Jordan Decomposition

Let k be an algebraically closed field.

Let V be a vector space over k of dimension $n \geq 1$.

$\boxed{\text{Def}}$ We say that $g \in \mathrm{End}(V)$ is:

- Semisimple (or diagonalizable) if V has a basis consisting of eigenvectors of g.
- nilpotent if $g^k = 0$ for $k >> 0$
  ($\Leftrightarrow$ all eigenvalues of g are equal to 0)
- Unipotent if $g - id_V$ is nilpotent
  ($\Leftrightarrow$ all eigenvalues of g are equal to 1)

$\boxed{\text{Ex}}$ Fix an isomorphism $V \simeq k^n$, so $\mathrm{End}(V) \simeq \mathrm{Mat}_{n,n}$.

- Any diagonal matrix is semisimple
- Any matrix in $U_n = \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \right\}$ is unipotent
- Any matrix in $U_n - Id_n = \left\{ \begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix} \right\}$ is nilpotent

Recall:

① A Jordan block is a square matrix of the form

$$J_{\lambda,m} = \begin{bmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ 0 & & & & \lambda \end{bmatrix} \in Mat_{m,m}$$

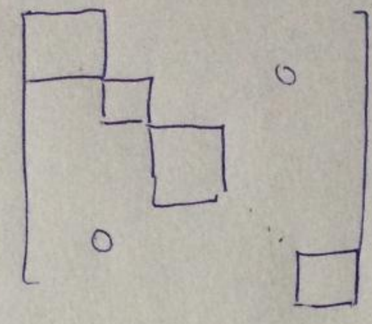
- Single eigenvalue $\lambda \in k$ of multiplicity m.
- Single generalized eigenspace of dimension m.
- $(J_{\lambda,m} - \lambda \cdot id)^m = 0$
- Observe:

$$J_{\lambda,m} = \underbrace{\begin{bmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{bmatrix}}_{\text{semisimple}} + \underbrace{\begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ 0 & & & & 0 \end{bmatrix}}_{\text{nilpotent}} \quad (*)$$

$$J_{\lambda,m} = \underbrace{\begin{bmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{bmatrix}}_{\text{semisimple}} \cdot \underbrace{\begin{bmatrix} 1 & \lambda^{-1} & & & 0 \\ & 1 & \lambda^{-1} & & \\ & & \ddots & \ddots & \\ & & & 1 & \lambda^{-1} \\ 0 & & & & 1 \end{bmatrix}}_{\text{unipotent}} \quad (**)$$

② [Jordan decomposition] Given $g \in \mathrm{End}(V)$, there is a basis of V in which the matrix corresponding to g is in Jordan normal form:

$g =$



- block diagonal matrix, each block is a Jordan block
- eigenvalues of g (counted with multiplicity) are the diagonal entries.
- Unique up to changing the basis based on a reordering of the blocks
- <u>Note</u>: Every Jordan block has a single eigenvalue, but a given eigenvalue can appear in multiple Jordan blocks.

$\boxed{\text{Lemma}}$ For any $g \in \mathrm{End}(V)$, there exist $g_s, g_n \in \mathrm{End}(V)$ such that:

- g_s is semisimple
- g_n is nilpotent
- $g = g_s + g_n$
- $g_s g_n = g_n g_s$

Proof. In the basis yielding the Jordan normal form of g, apply observation (*) above $\square$

Note: Eigenvalues of g $\underset{\text{(with multiplicities)}}{=}$ Eigenvalues of g_s

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$\boxed{\text{Prop}}$ Let $g \in \mathrm{End}(V)$. Let g_s and g_n be as in the Lemma. Then:

① $\exists$ polynomials $P, Q \in k[T]$ with $P(0)=0$ and $Q(0)=0$ such that

$$P(g) = g_s \quad \text{and} \quad Q(g) = g_n$$

② The elements g_s and g_n are uniquely determined satisfying the conditions of the Lemma.

