(1) Find the solutions of the following nonlinear DEs.

(a) $y''(x)y'(x)=1, \quad y'(0)=0$

#1 $u=\frac{d}{dx}y=y'$

$\frac{d}{dx}u \times u = 1 \Rightarrow u\,du = dx \Rightarrow \frac{u^2}{2} = x+C, C=0$

$u^2 = 2x \Rightarrow u = \sqrt{2x}$

#2 $\frac{dy}{dx} = \sqrt{2x} \Rightarrow dy = \sqrt{2x}\,dx \Rightarrow y = \frac{2}{3}\sqrt{2}\,x^{\frac{3}{2}} + C$ #

(b) $y''(x) = -3y'(x)y^2(x), \quad y(1)=2^{-1/2}, \quad y'(1)=-2^{-3/2}$

#1 $u=y'$, $u\frac{d}{dy}u$

#2 $u\frac{d}{dy}u = -3uy^2 \Rightarrow \frac{du}{-3} = y^2dy \Rightarrow -\frac{u}{3} = \frac{1}{3}y^3 + C_1$

$-3C_1 - u = y^3 \because (2^{-\frac{1}{2}})^3 = 2^{-\frac{3}{2}} - 3C_1 \therefore -3C_1 = 0$

$y^3 = -u$

#3 $y^3 = -\frac{dy}{dx} \rightarrow dx = -\frac{dy}{y^3} \rightarrow x + C = \frac{1}{2}y^{-2} \because 1 + C = \frac{1}{2}(2^{-\frac{1}{2}})^2$

$\therefore C = -\frac{3}{4}$

$y = \frac{1}{\sqrt{2x - \frac{3}{2}}}$ #

(c) $y''(x) = \exp(y(x))$, $y(0) = 0$ $y'(0) = \sqrt{2}$

#1 $u = \frac{d}{dx}y$, $\frac{d}{dx}u = u\frac{d}{dy}u$

$$u\frac{d}{dy}u = e^y$$

#2 $u\,du = e^y\,dy \rightarrow \frac{1}{2}u^2 = e^y + C \rightarrow u^2 = 2e^y + C$

$\because (\sqrt{2})^2 = 2e^0 + C \therefore C = 0$

#3 $\frac{dy}{dx} = \sqrt{2e^y} \rightarrow \frac{dy}{\sqrt{2e^y}} = dx$

$$x + C = -\sqrt{\frac{2}{e^y}}$$

$\because 0 + C = -\sqrt{\frac{2}{e^0}} \therefore C = -\sqrt{2}$

$$(x - \sqrt{2})^2 = -\frac{2}{e^y}$$

$$e^y = -\frac{2}{x^2 - 2\sqrt{2}x + 2}$$

$$y = -\ln\left(\frac{2}{x^2 - 2\sqrt{2}x + 2}\right)$$ #

(2) Solve the following PDEs.

(a) $x^2 \frac{\partial}{\partial x} u(x,y) = y \frac{\partial}{\partial y} u(x,y)$

#1 $u = XY$

#2 $x^2 X' Y = y X Y'$

$$\frac{x^2 X'}{X} = \frac{y Y'}{Y} = -\lambda \Rightarrow \begin{aligned} x^2 X' + \lambda X = 0 \\ y Y' + \lambda Y = 0 \end{aligned}$$

#3 $\frac{X'}{X} = \frac{-\lambda}{x^2} \rightarrow \ln|X| = \int \frac{-\lambda}{x^2} dx = \lambda x^{-1} + c_1$

$$\Rightarrow |X| = e^{\lambda x^{-1} + c_1} \Rightarrow |X| = e^{c_1} e^{\lambda x^{-1}} \Rightarrow X = \pm e^{c_1} e^{\lambda x^{-1}} = C_2 e^{\lambda x^{-1}}$$

$Y = C_3 y^{-\lambda}$

$$u = C_2 e^{\lambda x^{-1}} C_3 y^{-\lambda} = A_1 e^{\lambda x^{-1}} y^{-\lambda}$$

$$\Rightarrow u = \sum_{\lambda} A_1 e^{\lambda x^{-1}} y^{-\lambda} \quad \#$$

(b) $\frac{\partial^2}{\partial x^2}u(x,y)=u(x,y)+\frac{\partial}{\partial y}u(x,y) \qquad 0<x<2, \quad y>0,$

