The Adjoint Action

Let G be a linear algebraic group. Let $\mathfrak{g} = T_1 G = \mathrm{Der}(k[G], k_1)$ be its Lie algebra, with bracket coming from the associative algebra structure on $k[G]^*$.

Def The *adjoint action* of G on $\mathfrak{g}$ is:

$$G \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

$$g, \xi \longmapsto Ad(g)(\xi) := d_1(c_g)(\xi)$$

where $c_g : G \to G$, $x \mapsto gxg^{-1}$ is the "conjugation by g" map.

So $Ad(g)(\xi) = \xi \circ c_g^* : k[G] \to k_1$.

The adjoint action is a representation of G on $\mathfrak{g}$. We also have the following representations:

$$G \times \mathrm{Lie}(G)_\ell \longrightarrow \mathrm{Lie}(G)_\ell \qquad G \times \mathrm{Lie}(G)_r \longrightarrow \mathrm{Lie}(G)_r$$

$$g, \theta \longmapsto \rho_g \circ \theta \circ \rho_{g^{-1}} \qquad g, \theta \longmapsto \lambda_g \circ \theta \circ \lambda_{g^{-1}}$$

Prop ① These two actions, as well as the adjoint action, are compatible with the Lie bracket:

$$g([-,-]) = [g(-), g\cdot(-)]$$

② These actions are identified with each other under the isomorphisms:

$$\begin{tikzcd} \mathrm{Lie}(G)_\ell \arrow[rr, "\partial \mapsto S\circ\partial\circ S"] \arrow[dr, "\partial \mapsto \eta\circ\partial"'] & & \mathrm{Lie}(G)_r \arrow[dl, "\partial \mapsto -\eta\circ\partial"] \\ & \mathfrak{g} & \end{tikzcd}$$

<u>Note</u>: These actions on $\mathrm{Lie}(G)_\ell$ and $\mathrm{Lie}(G)_r$ are well-defined since the λ's and the ρ's commute.

<u>Proof</u>. ① It is easy to see the compatibility with the Lie bracket in the cases of $\mathrm{Lie}(G)_\ell$ and $\mathrm{Lie}(G)_r$. For the adjoint action, first note that

$$\begin{tikzcd} G\times G \arrow[r, "\mu"] \arrow[d, "c_g\times c_g"'] & G \arrow[d, "c_g"] \\ G\times G \arrow[r, "\mu"] & G \end{tikzcd}$$

commutes for any $g\in G$.

Therefore, the following diagram commutes for any $g\in G$:

$$\begin{tikzcd} k[G] \arrow[r, "\Delta"] \arrow[d, "c_g^*"'] & k[G]\otimes k[G] \arrow[d, "c_g^*\otimes c_g^*"] \\ k[G] \arrow[r, "\Delta"] & k[G]\otimes k[G] \end{tikzcd}$$

Now let $\xi, \xi' \in \mathfrak{g}$. We claim that the product $Ad(g)(\xi) \cdot Ad(g)(\xi')$ in $k[G]^*$ coincides with $(\xi \cdot \xi') \circ c_g^*$. This claim follows from the commutativity of the following diagram:

$$\begin{tikzcd}
k[G] \arrow[r, "\Delta"] \arrow[d, "c_g^*"'] & k[G] \otimes k[G] \arrow[rr, "Ad(g)(\xi) \otimes Ad(g)(\xi')"] \arrow[d, "c_g^* \otimes c_g^*"] & & k \otimes k \simeq k \arrow[d, leftrightarrow, "="] \\
k[G] \arrow[r, "\Delta"] & k[G] \otimes k[G] \arrow[rr, "\xi \otimes \xi'"] & & k
\end{tikzcd}$$

(The top horizontal composition is $Ad(g)(\xi) \cdot Ad(g)(\xi')$ while going down then right along the bottom composition gives $\xi \cdot \xi' \circ c_g^*$.)

Hence,

$$\begin{aligned}
[Ad(g)(\xi), Ad(g)(\xi')] &= (\xi \cdot \xi') \circ c_g^* - (\xi' \cdot \xi) \circ c_g^* \\
&= [\xi, \xi'] \circ c_g^* = Ad(g)([\xi, \xi']).
\end{aligned}$$

② To show that $Lie(G)_e \longrightarrow \mathfrak{g}$; $\partial \longmapsto \eta \circ \partial$, is G-equivariant, first observe that the diagram on the left commutes, which implies that the diagram on the right commutes.

$$\begin{tikzcd}
\{1\} \arrow[r, hook] \arrow[d] & G \arrow[d, "r_{g^{-1}}"] \\
G \arrow[r, "l_g"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
k[G] \arrow[r, "\rho_g = r_{g^{-1}}^*"] \arrow[d, "\lambda_{g^{-1}} = l_g^*"'] & k[G] \arrow[d, "\eta"] \\
k[G] \arrow[r, "\eta"] & k
\end{tikzcd}$$

