

# Diagonalizable Groups and Tors

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**Def** A linear algebraic  $G$  is diagonalizable if it is isomorphic to a closed subgroup of  $(\mathbb{G}_m)^n$  for some  $n \geq 0$ . The group  $G$  is an algebraic torus (or just a torus) if  $G \approx (\mathbb{G}_m)^n$  for some  $n \geq 0$ .

Note:  $(\mathbb{G}_m)^\circ = \{1\}$  if trivial group.

We'll see that there is a dictionary:

$$\begin{array}{ccc}
 \left\{ \begin{array}{l} \text{diagonalizable} \\ \text{groups} \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{l} \text{finitely generated} \\ \text{abelian groups} \end{array} \right\} \\
 \psi & & \psi \\
 \mathbb{G}_m & \longmapsto & \mathbb{Z}
 \end{array}$$

This dictionary is defined by taking characters.

**Def** Let  $G$  be a linear algebraic group.

① A character of  $G$  is a homomorphism of algebraic groups

$$\chi: G \longrightarrow \mathbb{G}_m$$

We set  $\chi^*(G) = \{\text{characters of } G\}$

[2]

② A cocharacter (or multiplicative one-parameter subgroup) of  $G$  is a group homomorphism

$$\chi : \mathbb{G}_m \longrightarrow G$$

We set

$$X^*(G) = \{ \text{cocharacters of } G \}.$$

Remarks:  $X^*(G) \subseteq k[G]$  and  $X^*(G)$  is an abelian group,

written additively:

$$(\chi_1 + \chi_2)(g) = \chi_1(g) \cdot \chi_2(g) \quad \forall g \in G$$

If  $G$  is commutative, then

Similarly,  $X^*(G)$  is an abelian group, written additively:

$$(\lambda_1 + \lambda_2)(x) = \lambda_1(x) \lambda_2(x) \quad \forall x \in \mathbb{G}_m$$

### Example

①  $G = \mathbb{G}_m$ . Then  $X^*(G) = \text{Hom}_{\text{Grps}}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$

and  $X^*(\mathbb{G}_n) = \mathbb{Z}$ .

②  $G = (\mathbb{G}_m)^n$ . Then  $X^*(G) = \text{Hom}_{\text{Grps}}((\mathbb{G}_m)^n, \mathbb{G}_m)$

$$= \left( \text{Hom}_{\text{Grps}}(\mathbb{G}_m, \mathbb{G}_m) \right)^n = \mathbb{Z}^n$$

with  $\mathbb{Z}$ -basis given by

$$\chi_i : (\mathbb{G}_m)^n \longrightarrow \mathbb{G}_m$$

$$x = (x_1, \dots, x_n) \longmapsto x_i$$

Note: Each  $x_i \in k[(\mathbb{G}_m)^n]$  and we have

[3]

$$k[(\mathbb{G}_m)^n] = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

In fact,  $X^*((\mathbb{G}_m)^n)$  forms a basis of  $k[(\mathbb{G}_m)^n]$ .

Similarly,  $X_*(\mathbb{G}_m) \cong \mathbb{Z}^n$  with  $\mathbb{Z}$ -basis given by

$$\lambda_i : \mathbb{G}_m \longrightarrow (\mathbb{G}_m)^n$$

$$a \longmapsto (1, \dots, 1, a, 1, \dots, 1)$$

(ith coordinate,

③ Let  $\zeta_e$  be primitive  $e$ th root of unity in  $k$ . Let  $G = \langle \zeta_e \rangle$

$\cong \mathbb{Z}/e\mathbb{Z}$ , so  $G$  is a diagonalizable group. We have

$$X^*(G) = \text{Hom}_{\text{Grp}}(\langle \zeta_e \rangle, \mathbb{G}_m)$$

$$= [\zeta_e \longmapsto \zeta_e^i \text{ for } i \in \{0, 1, \dots, e-1\}]$$

$$\cong \mathbb{Z}/e\mathbb{Z}$$

Meanwhile,

$$X_*(G) = \text{Hom}_{\text{Grp}}(\mathbb{G}_m, \langle \zeta_e \rangle) = \{1\}$$

since the map  $\mathbb{G}_m \rightarrow \mathbb{G}_m$ ,  $x \mapsto x^n$  is in  $\langle \zeta_e \rangle$  iff  $n=0$ .

[4]

Theorem Let  $G$  be a linear algebraic group. The following are equivalent:

(a)  $G$  is diagonalizable.

(b)  $X^*(G)$  is a fin. gen. abelian group and its elements form a  $k$ -basis of  $k[G]$ .

(c) Any representation of  $G$  is the direct sum of one-dimensional representations.

Proof  $\underline{(a) \Rightarrow (b)}$

$G$  diagonalizable  $\Rightarrow G \simeq (\mathbb{G}_m)^n$  for some  $n$

$\Rightarrow k[G] = k[(\mathbb{G}_m)^n] / I$  for some ideal  $I$

$\Rightarrow X^*((\mathbb{G}_m)^n) \xrightarrow{\text{restrict}} k[\mathbb{G}_m]$  is surjective.