③ If $W \subseteq V$ is g-stable (i.e. $g(W) \subseteq W$), then W is stable for g_s and g_n. Moreover

$$(g|_W)_s = g_s|_W \quad \text{and} \quad (g|_W)_n = g_n|_W$$

proof ① Notation:

- Characteristic polynomial of g:

$$\Phi_g(T) = \prod_{i=1}^{r} (T - \lambda_i)^{n_i} \quad \in k[T]$$

where $\lambda_1, \dots, \lambda_r$ are the distinct eigenvalues of g ($\lambda_i \neq \lambda_j$ if $i \neq j$) and $n_i \geq 1 \quad \forall i$.

- For $i = 1, \dots, r$, set

$$V_i = \{ v \in V \;:\; (g - \lambda_i)^{n_i} v = 0 \}$$

to be the generalized eigenspace of λ_i.

So $V = \bigoplus_{i=1}^{r} V_i$.

Now, by the Chinese Remainder Theorem for polynomial rings, we have

$$k[T] / \langle \Phi \rangle \simeq \bigoplus_{i=1}^{r} k[T] / \langle (T - \lambda_i)^{n_i} \rangle$$

Thus, there exists $P(T) \in k[T]$ such that

$$P(T) \equiv \lambda_i \quad \bmod (T - \lambda_i)^{n_i}$$

for each $i = 1, \dots, r$.

Claim $P(T)$ can be chosen so that $P(0) = 0$.

pf of Claim. Case I: $\Phi(0) = 0$.

Then 0 is an eigenvalue of g, say $\lambda_{i'} = 0$.
Then $P(T) \equiv 0 \mod T^{n_{i'}}$ by construction. Thus $P(0) = 0$.

Case II. $\Phi(0) \neq 0$.

Replace P by $P' = P - \frac{P(0)}{\Phi(0)} \cdot \Phi$.

Note that $P'(0) = 0$ and $P \equiv P' \mod \Phi$. [Claim is proven]

Claim $P(g) = g_s$.

Note: What do we mean by $P(g)$? We have an algebra homomorphism

$$\begin{aligned} \rho: k[T] &\longrightarrow \mathrm{End}(V) \\ T &\longmapsto g \end{aligned}$$

Then $P(g) := \rho(P(T))$. Since $\Phi(g) = 0$ (i.e. g satisfies its characteristic polynomial), we have that ρ factors through $k[T]/\langle \Phi \rangle$.

Now we prove the claim that $P(g) = g_s$.

First note that $g(V_i) \subseteq V_i$ for $i = 1, \dots, r$

$$\left[\underline{\text{Reason}}: \ (g-\lambda_i)^{n_i} g(v) = g(g-\lambda_i)^{n_i} v = 0 \right]$$
for $v \in V_i$

Thus, we have a well-defined algebra homomorphism

$$\begin{aligned} k[T] &\xrightarrow{\ \rho_i\ } \mathrm{End}(V_i) \qquad i = 1, \dots, r \\ T &\longmapsto g|_{V_i} \end{aligned}$$

Now, $\rho_i((T-\lambda_i)^{n_i}) = (g|_{V_i} - \lambda_i)^{n_i} \equiv 0$ on V_i.

Thus, $\exists\ \bar{\rho}_i : k[T]/\langle (T-\lambda_i)^{n_i} \rangle \longrightarrow \mathrm{End}(V_i)$ making the following diagram commute:

$$\begin{tikzcd} k[T] \arrow[r, "\rho_i"] \arrow[d, "\pi_i"'] & \mathrm{End}(V_i) \\ k[T]/\langle (T-\lambda_i)^{n_i} \rangle \arrow[ur, dashed, "\bar{\rho}_i"'] & \end{tikzcd}$$

where π_i is the quotient map.

