Algebraic Groups

Topic: Algebraic Geometry Review.

The Zariski Topology

Let $k = \bar{k}$ be an algebraically closed field.

Let $V = k^n$ be a vector space of dimension n over k.

Elements: $v = (v_1, \dots, v_n)$ with $v_i \in k$.

Let $S = k[T_1, \dots, T_n]$ be the polynomial algebra in n variables over k.

Evaluation map:

$$\begin{aligned} S \times V &\longrightarrow k \\ f, v &\longmapsto f(v) \end{aligned}$$

"Plug in v_i for T_i and multiply out".

Zariski topology:

- <u>Closed subsets</u>:

$$\mathcal{V}(I) = \{x \in V : f(x) = 0 \quad \forall f \in I\}$$

as I ranges over the ideals of S

$\rightsquigarrow$ These are also known as the <u>algebraic</u> subsets of V

- <u>Basis for open subsets</u>:

$$D(f) = \{x \in V : f(x) \neq 0\}$$

as f ranges over the elements of S

$\rightsquigarrow$ These are also known as the <u>principal</u> open sets of V.

<u>Reminder</u>: An ideal $I \subseteq S$ is a <u>radical ideal</u> if

$$I = \{ f \in S : f^n \in I \text{ for some } n \geq 1 \}$$

$$(\text{i.e. } f^n \in I \Rightarrow f \in I)$$

For any subset $X \subseteq V$, set

$$\mathcal{I}(X) = \{ f \in S : f(x) = 0 \ \forall x \in X \}$$

For any ideal $I \subseteq S$, set

$$\mathcal{V}(I) = \{ x \in V : f(x) = 0 \ \forall f \in I \}$$

Diagram:

$$\begin{tikzcd}
\left\{\begin{matrix}\text{subsets}\\ \text{of } V\end{matrix}\right\} \arrow[dr, leftrightarrow, "\mathcal{I}"] & & \left\{\begin{matrix}\text{ideals of}\\ S\end{matrix}\right\} \arrow[dl, leftrightarrow, "\mathcal{V}"'] \\
\left\{\begin{matrix}\text{algebraic}\\ \text{subsets of } V\end{matrix}\right\} \arrow[u, hook] \arrow[rr, leftrightarrow, "\text{bijection}"'] & & \left\{\begin{matrix}\text{radical ideals}\\ \text{of } S\end{matrix}\right\} \arrow[u, hook]
\end{tikzcd}$$

(order-reversing)

Affine algebras

<u>Idea</u>: Characterize an algebraic set through its algebra of functions.

Let $X \subseteq V = k^n$ be an algebraic set.

Def The <u>affine algebra</u> of X is defined as

$$k[X] = k[T_1, \dots, T_n] / I(X)$$

<u>Claim</u>: X and its Zariski topology are determined by $k[X]$

$\rightsquigarrow$ intrinsic description of X

<u>Idea</u>:

① There is a bijection

$$X \longrightarrow \mathrm{Max}(k[X]) = \left\{ \begin{array}{c} \text{maximal ideals} \\ \text{of } k[X] \end{array} \right\}$$

$$x \longmapsto M_x = \{ f \in k[X] : f(x) = 0 \}$$

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② The closed sets of X are

$$\mathcal{V}_X(I) = \{x \in X : f(x) = 0 \quad \forall f \in I\}$$

as I runs over the ideals of $k[X]$.

<u>Note</u>: A basis for the open sets of X is given by the principal open sets

$$D(f) = \{x \in X : f(x) \neq 0\}$$

as f ranges over the elements of $k[X]$.

<u>Terminology</u>:

$\boxed{\text{Def}}$ An <u>affine k-algebra</u> is a k-algebra A satisfying:

① A is commutative and finitely generated as a k-algebra.

② A is reduced, i.e. has no nonzero nilpotent elements.

<u>Claim:</u>

① The affine algebra of any algebraic set is an affine k-algebra.

(Pick an embedding to see finite generation.

$I(X)$ radical $\Rightarrow$ $k[X]$ reduced.)

② If A is an affine k-algebra, then $\exists$ an algebraic set X s.t. $A \simeq k[X]$.

$$\left\{\begin{matrix}\text{algebraic} \\ \text{sets over } k\end{matrix}\right\} \xrightarrow{\text{bijection}} \left\{\begin{matrix}\text{affine} \\ k\text{-algebras}\end{matrix}\right\}$$

$$X \longmapsto k[X]$$

<u>Later</u>: equivalence of categories.