$u(0,y)=u(2,y)=0, \quad u(x,0)=\cos(\pi x)\sin(2\pi x)$

#1 $u(x,y)=X(x)Y(y)$

#2 $X''Y = XY+XY' \Rightarrow \frac{X''}{X} = 1+\frac{Y'}{Y} = -\lambda$

$X''+\lambda X=0$, $Y'+Y(1+\lambda)=0$

$X(0)=X(2)=0$, $Y(0)=\cos(\pi x)\sin(2\pi x)$

#3 Case1: $\lambda=0$

$X''=0$, $m^2=0 \Rightarrow X=C_1+C_2x$

$X(0)=X(2)=0$, $C_1=C_2=0 \Rightarrow u=XY=0 \Rightarrow$ trivial

Case2 $\lambda<0$, $\lambda=-\alpha^2$,

$X''-\alpha^2X=0 \rightarrow m^2=\alpha^2 \rightarrow m=\pm\alpha$

$X=d_1e^{\alpha x}+d_2e^{-\alpha x} \Rightarrow X(x)=C_3\cosh(\alpha x)+C_4\sinh(\alpha x)$

$X(0)=X(2)=0 \Rightarrow 0=C_3\cosh(0)+C_4\sinh(0) \Rightarrow C_3\cdot1+0=0 \Rightarrow C_3=0$

$C_3\cosh(2\alpha)+C_4\sinh(2\alpha)=0 \rightarrow C_4\sinh(2\alpha)=0 \because \sinh(2\alpha)\neq0 \therefore C_4=0$

$\rightarrow$ trival

Case3 $\lambda>0$, $\lambda=\alpha^2$

$X''+\alpha^2X=0 \rightarrow m^2=-\alpha^2 \Rightarrow m=\pm j\alpha$

$X(x)=C_5\cos(\alpha x)+C_6\sin(\alpha x)$

$X(0)=C_5\cos(0)=0 \because \cos(0)=1 \therefore C_5=0$

$X(2)=0\cdot\cos(2\alpha)+C_6\sin(2\alpha)=0 \quad \sin(2\alpha)=0$

$\rightarrow \begin{cases} C_6 = \text{非0常數} \\ \alpha=\frac{n\pi}{2}, n\in N \end{cases}$

$X(x)=C_6\sin\frac{n\pi}{2}x$, $\lambda=\frac{n^2\pi^2}{4}$

$Y'+(1+\lambda)Y=0 \rightarrow Y'+(1+\frac{n^2\pi^2}{4})Y=0 \rightarrow m=-1-\frac{n^2\pi^2}{4}$

$Y(y)=C_7e^{(-1-\frac{n^2\pi^2}{4})y}$

$\therefore u(x,y)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{2}x\, e^{(-1-\frac{n^2\pi^2}{4})y}$

$u(x,0)=\cos(\pi x)\sin(2\pi x)$

$\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{2}x=\cos(\pi x)\sin(2\pi x) \xrightarrow{\text{Fourier series}} A_n=\int_0^2\cos(\pi x)\sin(2\pi x)\sin\frac{n\pi}{2}x\,dx$

$= \frac{8(n^2-12)\sin(\pi n)}{\pi(n^4-40n^2+144)}$

$u(x,y)=\sum_{n=1}^{\infty}\frac{8(n^2-12)\sin(\pi n)}{\pi(n^4-40n^2+144)}\sin\frac{n\pi}{2}x\, e^{(-1-\frac{n^2\pi^2}{4})y}$ #

(c) $\frac{\partial^2}{\partial x^2}u(x,y)+\frac{\partial^2}{\partial y^2}u(x,y)=0 \qquad 0<x<1, \quad 0<y<1,$

$u(0,y)=u(1,y)=u(x,0)=0, \quad u(x,1)=1-2|x-1/2|$

#1 $u(x,y)=X(x)Y(y)$

$X''Y+XY''=0$

$\rightarrow \frac{X''}{X}=-\frac{Y''}{Y}=-\lambda \rightarrow X''+\lambda X=0,\ Y''-\lambda Y=0$

#2 $X(0)=X(1)=0,\ Y(0)=0,\ Y(1)=1-2|\frac{x-1}{2}|$

#3 Case1: $\lambda=0$

$X''=0 \rightarrow X=C_1+C_2x \rightarrow X(0)\rightarrow C_1=0 \quad X(1)=C_2\cdot 1=0 \quad C_2=0$

$Y''=0 \rightarrow Y=C_3+C_4y \rightarrow Y(0)\rightarrow C_3=0$, $Y(1)$ should be const

$u(x,y)=X(x)Y(y) \Rightarrow$ trival

Case2: $\lambda<0,\ \lambda=-\alpha^2$

$X=C_5\cosh(\alpha x)+C_6\sinh(\alpha x)\rightarrow X(0)=0=C_5\cosh(0)+C_6\sinh(0)\Rightarrow C_5=0$

