=1/5e5;%k wavelength
l=1/5e5;%l wavelength
delX=6e6;%Delta X
lat=30;%Latitude
Omega=7.2921169e-5;%earth angular velocity
a=6.356e6;%Earth radius
f=2*Omega*sind(lat);%cCoriolis parameter
g=9.81;%gravity
beta=2*Omega*cosd(lat)/a;%Beta parameter
lambda0=sqrt(g*H)/f;%Lambda for barotropic mode
Cx0=-beta/(k^2+l^2+(1/lambda0)^2);
delT0=delX/abs(Cx0);
delT0=delT0/(24*60*60); % [days]
```

$\lambda_0 = \frac{\sqrt{gH}}{f} = 2.7165 \times 10^6$

$C_x = -2.44[ms^{-1}]$

$\Delta t = \frac{\Delta x}{C_x} = 2.4567 \times 10^6[s] = 28.43[days]$

- for $i = 1$ (baroclinic mode)

```
N=2e-3;
g_prime=N;
lambda1=sqrt(g_prime.*H)/f;
Cx1=-beta/(k^2+l^2+(1/lambda1)^2);
delT1=delX/abs(Cx1);
delT1=delT1/(24*60*60); % [days]
```

$N^2 = -\frac{g}{\rho_0}\frac{\partial \rho(z)}{\partial z} = g^{'}$

$\lambda_1 = \frac{1}{f}\frac{\sqrt{g^{'}H_1H_2}}{H_1 + H_2} \approx \frac{\sqrt{g^{'}H_1}}{f}$, if $H_1 << H_2$

then:

$\lambda_1 = \frac{\sqrt{N^2H}}{f}$

$C_x = -0.0295[ms^{-1}]$

$\Delta t = \frac{\Delta x}{C_x} = 2.0311 \times 108[s] = 6.44[years]$