Final Lecture: Vistas Toward Geometric Representation Theory

Structure of reductive groups

Let G be a reductive group. Let T be a maximal torus of G and B a Borel subgroup of G containing T: $T \subseteq B \subseteq G$.

Fact: There exists a Borel subgroup B^- s.t. $B \cap B^- = T$.
This is known as the Borel subgroup opposite to B.

(Let w_0 be the "longest element" in the Weyl group $W = N_G(T)/T$.
Then $B^- = \tilde{w}_0 B \tilde{w}_0^{-1}$ for any lift $\tilde{w}_0$ of w_0.)

$\boxed{\text{Ex}}$ $G = GL_n$. $T = \{$diagonal matrices in $GL_n\}$

$B = \left\{ \begin{array}{l} \text{upper-triangular} \\ \text{matrices in } GL_n \end{array} \right\}$. $B^- = \left\{ \begin{array}{l} \text{lower-triangular} \\ \text{matrices in } GL_n \end{array} \right\}$

Recall: The root system:

$$\Phi = \Phi_{(G,T)} = \{ \chi \in X^*(T) : \mathfrak{g}_\chi \neq 0 \} \setminus \{0\}$$

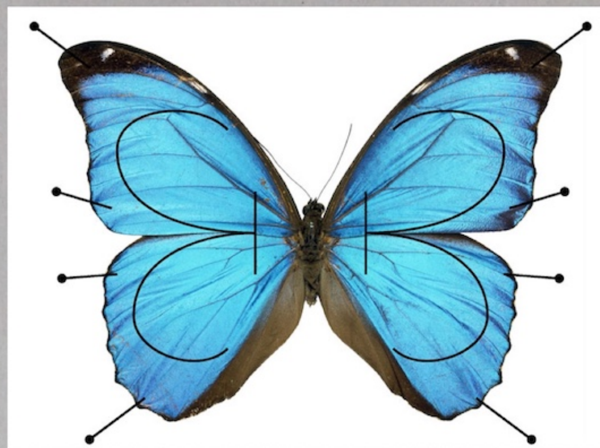
Let $\Phi^+ := \{ \chi \in \Phi : \mathrm{Lie}(B)_\chi \neq 0 \} \subseteq \Phi$

Facts: ① Φ^+ is a set of positive roots

② $-(\Phi^+) = \{ \chi \in \Phi : \mathrm{Lie}(B^-)_\chi \neq 0 \}$

Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be the simple roots in Φ^+

Then $-\Delta = \{-\alpha_1, \ldots, -\alpha_r\}$ are the simple roots in $-(\Phi^+)$.



Following Grothendieck's vision (c.f. Milne):

- The body of the butterfly is a maximal torus T.
- The wings are opposite Borels B and B^- (and $B \cap B^- = T$).
- The pins are the root system, which rigidify this situation.

Tannaka Reconstruction (§2.5 in Springer)

Let G be a linear algebraic group.
Let $\mathrm{Rep}(G)$ be the category of finite-dimensional representations of G over k.

Idea: One can "reconstruct" G as an algebraic group from the category $\mathrm{Rep}(G)$.

Set up: Let $\rho_V : G \longrightarrow GL(V)$ be a fin. dim'l representation.

Then:

- Let V^* be the dual vector space of V. Let

$$\langle , \rangle : V^* \otimes V \longrightarrow k$$

be the evaluation map. Then V^* is a representation of G via:

$$\rho_{V^*} : G \longrightarrow GL(V^*)$$

$$g \longmapsto \Big[\underset{V^*}{\underset{\cap}{v^*}} \longmapsto \Big[\underset{V}{\underset{\cap}{v}} \longmapsto \langle v^*, \underbrace{g^{-1} \cdot v}_{\rho_V(g^{-1})(v)} \rangle \Big] \Big]$$

So

$$\langle g \cdot v^*, g \cdot v \rangle = \langle v^*, v \rangle$$

for all $v^* \in V^*$ and $v \in V$.

- Let $\rho_W : G \longrightarrow GL(W)$ be another representation of G.

