1. Solve the following nonlinear DE: (7 %)

$$y^{-3}(x)\,y''(x) = 1, \qquad y(1) = -\sqrt{2},\ \ y'(1) = \sqrt{2}$$

Sol:

$$u = y' \quad \frac{d^2}{dx^2}y = \frac{d}{dx}u = \frac{dy}{dx}\frac{d}{dy}u = u\frac{d}{dy}u$$

$$\Rightarrow \quad y^{-3}u\frac{d}{dy}u = 1$$

$$\Rightarrow \quad u\frac{d}{dy}u = y^3 \Rightarrow u\,du = y^3\,dy$$

$$\Rightarrow \quad \frac{y^4}{4} = \frac{1}{2}u^2 + C, \quad \frac{(-\sqrt{2})^4}{4} = \frac{1}{2}(\sqrt{2})^2 + C \Rightarrow C = 0$$

$$\frac{y^4}{4} = \frac{1}{2}(y')^2 \Rightarrow y^4 = 2(y')^2 = 2(x+f)^2 \Rightarrow 4 = 2(1+f)^2$$

$$f = \sqrt{2} - 1$$

$$y^4 = 2(x+\sqrt{2}-1)^2 \Rightarrow y^2 = \sqrt{2}(x+\sqrt{2}-1)$$

$$\therefore\ y = \sqrt{\sqrt{2}x - \sqrt{2} + 2}$$

2. Solve the following PDEs:

(a) $16\dfrac{\partial^2}{\partial x^2}u(x,t)=\dfrac{\partial}{\partial t}u(x,t), \quad 0<x<2, \quad t>0, \quad u(0,t)=u(2,t)=0,$

$u(x,0)=\sin(\pi x)$ for $0<x<1$, $u(x,0)=0$ for $1<x<2$

Sol:

$$u(x,t)=X(x)T(t)$$

$$16X''T=XT' \Rightarrow \frac{X''}{X}=\frac{T'}{16T}=-\lambda$$

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ T'(t)+16\lambda T(t)=0 \end{cases}$$

Case 1 $\lambda=0$

$$\begin{cases} X''(x)=0 \\ T'(t)=0 \end{cases} \Rightarrow \begin{cases} X=C_1+C_2x \\ T=C_3 \end{cases}$$

$X'(0)=X'(2)=0 \quad u=XT=A_1$

Case 2 $\lambda<0$ $\lambda=-\alpha^2$

$$\begin{cases} X''(x)-\alpha^2X(x)=0 \\ T'(t)-16\alpha^2T(t)=0 \end{cases}$$

$$\begin{cases} X=C_4e^{\alpha x}+C_5e^{-\alpha x} \\ T=C_6e^{16\alpha t} \end{cases}$$

$X'(0)=X'(2)=0$

$C_4=C_5=0$

$u=0$

Case 3 $\lambda>0$ $\lambda=\alpha^2$

$$\begin{cases} X''(x)+\alpha^2X(x)=0 \\ T'(t)+16\alpha^2T(t)=0 \end{cases}$$

$$\begin{cases} X=C_7\cos(\alpha x)+C_8\sin(\alpha x) \\ T=C_9e^{-16\alpha^2t} \end{cases}$$

$X'(0)=X'(2)=0 \Rightarrow X=C_7\cos(\frac{n\pi}{2}x)$

$\lambda=\frac{n^2\pi^2}{4}$

$T=C_9\exp(-16\frac{n^2\pi^2}{4}t)=C_9e^{-4n^2\pi^2t}$

$$u=XT=\sum_{n=0}^{\infty}A_ne^{-4n^2\pi^2t}\cos(\frac{n\pi}{2}x)$$

當 $u(x,0)=\sin(\pi x)$

$$\sum_{n=0}^{\infty}A_n\cos(\frac{n\pi}{2}x)=\sin(\pi x)$$

$a_0=\frac{1}{2}\int_0^2\sin(\pi x)dx=0 \quad a_n=\frac{2}{2}\int_0^2\sin(\pi x)\cos(\frac{n\pi}{2}x)dx=\frac{4(\cos(n\pi)-1)}{\pi(n^2-4)}$

$$u=\sum_{n=1}^{\infty}\frac{4(\cos(n\pi)-1)}{\pi(n^2-4)}e^{-4n^2\pi^2t}\cos(\frac{n\pi}{2}x)$$

