Chapter 2: First properties of linear algebraic groups

- Let k be an algebraically closed field.
- Let $\{pt\}$ denote the k-variety corresponding to k as an affine algebra over itself. We have:
  - $k[\{pt\}] = k$
  - $\forall$ k-variety X, $\exists!$ $X \to \{pt\}$.

Def ① An <u>algebraic group</u> is an algebraic variety G equipped with a morphism

$$\mu : G \times G \longrightarrow G \quad , \quad x, y \longmapsto xy$$

satisfying the group axioms. In particular we have maps

$$1_G : \{pt\} \longrightarrow G \qquad\qquad i : G \longrightarrow G$$

$$pt \longmapsto \text{identity in } G \qquad\qquad x \longmapsto x^{-1}$$

The maps μ, 1_G, and i are required to be morphisms of varieties over k.

② If the underlying algebraic variety of G is affine, we say that G is a <u>linear algebraic group</u>

<u>Recall</u>: An affine variety X is completely determined by its affine algebra $k[X]$.

<u>In fact</u>: the group structure on a linear algebraic group G can be expressed in terms of its affine algebra $k[G]$.

<u>Recall the group axioms</u>:

"associativity"

$g(hk) = (gh)k \quad \forall g,h,k \in G$

$$\begin{tikzcd}
G\times G\times G \arrow[r, "\mu\times id"] \arrow[d, "id\times\mu"'] \arrow[dr, phantom, "\circlearrowleft"] & G\times G \arrow[d, "\mu"] \\
G\times G \arrow[r, "\mu"] & G
\end{tikzcd}$$

"Unit axiom"

$\exists 1_G \in G$ s.t. $1_G\cdot g = g = g\cdot 1_G$ $\forall g\in G$

$$\begin{tikzcd}
G \arrow[r, "g\mapsto (g{,}1)"] \arrow[d, "\substack{g\\ \downarrow \\ (1{,}g)}"'] \arrow[dr, "id"] & G\times G \arrow[d, "\mu"] \\
G\times G \arrow[r, "\mu"] & G
\end{tikzcd}$$

"inverse axiom":

$g\cdot g^{-1} = 1_G = g^{-1}\cdot g \quad \forall g\in G$

$$\begin{tikzcd}
 & G\times G \arrow[r, "i\times id"] & G\times G \arrow[dr, "\mu"] & \\
G \arrow[ur, "diag"] \arrow[r] \arrow[dr] & \{pt\} \arrow[r, "1_G"] & & G \\
 & G\times G \arrow[r, "id\times i"'] & G\times G \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Now:

$\mu : G \times G \longrightarrow G \quad \overset{\text{induces}}{\rightsquigarrow} \quad \Delta : k[G] \longrightarrow k[G] \otimes k[G]$ "coproduct"

$1_G : \{pt\} \longrightarrow G \quad \rightsquigarrow \quad \eta : k[G] \longrightarrow k$ "counit"

$i : G \longrightarrow G \quad \rightsquigarrow \quad S : k[G] \longrightarrow k[G]$ "antipode"

associativity axiom $\rightsquigarrow$ "coassociativity axiom"

$$\begin{tikzcd}
k[G] \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowleft"] & k[G] \otimes k[G] \arrow[d, "id \otimes \Delta"] \\
k[G] \otimes k[G] \arrow[r, "\Delta \otimes id"'] & k[G] \otimes k[G] \otimes k[G]
\end{tikzcd}$$

Unit axiom $\rightsquigarrow$ "counit axiom":

$$\begin{tikzcd}
k[G] \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "id"] & k[G] \otimes k[G] \arrow[d, "id \otimes \eta"] \\
k[G] \otimes k[G] \arrow[r, "\eta \otimes id"] & k[G]
\end{tikzcd}$$

inverse axiom $\rightsquigarrow$ "hexagon axiom":

$$\begin{tikzcd}
 & k[G] \otimes k[G] \arrow[r, "S \otimes id"] & k[G] \otimes k[G] \arrow[dr, "m"] & \\
k[G] \arrow[ur, "\Delta"] \arrow[r, "\eta"] \arrow[dr, "\Delta"'] & k \arrow[rr] & & k[G] \\
 & k[G] \otimes k[G] \arrow[r, "id \otimes S"'] & k[G] \otimes k[G] \arrow[ur, "m"'] &
\end{tikzcd}$$

