Reductive Groups

Let G be a connected linear algebraic group.

Recall: A Borel subgroup is a maximal closed connected solvable subgroup.

Def The radical $R(G)$ of G is the (unique) maximal closed connected normal solvable subgroup of G.

The unipotent radical $R_u(G)$ is the maximal closed connected normal unipotent subgroup of G.

Note: We use the fact that the product $H.K = \{hk : h \in H, k \in K\}$ of normal solvable (resp. unipotent) subgroups H and K is also solvable (resp. unipotent).

Def A connected linear algebraic group G is

- semisimple if $R(G) = \{1\}$
- reductive if $R_u(G) = \{1\}$.

Note: $R_u(G) = R(G)_u \subseteq R(G)$, so semisimple $\Rightarrow$ reductive.

$\boxed{\text{Exercise}}$ Suppose G admits a faithful fin. dim'l irreducible representation. Then G is reductive.

(*Hint*: 2.4.15)

Observe:
- Gl_n, SL_n, $SO(n)$ are reductive (use the exercise)
- Tori are semisimple (hence reductive)
- $B_n \subseteq Gl_n$ is neither semisimple nor reductive for $n \geq 2$.

  $(R(B_n) = B_n$ and $R_u(B_n) = U_n)$.
- Let G be any connected linear algebraic group. Then

  $G/R(G)$ is semisimple

  $G/R_u(G)$ is reductive.

$\boxed{\text{Lemma}}$ Let G be a connected and reductive linear algebraic group. Then:

1. $R(G)$ is a central torus.
2. $R(G) \cap [G,G]$ is finite
3. $[G,G]$ is semisimple

Proof. ① Recall the following facts from earlier lectures:

(I) (6.3.5) If H is connected and solvable, and $T \subseteq H$ is a maximal torus, then the multiplication map

$$T \times H_u \longrightarrow H$$

is an isomorphism of varieties.

(II) (3.2.9) If $T \subseteq G$ is a torus, then

$$N_G(T)^\circ = Z_G(T)^\circ$$

[Follows from the "rigidity of tori" theorem.]

Now let G be connected reductive. Then $R(G)_u = R_u(G) = \{1\}$, so by (I), $R(G)$ is a torus. The fact that $R(G)$ is normal in G implies that $N_G(R(G)) = G$ and so $N_G(R(G))^\circ = N_G(R(G))$. Using (II) we deduce that $R(G)$ is central.

② Embed $G \hookrightarrow GL(V)$. Since $R(G)$ is a torus, we can decompose V as

$$V = \bigoplus_{\chi \in X^*(R(G))} V_\chi$$

where $V_\chi = \{ v \in V : t \cdot v = \chi(t) v \quad \forall t \in R(G) \}$.

Note that $V_\chi = 0$ for all but finitely many χ.

Since $R(G)$ is central, each V_χ is a representation of G. Thus we have

$$\rho_\chi : G \longrightarrow GL(V_\chi) \qquad \forall \chi \in X^*(R(G)) \text{ s.t. } V_\chi \neq 0.$$

and an embedding

$$G \hookrightarrow \prod_{\substack{\chi \in X^*(R(G)) \\ V_\chi \neq 0}} GL(V_\chi)$$

$$g \longmapsto (\rho_\chi(g))_\chi$$

For $n \geq 1$, let $\mu_n \subseteq k$ be the finite subgroup of nth roots of unity. For each χ, we identify $\mu_{\dim(V_\chi)} \subseteq GL(V_\chi)$ as a subgroup of scalar matrices

$\boxed{\text{Claim}}$ For each χ with $V_\chi \neq 0$,

$$\rho_\chi([G,G] \cap R(G)) \subseteq \mu_{\dim(V_\chi)}$$

pf of Claim. Let $g \in [G,G] \cap R(G)$. Since $g \in R(G)$, we have

$$\det(\rho_\chi(g)) = \det(\chi(g) \cdot \mathrm{id}_{\dim(V_\chi)}) = \chi(g)^{\dim(V_\chi)}$$

OTOH, since $g \in [G,G]$, we can write $g = xyx^{-1}y^{-1}$ for some $x, y \in G$. Then

$$\det(\rho_\chi(g)) = \det(\rho_\chi(x) \circ \rho_\chi(y) \circ \rho_\chi(x)^{-1} \circ \rho_\chi(y)^{-1}) = 1$$

The claim follows (Claim) $\square$

The claim implies that $[G,G] \cap R(G)$ embeds into $\prod_{\substack{\chi \in X^*(R(G)) \\ V_\chi \neq 0}} \mu_{\dim(V_\chi)}$

and hence is finite. $\square$

③ Exercise

Recall: The center of GL_n is $Z(GL_n) = \{\lambda \cdot id_n : \lambda \in k^\times\} \simeq \mathbb{G}_m$.

