(1) Determine the circular convolution of $x[n]$ and $h[n]$ where (10 scores)

$$x[n]=[1,0,2,3,-1,2,2,1,0], \qquad h[n]=[2,2,1,1,0,0,0,0,0]$$

and $N = 9$.

Sol:

$$y[n] = x[n] \circledast_c h[n] = \sum_{k=0}^{N-1} x[k]h[((n-k))_N]$$

$$\begin{aligned}
y[0] &= x[0]h[0]+x[1]h[8]+x[2]h[7]+x[3]h[6]+ \\
&\quad x[4]h[5]+x[5]h[4]+x[6]h[3]+x[7]h[2]+x[8]h[1] \\
&= 1\times2+2\times1+1\times1=5
\end{aligned}$$

$$\begin{aligned}
y[1] &= x[0]h[1]+x[1]h[0]+x[2]h[8]+x[3]h[7]+ \\
&\quad x[4]h[6]+x[5]h[5]+x[6]h[4]+x[7]h[3]+x[8]h[2] \\
&= 1\times2+1\times1=3
\end{aligned}$$

$$\begin{aligned}
y[2] &= x[0]h[2]+x[1]h[1]+x[2]h[0]+x[3]h[8]+ \\
&\quad x[4]h[7]+x[5]h[6]+x[6]h[5]+x[7]h[4]+x[8]h[3] \\
&= 1\times1+2\times2=5
\end{aligned}$$

$$\begin{aligned}
y[3] &= x[0]h[3]+x[1]h[2]+x[2]h[1]+x[3]h[0]+ \\
&\quad x[4]h[8]+x[5]h[7]+x[6]h[6]+x[7]h[5]+x[8]h[4] \\
&= 1\times1+2\times2+3\times2=11
\end{aligned}$$

$$\begin{aligned}
y[4] &= x[0]h[4]+x[1]h[3]+x[2]h[2]+x[3]h[1]+ \\
&\quad x[4]h[0]+x[5]h[8]+x[6]h[7]+x[7]h[6]+x[8]h[5] \\
&= 2\times1+3\times2+-1\times2=6
\end{aligned}$$

$$\begin{aligned}
y[5] &= x[0]h[5]+x[1]h[4]+x[2]h[3]+x[3]h[2]+ \\
&\quad x[4]h[1]+x[5]h[0]+x[6]h[8]+x[7]h[7]+x[8]h[6] \\
&= 2\times1+3\times1+(-1)\times2+2\times2=7
\end{aligned}$$

$$\begin{aligned}
y[6] &= x[0]h[6]+x[1]h[5]+x[2]h[4]+x[3]h[3]+ \\
&\quad x[4]h[2]+x[5]h[1]+x[6]h[0]+x[7]h[8]+x[8]h[7] \\
&= 3\times1+(-1)\times1+2\times2+2\times2=10
\end{aligned}$$

$$\begin{aligned}
y[7] &= x[0]h[7]+x[1]h[6]+x[2]h[5]+x[3]h[4]+ \\
&\quad x[4]h[3]+x[5]h[2]+x[6]h[1]+x[7]h[0]+x[8]h[8] \\
&= -1\times1+2\times1+2\times2+1\times2=7
\end{aligned}$$

$$\begin{aligned}
y[8] &= x[0]h[8]+x[1]h[7]+x[2]h[6]+x[3]h[5]+ \\
&\quad x[4]h[4]+x[5]h[3]+x[6]h[2]+x[7]h[1]+x[8]h[0] \\
&= 2\times1+2\times1+1\times2=6
\end{aligned}$$

$$y[n]=[5,3,5,11,6,7,10,7,6]$$

(2) Suppose that the PDF of X is (20 scores)

$$f_X(x) = x/2 \quad for\ 0 < x < 2, \qquad f_X(x) = 0 \quad otherwise.$$

(a) Determine the CDF and the mean of X.

Sol:

(1) CDF

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \int_{-\infty}^{x} f_X(x)\,dx = \int_0^x \frac{x}{2}\,dx, & 0 < x < 2 \\ 1, & x > 2 \end{cases}$$

$$\Rightarrow \left[\frac{x^2}{4}\right]_0^x = \frac{x^3}{4}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x^3}{4} & 0 < x < 2 \\ 1, & x > 2 \end{cases}$$

(2) Mean

$$\mu_x = \int_0^2 x\,\frac{x}{2}\,dx = \left[\frac{x^3}{6}\right]_0^2$$

$$= \frac{8}{6} = \frac{4}{3}$$

(b) Determine the standard deviation, the variance, and the skewness of X.