$\hookrightarrow X(1)=0\cdot\cosh(\alpha)+C_6\sinh(\alpha)=0 \because \sinh(\alpha)\neq 0\ C_6=0$

$Y=C_7\cosh(\alpha y)+C_8\sinh(\alpha y)$

$u(x,y)=X(x)Y(y)\rightarrow$ trival

Case3: $\lambda>0,\ \lambda=\alpha^2$

$X=C_9\cos(\alpha x)+C_{10}\sin(\alpha x)\rightarrow X(0)=C_9\cdot 1=0\ C_9=0, X(1)=C_{10}\sin(\alpha)=0 \Rightarrow \sin(\alpha)=0\Rightarrow \alpha=n\pi$

$n\in N.$

$Y=C_{11}\cosh(\alpha y)+C_{12}\sinh(\alpha y)$

$=C_{11}\cosh(n\pi y)+C_{12}\sinh(n\pi y)$

$Y(0)\Rightarrow C_{11}=0$

$Y=C_{12}\sinh(n\pi y)$

$u(x,y)=\sum_{n=1}^{\infty}A_n\sin(n\pi x)\sinh(n\pi y)$

#4 $\because u(x,1)=1-2|\frac{x-1}{2}| \therefore A_n\sinh(n\pi)=2\int_0^1(1-2|\frac{x-1}{2}|)\sin(n\pi x)dx$

$A_n=\frac{2}{\sinh(n\pi)}\int_0^1(1-2|\frac{x-1}{2}|)\sin(n\pi x)dx=\frac{2}{\sinh(n\pi)}\times\frac{(\sin(\pi n)-\pi n\cos(\pi n))}{\pi^2n^2}$

$\therefore u(x,y)=\sum_{n=1}^{\infty}\frac{2}{\sinh(n\pi)}\times\frac{(\sin(\pi n)-\pi n\cos(\pi n))}{\pi^2n^2}\sin(n\pi x)\sinh(n\pi y)$ *

(d) $(x+1)\frac{\partial}{\partial x}u(x,y)=\frac{\partial}{\partial y}u(x,y)+\cos y$

$G=\cos y$

$u(x,y)=v(x,y)+\psi(y)$

$$(x+1)\frac{\partial v(x,y)}{\partial x}+(x+1)\frac{d\psi(y)}{dx}=\frac{\partial v(x,y)}{\partial y}+\frac{d\psi(y)}{dy}+\cos y$$

$$\Rightarrow (x+1)\frac{\partial v(x,y)}{\partial x}=\frac{\partial v(x,y)}{\partial y}+\frac{d\psi(y)}{dy}+\cos y$$

PA: $\psi'(y)+\cos y=0$

$\psi(y)=-\sin y+C_1$

$v(x,y)=X(x)Y(y)$, $(x+1)X'Y=XY'\to\frac{(x+1)X'}{X}=\frac{Y'}{Y}=-\lambda$

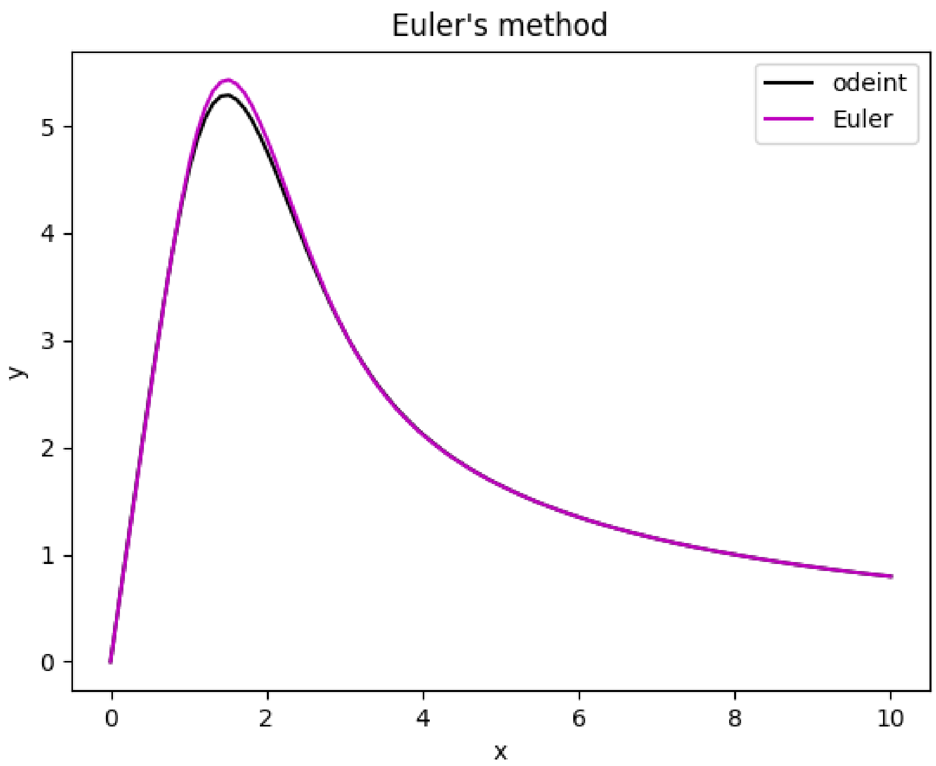
$$\begin{cases}(x+1)X'+\lambda X=0\\ Y'+\lambda Y=0\end{cases}\longrightarrow\begin{cases}X=C_2e^{-\lambda\frac{x}{x+1}}\\ Y=C_3e^{-\lambda y}\end{cases}$$