Consequently,

$$P(g)\Big|_{V_i} = g_i\big(P(T)\big) = \bar{g}_i\big(\pi_i(P(T))\big)$$

$$= \bar{g}_i(\lambda_i) \qquad \left[\begin{array}{c}\text{by our choice}\\ \text{of } P(T)\end{array}\right]$$

$$= \lambda_i \cdot \mathrm{id}_{V_i}$$

$$= g_s\Big|_{V_i}$$

Since $V = \bigoplus_{i=1}^{r} V_i$, we have that $P(g) = g_s$. [claim is proven].

Set $Q = T - P$. Then

- $Q(0) = 0 - P(0) = 0$
- $Q(g) = g - g_s = g_n$

[① is proven].

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② Let $g = h_s + h_n$ be another decomposition with

- h_s semisimple
- h_n nilpotent
- $h_s h_n = h_n h_s$

Claim $g_s - h_s$ is semisimple.

pf. First we show that g_s and h_s commute. Indeed:

$$g_s h_s = P(g) h_s = h_s P(g) = h_s g_s .$$

since g and h_s commute

Since g_s and h_s are commuting diagonalizable operators, there is a basis of V consisting of eigen~~values~~ vectors for both g_s and h_s. This basis is also a basis of eigenvectors for $g_s - h_s$. □ [Claim]

Claim $g_n - h_n$ is nilpotent.

pf. One shows that $g_n h_n = h_n g_n$ using Q.

Let $k >> 0$ be such that $g_n^k = 0 = h_n^k$. Then

$$(g_n + h_n)^{2k} = \left(\sum_{j=0}^{k} \binom{2k}{j} g_n^j h_n^{k-j} \right) h_n^k$$

$$+ g_n^k \left(\sum_{j \geq 1}^{k} \binom{2k}{k+j} g_n^j h_n^{k-j} \right)$$

$$= 0 + 0 = 0 .$$

(argument is similar for g_n - h_n)

□ [Claim]

Now,

$$g_s - h_s = g - g_n - g + h_n = h_n - g_n$$

The LHS is semisimple and the RHS is nilpotent.

$\Rightarrow$ both sides are zero

$\Rightarrow g_s = h_s$ and $g_n = h_n$ $\square$ [② is proven].

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③ First,

$$g(W) \subseteq W \Rightarrow P(g)(W) \subseteq W \text{ and } Q(g)(W) \subseteq W$$

$$\Rightarrow g_s(W) \subseteq W \text{ and } g_n(W) \subseteq W.$$

We have the diagram:

$$\begin{tikzcd}
k[T] \arrow[rr, "\rho"] \arrow[ddrr, "\rho_W"'] & & \mathrm{End}(V) \arrow[d, "f \mapsto f|_W"] \\
& & \mathrm{Hom}(W, V) \\
& & \mathrm{End}(W) \arrow[u, hook]
\end{tikzcd}$$

Thus, for any $f \in k[T]$:

$$f(g)|_W = \rho(f)|_W = \rho_W(f) = f(g|_W)$$

In particular,

$$\left(g\big|_W\right)_s = \rho_W(P(T)) = \rho(P(T))\big|_W = P(g)\big|_W = g_s\big|_W$$

Since the characteristic polynomial of $g|_W$ divides that of g

and similarly for g_n. $\square$

<u>Follow-up</u>: If $W \subseteq V$ and $g(W) \subseteq W$, then there are induced maps

$$\tilde{g} : V/W \longrightarrow V/W \qquad \tilde{g}_s : V/W \to V/W \qquad \tilde{g}_n : V/W \to V/W.$$

Similar arguments to those given above show that

$$(\tilde{g})_s = \tilde{g}_s \quad \text{and} \quad (\tilde{g})_n = \tilde{g}_n$$

Lemma Let $\phi : V \longrightarrow W$ be a linear map b/w fin. dim'l vector spaces V, W. Suppose $g \in$ End(V) and $h \in$ End(W) fit into a commutative diagram:

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "g"'] & W \arrow[d, "h"] \\ V \arrow[r, "\phi"] & W \end{tikzcd}$$

Then the following diagrams also commute:

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "g_s"'] & W \arrow[d, "h_s"] \\ V \arrow[r, "\phi"] & W \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "g_n"'] & W \arrow[d, "h_n"] \\ V \arrow[r, "\phi"] & W \end{tikzcd}$$

<u>proof</u>. Factor ϕ as the composition:

$$V \hookrightarrow V \oplus W \twoheadrightarrow W$$

where $v \longmapsto (v, \phi(v))$ and $(v,w) \longmapsto w$. Then both squares of the following diagram commute:

$$\begin{tikzcd} V \arrow[r] \arrow[d, "g"'] & V \oplus W \arrow[r] \arrow[d, "(g,h)"] & W \arrow[d, "h"] \\ V \arrow[r] & V \oplus W \arrow[r] & W \end{tikzcd}$$

Argue that $(g,h)_s = (g_s, h_s)$ in $\mathrm{End}(V \oplus W)$. Then

$$\begin{tikzcd} V \arrow[r] \arrow[d, "g_s"'] & V \oplus W \arrow[r] \arrow[d, "(g_s,h_s)"] & W \arrow[d, "h_s"] \\ V \arrow[r] & V \oplus W \arrow[r] & W \end{tikzcd}$$

commutes

$\square$

Similarly for g_u.

$\boxed{\text{Prop}}$ (The Multiplicative Jordan Decomposition). Let $g \in GL(V)$, where $\dim(V) < \infty$. Then:

① $\exists!$ $g_s, g_u \in GL(V)$ s.t.

- g_s is semisimple,
- g_u is unipotent
- $g = g_s g_u = g_u g_s$.

② $\exists$ polynomials $P, R \in k[T]$ s.t.

- $P(0)=0$ and $P(g) = g_s$
- $R(0)=0$ and $R(g) = g_u$.

③ If $W \subseteq V$ is g-stable (so $g(W) \subseteq W$), then it is also stable for g_s and g_u. Moreover:

$$(g|_W)_s = g_s|_W \quad \text{and} \quad (g|_W)_u = g_u|_W$$

and $$(\tilde{g})_s = \tilde{g}_s \quad \text{and} \quad (\tilde{g})_u = \tilde{g}_u \quad \text{on } V/W.$$

Idea of proof: Take $g_u = id_V + g_s^{-1} g_n$ $\square$
(c.f. Observation $(**)$ from above).

Extension to the infinite-dimensional case:

Let V be a vector space over k, not necessarily finite dimensional.

$\boxed{\text{Def}}$ An element $g \in GL(V)$ is locally finite if V is a union of g-stable subspaces which are fin. dim'l
[so $V = \bigcup_i W_i$ with $g(W_i) \subseteq W_i$ and $\dim(W_i) < \infty$]

Such an element is said to be:

- semisimple if $g|_W$ is semisimple for any finite-dim'l g-stable $W \subseteq V$
- locally unipotent if $g|_W$ is unipotent for any fin. dim'l g-stable $W \subseteq V$.

Lemma Let $g \in GL(V)$ be a locally finite element. Then

① $\exists$ uniquely determined $g_s, g_u \in GL(V)$ s.t.

- g_s is semisimple
- g_u is locally unipotent
- $g_s g_u = g = g_u g_s$

② If $W \subseteq V$ is g-stable and fin. dim'l, then it is g_s- and g_u- stable. Moreover,

$$(g|_W)_s = g_s|_W \quad \text{and} \quad (g|_W)_u = g_u|_W.$$

proof (Idea). Write $V = \bigcup_{i \in I} W_i$ where $\dim(W_i) < \infty$ and $g(W_i) \subseteq W_i \quad \forall i$.

