§5.5 Quotients

Let G be a linear algebraic group and H a closed subgroup.

Def A quotient of G by H is a pair (X, x) consisting of a homogeneous space X for G, together with a point $x \in X$, such that the following universal property holds:

$\forall (Y, y)$ where
- Y is a homogeneous G-space
- $y \in Y$
- $H \subseteq \mathrm{Stab}_G(y)$

$\exists!\ f: X \to Y$ with $f(x) = y$ (f is G-equivariant)

"(X,x) is initial in the category of pointed homogeneous G-spaces with H contained in the stabilizer of the point."

Thm Quotients by closed subgroups exist and are unique up to unique isomorphism. We write G/H for this quotient.

Idea of the construction:

① We have that $k[H] = k[G]/I$ for an ideal I.

Fix $V \subseteq k[G]$ a fin. dim'l subspace containing a set of generators of $k[G]$ as an algebra, and ρ_g-stable $\forall g \in G$.

Let $W = V \cap I$. Argue that

$$H = \{ x \in G : \rho_x(W) = W \}.$$

(2) Let $d = \dim W$. Consider the exterior powers:

$$\Lambda^d V \supseteq \Lambda^d W =: L$$

1-dimensional.

We have a representation $\phi = \Lambda^d \rho_* : G \longrightarrow GL(\Lambda^d V)$ of G on $\Lambda^d V$.

$$[\ \phi(g)(v_1 \wedge \cdots \wedge v_d) = \rho_g(v_1) \wedge \cdots \wedge \rho_g(v_d)\ ]$$

Argue that

$$H = \{x \in G : \phi(x)(L) = L\}.$$

(3) Consider the projective space of $\Lambda^d V$:

$$\mathbb{P}(\Lambda^d V) = \{\text{1-dimensional subspaces of } \Lambda^d V\}$$

The line L in $\Lambda^d V$ defines a point x in $\mathbb{P}(\Lambda^d V)$. Note that G acts on $\mathbb{P}(\Lambda^d V)$. Let X be the orbit of x.

Argue that $\mathrm{Stab}_G(x) = H$.

Facts:

- G/H is a quasi-projective variety of dimension $\dim G - \dim H$.
- If H is normal in G, then G/H is affine, and has the natural structure of a linear algebraic group.

[Show that $G/H \hookrightarrow GL(W)$, where $W = V \cap I$ is from above]

§6.1 Complete varieties

Def An algebraic variety X is <u>complete</u> if for any algebraic variety Y, the projection

$$X \times Y \longrightarrow Y$$

is a closed morphism [i.e. the image of any closed set is closed].

<u>Remark</u>: Completeness in the category of algebraic varieties is an analogue to compactness in the category of locally compact top. spaces.

Ex $\mathbb{A}^1$ is not complete. For example,

$$\begin{array}{ccc} X = \{(x,y) : xy = 1\} & \xhookrightarrow{\text{closed}} & \mathbb{A}^1 \times \mathbb{A}^1 \\ & & \downarrow \text{proj}_2 \\ & & \mathbb{A}^1 \end{array}$$

The image of X under the projection onto the 2nd factor is $\mathbb{A}^1 \setminus 0$, which is not closed.

Similarly: $\mathbb{A}^n$ is not complete $\forall n \geq 1$.

<u>Fact</u>: Any projective variety is complete

$\rightsquigarrow$ Enough to show that $\mathbb{P}^n$ is complete (see 6.1.3 for a full proof)

This is analogous to $\mathbb{P}^n$ being compact as a manifold.

$\boxed{\text{Prop}}$ Let X be a complete variety.

(1) Any closed subvariety of X is complete.

(2) If $f: X \longrightarrow Y$ is a morphism to any variety Y, then the image $f(X)$ is complete, and closed in Y.

(3) If X is affine, then X is finite.

pf. Claim (1) follows from definitions. For (2), think about

$$\begin{tikzcd}
X \arrow[r, hook, "x \mapsto (x, f(x))"] \arrow[dr, "f"'] & X \times Y \arrow[d, "\mathrm{proj}_2"] \\
& Y
\end{tikzcd}$$

The image of $x \mapsto (x, f(x))$ is the graph of f, which is closed in $X \times Y$ and isomorphic to X (hence complete). The image $f(X)$ is the same as the image of the graph $\Gamma \subseteq X \times Y$ under proj_2. Hence it is closed.
To see that $f(X)$ is complete, let Z be any variety and $V \subseteq X \times Z$ closed. Then the image of V under $\mathrm{proj}_2 : X \times Z \to Z$ is the image of $(f \times 1)^{-1}(V)$ under $\mathrm{proj}_2 : X \times Z \longrightarrow Z$, where $f \times 1 : X \times Z \to f(X) \times Z$. Thus it is closed.