$$u(x,y)=\sum_{\lambda}C_\lambda e^{-\lambda(\frac{x}{x+1}+y)}-\sin y \quad \#$$

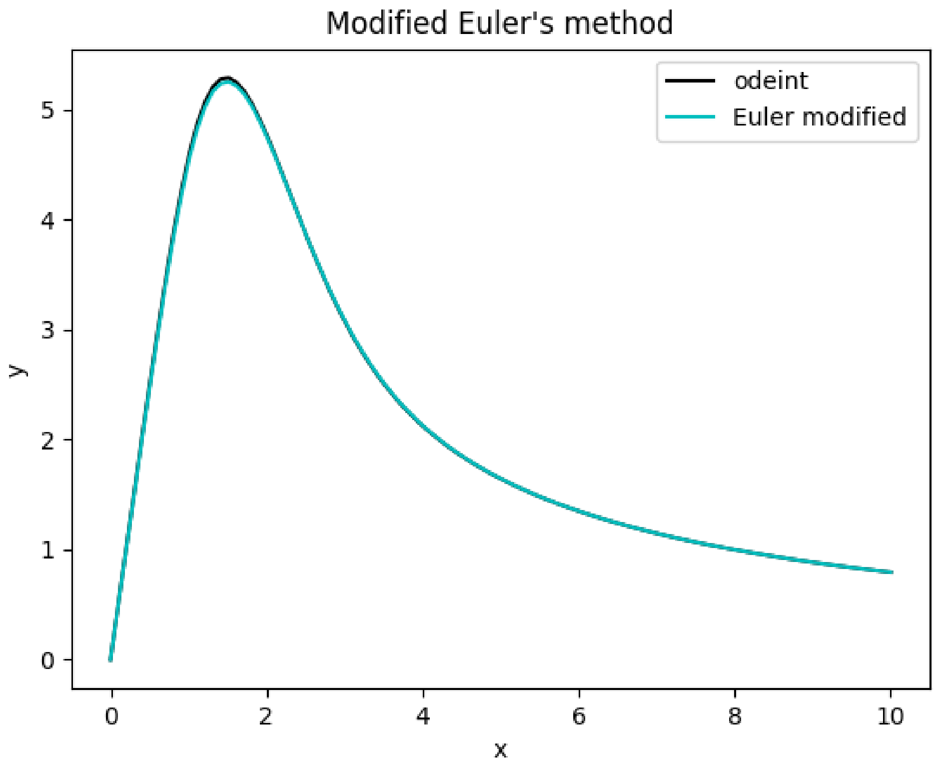
(3) Solve the following 1st order nonlinear DE numerically. Plot the result $y(x)$. The Matlab (or Python) code should also be handed out.

$$\frac{\partial y(x)}{\partial x} = 5\cos\left(-\frac{1}{5}\left|xy\right|\right), \quad y(0)=0, \quad 0 \le x \le 10, \quad x_{n+1} - x_n = 0.01$$