Argue (using previous results) that:

$$(g|_{W_i})_s \Big|_{W_i \cap W_j} = (g|_{W_i \cap W_j})_s = (g|_{W_j})_s \Big|_{W_i \cap W_j}$$

for all $i, j \in I$. Consequently,

$\{(g|_{W_i})_s\}_{i \in I}$ glue together to give a well-defined elem't $g_s \in GL(V)$. Similarly for g_u. $\square$

Generalization to arbitrary linear algebraic groups

Let G be a linear algebraic group. Recall that, for any $g \in G$, we have a k-algebra automorphism

$$\rho_g : k[G] \longrightarrow k[G]$$
$$f \longmapsto [x \longmapsto f(xg)]$$

In particular, $\rho_g \in GL(k[G])$. We argued that ρ_g is locally finite. Thus we have a Jordan decomposition

$$\rho_g = (\rho_g)_s (\rho_g)_u$$

We also have, for $g \in G$:

$$\lambda_g : k[G] \longrightarrow k[G]$$
$$f \longmapsto [x \longmapsto f(g^{-1}x)]$$

Note: For any $g, h \in G$, we have

$$\lambda_g \circ \rho_h = \rho_h \circ \lambda_g$$

as automorphisms of $k[G]$.

$\boxed{\text{Theorem}}$ Let G be a linear algebraic group.

① For any $g \in G$, there are unique elements g_s and g_u in G such that:

- $(\rho_g)_s = \rho_{g_s}$
- $(\rho_g)_u = \rho_{g_u}$
- $g = g_s g_u = g_u g_s$.

② If $\phi : G \longrightarrow G'$ is a homomorphism of algebraic groups, then $\phi(g)_s = \phi(g_s)$ and $\phi(g)_u = \phi(g_u)$

$\boxed{\text{Def}}$
- g_s is known as the <u>semisimple part</u> of g
- g_u " " " <u>unipotent part</u> of g

- g is <u>semisimple</u> if $g = g_s$
- g is <u>unipotent</u> if $g = g_u$

<u>proof of Theorem</u>. Since ρ_g is an algebra automorphism, we have the following commutative diagram:

$$\begin{tikzcd}
k[G] \otimes k[G] \arrow[r, "\rho_g \otimes \rho_g"] \arrow[d, "m"'] & k[G] \otimes k[G] \arrow[d, "m"] \\
k[G] \arrow[r, "\rho_g"] & k[G]
\end{tikzcd}$$

where m is the multiplication map.

As we saw above, we can replace all instances of ρ_g by $(\rho_g)_s$ to obtain another commutative diagram. It follows that $(\rho_g)_s$ is an algebra automorphism of $k[G]$, so

$$\begin{aligned} k[G] &\longrightarrow k \\ f &\longmapsto \left[(\rho_g)_s(f)\right](e) \end{aligned}$$

is a homomorphism of k-algebras. This homomorphism defines a point in G, call it g_s. We have that

$$f(g_s) = \left[(\rho_g)_s(f)\right](e)$$

for all $f \in k[G]$. Then, for any $x \in G$, we compute:

$$\left[(\rho_g)_s(f)\right](x) = \left[\lambda_{x^{-1}} \circ (\rho_g)_s(f)\right](e)$$

$$= \left[(\rho_g)_s \circ \lambda_{x^{-1}}(f)\right](e)$$

{ $\lambda_{x^{-1}}$ commutes with ρ_g, hence it commutes with its semisimple part }

$$= \left[(\rho_g)_s\left(\lambda_{x^{-1}}(f)\right)\right](e)$$

$$= \left[\lambda_{x^{-1}}(f)\right](g_s) = f(xg_s) = \left[\rho_{g_s}(f)\right](x).$$

Hence $(\rho_g)_s = \rho_{g_s}$. Similarly, we obtain an element $g_u \in G$ and $(\rho_g)_u = \rho_{g_u}$.

Finally, we argue that $g = g_s g_u = g_u g_s$. This follows from the following computation, together with the fact that $g \mapsto \rho_g$ is a faithful representation (i.e. an injection):

$$\rho_{g_s g_u} = \rho_{g_s} \circ \rho_{g_u} = (\rho_g)_s \circ (\rho_g)_u = \rho_g$$

$$= (\rho_g)_u \circ (\rho_g)_s = \rho_{g_u} \circ \rho_{g_s} = \rho_{g_u g_s} \qquad \square \text{ ①}$$

<u>proof of ②</u>: Factor ϕ as

$$G \twoheadrightarrow \mathrm{Im}\,\phi \hookrightarrow G'$$

where the first map is surjective and the last is injective, in fact a closed embedding. It suffices to prove the claim in two cases:

<u>Case I</u>: ϕ is a closed embedding:

Then $k[G] = k[G']/I$ for some ideal I.

