(1) Determine the Fourier transform of the following functions.

(a) $g(x) = \exp(-\pi x^2/2)(x^3+x)$

Sol:

$$x^3+x \Rightarrow H_3(x) = 8x^3-12x = 8\left(x^3-\frac{6}{4}x\right)$$

$$H_1(x) = 2x$$

$$\frac{5}{4}H_1(x) = \frac{5}{2}x$$

$$x^3+x = \frac{1}{8}H_3(x)+\frac{5}{4}H_1(x)$$

$$\mathcal{F}\{x\,g(x)\} = \frac{j}{2\pi}G'(f)$$

$$g_1(x) = \exp\left(\frac{-\pi x^2}{2}\right)$$

$$\mathcal{F}(g_1(x)) = G_1(X) = \sqrt{2}\exp(-2\pi f^2)$$

$$\mathcal{F}\left\{\exp\left(\frac{-\pi x^2}{2}\right)(x^3+x)\right\} = \mathcal{F}\left\{\exp\left(\frac{-\pi x^2}{2}\right)(x^3+x)\right\}$$

$$= \left(\frac{j}{2\pi}\right)^3 G_1'''(f) + \frac{j}{2\pi}G'(f)$$

$$= \frac{-j}{2\pi^3}\sqrt{2}\left(-64\pi^3 f^3 e^{-2\pi f^2}\right) + \frac{j}{2\pi}\sqrt{2}\left(-4\pi f e^{-2\pi f^2}\right)$$

$$= j\sqrt{2}\,32f^3e^{-2\pi f^2} - j\sqrt{2}\,2fe^{-2\pi f^2}$$

$$= \left(2\sqrt{2}\,jf\exp(-2\pi f^2)\right)(16f^2-1)$$

$$e^{-2\pi f^2} \underset{1d}{=} -4\pi f e^{-2\pi f^2} \underset{2d}{=} 16\pi^2 f^2 e^{-2\pi f^2} \underset{3d}{=} -64\pi^3 f^3 e^{-2\pi f^2}$$

(b) $g(x) = \sin(\pi x/6)$ for $0 < x < 6$, $g(x) = 0$ otherwise

Sol:

$$\prod\left(\frac{x-b}{a}\right)\begin{cases}1 & b-\frac{a}{2} < x < b+\frac{a}{2}\\ 0 & \text{otherwise}\end{cases}$$

center $b = 3$, width $a = 6$

$$\sin\left(\frac{\pi x}{6}\right)\prod\left(\frac{x-3}{6}\right) = \frac{1}{2j}\exp\left(\frac{j\pi x}{6}\right)\prod\left(\frac{x-3}{6}\right) - \frac{1}{2j}\exp\left(\frac{-j\pi x}{6}\right)\prod\left(\frac{x-3}{6}\right)$$

$$\mathcal{F}\left\{\prod\left(\frac{x-3}{6}\right)\right\} = 6e^{-j6\pi f}\operatorname{sinc}(6f)$$

$$\mathcal{F}\left\{\exp\left(\frac{j\pi x}{6}\right)\prod\left(\frac{x-3}{6}\right)\right\} = 6e^{-j6\pi(f-\frac{1}{3})}\operatorname{sinc}(6(f-\tfrac{1}{3}))$$

$$\mathcal{F}\left\{\exp\left(\frac{-j\pi x}{6}\right)\prod\left(\frac{x-3}{6}\right)\right\} = 6e^{-j6\pi(f+\frac{1}{3})}\operatorname{sinc}(6(f+\tfrac{1}{3}))$$

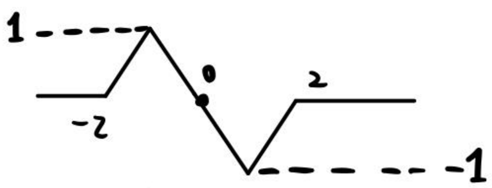
$$\therefore \mathcal{F}[g(x)] = \frac{1}{2j}\mathcal{F}\left\{\exp\left(\frac{j\pi x}{6}\right)\prod\left(\frac{x-3}{6}\right) - \exp\left(\frac{-j\pi x}{6}\right)\prod\left(\frac{x-3}{6}\right)\right\}$$

$$= 3je^{-j6\pi(f-\frac{1}{3})}\operatorname{sinc}(6(f-\tfrac{1}{3})) - 3je^{-j6\pi(f+\frac{1}{3})}\operatorname{sinc}(6(f+\tfrac{1}{3})) \quad \#$$

(c) $g(x) = -x$ for $-1 < x < 1$, $g(x) = x-2$ for $1 < x < 2$,

$g(x) = 2+x$ for $-2 < x < -1$, $g(x) = 0$ otherwise.