Since  $X^*((\mathbb{G}_m)^n) \simeq \mathbb{Z}^n$  and its elements form a  $k$ -basis of  $k[(\mathbb{G}_m)^n]$ , we obtain (b).

$(b) \Rightarrow (c)$

Assume (b). Let  $g: G \longrightarrow GL(V)$  be a representation. Pick a basis of  $V$  and extend  $g$  to a map

$\bar{g}: G \longrightarrow \text{Mat}_{n,n}$

Aim to show:

$$V = \bigoplus_{x \in X^*(G)} V_x$$

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where  $V_x = \{v \in V : g \cdot v = \chi(g)v \quad \forall g \in G\}$

Let  $\bar{g}^* : k[\text{Mat}_{\{n,n\}}] \longrightarrow k[G]$

be the algebra homomorphism corresponding to  $\bar{g}$ . Since  $X = X^*(G)$  forms a  $k$ -basis of  $k[G]$ , we have:

$$\bar{g}^*(f) = \sum_{x \in X} f_x \cdot x$$

for some  $f_x \in k$ . These  $f_x$  are uniquely determined, and only finitely many are nonzero.

For  $i, j \in \{1, \dots, n\}$ , let  $T_{ij}$  be the  $i$ th column of  $k[\text{Mat}_{n,n}]$ ,

and let  $E_{ij} \in \text{Mat}_{n,n}$  be the corresponding elementary matrix. Then:

$$\bar{g}(g) = \sum_{i=1}^n \sum_{j=1}^n T_{ij}(\bar{g}(g)) E_{ij}$$

$$= \sum_{i,j} \bar{g}^*(T_{ij})(g) E_{ij}$$

$$= \sum_{i,j} \sum_x (T_{ij})_x \chi(g) E_{ij}$$

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$$= \sum_x \chi(g) \left( \underbrace{\sum_{i,j} (T_{ij})_x E_{ij}}_{A_x \in \mathbb{P}GL_{n,n}} \right) = \sum_x \chi(g) A_x$$

To check:

$$\cdot A_x \circ A_\psi = \begin{cases} A_x & \text{if } x = \psi \\ 0 & \text{otherwise.} \end{cases}$$

$$\cdot \sum_x A_x = \text{id}_n$$

$$\cdot \text{Im}(A_x) = V_x \quad (= \{v \in V : g \cdot v = \chi(g)v \ \forall g \in G\})$$

From these finds it follows that  $V = \bigoplus_x V_x$ , as  $G$ -representations.

For each  $x$ ,  $V_x$  is a direct sum of the subdirect representations given by  $\chi$ :

$$G \longrightarrow \text{GL}_1 = \mathbb{G}_m. \quad \text{This proves (c).}$$

(c)  $\Rightarrow$  (a) Embed  $G$  into a closed subgroup of  $\text{GL}(V)$  for some  $V$ .

By assumption,  $\exists$  basis of  $V$  in which  $G$  acts diagonally.

Thus  $G \hookrightarrow D_n \cong (\mathbb{G}_m)^n$ , so  $G$  is diagonalizable.

□

Cor If  $G$  is diagonalizable, then the group algebra of the abelian group  $X^*(G)$  is isomorphic to  $k[G]$ .

(char( $k$ ) = 0)

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$$\left\{ \begin{array}{l} \text{diagonalizable} \\ \text{groups} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{finitely generated} \\ \text{abelian groups} \end{array} \right\}$$

$$G \longmapsto X^*(G)$$

Claim: This is a bijection.

Construction: Let  $M$  be a fin. gen. abelian group. Let  $k[M]$  denote its group algebra, with basis  $\{e_m : m \in M\}$ . Define a Hopf algebra structure on  $k[M]$  via:

$$\Delta: k[M] \longrightarrow k[M] \otimes k[M]$$

$$e_m \longmapsto e_m \otimes e_m$$

$$\gamma: k[M] \longrightarrow k$$

$$e_m \longmapsto 1$$

$$\delta: k[M] \longrightarrow k[M]$$

$$e_m \longmapsto e_{-m}$$

One argues that  $k[M]$  is reduced. Thus,  $\exists$  a linear algebra grp  $G$  with  $k[G] = k[M]$  as Hopf algebras.

To check:

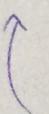
- $G$  is diagonalizable
- $X^*(G) \cong M$

Idea:

$$X^*(G) = \text{Hom}_{\text{Grps}}(G, \mathbb{G}_m) = \text{Hom}_{\text{Hopf-Alg}}(k[T^{\pm 1}], k[G])$$

$$= \text{Hom}_{\text{Hopf-Alg}}(k[T^{\pm 1}], k[M])$$

$$= \{ T \mapsto e_m : m \in M \} \cong M$$

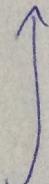
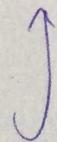


reduces to the argument given to show that  $\text{Hom}_{\text{Grps}}(\mathbb{G}_n, \mathbb{G}_n) = \mathbb{Z}$ .