Reducibility:

- A topological space is <u>reducible</u> if it is the union of two proper closed subsets.

- The (irreducible) components of a top. space are its [redacted] maximal irreducible subsets

- For X an algebraic space [redacted]:

① X has finitely many irreducible components. They are all closed in X, and together cover X

② X is irreducible iff $k[X]$ is an integral domain. (iff $I(X)$ is a prime ideal)

③ X Irreducible $\Rightarrow$ X connected (but not the other way around)

↳ <u>Ex.</u> $X = \{(x,y) \in k^2 : xy = 0\}$.

<u>Reminder</u>: localization of rings:

Let R be a commutative ring.

Let $S \subseteq R$ be a multiplicative set

$(a \in S$ and $b \in S \Rightarrow ab \in S)$ (also $1 \in S$)

The localization of R at S is defined as

$R \times S / \sim$ where $(r_1, s_1) \sim (r_2, s_2)$

if $t(r_1 s_2 - r_2 s_1) = 0$

for some $t \in S$

$\rightsquigarrow$ think of $(r,s) \longleftrightarrow$ "$\frac{r}{s}$"

<u>Notation</u>:

① For $r \in R \setminus \{0\}$, set

$$R[r^{-1}] := R[\{1, r, r^2, r^3, \dots\}^{-1}]$$

② For $P \subseteq R$ a prime ideal,

$$R_P := R[(R \setminus P)^{-1}]$$

Recall:

topology / geometry		commutative algebra
$\left\{ \begin{array}{c} \text{algebraic sets} \\ \text{over } k \end{array} \right\}$	$\xrightarrow{\text{bijection}}$	$\left\{ \begin{array}{c} \text{affine} \\ k\text{-algebras} \end{array} \right\}$
X	$\longmapsto$	$k[X]$

Two motivating questions:

$\boxed{Q1}$ How does this bijection interact with morphisms?

Note: If $\gamma : k[Y] \longrightarrow k[X]$ is a k-algebra homomorphism, then there is an induced continuous map

$$X \longrightarrow Y$$

(Key observations: ① $X \hookrightarrow \mathrm{Max}(k[X])$

② If $M \subseteq k[X]$ is a maximal ideal of $k[X]$, then $\gamma^{-1}(M)$ is a maximal ideal of $k[Y]$.)

However: Not every continuous map $X \to Y$ arises in this way. We want to characterize those that do.

Q2 Just as manifolds are <u>locally</u> Euclidean, how can we consider sets that are <u>locally</u> algebraic?

The key to addressing both of these questions is to use the language of sheaves.

<u>Next</u>: Given an algebraic set X over k, we will define the sheaf of k-valued regular functions on X.

<u>Idea</u>: The algebra $k[X]$ comprises the <u>global</u> functions on X, but open subsets of X have more functions.

Ex
- $X = k^2 \rightsquigarrow k[X] = k[z, w]$
- $U = D(z) \underset{\text{open}}{\subseteq} X$, i.e. $U = \{(z,w) \in X : z \neq 0\}$
- Functions on U are given by

$$k[z^{\pm 1}, w] = \underbrace{k[X][z^{-1}]}_{\text{localization}}$$

- Notation:

$$\mathcal{O}_X(U) = k[X][z^{-1}]$$

Def Let X be an algebraic set and $f \in k[X]$ nonzero.

Set

$$\mathcal{O}_X(D(f)) := \underbrace{k[X][f^{-1}]}_{\text{localization of } k[X] \text{ at } f.}$$

$D(f) = \{x \in X : f(x) \neq 0\}$

<u>Claim</u> (exercise / reminder / reading)

① The definition above naturally extends to a well-defined assignment

$$U \longmapsto \mathcal{O}_X(U)$$

for any open $U \subseteq X$. We call $\mathcal{O}_X(U)$ the k-algebra of regular functions on U.

