# Derivations, Differentials, and Lie Algebras

Let $R$ be a commutative ring. Let $A$ be an $R$-algebra and $M$ a left $A$-module.

**Def** An <u>$R$-derivation</u> of $A$ in $M$ is an $R$-linear map

$$D : A \longrightarrow M$$

such that $\quad D(ab) = a\,D(b) + b\,D(a) \quad \forall a, b \in A.$

<u>Notation</u>: $\mathrm{Der}_R(A, M)$

**Ex** Suppose $R = k$ and $A = k[T]$. Then the derivative map

$$\frac{d}{dt} : A \longrightarrow A$$

is a derivation by the product rule $(fg)' = f'g + fg'$.

**Ex** Suppose $R = k$ and $A = k[\mathbb{A}^n] = k[T_1, \dots, T_n]$. Then the partial derivative

$$D_i := \frac{\partial}{\partial T_i} : k[\mathbb{A}^n] \longrightarrow k[\mathbb{A}^n]$$

is a derivation for $i = 1, \dots, n$.

<u>Facts</u>: ① $D(r) = 0 \quad \forall r \in R$ whenever $D$ is an $R$-derivation.

② $\mathrm{Der}_R(A, M)$ is a left $A$-module.

(3) Given an $R$-algebra homomorphism $\phi : A \longrightarrow B$ and a left $B$-module $N$, we have an exact sequence:

$$0 \longrightarrow \mathrm{Der}_A(B, N) \longrightarrow \mathrm{Der}_R(B, N) \longrightarrow \mathrm{Der}_R(A, N)$$

In particular, if $\phi$ is surjective, then $\mathrm{Der}_R(B,N) \hookrightarrow \mathrm{Der}_R(A,N)$.

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$\boxed{\text{Ex}}$ $\mathrm{Der}_k(k[\mathbb{A}^n], k[\mathbb{A}^n])$ is the free $k[\mathbb{A}^n]$-module generated by the $\{D_i\}_{i=1}^n$.

$\boxed{\text{Ex}}$ Let $X = V(I) \subseteq \mathbb{A}^n$ for some $I = \sqrt{I} \subseteq k[\mathbb{A}^n]$, so that $k[X] = k[\mathbb{A}^n]/I$. Then

$$0 \longrightarrow \mathrm{Der}_k(k[X], N) \hookrightarrow \mathrm{Der}_k(k[\mathbb{A}^n], N)$$

for any $k[X]$-module $N$.

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<u>Motivating Question</u>:

Given $X \underset{\text{closed}}{\subseteq} \mathbb{A}^n$ and $x \in X$, what is the tangent space of $X$ at $x$?

<u>Heuristic</u>: $v \in k^n$ is a tangent vector to $X$ at $x$ iff $\nabla_x f \cdot v = 0 \quad \forall f \in I = I(X)$

iff $$\sum_{i=1}^{n} \left[ D_i(f)(x) \right] v_i = 0 \qquad \forall\, f \in I$$

iff $$\left( \sum_{i=1}^{n} v_i D_i \right)(f) \in \tilde{M}_x \qquad \forall\, f \in I$$

$\tilde{M}_x$ ↳ maximal ideal of $k[\mathbb{A}^n]$ corresponding to $x$.

iff $\sum_{i=1}^{n} v_i D_i$ defines a derivation $k[X] \longrightarrow k_x = k[X]/M_x$

$M_x$: maximal ideal of $k[X]$ corresponding to $x$

"$\Rightarrow$" is clear, "$\Leftarrow$" requires a bit of thought.

$\boxed{\text{Def}}$ The tangent space of an affine variety $X$ at $x \in X$ is defined as

$$T_x X := \mathrm{Der}_k (k[X], k_x)$$

Facts: ① If $\phi : X \longrightarrow Y$ is a morphism of affine varieties, we obtain a $k$-linear map

$$d_x \phi : T_x X \longrightarrow T_{\phi(x)} Y$$

(consider $\phi^* : k[Y] \longrightarrow k[X]$).

② [Chain rule]. Given morphisms $\phi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$ of affine varieties, we have

$$d_x(\psi \circ \phi) = d_{\phi(x)}\psi \circ d_x \phi$$

as $k$-linear maps $T_x X \longrightarrow T_{\psi\circ\phi(x)} Z$.