Argue that
$$G = \{ g \in G' : \rho_g(I) = I \}.$$

Let $g \in G$. Thinking of it as an elt of G, we have an automorphism

$$\rho_g^{(G)} : k[G] \longrightarrow k[G]$$

Thinking of it as an element of G', we have an automorphism

$$\rho_g^{(G')} : k[G'] \longrightarrow k[G']$$

We have a commutative diagram

$$\begin{tikzcd} k[G'] \arrow[r, "\rho_g^{(G')}"] \arrow[d] & k[G'] \arrow[d] \\ k[G] = k[G']/I \arrow[r, "\rho_g^{(G)}"] & k[G] = k[G']/I \end{tikzcd}$$

So $\rho_g^{(G)} = \overline{\rho_g^{(G')}}$. Now,

$$\underbrace{\rho_{g_s}^{(G)}}_{\substack{\text{semisimple part of} \\ g\text{, thought of as an} \\ \text{element of } G}} = \left(\rho_g^{(G)}\right)_s = \left(\overline{\rho_g^{(G')}}\right)_s \underset{\substack{\text{result about} \\ V \longrightarrow V/W}}{=} \overline{\left(\rho_g^{(G')}\right)_s} = \underbrace{\overline{\rho_{g_s}^{(G')}}}_{\substack{\text{Semisimple part} \\ \text{of } g\text{, thought} \\ \text{of as an elt of } G'}}$$

Thus, g_s is well-defined, whether we think of g as an elt of G or of G'.

$\square$ (Case II)

<u>Case II</u>: ϕ is surjective.

Then $k[G'] \subseteq k[G]$ and $\rho_g(k[G']) \subseteq k[G']$

for all $g \in G$. Use results from above about $g_s|_W = (g|_W)_s$,

etc. (details omitted).

$\square$ (Case 2)

$\square$ ②

Prop ① Suppose $G = GL_n$ and $g \in G$. Then the elements g_s and g_u appearing in the theorem above coincide with the elements g_s and g_u coming from the Jordan decomposition.

② Let G be a linear algebraic group and $g \in G$. Then g is semisimple (i.e. $g = g_s$) if and only if, for any closed embedding $\phi: G \hookrightarrow GL_n$, we have that $\phi(g)$ is a semisimple matrix.

③ g is unipotent ($g = g_u$) iff $\phi(g)$ is a unipotent matrix for any $\phi: G \hookrightarrow GL_n$.

<u>proof</u>. These are straightforward consequences of previous results. See Springer for details. $\square$

Def A linear algebraic group is said to be <u>unipotent</u> if all its elements are unipotent.

Ex ① The subgroup $U_n = \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \right\} \subseteq GL_n$ is unipotent.

② Any conjugate $x U_n x^{-1}$ is also unipotent $(x \in GL_n)$. (Note that U_n is not normal in GL_n).

Prop Let G be a subgroup of GL_n consisting of unipotent matrices. Then $\exists x \in GL_n$ s.t. $x G x^{-1} \subseteq U_n$.

<u>pf</u>. Proceed by induction on n.

<u>Base step</u>: $n=1$. The result is clear since 1 is the only unipotent element of $GL_1 = \mathbb{G}_m = k^*$, and $U_1 = \{1\}$.

<u>Induction step</u>: Let $V = k^n$ with action of G induced from $G \hookrightarrow GL_n$.

<u>Case 1</u>: The action of G on V is <u>not</u> irreducible.

