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1. Prove that any sorting network on n inputs has depth at least $\lg n$.

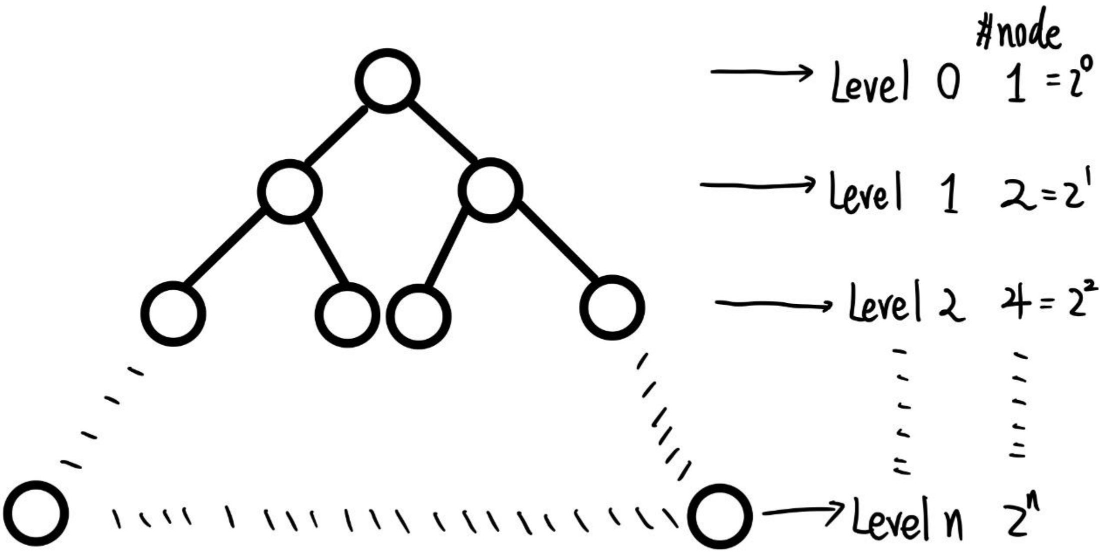
Sol:

假設在排序網路中的任一個輸入元素設為 x。

在 Network 的第 1 層之後，x 最多可能在 2 個不同的位置，經過第 2 層後最多

可能在 4 個位置，以此類推。

可以用一個決策樹來看，



Level0 代表還沒開始比較，Level1 是網路的第一層以此類推，

n 個 input 代表 x 會有 n 個可能的位置，

令樹高為 h，當2^h有 n 個 node，

所以代表 h = lgn，故至少深度為 lgn。 #QED

2. Prove that the number of comparators in any sorting network is $\Omega(n \lg n)$.

Sol:

設 T 為 sortting network 的決策樹，h 為 T 的高度，

假設有 n 個 input，代表說排列數有 n!種，

所以 T 的 leaf 數至少為 n!，高度 h = depth = Runtime = comparators 的次數，所以比較器的數量 = h，

$2^h \geq n!$

$\mathrm{h} \geq \log n!$

$\log n! = \Omega(\mathrm{n}\log n)$ #QED

3. Applying the following keys to the shuffling-based Batcher's odd-even merge network, answer the following questions:

   (a) Keys: A E Q S U Y E I N O S T.

   (b) Keys: 1 0 0 1 1 1 0 0 0 0 0 1 0 1 0 0.

a.

AEQSUYEINOST

#1 AQUENS|ESYIOT

#2 AUN|QES

#3 ANU

#4 AN

#5 U

#6 AUN

#7 ANU

#8 EQ

#9 S

#10 ESQ

#11 EQS

#12 AENQUS

#13 AENQSU

#14 EYOSIT

#15 EY

#16 O

#17 EOY

#18 IS

T

#19 ITS

#20 IST

#21 EIOSYT

#22 EIOSTY

#23 AEEINOQSSTUY*

b.

