Diagonalizable Groups and Tori

Def A linear algebraic G is <u>diagonalizable</u> if it is isomorphic to a closed subgroup of $(\mathbb{G}_m)^n$ for some $n \geq 0$. The group G is an <u>algebraic torus</u> (or just a <u>torus</u>) if $G \simeq (\mathbb{G}_m)^n$ for some $n \geq 0$.

<u>Note</u>: $(\mathbb{G}_m)^0 = \{1\}$ trivial group.

We'll see that there is a dictionary:

$$\left\{ \begin{array}{c} \text{diagonalizable} \\ \text{groups} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{finitely generated} \\ \text{abelian groups} \end{array} \right\}$$

$$\begin{array}{ccc} \cup & & \cup \\ \mathbb{G}_m & \longleftrightarrow & \mathbb{Z} \end{array}$$

This dictionary is defined by taking characters.

Def Let G be a linear algebraic group.

① A <u>character</u> of G is a homomorphism of algebraic groups

$$\chi : G \longrightarrow \mathbb{G}_m$$

We set $X^*(G) = \{\text{characters of } G\}$

② A cocharacter (or multiplicative one-parameter subgroup) of G is a group homomorphism

$$\lambda : \mathbb{G}_m \longrightarrow G$$

We set

$$X_*(G) = \{\text{cocharacters of } G\}.$$

Remarks: $X^*(G) \subseteq k[G]$ and $X^*(G)$ is an abelian group, written additively:

$$(\chi_1 + \chi_2)(g) = \chi_1(g) \cdot \chi_2(g) \qquad \forall g \in G$$

If G is commutative, then ~~S~~similarly, $X_*(G)$ is an abelian group, written additively:

$$(\lambda_1 + \lambda_2)(x) = \lambda_1(x)\, \lambda_2(x) \qquad \forall x \in \mathbb{G}_m.$$

$\boxed{\text{Examples}}$

① $G = \mathbb{G}_m$. Then $X^*(G) = \mathrm{Hom}_{\mathrm{Grps}}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$.

and $X_*(\mathbb{G}_m) = \mathbb{Z}$.

② $G = (\mathbb{G}_m)^n$. Then $X^*(G) = \mathrm{Hom}_{\mathrm{Grps}}((\mathbb{G}_m)^n, \mathbb{G}_m)$

$$= \left(\mathrm{Hom}_{\mathrm{Grps}}(\mathbb{G}_m, \mathbb{G}_m)\right)^n = \mathbb{Z}^n$$

with $\mathbb{Z}$-basis given by

$$\chi_i : (\mathbb{G}_m)^n \longrightarrow \mathbb{G}_m$$

$$x = (x_1, \dots, x_n) \longmapsto x_i$$

Note: Each $x_i \in k[(\mathbb{G}_m)^n]$ and we have

$$k[(\mathbb{G}_m)^n] = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

In fact, $X^*((\mathbb{G}_m)^n)$ forms a basis of $k[(\mathbb{G}_m)^n]$.

Similarly, $X_*((\mathbb{G}_m)^n) \simeq \mathbb{Z}^n$ with $\mathbb{Z}$-basis given by

$$\lambda_i : \mathbb{G}_m \longrightarrow (\mathbb{G}_m)^n$$

$$a \longmapsto (1, \ldots, 1, \underset{\uparrow\, i\text{th coordinate}}{a}, 1, \ldots, 1)$$

③ Let ζ_e be primitive eth root of unity in k. Let $G = \langle \zeta_e \rangle \simeq \mathbb{Z}/e\mathbb{Z}$, so G is a diagonalizable group. We have

$$X^*(G) = \mathrm{Hom}_{Grp}(\langle \zeta_e \rangle, \mathbb{G}_m)$$

$$= [\zeta_e \longmapsto \zeta_e^i \text{ for } i \in \{0, 1, \ldots, e-1\}]$$

$$\cong \mathbb{Z}/e\mathbb{Z}.$$

Meanwhile,

$$X_*(G) = \mathrm{Hom}_{Grp}(\mathbb{G}_m, \langle \zeta_e \rangle) = \{1\}$$

since the image of $\mathbb{G}_m \to \mathbb{G}_m$, $x \longmapsto x^n$ is in $\langle \zeta_e \rangle$ iff $n=0$.

Theorem Let G be a linear algebraic group. The following are equivalent:

(a) G is diagonalizable.

(b) $X^*(G)$ is a fin. gen. abelian group and its elements form a k-basis of $k[G]$.

(c) Any representation of G is the direct sum of one-dimensional representations

proof (a) $\Rightarrow$ (b).

G diagonalizable $\Rightarrow$ $G \subseteq (\mathbb{G}_m)^n$ for some n

$\Rightarrow$ $k[G] = k[(\mathbb{G}_m)^n] / I$ for some ideal I

$\Rightarrow$ $X^*((\mathbb{G}_m)^n) \xrightarrow{\text{restrict}} k[G]$ is surjective.

