(1) Solve the following PDEs. (30 scores)

(a) $\frac{\partial u(x,y,z)}{\partial x}+y\frac{\partial u(x,y,z)}{\partial y}+z^2\frac{\partial u(x,y,z)}{\partial z}=0$

$$X'YZ + Y(XY'Z) + Z^2(XYZ') = 0$$

$$X'YZ = -y(XY'Z) - z^2(XYZ') = 0$$

$$\frac{X'}{X} = -y\frac{Y'}{Y} - z^2\frac{Z'}{Z} = -\lambda$$

$$-y\frac{Y'}{Y} = z^2\frac{Z'}{Z} + \lambda = -\mu$$

$$X' + \lambda X = 0, \quad -yY' + \mu Y = 0, \quad z^2 Z' + (\lambda+\mu)Z = 0$$

$$X' + \lambda X = 0 \Rightarrow X(x) = C_1 e^{-\lambda x}$$

$$-yY' + uY = 0 \Rightarrow Y(y) = C_2 y^u$$

$$z^2 Z' + (\lambda + u)Z = 0 \Rightarrow Z(z) = C_3 z^{-2(\lambda+u)}$$

$$\Rightarrow u(x,y,z) = \sum_\lambda \sum_u C_{\lambda,u}\, e^{-\lambda x}\, y^u\, z^{-2(\lambda+u)}$$

(b) $\dfrac{\partial u(x,y,z)}{\partial x}+\dfrac{\partial u(x,y,z)}{\partial y}+\dfrac{\partial u(x,y,z)}{\partial z}=x+y+z$

$$u(x,y,z) = v(x,y,z)+\psi_1(x)+\psi_2(y)+\psi_3(z)$$

$$\frac{\partial v(x,y,z)}{\partial x}+\frac{\partial \psi_1(x)}{\partial x}+\frac{\partial v(x,y,z)}{\partial y}+\frac{\partial \psi_2(y)}{\partial y}$$

$$+\frac{\partial v(x,y,z)}{\partial z}+\frac{\partial \psi_3(z)}{\partial z}=x+y+z$$

A: $\psi_1'(x)=x \Rightarrow \psi_1(x)=\frac{1}{2}x^2+C_1$

B: $\psi_2'(y)=y \Rightarrow \psi_2(y)=\frac{1}{2}y^2+C_2$

C: $\psi_3'(z)=z \Rightarrow \psi_3(z)=\frac{1}{2}z^2+C_3$

D: $\dfrac{\partial v(x,y,z)}{\partial x}+\dfrac{\partial v(x,y,z)}{\partial y}+\dfrac{\partial v(x,y,z)}{\partial z}=0$

$$\Rightarrow \frac{X'}{X}+\frac{Y'}{Y}+\frac{Z'}{Z}=0 \Rightarrow \frac{X'}{X}=-\frac{Y'}{Y}-\frac{Z'}{Z}=-\lambda$$

$$\frac{Y'}{Y}=-\frac{Z'}{Z}+\lambda=-u$$

$$\Rightarrow X'+\lambda X=0,\ Y'+uY=0,\ Z'-(\lambda+u)Z=0$$

$$\Rightarrow X=C_4e^{-\lambda x},\ Y=C_5e^{-uy}\ \ Z=C_6e^{(\lambda+u)z}$$

$$v(x,y,z)=\sum_{\lambda}\sum_{u}C_{\lambda,u}e^{-\lambda x-uy+(\lambda+u)z}$$

$$u(x,y,z)=\sum_{\lambda}\sum_{u}C_{\lambda,u}e^{-\lambda x-uy+(\lambda+u)z}+\frac{1}{2}x^2+\frac{1}{2}y^2+\frac{1}{2}z^2+A$$

$(A=C_1+C_2+C_3)$ ※

(c) $\dfrac{\partial^2 u(x,y,t)}{\partial x^2}+\dfrac{\partial^2 u(x,y,t)}{\partial y^2}=\dfrac{\partial u(x,y,t)}{\partial t} \qquad 0\le x\le 2,\quad 0\le y\le 2,\quad t\ge 0$

$u(0,y,t)=u(2,y,t)=u(x,0,t)=u(x,2,t)=0$

$u(x,y,0)=\left(2x-x^2\right)\left(2y-y^2\right)$

$u=X(x)Y(y)T(t)$

$X''YT+XY''T=XYT'$

$\dfrac{X''}{X}+\dfrac{Y''}{Y}=\dfrac{T'}{T} \qquad X(0)=X(2)=Y(0)=Y(2)=0$

$\dfrac{X''}{X}=-\lambda=-\alpha^2,\ \alpha=\dfrac{n\pi}{2},\ X(x)=C_1\sin\left(\dfrac{n\pi x}{2}\right)$