③ If $X$ is a general variety, we define for $x \in X$:

$$T_x X := \lim_{\substack{U \ni x \\ U \text{ open affine in } X}} T_x U$$

Facts ① and ② hold for non-affine varieties as well.

④(a) If $\phi : X \longrightarrow Y$ is an open inclusion of a subvariety, then

$$d_x \phi \text{ is an isomorphism } \forall x \in X$$

(b) If $\phi : Z \hookrightarrow X$ is a closed inclusion, then

$$d_z \phi \text{ is injective } \forall z \in Z$$

⑤ Equivalent definitions of $T_x X$:

(a) $T_x X = \left( M_x / M_x^2 \right)^*$

$$\left[\begin{tikzcd} k[X] \arrow[r, "\text{derivation}"] & k_x \\ M_x \arrow[u, hook] \arrow[r] & M_x/M_x^2 \arrow[u, dotted, "k\text{-linear}"'] \end{tikzcd}\right]$$

(b) $T_x X = \mathrm{Der}_k(\mathcal{O}_x, k)$

**Def** Let $X$ be an algebraic variety. We say that $X$ is
<u>smooth at $x \in X$</u> if $\dim T_x X = \dim X$.
We say that $X$ is <u>smooth</u> if $X$ is smooth at all points.

**Ex** ① $V(x^2 + y^2 - z^2) \subseteq k^3$ is <u>not</u> smooth at $(0,0,0)$, but smooth at all other points.

② $V(x^2 + y^2 - z^2 + 1) \subseteq k^3$ is smooth.

**Theorem** Let $X$ be an irreducible variety of dimension $d$.

① The smooth points of $X$ form a nonempty open subset of $X$.

② $\forall x \in X$, $\dim T_x X \geq \dim X$.

We will not prove this theorem, but will give some idea of the necessary tools later on (see also Springer § 4.3).

**Corollary** Let $G$ be a connected algebraic group.

① Any homogeneous space is smooth. In particular, $G$ is smooth.

② Let $\phi: X \longrightarrow Y$ be a $G$-equivariant map between homogeneous spaces. Then $d_x \phi$ has the same rank $\forall x \in X$

Pf. ① We know that the smooth locus of $X$ is nonempty, say $x \in X$ is smooth. Now, $\forall x' \in X$, $\exists g \in G$ s.t. $x' = g \cdot x$.

Then

$$d_x g : T_x X \longrightarrow T_{x'} X$$

is an isomorphism (the inverse is given by $d_{x'} g^{-1}$).

In particular, $\dim(T_{x'}X) = \dim(T_x X) = \dim(X)$, so $x'$ is a smooth point.

② Let $x, x' \in X$ and let $g \in G$ be such that $g \cdot x = x'$.

Then

$$\begin{array}{ccc} T_x X & \xrightarrow{d_x \phi} & T_{\phi(x)} Y \\ {\scriptstyle d_x g}\downarrow \simeq & & {\scriptstyle d_{\phi(x)} g}\downarrow \simeq \\ T_{x'} X & \xrightarrow{d_{x'} \phi} & T_{\phi(x')} Y \end{array}$$

commutes since $\phi(g \cdot x) = g \cdot \phi(x)$. Hence $d_x \phi$ and $d_{x'} \phi$ have the same rank. ∎

## Kähler differentials

Let $R$ be a commutative ring and $A$ a commutative $R$-algebra.

**Def** Let $m: A \otimes_R A \longrightarrow A$ be the multiplication map and let $I \subseteq A \otimes_R A$ be its kernel. Define the <u>Kähler differentials of $A$ over $R$</u> as $\Omega_{A/R} = I/I^2$

Notes: ① $I$ is generated by $\{a \otimes 1 - 1 \otimes a \ : \ a \in A\}$.

② $\Omega_{A/R}$ is constructed as an $A \otimes_R A$-module, but has a natural induced $A$-module structure. (Check this!)

We henceforth regard $\Omega_{A/R}$ as an $A$-module.

③ For $a \in A$, write $da$ for the image of $a \otimes 1 - 1 \otimes a \in I$ in $I/I^2 = \Omega_{A/R}$.

The $da$ generate $\Omega_{A/R}$ as an $A$-module.

④ The map
$$A \longrightarrow \Omega_{A/R}$$
$$a \longmapsto da$$
is an $R$-derivation.

(check this!)