1 0 0 1 1 1 0 0 0 0 0 1 0 1 0 0

#1 1 0 1 0 0 0 0 0 0 1 1 0 0 1 1 0

#2 1 1 0 0 0 0 0 0

#3 1 0 1 0

#4 0 1

#5 0 1

#6 0 0 1 1

#7 0 0 0 0

#8 0 0 0 0 1 0 1 0

#9 0 0 0 0 0 1 0 1

#10 0 1 0 1 1 0 1 0

#11 0 0 1 1

#12 0 1

#13 0 1

#14 0 0 1 1

#15 0 0 0 0 1 1 1 1

#16 0 0 0 0 0 0 0 0 0 1 1 1 0 1 1 1

#17 0 0 0 0 0 0 0 0 0 1 1 0 1 1 1 1

4. Considering the sorting network SORTER$[n]$, answer the following questions:

(a) How many comparators are there in SORTER$[n]$?

(b) Show that the depth of SORTER$[n]$ is exactly $(\lg n)(\lg n+1)/2$.

a.

MERGER[n] 由 $\frac{n}{2}\lg n$ 個 comparators 組成

$\because$ $\lg n$ 階層，每層有 $n/2$ 個 comparators

令 SORTER[n] 為 $C(n)$ $\Rightarrow$ 由 2 個 SORTER$[\frac{n}{2}]$ + 1 個 MERGER[n] 組成

$$C(n)=2C(\tfrac{n}{2})+\tfrac{n}{2}\lg n \quad C(1)=0$$

$$=2\left[2C(\tfrac{n}{4})+\tfrac{n}{4}\lg\tfrac{n}{2}\right]+\tfrac{n}{2}\lg n=4C(\tfrac{n}{4})+\tfrac{n}{2}(\lg n-1)+\tfrac{n}{2}(\lg n)$$

$$=4\left[2C(\tfrac{n}{8})+\tfrac{n}{8}\lg\tfrac{n}{4}\right]+\tfrac{n}{2}(\lg n-1)+\tfrac{n}{2}\lg n$$

$$=8C(\tfrac{n}{8})+\tfrac{n}{2}(\lg n-2)+\tfrac{n}{2}(\lg n-1)+\tfrac{n}{2}\lg n$$

$$\vdots$$

$$=n\cdot C(1)+\tfrac{n}{2}(1+2+\cdots+(\lg n-1)+\lg n)$$

$$=\frac{n}{2}\times\frac{(\lg n+1)\lg n}{2}=\frac{n}{4}\lg^2 n+\frac{n}{4}\lg n\Rightarrow\Theta(n\lg^2 n) \quad \#$$

b.

$$D(n)=\begin{cases}0 & \text{if } n=1\\ D(\frac{n}{2})+\lg n & \text{if } n=2^k \text{ 且 } k\geq 1\end{cases}$$

By induction on k

step1 當 $k=0$, $n=1$ 則 $\lg 1(\lg 1+1)/2=0=D(1)$ 成立

step2 設 $k-1$ 成立

$$D(\tfrac{n}{2})=\frac{\lg(\frac{n}{2})(\lg(\frac{n}{2})+1)}{2}=\frac{(\lg n-1)(\lg n)}{2}$$

