Clarifications + Examples

Recall: We had an isomorphism of Lie algebras:

$$\begin{array}{ccc} T_1 GL_n & \xrightarrow{\sim} & Mat_{n,n} = gl_n \\ \Psi & & \Psi \\ \xi_{ij} = \eta \circ \frac{\partial}{\partial T_{ij}} & \longmapsto & E_{ij} \end{array} \qquad \begin{array}{l} i = 1, \dots, n \\ j = 1, \dots, n \end{array}$$

$\boxed{\text{Claim}}$ Under this isomorphism, the Lie subalgebra $T_1 SL_n \subseteq T_1 GL_n$ is identified with the Lie subalgebra

$$sl_n = \{ \text{trace} = 0 \} \subseteq gl_n.$$

pf (Idea). Let $\det \in k[T_{ij} : \begin{smallmatrix} i=1 \dots n \\ j=1 \dots n \end{smallmatrix}]$ be the determinant.

Thus, $k[GL_n] = k[T_{ij}][\det^{-1}]$

and $k[SL_n] = k[GL_n] / (\det - 1) = k[T_{ij}] / (\det - 1)$.

One argues by induction on n that

$$\eta \circ \frac{\partial}{\partial T_{ij}} (\det) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

η ↳ evaluation at the $n \times n$ identity matrix

(*Hint*: Use 1^{st} row expansion:

$$\det_n\left(T_{ij} : \begin{smallmatrix} i=1,\dots,n \\ j=1,\dots,n \end{smallmatrix}\right) = \sum_{l=1}^{n} (-1)^{l+1} T_{1l} \det_{n-1}\left(\left\{T_{uv} : \begin{smallmatrix} u \neq 1 \\ v \neq l \end{smallmatrix}\right\}\right)\Bigg)$$

Thus, given $\xi \in T_1 GL_n = \mathrm{Der}(k[GL_n], k_1)$, we write $\xi = \sum_{ij} a_{ij}\, \xi_{ij}$ as a linear combination of the ξ_{ij} and observe that ξ factors through $k[SL_n]$ iff

$$\xi(\det - 1) = 0 \quad \text{iff} \quad \sum_{ij} a_{ij}\, \xi_{ij}(\det - 1) = 0$$

$$\text{iff} \quad \sum_{ij} a_{ij}\left(\eta \circ \frac{\partial}{\partial T_{ij}}(\det - 1)\right) = 0$$

iff $\sum_i a_{ii} = 0$ iff ξ corresponds to a traceless matrix

$\square$

$\boxed{\text{Ex}}$ ① $\mathfrak{sl}_2 = \mathrm{Span}_k \left\{ E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

Relations:

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H.$$

② $gl_2 = \mathrm{Span}_{\mathbb{R}} \left\{ E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, H_+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, H_- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Relations:

$$[H_\pm, E] = \pm E \quad , \quad [H_\pm, F] = \mp F$$

$$[H_+, H_-] = 0 \qquad [E, F] = H_+ - H_-$$

③ <u>Recall</u>: $T_1 SO(3) = \{ M \in \mathrm{Mat}_{3,3} : M + M^T = 0 \}$

This Lie algebra is denoted $\mathfrak{so}_3$. We have

$$\mathfrak{so}_3 = \mathrm{Span}_{\mathbb{R}} \left\{ A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$$

<u>Relations</u>:

$$[B, A] = C \quad , \quad [A, C] = B, \quad [C, B] = A$$

<u>Note</u>: There is an isomorphism $sl_2 \longrightarrow \mathfrak{so}_3$

$$E \longmapsto B + iC$$

$$F \longmapsto -B + iC$$

$$H \longmapsto 2iA$$