$\boxed{\text{Def}}$ A <u>Hopf algebra</u> over k is a k-algebra H equipped with

· k-algebra homomorphisms:

$$\Delta : H \longrightarrow H \otimes H$$
$$\eta : H \longrightarrow k$$
$$S : H \longrightarrow H$$

satisfying the coassociativity, counit, and hexagon axioms.

We have:

$$\left\{ \begin{matrix} \text{linear algebraic} \\ \text{groups} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{affine Hopf} \\ k\text{-algebras} \end{matrix} \right\}$$

$$G \longmapsto k[G]$$

<u>Aside</u>: This is a bijection that reflects an equivalence of categories.

$\boxed{\text{Examples}}$

① $G = \mathbb{A}^1 = k$ with addition. Also known as $\mathbb{G}_a$, the "additive group".

- $k[G] = k[T]$ polynomials in one variable.
- $\Delta : k[T] \longrightarrow k[T] \otimes k[T] \simeq k[T_1, T_2]$
$$T \longmapsto T_1 + T_2$$
- $\eta : k[T] \longrightarrow k$
$$T \longmapsto 0$$
- $S : k[T] \longrightarrow k[T]$
$$T \longmapsto -T$$

(Check axioms!)

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② $G = \mathbb{A}^1 \setminus \{0\} = k^\times$ with multiplication, a.k.a. $\mathbb{G}_m$ "the multiplicative group".

- $k[G] = k[T^{\pm 1}]$ Laurent polynomials in one variable
- $\Delta : k[T^{\pm 1}] \longrightarrow k[T^{\pm 1}] \otimes k[T^{\pm 1}] = k[T_1^{\pm 1}, T_2^{\pm 1}]$
$$T \longmapsto T_1 T_2$$
- $\eta : k[T^{\pm 1}] \longrightarrow k$
$$T \longmapsto 1$$
- $S : k[T^{\pm 1}] \longrightarrow k[T^{\pm 1}]$
$$T \longmapsto T^{-1}$$

Side note:
k^* is an affine algebraic variety via:

k^* = V(xy -1) inside of k^2

where k[k^2] = k[x,y].

$\boxed{\text{Def}}$ ① A <u>homomorphism of algebraic groups</u> $G \to H$ is a group homomorphism that is also a map of varieties.

② A <u>closed subgroup</u> of G is a subgroup that is closed in the Zariski topology.

$\boxed{\text{Claim}}$ For every $n \in \mathbb{Z}$, we have a group homomorphism

$$\phi_n : \mathbb{G}_m \longrightarrow \mathbb{G}_m$$
$$x \longmapsto x^n$$

• group homomorphism: $\phi_n(xy) = (xy)^n = x^n y^n = \phi_n(x) \cdot \phi_n(y)$

$\mathbb{G}_m$ is abelian

• map of varieties: yes, since it is induced from

$$k[T^{\pm 1}] \longrightarrow k[T^{\pm 1}]$$
$$T \longmapsto T^n$$

(Check this!)

<u>Later</u>: we'll show that there is a bijection

$$\mathrm{Hom}_{\mathrm{Grp}}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}$$
$$\phi_n \longleftrightarrow n$$

(in fact an isomorphism of groups)

$\boxed{\text{Examples (cont'd)}}$

$\boxed{\text{Ex 3}}$ The general linear group GL_n.

Let $Mat_{n,n} \simeq k^{n^2}$ be the affine algebraic variety of n by n matrices. The determinant

(GL_n = V(det*x -1) inside k^{n^2 + 1})

$$\det : Mat_{n,n} \longrightarrow k$$

is a regular function, i.e. $\det \in k[Mat_{n,n}]$. Set

$$GL_n = \{ X \in Mat_{n,n} : \det(X) \neq 0 \} \quad \left(\begin{matrix}\text{principal}\\ \text{open set}\end{matrix}\right)$$

$\rightsquigarrow$ structure of an affine algebraic variety

$\rightsquigarrow$ group structure with operation given by matrix mult.

