Weyl Groups and Root Data

Let G be a connected reductive group.

Let T be a maximal torus of G and $X = X^*(T)$ the character lattice of T.

Def The Weyl group of G is defined as

$$W = N_G(T)/T$$

Rmk: ① Since any two maximal tori are conjugate, this definition does not depend on the choice of T.

② For a general connected linear algebraic group, W is defined as $W = N_G(T)/Z_G(T)$.

But $Z_G(T) = T$ for reductive groups.

③ W is finite by 3.2.9 (rigidity of tori).

Ex Let $G = GL_n$ and $T = \left\{ \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} : a_i \in k^\times \right\}$ be the maximal torus of diagonal matrices. There is an inclusion of groups

$$\begin{aligned} S_n &\longrightarrow GL_n \\ \sigma &\longmapsto M_\sigma \end{aligned}$$

where $(M_\sigma)_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i) \\ 0 & \text{otherwise.} \end{cases}$

It is easy to see that $M_\sigma \in N_G(T)$ for any $\sigma \in S_n$.

In fact, $S_n \simeq N_G(T)/T$. (More on this later; the $n=2$ case is easy to prove).

$\boxed{\text{Lemma}}$ There is a well-defined action of W on $X^*(T)$ given by:

$$(w \cdot \chi)(t) = \chi(\tilde{w}\, t\, \tilde{w}^{-1})$$

where $t \in T$, $\chi \in X^*(T)$, and $\tilde{w} \in N_G(T)$ is any lift of $w \in W$. This action extends to a representation of W on the real vector space $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

pf. Well-defined because:

- T is commutative, so $sts^{-1} = t \quad \forall s, t \in T$
- $\tilde{w}\, t\, \tilde{w}^{-1} \cdot \tilde{w}\, s\, \tilde{w}^{-1} = \tilde{w}\, ts\, \tilde{w}^{-1} \quad \forall s, t \in T$.

$\square$

<u>Technical point</u>: Fix a positive definite symmetric bilinear form $(\,,)$ on $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ that is <u>W-invariant</u>:

$$(w \cdot x,\ w \cdot y) = (x, y) \quad \forall\ x, y \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$$

[Such a form exists: take any positive definite symmetric bilinear form f and "average" it:

$$(x,y) := \sum_{w \in W} f(wx, wy) \qquad x, y \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}.$$
]

Consider the adjoint action of T on $\mathfrak{g}$:

$$T \longrightarrow GL(\mathfrak{g})$$

$$t \longmapsto [\mathrm{Ad}(t) : \mathfrak{g} \longrightarrow \mathfrak{g}]$$

$\boxed{\text{Def}}$ For $\chi \in X^*(T)$, set

$$\mathfrak{g}_\chi = \{ \xi \in \mathfrak{g} : \mathrm{Ad}(t)(\xi) = \chi(t)\xi \quad \forall t \in T \}$$

The <u>roots</u> of (G,T) are:

$$\Phi = \Phi_{(G,T)} = \{ \chi \in X^*(T) : \mathfrak{g}_\chi \neq 0 \} \setminus \{0\}$$

Let $V \subseteq X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ be the $\mathbb{R}$-span of Φ.

$\boxed{\text{Thm}}$ The set $\Phi_{(G,T)}$ is a root system in V whose isomorphism class does not depend on the choice of $T \subseteq G$. The associated reflection group is $W = N_G(T)/T$.

Let's first discuss examples; later I'll say something about the proof.

$\boxed{\text{Ex}}$ $G = SL_2$, $T = \{ \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} : a \in k^\times \} \simeq \mathbb{G}_m$.

Then we have an isomorphism

$$\mathbb{Z} \xrightarrow{\sim} X^*(T)$$

$$n \longmapsto \left[\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} \longmapsto a^n \right]$$

Recall: $\mathfrak{g} = \mathfrak{sl}_2 = \mathrm{Span}_k \{ E, F, H \}$ with $[H, E] = 2E$, $[H, F] = -2F$, $[E, F] = H$.