(b) $\frac{\partial}{\partial x}u(x,y)+y\frac{\partial}{\partial y}u(x,y)=\cos x+y$

Sol: $dy=ydx \Longrightarrow \ln(y)=x+C_1 \Longrightarrow y=e_{x+C_1}=C_2e_x$

$\eta=x+\ln y\eta=x+\ln y$，這意味著 $y=e\eta-xy=e_{\eta-x}$

把 $\eta\eta$ 帶入原方程： $\partial u\partial x+y\partial u\partial y=\cos(x+y)_{\partial x\partial u}+y_{\partial y\partial u}=\cos(x+y)$ 現在我們將 uu 表示為 $u=u(\eta)u=u(\eta)$，並且計算：

$\partial u\partial x=dud\eta\partial\eta\partial x=dud\eta(1-1y\partial y\partial x)_{\partial x\partial u=d\eta du\partial x\partial\eta=d\eta du}(1-_{y1\partial x\partial y})$ 由於 $\partial y\partial x=y_{\partial x\partial y}=y$， $\partial\eta\partial x=1-1=0_{\partial x\partial\eta}=1-1=0$ 所以 $\partial u\partial x=dud\eta_{\partial x\partial u=d\eta du}$

我們可以簡化為常數係數微分方程： $dud\eta=\cos(x+e\eta-x)_{d\eta du}=\cos(x+e_{\eta-x})$

$u(\eta)=\int\cos(x+y)d\eta$ $u(\eta)=\sin(x+y)+Cu(\eta)=\sin(x+y)+C$

$u(x,y)=\sin(x+y)+C$

(c) $\frac{\partial}{\partial x}u(x,y,z)+\tan y\frac{\partial}{\partial y}u(x,y,z)+\cot z\frac{\partial}{\partial z}u(x,y,z)=0$

Sol:

3. Suppose that (6 %)

$$\phi_0(x)=1,\qquad \phi_1(x)=x+c_0,\qquad \phi_2(x)=x^2+d_1x+d_0,\qquad x\in[0,10]$$

Find c_0, d_0, and d_1 such that $\phi_0(x)$, $\phi_1(x)$, and $\phi_2(x)$ form an orthogonal set (unnecessary to be orthonormal) within $x\in[0,10]$ and with respect to the weight function of $w(x)=x(10-x)$.

Sol:

$$\int_0^{10} w(x)\phi_0(x)\phi_1(x)\,dx = 0$$

$$\int_0^{10} x(10-x)(x+C_0) = 0 \Rightarrow C_0 = -5 \Rightarrow \phi_1(x) = x-5$$

$$\int_0^{10} w(x)\phi_0(x)\phi_2(x)\,dx = 0$$

$$\Rightarrow \int_0^{10} (10x-x^2)(x^2+d_1x+d_0) = 0$$

$$\Rightarrow 5d_1 + d_0 = -30$$

$$\int_0^{10} (10x-x^2)(x-5)(x^2+d_1x+d_0) = 0$$

$$\Rightarrow d_1 = -10 \Rightarrow d_0 = 20$$

4. Find the Fourier transforms of the following functions:

(a) $g(x) = x \quad for\ 10 < |x| < 20, \qquad g(x) = 0 \quad otherwise$

(Please express the result in terms of the sinc function)

