1. (10 points) The matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 \\ 62/25 & 38/25 & 9/25 \\ 16/25 & -16/25 & 62/25 \end{bmatrix}. \tag{1}$$

Find the Jordan canonical form $\mathcal{J}$ of the matrix $\mathbf{A}$. The diagonal elements of $\mathcal{J}$ are sorted in the descending order:

$$[\mathcal{J}]_{1,1} \geq [\mathcal{J}]_{2,2} \geq [\mathcal{J}]_{3,3}\,. \tag{2}$$

Sol:

$$\det(A-\lambda I)=0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 0 & 0 \\ \frac{62}{25} & \frac{38}{25}-\lambda & \frac{9}{25} \\ \frac{16}{25} & -\frac{16}{25} & \frac{62}{25}-\lambda \end{vmatrix}$$

$$=(4-\lambda)\left(\frac{38}{25}-\lambda\right)\left(\frac{62}{25}-\lambda\right)-\left[\frac{9}{25}\times\frac{-16}{25}(4-\lambda)\right]$$

$$=(4-\lambda)\left(\frac{2356}{625}-4\lambda+\lambda^2\right)-\left(\frac{144}{625}\lambda-\frac{576}{625}\right)$$

$$=\frac{9424}{625}-16\lambda+4\lambda^2-\frac{2356}{625}\lambda+4\lambda^2-\lambda^3-\frac{144}{625}\lambda+\frac{576}{625}$$

$$=16-20\lambda+8\lambda^2-\lambda^3$$

$$=-(\lambda-4)(\lambda-2)^2$$

$$\Rightarrow -(\lambda-4)(\lambda-2)^2=0 \Rightarrow \lambda=4,2,2$$

$$\therefore J_1=\lambda_1 I_1+U_1=\lambda_1=4$$

$$J_2=\lambda_2 I_2+U_2=\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$J=\mathrm{blkdiag}\left(4,\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\right)=\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

2. (10 points) Let $\mathbf{X} \in \mathbb{C}^{2\times 2}$. Solve the system of equations

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \mathbf{X} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} + \mathbf{X} = \begin{bmatrix} 0 & 8 \\ -1 & 22 \end{bmatrix}. \tag{3}$$

for the matrix $\mathbf{X}$.

Hint: Take the vectorization operator $\mathrm{vec}(\cdot)$ on both sides of (3).

Sol:

$$X = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$$

$$= \begin{bmatrix} \alpha+2\beta & \gamma+2\delta \\ 3\alpha+4\beta & 3\gamma+4\delta \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$$

$$= \begin{bmatrix} 3\alpha+6\beta & \gamma+2\delta \\ 9\alpha+12\beta & 3\gamma+4\delta \end{bmatrix} + \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$$

$$= \begin{bmatrix} 3\alpha+6\beta+\alpha & \gamma+2\delta+\gamma \\ 9\alpha+12\beta+\beta & 3\gamma+4\delta+\delta \end{bmatrix}$$

$$= \begin{bmatrix} 4\alpha+6\beta & 2\gamma+2\delta \\ 9\alpha+13\beta & 3\gamma+5\delta \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ -1 & 22 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 4\alpha+6\beta = 0 \\ 9\alpha+13\beta = -1 \end{cases} \quad \begin{cases} 2\gamma+2\delta = 8 \\ 3\gamma+5\delta = 22 \end{cases}$$

$$\Rightarrow \begin{matrix} \alpha = -3 \\ \beta = 2 \end{matrix} \qquad \Rightarrow \begin{matrix} \gamma = -1 \\ \delta = 5 \end{matrix}$$

$$X = \begin{bmatrix} -3 & -1 \\ 2 & 5 \end{bmatrix} \#$$

3. (10 points) The matrix $\mathbf{B}$ is defined as

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Find the matrix power $\mathbf{B}^{10}$.

Sol:

$$B = \left[\begin{array}{ccc|cc} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right] = \left[\begin{array}{c|c} J_1 & 0 \\ \hline 0 & J_2 \end{array}\right]$$

$$B^{10} = \text{blkdiag}(J_1^{10}, J_2^{10})$$

$$\alpha = 10 \quad L_k = 3 \quad k = 1$$

$$J_1^{10} = \begin{bmatrix} 1 & \binom{10}{1} & \binom{10}{2} \\ 0 & 1 & \binom{10}{1} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 10 & 45 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_k = 2$$

$$J_2^{10} = \begin{bmatrix} 2^{10} & 2^{9}\binom{10}{1} \\ 0 & 2^{10} \end{bmatrix} = \begin{bmatrix} 1024 & 5120 \\ 0 & 1024 \end{bmatrix}$$

$$\therefore B^{10} = \begin{bmatrix} 1 & 10 & 45 & 0 & 0 \\ 0 & 1 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1024 & 5120 \\ 0 & 0 & 0 & 0 & 1024 \end{bmatrix}$$

4. (20 points) Let $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathsf{H}}$ be the SVD of the matrix $\mathbf{A}$. We assume that $[\mathbf{U}]_{1,n} > 0$ for all n. Determine the matrices $\mathbf{U}$, $\boldsymbol{\Sigma}$, and $\mathbf{V}$ for the following matrices:

(a) (10 points)

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}. \tag{5}$$

Sol:

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \lambda_1 = 3,\ \lambda_2 = 2, \quad \sigma_1 = \sqrt{3},\ \sigma_2 = \sqrt{2}$$

$$\longrightarrow \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$V(3) = \ker(A^T A - 3I) = \ker\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}, \quad \underline{v_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$V(2) = \ker\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}, \quad \underline{v_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\underline{u_1} = \frac{1}{\sigma_1} A \underline{v_1} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\underline{u_2} = \frac{1}{\sigma_2} A \underline{v_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$N(A^T) = N\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}\right\}$$

$$\underline{u_3} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

$$\therefore A = U\Sigma V^H = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(b) (10 points)

$$\mathbf{A} = \begin{bmatrix} 1 \\ \jmath \end{bmatrix},$$

where $\jmath = \sqrt{-1}$.

Sol:

$$A^H A = \begin{bmatrix} 1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ j \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$$

$$\lambda = 2, \quad \sigma = \sqrt{2}$$

$$V(2) = \ker(A^H A - 2I) = \ker[0] = \operatorname{span}\{[1]\}$$

$$\underline{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ j \end{bmatrix} [1] = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix}$$

$$N(A^H) = N(\begin{bmatrix} 1 & -j \end{bmatrix}) = \operatorname{span}\left\{ \begin{bmatrix} j \\ 1 \end{bmatrix} \right\} \Rightarrow \underline{u_2} = \begin{bmatrix} \frac{j}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{j}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A = U \Sigma V^H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{j}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

5. (10 points) We consider the matrix $\mathbf{M}$ as follows:

$$\mathbf{M} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Determine the matrix p-norms

$$\|\mathbf{M}\|_1, \quad \|\mathbf{M}\|_2, \quad \|\mathbf{M}\|_\infty,$$

and the nuclear norm

$$\|\mathbf{M}\|_*.$$

Sol:

$$M = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ -1 & 5 \\ 0 & 0 \end{bmatrix}$$

$$\|M\|_1 = \max_{1 \le j \le N} \sum_{i=1}^{M} \left| [M]_{i,j} \right| = \max\{6, 0\} = 6$$

$$\|M\|_\infty = \max_{1 \le i \le N} \sum_{j=1}^{N} \left| [M]_{i,j} \right| = \max\{2, 4, 0\} = 4$$

$$\|M\|_2 = \sigma_1 = 3$$

$$\|M\|_* = \|\sigma\|_1 = \sum_{i=1}^{q} \sigma_i = 3 + 1 = 4$$

6. (10 points) Let the data vectors be

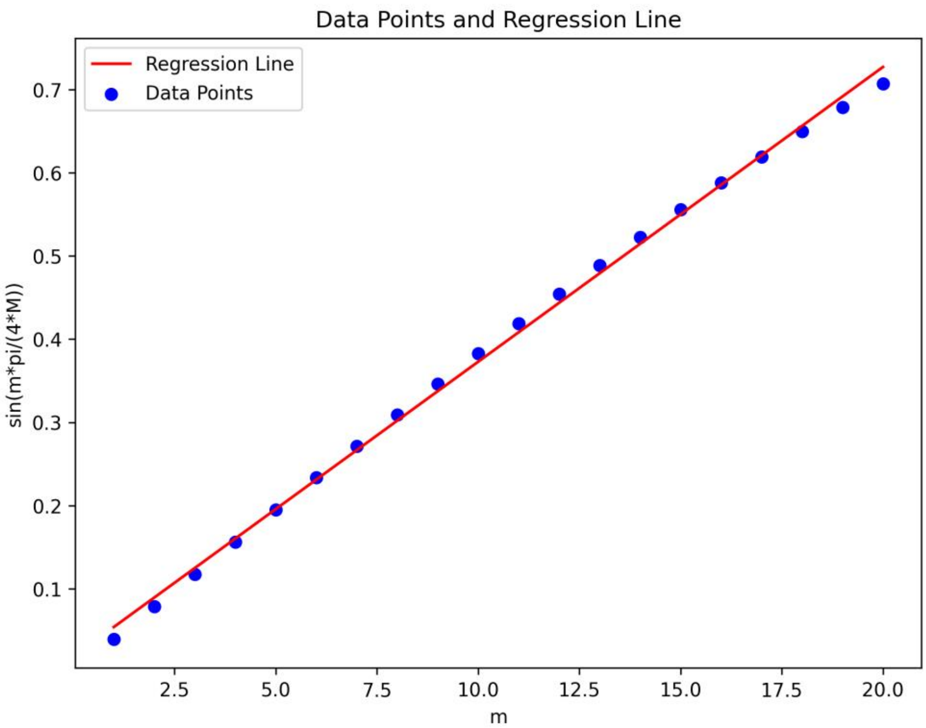
$$\mathbf{x}_m \triangleq \begin{bmatrix} m & \sin\left(\frac{m\pi}{4M}\right) \end{bmatrix}, \tag{10}$$

where $m = 1, 2, \ldots, M$. We suppose that $M = 20$. We can derive a regression line in $\mathbb{R}^2$ based on (10) and the PCA with $L = 1$. Use MATLAB or Python to plot these data vectors and the this regression line. The code should be handed out by NTUCool.
Hint: The horizontal axis is $[\mathbf{x}]_1$ and the vertical axis is $[\mathbf{x}]_2$.

