Lie Algebras

Def A <u>Lie algebra</u> over k is a vector space $\mathfrak{g}$ over k endowed with a bilinear operation

$$[\,,\,] : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$$

satisfying:

- [skew-symmetry]: $\quad [x,y] = -[y,x] \quad \forall\, x,y \in \mathfrak{g}$
- [Jacobi identity]: $\quad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad \forall\, x,y,z \in \mathfrak{g}.$

Def A Lie algebra is <u>commutative</u> if $[\,,\,] = 0$

<u>Idea</u>: To every linear algebraic group we will associate a Lie algebra. We can understand the structure and representation theory of the group through the Lie algebra.

$$\left\{\begin{matrix}\text{linear algebraic}\\ \text{groups}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{finite-dimensional}\\ \text{Lie algebras}\end{matrix}\right\}$$

$$\begin{matrix}\text{geometric}\\ \text{methods}\end{matrix} \rightsquigarrow \text{algebraic methods}$$

[the map is roughly finite-to-one].

Lemma Let E be an associative algebra over k. We can endow E with the structure of a Lie algebra by:

$$[\,,\,] : E \otimes E \longrightarrow E$$
$$x, y \longmapsto xy - yx \qquad \text{"commutator"}$$

So E is commutative as an associative algebra iff it is commutative as a Lie algebra.

Pf. Easy verification. $\square$

[Aside: This is a forgetful functor $\left\{ \begin{matrix} \text{assoc.} \\ \text{algebras} \end{matrix} \right\} \longrightarrow \{\text{Lie algebras}\}$

We'll see the adjoint later.]

Examples.

① Let V be a vector space$/k$. The endomorphism algebra $E = \mathrm{End}(V)$ has a Lie algebra structure via the commutator. We will write $\mathfrak{gl}(V)$ instead of $\mathrm{End}(V)$ when we want to regard it as a Lie algebra.

② Let V be as above. The subspace

$$\{ f \in \mathrm{End}(V) : \mathrm{tr}(f) = 0 \}$$

is a Lie subalgebra of $\mathrm{End}(V)$ $(= \mathfrak{gl}(V))$ since $\mathrm{tr}(fg) = \mathrm{tr}(gf)$

for all $f, g \in \mathrm{End}(V)$.

(Note: traceless endomorphisms are defined independent of any basis.

They are not an associative subalgebra of $\mathrm{End}(V)$.)

We denote this Lie subalgebra by $\mathfrak{sl}(V)$.

③ Let R be a commutative ring and A an R-algebra, also commutative.

Then $\mathrm{Der}_R(A,A)$ is a Lie subalgebra of $\mathrm{End}_R(A)$.

To check: Given $D_1, D_2 \in \mathrm{Der}_R(A,A)$, show that

$D_1 \circ D_2 - D_2 \circ D_1$ is again a derivation.

(Note: $\mathrm{Der}_R(A,A)$ is not an associative subalgebra of $\mathrm{End}_R(A)$).

④ Let R, A be as in ③. Suppose a group G acts on A by R-algebra automorphisms. So

$$\rho : G \longrightarrow GL(A)$$

with $\rho(g) : A \to A$ an R-algebra automorphism.

Then G acts on $\mathrm{End}_R(A)$ by:

$$(g \cdot f)(a) = g\, f(g^{-1} a)$$

Claims (a) $\mathrm{Der}_R(A,A)$ is stable under the G-action

[so $D \in \mathrm{Der}_R(A,A) \Rightarrow g \cdot D \in \mathrm{Der}_R(A,A)$]
$\forall g \in G$

(b) The invariants

$$\mathrm{End}_R(A)^G = \{ f \in \mathrm{End}_R(A) : \quad g \cdot f = f \quad \forall g \in G \}$$

$$\mathrm{Der}_R(A,A)^G = \{ D \in \mathrm{Der}_R(A,A) : \quad g \cdot D = D \quad \forall g \in G \}$$

$$= \mathrm{Der}_R(A,A) \cap \mathrm{End}_R(A)^G$$

are Lie-subalgebras. (Check this!)