So for $\theta \in \mathrm{Lie}(G)_\ell$, we have

$$\eta\circ(g\cdot\theta) = \eta\circ(\rho_g\circ\theta\circ\rho_{g^{-1}}) = \eta\circ\lambda_{g^{-1}}\circ\theta\circ\rho_{g^{-1}}$$

$$= \eta\circ\theta\circ\lambda_{g^{-1}}\circ\rho_{g^{-1}} = \eta\circ\theta\circ l_g^*\circ r_g^*$$

$$= \eta\circ\theta\circ c_g^* = \mathrm{Ad}(g)(\eta\circ\theta).$$

The argument for $\mathrm{Lie}(G)_r \longrightarrow \mathfrak{g}$ is analogous. The G-equivariance of $\mathrm{Lie}(G)_\ell \longrightarrow \mathrm{Lie}(G)_r$, $\theta \mapsto S\circ\theta\circ S$ now follows from the G-equivariance of the other two maps (or it can be shown directly using $S\circ\lambda_g = \rho_g\circ S$). $\square$

Remark. The proof that $\mathrm{Ad}(g)$ acts on $\mathfrak{g}$ by Lie algebra automorphisms [illegible] extends to show more generally that G acts on $k[G]^*$ by algebra automorphisms:

$$G\times k[G]^* \longrightarrow k[G]^*$$
$$g,\ \psi \longmapsto \psi\circ c_g^*$$

$\boxed{\text{Examples}}$ ① If $G = GL_n$, we identify $\mathfrak{gl}_n = \left\{ \begin{matrix} n\times n \text{ matrices} \\ \text{with commutator} \end{matrix} \right\}$

The adjoint action becomes matrix multiplication:

$$GL_n \times \mathfrak{gl}_n \longrightarrow \mathfrak{gl}_n$$
$$g,\ \xi \longmapsto g\xi g^{-1}$$

(The idea is to look at $E_{ij} \hookrightarrow \eta \circ \frac{\partial}{\partial T_{ij}}$ from before)

② If $G = SL_n$, we identify its Lie algebra with

$sl_n = \{ \xi \in gl_n : \mathrm{trace}(\xi) = 0 \}$.

The adjoint action is again given by conjugation:

$$Ad : SL_n \times sl_n \longrightarrow sl_n$$
$$g, \xi \longmapsto g \xi g^{-1}$$

Note: $\mathrm{trace}(g \xi g^{-1}) = \mathrm{trace}(\xi)$ $\forall g \in SL_n$ (or even GL_n)
$\forall \xi \in sl_n$ (or even gl_n)

③ If $G = O(n)$, we have the identification $\mathfrak{o}(n) = \{ \xi \in gl_n : \xi + \xi^T = 0 \}$.
The adjoint action is again matrix conjugation.

Note. $g \xi g^{-1} + (g \xi g^{-1})^T = g(\xi + \xi^T) g^{-1}$

for any $\xi \in gl_n$ and $g \in O(n)$ (but not for a general $g \in GL_n$).

④ If G is connected, then:

adjoint action is trivial $\Longleftrightarrow$ G is commutative.

Remark. We also have the coadjoint action of G on $\mathfrak{g}^* = (T_1 G)^*$:

$$G \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$$

$$g, \xi^* \longmapsto [\xi \mapsto \langle \xi^*, Ad(g^{-1})\xi \rangle]$$

where $\langle , \rangle : \mathfrak{g}^* \otimes \mathfrak{g} \longrightarrow k$ is the evaluation pairing.

⌈Aside: $\mathfrak{g}^*$ has a natural symplectic structure whose symplectic leaves are precisely the G-orbits.⌋

Recall: We have isomorphisms of Lie algebras

$$\mathfrak{g} \xrightarrow{\sim} Lie(G)_e \qquad \mathfrak{g} \xrightarrow{\sim} Lie(G)_r \qquad (*)$$

taking $\xi \in \mathfrak{g} = T_1 G$ to the vector field whose value at $g \in G$ is $d_1(l_g)(\xi)$, respectively $d_1(r_g)(\xi)$.

Proposition Let

$$\alpha_l : \mathfrak{g} \longrightarrow \mathcal{T}_G \qquad \alpha_r : \mathfrak{g} \longrightarrow \mathcal{T}_G$$

be the compositions of the maps in $(*)$ with the inclusions into $\mathcal{T}_G$. These extend to $k[G]$-linear isomorphisms

$$\tilde{\alpha}_l : k[G] \otimes \mathfrak{g} \longrightarrow \mathcal{T}_G \qquad \tilde{\alpha}_r : k[G] \otimes \mathfrak{g} \longrightarrow \mathcal{T}_G$$

Note: The $k[G]$-action is in the first factor of $k[G] \otimes \mathfrak{g}$.