Since $X^*((\mathbb{G}_m)^n) \simeq \mathbb{Z}^n$ and its elements form a k-basis of $k[(\mathbb{G}_m)^n]$, we obtain (b).

(b) $\Rightarrow$ (c)

Assume (b). Let $\rho : G \longrightarrow GL(V)$ be a representation. Pick a basis of V and extend ρ to a map

$$\bar{\rho} : G \longrightarrow \mathrm{Mat}_{n,n}$$

Aim to show: $$V = \bigoplus_{\chi \in X^*(G)} V_\chi$$

where $V_\chi = \{ v \in V : g \cdot v = \chi(g) v \ \ \forall g \in G \}$

Let $\bar{\rho}^* : k[\text{Mat}_{n,n}] \longrightarrow k[G]$

be the algebra homomorphism corresponding to $\bar{\rho}$. Since $X = X^*(G)$ forms a k-basis of $k[G]$, we have:

$$\bar{\rho}^*(f) = \sum_{\chi \in X} f_\chi \cdot \chi$$

for some $f_\chi \in k$. These f_χ are uniquely determined, and only finitely many are nonzero.

For $i, j \in \{1, .., n\}$, let T_{ij} be the corresponding generator of $k[\text{Mat}_{n,n}]$, and let $E_{ij} \in \text{Mat}_{n,n}$ be the corresponding elementary matrix. Then:

$$\bar{\rho}(g) = \sum_{i=1}^{n} \sum_{j=1}^{n} T_{ij}(\bar{\rho}(g)) \, E_{ij}$$

$$= \sum_{i,j} \bar{\rho}^*(T_{ij})(g) \, E_{ij}$$

$$= \sum_{i,j} \sum_{\chi} (T_{ij})_\chi \, \chi(g) \, E_{ij}$$

$$= \sum_{\chi} \chi(g) \underbrace{\left(\sum_{i,j} (T_{ij})_{\chi} E_{ij} \right)}_{A_\chi \in \mathrm{Mat}_{n,n}} = \sum_{\chi} \chi(g) A_\chi$$

To check:

- $A_\chi \cdot A_\psi = \begin{cases} A_\chi & \text{if } \chi = \psi \\ 0 & \text{otherwise.} \end{cases}$

- $\sum_\chi A_\chi = id_n$

- $\mathrm{Im}(A_\chi) = V_\chi \qquad (= \{ v \in V : g \cdot v = \chi(g) v \ \forall g \in G \})$

From these facts it follows that $V = \bigoplus_\chi V_\chi$, as G-representations.

For each χ, V_χ is a direct sum of the one-dimensional representation given by χ:

$$G \longrightarrow GL_1 = \mathbb{G}_m. \quad \text{This proves (c).}$$

(c) $\Rightarrow$ (a) Embed G into a closed subgroup of $GL(V)$ for some V.

By assumption, $\exists$ basis of V in which G acts diagonally.

Thus $G \hookrightarrow D_n \simeq (\mathbb{G}_m)^n$, so G is diagonalizable.

$\square$

Cor If G is diagonalizable, then the group algebra of the abelian group $X^*(G)$ is isomorphic to $k[G]$.

($\mathrm{char}(k) = 0$)

$$\left\{ \begin{matrix} \text{diagonalizable} \\ \text{groups} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{finitely generated} \\ \text{abelian groups} \end{matrix} \right\}$$

$$G \longmapsto X^*(G)$$

$\boxed{\text{Claim}}$ This is a bijection.

Construction: Let M be a fin. gen. abelian group. Let $k[M]$ denote its group algebra, with basis $\{e_m : m \in M\}$. Define a Hopf algebra structure on $k[M]$ via:

$$\Delta : k[M] \longrightarrow k[M] \otimes k[M] \qquad \eta : k[M] \longrightarrow k$$

$$e_m \longmapsto e_m \otimes e_m \qquad e_m \longmapsto 1$$

$$S : k[M] \longrightarrow k[M]$$

$$e_m \longmapsto e_{-m}$$

One argues that $k[M]$ is reduced. Thus, $\exists$ a linear algebraic group G with

$$k[G] = k[M] \quad \text{as Hopf algebras.}$$

To check:

- G is diagonalizable
- $X^*(G) \simeq M$

Idea:

$$X^{\vee}(G) = \mathrm{Hom}_{\text{grps}}(G, \mathbb{G}_m) = \mathrm{Hom}_{\text{Hopf-Alg}}(k[T^{\pm 1}], k[G])$$

$$= \mathrm{Hom}_{\text{Hopf-Alg}}(k[T^{\pm 1}], k[M])$$

$$= \{ T \longmapsto e_m \ : m \in M \} \simeq M$$

reduces to the argument given to show that $\mathrm{Hom}_{\text{Grp}}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$..

Use part (b) of the theorem to show that G is diagonalizable.