② If $W \subseteq V \subseteq U$ are open subsets, then

$$\begin{tikzcd}
\mathcal{O}_X(U) \arrow[r, "\text{restrict from } U \text{ to } V"] \arrow[dr, bend right, "\text{restrict from } U \text{ to } W"'] & \mathcal{O}_X(V) \arrow[d, "\text{restrict from } V \text{ to } W"] \\
& \mathcal{O}_X(W)
\end{tikzcd}$$

③ Let $U \subseteq X$ be open and $\mathcal{U} = \bigcup_{\alpha \in A} \mathcal{U}_\alpha$ be an open cover of U. Then the following sequence is exact:

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\psi_1} \bigoplus_{\alpha \in A} \mathcal{F}(\mathcal{U}_\alpha) \xrightarrow{\psi_2} \bigoplus_{\alpha, \beta \in A} \mathcal{F}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

where $\psi_1(f)_\alpha = f|_{\mathcal{U}_\alpha}$

$$\psi_2\left((f_\gamma)_{\gamma \in A}\right)_{\alpha, \beta} = f_\alpha|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} - f_\beta|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta}$$

$\Rightarrow \mathcal{O}_X$ is a <u>sheaf</u> (of k-algebras) on X.

The pair $(X, \mathcal{O}_X)$ is an example of a <u>ringed space</u>.
Ringed spaces that arise in this way from algebraic sets with regular functions are called

<u>affine algebraic varieties over k</u> or <u>affine k-varieties</u>

One can show (Theorem 1.4.5) that

$$\mathcal{O}_X(X) \simeq k[X]$$

Morphisms :

Note : Let $\gamma : k[Y] \longrightarrow k[X]$ be a k-algebra homomorphism, and $\varphi : X \longrightarrow Y$ the induced map.

Then, for all $f \in k[Y]$, we have :

- $\varphi^{-1}(D(f)) = D(\gamma(f))$

- $$\begin{tikzcd} k[Y] \arrow[r, "\gamma"] \arrow[d, "loc."'] \arrow[dr, phantom, "\circlearrowright"] & k[X] \arrow[d, "loc."] \\ k[Y][f^{-1}] \arrow[r, dashed] & k[X][\gamma(f)^{-1}] \end{tikzcd}$$

Therefore :

$$\begin{tikzcd} \mathcal{O}_Y(Y) \arrow[r, "\gamma"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{O}_X(X) \arrow[d] \\ \mathcal{O}_Y(D(f)) \arrow[r, dashed] & \mathcal{O}_X(\varphi^{-1}(D(f))) \end{tikzcd}$$

More generally, for any $V \subseteq Y$ open, we have

$$\begin{tikzcd} \mathcal{O}_Y(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{O}_X(X) \arrow[d] \\ \mathcal{O}_Y(V) \arrow[r, dashed] & \mathcal{O}_X(\varphi^{-1}(V)) \end{tikzcd} \qquad (*)$$

Def A morphism of ringed spaces $(X, \mathcal{O}_X) \xrightarrow{\varphi} (Y, \mathcal{O}_Y)$ is a continuous map $\varphi : X \longrightarrow Y$ such that

$$\begin{tikzcd}
\mathcal{O}_Y(Y) \arrow[r, "\varphi^*"] \arrow[d, "res"'] & \mathcal{O}_X(X) \arrow[d, "res"] \\
\mathcal{O}_Y(V) \arrow[r] & \mathcal{O}_X(\varphi^{-1}(V))
\end{tikzcd}$$

commutes for every $V \subseteq Y$ open.

<u>Upshot</u>

$$\mathrm{Mor}\big((X, \mathcal{O}_X), (Y, \mathcal{O}_X)\big) \xrightarrow{bij} \left\{ \begin{array}{c} k\text{-algebra homomorphisms} \\ k[Y] \longrightarrow k[X] \end{array} \right\}$$

$$\varphi \longmapsto \varphi^* : k[Y] \longrightarrow k[X]$$

$$\big[x \mapsto \mu_x \mapsto \gamma^{-1}(\mu_x) \mapsto y\big] \longleftarrow\!\shortmid \gamma$$

For those familiar with category theory:

$$\left\{\begin{array}{l}\text{category of}\\ \text{affine } k\text{-varieties.}\end{array}\right\} \xleftrightarrow[\text{of categories}]{\text{equivalence}} \left\{\begin{array}{l}\text{Category of}\\ \text{affine } k\text{-algebras}\end{array}\right\}$$

$$X \longmapsto \mathcal{O}_X(X) = k[X]$$

$$\mathrm{Spec}(A) \longleftarrow\!\shortmid A$$

<u>Final notes on ringed spaces:</u>

For $x \in X$, the <u>ring of regular functions at x</u> is

$$\mathcal{O}_{X,x} = \lim_{U \ni x} \mathcal{O}_X(U)$$

This is a local ring, and can also be described as

$$\mathcal{O}_{X,x} = k[X]\left[\left(k[X] \setminus M_x\right)^{-1}\right] = k[X]_{M_x}$$

maximal ideal corresponding to x

$\rightarrow$ Exercise 1.4.4.