- $k[GL_n] = k[T_{ij}, D^{\pm 1}]_{i,j \in \{1,\dots,n\}} \Big/ \det(T_{ij}) = D$

$$= k[Mat_{n,n}][\det^{-1}]$$

- $\Delta : T_{ij} \longmapsto \sum_{l=1}^{n} T_{il} \otimes T_{lj}$

- $\eta : T_{ij} \longmapsto \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

- $S : T_{ij} \longmapsto$ i,j entry of the inverse matrix of $(T_{ab})_{a,b \in \{1,\dots,n\}}$

$\boxed{\text{Ex 3a}}$ $n = 2$. $k[\mathrm{Mat}_{2,2}] = k[a,b,c,d] \ni \det = ad - bc$

$$k[GL_2] = k[a,b,c,d]\left[(ad-bc)^{-1}\right]$$

- $\Delta : k[GL_2] \longrightarrow k[GL_2] \otimes k[GL_2]$

$$a \longmapsto a \otimes a + b \otimes c$$
$$b \longmapsto a \otimes b + b \otimes d$$
$$c \longmapsto c \otimes a + d \otimes c$$
$$d \longmapsto c \otimes b + d \otimes d$$

- $\eta : k[GL_2] \longrightarrow k$

$$a \longmapsto 1$$
$$b \longmapsto 0$$
$$c \longmapsto 0$$
$$d \longmapsto 1$$

- $S : k[GL_2] \longrightarrow k[GL_2]$

$$a \longmapsto d/_{ad-bc} = d(ad-bc)^{-1}$$
$$b \longmapsto -b/_{ad-bc} = -b(ad-bc)^{-1}$$
$$c \longmapsto -c/_{ad-bc} = -c(ad-bc)^{-1}$$
$$d \longmapsto \frac{a}{ad-bc} = a(ad-bc)^{-1}$$

Ex 4 Closed subgroups of GL_n:

Ex 4a Finite subgroups, e.g. the symmetric group $S_n \hookrightarrow GL_n$ which includes as permutation matrices.

Ex 4b Diagonal matrices:

$$D_n = \left\{ \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} : \lambda_i \in k^\times \right\} \simeq (\mathbb{G}_m)^n$$

aka torus T.

Ex 4c Unipotent upper triangular matrices:

$$U_n = \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \right\}$$

As algebraic varieties, $U_n \simeq k^{\frac{n(n-1)}{2}}$, but the group structure is different.

Ex 4d Upper triangular matrices:

$$B_n = \left\{ \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} : \lambda_i \in k^\times \right\} \simeq D_n \ltimes U_n$$

aka standard Borel (more later)

Ex 4e Special linear group: $SL_n = \{ X \in GL_n : \det(X) = 1 \}$

Ex 4f Orthogonal group: $O_n = \{ X \in GL_n : X^T X = 1_n \}$

$\rightsquigarrow$ columns are orthonormal

transpose

identity $n \times n$ matrix

Ex 4g Special orthogonal group: $SO_n = SL_n \cap O_n$

Ex 4h) The symplectic group:

$$Sp_n = \{ X \in GL_n : X^T J X = J \}$$

where
$$J = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}.$$

Ex 5) Elliptic curves are examples of non-linear algebraic groups. They are defined as certain closed subsets of $\mathbb{P}^2$.

(We might discuss elliptic curves at the end of this course, but for the most part we will focus on <u>linear</u> algebraic groups)

Exercises 2.1.5 (1) (c)
(3) (a), (b), (c)
(5)

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§2.2 Basic results

Prop ① $\exists!$ irreducible component G° of G that contains the identity element 1_G.