Sol:

$$g(x) = \begin{cases} x & \text{for } 10 < |x| < 20 \\ 0 & \text{otherwise} \end{cases}$$

$$G(f) = \int_{-\infty}^{\infty} g(x) e^{-j2\pi f x} dx$$

$$= \int_{10}^{20} g(x) e^{-j2\pi f x} dx$$

$$\downarrow$$

$$\mathcal{F}\left\{ \Pi\left(\frac{x-15}{10}\right) \right\} = 10 e^{-j30\pi f} \operatorname{sinc}(10f)$$

$$\mathcal{F}\left\{ x \cdot \Pi\left(\frac{x-15}{10}\right) \right\} = \frac{1}{-j2\pi} \frac{d}{df} \left(10 e^{-j30\pi f} \operatorname{sinc}(10f) \right)$$

$$= 150 e^{-j30\pi f} \operatorname{sinc}(10f) + \frac{j20}{\pi} e^{-j30\pi f} \operatorname{sinc}'(10f)$$

(b) $g(x) = \exp\left(-x^2 - 2x - 3\right) + \delta(4x)$

Sol:

$$\mathcal{F}\{\exp(-x^2-2x-3)+\delta(4x)\}$$
$$= 4 + \int_{-\infty}^{\infty} e^{-x^2-2x-3} e^{-j2\pi f x}\,dx$$
$$= 4 + e^{-2}e^{-\pi^2 f^2} e^{j2\pi f} \int_{-\infty}^{\infty} e^{-(x+1+j\pi f)^2}\,dx$$
$$= 4 + e^{-2}e^{-\pi^2 f^2} e^{j2\pi f} \int_{-\infty}^{\infty} (x+1+j\pi f) e^{-2}\,dx$$
$$= 4 + e^{-2}e^{-\pi^2 f^2} e^{j2\pi f} \left[\int_{-\infty}^{\infty} x e^{-2}\,dx - (1+j\pi f)\sqrt{\pi}\right]$$
$$= \sqrt{\pi}\, e^{-2}(-j\pi f - 1) e^{-\pi^2 f^2} e^{j2\pi f} + 4$$

(c) $g(x,y)=1 \quad if\ 1 \le \sqrt{(x/2)^2+y^2} \le 2, \quad g(x,y)=0 \quad otherwise.$

(Please determine the two-dimensional FT of $g(x, y)$)

Sol:

5. Find the following convolutions:

(a) $\operatorname{sinc}(3x) * \operatorname{sinc}(6x) * \operatorname{sinc}(12x) * \left(\cos(2\pi x) + \sin(4\pi x) + \cos(8\pi x)\right)$

Sol:

$$g_1(x) = \text{sinc}(3x) \Rightarrow G_1(f) = \frac{1}{3}$$
$$g_2(x) = \text{sinc}(6x) \Rightarrow G_2(f) = \frac{1}{6}$$
$$g_3(x) = \text{sinc}(12x) \Rightarrow G_3(f) = \frac{1}{12}$$

$$G_1(f)G_2(f)G_3(f) = \frac{1}{216}$$

$$g_4(x) = \cos(2\pi x) + \sin(4\pi x) + \cos(8\pi x)$$
$$\downarrow$$
$$G_4(x) = \frac{1}{2}\delta(f+1) + \frac{1}{2}\delta(f-1) + \frac{-j}{2}\delta(f-2) + \frac{j}{2}\delta(f+2) + \frac{1}{2}\delta(f+4) + \frac{1}{2}\delta(f-4)$$
$$= \frac{1}{2}\delta(f+4) + \frac{1}{2}\delta(f-4)$$

$$G_1(f)G_2(f)G_3(f)G_4(f) = \frac{1}{216}\left(\frac{1}{2}\delta(f+4) + \frac{1}{2}\delta(f-4)\right)$$
$$= \frac{1}{216}\cos(8\pi x)$$

(b) $\delta(x-1) * \delta''(x) * \delta(3x) * \left(x^3 + x^2 + x + 1\right)$

Sol:

δ(3x)=1/3δ(x)
δ(x−1)·δ′′(x)·1/3δ(x)=0

0(x^3+x^2+x+1)=0

6. Suppose that the PMF of X is (10 %)

$$P_X(1) = P_X(5) = 0.1, \quad P_X(2) = P_X(4) = 0.2, \quad P_X(3) = 0.4$$

Determine the variance, the skewness, the kurtosis, and the entropy of X.
Please express the entropy in terms of ln10 and ln2.