Products

Lemma 1.5.2 Let A and B be k-algebras of finite type.

If A and B are reduced, then so is $A \otimes_k B$.

If A and B are integral domains, then so is $A \otimes_k B$.

<u>Note</u>: $A \otimes_k B$ is a k-algebra with operation given on simple tensors by:

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$$

and extends easily to all tensors.

<u>Upshot</u>: Given affine algebraic varieties X and Y, we can construct a new variety $Z_{X,Y}$ corresponding to $k[X] \otimes_k k[Y]$.

Induced maps:

- $k[X] \longrightarrow (k[X] \otimes_k k[Y]) = k[Z_{XY}]$ induces $Z_{X,Y} \xrightarrow{p} X$

  $f \longmapsto f \otimes 1$

- $k[Y] \longrightarrow k[X] \otimes_k k[Y] = k[Z_{X \times Y}]$ induces $Z_{X,Y} \xrightarrow{q} Y$

  $g \longmapsto 1 \otimes g$

Claim The variety $Z_{X,Y}$ satisfies the following <u>universal property</u>:

For any affine variety Z' and equipped with maps $Z' \xrightarrow{p'} X$ and $Z' \xrightarrow{q'} Y$, there exists a unique morphism

$$Z' \xrightarrow{r} Z$$

such that the following diagram commutes:

$$\begin{tikzcd}
Z' \arrow[dr, dashed, "r"] \arrow[drr, bend left, "q'"] \arrow[ddr, bend right, "p'"'] & & \\
& Z_{X,Y} \arrow[r, "q"'] \arrow[d, "p"] & Y \\
& X &
\end{tikzcd}$$

Thus, $Z_{X,Y}$ is the <u>product</u> of X and Y (in the category of affine k-varieties)

and ~~we~~ denote it as $Z_{X,Y} = X \times Y$. So

$$k[X \times Y] = k[X] \otimes_k k[Y]$$

$\boxed{\text{Exercise 1.6.5 (1) and (2)}}$

Prevarieties and Varieties

<u>Recall</u>. A <u>manifold</u> is a Hausdorff topological space that is locally homeomorphic to Euclidean space $\mathbb{R}^n$ for some n.

Def A <u>variety</u> over k is a quasi-compact ringed space $(X, \mathcal{O}_X)$ satisfying the following conditions:

① [Prevariety] The ringed space $(X, \mathcal{O}_X)$ is locally an affine k-variety:

$\forall x \in X$, $\exists\, U \ni x$ open nbhd s.t. $(U, \mathcal{O}_X|_U)$ is isomorphic to an affine k-variety.

② [Separation axiom] The diagonal

$$\Delta_X = \{(x,x) : x \in X\} \text{ is closed in } X \times X.$$

<u>Notes</u>:

- quasi-compact means every open cover has a finite refinement (usually "compact" is reserved for Hausdorff spaces)
- The separation axiom is an analogue to the Hausdorff condition. However, a variety that is Hausdorff is just a finite set of points (exercise).

Prop Let X be a variety and Y be a prevariety.

① If $\phi : Y \longrightarrow X$ is a morphism, then

$$\Gamma_\phi = \{ (y, \phi(y)) : y \in Y \}$$ is closed in $Y \times X$

② If $\phi, \psi : Y \longrightarrow X$ are morphisms that coincide on a dense set, then $\phi = \psi$.

Criterion Let X be a prevariety and $X = \bigcup_{i=1}^{m} U_i$ an open covering by affine open sets. Then X is a variety iff the following holds:

- $\forall\, i, j$, the intersection $U_i \cap U_j$ is an affine open set, and the images of

$$\mathcal{O}_X(U_i) \longrightarrow \mathcal{O}_X(U_i \cap U_j)$$

$$\mathcal{O}_X(U_j) \longrightarrow \mathcal{O}_X(U_i \cap U_j)$$

(together) generate $\mathcal{O}_X(U_i \cap U_j)$.