step3
$$\begin{aligned}D(n)&=D(\tfrac{n}{2})+\lg n\\&=\frac{(\lg n-1)(\lg n)}{2}+\lg n\\&=\frac{\lg^2 n+\lg n}{2}\\&=\frac{(\lg n)(\lg n+1)}{2}\end{aligned}$$
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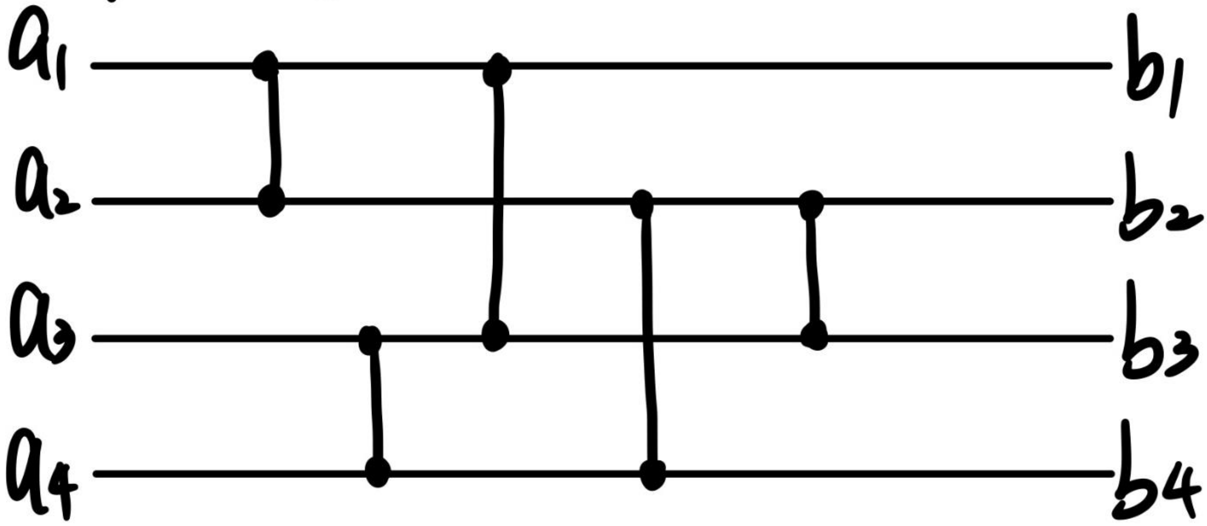
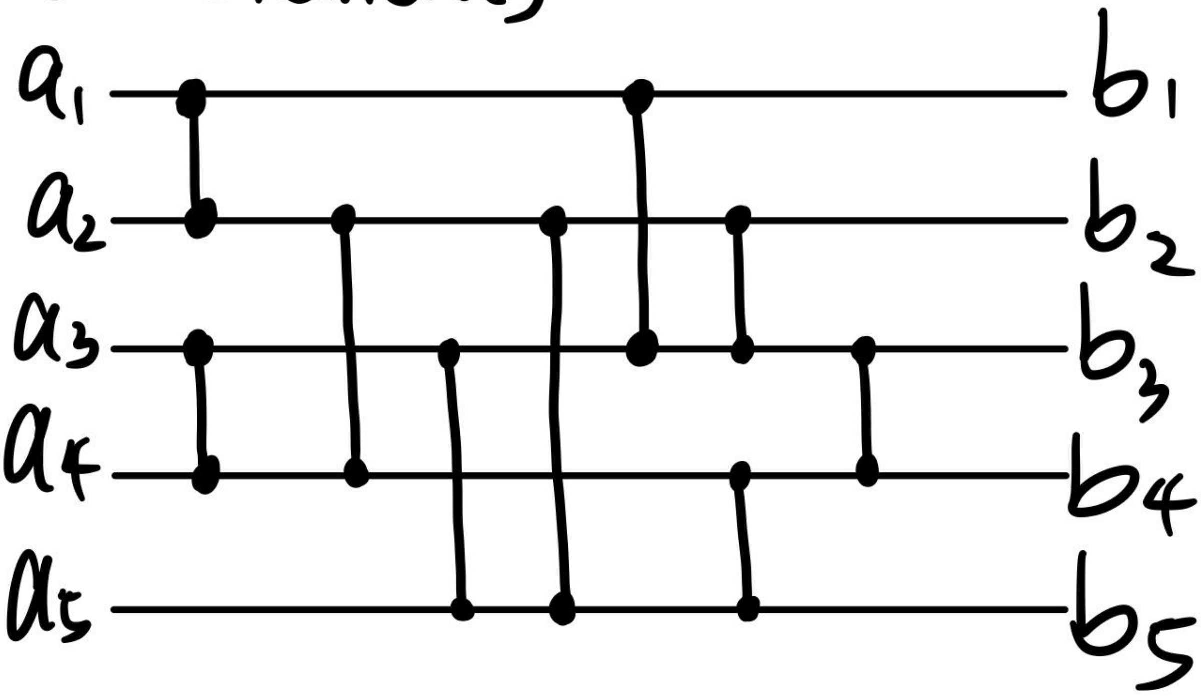
5. Give sorting networks for four, five, and six elements. Use as few comparators as possible.