Sol:

(1) standard deviation

$\because \sigma_X = \sqrt{Var_X} = \sqrt{E((X-\mu_X)^2)} = \sqrt{\int_0^2 (x-\frac{4}{3})^2 \frac{x}{2} dx}$

$\therefore \sigma_X = \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}$

(2) variance

$$Var_X = \int_0^2 (x-\frac{4}{3})^2 \frac{x}{2} dx$$

$$= \int_0^2 \frac{x^3}{2} - \frac{4}{3}x^2 + \frac{8}{9}x \, dx$$

$$= \left[\frac{x^4}{8} - \frac{4}{9}x^3 + \frac{4}{9}x^2 \right]_0^2 = \frac{2}{9} = 0.2222$$

(3) skewness $k=3$

$$\Rightarrow \frac{\int_0^2 (x-\frac{4}{3})^3 \frac{x}{2} dx}{\left[\int_0^2 (x-\frac{4}{3})^2 \frac{x}{2} dx\right]^{3/2}} = \frac{\left[\frac{x^5}{10} - \frac{x^4}{2} + \frac{8x^3}{9} - \frac{16x^2}{27} \right]_0^2}{(\frac{2}{9})^{\frac{3}{2}}}$$

$$= \frac{(-\frac{8}{135})}{\frac{2}{9}\sqrt{\frac{2}{9}}} = \frac{-\frac{4}{15}}{\sqrt{\frac{2}{9}}} = \frac{-4}{5\sqrt{2}} \doteqdot -0.5657$$

(3) Suppose that the mean, the variance, the skewness, and the kurtosis of X are μ, v, s, and k, respectively. Also suppose that $Y = X/5 + 5$.

(a) Determine the correlation between X and Y.

Sol:

$$\mathrm{corr}_{X,Y} = \frac{E((X-\mu_X)(Y-\mu_Y))}{\sigma_X \sigma_Y}$$

$$\because Y = \frac{1}{5}X + 5$$

$$\sigma_Y = \frac{1}{5}\sigma_X \quad , \quad \mu_Y = E[Y] = E[\tfrac{X}{5}+5] = \frac{1}{5}E[X]+5 = \frac{\mu}{5}+5$$

$$E((X-\mu)(\tfrac{X}{5}+5-(\tfrac{\mu}{5}+5))) = \frac{1}{5}E((X-\mu)^2) = \frac{1}{5}V$$

$$\mathrm{corr}_{X,Y} = \frac{\frac{1}{5}V}{\frac{1}{5}\sigma_X^2} = 1 \quad *$$

(b) Determine the mean, the variance, the skewness, and the kurtosis of Y.

(20 scores)

Sol:

(b) $Y = \frac{X}{5} + 5$

(1) mean

$$\mu_Y = E[Y] = E[\tfrac{X}{5}+5] = \tfrac{1}{5}E[X]+5 = \tfrac{\mu}{5}+5$$

(2) variance

$$Var(Y) = E[(Y-\mu_Y)^2] = E[((\tfrac{X}{5}+5)-(\tfrac{\mu}{5}+5))^2] = E[\tfrac{(X-\mu)^2}{25}]$$

$$= \tfrac{1}{25}V$$

(3) skewness

$$S(Y) = \frac{E[(Y-\mu_Y)^3]}{(Var(Y))^{3/2}} = \frac{E[((\frac{X}{5}+5)-(\frac{\mu}{5}+5))^3]}{(\frac{1}{25}V)^{3/2}} = \frac{E[\frac{1}{125}(X-\mu)^3]}{(\frac{1}{25}V)^{3/2}}$$

$$= \frac{\frac{1}{125}E[(X-\mu)^3]}{\frac{1}{125}V^{\frac{3}{2}}} = S(X) = s$$

(4) kurtosis

$$K(Y) = \frac{E[(Y-\mu_Y)^4]}{(Var(Y))^2} = \frac{E[((\frac{X}{5}+5)-(\frac{\mu}{5}+5))^4]}{(\frac{1}{25}V)^2} = \frac{E[\frac{1}{625}(X-\mu)^4]}{(\frac{1}{25}V)^2}$$

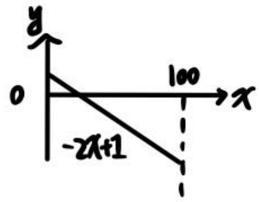
$$= \frac{\frac{1}{625}E[(X-\mu)^4]}{\frac{1}{625}V^2} = \frac{E[(X-\mu)^4]}{V^2} = K(X) = k$$

(4) Suppose that the joint PDF of X and Y is: (10 scores)

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{100}\delta(y+2x-1) & for \quad 0<x<100\\ 0 & otherwise\end{cases}$$

Determine the correlation of X and Y in terms of a.