The tensor product $V \otimes W$ of the vector spaces V and W carries the structure of a representation of G via:

$$\rho_{V\otimes W} : G \longrightarrow GL(V \otimes W)$$

$$g \longmapsto [\, v \otimes w \longmapsto (g \cdot v) \otimes (g \cdot w)\,]$$

So $\rho_{V\otimes W}(g) = \rho_V(g) \otimes \rho_W(g)$.

<u>Remark</u>: In categorical language, $Rep(G)$ is a "rigid tensor category" over k.

$\boxed{\text{Thm}}$ (Tannaka). Suppose we are given an element $\alpha_V \in GL(V)$ for every $V \in Rep(G)$, and suppose that the $\{\alpha_V\}$ satisfy:

(a) $\alpha_V \otimes \alpha_W = \alpha_{V\otimes W}$ in $GL(V\otimes W)$ for all $V, W \in Rep(G)$

(b) $\phi \circ \alpha_V = \alpha_W \circ \phi$ in $Hom(V, W)$ for any $\phi : V \longrightarrow W$ G-linear

(c) $\alpha_{V_{triv}} = 1$ in $GL(V_{triv}) = \mathbb{G}_m$ where $V_{triv} \cong k$ is the trivial representation

[Then $\exists!\ g \in G$ s.t. $\alpha_V = \rho_V(g)$ for any $V \in Rep(G)$.

Idea of proof:

- Recall that $k[G]$ is a union of finite-dimensional G-representations. Use this fact to obtain a well-defined element

$$\alpha_{k[G]} \in GL(k[G])$$

- Argue that

$$\alpha_{k[G]} : k[G] \longrightarrow k[G]$$

is an algebra automorphism.

- Let $\phi : G \longrightarrow G$ be the associated isomorphism of varieties induced by $\alpha_{k[G]}$. Take $g = \phi(1_G)$. $\square$

Aside: Here is the categorical interpretation: Let

$$F : Rep(G) \longrightarrow Vec_k$$

be the forgetful functor to the category of k-vector spaces. Any $g \in G$ defines a natural isomorphism from the functor F to itself:

$$\eta_g : F \longrightarrow F$$

via

$$(\eta_g)_V := \rho_V(g) : F(V) (= V) \longrightarrow F(V) (= V)$$

The η_g respect the tensor structure on $\mathrm{Rep}(G)$, and $\eta_g \circ \eta_h = \eta_{gh}$ for any $g, h \in G$. Hence we obtain a map of groups:

$$G \longrightarrow \mathrm{Aut}^{\otimes}(F)$$

Tannaka's theorem asserts that this map is an isomorphism.

<u>Note</u>: One can also recover the affine algebra $k[G]$ from the category $\mathrm{Rep}(G)$ as a quotient of $\bigoplus_{V \in \mathrm{Rep}(G)} V^* \otimes V$.

See Springer §2.5.5 – 2.5.7 or Chapter 9 of Milne.

<u>Application</u>: Let G be a semisimple linear algebraic group. Let $\mathfrak{g} = \mathrm{Lie}(G)$ be its Lie algebra. Consider the category $\mathrm{Rep}(\mathfrak{g})$ of finite-dimensional representations of $\mathfrak{g}$.

Thm There is a fully faithful functor

$$\mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(\mathfrak{g})$$

If G is simply connected, then this is an equivalence.

("Fully faithful" means that $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_{\mathfrak{g}}(V, W)$ $\forall\, V, W$ reps of G)

Idea of proof:

- The functor is given by differentiation. Specifically, a representation $\rho : G \longrightarrow GL(V)$ of G is sent to the representation of $\mathfrak{g}$ given by
$$d_1 \rho : T_1 G \simeq \mathfrak{g} \longrightarrow T_{Id_V} GL(V) \simeq \mathfrak{gl}(V).$$
- Verify that $Rep(\mathfrak{g})$ is a rigid tensor category. Consider the forgetful functor
$$F : Rep(\mathfrak{g}) \longrightarrow Vec_k$$
- Argue that $Aut^{\otimes}(F)$ is a semisimple and simply connected algebraic group; call it $\tilde{G}$. Thus $Rep(\tilde{G}) \xrightarrow{\sim} Rep(\mathfrak{g})$.
- Show that there is a finite covering map of groups
$$\tilde{G} \longrightarrow G.$$
$\square$

Chevalley Restriction Theorem

Let G be a semisimple algebraic group.