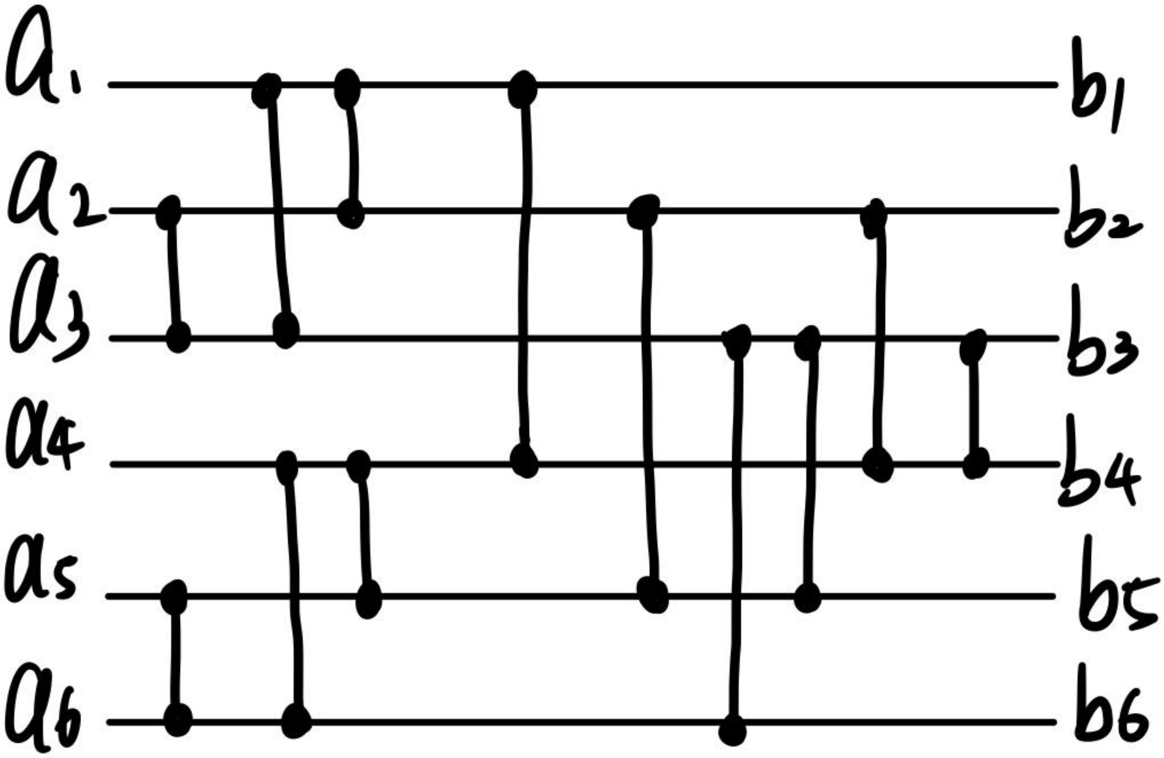
4 elements



5 elements



6 elements



6. The conversion of a Gray codeword into its binary equivalent can be carried out as in the following operations:

$$b_{n-1} = g_{n-1}$$
$$b_i = b_{i+1} \oplus g_i \qquad \text{where } 0 \le i \le n-2$$

(a) Represent each b_i as a function of input g only, where $g = (g_{n-1} \cdots g_1 g_0)$.

