Semisimple and Unipotent Elements of SL_2

Let $G = SL_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Mat_{2,2} : ad - bc = 1 \right\}$.

Notation: For $g \in G$, denote the conjugacy class of g by

$$\mathcal{C}_g = \{ x g x^{-1} : x \in G \} \qquad (= \{ x g x^{-1} : x \in GL_2 \}).$$

$\boxed{Q}$. How to describe G_s and G_u?

$\boxed{\text{Tool \# 1}}$ Jordan decomposition

Observation 1A: $G_u = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \amalg \mathcal{C}_{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$

Reason: $g \in G_u \iff$ the Jordan normal form of g is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Observation 1B:

$$g \in G$$

$g \in G_s$:

Jordan normal form of g is $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$, $\lambda\mu = 1$

$\Rightarrow g \sim \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$

$g \notin G_s$:

Jordan normal form of g is $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, $\lambda^2 = 1$

$\Rightarrow g \in \mathcal{C}_{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$ or $g \in \mathcal{C}_{\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}}$

Note: $\lambda = \lambda^{-1}$ iff $\lambda \in \{\pm 1\}$ iff $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \in Z(SL_2)$.

Assume henceforth that $\mathrm{char}(k) \neq 2$, so $1 \neq -1$.

Then:

$$G_s = \left(\coprod_{\lambda \in k^\times \setminus \{\pm 1\}} \mathcal{C}_{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}} \right) \amalg \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \amalg \left\{ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

Set

- $G_s^\circ = \left(\coprod_{\lambda \in k^\times \setminus \{\pm 1\}} \mathcal{C}_{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}} \right)$ These are the "regular semisimple elements".

- $-G_u = \left\{ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\} \amalg \mathcal{C}_{\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}}$

We have:

$$\boxed{SL_2 = G_s^\circ \amalg G_u \amalg -G_u}$$

We'll see that G_s° is open and G_u and $-G_u$ are each closed.

$\boxed{\text{Tool \#2}}$ The trace map

$$\mathrm{trace} : SL_2 \longrightarrow k$$
$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \longmapsto x + w$$

Observation 2A: The trace map is conjugation invariant:

$$\mathrm{trace}(xgx^{-1}) = \mathrm{trace}(g) \qquad \forall g, x \in G$$

Also:

- trace (g) equals the sum of the eigenvalues of g.
- the characteristic polynomial of g is

$$T^2 - (\text{trace}(g))\,T + 1 \qquad \in k[T].$$

which has multiple roots iff $(\text{trace}(g))^2 - 4 = 0$

[discriminant in the quadratic formula]

iff trace $(g) \in \{\pm 2\}$.

<u>Observation 2B</u>:

$$G_s^\circ = \text{trace}^{-1}(k \setminus \{\pm 2\})$$

$$G_u = \text{trace}^{-1}(2)$$

$$-G_u = \text{trace}^{-1}(-2)$$

This follows from previous observations $\left(\lambda = \lambda^{-1} \iff \lambda = \pm 1 \iff \lambda + \lambda^{-1} = \pm 2\right)$

and shows that G_s° is open while G_u and $-G_u$ are each closed.

<u>Observation 2C</u>: For any $t \in k$, there is an isomorphism of algebraic varieties

$$\text{trace}^{-1}(t) = V\left(x^2 + y^2 - z^2 - (t-2)(t+2)\right)$$

(RHS is in k^3 with coordinates x, y, z)

Proof. We compute:

$$\mathrm{trace}^{-1}(t) = \{ (a,b,c,d) \in k^4 : ad - bc = 1 , \quad a + d = t \}$$

$$= \{ (a, b, c) \in k^3 : a(t-a) - bc = 1 \}$$

$$= \{ (x,y,z) \in k^3 : \left(\frac{t+x}{2}\right)\left(\frac{t-x}{2}\right) - \left(\frac{y+z}{2}\right)\left(\frac{y-z}{2}\right) = 1 \}$$

using the change of variables

(Also $\mathrm{char}(k) \neq 2$)

$$x = 2a - t , \quad y = b + c , \quad z = b - c$$

$$= \{ (x,y,z) \in k^3 : t^2 - x^2 - y^2 + z^2 = 4 \}$$

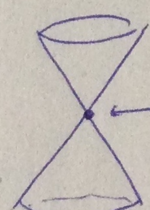
$$= V\left(x^2 + y^2 - z^2 - (t^2 - 4) \right)$$

$$= V\left(x^2 + y^2 - z^2 - (t-2)(t+2) \right) \qquad \square$$

Therefore:

- $t = \pm 2 \rightsquigarrow \mathrm{trace}^{-1}(2) = V(x^2 + y^2 - z^2) \subseteq k^3$

Picture over $\mathbb{R}$: $t = 2$



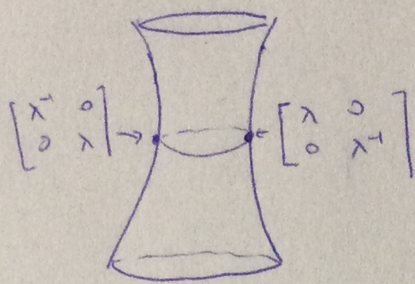
$\leftarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ singularity

(when $t = -2$ the singularity is at $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$)

• $t \notin \{\pm 2\}$ $\rightsquigarrow$ $\mathrm{trace}^{-1}(t) = V(x^2+y^2-z^2-1) \subseteq k^3$

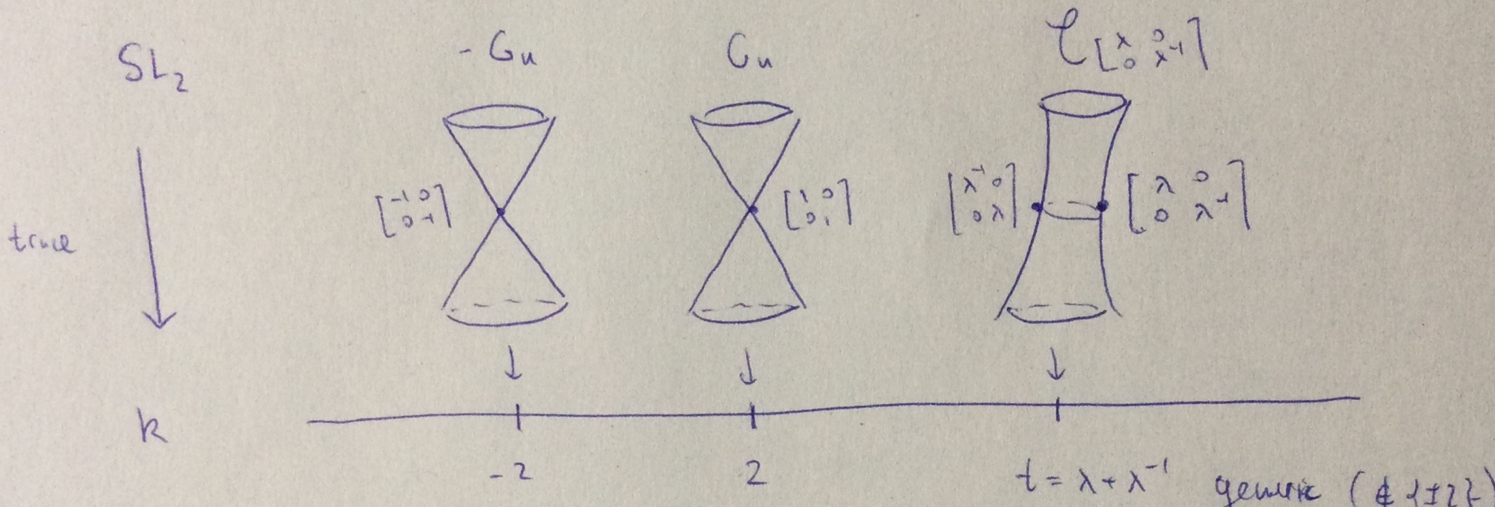
Picture over $\mathbb{R}$

$\lambda + \lambda^{-1} = t$



hyperboloid, no singularities

$\boxed{\text{Summary}}$:



<u>Follow-up</u>: What does this picture look like for PSL_2 and GL_2?