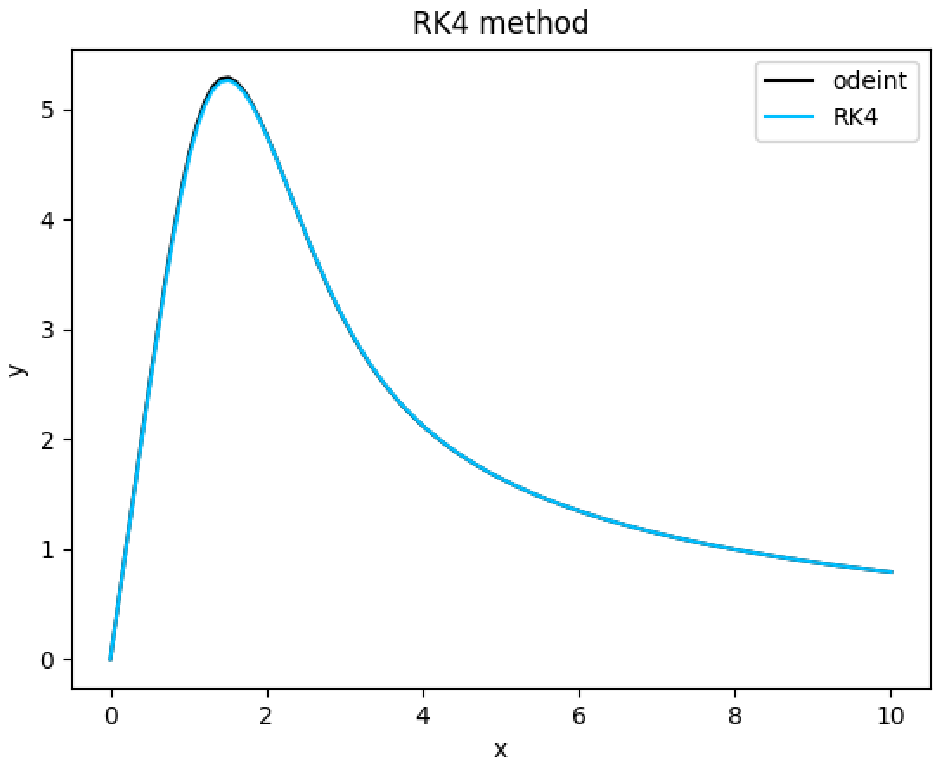
(a) By Euler's method.



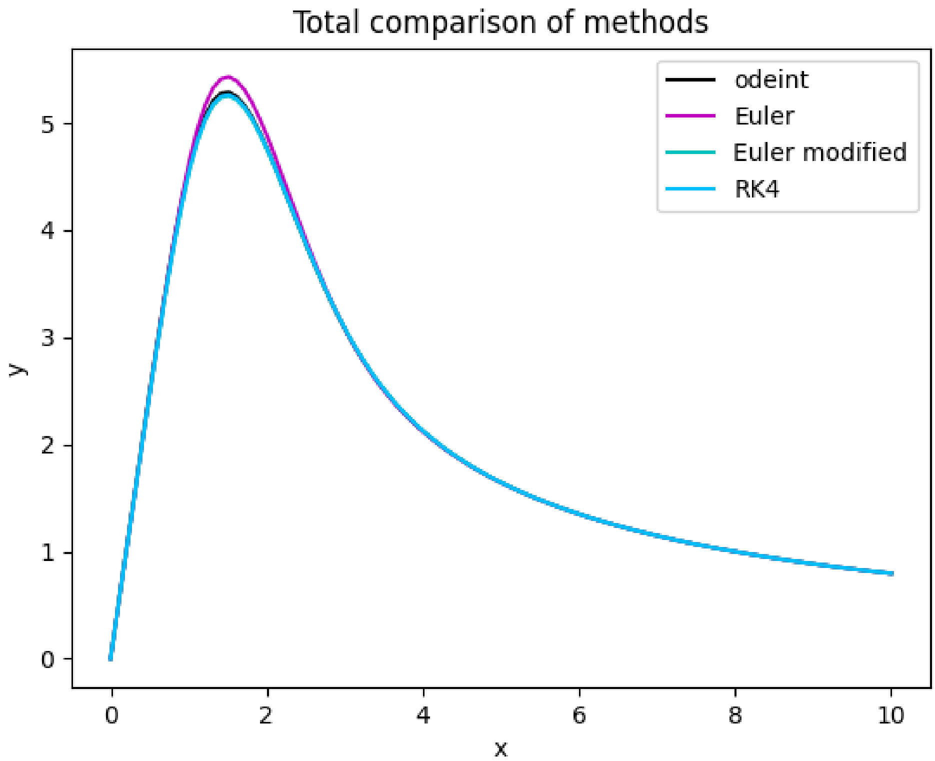
(b) By modified Euler's method.



(c) By the RK4 method.



Extra: 一起做比較:



Total comparison of methods
odeint
Euler
Euler modified
RK4
y
x
0
1
2
3
4
5
0
2
4
6
8
10