$\boxed{\text{Theorem}}$ Let $M$ be an $A$-module. The map

$$\Phi : \mathrm{Hom}_A(\Omega_{A/R}, M) \longrightarrow \mathrm{Der}_R(A, M)$$

$$\phi \longmapsto \phi \circ d$$

is an isomorphism of $A$-modules.

<u>Aside</u>: This means that $\Omega_{A/R}$ represents the functor

$$\mathrm{Der}_R(A, -) : A\text{-mod} \longrightarrow A\text{-mod}$$

$$M \longmapsto \mathrm{Der}_R(A, M)$$

<u>proof of Theorem</u>. It is easy to check that $\Phi$ is an $A$-module homomorphism, and that it is injective (since $d$ is injective).
To show surjectivity, let $D \in \mathrm{Der}_R(A, M)$ and define:

$$\psi : A \otimes_R A \longrightarrow M$$

$$a \otimes b \longmapsto b \cdot D(a)$$

<u>To check</u>:

- $\psi$ is $R$-linear, i.e. $\psi \in \mathrm{Hom}_R(A \otimes A, M)$.
- For any $x, y \in A \otimes_R A$, we have

$$\psi(xy) = m(x) \cdot \psi(y) + m(y) \cdot \psi(x)$$

[prove this first for $x = a_1 \otimes b_1$ and $y = a_2 \otimes b_2$ ]

- Deduce that $\psi$ vanishes on $I^2$ (since $I = \ker(m)$).

Hence $\psi$ defines an $R$-linear map $\overline{\psi} : I/I^2 \longrightarrow M$

$$\underset{\Omega_{A/R}}{\overset{\shortparallel}{I/I^2}}$$

- Argue that this map is $A$-linear, and $\overline{\psi} \circ d = D$

[for the latter, check for the elements $da \mapsto a \otimes 1 - 1 \otimes a \mod I^2$].

$\square$

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<u>Observe</u>: Given an $R$-algebra homomorphism $\phi : A \to B$, there is a unique map of $A$-modules

$$\phi : \Omega_{A/R} \longrightarrow \Omega_{B/R}$$

making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "d"'] & B \arrow[d, "d"] \\ \Omega_{A/R} \arrow[r, "\phi^{\circ}"] & \Omega_{B/R} \end{tikzcd}$$

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$\boxed{\text{Lemma}}$ Let $M$ be a module for $R[T_1, \dots, T_n]$. There is an isomorphism

$$\varphi : M^{\oplus n} \longrightarrow \mathrm{Der}_R(R[T_1, \dots, T_n], M)$$

$$(m_i)_{i=1}^{n} \longmapsto \left[ f \mapsto \sum_{i=1}^{n} D_i(f) \cdot m_i \right]$$

where $D_i = \dfrac{\partial}{\partial T_i}$.

Pf. Easy computation. The inverse is given by $D \mapsto (D(T_i))_{i=1}^{m}$.

[Also verify that $\varphi((m_i)_{i=1}^{n})$ is actually a derivation]. $\square$

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Let $A$ be a finitely generated $R$-algebra, so there is a quotient map $\pi : R[T_1, \dots, T_n] \longrightarrow A$ for some $n \geq 1$.

$\boxed{\text{Prop}}$ Given $A$ and $\pi$ as above, there is a surjection of $A$-modules

$$A^n \longrightarrow \Omega_{A/R}$$

$$e_i \longmapsto d(\pi(T_i))$$

(where $\{e_i\}$ is the standard basis of $A^n$), whose kernel is the $A$-submodule of $A^n$ generated by

$$\left\{ \sum_{i=1}^{n} \pi(D_i(f)) \cdot e_i \ : \ f \in \ker(\pi) \right\}$$

proof. Let $K$ be the $A$-submodule of $A^n$ in question. We claim that the isomorphism $\varphi$ from the above lemma induces a map $\mathrm{Hom}_A(A^n/K, M) \longrightarrow \mathrm{Der}_R(A, M)$ that fits into:

$$\begin{array}{ccc} M^{\oplus n} & \xrightarrow[\simeq]{\varphi} & \mathrm{Der}_R(R[T_1, \dots, T_n], M) \\ \big\uparrow & & \big\uparrow \\ \mathrm{Hom}_A(A^n/K, M) & \dashrightarrow & \mathrm{Der}_R(A, M) \end{array}$$

To this end, note that $(m_i)_{i=1}^n \in M^{\oplus n}$ defines a homomorphism

$$A^n/K \longrightarrow M$$
$$[e_i] \longmapsto m_i$$

iff $\sum_{i=1}^{n} \pi(D_i(f))\, m_i = 0 \quad \forall f \in \ker(\pi)$.