② G° is a closed subgroup of finite index.

proof ① (sketch) Let X and Y be irreducible components of G, both containing 1_G. Argue that $\overline{\mu(X \times Y)}$ is irreducible (using results from Ch 1). Then

$$\left.\begin{array}{l} \cdot\ X \subseteq \overline{\mu(X\times Y)} \Rightarrow X = \overline{\mu(X\times Y)} \\ \cdot\ Y \subseteq \overline{\mu(X\times Y)} \Rightarrow Y = \overline{\mu(X\times Y)} \end{array}\right\} \Rightarrow X = Y$$

② • G° is a subgroup:

- $\left.\begin{array}{l}\mu(G^\circ \times G^\circ) \\ \text{is irreducible and} \\ 1_G \in \mu(G^\circ\times G^\circ)\end{array}\right\} \overset{①}{\Longrightarrow} \mu(G^\circ \times G^\circ) = G^\circ$

  $\Rightarrow G^\circ$ is closed under multiplication

- $i(G^\circ)$ irreducible and $1_G \in i(G^\circ)$ $\overset{①}{\Longrightarrow} i(G^\circ) = G^\circ \Rightarrow G^\circ$ is closed under inverses.

• G° is closed since irreducible components are closed.

• To show G° is normal; observe that $\forall x \in G$, $xG^\circ x^{-1}$ is irreducible and contains 1_G. Thus $xG^\circ x^{-1} = G^\circ$ $\forall x \in G$.

• $\#(G/G^\circ) = \#$ irreducible components of G $< \infty$

follows from ①

true for any algebraic variety $\square$

$\boxed{\text{Corollary}}$ ① Irreducible components of G are mutually disjoint.

② $\{\text{irred. components of } G\} = G/G^\circ = \pi_0(G) = \{\text{connected components of } G\}$

Pf. ① Suppose X and Y are irreducible components of G and $g \in X \cap Y$. Then:

$$1_G \in (g^{-1} X) \cap (g^{-1} Y)$$

$$\Rightarrow g^{-1}X = g^{-1}Y \quad \Rightarrow \quad X = Y$$

② Clear from ①. $\square$

$\boxed{\text{Exercises } 2.2.2}$ (1), (2), (4)

$\boxed{\text{Lemma}}$ ① If U, V are open and dense subsets of G, then $UV := \mu(U \times V)$ is dense of G.

② If H is a subgroup of G, then the closure $\overline{H}$ is also a subgroup.

③ Let $\phi : G \longrightarrow G'$ be a homomorphism of algebraic groups. Then:

(i) $\ker \phi$ is a closed normal subgroup of G.

(ii) $\phi(G)$ is a closed subgroup of G'.

(iii) $\phi(G^\circ) = [\phi(G)]^\circ$

There are more useful results in § 2.2 that we'll go back to as necessary.

§2.3 G-spaces.

Def ① A <u>G-variety</u> (or <u>G-space</u>) is an algebraic variety X with an action of G, i.e. a morphism of varieties

$$a : G \times X \longrightarrow X$$

such that

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mu \times id"] \arrow[d, "id \times a"'] \arrow[dr, phantom, "\circlearrowright"] & G \times X \arrow[d, "a"] \\ G \times X \arrow[r, "a"] & X \end{tikzcd}$$

commutes (i.e. $g(hx) = (gh)x$ $\forall g, h \in G,\ x \in X$)

and

$$\begin{tikzcd} X \simeq \{pt\} \times X \arrow[r, "1_G \times id"] \arrow[dr] & G \times X \arrow[d, "a"] \arrow[dl, phantom, "\circlearrowright"] \\ & X \end{tikzcd}$$

commuts

(i.e. $1_G \cdot x = x \quad \forall\ x \in X$)

② A morphism $\phi : X \longrightarrow Y$ between G-spaces is <u>equivariant</u> if

$$\begin{tikzcd} G \times X \arrow[r, "a_X"] \arrow[d, "id \times \phi"'] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d, "\phi"] \\ G \times Y \arrow[r, "a_Y"] & Y \end{tikzcd}$$

commutes.