One computes directly that

$$\mathrm{Ad}\left(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}\right)(E) = a^2 E$$

$$\mathrm{Ad}\left(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}\right)(F) = a^{-2} F$$

$$\mathrm{Ad}\left(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}\right)(H) = H$$

i.e. $\mathrm{Ad}\left(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}\right) = \begin{bmatrix} a^2 & & \\ & a^{-2} & \\ & & 1 \end{bmatrix}$ in the basis $\{E, F, H\}$.

Thus $\mathfrak{g}_0 = \mathrm{Span}_k \{ H \}$, $\mathfrak{g}_2 = \mathrm{Span}_k \{ E \}$, $\mathfrak{g}_{-2} = \mathrm{Span}_k \{ F \}$.

and $\Phi = \{ 2, -2 \}$.

$\boxed{\text{Ex}}$ $G = PSL_2 = SL_2 / \{\pm 1\}$. Let $T = \{ \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} : a \in k^\times \} \subseteq SL_2$ as above. Observe:

- $T/\pm 1$ is a maximal torus of PSL_2

- The map $T \longrightarrow T/\pm 1$ can be identified with the map

$$\mathbb{G}_m \longrightarrow \mathbb{G}_m$$
$$a \longmapsto a^2$$

So on the level of characters we have

$$\mathbb{Z} = X^*(T/\pm 1) \longrightarrow X^*(T) = \mathbb{Z}$$
$$n \longmapsto 2n$$

- $\mathrm{Lie}(PSL_2) = \mathrm{Lie}(SL_2) = \mathfrak{sl}_2$, and $\mathrm{Ad}\big|_T$ factors through $T/\pm 1$

$$\left(\mathrm{Ad}\left(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}\right) = \begin{bmatrix} a^2 & & \\ & a^{-2} & \\ & & 1 \end{bmatrix} \right)$$

Thus, the root system of PSL_2 is $\{\pm 1\} \subseteq X^*(T/\pm 1) = \mathbb{Z}$.

$\boxed{\text{Ex}}$ $G = GL_n$, $T = \left\{ \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} : a_i \in k^\times \right\} \simeq (\mathbb{G}_m)^n$.

Identify

$$\mathbb{Z}^n \xrightarrow{\sim} X^*(T)$$
$$e_i \longmapsto \left[\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \longmapsto a_i \right]$$

where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{Z}^n$.

(So $\sum_{i=1}^n c_i e_i \in \mathbb{Z}^n$ corresponds to the character $\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \mapsto \prod a_i^{c_i}$.)

For $i,j = 1, \ldots, n$, let $E_{ij} \in \mathfrak{gl}_n$ be the elementary matrix. One computes:

$$Ad(t)\,(E_{ij}) = (e_i - e_j)(t)\, E_{ij}$$

for $t \in T$. Thus,

$$\Phi = \{ e_i - e_j \ : \ i,j = 1, \ldots, n \} \setminus \{0\}$$

and $$V = \{ \sum_{i=1}^{n} c_i e_i \ : \ \sum c_i = 0 \} \subseteq X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^n$$

(note that $\dim(V) = n-1$).

Some ingredients of the proof of the theorem:

① Recall that $R(G)$ is a central torus. As a result, $Ad|_{R(G)}$ is trivial and $Lie(R(G)) \subseteq \mathfrak{g}_0$. It follows that the roots of G and $G/R(G)$ are identified. Thus we can assume G is semisimple.

② Argue that $\mathfrak{g}$ is semisimple as a Lie algebra, and $\mathfrak{t} = Lie(T)$ is a Cartan subalgebra. Then proceed as follows (see e.g. Fulton + Harris, Representation Theory: A First Course, Chapter 14).

- $\mathfrak{g}_\alpha$ is one-dimensional $\forall \alpha \in \Phi$

- $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_0$ is also 1-dimensional, and

$$\mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}$$

is isomorphic to $\mathfrak{sl}_2$.

- For each $\alpha \in \Phi$, let $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ correspond to $H \in \mathfrak{sl}_2$. Use the integrality and symmetry of the eigenvalues of H_α; as defining an operator $[H_\alpha, -] : \mathfrak{g} \longrightarrow \mathfrak{g}$.
(Comes down to basic $\mathfrak{sl}_2$-representation theory).