Sol:



$$g(x) = \begin{cases} x+2 & \text{for } -2<x<-1 \\ x-2 & \text{for } 1<x<2 \\ -x & \text{for } -1<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$g(x) = \Lambda\left(\frac{x+1}{2}\right) - \Lambda\left(\frac{x-1}{2}\right)$$

$$\because \mathcal{F}\{\Lambda(x)\} = \mathcal{F}\{\Pi(x) * \Pi(x)\} = \operatorname{sinc}^2 x$$

$$\therefore \mathcal{F}\{g(x)\} = \mathcal{F}\left\{\Lambda\left(\frac{x+1}{2}\right) - \Lambda\left(\frac{x-1}{2}\right)\right\}$$

$$= 4e^{j2\pi f}\operatorname{sinc}^2(2f) - 4e^{-j2\pi f}\operatorname{sinc}^2(2f)$$

(d) $g(x) = \delta(\sin(x))$

Sol:

$$\because \delta(g(x)) = \frac{\delta(x - x_0)}{|g'(x_0)|}$$

$\Rightarrow$ In general $\delta(g(x)) = \sum_{n=1}^{N} \frac{\delta(x - x_n)}{|g'(x_n)|}$

$$\therefore \delta(\sin(x)) = \sum_{n=-\infty}^{\infty} \delta(x - n\pi)$$

$$\mathcal{F}\{g(x)\} = \mathcal{F}\{\delta(\sin(x))\}$$

$$= \sum_{n=-\infty}^{\infty} \mathcal{F}\{\delta(x - n\pi)\}$$

$$= \sum_{n=-\infty}^{\infty} \exp(-j2n\pi f) \quad \mathcal{F}\{\sum_n \delta(x - n\pi)\} = \sum_n \delta(f - n\pi)$$

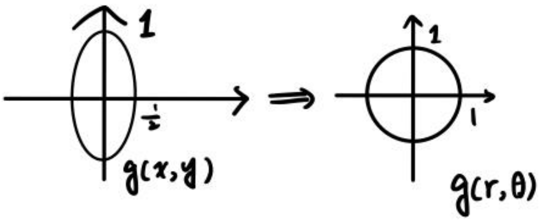
(2) Determine the 2D Fourier transform of

$$g(x,y)=1 \quad for\ (x-1)^2+\frac{y^2}{4}<1, \quad g(x,y)=0 \quad otherwise.$$

Sol:

$g(x,y) \Rightarrow g(r,\theta)$ $\quad x = r\cos\theta,\ y = 2r\sin\theta$

$G(f,h) \Rightarrow G(s,\phi)$ $\quad f = 2s\cos\phi,\ h = s\sin\phi$



$$g(r,\theta) = g(\ x = r\cos\theta,\ y = 2r\sin\theta) = \begin{cases} 1, & |r|<1 \\ 0, & \text{otherwise} \end{cases}$$

$$g(r) = \delta(r-1)$$

$$G(s) = 2\pi\int_0^\infty g(r) J_0(2\pi sr)\, r\, \delta(r-1)\, dr$$

$$= 2\pi J_0(2\pi s) \quad \text{: if } g(r) = \Pi(r) = \begin{cases} 1, & -a \le r \le a \\ 0, & \text{otherwise} \end{cases}, \text{ then } G(s) = \frac{a \cdot J_1(2\pi s)}{s}$$

$$G(f,h) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi fx} e^{-j2\pi fy}\, g(x,y)\, dx\, dy = G(\ f = 2s\cos\phi,\ h = s\sin\phi,$$

$$= \int_0^1\int_0^{2\pi} e^{-j2\pi sr\cos(\phi-\theta)}\, r\, d\theta\, dr = 2\pi\int_0^1 J_0(2\pi sr)\, r\, dr = \frac{J_1(2\pi s)}{s} = \frac{J_1(2\pi\sqrt{f^2+4h^2})}{\sqrt{f^2+4h^2}}$$

where $s = \sqrt{f^2+4h^2}$

(3) Determine the 30-point DFT of $g[n]$ where

$g[n]=1$ *when* n is a multiple of 3 or 5, $g[n]=0$ *otherwise.*

$$G[m] = \sum_{n=0}^{N-1} g[n] e^{-j2\pi \frac{mn}{N}}, \ N=30, \ 0-29$$

$$= \sum_{k=0}^{9} e^{-j2\pi \frac{m3k}{30}} + \sum_{k=0}^{5} e^{-j2\pi \frac{m5k}{30}} - \sum_{k=0}^{1} e^{-j2\pi \frac{m15k}{30}}$$

$$= 10 P_{10}[m] + 6 P_6[m] - 2 P_2[m], \ 0 \le m < 30$$

(4) Determine the following convolutions.