③ Let X be a G-space and $x \in X$. The <u>orbit</u> of x is

$$G.x = \{ gx : g \in G \} = \mathrm{Image}\left(G \times \{x\} \longrightarrow X \right) \subseteq X$$

The <u>isotropy group</u> (or <u>stabilizer</u>) of x is the closed subgroup

$$G_x = \{ g \in G : gx = x \} \subseteq G$$

(check that it is closed)

④ A G-variety is <u>homogeneous</u> if G acts transitively.

$\boxed{\text{Ex 1}}$ $G \circlearrowright G$ by conjugation:

$$a(g, x) = gxg^{-1}$$

$\boxed{\text{Ex 2a}}$ $G \circlearrowright G$ by left translations:

$$a(g, x) = gx$$

$\boxed{\text{Ex 2b}}$ $G \circlearrowright G$ by right translations:

$$a(g, x) = xg^{-1}$$

<u>Note</u>: Ex 2a + 2b are instances of <u>principal homogeneous spaces</u>, or <u>torsors</u>, i.e. the action is transitive and all isotropy groups are trivial.

Ex 3 Let V be a finite dimensional vector space over k. A <u>rational representation</u> of G is a homomorphism of algebraic groups

$$\rho : G \longrightarrow GL(V)$$

In particular, V is a G-variety with $g \cdot v = \rho(g)(v)$

Choosing a basis for V, we obtain a map of algebraic groups

$$G \longrightarrow GL_n$$

where $n = \dim(V)$.

Exercises 2.3.4 (1), (2), (3).

Lemma Any finite group is a linear algebraic group.

<u>proof</u>. Let G be a finite group and let A_G be the group algebra, which is a finite dimensional vector space with standard basis $\{ e_g : g \in G \}$. The <u>regular representation</u> of G is

$$\rho : G \longrightarrow GL(A_G) \simeq GL_{|G|}$$

$$g \longmapsto [e_h \longmapsto e_{gh}]$$

The result follows from the fact that ρ is injective with Zariski closed image. $\square$

(<u>Note</u>: $k[G]$ is the "Hopf dual" of A_G.)

Aiming to prove:

$\boxed{\text{Theorem}}$ (Embedding theorem) Each linear algebraic group is isomorphic to a closed subgroup of GL_n for some $n > 0$

Idea: Take inspiration from the proof of the lemma above.
Let G be a linear algebraic group and $g \in G$. Set

$$\rho_g : k[G] \longrightarrow k[G]$$
$$f \longmapsto [x \longmapsto f(xg)]$$

This is a k-algebra homomorphism.

$\boxed{\text{Lemma}}$ Let $V \subseteq k[G]$ be a subspace.

① $\rho_g(V) \subseteq V$ iff $\Delta(V) \subseteq V \otimes k[G]$.

② If $\dim(V) < \infty$, then $\exists$ fin. dim'l v.s. $W \subseteq k[G]$ s.t.
- $V \subseteq W$
- $\rho_g(W) \subseteq W$

proof. ①($\Leftarrow$) Assume $\Delta(V) \subseteq V \otimes k[G]$. Then, for any $f \in V$, we have

$$\Delta(f) = \sum_i f_i \otimes a_i$$

for some $f_i \in V$ and some $a_i \in k[G]$.

Then: for $x, y \in G$:

$$\rho_g(f)(x) = f(xg) = f(\mu(x,g)) = \mu^*(f)(x,g)$$

$$= \Delta(f)(x,g) = \sum_i f_i(x)\,\varphi_i(g)$$

Therefore:

$$\rho_g(f) = \sum_i \varphi_i(g) f_i \in V$$

① ($\Rightarrow$) Suppose $\rho_g(V) \subseteq V$. Let $\{f_i : i \in I\}$ be a basis of V and extend this by $\{g_j : j \in J\}$ so that

$\{f_i, g_j : i \in I, j \in J\}$ is a basis of $k[G]$.