$\dfrac{Y''}{Y}=-\mu=-\beta^2,\ \beta=\dfrac{m\pi}{2},\ Y(y)=C_2\sin\left(\dfrac{m\pi y}{2}\right)$

$\dfrac{T'}{T}=-\mu-\lambda \Rightarrow T'+(\mu+\lambda)T=0,\ T(t)=C_3e^{(-\mu-\lambda)t}=C_3e^{(-n^2-m^2)\frac{\pi^2}{4}t}$

$u(x,y,t)=\sum_m\sum_n A_{m,n}\sin\left(\dfrac{n\pi x}{2}\right)\sin\left(\dfrac{m\pi y}{2}\right)e^{(-n^2-m^2)\pi^2 t}$

If $t=0$

$\Rightarrow (2x-x^2)(2y-y^2)=\sum_m\sum_n A_{m,n}\sin\left(\dfrac{n\pi x}{2}\right)\sin\left(\dfrac{m\pi y}{2}\right)$

$\Rightarrow A_{mn}=\frac{2}{2}\int_0^2\frac{2}{2}\int_0^2(2x-x^2)(2y-y^2)\sin\left(\frac{n\pi x}{2}\right)dx\sin\left(\frac{m\pi y}{2}\right)dy$

$=\int_0^2\int_0^2(2x-x^2)(2y-y^2)\sin\left(\frac{n\pi x}{2}\right)dx\sin\left(\frac{m\pi y}{2}\right)dy$

$=\dfrac{64\left(\left(2\cdot(-1)^m-2\right)\left(2\cdot(-1)^n-2\right)\right)}{\pi^6 m^3 n^3}$

$u(x,y,t)=\sum_m\sum_n\dfrac{64\left(\left(2\cdot(-1)^m-2\right)\left(2\cdot(-1)^n-2\right)\right)}{\pi^6 m^3 n^3}\sin\left(\dfrac{n\pi x}{2}\right)\sin\left(\dfrac{m\pi y}{2}\right)$ #

(2) Solve the steady temperature $u(r,\theta)$ in a fan-shape plane where

$0\le r\le 1,\quad 0\le\theta\le\pi/3,$

$u(1,\theta)=\sin(6\theta)+\sin(12\theta),\quad 0<\theta<\pi/3$

$u(r,0)=0,\quad u(r,\pi/3)=0,\quad 0<r<1$ (10 scores)

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$$

#1 令 $u(r,\theta)=R(r)\Theta(\theta)$

$$R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}R\Theta''=0$$

$$r^2\frac{R''}{R}+r\frac{R'}{R}+\frac{\Theta''}{\Theta}=0$$

#2 $\frac{r^2R''+rR'}{R}=-\frac{\Theta''}{\Theta}=\lambda \Rightarrow \begin{array}{l} r^2R''+rR'-\lambda R=0 \\ \Theta''+\lambda\Theta=0 \end{array}$

#3 $\because$ only when $\lambda=\alpha^2>0$, we will have non-trivial sol

$\therefore\ \Theta''(\theta)+\alpha^2\Theta(\theta)=0 \Rightarrow \Theta(\theta)=C_1\cos\alpha\theta+C_2\sin\alpha\theta$

$\Theta(0)=0,\ \Theta(\frac{\pi}{3})=0 \Rightarrow C_1=0,\ C_2\sin\frac{\alpha\pi}{3}=0 \quad \frac{\alpha\pi}{3}=n\pi \Rightarrow \alpha=3n,\ n\in N$

$\Rightarrow \Theta(\theta)=C_2\sin 3n\theta,\ n\in N$

$\lambda=9n^2$

#4 $r^2R''+rR'-\lambda R=0 \Rightarrow r^2R''+rR'-9n^2R=0$

$R(r)=C_3r^{3n}+C_4r^{-3n} \Rightarrow r=0,\ C_4=0 \therefore R(r)=C_3r^{3n},\ n\in N$

#5 $U_n(r,\theta)=R(r)\Theta(\theta)=A_nr^{3n}\sin(3n\theta) \Rightarrow u(r,\theta)=\sum_{n=1}^{\infty}A_nr^{3n}\sin 3n\theta$