We have:

$$\begin{tikzcd}
\left\{ \begin{array}{c} \text{diagonalizable} \\ \text{groups} \end{array} \right\} \arrow[r, leftrightarrow] & \left\{ \begin{array}{c} \text{fin. gen.} \\ \text{abelian groups} \end{array} \right\} \\
\{\text{tori}\} = \left\{ \begin{array}{c} \text{connected} \\ \text{diagonalizable} \\ \text{groups} \end{array} \right\} = \{ (\mathbb{G}_m)^n : n \geq 0 \} \arrow[u, hook] \arrow[r, Leftrightarrow] & \{ \text{free f.g. abelian grp} \} \arrow[u, hook]
\end{tikzcd}$$

$$= \{ \mathbb{Z}^n : n \geq 1 \}$$

$$= \left\{ \begin{array}{c} \text{torsion-free f.g.} \\ \text{abelian groups} \end{array} \right\}$$

Prop [Rigidity of diagonalizable groups].

Let G and H be diagonalizable groups, and V a connected affine variety. Suppose

$$\phi : V \times G \longrightarrow H$$

is a morphism of varieties s.t. $g \longmapsto \phi(v,g)$ is a map of algebraic groups $G \to H$, $\forall v \in V$. Then $\phi(v,g)$ is independent of $v \in V$.

pf. Let $$\phi^* : k[H] \longrightarrow k[V] \otimes k[G]$$
be the corresponding algebra homomorphism. For $\psi \in X^*(H)$, write

$$\phi^*(\psi) = \sum_{\chi \in X^*(G)} f_{\psi,\chi} \otimes \chi$$

for some $f_{\psi,\chi} \in k[V]$. Argue that

① The map $g \longmapsto \sum_{\chi} f_{\psi,\chi}(v)\, \chi(g)$ is a character of G for all $v \in V$ (and fixed $\psi \in X^*(H)$)

② For fixed $\psi \in X^*(H)$, $\exists! \chi_\psi \in X^*(G)$ s.t.

$$f_{\psi,\chi} \equiv \begin{cases} 1 & \text{if } \chi = \chi_\psi \\ 0 & \text{otherwise} \end{cases} \quad \text{in } k[V].$$

Therefore, $\phi^*(\psi) = 1 \otimes \chi_\psi \qquad \forall \psi \in X^*(H)$.

Since $X^*(H)$ is a basis of $k[H]$, this implies the result □

Cor Let H be a diagonalizable subgroup of G. Then

$$N_G(H)^\circ = Z_G(H)^\circ \quad \text{and} \quad N_G(H)/Z_G(H) \text{ is finite.}$$

Pf. Apply the previous result to

$$N_G(H)^\circ \times H \longrightarrow H$$

$$g, h \longmapsto ghg^{-1}.$$

□

In exercises 3.2.10, assume char(k) = 0.

Let T be a torus, and set $X = X^*(T)$ and $Y = X_*(T)$.

Observe: There is a pairing

$$\langle\, , \rangle : Y \times X \longrightarrow \mathbb{Z}$$

given by

$$Y \times X = \mathrm{Hom}_{\mathrm{Grp}}(\mathbb{G}_m, T) \times \mathrm{Hom}_{\mathrm{Grp}}(T, \mathbb{G}_m) \xrightarrow{\text{compose}} \mathrm{Hom}_{\mathrm{Grp}}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}$$

$$\lambda \quad , \quad \chi \longmapsto \chi \circ \lambda$$

so $\chi(\lambda(a)) = a^{\langle \lambda, \chi \rangle}$ for all $a \in \mathbb{G}_m$.

This pairing induces maps

$$X \longrightarrow \mathrm{Hom}_{\mathrm{Grp}}(Y, \mathbb{Z}) \qquad Y \longrightarrow \mathrm{Hom}_{\mathrm{Grp}}(X, \mathbb{Z})$$

$$\chi \longmapsto \langle -, \chi \rangle \qquad \lambda \longmapsto \langle \lambda, - \rangle$$

Lemma ① $\langle\, , \rangle$ is a perfect pairing

(i.e. $\chi \mapsto \langle -, \chi \rangle$ and $\lambda \mapsto \langle \lambda, - \rangle$ are bijections.)

② There is a canonical isomorphism

$$k^\times \otimes_{\mathbb{Z}} Y \xrightarrow{\sim} T$$

given by $a \otimes \lambda \longmapsto \lambda a$ $\forall a \in k^\times \simeq \mathbb{G}_m$ and $\lambda \in Y$.

pf (idea). It is enough to prove this in the case that $T \simeq (\mathbb{G}_m)^n$, so $X \simeq \mathbb{Z}^n$ and $Y \simeq \mathbb{Z}^n$. Then

$$\langle y, x \rangle = \sum_{i=1}^{n} x_i y_i$$

is the dot product of $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots y_n)$.

Also, $k^\times \otimes_{\mathbb{Z}} Y \simeq k^\times \otimes_{\mathbb{Z}} \mathbb{Z}^n \simeq (k^\times)^n \simeq (\mathbb{G}_m)^n \simeq T$. $\square$

Rmk. In the case of a torus, Y (in addition to X) is a free abelian group.