Proof (sketch). We consider α_e. The extension to $k[G]\otimes\mathfrak{g}$ is given by

$$\Big[\underset{\substack{\uparrow\\ k[G]}}{\tilde{\alpha}_e}(f\otimes \underset{\substack{\uparrow\\ \mathfrak{g}}}{\xi})\Big](\underset{\substack{\uparrow\\ k[G]}}{h})(\underset{\substack{\uparrow\\ G}}{g}) = f(g)\Big[d_1(l_g)(\xi)\Big](h)(g)$$

To see injectivity, set the RHS equal to zero for all $h\in k[G]$ and $g\in G$. Then either $f=0$ (so $f\otimes\xi=0$) or $d_1(l_g)(\xi)=0$ $\forall g\in G$. Since $d_1(l_g)$ is an isomorphism for all $g\in G$, the latter happens iff $\xi=0$.

To show surjectivity, pick a basis $\xi_1,\dots,\xi_n$ for $\mathfrak{g}$. Then for all $g\in G$, $\{d_1(l_g)(\xi_i)\}_{i=1}^n$ is a basis of T_gG. Hence, given $\theta\in\mathcal{T}_G$, $\exists\ \theta_g^i\in k$ $i=1,\dots,n$, s.t.

$$\theta_g = \sum_{i=1}^n \theta_g^i\cdot d_1(l_g)(\xi_i)$$

One argues that $g\longmapsto\theta_g^i$ belongs to $k[G]$. Hence

$$\theta = \tilde{\alpha}_e\Big(\sum_{i=1}^n \theta^i\otimes\xi_i\Big).$$

□

Slogan: $\alpha_\ell(\xi)$ and $\alpha_r(\xi)$ differ at $g\in G$ by $Ad_g(\xi)$.

More precisely, the following diagram commutes $\forall g\in G$:

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr, "Ad(g)"] \arrow[d, "\alpha_\ell"'] & & \mathfrak{g} \arrow[d, "\alpha_r"] \\
\mathcal{T}_G \arrow[dr, "ev_g"'] & & \mathcal{T}_G \arrow[dl, "ev_g"] \\
& T_g G &
\end{tikzcd}$$

Justification:

$$\alpha_r(Ad(g)\xi)_g = d_1(r_g)(Ad(g)\xi) = (\xi \circ c_g^*) \circ r_g^*$$

$$= \xi \circ (r_g \circ c_g)^* = d_1(r_g \circ c_g)(\xi)$$

$$= d_1(\ell_g)(\xi) = \alpha_\ell(\xi)_g .$$

Facts: ① $\dim \mathfrak{g} = \dim G$ (G smooth)

② Let H be a closed subgroup of G. Then $\mathfrak{h} = T_1 H$ is a Lie subalgebra of $\mathfrak{g} = T_1 G$.

(**Idea**: We have that $k[H] = k[G]/I$ for some ideal I.

Then $T_1 H = \{ \xi \in T_1 G : \xi(I) = 0 \}$)

(3a) Let $\phi : G \longrightarrow H$ be a morphism of linear algebraic groups.

Then

$$d_1 \phi : \mathfrak{g} = T_1 G \longrightarrow \mathfrak{h} = T_1 H$$

is a Lie algebra homomorphism.

(**Idea**: the fact that $\phi^* : k[H] \longrightarrow k[G]$ is a map of coalgebras implies that $\phi^{**} : k[G]^* \longrightarrow k[H]^*$ is a map of algebras. Observe that $d_1 \phi = \phi^{**}\Big|_{T_1 G}$)

(3b) If $\rho : G \longrightarrow GL(V)$ is a representation of G,

then $d_1 \rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a representation of $\mathfrak{g} = T_1 G$.

[**Aside**: There is an equivalence of categories

$$\mathrm{Rep}(G) \xrightarrow{\sim} \mathrm{Rep}(\mathfrak{g})^{\text{locally-finite}}$$

for G simply connected]

(4) $\mathrm{Lie}(G \times H) = \mathrm{Lie}(G) \oplus \mathrm{Lie}(H)$ (direct sum)

(*Idea*: use the general fact that $T_{(x,y)}(X \times Y) = T_x X \oplus T_y Y$ for any varieties X, Y and any $x \in X$, $y \in Y$.)

The bracket on the RHS is

$$[(a,b),(c,d)] = ([a,c],[b,d]).$$

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(5) Let $\mu : G \times G \longrightarrow G$ and $i : G \longrightarrow G$ be the multiplication and inverse maps. Then

$$d_{(1,1)}\mu(\xi, \xi') = \xi + \xi' \qquad d_1 i(\xi) = -\xi$$

for $\xi, \xi' \in \mathfrak{g} = T_1 G$.