Exercises 1.6.13 (1), (2).

Projective varieties

↳ most important examples of non-affine varieties.

Recall:

Projective space of dimension n over k is:

$$\mathbb{P}^n = \{\text{lines in } k^{n+1}\} = \left(k^{n+1} \setminus \{0\}\right) / k^\times$$

Goal: Define the structure of an algebraic variety on $\mathbb{P}^n$.

① **Notation**:

- $k^{n+1} \setminus \{0\} \twoheadrightarrow \mathbb{P}^n$

$$x = (x_0, \dots, x_n) \longmapsto [x] = [x_0 : \dots : x_n] = x^*$$

- For $i = 0, \dots, n$, set:

$$\mathcal{U}_i = \{[x_0 : \dots : x_n] \in \mathbb{P}^n : x_i \neq 0\}$$

$$\phi_i : \mathcal{U}_i \longrightarrow \mathbb{A}^n$$

$$[x_0 : \dots : x_n] \longmapsto (x_i^{-1} x_0, x_i^{-1} x_1, \dots, x_i^{-1} x_{i-1}, x_i^{-1} x_{i+1}, \dots, x_i^{-1} x_n)$$

② <u>Claim</u>: ϕ_i is well-defined and bijective.

$\rightsquigarrow$ transport the structure of an affine alg. variety on $\mathbb{A}^n$ to each $\mathcal{U}_i$.

$\rightsquigarrow$ Identifying $k[\mathbb{A}^n]$ with $k[T_1, \dots, T_n]$, we have that

$$\phi_i(\mathcal{U}_i \cap \mathcal{U}_j) = \begin{cases} D(T_{j+1}) & \text{if } j < i \\ k[\mathbb{A}^n] & \text{if } j = i \\ D(T_j) & \text{if } j > i \end{cases}$$

$\Rightarrow$ $\mathcal{U}_i \cap \mathcal{U}_j$ is affine and open in $\mathcal{U}_i$.

③ Using the fact that the $\mathcal{U}_i$'s cover $\mathbb{P}^n$, define a topology on $\mathbb{P}^n$ and a sheaf of regular functions $\mathcal{O}_{\mathbb{P}^n}$ [details omitted]

$\rightsquigarrow$ $\mathbb{P}^n$ is a prevariety

④ Use the criterion 1.6.12 (together with the observations from step ②) to show that $\mathbb{P}^n$ is a variety.

$\boxed{\text{Exercises 1.7.2 (1) and (2)}}$

Let $S = k[T_0, \dots, T_n]$.

The <u>degree</u> of a monomial $T_0^{m_0} T_1^{m_1} \cdots T_n^{m_n} \in S$

is $d = \sum_{i=1}^{n} m_i$.

We have:

$$S = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} S_d$$

where

$$S_d = \mathrm{Span}_k \left\{ \begin{array}{c} \text{all monomials} \\ \text{of degree } d \end{array} \right\}$$

$\boxed{\text{Ex}}$ $n = 2$:

$S_0 = k$

$S_1 = k\{T_0, T_1\}$

$S_2 = k\{T_0^2, T_0T_1, T_1^2\}$

$S_3 = k\{T_0^3, T_0^2T_1, T_0T_1^2, T_1^3\}$

$\vdots$

⌈ <u>Aside</u>: S is a graded algebra:

mult: $S_{d_1} \times S_{d_2} \longrightarrow S_{d_1+d_2}$ ⌋

Def ① $f \in S$ is homogeneous if $f \in S_d$ for some d

② An ideal $I \subseteq S$ is homogeneous if it is generated by homogeneous elements.