(3) To connect with the Weyl group, set:

$\boxed{\text{Def}}$ For $\alpha \in \Phi$, set

$$G_\alpha = Z_G(\underbrace{\ker(\alpha)^\circ}_{\text{subtorus of } T, \text{ hence of } G})$$

$\boxed{\text{Ex}}$ $G = SL_2$, $T = \left\{ \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} : a \in k^\times \right\}$, $\alpha = 2 : \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} \longmapsto a^2$,

So $\ker(\alpha) = \mathbb{1}$ and $G_\alpha = G$.

$\boxed{\text{Ex}}$ $G = GL_n$, $T = \left\{ \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} : a_i \in k^\times \right\}$. The roots are all of the form

$$\alpha = (e_i - e_j) : \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \longmapsto a_i a_j^{-1}$$

So

$$\ker(\alpha) = \left\{ \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} : a_i = a_j \right\}$$

and

$$G_{e_i - e_j} = \left\{ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} \begin{bmatrix} a_1 & & & & \\ & \ddots & & & \\ & & x & \cdots & y \\ & & \vdots & \ddots & \vdots \\ & & z & \cdots & w \\ & & & & \ddots & \\ & & & & & a_n \end{bmatrix} : \begin{array}{l} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in GL_2, \\ a_s \in k^\times \text{ for } s \notin \{i, j\} \end{array} \right\}.$$

So $G_{e_i - e_j} \simeq Gl_2 \times (\mathbb{G}_m)^{n-2}$

Note: For each α, we have $T \subseteq G_\alpha$ and $T/_{\ker(\alpha)^\circ} \simeq \mathbb{G}_m$.

$\boxed{\text{Prop}}$ (G continues to be reductive). For any $\alpha \in \Phi$, we have:

① G_α is connected and reductive

② $[G_\alpha, G_\alpha]$ is isomorphic to SL_2 or PSL_2

(and $\mathrm{Lie}([G_\alpha, G_\alpha]) \simeq \mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}$ from above)

③ The G_α generate G

($\boxed{\text{Cor}}$ If G is semisimple, then $G = [G, G]$.)

To connect with the Weyl group, one argues that:

- $N_G(T)/_T$ is generated by $N_{G_\alpha}(T)/_{Z_{G_\alpha}(T)}$

- For each $\alpha \in T$, we have:

$$N_{G_\alpha}(T)/_{Z_{G_\alpha}(T)} \simeq N_{G_\alpha/\ker(\alpha)^\circ}(T/\ker(\alpha)^\circ) \Big/ Z_{G_\alpha/\ker(\alpha)^\circ}(T/\ker(\alpha)^\circ)$$

$$\simeq N_{G_\alpha/\ker(\alpha)^\circ}(\mathbb{G}_m) \Big/ Z_{G_\alpha/\ker(\alpha)^\circ}(\mathbb{G}_m)$$

- This last quotient can be shown to be isomorphic to

$$N_{SL_2}\left(\left\{\begin{bmatrix} a & \\ & a^{-1}\end{bmatrix} : a \in k^\times\right\}\right) \Big/ Z_{SL_2}\left(\left\{\begin{bmatrix} a & \\ & a^{-1}\end{bmatrix} : a \in k^\times\right\}\right)$$

which is easily shown to be $\mathbb{Z}/2\mathbb{Z}$. The generator of this copy of $\mathbb{Z}/2\mathbb{Z}$ in $N_G(T)/T$ corresponds to the reflection s_α.

So we have an assignment:

$$\left\{\begin{matrix}\text{reductive} \\ \text{algebraic groups}\end{matrix}\right\} \longrightarrow \{\text{root systems}\}$$

$$G \longmapsto \Phi_G = \{\chi \in X^*(T) : \mathfrak{g}_\chi \neq 0\} \setminus \{0\}$$

for $T \subseteq G$ a maximal torus.

(See the appendix to these notes about what happens for non-reductive groups)

<u>Facts</u>: ① $G_1 \times G_2 \longmapsto \Phi_{G_1} \amalg \Phi_{G_2}$

② $\Phi_G \simeq \Phi_{G/Z}$ where Z is the center of G.