Sol:

$$\mu_x = E(X) = \sum_n n P_x(n) = 1\times0.1 + 2\times0.2 + 3\times0.4 + 4\times0.2 + 5\times0.1 = 3$$

$$\text{variance } (\sigma^2) = E((X-\mu_x)^2) = \sum_n (n-\mu_x)^2 P_x(n)$$
(k=2)
$$= 4\times0.1 + 0.2 + 0.2 + 4\times0.1 = 1.2$$

$$\text{Skewness } (k=3) = \frac{\sum_n (n-\mu_x)^3 P_x(n)}{\left(\sum_n (n-\mu_x)^2 P_x(n)\right)^{3/2}}$$

$$= \frac{-8\times0.1 + (-1)\times0.2 + 0.2 + 8\times0.1}{(1.2)^{3/2}} = 0$$

$$\text{kurtosis } (k=4) = \frac{\sum_n (n-\mu_x)^4 P_x(n)}{\left(\sum_n (n-\mu_x)^2 P_x(n)\right)^2}$$

$$= \frac{16\times0.1 + 0.2 + 0.2 + 16\times0.1}{(1.2)^2} = \frac{3.6}{1.44} = 2.5$$

$$\text{Entropy} = -\sum_n P_x(n)\ln[P_x(n)] = -0.1\ln0.1 - 0.2\ln0.2 - 0.4\ln0.4 - 0.2\ln0.2 - 0.1\ln0.1$$

$$\Rightarrow \ln(0.1) = \ln(\tfrac{1}{10}) = -\ln(10),\ \ln(0.2) = \ln(2) - \ln(10)\ \ \ln(0.4) = 2\ln(2) - \ln(10)$$

$$\therefore -(0.1(-\ln10) + 0.2(\ln2 - \ln10) + 0.4(2\ln2 - \ln10) + 0.2(\ln2 - \ln10) + 0.1(-\ln10))$$

$$= -(1.2\ln2 - \ln10) = \ln10 - 1.2\ln2$$

7. The matrix **A** and the vector **b** are

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & -2 \\ 2 & -2 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -17 \\ 9 \end{bmatrix}.$$

(a) Find the singular values of **A**. (4 %)

Sol:

$$\underline{\underline{A}} = \begin{bmatrix} 3 & 2 & -2 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow \underline{\underline{B}} = \underline{\underline{A}}^T = \begin{bmatrix} 3 & 2 \\ 2 & -2 \\ -2 & 1 \end{bmatrix} = \underline{\underline{U}}\,\underline{\underline{\Sigma}}\,\underline{\underline{V}}^T$$

$$\underline{\underline{B}}^T\underline{\underline{B}} = \begin{bmatrix} 3 & 2 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ 0 & 9 \end{bmatrix}$$

$$\rightarrow \lambda_1 = 17 \quad \lambda_2 = 9$$

$$\sigma_1 = \sqrt{17} \quad \sigma_2 = 3$$

(b) Find the pseudo-inverse of **A**. (5 %)

Sol:

$\because$ $\underline{\underline{A}}$列獨立，用 $\underline{\underline{B}} = \underline{\underline{A}}^T$

$$\underline{\underline{B}}^+ = (\underline{\underline{B}}^T\underline{\underline{B}})^{-1}\underline{\underline{B}}^T = (\underline{\underline{A}}\,\underline{\underline{A}}^T)^{-1}\underline{\underline{A}}$$

$$\rightarrow \underline{\underline{A}}^+ = (\underline{\underline{B}}^T)^+ = (\underline{\underline{B}}^+)^T = \left[(\underline{\underline{A}}\,\underline{\underline{A}}^T)^{-1}\underline{\underline{A}}\right]^T = \underline{\underline{A}}^T(\underline{\underline{A}}\,\underline{\underline{A}}^T)^{-1}$$

$$\begin{bmatrix} 3 & 2 \\ 2 & -2 \\ -2 & 1 \end{bmatrix}\left(\begin{bmatrix} 17 & 0 \\ 0 & 9 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 3 & 2 \\ 2 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{17} & 0 \\ 0 & \frac{1}{9} \end{bmatrix} = \begin{bmatrix} \frac{3}{17} & \frac{2}{9} \\ \frac{2}{17} & \frac{-2}{9} \\ \frac{-2}{17} & \frac{1}{9} \end{bmatrix}$$