(True through the isomorphism $T_{(1,1)}(G \times G) \simeq T_1 G \oplus T_1 G$.)

Universal Enveloping Algebras

Let V be a vector space $/k$. The tensor algebra of V is

$$T^{\bullet} V = k \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \cdots$$
$$= \bigoplus_{i=0}^{\infty} V^{\otimes i}$$

with multiplication given by concatenation of vectors:

$$(v_1 \otimes \cdots \otimes v_i) \cdot (w_1 \otimes \cdots \otimes w_j) = v_1 \otimes v_2 \otimes \cdots \otimes v_i \otimes w_1 \otimes \cdots \otimes w_j$$

$$\underset{V^{\otimes i}}{\cap} \qquad \underset{V^{\otimes j}}{\cap} \qquad \underset{V^{\otimes i+j}}{\cap}$$

Note: If $\{v_1, \ldots, v_n\}$ is a basis for V, then $T^\bullet V$ is the free algebra on the elements $v_1, \ldots, v_n$.

$\boxed{\text{Def}}$ The symmetric algebra of V is

$$\mathrm{Sym}^\bullet V = T^\bullet V / \langle v \otimes w - w \otimes v \ : \ v, w \in V \rangle$$

The exterior algebra of V is

$$\Lambda^\bullet V = T^\bullet V / \langle v \otimes w + w \otimes v \ : \ v, w \in V \rangle$$

Note: Both $\mathrm{Sym}^\bullet V$ and $\Lambda^\bullet V$ are graded algebras. Let $\mathrm{Sym}^d V$ and $\Lambda^d V$ be the images of $T^{\otimes d}$.

Note: $\mathrm{Sym}^\bullet V$ is a commutative algebra. If V is finite-dimensional, then $\mathrm{Sym}^\bullet V$ is the affine algebra of the affine variety V^*. (it's just a polynomial algebra on $\dim V$ variables).

Def Let $\mathfrak{g}$ be a Lie algebra. Its <u>universal enveloping algebra</u> is

$$\mathcal{U}\mathfrak{g} = T^{\bullet}\mathfrak{g} \Big/ \underbrace{\xi \otimes \xi' - \xi' \otimes \xi}_{\in\, \mathfrak{g}^{\otimes 2}} = \underbrace{[\xi, \xi']}_{\in\, \mathfrak{g}}$$

We often drop the tensor sign.

Ex ① If $\mathfrak{g}$ is commutative, then $\mathcal{U}\mathfrak{g} = \mathrm{Sym}^{\bullet}\mathfrak{g}$.

② $\mathcal{U}\mathfrak{g}$ is not graded in general (but it is filtered).

③ $$\mathcal{U}\mathfrak{sl}_2 = \mathbb{R}\langle E, F, H\rangle \Big/ \begin{array}{l} HE - EH = 2E \\ HF - FH = -2F \\ EF - FE = H \end{array}$$

One can easily show that $EF + FE + \frac{1}{2}H^2$ is central in $\mathcal{U}\mathfrak{sl}_2$.

<u>Facts</u>: ① $$\mathcal{U} : \{\text{Lie algebras}\} \longrightarrow \{\text{associative algebras}\}$$

$$\mathfrak{g} \longmapsto \mathcal{U}\mathfrak{g}$$

is the left adjoint to the forgetful functor

$$F : \{\text{associative algebras}\} \longrightarrow \{\text{Lie algebras}\}$$

In particular, for any vector space V, we have

$$\mathrm{Hom}_{\text{Lie-alg}}(\mathfrak{g}, \mathfrak{gl}(V)) = \mathrm{Hom}_{\substack{\text{assoc-}\\ \text{alg}}}(\mathcal{U}\mathfrak{g}, \mathrm{End}(V)).$$

$\rightsquigarrow$ Equivalence of categories $\mathrm{Rep}(\mathfrak{g}) \simeq \mathrm{Rep}(\mathcal{U}\mathfrak{g})$.

(2) As a vector space, $\mathcal{U}\mathfrak{g} \simeq \mathrm{Sym}^\bullet \mathfrak{g}$ (<u>not</u> as algebras)

In particular, $\mathfrak{g} \hookrightarrow \mathcal{U}\mathfrak{g}$.

(3) The adjoint action of G on $\mathfrak{g}$ extends to an action on $\mathcal{U}\mathfrak{g}$ by algebra automorphisms.

(<u>Note</u>: $g \cdot (\xi \otimes \xi') = (g \cdot \xi) \otimes (g \cdot \xi')$).

The <u>center</u> of $\mathcal{U}\mathfrak{g}$ consists of the adjoint-invariant elements:

$$Z(\mathcal{U}\mathfrak{g}) = (\mathcal{U}\mathfrak{g})^G.$$