Sol:



$\Rightarrow Y = 1-2X$

$$f_X(x)=\frac{1}{100}\int_0^{100}\delta(y+2x-1)\,dy \qquad f_Y(y)=\frac{1}{100}$$

$$=\frac{1}{100}\int_{-\infty}^{\infty}\delta(y)\,dy=\frac{1}{100}$$

$$u_X=\frac{1}{100}\int_0^{100}x\,dx=\frac{10000}{2}\times\frac{1}{100}=50$$

$$u_Y=\frac{1}{100}\int_0^{100}(1-2x)\,dx=\frac{1}{100}\left[x-x^2\right]_0^{100}$$

$$=\frac{-9900}{100}=-99$$

$$COV_{X,Y}=\int_0^{100}\int_{-\infty}^{\infty}(x-50)(y+99)\frac{1}{100}\delta(y+2x-1)\,dy\,dx$$

$$=\frac{1}{100}\int_0^{100}(x-50)(-2x+100)\,dx$$

$$=\frac{-1}{50}\int_0^{100}(x-50)^2\,dx$$

$$=-\frac{5000}{3}$$

$$\sigma_X=\sqrt{\frac{1}{100}\int_0^{100}(x-50)^2\,dx}=\sqrt{\frac{2500}{3}}$$

$$\sigma_Y=\sqrt{\frac{1}{100}\int_0^{100}(y+99)^2\,dy}=\sqrt{\frac{69103}{3}}$$

$$corr_{X,Y}=\frac{-\frac{5000}{3}}{\sqrt{\frac{2500}{3}}\sqrt{\frac{69103}{3}}}=\frac{-\frac{5000}{3}}{\frac{50}{3}\sqrt{69103}}=\frac{-100}{\sqrt{69103}}=-0.3804$$

(5) We assume that (10 scores)

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}$$

Find the permutation matrices $\mathbf{J}_1$ and $\mathbf{J}_2$ such that

$$\mathbf{A} \otimes \mathbf{B} = \mathbf{J}_1 (\mathbf{B} \otimes \mathbf{A}) \mathbf{J}_2$$

Hints: Expand the Kronecker products and match their entries.

Sol:

$$\underline{A} \otimes \underline{\underline{B}} = \left[(a_{1,1})\underline{\underline{B}},\ (a_{1,2})\underline{\underline{B}},\ (a_{1,3})\underline{\underline{B}} \right]_{2\times 6}$$

$$= \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} & a_{1,3}b_{1,1} & a_{1,3}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} & a_{1,3}b_{2,1} & a_{1,3}b_{2,2} \end{bmatrix}$$

$$\underline{\underline{B}} \otimes \underline{A} = \begin{bmatrix} (b_{1,1})\underline{A}, & (b_{1,2})\underline{A} \\ (b_{2,1})\underline{A} & (b_{2,2})\underline{A} \end{bmatrix}_{2\times 6}$$

$$= \begin{bmatrix} b_{1,1}a_{1,1} & b_{1,1}a_{1,2} & b_{1,1}a_{1,3} & b_{1,2}a_{1,1} & b_{1,2}a_{1,2} & b_{1,2}a_{1,3} \\ b_{2,1}a_{1,1} & b_{2,1}a_{1,2} & b_{2,1}a_{1,3} & b_{2,2}a_{1,1} & b_{2,2}a_{1,2} & b_{2,2}a_{1,3} \end{bmatrix}$$

$$\therefore \underline{A} \otimes \underline{\underline{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} (\underline{\underline{B}} \otimes \underline{A}) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{6\times 6}$$

$$\underline{\underline{J_1}} = \underline{\underline{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

$$\underline{\underline{J_2}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{6\times 6}$$

(6) For a real number α and an integer $N \geq 2$, we define the vector

$$\mathbf{v}(\alpha) = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{N-1} \end{bmatrix}^T$$

Find the value of the following norms:

$$\|\mathbf{v}(\alpha)\|_0, \quad \|\mathbf{v}(\alpha)\|_1, \quad \|\mathbf{v}(\alpha)\|_2, \quad \|\mathbf{v}(\alpha)\|_\infty, \quad \|\mathbf{v}(\alpha)\mathbf{v}^T(\alpha)\|_F$$

Simplify and express your answers in terms of N and α. Do not use dots (. . .) or summation (Σ) in your expression. (10 scores)