#6 From $u(1,\theta)=\sin(6\theta)+\sin(12\theta)=\sum_{n=1}^{\infty}A_n\sin 3n\theta$

$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{P}x,\ 0<x<P \quad \leftarrow A_n=b_n,\ P=\frac{\pi}{3}$

$b_n=\frac{2}{P}\int_0^P f(x)\sin\frac{n\pi}{P}x\,dx \Rightarrow A_n=\frac{2}{\pi/3}\int_0^{\frac{\pi}{3}}(\sin(6\theta)+\sin(12\theta))\sin(3n\theta)\,d\theta$

$=\frac{-6}{\pi}\cdot\frac{1}{12}(-6\cos(6\theta)-\cos(12\theta))\frac{-1}{3n}\cos(3\theta n)\Big|_0^{\frac{\pi}{3}}$

$=\frac{-1}{2\pi}(6+1)\times\frac{-1}{3n}\cos(\pi n)-(-\frac{3}{n\pi})=\frac{7(-1)^n+18}{6n\pi}$

#7 $u(r,\theta)=\sum_{n=1}^{\infty}\frac{7(-1)^n+18}{6n\pi}\sin 3n\theta$ #

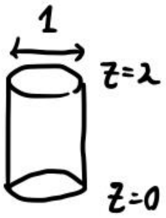
(3) Solve the steady temperature $u(r, z)$ in a cylinder region where

$0\le r\le 1,\quad 0\le z\le 2,$

$u(1,z)=z$ for $0<z<1,\quad u(1,z)=2-z$ for $1<z<2,$

$u(r,0)=0,\ \ u(r,2)=0\ \ 0<r<1$

Suppose that $u(r, z)$ is independent of θ. (10 scores)



$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0$$

$$Z(0)=0,\ Z(2)=0$$

#1 $u(r,z)=R(r)Z(z)$

$R''(r)Z(z)+\frac{1}{r}R'(r)Z(z)+R(r)Z''(z)=0$

#2 $\dfrac{R''+\frac{1}{r}R'}{R}=-\dfrac{Z''}{Z}=-\lambda$

$rR''+R'+\lambda rR=0,\quad Z''-\lambda Z=0$

From $u(r,0)=0,\ u(r,2)=0$

$Z(0)=0,\ Z(2)=0$

#3 $\lambda<0,\ \lambda=-\alpha^2$

$Z''+\alpha^2 Z=0$

$Z(z)=C_1\cos\alpha z+C_2\sin\alpha z$, $Z(2)=0$, $C_2\sin 2\alpha=0$, $\alpha=\frac{n\pi}{2}$

$\Rightarrow Z(z)=C_2\sin\alpha_n z \rightarrow \alpha_n=\frac{n\pi}{2},\ n=1,2,3\cdots,\ \lambda_n=-\alpha_n^2=-\frac{n^2\pi^2}{4}$

#4 $rR''+R'+\lambda rR=0$

$rR''+R'-\frac{n^2\pi^2}{4}rR=0 \rightarrow r^2R''+rR'-\frac{n^2\pi^2}{4}r^2R=0$

$x^2y''+xy'-(\alpha^2x^2+v^2)y=0 \rightarrow C_1I_v(\alpha x)+C_2K_v(\alpha x)$

$\Rightarrow R(r)=C_3I_0(\frac{n\pi}{2}r)+C_4K_0(\frac{n\pi}{2}r)$

since $K_0(0)\rightarrow\infty \Rightarrow R(r)=C_3I_0(\frac{n\pi}{2}r)$

#5 $u(r,z)=R(r)Z(z)\quad u_n(r,z)=A_nI_0(\frac{n\pi}{2}r)\sin(\frac{n\pi}{2}z)$

#6 $u(r,z)=\sum_{n=1}^{\infty}A_nI_0(\frac{n\pi}{2}r)\sin(\frac{n\pi}{2}z)$

#7 $u(1,z)=z,\ u(2,z)=2-z$

$A_nI_0(\frac{n\pi}{2})=\int_0^2 z\sin(\frac{n\pi}{2}z)\,dz=\frac{2}{n\pi}\cos(\frac{n\pi}{2}z)\Big|_0^2=\frac{2}{n\pi}\cos(n\pi)$

$A_nI_0(n\pi)=\int_0^2(2-z)\sin(\frac{n\pi}{2}z)\,dz=(z-2)\frac{2}{n\pi}\cos(\frac{n\pi}{2}z)\Big|_0^2=\frac{4}{n\pi}$