(c) Determine the LS solution $\mathbf{x}_{LS}$ to $\mathbf{Ax} = \mathbf{b}$. (3 %)

Sol:

$$\underline{X}_{LS} = \underline{\underline{A}}^{\dagger}\underline{b} = \begin{bmatrix} \frac{3}{17} & \frac{2}{9} \\ \frac{2}{17} & \frac{-2}{9} \\ \frac{-2}{17} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} -17 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix}$$

(d) Find ρ_{LS}, which is the size of the minimum residual associated with $\mathbf{Ax} = \mathbf{b}$. (2 %)

Sol:

$$\rho_{LS} = \|(\underline{\underline{I}} - \underline{\underline{A}}\underline{\underline{A}}^{\dagger})\underline{b}\|_2 = \left\|\left(\underline{\underline{I}} - \begin{bmatrix} 3 & 2 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{17} & \frac{2}{9} \\ \frac{2}{17} & \frac{-2}{9} \\ \frac{-2}{17} & \frac{1}{9} \end{bmatrix}\right) \begin{bmatrix} -17 \\ 9 \end{bmatrix}\right\|_2$$

$$= \left\|\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -17 \\ 9 \end{bmatrix}\right\|_2 = 0$$

8. Write **True** or **False** for these statements. There is no need to justify your answer. (6 %)

(a) If $\mathbf{J}$ is a Jordan block, then $\mathbf{J}^{1000}$ is also a Jordan block.

(b) Let $\mathbf{A} \in \mathbb{C}^{3\times3}$. The matrix 2-norm $\|\mathbf{A}\|_2$ and the nuclear norm $\|\mathbf{A}\|_*$ satisfy $\|\mathbf{A}\|_2 = |\mathbf{A}\|_*$ if and only if rank($\mathbf{A}$) = 1.

(c) Let $\mathbf{A} \in \mathbb{C}^{M\times N}$. If σ is a singular value of $\mathbf{A}$, then σ^2 is an eigenvalue of $\mathbf{AA}^{\mathrm{H}}$.

Sol:

a.True

b.Ture

c.Ture

9. The matrix $\mathbf{A}$ is (5%)

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

The integer N satisfies $N \geq 1000$. Find the value of the entry-wise L_∞ norm

$$\left\| \left(\mathbf{A} \otimes \left(\mathbf{A} - 3\mathbf{I}_5 \right) \right)^N \right\|_\infty .$$

Simplify and express your answer in terms of N.

Sol:

$$B = (A - 3I_5) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} (a_{1,1})B & (a_{1,2})B & (a_{1,3})B & (a_{1,4})B & (a_{1,5})B \\ (a_{2,1})B & (a_{2,2})B & (a_{2,3})B & (a_{2,4})B & (a_{2,5})B \\ (a_{3,1})B & (a_{3,2})B & (a_{3,3})B & (a_{3,4})B & (a_{3,5})B \\ (a_{4,1})B & (a_{4,2})B & (a_{4,3})B & (a_{4,4})B & (a_{4,5})B \\ (a_{5,1})B & (a_{5,2})B & (a_{5,3})B & (a_{5,4})B & (a_{5,5})B \end{bmatrix}$$

$$= \begin{bmatrix} 4\mathbf{B} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{B} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & \mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B} & \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 4\mathbf{B} \end{bmatrix}$$

$$\Rightarrow \left\| \begin{bmatrix} 4\mathbf{B} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{B} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & \mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B} & \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 4\mathbf{B} \end{bmatrix}^{1000} \right\|_\infty \triangleq \max_{(m,n)\in[M]\times[N]} |a_{m,n}| = 8$$