③ Every root system appears as the root system of a reductive group. So reductive groups form families depending on the type of their root system.

Type of (irreducible) root system	"simply connected" * representative	"adjoint" * representative	other members
A_n	SL_{n+1}	PSL_{n+1}	GL_{n+1}
B_n	$Spin(2n+1)$	$SO(2n+1)$	
C_n	$Sp(2n)$	$Sp(2n)/\pm 1$	
D_n	$Spin(2n)$	$PSO(2n)$	$SO(2n)$

- G_2, F_4, and E_8 each have a single semisimple group in their families, and it is both simply-connected and adjoint.
- E_6 and E_7 each have two semisimple groups in their families.

$\boxed{Q}$ How do we enhance the root system in order to distinguish different members of the same family?

$\boxed{A}$: Root datum.

$\boxed{Def}$ A <u>root datum</u> is a quadruple $\Psi = (X, R, X^\vee, R^\vee)$

where: (a) X and $X^\vee$ are free abelian groups of finite rank, in duality via a pairing

$$\langle , \rangle : X \times X^\vee \longrightarrow \mathbb{Z}.$$

(b) R and $R^\vee$ are finite subsets of X and $X^\vee$, respectively, and we are given a bijection:

$$R \longrightarrow R^\vee$$

$$\alpha \longmapsto \alpha^\vee$$

The quadruple satisfies the following axioms:

(1) $\langle \alpha, \alpha^\vee \rangle = 2 \quad \forall\, \alpha \in R$

(2) For $\alpha \in R$, we have $\quad s_\alpha(R) = R \quad$ and $\quad s_\alpha^\vee(R^\vee) = R^\vee$

where

$$s_\alpha : X \longrightarrow X \qquad\qquad s_\alpha^\vee : X^\vee \longrightarrow X^\vee$$

$$x \longmapsto x - \langle x, \alpha^\vee \rangle \alpha \qquad\qquad y \longmapsto y - \langle \alpha, y \rangle \alpha^\vee$$

(3) For $\alpha \in R$, $\quad \mathrm{Span}_{\mathbb{Z}}(\alpha) \cap R = \{\pm \alpha\}$.

<u>Rmk</u>: ① It follows that $\mathrm{Span}_{\mathbb{Z}}(\alpha^\vee) \cap R^\vee = \{\pm \alpha^\vee\} \quad \forall\, \alpha^\vee \in R$.

② Let $Q = \langle \alpha \,:\, \alpha \in R \rangle \subseteq X$ be the subgroup of X generated by the roots. Then R is a root system in the real vector space $Q \otimes_{\mathbb{Z}} \mathbb{R}$. Similarly for $R^\vee$.

③ For some authors, axiom (3) is omitted, and a root datum that satisfies axiom (3) is called "reduced".

Ex ① $(\mathbb{Z}, \{\pm 2\}, \mathbb{Z}, \{\pm 1\})$

with $\langle , \rangle : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ being multiplication.

(we will see that this root datum corresponds to SL_2)

② $(\mathbb{Z}, \{\pm 1\}, \mathbb{Z}, \{\pm 2\})$

(we will see that this corresponds to PSL_2)

③ $(\mathbb{Z}^2, \{\pm(e_1 - e_2)\}, \mathbb{Z}^2, \{\pm(e_1 - e_2)\})$

where $\langle , \rangle : \mathbb{Z}^2 \times \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ is the dot product.

(this corresponds to GL_2)

<u>The root datum of a reductive group</u>

Let G be a reductive group with maximal torus T.

Set $X = X^*(T)$ and $X^\vee = X_*(T)$.

Let $V = \mathrm{Span}_{\mathbb{R}}(\Phi) \subseteq X \otimes_{\mathbb{Z}} \mathbb{R}$.

Will be interested in the image of $\cancel{X}^\vee \subseteq X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} \simeq (X^*(T) \otimes_{\mathbb{Z}} \mathbb{R})^*$

under the surjective map $(X^*(T) \otimes_{\mathbb{Z}} \mathbb{R})^* \longrightarrow V^*$.