$$b_{n-1} = g_{n-1}$$

$$b_{n-2} = b_{n-1} \oplus g_{n-2} = g_{n-1} \oplus g_{n-2}$$

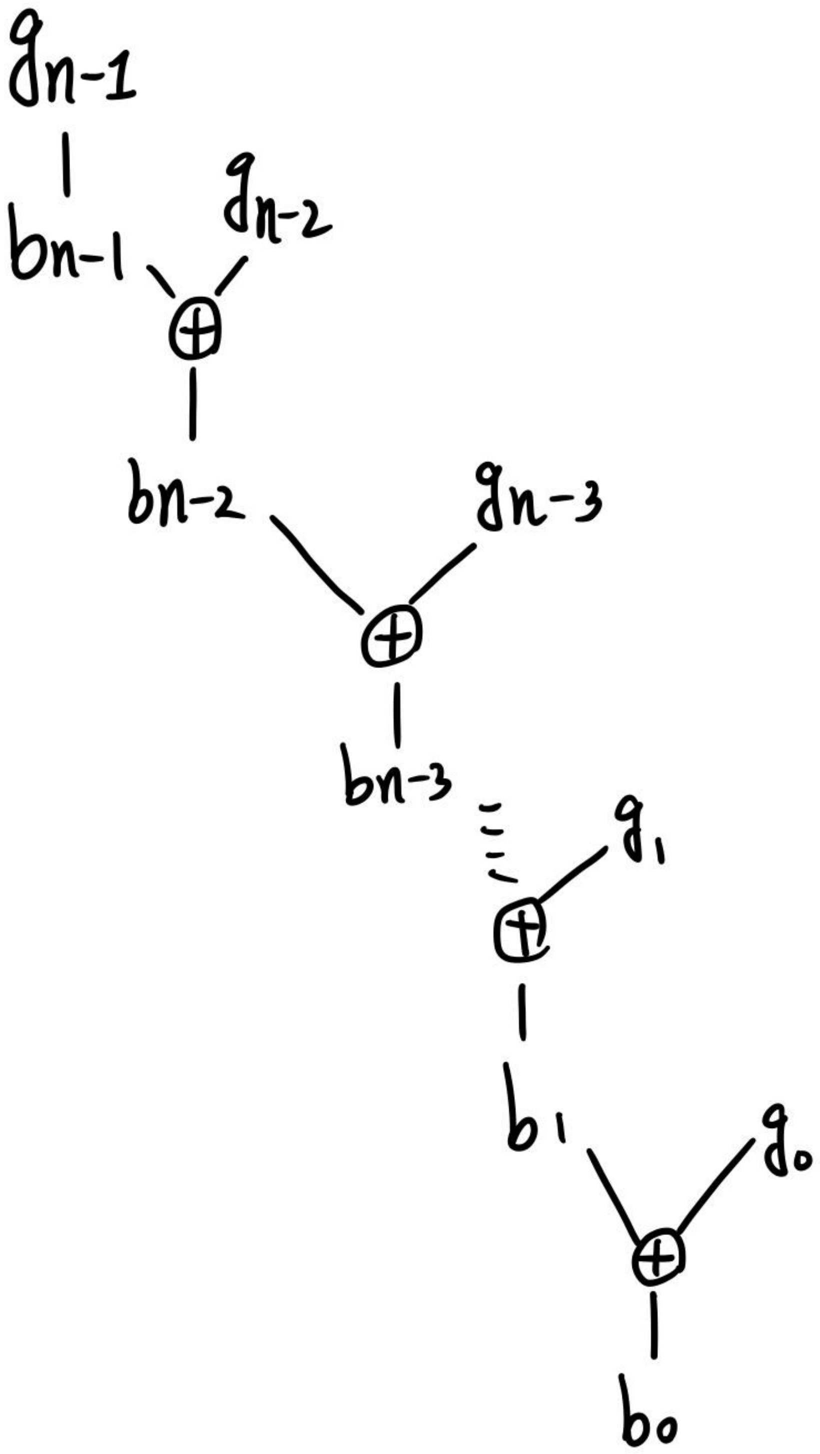
$$b_{n-3} = b_{n-2} \oplus g_{n-3}$$

$$= (g_{n-1} \oplus g_{n-2}) \oplus g_{n-3}$$

$$\vdots$$

$$b_0 = g_{n-1} \oplus g_{n-2} \oplus \cdots \oplus g_0 \quad \#$$

(b) Realize (a) with possible binary trees for each b_i. What is the time and space complexity of (a)?