(a) $\sin(5\pi x)\cos(3\pi x) * \mathrm{sinc}(5x) * \mathrm{sinc}(10x)$

Sol:

$$\sin(a)\cos(b) = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$$

$$\Rightarrow g_1(x) = \sin(5\pi x)\cos(3\pi x) = \frac{1}{2}(\sin(\underset{\substack{\downarrow \\ 2\cdot 4}}{8\pi x}) + \sin(\underset{\substack{\downarrow \\ 2\cdot 1}}{2\pi x}))$$

$$G_1(f) = \mathcal{F}\{g_1(x)\} = \frac{1}{2}\{\frac{1}{2j}[\delta(f-4) - \delta(f+4) + \delta(f-1) - \delta(f+1)\}$$

$$g_2(x) = \mathrm{sinc}(5x) \quad g_3(x) = \mathrm{sinc}(10x)$$

$$G_2(f) = \mathcal{F}\{g_2(x)\} = \frac{1}{5}\Pi(\frac{f}{5}) = \begin{cases} \frac{1}{5}, & -\frac{5}{2} \le x \le \frac{5}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$G_3(f) = \mathcal{F}\{g_3(x)\} = \frac{1}{10}\Pi(\frac{f}{10}) = \begin{cases} \frac{1}{10}, & -5 \le x \le 5 \\ 0, & \text{otherwise} \end{cases}$$

$$g(x) = g_1(x) * g_2(x) * g_3(x) \Rightarrow G(f) = \mathcal{F}\{g(x)\}$$

$$\Rightarrow G_1(f)G_2(f)G_3(f)$$

$$= \frac{1}{100}\{\frac{1}{2j}[\delta(f-1) - \delta(f+1)]\}$$

$$g(x) = \mathcal{F}^{-1}\{G(f)\} = \frac{1}{100}\sin(2\pi x) \quad *$$

(b) $\delta'(x) * \delta(2x) * \delta(x-3) * \exp(-x^2)$

Sol:

$$\delta(2x) = \frac{1}{2}\delta(x)$$

$$\delta(2x) * \delta(x-3)$$

$$= \frac{1}{2}\delta(x-3)$$

$$\delta'(x) * \exp(-x^2) = \frac{de^{-x^2}}{dx} = -2xe^{-x^2}$$

$$\therefore \delta'(x) * \delta(2x) * \delta(x-3) * \exp(-x^2)$$

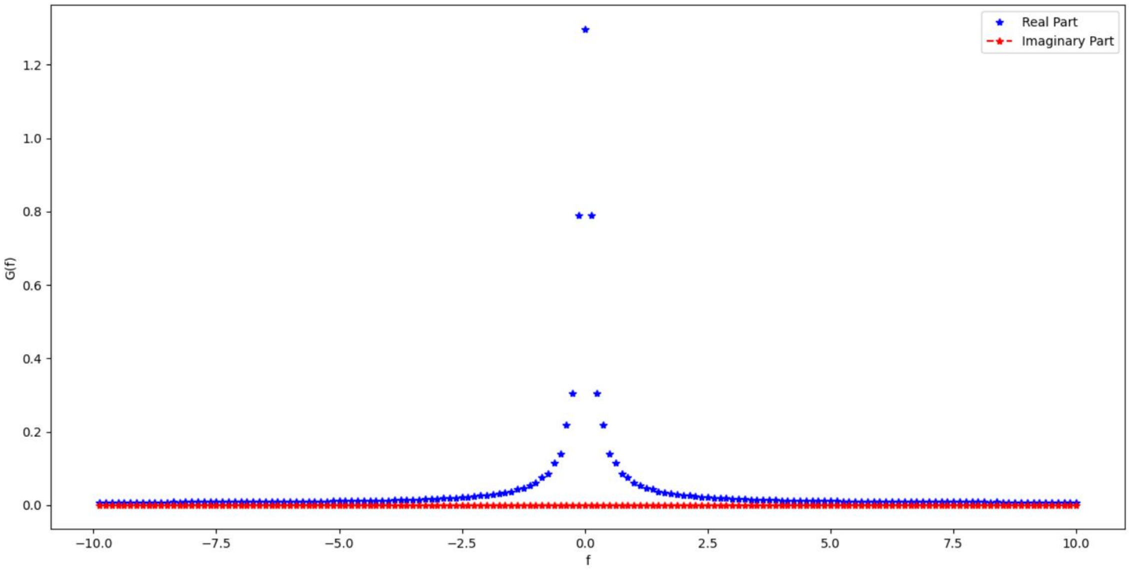
$$= -2xe^{-x^2} * \frac{1}{2}(\delta(x-3))$$

$$= -(x-3)e^{-(x-3)^2} \#$$

(5) Using a Matlab or Python code to determine the continuous Fourier transform of the following functions by the DFT.

(a) $g(x) = \exp(-|x|^{0.5}) - \exp(-2)$, for $-4 < x < 4$,

$g(x) = 0$ otherwise, $\Delta_x = 0.05$.



(b) $g(x) = \sin(\pi x^2/9)$ for $0 < x < 3$, $g(x) = 0$ otherwise, $\Delta_x = 0.1$.