Sol:

$$\text{supp}(V(\alpha)) = \{1, 2, 3, \cdots, N\}$$

$$\|v(\alpha)\|_0 \triangleq \text{card}(\text{supp}(V(\alpha))) = \text{card}(\{1, 2, 3, \cdots, N\}) = N$$

$$\|v(\alpha)\|_1 \triangleq \left(\sum_{i=1}^{N} |v(\alpha)_i|^1\right)$$

$$= 1 + \alpha + \alpha^2 + \cdots + \alpha^{N-1} \Rightarrow \text{公比 } \alpha$$

$$= \frac{1(1-\alpha^N)}{1-\alpha} = \frac{1-\alpha^N}{1-\alpha} = \frac{\alpha^N - 1}{\alpha - 1}$$

$$\|V(\alpha)\|_2 \triangleq \left(\sum_{i=1}^{N} |V(\alpha)_i|^2\right)^{\frac{1}{2}}$$

$$= \sqrt{1^2 + |\alpha|^2 + |\alpha^2|^2 + \cdots + |\alpha^{N-1}|^2}$$

$$= \sqrt{V(\alpha)^H V(\alpha)}$$

$$= \sqrt{tr(V(\alpha)^H V(\alpha))}$$

$$= \sqrt{tr\left(\begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{N-1} \\ \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^N \\ \alpha^2 & \alpha^3 & \alpha^4 & \cdots & \alpha^{N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha^{N-1} & \alpha^N & \alpha^{N+1} & \cdots & \alpha^{2N-2} \end{bmatrix}\right.}$$

$$= \sqrt{1 + \alpha^2 + \alpha^4 + \alpha^6 + \cdots + \alpha^{2N-2}}$$

$$= \sqrt{\frac{\alpha^{2N} - 1}{\alpha^2 - 1}}$$

$$\|V(\alpha)\|_\infty \triangleq \lim_{p\to\infty}\left(\sum_{i=1}^{N}|V(\alpha)_i|^p\right)^{1/p}$$

$$= \max_{i\in|N|}|V(\alpha)_i|$$

$$= \alpha^{N-1}$$

$\mathrm{tr}(AA^H)$

$$= \mathrm{tr}\left(\begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{N-1} \\ \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^{N} \\ \alpha^2 & \alpha^3 & \alpha^4 & \cdots & \alpha^{N+1} \\ \vdots & \vdots & \vdots & \ddots & \\ \alpha^{N-1} & \alpha^{N} & \alpha^{N+1} & \cdots & \alpha^{2N-2} \end{bmatrix}\begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{N-1} \\ \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^{N} \\ \alpha^2 & \alpha^3 & \alpha^4 & \cdots & \alpha^{N+1} \\ \vdots & \vdots & \vdots & \ddots & \\ \alpha^{N-1} & \alpha^{N} & \alpha^{N+1} & \cdots & \alpha^{2N-2} \end{bmatrix}\right)$$

$$= \left(1^2+\alpha^2+\cdots+(\alpha^2)^{N-1}\right)+\left(\alpha^2+\cdots+(\alpha^N)^2\right)$$

$$+\left((\alpha^{N-1})^2+\cdots+(\alpha^{2N-2})^2\right)$$

$$= \frac{1(\alpha^{2N}-1)}{\alpha^2-1}+\frac{\alpha^2(\alpha^{2N}-1)}{\alpha^2-1}+\frac{\alpha^4(\alpha^{2N}-1)}{\alpha^2-1}+\cdots+\frac{\alpha^{2N-2}(\alpha^{2N}-1)}{\alpha^2-1}$$

$$= \frac{(\alpha^{2N}-1)}{\alpha^2-1}\left(1+\alpha^2+\alpha^4+\cdots+\alpha^{2N-2}\right)$$

$$= \frac{(\alpha^{2N}-1)}{\alpha^2-1}\,\frac{(\alpha^{2N}-1)}{\alpha^2-1}$$

令 $A = V(\alpha)V^T(\alpha)$

$$= \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \\ \vdots \\ \alpha^{N-1} \end{bmatrix} \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{N-1} \\ \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^N \\ \alpha^2 & \alpha^3 & \alpha^4 & \cdots & \alpha^{N+1} \\ \vdots & \vdots & \vdots & \ddots & \\ \alpha^{N-1} & \alpha^N & \alpha^{N+1} & \cdots & \alpha^{2N-2} \end{bmatrix}_{N \times N}$$

$$\|A\|_F \overset{\Delta}{=} \sqrt{\sum_{m=1}^{M}\sum_{n=1}^{N} |a_{m,n}|^2}$$

$$= \sqrt{tr(AA^H)}$$

$$= \sqrt{\left[\frac{(\alpha^{2N}-1)}{\alpha^2-1}\right]^2}$$

$$= \frac{\alpha^{2N}-1}{\alpha^2-1}$$

(7) Suppose that (20 scores)

$$P_{X,Y}(m,n)=\frac{100-|m-n|}{666700} \qquad for\ \ m=1,2,\cdots,100,\ \ n=1,2,\cdots,100$$

Use a Matlab or Python code to determine the cross entropy of X and Y if

(a) The definition of the cross entropy on page 532 is applied.

(b) The definition of the cross entropy on page 534 is applied.

The code should be handed out by NTUCool.

Sol:

a.

Cross entropy H(X, Y): 4.5987445992167331

b.

Cross entropy H(X, Y): 4.206181