Commutative Algebraic Groups

Lemma Let $S \subseteq Mat_{n,n}$ be a set of pairwise commuting matrices. Then:

① $\exists x \in GL_n$ such that xSx^{-1} consists of upper triangular matrices.

② If all matrices in S are semisimple, then $\exists x \in GL_n$ such that xSx^{-1} consists of diagonal matrices.

proof (sketch of ①) By induction on n. Base step: $n=1$ ✓

Induction step:

Case I: $S \subseteq \{\lambda \cdot id_n : \lambda \in k\}$

Then all matrices in S are already diagonal.

Case II: $S \nsubseteq \{\lambda \cdot id_n : \lambda \in k\}$

Then $\exists s \in S$ and $\lambda \in k$ such that

$$0 < \dim\left(\{v \in k^n : (s-\lambda)v = 0\}\right) < n$$

Let $W = \{v \in V : (s-\lambda)v = 0\} \subseteq V$. Pick a basis of W and extend it to V.

In this basis,

$$S \subseteq \left\{ \begin{bmatrix} \overbrace{\ast}^{\dim(W)} & \ast \\ 0 & \ast \end{bmatrix} \right\}$$

since W is stable under any $t \in S$. Apply induction to the endomorphisms of W and V/W induced by S.

$$\rightsquigarrow \quad xSx^{-1} \subseteq \left\{ \left[\begin{array}{c|c} \begin{matrix} \ast & & \ast \\ & \ddots & \\ 0 & & \ast \end{matrix} & \ast \\ \hline 0 & \begin{matrix} \ast & & \ast \\ & \ddots & \\ 0 & & \ast \end{matrix} \end{array} \right] \right\}$$

$\square$

$\boxed{\text{Def}}$ Let G be a group. The commutator group of G is

$$[G, G] := \langle xyx^{-1}y^{-1} \; : \; x, y \in G \rangle$$

Note that $[G,G] \trianglelefteq G$ and $G/[G,G]$ is commutative.

Also, G is commutative iff $[G,G] = \{e\}$.

$\boxed{\text{Lemma}}$ Let $U_n = \left\{ \begin{bmatrix} 1 & & \ast \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \right\} \subseteq GL_n$. Set $U_n^{[0]} = U_n$

and $U_n^{[k]} = [U_n^{[k-1]}, U_n^{[k-1]}]$ for $k \geq 1$. Then

$$U_n^{[n]} = \{1\}$$

<u>proof</u> (sketch) Argue that

$$U_n^{[k]} = \left\{ \begin{bmatrix} 1 & \overbrace{0 & \cdots & 0}^{k} & * & \cdots & * \\ & 1 & 0 & \cdots & 0 & * & \vdots \\ & & \ddots & & & \ddots & * \\ & & & & & & 0 \\ & 0 & & & \ddots & & \vdots \\ & & & & & & 0 \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{bmatrix} \right.$$

(the last column ends in k zeros above the final 1)

$\square$

$\boxed{\text{Theorem}}$ Let G be a commutative linear algebraic group.

① The sets G_s and G_u of semisimple and unipotent elements are closed subgroups of G.

② The product map

$$\mu : G_s \times G_u \longrightarrow G$$

is an isomorphism of algebraic groups.

<u>proof</u>. The set G_u is closed in any linear algebraic group (exercise 2.4.10 (2)). To show that G_u is a subgroup, let $g, h \in G_u$ and pick $m \gg 0$ so that $(g-1)^m = 0 = (h-1)^m$. Then use binomial expansion (together with the fact that g and h commute) to show that $(gh-1)^{2m} = 0$

[<u>hint</u>: write $gh - 1 = (g-1)h + h - 1$]

Next we turn our attention to G_s. We may assume that G is a closed subgroup of GL_n, for some n.

If $g, h \in G_s$, then they are commuting diagonalizable (matrices) elements, hence simultaneously diagonalizable. Hence $gh \in G_s$.