時間複雜度為 O(n)

空間複雜度為 O(n)

(c) Write each b_i as a prefix-sum expression.

$$s_{n-1} = b_{n-1}$$

$$s_{n-1} \oplus b_{n-2} = s_{n-2} = b_{n-1} \oplus b_{n-2}$$

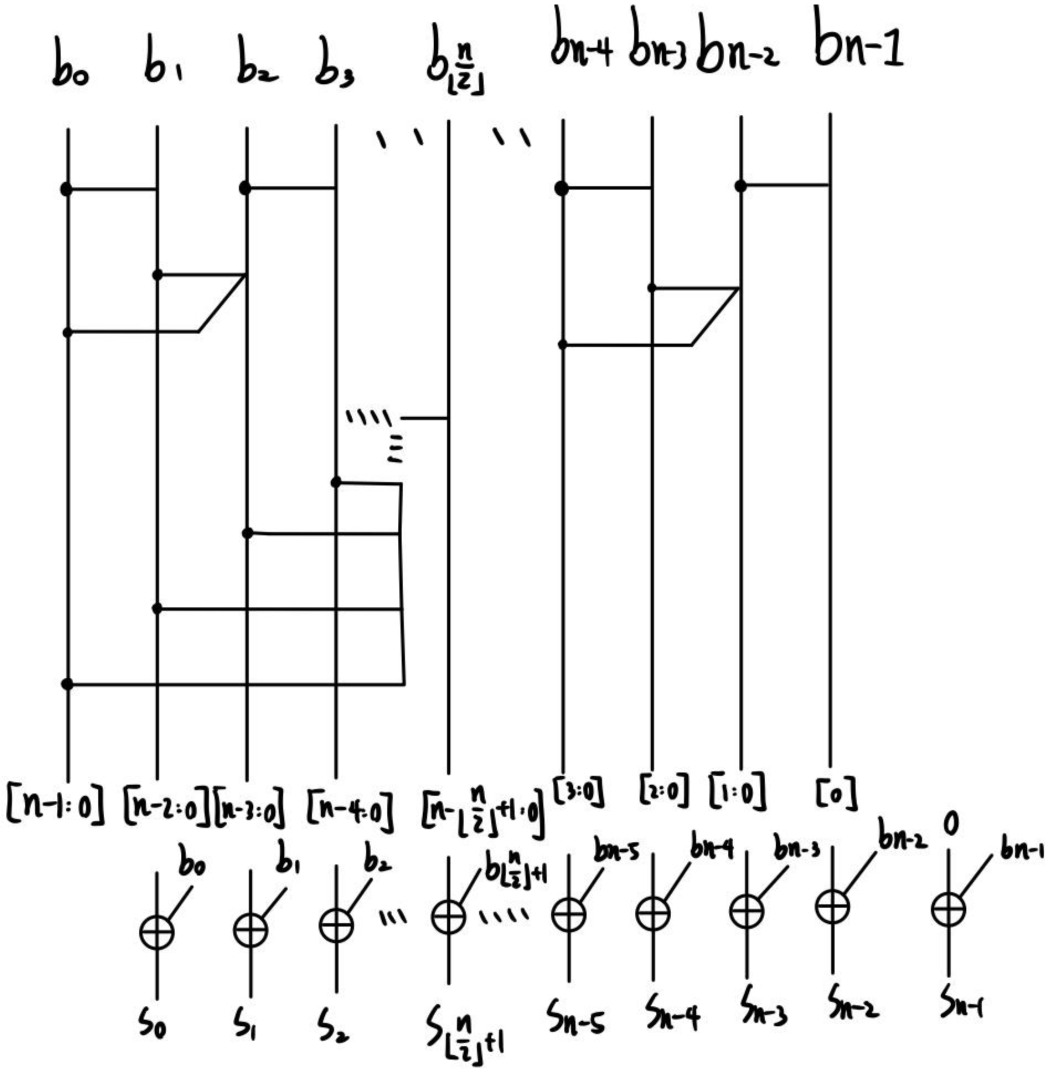
$$s_{n-2} \oplus b_{n-3} = s_{n-3} = b_{n-1} \oplus b_{n-2} \oplus b_{n-3}$$

$$\vdots$$

$$s_2 \oplus b_1 = s_1 = b_{n-1} \oplus \cdots \oplus b_2 \oplus b_1$$

$$s_1 \oplus b_0 = s_0 = b_{n-1} \oplus \cdots \oplus b_2 \oplus b_1 \oplus b_0$$

(d) Implement (c) with the Ladner-Fischer parallel-prefix network. Compare the time and space complexity with (a).