Sol:

```
Data point coordinates:
(1.0, 0.03925981575906861)
(2.0, 0.07845909572784494)
(3.0, 0.11753739745783764)
(4.0, 0.15643446504023087)
(5.0, 0.19509032201612825)
(6.0, 0.2334453638559054)
(7.0, 0.27144044986507426)
(8.0, 0.3090169943749474)
(9.0, 0.34611705707749296)
(10.0, 0.3826834323650898)
(11.0, 0.418659737537428)
(12.0, 0.45399049973954675)
(13.0, 0.4886212414969549)
(14.0, 0.5224985647159488)
(15.0, 0.5555702330196022)
(16.0, 0.5877852522924731)
(17.0, 0.619093949309834)
(18.0, 0.6494480483301837)
(19.0, 0.6788007455329417)
(20.0, 0.7071067811865476)

Mean of xm: (10.5, 0.3905529723350541)
```



7. (10 points) Let $\mathbf{x} \in \mathbb{C}^2$. We consider the system of equations

$$\begin{bmatrix} 1 & 0 \\ \jmath & \jmath \\ 0 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

where $\jmath = \sqrt{-1}$. Determine the LS solution $\mathbf{x}_{\mathrm{LS}}$ to (11).

Sol:

$$\Rightarrow \underline{\underline{A}} = \begin{bmatrix} 1 & 0 \\ j & j \\ 0 & 2 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\underline{\underline{A}}^{+} = (A^H A)^{-1} A^H = \left(\begin{bmatrix} 1 & -j & 0 \\ 0 & -j & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ j & j \\ 0 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & -j & 0 \\ 0 & -j & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -j & 0 \\ 0 & -j & 2 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -j & 0 \\ 0 & -j & 2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5 & -4j & -2 \\ -1 & -j & 4 \end{bmatrix}$$

$$X_{LS} = A^{+} b = \frac{1}{9} \begin{bmatrix} 5 & -4j & -2 \\ -1 & -j & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 7-4j \\ -5-j \end{bmatrix} \#$$

8. (10 points) The matrix $\mathbf{A}$ is defined as

$$\mathbf{A} \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

Determine the pseudo-inverse of $\mathbf{A}$.

Sol:

$$\underline{\underline{A}} \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 4 & 0 \\ 0 & 5 \\ 0 & 10 \\ 0 & 20 \end{bmatrix}$$

$$A^{+} = (A^{H}A)^{-1}A^{H}$$

$$= \left(\begin{bmatrix} 1 & 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 10 & 20 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 4 & 0 \\ 0 & 5 \\ 0 & 10 \\ 0 & 20 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 10 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 \\ 0 & 525 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 10 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{21} & 0 \\ 0 & \frac{1}{525} \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 10 & 20 \end{bmatrix}$$

$$= \frac{1}{105} \begin{bmatrix} 5 & 10 & 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{bmatrix}$$

9. (10 points) Find the optimal solution to the optimization problem

$$\begin{array}{ll} \underset{\mathbf{x}\in\mathbb{C}^3}{\text{minimize}} & \|\mathbf{x}\|_2^2 \\ \text{subject to} & 3 + \begin{bmatrix} 2 & \jmath & 1 \end{bmatrix} \mathbf{x} = 0, \end{array}$$

where $\jmath = \sqrt{-1}$.

Sol:

$$\min_{x\in\mathbb{C}^3} \|X\|_2^2 \Rightarrow X = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in C^3 \;\middle|\; 2x_1 + jx_2 + x_3 = -3 \right\}$$

$$A = [2 \; j \; 1] \quad b = -3$$

$\downarrow$

SVD

$$A = U\Sigma V^T = [1][\sqrt{6} \;\; 0 \;\; 0] \begin{array}{c} \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} \\ \begin{bmatrix} \frac{\sqrt{6}}{3} & \frac{j\sqrt{6}}{6} & \frac{\sqrt{6}}{6} \\ \frac{j\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} & 0 \\ \frac{-\sqrt{30}}{15} & \frac{-j\sqrt{30}}{6} & \frac{\sqrt{30}}{6} \end{bmatrix} \end{array}$$

$$U_1 = 1 \quad \sigma_1 = \sqrt{6}$$

$$\Rightarrow \frac{1^H x - 3}{\sqrt{6}} = \frac{3}{\sqrt{6}}$$

$$X_{LS} = \frac{3}{\sqrt{6}} \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \frac{j\sqrt{5}}{5} \\ \frac{-\sqrt{30}}{15} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3j\sqrt{5}}{5\sqrt{6}} \\ \frac{-\sqrt{5}}{5} \end{bmatrix}$$