Def ① Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras. A <u>morphism of Lie algebras</u> is a linear map $\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ s.t.

$$\phi([x,y]) = [\phi(x), \phi(y)] \qquad \forall x, y \in \mathfrak{g}.$$

② A <u>representation of a Lie algebra</u> $\mathfrak{g}$ is a morphism of Lie algebras

$$\mathfrak{g} \longrightarrow \mathfrak{gl}(V) \qquad \text{for some v.s. } V.$$

Def Let X be an affine variety. The Lie algebra of vector fields on X is defined as

$$T_X := \mathrm{Der}_k(k[X], k[X])$$

The space of 1-forms on X is

$$\Omega_X := \Omega_{k[X]}$$

Note: For any $x \in X$, we have that

$$T_x X = \mathrm{Der}(k[X], k_x) = \mathcal{T}_X \otimes_{k[X]} k_x$$

$$T_x^* X = \Omega_{k[X]} \otimes_{k[X]} k_x = \Omega_X \otimes_{k[X]} k_x$$

Thus we have maps

$$\mathcal{T}_X \longrightarrow T_x X \qquad \Omega_X \longrightarrow T_x^* X$$

Also $T_x^* X = (T_x X)^*$ as vector spaces.

Now suppose a linear algebraic group G acts on an affine variety X.

We have:

- Action map $\quad a : G \times X \longrightarrow X$
- Pullback of $a \quad a^* : k[X] \longrightarrow k[G] \otimes k[X] \qquad$ (coaction)

$$f \longmapsto [g, x \longmapsto f(gx)]$$

-
- Representation of G on $k[X]$:

$$\rho : G \longrightarrow GL(k[X])$$

$$g \longmapsto \rho_g : [f \longmapsto [x \longmapsto f(g^{-1}x)]]$$

$\hookrightarrow$ algebra automorphism of $k[X]$

Observe: $\quad \rho_g = a_{g^{-1}}^* \quad$ where $\quad a_g : X \longrightarrow X$

$$x \longmapsto gx$$

Def The Lie algebra of G-equivariant vector fields on X is defined as

$$(\mathcal{T}_X)^G = \mathrm{Der}\,(k[X], k[X])^G \subseteq \mathcal{T}_X$$

Explicitly,

$$(\mathcal{T}_X)^G = \{\, \partial \in \mathrm{Der}\,(k[X], k[X]) : \theta \circ \rho_g = \rho_g \circ \theta \text{ for all } g \in G \,\}$$

Ex $G = \mathbb{G}_m \curvearrowright X = \mathbb{C}$ by scalar multiplication: $\lambda \cdot a = \lambda a$ for $\lambda \in \mathbb{G}_m$, $a \in k$

Then the action of $\lambda \in \mathbb{G}_m$ on $k[X] = k[T]$ is given by:

$$\rho_\lambda : k[T] \longrightarrow k[T]$$
$$T \longmapsto \lambda^{-1} T$$

Observe that $\underset{\forall \lambda \in \mathbb{G}_m}{\rho_\lambda(f) = \lambda^{-n} f}$ iff f is homogeneous of degree n.

Now, any derivation $k[T] \longrightarrow k[T]$ has the form $v \frac{\partial}{\partial T}$ for some $v \in k[T]$. We compute:

$D = v \frac{\partial}{\partial T}$ is in $\mathrm{Der}(k[X], k[X])^G$

iff $D \circ \rho_\lambda = \rho_\lambda \circ D \quad \forall \lambda \in \mathbb{G}_m$

iff $D(\lambda^{-1} T) = \rho_\lambda(D(T)) \quad \forall \lambda \in \mathbb{G}_m$

iff $\quad v \frac{\partial}{\partial T}(x^{-1}T) = \rho_x\left(v \frac{\partial}{\partial T}(T)\right) \qquad \forall x \in \mathbb{G}_m$

iff $\quad x^{-1} v = \rho_x(v) \qquad \forall x \in \mathbb{G}_m$

iff $\quad v \in k[T]$ is homogeneous of degree 1

iff $\quad v \in k\{T\} \subseteq k[T]$

iff $\quad D \in k\{T \cdot \frac{\partial}{\partial T}\}$

Thus, the space of G-equivariant vector fields is the span of $\overbrace{T \cdot \frac{\partial}{\partial T}}^{\text{Euler operator}}$ (one-dimensional).

<u>Extension</u>: Compute

$$\left(\mathcal{T}_{k^n}\right)^{\mathbb{G}_m} \subseteq \mathcal{T}_{k^n} = k[T_1, \dots, T_n]\left\{\frac{\partial}{\partial T_1}, \dots, \frac{\partial}{\partial T_n}\right\}.$$

where $\mathbb{G}_m \curvearrowright k^n$ by scalar multiplication. $\boxtimes$

$\boxed{\text{Prop}}$ Let G be a linear algebraic group and X an affine G-variety.