Use part (b) of the theorem to show that  $G$  is diagonalizable.

We have:

$$\left\{ \begin{array}{l} \text{diagonalizable} \\ \text{groups} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{fin. gen.} \\ \text{abelian groups} \end{array} \right\}$$



$$\text{char} = \left\{ \begin{array}{l} \text{connected} \\ \text{diagonalizable} \end{array} \right\} = \left\{ (\mathbb{G}_m)^n : n \geq 0 \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{free f.g. abelian gp} \\ \mathbb{Z}^n \end{array} \right\} = \left\{ \mathbb{Z}^n : n \geq 1 \right\}$$

$$= \left\{ \begin{array}{l} \text{torsion-free f.g.} \\ \text{abelian groups} \end{array} \right\}$$

Q

**Prop** [Rigidity of diagonalizable groups].

Let  $G$  and  $H$  be diagonalizable groups, and  $V$  a connected affine variety. Suppose

$$\phi: V \times G \longrightarrow H$$

is a morphism of varieties s.t.  $g \mapsto \phi(v, g)$  is a map of algebraic groups  $G \rightarrow H$ ,  $\forall v \in V$ . Then  $\phi(v, g)$  is independent of  $v \in V$ .

pf. Let  $\phi^*: k[H] \longrightarrow k[V] \otimes k[G]$

be the corresponding algebra homomorphism. For  $\psi \in X^*(H)$ , write

$$\phi^*(\psi) = \sum_{x \in X^*(G)} f_{\psi, x} \otimes x$$

for some  $f_{\psi, x} \in k[V]$ . Argue that

① The map  $g \mapsto \sum_x f_{\psi, x}(v) X(g)$  is a character

of  $G$  for all  $v \in V$  (and fixed  $\psi \in X^*(H)$ )

② For fixed  $\psi \in X^*(H)$ ,  $\exists! x_\psi \in X^*(G)$  s.t.

$$f_{\psi, x} = \begin{cases} 1 & \text{if } x = x_\psi \\ 0 & \text{otherwise} \end{cases} \quad \text{in } k[V]$$

Therefore,  $\phi^*(\psi) = 1 \otimes \chi_\psi \quad \forall \psi \in X^*(H)$ . □

Since  $X^*(H)$  is a basis of  $k[H]$ , this implies the result □

Cn Let  $H$  be a diagonalizable subgroup of  $G$ . Then

$$N_G(H)^\circ = Z_G(H)^\circ \text{ and } N_G(H) / Z_G(H) \text{ is finite.}$$

Pf. Apply the previous result to

$$N_G(H)^\circ \times H \longrightarrow H$$

$$g, h \longmapsto g h g^{-1}.$$

□

In exercises 3.2.10, assume  $\text{char}(k) = 0$ .

Let  $T$  be a torus, and set  $X = X^*(T)$  and  $Y = X_*(T)$ . 11

Observe: There is a pairing

$$\langle , \rangle : Y \times X \longrightarrow \mathbb{Z}$$

given by

$$Y \times X = \text{Hom}_{\text{Grp}}(G_m, T) \times \text{Hom}_{\text{Grp}}(T, G_n) \xrightarrow{\text{compose}} \text{Hom}_{\text{Grp}}(G_n, G_m) \cong \mathbb{Z}$$

$$\lambda, \chi \longmapsto \chi \circ \lambda$$

$$\text{so } \chi(\lambda(a)) = a^{\langle \lambda, \chi \rangle} \text{ for all } a \in G_m.$$

This pairing induces maps

$$X \longrightarrow \text{Hom}_{\text{Grp}}(Y, \mathbb{Z}) \quad Y \longrightarrow \text{Hom}_{\text{Grp}}(X, \mathbb{Z})$$

$$\chi \longmapsto \langle -, \chi \rangle \quad \lambda \longmapsto \langle \lambda, - \rangle$$

Lemma ①  $\langle , \rangle$  is a perfect pairing

(i.e.  $\chi \mapsto \langle -, \chi \rangle$  and  $\lambda \mapsto \langle \lambda, - \rangle$  are bijections.)

② There is a canonical isomorphism

$$k^* \otimes_{\mathbb{Z}} Y \xrightarrow{\sim} T$$

given by  $a \otimes \lambda \longmapsto \lambda a$   $\forall a \in k^* \cong G_m$  and  $\lambda \in Y$ .

pf (ide). It is enough to prove this in the case that

$T \simeq (\mathbb{G}_m)^n$ , so  $X \simeq \mathbb{Z}^n$  and  $Y = \mathbb{Z}^n$ . Then

$$\langle y, x \rangle = \sum_{i=1}^n x_i y_i$$

is the dot product of  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$ .

Also,  $k^* \otimes_{\mathbb{Z}} Y \simeq k^* \otimes_{\mathbb{Z}} \mathbb{Z}^n \simeq (k^*)^n \simeq (\mathbb{G}_m)^n \simeq T$ .

□

Rank. In the case of a torus,  $Y$  (in addition to  $X$ ) is

a free abelian group.