$A_n=\dfrac{2\cos(n\pi)}{n\pi I_0(\frac{n\pi}{2})}=\dfrac{4}{n\pi I_0(n\pi)}\Rightarrow\dfrac{1}{2}\dfrac{\cos(n\pi)I_0(n\pi)}{I_0(\frac{n\pi}{2})}$

$u(r,z)=\dfrac{1}{2}\sum_{n=1}^{\infty}\dfrac{\cos(n\pi)I_0(n\pi)}{I_0(\frac{n\pi}{2})}\sin(\frac{n\pi}{2}z)$ #

(4) Solve the following PDE by the 1-sided Laplace transform.

(10 scores)

$$\frac{\partial u(x,t)}{\partial x}+\frac{\partial u(x,t)}{\partial t}=1, \quad x>0, \quad t>0$$

$$u(0,t)=t^2+t, \quad u(x,0)=0$$

(Hint): When solving the ODE of x, the constant may be a function of s.

$$\mathcal{L}_{t\to s}\left\{\frac{\partial u(x,t)}{\partial x}\right\}+\mathcal{L}_{t\to s}\left\{\frac{\partial u(x,t)}{\partial t}\right\}=\mathcal{L}_{t\to s}\{1\}$$

$$\frac{d}{dx}\mathcal{L}_{t\to s}\left\{\frac{\partial u(x,t)}{\partial x}\right\}=U(x,s)$$

$$\frac{\partial U(x,s)}{\partial x}+\frac{\partial U(x,s)}{\partial t}=\frac{1}{s} \Rightarrow \frac{\partial U(x,s)}{\partial x}+sU(x,s)-u(x,0)=\frac{1}{s}$$

$$\underline{\frac{\partial}{\partial x}U(x,s)+sU(x,s)=\frac{1}{s}}$$

$\uparrow$

$U_c(x,s)=C_1e^{-sx}$ $\quad U_p(x,s)=C_2 \quad C_2=\frac{1}{s^2}$

$\therefore U=C_1e^{-sx}+\frac{1}{s^2}$

by $u(0,t)=t^2+t$, $U(0,s)=\mathcal{L}_{t\to s}\{u(0,t)\}=C_1+\frac{1}{s^2}=\frac{2}{s^3}+\frac{1}{s^2}\Rightarrow C_1=\frac{2}{s^3}$

$\therefore U=\frac{2}{s^3}e^{-sx}+\frac{1}{s^2}$

$u(x,t)=\mathcal{L}^{-1}_{s\to t}\left\{\frac{2}{s^3}e^{-sx}+\frac{1}{s^2}\right\}=(t-x)^2\,U(t-x)+t$ #

(5) (a) Convert 1, x, and x^2 into an orthonormal function set for $x \in [0, 4]$.

a. $f_0(x) = 1$, $f_1(x) = x$, $f_2(x) = x^2$

$h_0(x) = \langle h_1, h_1 \rangle = \int_0^4 1\, dx = 4$

$h_1(x) = x - \frac{\langle x, 1 \rangle}{4} 1 = x - \frac{1}{4}\int_0^4 x\, dx = x - \frac{1}{4}\left[\frac{1}{2}x^2 + c\right]_0^4 = x - 2 \Rightarrow \langle h_1, h_1 \rangle = \frac{16}{3}$

$h_2(x) = x^2 - \frac{\langle x^2, 1 \rangle}{4} - \frac{\langle x^2, x \rangle}{16/3} \cdot x = x^2 - \frac{1}{4}\int_0^4 x^2 dx - \frac{x}{16/3}\int_0^4 x^3 dx$

$= x^2 - \frac{1}{4}\left[\frac{1}{3}x^3 + c\right]_0^4 - \frac{x}{16/3}\left[\frac{1}{4}x^4 + c\right]_0^4$

$= x^2 - \frac{16}{3} - 12x \Rightarrow \langle h_2, h_2 \rangle = \frac{119296}{45}$

$\text{Ans} = \left\{ \frac{1}{2}, \frac{x-2}{\sqrt{\frac{16}{3}}}, \frac{x^2 - 12x - \frac{16}{3}}{\sqrt{\frac{119296}{45}}} \right\}$ #