Let $\mathrm{Irrep}(\mathfrak{g})$ denote the set of isomorphism classes of irreducible representations of $\mathfrak{g} = \mathrm{Lie}(G)$.

$\boxed{Q}$ How can we parametrize $\mathrm{Irrep}(\mathfrak{g})$?

E.g. using geometric or combinatorial data?

Observations:

(1) There is an equivalence of categories

$$\mathrm{Rep}(\mathfrak{g}) \simeq \mathrm{Rep}(\mathcal{U}\mathfrak{g}).$$

(2a) Let A be a (non-commutative) algebra over k. Let $\rho: A \to \mathrm{End}(V)$ be an irreducible representation of A (aka a simple A-module). By Schur's lemma, the center $Z(A)$ of A acts by scalars on V, i.e. there is an algebra homomorphism

$$\chi_V : Z(A) \longrightarrow k$$

s.t. $\rho_V(z)(v) = \chi_V(z)\, v \qquad \forall\, z \in Z(A),\ v \in V$

The map χ_V is called the "central character" of V

Hence we have a map, known as the "central character map"

$$\chi : \mathrm{Irrep}(A) \longrightarrow \mathrm{Hom}_{Alg}(Z(A), k)$$

$$V \longmapsto \chi_V$$

(2b) Suppose $Z(A) = k[X]$ is the affine algebra of an affine algebraic variety X. Then the central character map becomes:

$$\chi : \mathrm{Irrep}(A) \longrightarrow X$$

(For many algebras of interest, χ has finite fibers)

Based on these observations, we want to understand $Z(\mathcal{U}\mathfrak{g})$ in order to parametrize $\mathrm{Irrep}(\mathfrak{g})$.

Let $T \subseteq G$ be a maximal torus and $\mathfrak{t} = \mathrm{Lie}(T)$ (note that $\mathfrak{t}$ is an abelian Lie algebra, hence nothing more than a vector space). Let $\mathfrak{t}^*$ be the dual.

Recall: (1) (4.4.10 and 4.4.11)

$$\mathfrak{t} = X_*(T) \otimes_{\mathbb{Z}} k \qquad \mathfrak{t}^* = X^*(T) \otimes_{\mathbb{Z}} k$$

(2) (Last week's lecture)

The Weyl group $W = N_G(T)/T$ acts on $X^*(T)$

Hence: $W \curvearrowright \mathfrak{t}^*$ and $W \curvearrowright k[\mathfrak{t}^*]$.

Thm (Chevalley)

(1) There is an isomorphism of (commutative) algebras:

$$Z(U_\mathfrak{g}) \simeq k[\mathfrak{t}^*]^W$$

(2) The orbit space $\mathfrak{t}^*/W$ is an affine algebraic variety (with the quotient topology) whose affine algebra is $k[\mathfrak{t}^*]^W$.

(3) The central character map is injective:

$$\mathrm{Irrep}(\mathfrak{g}) \longrightarrow \mathfrak{t}^*/W$$

Rmks: Part (1): This is a generalization of the fact that conjugacy classes in $\mathfrak{gl}_n$ can be described by Jordan normal forms, which are almost diagonal matrices. Also, Jordan blocks are only well-defined up to permutations ($W = S_n$ for $\mathfrak{gl}_n$).

Part (2): In fact, $\mathfrak{t}^*/W$ is a vector space of the same dimension as $\mathfrak{t}^*$.

Part (3): One can describe the image of $\mathrm{Irrep}(\mathfrak{g})$ in $\mathfrak{t}^*/W$ explicitly using "dominant weights".