① Let $\theta \in \mathcal{T}_X$. Then $\theta \in (\mathcal{T}_X)^G$ iff

$$d_x(a_g)(\theta_x) = \theta_{gx} \qquad \forall g \in G,\ x \in X$$

② $\mathcal{T}_X{}^G$ is a Lie subalgebra of $\mathcal{T}_X$.

③ If X is a principal homogeneous space (torsor) for G, then, for any $x \in X$, the map

$$(\mathcal{T}_X)^G \longrightarrow T_x X$$

$$\theta \longmapsto \theta_x \quad (= \pi_x \circ \theta \text{, where } \pi_x : k[X] \to k_x)$$

is an isomorphism of vector spaces.

Pf. ① Let $\theta \in \mathcal{T}_X$ be a vector field. Then

$$\theta \in (\mathcal{T}_X)^G \quad \text{iff} \quad \rho_g \circ \theta = \theta \circ \rho_g \quad \forall g \in G$$

$$\text{iff} \quad a_g^* \circ \theta = \theta \circ a_g^* \quad \forall g \in G \quad (\text{or } g^{-1} \in G)$$

$$\text{iff} \quad [(a_g^* \circ \theta)(f)](x) = [(\theta \circ a_g^*)(f)](x) \quad \forall g \in G,\ \forall f \in k[X],\ \forall x \in X$$

$$\text{iff} \quad \theta(f)(gx) = d_x(a_g)(\theta_x)(f) \quad \forall g \in G,\ \forall f \in k[X],\ \forall x \in X$$

$$\text{iff} \quad \theta_{gx}(f) = d_x(a_g)(\theta_x)(f) \quad \sim \text{"} \sim$$

$$\text{iff} \quad \theta_{gx} = d_x(a_g)(\theta_x) \quad \forall g \in G,\ x \in X.$$

② Immediate from the fact that $\mathrm{Der}(k[X], k[X])^G$ is a Lie subalgebra of $\mathrm{Der}(k[X], k[X])$.

③ The choice of $x \in X$ gives an isomorphism of varieties

$$\begin{aligned} G &\longrightarrow X \\ g &\longmapsto g \cdot x \end{aligned}$$

Let $\psi : k[X] \longrightarrow k[G]$ be the induced isomorphism.

Define:

$$s : T_x X \longrightarrow \mathrm{End}_k(k[X])$$

as follows, for $\xi \in T_x X = \mathrm{Der}(k[X], k_x)$:

$$s(\xi) = \left[k[X] \xrightarrow{a^*} k[G] \otimes k[X] \xrightarrow{\psi^{-1} \otimes 1} k[X] \otimes k[X] \xrightarrow{\xi \otimes 1} k_x \otimes k[X] \simeq k[X] \right]$$

More concretely:

$$\left[s(\xi)(f) \right](g \cdot x) = \left[d_x(a_g)(\xi) \right](f) \qquad (*)$$

To check:

- $s(\xi)$ is a derivation $\forall \xi \in T_x X$
- $s(\xi)$ is G-equivariant $\forall \xi \in T_x X$. □

Remark: Part ③ is true whenever $X = G/H$ for a normal subgroup H of G; and essentially the same proof works (with some modifications to check that s is well-defined).

Let G be a linear algebraic group. Recall the actions of left and right translation:

$$\begin{aligned} l_g : G &\longrightarrow G \\ x &\longmapsto gx \end{aligned} \qquad\qquad \begin{aligned} r_g : G &\longrightarrow G \\ x &\longmapsto xg^{-1} \end{aligned}$$

Denote the Lie algebras of left- and right-invariant vector fields on G by:

$$\mathrm{Lie}(G)_l = \left(\mathcal{T}_G\right)^{\text{left-invariant}} = \{\theta \in \mathcal{T}_G : l_g^* \circ \theta = \theta \circ l_g^*\}$$

$$\mathrm{Lie}(G)_r = \left(\mathcal{T}_G\right)^{\text{right-invariant}} = \{\theta \in \mathcal{T}_G : r_g^* \circ \theta = \theta \circ r_g^*\}$$

$\boxed{\text{Lemma}}$ Let $S : k[G] \to k[G]$ be the antipode. The map

$$\begin{aligned} \mathrm{Lie}(G)_r &\longrightarrow \mathrm{Lie}(G)_l \\ \theta &\longmapsto S \circ \theta \circ S \end{aligned}$$

is an isomorphism of Lie algebras.