Then there exists a G-stable subspace $W \subseteq V$ $\left(\begin{matrix} g(W) \subseteq W \\ \forall g \in G. \end{matrix}\right)$ with $0 < \dim(W) < n$.

Let $m = \dim(W)$.

Fixing a basis for W and extending it to a basis of V, we see that G includes into the subgroup of all matrices of the form

$$\begin{array}{cc} & \begin{array}{cc} m & n-m \end{array} \\ \begin{array}{c} m \\ n-m \end{array} & \left[\begin{array}{c|c} X & * \\ \hline 0 & * \end{array}\right] \end{array}.$$

Call this subgroup H.

Let $\phi: G \longrightarrow GL_m$ and $\psi: G \longrightarrow GL_{n-m}$ be defined as in the diagram:

$$\begin{tikzcd} & & GL_m \\ G \arrow[urr, bend left, "\phi"] \arrow[drr, bend right, "\psi"'] \arrow[rr, hook] & & H \arrow[u] \arrow[d] \\ & & GL_{n-m} \end{tikzcd}$$

By induction, $\exists\ x_1 \in GL_m$, $x_2 \in GL_{n-m}$ s.t.

$$x_1 \operatorname{Im}(\phi) x_1^{-1} \subseteq U_m \qquad \text{and} \qquad x_2 \operatorname{Im}(\psi) x_2^{-1} \subseteq U_{n-m}.$$

Let $x = \left[\begin{array}{c|c} x_1 & 0 \\ \hline 0 & x_2 \end{array}\right] \in GL_n$. Then $x G x^{-1} \subseteq \left[\begin{array}{c|c} U_m & * \\ \hline 0 & U_{n-m} \end{array}\right] = U_n$.

<u>Case II</u>. G acts irreducibly on V.

Consider

$$\begin{tikzcd} G \arrow[r, hook] \arrow[dr, "\bar{\rho}"'] & GL(V) \arrow[d, hook] \\ & \mathrm{End}(V) \simeq k^{n^2} \end{tikzcd}$$

$\boxed{\text{Burnside's Theorem}}$ If V is irreducible, then

$$\mathrm{Span}_k \{ \bar{\rho}(g) \ : \ g \in G \} = \mathrm{End}(V)$$

__Note__: $g \in G \underset{\text{by hypothesis}}{\Rightarrow} g$ is a unipotent matrix $\underset{\substack{\text{all eigenvalues are} \\ \text{equal to 1.}}}{\Rightarrow} \mathrm{Tr}(g) = n$

Thus, $\mathrm{Tr}(gh) = \mathrm{Tr}(g) \quad \forall \ g, h \in G$

$\Rightarrow \mathrm{Tr}((g-1)h) = 0 \quad \forall \ g, h \in G$

$\boxed{\text{Claim}}$ $\mathrm{Tr}((g-1)\phi) = 0 \quad \forall \ g \in G$ and $\phi \in \mathrm{End}(V)$.

To see this, write ϕ as a linear combination of elements of the form $\bar{\rho}(g)$, and use that trace is additive.

$\boxed{\text{Lemma}}$ Let $A \in \mathrm{End}(V)$ and suppose $\mathrm{Tr}(AB) = 0$ for all $B \in \mathrm{End}(V)$. Then $A = 0$

pf of Lemma: (idea) Pick a basis of V and let B run through the elementary matrices in the basis. $\square$ (Lemma)

Using the Lemma and the Claim, we conclude that $g - 1 = 0$ $\forall g \in G$. Thus $G = \{1\}$ and so $G \subseteq U_n$. $\square$

Remark: What we are really proving in the above argument is that any finite-dimensional irreducible representation of a unipotent group is trivial.

$\boxed{\text{Prop}}$ Let G be a unipotent linear algebraic group and X an affine G-space. Then all orbits of G in X are closed.

pf. See Springer 2.4.14. $\square$

(2.4.15)
Exercise: Let G be a subgroup of GL_n which acts irreducibly on $V = k^n$. Show that the only normal unipotent subgroup of G is the trivial one.