Cor The radical of GL_n is its center.

pf. Follows from the lemma (part ①) and the fact that $Z(GL_n)$ is closed connected normal and solvable. $\square$

Thm If G is semisimple, then $G = [G, G]$.

One ingredient of the proof: $SL_2 = [SL_2, SL_2]$ (HW exercise)

More ingredients will be given later on.
(most likely we will not have time for a full proof).

Cor If G is reductive, then $G = R(G) . [G, G]$.

pf. HW exercise $\square$

Connection to representation theory

($k = \overline{k}$, $\mathrm{char}(k) = 0$.
All representations are finite-dimensional unless specified otherwise.)

Def Let G be a linear algebraic group. A representation of G (over k) is <u>semisimple</u> if it decomposes as a direct sum of irreducible representations.

Ex
- Any representation of a finite group is semisimple.
- Any representation of a diagonalizable group is semisimple.
- The representation
$$\begin{aligned} \mathbb{G}_a &\longrightarrow GL_2 \\ x &\longmapsto \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \end{aligned}$$
is <u>not</u> semisimple.

Thm Let G be a semisimple linear algebraic group. Then any finite-dimensional representation of G is semisimple.

<u>Key construction of the proof</u>: Let V be a representation of G. Let $\rho: \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be the corresponding Lie algebra representation.

Define a symmetric bilinear form on $\mathfrak{g}$ as follows:

$$B_V : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow k$$
$$\xi \otimes \xi' \longmapsto \mathrm{Tr}(\rho(\xi) \circ \rho(\xi'))$$

Argue that the fact that $R(G) = \{1\}$ implies that $\ker(B_V) = 0$.
(Need some Lie algebra theory here, e.g. the notion of a solvable ideal)

Then B_V is nondegenerate, so $\exists$ bases $\{e_i\}$ and $\{e_i'\}$ of $\mathfrak{g}$ such that $B_V(e_i, e_j') = \delta_{ij}$

Let
$$C_V = \sum_{i=1}^{n} \rho(e_i) \circ \rho(e_i') \in \mathrm{End}_{\mathfrak{g}}(V)$$

If $W \subseteq V$ is a subrepresentation, one uses the $\mathfrak{g}$-linear endomorphism

$$C_V : V \longrightarrow V$$

to construct a complement to W
(See Ch. 22 of Milne for a full proof).

<u>Note</u>: The operator C_V on V is called a <u>Casimir operator</u>.
If $V = \mathfrak{g}$ is the adjoint representation, then B is called the <u>Killing form</u>.

(For $\mathfrak{g} = \mathfrak{sl}_2$ and V = adjoint representation, we have

$$C_V = Ad(E) \circ Ad(F) + Ad(F) \circ Ad(E) + \frac{1}{2} Ad(H) \circ Ad(H))$$

Cor Let G be a connected reductive group. TFAE:

(a) G is reductive

(b) Any finite-dimensional representation of G is semisimple.

(c) G has a faithful finite dimensional representation that is semisimple.

Pf. (idea)

(a) $\Rightarrow$ (b). Suppose G is reductive. By a previous lemma, we know that $R(G)$ is a central torus. Therefore, given a representation V of G, decompose it as:

$$V = \bigoplus_{\chi \in X^*(R(G))} V_\chi$$

where

$$V_\chi = \{ v \in V : t \cdot v = \chi(t) v \quad \forall t \in R(G) \}.$$

Since $R(G)$ is central, each V_χ is a representation of G. In particular, each V_χ is a representation of the commutator $[G,G]$, which is a semisimple group.

So each V_χ decomposes as a direct sum of irreducible representations of $[G,G]$:

$$V_\chi = \bigoplus_i V_{\chi,i}$$

The fact that $G = [G,G] \cdot R(G)$ implies that each $V_{\chi,i}$ is an irreducible representation of G.

(b) $\Rightarrow$ (c) Immediate from the Embedding Theorem.

(c) $\Rightarrow$ (a) Follows from the exercise on page 2. $\square$

Classification Theorem (first pass)

Can classify reductive groups in terms of combinatorics:

$$\left\{ \begin{array}{c} \text{reductive} \\ \text{groups} \end{array} \right\}_{/\text{iso}} \longleftrightarrow \left\{ \text{root data} \right\}_{/\text{iso}}$$

We need more set-up to define the notion of root data.

Here's a preview: for a reductive group G, its root data is a quadruple $(X, R, X^\vee, R^\vee)$ where:

- $X = X^*(T)$ is the character group of a maximal torus T of G
- $X^\vee = \mathrm{Hom}_{\mathrm{Grps}}(X, \mathbb{Z})$.

- R and $R^\vee$ are root systems in $X \otimes_{\mathbb{Z}} \mathbb{R}$ and $X^\vee \otimes_{\mathbb{Z}} \mathbb{R}$. $\boxed{10}$