Ex $G = SL_2$, $T = \{ \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} : a \in k^\times \}$

$$\mathcal{U}\mathfrak{sl}_2 = \mathbb{C}\langle E, F, H \rangle \Big/ \left\langle \begin{matrix} HE - EH = 2E, \\ HF - FH = -2F, \end{matrix} \quad EF - FE = H \right\rangle$$

- The Casimir operator

$$\Delta = EF + FE + \tfrac{1}{2} H^2$$

belongs to $Z(\mathcal{U}\mathfrak{sl}_2)$ (easy to check).

- $W = \mathbb{Z}/2\mathbb{Z} = \{\pm 1\} \curvearrowright \mathfrak{t}^* = \mathbb{A}^1$ by multiplication by -1.

and so $k[\mathfrak{t}^*]^W = k[x]^{\{\pm 1\}} = k[x^2]$

- The Chevalley isomorphism is

$$k[x^2] \xrightarrow{\sim} Z(\mathcal{U}\mathfrak{sl}_2)$$

$$x^2 \longmapsto EF + FE + \tfrac{1}{2} H^2$$

[There will be a take-home exam problem on the finite-dimensional representations of $\mathfrak{sl}_2$.]

Line bundles on G/B

Let G be a semisimple algebraic group and B a Borel subgroup.

Recall: The flag variety G/B, a projective variety.

Let $T \subseteq B$ be a maximal torus of G contained in B.

Fact: There is a quotient map $\pi : B \longrightarrow T$.

(the kernel is B_u. See the lecture on solvable groups).

$\boxed{\text{Ex}}$ $G = GL_n$

$$B \ni \begin{bmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{bmatrix} \longmapsto \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} \in T$$

Let $\lambda : T \longrightarrow \mathbb{G}_m$ be a character of T.

Let $\lambda^{(B)} = \lambda \circ \pi : B \longrightarrow \mathbb{G}_m$ be the pullback to B.

Let k_λ be the resulting 1-dimensional B-module, so

$$b \cdot z = \lambda^{(B)}(b)\, z \qquad \forall z \in k,\ b \in B.$$

$\boxed{\text{Def}}$ For $\lambda \in X^*(T)$, define:

$$\mathcal{L}_\lambda := G \times_B k_\lambda = \left. G \times k \middle/ (gb, z) \sim (g, \lambda^{(B)}(b^{-1}) z) \right.$$
$$\forall g \in G,\ b \in B,\ z \in k$$

$\boxed{\text{Lemma}}$: $\mathcal{L}_\lambda$ has a natural action of G, and the map

$$\begin{array}{ccc} \mathcal{L}_\lambda & \ni & [g, z] \\ \downarrow & & \downarrow \\ G/B & \ni & gB \end{array}$$

is a G-equivariant line bundle on G/B.

Thm (Special case of Borel-Weil-Bott).

The global sections $\Gamma(G/B, \mathcal{L}_\lambda)$ is an irreducible representation of G, and all irreducible representations of G appear in this way.

Rmk: The central character of $\Gamma(G/B, \mathcal{L}_\lambda)$ is not λ, but is related to λ via a shifted W-action.

Ex $G = SL_2$, $G/B = \mathbb{P}^1$, $G \circlearrowright \mathbb{P}^1$ by Möbius transformations.

$X^*(T) \simeq \mathbb{Z} \ni n \rightsquigarrow \mathcal{L}_n \longrightarrow \mathbb{P}^1$.

$$\Gamma(\mathbb{P}^1, \mathcal{L}_n) = \begin{cases} 0 & \text{if } n > 0 \\ k[x,y]_{\text{degree} = -n} & \text{if } n \leq 0 \end{cases}$$

[More about this on the final exam].

Generalizations / Enhancements

(1) Borel-Weil-Bott Theorem: A description of the higher cohomologies $H^i(G/B, \mathcal{L}_\lambda)$ as irreducible representations, and their central characters.