On the other hand, $\varphi((m_i)_{i=1}^n)$ defines a derivation $A \to M$

iff $\varphi((m_i)_{i=1}^n)(f) = 0 \quad \forall f \in \ker(\pi)$

iff $\sum_{i=1}^{n} D_i(f) \cdot m_i = 0 \quad \forall f \in \ker(\pi)$ [use lemma from above]

iff $\sum_{i=1}^{n} \pi(D_i(f)) \cdot m_i = 0 \quad \forall f \in \ker(\pi)$ [the action of $R[T_1, \dots, T_n]$ on $M$ is via $\pi$]

Therefore $\mathrm{Hom}_A(A^n/K, M) \simeq \mathrm{Der}_R(A, M)$.

The universal property of $\Omega_{A/R}$ now implies the result (or Yoneda's lemma, if you prefer) $\square$

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<u>Note</u>: If $\ker(\pi)$ is generated by $f_1, \dots, f_s$, then the $A$-submodule of $A^n$ in question is generated by

$$\left\{ \sum_{i=1}^{n} \pi(D_i(f_j)) \cdot e_i \ : \ j = 1, \dots, s \right\}.$$

$\boxed{\text{Example}}$ What is $T_1 SL_2$ ?

$$T_1 SL_2 = \mathrm{Der}(k[SL_2], k_1) = \mathrm{Hom}_{k[SL_2]}(\Omega_{k[SL_2]}, k_1)$$

Recall: $k[SL_2] = k[a,b,c,d]/(ad-bc=1)$ and the action on $k_1$

is given by the counit map. So for $x \in k_1$, we have:

$$a \cdot x = x \quad , \quad b \cdot x = 0 \quad , \quad c \cdot x = 0 \quad , \quad d \cdot x = 1 .$$

$\boxed{\text{Claim}}$

$$\Omega_{k[SL_2]} = \frac{k[SL_2]\{e_a, e_b, e_c, e_d\}}{\langle d \cdot e_a - c e_b - b e_c + a \cdot e_d \rangle}$$

i.e. $\Omega_{k[SL_2]}$ is the quotient of the free $k[SL_2]$-module generated by elements $e_a, e_b, e_c, e_d$ by the relation

$$d e_a - c e_b - b e_c + a e_d .$$

proof. We use the lemma above. Let

$$\pi : k[a,b,c,d] \longrightarrow k[SL_2]$$

be the quotient map

[by abuse of notation, we use the same symbols for elements in $k[a,b,c,d]$ as for their images in $k[SL_2]$ ]

We compute:

$$\sum_{i=1}^{4} \pi\left(D_i(ad-bc-1)\right) e_i \qquad \begin{pmatrix} e_1 \longleftrightarrow e_a \\ \text{etc.} \end{pmatrix}$$

$$= \pi\left(\frac{\partial}{\partial a}(ad-bc-1)\right) e_a + \pi\left(\frac{\partial}{\partial b}(ad-bc-1)\right) e_b$$
$$+ \pi\left(\frac{\partial}{\partial c}(ad-bc-1)\right) e_c + \pi\left(\frac{\partial}{\partial d}(ad-bc-1)\right) e_d$$

$$= d e_a - c e_b - b e_c + a e_d. \qquad \text{(Claim)} \ \square$$

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Therefore:

$$T_1 SL_2 = \mathrm{Hom}_{k[SL_2]}\left(\frac{k[SL_2]\{e_a, e_b, e_c, e_d\}}{de_a - ce_b - be_c + ae_d}, k, \right)$$

$$= \{(\bar{e}_a, \bar{e}_b, \bar{e}_c, \bar{e}_d) \in k^4 : d\cdot\bar{e}_a - c\cdot\bar{e}_b - b\cdot\bar{e}_c + a\cdot\bar{e}_d = 0\}$$

$$= \{(\bar{e}_a, \bar{e}_b, \bar{e}_c, \bar{e}_d) \in k^4 : \bar{e}_a + \bar{e}_d = 0\}$$

$$= \{M \in Mat_{2,2} : tr(M) = 0\}$$

We'll see why this is the right identification in the next lecture.

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Similarly one can show that

$$T_1 O(n) = \{M \in Mat_{n,n} : M + M^T = 0\}.$$