Thus G_s is a subgroup, and it remains to show that G_s is closed. Since all elements in G_s are simultaneously diagonalizable (see lemma from the start of the lecture), $\exists$ basis $\{v_i\}$ of V and group homomorphisms $\phi_i : G_s \longrightarrow k^\times = \mathbb{G}_m$ s.t.

$$g \cdot v_i = \phi_i(g)\, v_i \qquad \forall i, \ \forall g \in G_s.$$

In the basis $\{v_i\}$, we have that

$$G_s = G \cap D_n.$$

Indeed, $G_s \subseteq G \cap D_n$ by construction. On the other hand, $G \cap D_n \subseteq G_s$ since D_n consists of semisimple elements (see 2.4.4).

Hence G_s is closed in G. ① $\square$

Claim ② follows since $G_s \cap G_u = \{1\}$. ② $\square$

$\boxed{\text{Proposition}}$ Let G be a connected linear algebraic group of dimension 1. Then:

① G is commutative

② Either $G = G_s$ or $G = G_u$.

"All students are enjoined in the strongest possible terms to eschew proofs by contradiction!"

– Royden, Real Analysis, 3rd edition.

<u>proof</u> ① Suppose that G is not commutative and $\dim G \leq 1$.

We show that $\dim G = 0$.

Since G is not commutative, $\exists g \in G$ whose conjugacy class

$$\mathcal{C}_g = \{ x g x^{-1} : x \in G \}$$

is not equal to $\{e\}$. Since $\mathcal{C}_g$ is closed and irreducible, and $\mathcal{C}_g \neq \{e\}$, dimension considerations imply that $\overline{\mathcal{C}_g} = G$.

Then $G \setminus \mathcal{C}_g$ is finite, say

$$G = \mathcal{C}_g \amalg \{g_1, \dots, g_m\}$$

Embed G into GL_n for some n, and let

$$\chi : G \hookrightarrow GL_n \longrightarrow k[T]_{\text{degree} \leq n} = \operatorname{Span}_k \{1, T, \dots, T^n\}$$

$$y \longmapsto \det(T - y)$$

be the map taking $y \in G$ to its characteristic polynomial.

Then

$$\operatorname{Image}(\chi) = \{\det(T - y) : y \in G\}$$

$$= \{\det(T - y) : y \in \mathcal{C}_g\} \amalg \{\det(T - g_i)\}_{i=1}^{m}$$

$$= \{\det(T - g)\} \amalg \{\det(T - g_1)\} \amalg \cdots \amalg \{\det(T - g_m)\}$$

Since the image of χ is finite, and since G is connected, we have that

$$\det(T - y) = \det(T - 1) = (T - 1)^n \quad \forall y \in G.$$

Thus, $G = G_u$. Changing the basis if necessary, we have that $G \subseteq U_n$.

Recall that $[\cdots[U_n, U_n]\cdots] = 0$ eventually.

As a consequence, $[G,G]$ must be proper in G. Since it is also connected, we have that $[G,G] = \{e\}$.

Then:

$$G = \{g_1, \dots, g_m\} \sqcup C_g = \{g_i\}_{i=1}^m \sqcup g(g^{-1}C_g)$$

$$= \{g_i\} \sqcup g\{g^{-1}xgx^{-1} : x \in G\} \subseteq \{g_i\} \cup g[G,G]$$

$$= \{g_i\} \sqcup g \cdot \{e\} = \{g, g_1, g_2, \dots, g_m\}$$

So G is finite $\Rightarrow \dim G = 0$. ① $\square$

Claim ② is clear from the previous theorem. $\square$

$\boxed{\text{Theorem}}$ Let G be a connected linear algebraic group of dimension 1. Then G is isomorphic to either $\mathbb{G}_a$ or $\mathbb{G}_m$.

We know from above that G is either semisimple or unipotent. In the next lecture we will show that if G is semisimple, then it is isomorphic to the multiplicative group. We will not cover the proof that if G is unipotent, then it is isomorphic to the additive group. See sections 3.3 and 3.4 in Springer.