<u>proof</u>. Recall that $S = i^*$ where $i : G \to G$ is the "inverse" $x \mapsto x^{-1}$. Note that the following diagrams commute:

$$\begin{tikzcd} G \arrow[r, "r_g"] \arrow[d, "i"'] & G \arrow[d, "i"] \\ G \arrow[r, "l_g"] & G \end{tikzcd} \qquad\qquad \begin{tikzcd} G \arrow[r, "l_g"] \arrow[d, "i"'] & G \arrow[d, "i"] \\ G \arrow[r, "r_g"] & G \end{tikzcd}$$

To check:

- φ is well-defined:

$$S \circ \partial \circ S \circ l_g^* = S \circ \partial \circ i^* \circ l_g^* = S \circ \partial \circ (l_g \circ i)^*$$
$$= S \circ \partial \circ (i \circ r_g)^* = S \circ \partial \circ r_g^* \circ i^*$$
$$= S \circ r_g^* \circ \partial \circ S = l_g^* \circ S \circ \partial \circ S .$$

- φ is a map of Lie algebras
  (use $i^2 = id_G$ so $S^2 = id_{k[G]}$)

- φ is an isomorphism. $\square$

Here's another way to think about this algebra.

Recall the coproduct map:

$$\Delta : k[G] \longrightarrow k[G] \otimes k[G]$$

and counit $\eta = ev_1 : k[G] \longrightarrow k$.

$\boxed{\text{Lemma}}$ The dual vector space $k[G]^* = \mathrm{Hom}_k(k[G], k)$ of $k[G]$ carries an associative algebra structure given by

$$k[G]^* \otimes k[G]^* \longrightarrow k[G]^*$$
$$\xi \otimes \xi' \longmapsto \xi \cdot \xi' = (\xi \otimes \xi') \circ \Delta$$

with unit $\eta \in k[G]^*$.

proof. Follows from the coassociativity of Δ and the counit axiom $\square$

As we've seen, we get a Lie algebra structure on $k[G]^*$ via the commutator:

$$[\xi, \xi'] = \xi \cdot \xi' - \xi' \cdot \xi \qquad \text{for } \xi, \xi' \in k[G]^*.$$

Lemma The subspace $T_1 G = \mathrm{Der}(k[G], k) \subseteq k[G]^*$ is a Lie subalgebra of $k[G]^*$

proof. We use Sweedler notation: $\Delta(a) = a_1 \otimes a_2$ for $a \in k[G]$

So the summation is implicit.

For $\xi, \xi' \in T_1 G$ and $a, b \in k[G]$ we have:

- $\xi \cdot \xi'(a) = \xi(a_1)\, \xi'(a_2)$

- $\xi \cdot \xi'(ab) = \xi(a_1 b_1)\, \xi'(a_2 b_2)$

$$= [a_1 \cdot \xi(b_1) + b_1 \cdot \xi(a_1)][a_2 \cdot \xi'(b_2) + b_2\, \xi'(a_2)]$$

$$= \dots\dots$$

$$= \eta(a)(\xi \cdot \xi'(b)) + \eta(b)(\xi \cdot \xi'(a))$$
$$+ \xi(a)\, \xi'(b) + \xi(b)\, \xi'(a)$$

Using the facts that:

- $\eta(a_1 a_2) = \eta(a)$
- $\eta(a_1) a_2 = a = a_1 \eta(a_2)$
- ξ and ξ' are k-linear.

) and similarly for b

It then follows easily that

$$[\xi, \xi'](ab) = a([\xi, \xi'](b)) + b([\xi, \xi'](a)). \qquad \square$$

<u>Upshot</u>: We get a Lie algebra structure on $T_1 G$.
We write $\mathfrak{g}$ for $T_1 G$ equipped with this Lie algebra structure.

Proposition The map

$$\begin{aligned} \mathrm{Lie}(G)_e &\longrightarrow \mathfrak{g} \\ \partial &\longmapsto \eta \circ \partial \end{aligned}$$

defines an isomorphism of Lie algebras.