For fixed $f \in V$,

$$\Delta(f) = \sum_{i \in I} f_i \otimes \varphi_i + \sum_{j \in J} g_j \otimes \psi_j$$

for some φ_i, ψ_j in $k[G]$. Then:

$$\rho_g(f) = \sum_{i \in I} \varphi_i(g) f_i + \sum_{j \in J} \psi_j(g)\, g_j$$

Since $\rho_g(V) \subseteq V$, we have that $\psi_j = 0$ $\forall j$. Thus

$$\Delta(f) = \sum f_i \otimes \varphi_i \in V \otimes k[G].$$

(2) Enough to prove the claim when $\dim(V) = 1$.

Assume $V = \mathrm{Span}_k\{f\}$ for some $f \in k[G]$.

Write
$$\Delta(f) = \sum_i f_i \otimes \varphi_i$$
for some $f_i, \varphi_i \in k[G]$. Let $W' = \mathrm{Span}_k\{f_i\}$, so W is finite-dimensional. Then
$$\rho_g(f) = \sum_i \varphi_i(g) f_i \quad \in W' \qquad \forall g \in G$$

Set $\quad W = \mathrm{Span}_k\{\rho_g(f) : g \in G\} \subseteq W'$

Then:

- $\dim(W) \leq \dim(W') < \infty$
- $V = \mathrm{Span}\{f\} = \mathrm{Span}\{\rho_e(f)\} \subseteq W$
- $\rho_g(W) \subseteq W \qquad \forall g \in G \qquad \left[\begin{array}{l}\text{since} \\ \rho_g \circ \rho_h = \rho_{gh}\end{array}\right]$

$\square$

$\boxed{\text{Proof of the Embedding Theorem}}$

<u>Recall</u>: $k[G]$ is finitely generated as an algebra.

Let $\{f_1, \dots, f_m\}$ be a set of generators.

Let $V = \mathrm{Span}_k\{f_1, \dots, f_m\} \underset{\text{fin. dim'l}}{\subseteq} k[G]$

Apply ② of the Lemma to get $W \underset{f.d.}{\subseteq} k[G]$ such that

- $W \supseteq V$
- $\rho_g(W) \subseteq W \quad \forall g \in G \quad \left[\overset{\text{Lemma ①}}{\Longleftrightarrow} \quad \Delta(W) \subseteq W \otimes k[G] \right]$

Let $\{f_1, \dots, f_m, f_{m+1}, \dots, f_n\}$ be a basis of W.

(note that this set still generates $k[G]$ as a k-algebra).

Since $\Delta(W) \subseteq W \otimes k[G]$, we have $\varphi_{ij} \in k[G]$ s.t.

$$\Delta(f_i) = \sum_{j=1}^{n} f_j \otimes \varphi_{ij} \qquad \forall i \in \{1, \dots, n\}$$

$$\Rightarrow \quad \rho_g(f_i) = \sum_{j=1}^{n} \varphi_{ij}(g) f_j \qquad \forall i \in \{1, \dots, n\},\ g \in G.$$

Define:

$$\Phi : G \longrightarrow GL(W) \simeq GL_n$$

$$g \longmapsto [\varphi_{ij}(g)]_{i,j=1}^{n} \qquad n \times n \text{ matrix}$$

To check:

- Φ is well-defined (the matrix $[\varphi_{ij}(g)]$ is invertible)
- Φ is a group homomorphism ($\rho_g \circ \rho_h = \rho_{gh}$)
- Φ is a injective map of varieties

(induced from $\Phi^*: k[GL_n] \longrightarrow k[G]$

$$T_{ij} \longmapsto a_{ij}$$

and we have $f_i = \Phi^*\left(\sum_{j=1}^{n} f_j(1)\, T_{ij} \right)$ in the image)

$\Rightarrow$ Φ is a closed embedding. $\square$

$\boxed{\text{Extension}}$ (Szamuely Notes, Prop. 4.3)

$\boxed{\text{Proposition}}$ Let H be a closed subgroup of a linear algebraic group G. Then $\exists$ fin. dim'l v.s. W and a morphism of algebraic groups $\rho: G \longrightarrow GL(W)$ s.t. $\ker(\rho) = H$.

Let X be an affine G-space, with action map

$$a: G \times X \longrightarrow X$$

Let $a^*: k[X] \longrightarrow k[G] \otimes k[X]$ be the induced map.