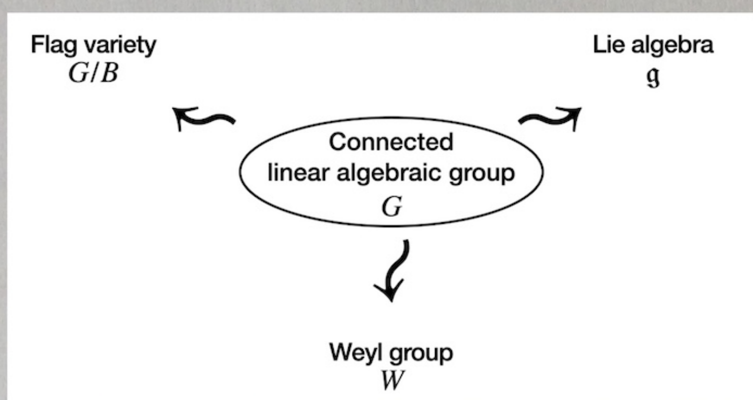
(2) Beilinson-Bernstein Localization: A description of all representations of $\mathfrak{g}$ in terms of sheaves on G/B (really D-modules)

There are functors:

$$\begin{tikzcd} \mathfrak{g}\text{-mod} \arrow[r, "Loc"] & Shv(G/B) \\ \left\{ \begin{array}{l} U\mathfrak{g}\text{-modules with} \\ \text{central character } [\lambda] \in \mathfrak{t}^*/W \end{array} \right\} \arrow[u, hook] \arrow[r, "Loc_\lambda"] & \left\{ \begin{array}{l} \mathcal{D}_\lambda\text{-twisted D-modules} \\ \text{on } G/B \end{array} \right\} \arrow[u, hook] \end{tikzcd}$$

The theorem describes conditions on λ for when Loc_λ is an equivalence of (abelian or derived) categories.

<u>Summary</u>:

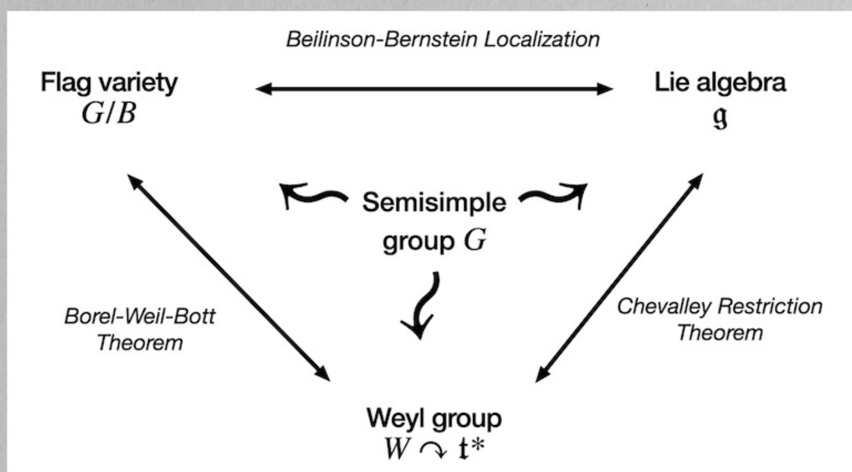


Given a connected linear algebraic group G, we associated several objects:

- $\mathfrak{g}$ - Lie algebra. Connections to:
  - derivations & differentials
  - noncommutative algebra via $U\mathfrak{g}$
- G/B - Flag variety. Connections to:
  - projective geometry
  - moduli spaces

• W - Weyl group. Connections to:
  - finite reflection groups
  - root systems.

When G is semisimple, there are strong interplays between these objects, and they can be used to understand the representation theory of G.



Specifically:

• $\mathrm{Rep}(\mathfrak{g}) \supseteq \mathrm{Rep}(G)$ with equality if G is simply connected.

• $Z(U\mathfrak{g}) \simeq k[\mathfrak{t}^*/W]$, so $\mathfrak{t}^*/W$ can be used to parametrize the irreducible representations of $\mathfrak{g}$ (and of G).

• Irreducible representations of G are geometrically realized as line bundles on G/B

• All representations of G appear as certain sheaves on G/B. (D-modules)