<u>proof</u>. Homework exercise (check website). $\square$

First step is to show that

$$\begin{tikzcd} k[G] \arrow[r, "\partial"] \arrow[d, "\Delta"'] & k[G] \arrow[d, "\Delta"] \\ k[G]\otimes k[G] \arrow[r, "1\otimes\partial"] & k[G]\otimes k[G] \end{tikzcd}$$

commutes for any $\partial \in \mathrm{Lie}(G)_e$

Remark: In summary, we have isomorphisms of Lie algebras:

$$\begin{tikzcd} \mathrm{Lie}(G)_\ell \arrow[rr, "\theta \mapsto S \circ \theta \circ S"] \arrow[dr, "\theta \mapsto \eta \circ \theta"'] & & \mathrm{Lie}(G)_r \arrow[dl, "\theta \mapsto -\eta \circ \theta"] \\ & \mathfrak{g} & \end{tikzcd}$$

⌈Aside: Any Lie algebra is isomorphic to its opposite via $x \longmapsto -x$ ⌋

$\boxed{\text{Question}}$ How can we describe $\mathfrak{g}$ in terms of generators and relations?

$\boxed{\text{(Partial) Answer}}$: Since any linear algebraic group embeds into GL_n the first step is to understand $T_1 GL_n$. Since $GL_n \hookrightarrow Mat_{n,n}$ is an open inclusion, we have a vector space isomorphism

$$T_1 GL_n \xrightarrow{\sim} T_1 Mat_{n,n}$$

Observe: Pulling back along matrix multiplication gives a coalgebra structure on $k[Mat_{n,n}]$:

$$\Delta : k[Mat_{n,n}] \longrightarrow k[Mat_{n,n}] \otimes k[Mat_{n,n}]$$

$$\underset{\overset{\shortparallel}{ev_1}}{\eta} : k[Mat_{n,n}] \longrightarrow k$$

(No antipode, so only a coalgebra, not a Hopf algebra)

So we have an induced algebra structure on $k[Mat_{n,n}]^*$:

$$\Delta^* : k[Mat_{n,n}]^* \otimes k[Mat_{n,n}]^* \longrightarrow k[Mat_{n,n}]^*$$

One can check that:

(1) $T_1 Mat_{n,n}$ is a Lie subalgebra.

(2) The isomorphism $(*)$ is one of Lie algebras.

Hence, we reduce the problem to understanding $T_1 Mat_{n,n}$.

<u>Notation</u>: $k[Mat_{n,n}] = k[T_{ij} : 1 \le i,j \le n]$.

Claim: We have an injective algebra homomorphism

$$Mat_{n,n} \longrightarrow k[Mat_{n,n}]^*$$

$$E_{ij} \longmapsto \xi_{ij} := \eta \circ \frac{\partial}{\partial T_{ij}}$$

where $\{E_{ij}\}$ are the elementary matrices. The image of this map is $T_1 Mat_{n,n}$.

pf. To see that this is an algebra homomorphism, first recall the elementary matrices multiply according to the rule:

$$E_{ij} E_{rs} = \delta_{jr} E_{is}$$

We compute:

$$(\xi_{ij} \cdot \xi_{rs})(T_{uv}) = (\xi_{ij} \otimes \xi_{rs})\left(\sum_{w=1}^{n} T_{uw} \otimes T_{wv}\right)$$

$$= \sum_{w=1}^{n} \xi_{ij}(T_{uw})\, \xi_{rs}(T_{wv}) = \begin{cases} 1 & \text{if } i=u \text{ and } s=v \text{ and } j=r \\ 0 & \text{otherwise} \end{cases}$$

$$= \delta_{jr}\, \xi_{is}(T_{uv})$$

Injectivity is clear. We know from previous discussions that $T_1 \mathrm{Mat}_{n,n} = \mathrm{Der}(k[\mathrm{Mat}_{n,n}], k_1)$ is spanned by $\varphi \circ \frac{\partial}{\partial T_{ij}}$ for $i,j = 1, \dots, n$. $\square$

Conclusions:

We identify the Lie algebra of Gl_n with the Lie algebra corresponding to the associative algebra of n by n matrices:

$$\mathfrak{gl}_n := \left\{ \begin{array}{l} n \text{ by } n \text{ matrices with Lie} \\ \text{bracket given by the commutator} \\ \text{of matrices} \end{array} \right\}$$

The Lie algebra of a general linear algebraic group can be identified with a Lie subalgebra of $\mathfrak{gl}_n$, but in general does not arise from an associative algebra.

Ex ① $\mathfrak{sl}_2 = \{ x \in \mathfrak{gl}_2 : \mathrm{trace}(x) = 0 \}$

$$= \mathrm{Span}_k \left\{ E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

Relations:

$$[H,E] = 2E \qquad [H,F] = -2F \qquad [E,F] = H$$