Recall: The Weyl group W acts on V and there is a W-invariant positive definite symmetric bilinear form $(\,,\,)$ on V. For $\alpha \in \Phi$, we have the reflection

$$S_\alpha : V \longrightarrow V$$

$$x \longmapsto x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha$$

$\boxed{\text{Def}}$ For $\alpha \in \Phi$, set

$$\alpha^\vee = 2\frac{(-,\alpha)}{(\alpha,\alpha)} \in V^*.$$

$\boxed{\text{Claim}}$ For $\alpha \in \Phi$, the element $\alpha^\vee$ belongs to the image of $X^\vee$ in V^*.

pf. (sketch) The argument reduces to the cases $G = SL_2$ or $G = PSL_2$.
In both cases we have $V = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

$\underline{SL_2}$: $\alpha^\vee = 2\dfrac{(-,2)}{(2,2)} = 1 \in X_*(T) = \mathbb{Z}$

$\underline{PSL_2}$: $\alpha^\vee = 2\dfrac{(-,1)}{(1,1)} = 2 \in X_*\left(T/_{\pm 1}\right) = \mathbb{Z}$ $\square$

Let $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\} \subseteq X^\vee$

$(= \Phi^\vee_{(G,T)})$

Thus, we have an assignment:

$$\left\{\begin{matrix}\text{reductive}\\ \text{algebraic groups}\end{matrix}\right\} \longrightarrow \{\text{root datum}\}$$

$$G \longmapsto \Big(X = X^*(T),\quad R = \Phi_{(G,T)},$$
$$X^\vee = X_*(T),\quad R^\vee = \Phi^\vee_{(G,T)} \Big)$$

Thm Two reductive groups are isomorphic iff they have isomorphic root data. Every root datum defines a unique reductive group

Note. Root data $(X, R, X^\vee, R^\vee)$ and $(Y, S, Y^\vee, S^\vee)$ are isomorphic if $\exists$ group isomorphism $f : X \to Y$ s.t. $f(R) = S$ and $f^\vee(S^\vee) = R^\vee$.

Df Let G be a reductive group with root datum $(X, R, X^\vee, R^\vee)$. The Langlands dual G^L of G is the reductive group determined by the root datum $(X^\vee, R^\vee, X, R)$

Ex ① $SL_n^L = PSL_n$

$PSL_n^L = SL_n$

② $GL_n^L = GL_n$

③ The Langlands dual of a simply-connected group is adjoint, and vice versa.

④ The Langlands dual of a group of type B_n is of type C_n, and vice versa. For all other types, the Langlands dual is of the same type.

Appendix

(I) Let G be a connected linear algebraic group. The associated root system is:

$$\Phi = \{ \alpha \in X^*(T) : \mathfrak{g}_\alpha \neq 0 \text{ and } G_\alpha \text{ is not solvable} \} \setminus \{0\}$$

But if G is not reductive, this does not capture enough information about the group to be useful.

(II) Let G be a reductive group and $\Phi \subseteq X^*(T)$ the root system, where $T \subseteq G$ is a maximal torus. Let $\Phi^\vee \subseteq X_*(T)$ be the coroots. Let $V \subseteq X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ be the span of Φ.

Def The <u>root lattice</u> of G is the subgroup of $X^*(T)$ generated by the roots:

$$Q = \langle \alpha : \alpha \in \Phi \rangle \subseteq X^*(T)$$

The <u>weight lattice</u> of G is:

$$P = \{ x \in V : \langle x, \Phi^\vee \rangle \subseteq \mathbb{Z} \}$$

We say that G is <u>simply-connected</u> if $X = P$

" " " " " <u>adjoint</u> if $X = Q$.

<u>Note</u>: $Q \subseteq X \subseteq P$.

$\boxed{\text{Ex}}$ ① SL_2: $P = X = \mathbb{Z}$, $Q = 2\mathbb{Z} \subseteq X$

So SL_2 is simply-connected

② PSL_2: $\underset{\text{index } 2}{P = \mathbb{Z}[\tfrac{1}{2}] \supseteq X}$, $Q = \mathbb{Z} = X$.

So PSL_2 is adjoint.