● ⇒ XOR associative binary operator

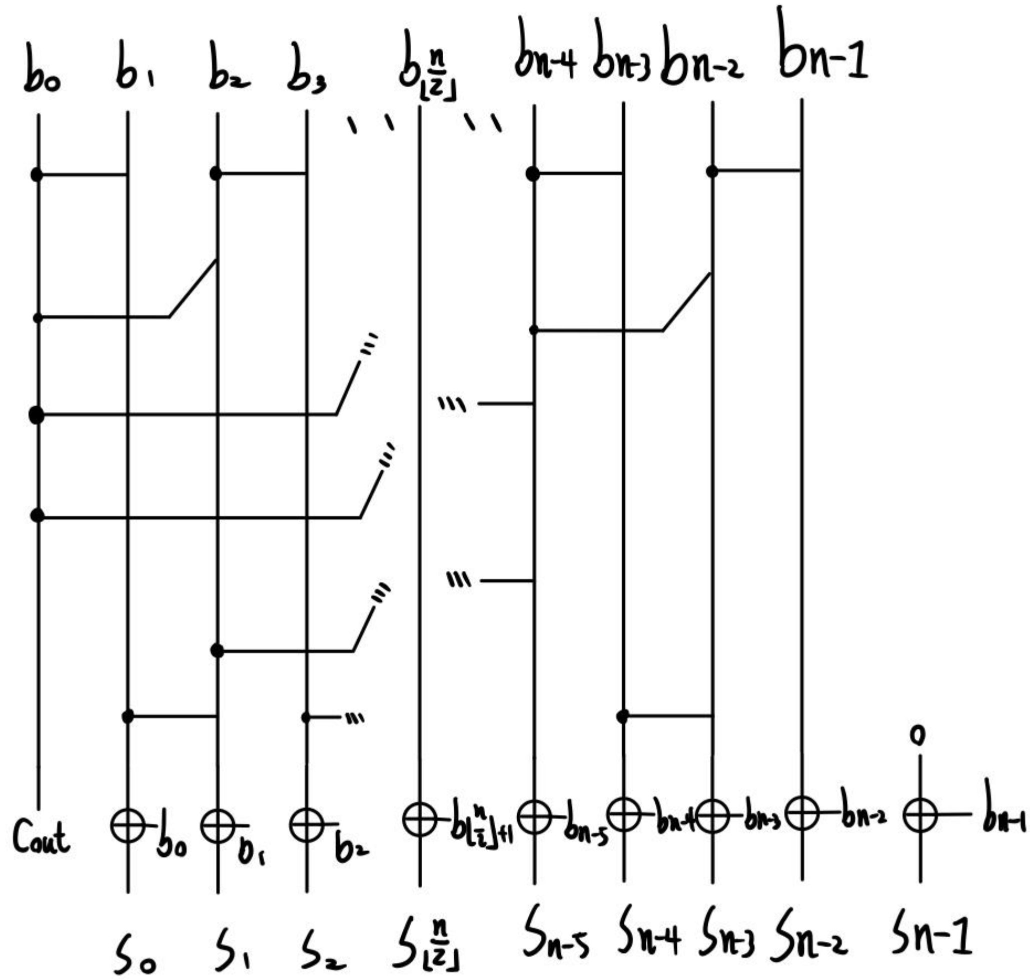
$\oplus$

跟二元樹做比較:

Time complexity O(lgn)

Space complexity O(n)

(e) Implement (c) with the Brent-Kung parallel-prefix network. Compare the time and space complexity with (a) and (d).



● → XOR

⊕

跟前兩者做比較:

Time complexity O(lgn)

Space complexity O(lgn)

跟 d 相比性能類似