③ If I is a (proper) homogeneous ideal, set

$$V^*(I) = \{ [x] \in \mathbb{P}^n : x \in V_{k^{n+1}}(I) \}$$

<u>Note</u>: If I is homogeneous and $x \in V(I) \subseteq k^{n+1}$, then

$$\lambda x \in V(I) \quad \forall \lambda \in k^\times.$$

<u>Diagram</u>:

$$\begin{tikzcd} V(I) \arrow[r, hook, "\text{closed}"] \arrow[d, two heads] & k^{n+1} \setminus \{0\} \arrow[d] \\ V^*(I) \arrow[r, hook] & \mathbb{P}^n \end{tikzcd}$$

($I \subseteq S$ proper homogeneous ideal)

Prop

$$\left\{ \begin{array}{c} \text{homogeneous} \\ \text{ideals in } S \\ \shortparallel \\ k[T_0, \dots, T_n] \end{array} \right\} \twoheadrightarrow \left\{ \begin{array}{c} \cancel{\text{homogen}} \text{ closed} \\ \text{subsets of } \mathbb{P}^n \end{array} \right\}$$

$$I \longmapsto V^*(I)$$

Def ① A <u>projective variety</u> is a closed subvariety of some $\mathbb{P}^n$.

② A <u>quasi-projective variety</u> is an open subvariety of a projective variety.

Dimension

- Suppose X is affine and irreducible. Then $k[X]$ is an integral domain, and has a field of fractions $k(X)$.

  Set:

  $$\dim(X) := \text{transcendence degree of } k(X) \text{ over } k.$$

- Suppose X is irreducible but not necessarily affine.

Claim ① $k(U)$ is an integral domain for any affine open $U \subseteq X$.

② If U, V are affine open sets in X, then the field of fractions $k(U)$ and $k(V)$ of $k[U]$ and $k[V]$, respectively, are canonically identified.

$\rightsquigarrow$ well-defined field of fractions $k(X)$.

$\dim(X) :=$ transcendence degree of $k(X)$ over k.

- Finally, suppose X is an arbitrary variety. Then X has irreducible components $X_1, X_2, \dots, X_n$.

and $$\dim(X) := \max_i \left(\dim(X_i) \right)$$

Aside: Suppose X is affine. Then $k[X]$ is finitely generated over k, say by elements $t_1, \dots, t_n$. Then $\dim X$ is the maximal number of these that form a linearly independent set. ⊥

Prop ① X irreducible and $Y \subseteq X$ proper, irreducible, and closed. Then $$\dim Y < \dim X$$

② If X and Y are irreducible, then $$\dim(X \times Y) = \dim X + \dim Y$$

Some results on morphisms

Let $\phi : X \to Y$ be a map of affine varieties and $\phi^* : k[Y] \to k[X]$ the associated algebra homomorphism. Then:

① ϕ^* surjective $\Rightarrow$ $\mathrm{Image}(\phi) \subseteq Y$ is closed

② ϕ^* injective $\Rightarrow$ $\mathrm{Image}(\phi) \subseteq Y$ is dense

③ X irreducible $\Rightarrow$
- $\overline{\phi X} \subseteq Y$ irred.
- $\dim \overline{\phi X} \leq \dim X$.

F-structures + F-varieties

Let $F \subseteq k$ be a subfield (e.g. $\mathbb{R} \subseteq \mathbb{C}$).

Let $A = k[X]$ be an affine algebra over k.

Def An <u>F-structure</u> on X is an F-subalgebra A_0 of A satisfying:

① A_0 is of finite type over F

② The multiplication map $k \otimes_F A_0 \longrightarrow k[X]$ is an isomorphism.

The set of F-rational points of X is

$$X(F) = \mathrm{Hom}_{F\text{-alg}}(A_0, F)$$

Ex (§1.3.9) Let $F = \mathbb{R} \subseteq \mathbb{C} = k$. Let $k[X] = \mathbb{C}[T,U] / (T^2 + U^2 - 1)$

Consider:

$$A_0^{(1)} = \mathbb{R}\langle a, b \rangle \subseteq k[X]$$

$$A_0^{(2)} = \mathbb{R}\langle ia, ib \rangle \subseteq k[X]$$

Claim $A_0^{(1)}$ and $A_0^{(2)}$ are two different $\mathbb{R}$-structures on X, with

$$\mathrm{Hom}_{\mathbb{R}\text{-alg}}(A_0^{(1)}, \mathbb{R}) \simeq S^1 \quad , \quad \mathrm{Hom}_{\mathbb{R}\text{-alg}}(A_0^{(2)}, \mathbb{R}) \simeq \emptyset .$$

- Can extend F-structure notions to affine F-varieties (§1.4.9) and to F-varieties (§1.6.14–17).