Then

$$\rho: G \longrightarrow GL(k[X])$$

$$g \longmapsto [f \mapsto [x \mapsto f(g^{-1}x)]]$$

defines a (generally infinite-dimensional) representation of G.

Note:
$$k[X] \times X \xrightarrow{\text{evaluate}} k$$

$\circlearrowleft$ G acts diagonally $\circlearrowleft$ G acts trivially

is equivariant: $\big(\rho(g)(f)\big)(g \cdot x) = f(g^{-1} \cdot g \cdot x) = f(x)$.

Generalization of Lemma from earlier:

Let $V \underset{f.d.}{\subseteq} k[X]$. Then $\rho_g(V) \subseteq V \ \forall g \in G$ iff $a^*(V) \subseteq k[G] \otimes V$.

In this case V is a rational representation of G.

Exercise: $k[X] = \bigcup_i V_i$ is a union of rational representations of G
2.3.9(1)

$\boxed{\text{Prop}}$ Any morphism of algebraic groups
$$\mathbb{G}_m \longrightarrow \mathbb{G}_m$$
is given by $x \longmapsto x^n$ for some $n \in \mathbb{Z}$.

$\boxed{\text{Lemma}}$ The units of the algebra $k[\mathbb{G}_m] = k[T^{\pm 1}]$ are given by

$$\{\lambda T^n : n \in \mathbb{Z},\ \lambda \in k^\times\}$$

pf. Let f be a unit, so $\exists\, g \in k[\mathbb{G}_m]$ s.t. $fg = 1$.

Write

$$f = \underbrace{f_{N_1} T^{N_1}}_{\substack{\text{lowest-order term,} \\ \text{so } f_{N_1} \neq 0}} + \cdots + \underbrace{f_{N_2} T^{N_2}}_{\substack{\text{highest-order term,} \\ f_{N_2} \neq 0}} \qquad N_2 \geq N_1$$

Similary,

$$g = g_{M_1} T^{M_1} + \cdots + g_{M_2} T^{M_2} \qquad M_2 \geq M_1$$

$$g_{M_1} \neq 0,\ g_{M_2} \neq 0.$$

Then we have

$$1 = fg = \underbrace{f_{N_1} g_{M_1} T^{N_1+M_1}}_{\text{lowest-order term}} + \cdots + \underbrace{f_{N_2} g_{M_2} T^{N_2+M_2}}_{\text{highest order term}}$$

So:

- $N_1 + M_1 = N_2 + M_2 \implies N_1 = N_2,\ M_1 = M_2$
- $f = f_N T^N$ for some $f_N \in k^\times$ and $N \in \mathbb{Z}$.

$\square$

<u>proof of the proposition</u>:

Let $\phi : \mathbb{G}_m \longrightarrow \mathbb{G}_m$ be a morphism of algebraic groups, and let $\phi^* : k[\mathbb{G}_m] = k[T^{\pm 1}] \longrightarrow k[T^{\pm 1}]$ be the induced map. Then

$$\phi^*(T) \text{ is a unit of } k[\mathbb{G}_m] = k[T^{\pm 1}]$$

$$\left(\text{since } 1 = \phi^*(1) = \phi^*(T \cdot T^{-1}) = \phi^*(T)\, \phi^*(T^{-1})\right)$$

By the lemma,

$$\phi^*(T) = \lambda T^n \quad \text{for some } \lambda \in k^*,\ n \in \mathbb{Z}.$$

Now,

$$\begin{tikzcd} k[\mathbb{G}_m] \arrow[rr, "\phi^*"] \arrow[dr, "\eta"'] & & k[\mathbb{G}_m] \arrow[dl, "\eta"] \\ & k & \end{tikzcd}$$

commutes,

So $\eta(\phi^*(T)) = \eta(T) \Rightarrow \eta(\lambda T^n) = 1 \Rightarrow \lambda = 1$.

Thus $\phi^*(T) = T^n$ for some $n \in \mathbb{Z}$

$\Rightarrow \phi(x) = x^n$, which we already know is a homomorphism.

$\square$