(b) Approximate min(x, 4-x) by a 2nd order polynomial with the least mean square error for $x \in [0, 4]$. (20 scores)

b. $g(x) = 4 - x$, $f(x) = C_0 P_0(x) + C_1 P_1(x) + C_2 P_2(x)$

$\int_0^4 (g(x) - f(x))^2 dx$ is minimize

$C_0 = \frac{1}{2}\int_{-1}^{1} g(x) P_0(x) dx = \frac{1}{2}\left[4x - \frac{1}{2}x^2\right]_{-1}^{1} = 4$

$C_1 = \frac{3}{2}\int_{-1}^{1} g(x) P_1(x) dx = \frac{3}{2}\left[2x^2 - \frac{1}{3}x^3\right]_{-1}^{1} = -1$

$C_2 = \frac{5}{2}\int_{-1}^{1} g(x) P_2(x) dx = \frac{5}{2}\left[-\frac{3}{8}x^4 + 2x^3 + \frac{1}{4}x^2 - 2x\right]_{-1}^{1}$

$= 0$

$f(x) = 4P_0(x)\left(\frac{2}{4}\left(x - \frac{4}{2}\right)\right) + P_1(x)\left(\frac{2}{4}\left(x - \frac{4}{2}\right)\right)$

$= 2x - 4 - \frac{1}{2}x^2 - x$

$= -\frac{1}{2}x^2 + 3x - 4$

(6) Suppose that there is a set of five 'discrete' basis.

$$b_k[n] = n^k \qquad n = -6, -5, \cdots, 5, 6$$

$$k = 0, 1, 2, 3, 4$$

(a) Write a code to use the Gram-Schmidt method to convert $b_k[n]$ ($k = 0\sim4$) into an orthonormal set.

(b) Write a code to use the Gram-Schmidt method to convert $b_k[n]$ ($k = 0\sim4$) into an orthonormal set if the weight is $w[n] = 1 - |n|/7$.

The codes should be handed out by NTUCool. (20 scores)

Sol.:

a.

```
[[ 2.77350098e-01  2.77350098e-01  2.77350098e-01  2.77350098e-01
   2.77350098e-01  2.77350098e-01  2.77350098e-01  2.77350098e-01
   2.77350098e-01  2.77350098e-01  2.77350098e-01  2.77350098e-01
   2.77350098e-01]
 [-4.44749590e-01 -3.70624658e-01 -2.96499727e-01 -2.22374795e-01
  -1.48249863e-01 -7.41249317e-02 -1.36947500e-17  7.41249317e-02
   1.48249863e-01  2.22374795e-01  2.96499727e-01  3.70624658e-01
   4.44749590e-01]
 [ 4.91689172e-01  2.45844586e-01  4.46990156e-02 -1.11747539e-01
  -2.23495078e-01 -2.90543602e-01 -3.12893109e-01 -2.90543602e-01
  -2.23495078e-01 -1.11747539e-01  4.46990156e-02  2.45844586e-01
   4.91689172e-01]
 [-4.59933106e-01  2.13634429e-16  2.50872603e-01  3.34496804e-01
   2.92684704e-01  1.67248402e-01  2.43838963e-17 -1.67248402e-01
  -2.92684704e-01 -3.34496804e-01 -2.50872603e-01 -1.82489361e-16
   4.59933106e-01]
 [ 3.79457988e-01 -2.52971992e-01 -3.67959261e-01 -2.06977084e-01
   4.21619987e-02  2.45306174e-01  3.21964353e-01  2.45306174e-01
   4.21619987e-02 -2.06977084e-01 -3.67959261e-01 -2.52971992e-01
   3.79457988e-01]]
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b.

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[[ 7.33799386e-01  7.33799386e-01  7.33799386e-01  7.33799386e-01
   7.33799386e-01  7.33799386e-01  7.33799386e-01  7.33799386e-01
   7.33799386e-01  7.33799386e-01  7.33799386e-01  7.33799386e-01
   7.33799386e-01]
 [-8.32050294e-01 -6.93375245e-01 -5.54700196e-01 -4.16025147e-01
  -2.77350098e-01 -1.38675049e-01  2.83887846e-17  1.38675049e-01
   2.77350098e-01  4.16025147e-01  5.54700196e-01  6.93375245e-01
   8.32050294e-01]
 [ 7.51067616e-01  3.75533808e-01  6.82788742e-02 -1.70697185e-01
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   7.51067616e-01]
 [-6.08434308e-01 -2.22542187e-16  3.31873259e-01  4.42497679e-01
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   6.08434308e-01]
 [ 4.48980746e-01 -2.99320497e-01 -4.35375269e-01 -2.44898589e-01
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