Finally, to show (3), it suffices to consider the case where X is connected. Let $f \in k[X]$. This f defines a map

$$f : X \longrightarrow \mathbb{A}^1$$

By ②, the image of f is complete and closed in $\mathbb{A}^1$. Since $\mathbb{A}^1$ is not complete, the image must be proper. The image is also connected. Thus f is constant, and $k[X] = k$, so X is a point. □

Thm Let G be a connected algebraic group that is complete as a variety. Then G is commutative.

⇝ Such groups are called <u>abelian varieties</u>

⇝ Linear algebraic groups are <u>never</u> abelian varieties (except the trivial group)

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§6.2 Parabolic subgroups + Borel subgroups

Let G be a linear algebraic group.

Def A closed subgroup $P \leq G$ is a <u>parabolic</u> subgroup if the quotient G/P is complete.

<u>Remark</u>. We already know that G/H is quasi-projective for any closed subgroup H. Note that [complete + quasi-projective] is equivalent to projective. Thus $P \leq G$ is parabolic if and only if G/P is projective.

$\boxed{\text{Ex}}$ ① Let $G = GL_2$ and $B = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : ad \neq 0 \right\} \subseteq GL_2$

be the subgroup of upper-triangular matrices.

Claim: $G/B = \mathbb{P}^1$ and hence B is parabolic.

Reason: Start with the usual action of GL_2 on k^2

This is a linear action, so takes lines to lines. Hence we get an action of GL_2 on $\mathbb{P}^1$. This action is transitive, and the stabilizer of the line $\mathrm{Span}_k \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \subseteq k^2$ is B.

② Similarly, $B_1 = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a \in k^\times \right\}$ is parabolic in SL_2 with quotient isomorphic to $\mathbb{P}^1$.

③ Let $G = GL_3$. Consider the following subgroups:

$$P_1 = \left\{ \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} : \det \neq 0 \right\} \qquad P_2 = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} : \det \neq 0 \right\}$$

$$B = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} : \det \neq 0 \right\}$$

Claim: ① $G/P_1 \simeq \underline{Gr(2,3)} \ (\simeq \mathbb{P}^2)$

↳ Grassmannian of 2-planes in k^3

② $G/P_2 \simeq \mathbb{P}^2 \ (\simeq Gr(1,3))$

③ $G/B \simeq \{ (V_1, V_2) \in \mathbb{P}^2 \times Gr(2,3) : V_1 \subseteq V_2 \}$

<u>Reason</u>: One checks that G acts transitively on each of

$$Gr(2,3), \quad \mathbb{P}^2, \quad \{(V_1, V_2) \in \mathbb{P}^2 \times Gr(2,3) : V_1 \subseteq V_2\}.$$

and that the stabilizers of

$$\mathrm{Span}_k\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right) \in Gr(2,3), \quad \mathrm{Span}_k\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \in \mathbb{P}^2$$

$$\left(\mathrm{Span}_k\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right), \mathrm{Span}_k\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right) \right) \in \{(V_1,V_2) \in \mathbb{P}^2 \times Gr(2,3) : V_1 \subseteq V_2\}$$

are P_1, P_2, B, respectively.

Note that $\{(V_1, V_2) \in \mathbb{P}^2 \times Gr(2,3) : V_1 \subseteq V_2\}$ is closed in $\mathbb{P}^2 \times Gr(2,3) \simeq \mathbb{P}^2 \times \mathbb{P}^2$, and is hence a projective variety.

We conclude that P_1, P_2, B are all parabolic subgroups of GL_3.

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④ Let $G = GL_n$. The <u>flag variety</u> of GL_n is

$$\mathcal{F}_n = \left\{ (V_1, V_2, \ldots, V_{n-1}) \in \prod_{i=1}^{n-1} Gr(i, n) : V_i \subseteq V_{i+1} \text{ for } i = 1, \ldots, n-2 \right\}$$

(note that $\dim(V_i) = i$)

<u>Claim</u>: ① Each $Gr(i,n)$ is a projective variety and $\mathcal{F}_n$ is closed in $\prod_{i=1}^{n-1} Gr(i,n)$.

② There is an isomorphism

$$\mathcal{F}_n \simeq G/B$$

where $B \leq GL_n$ is the subgroup of upper-triangular matrices

<u>Reason</u>: The first claim is basic algebraic geometry. For the second claim, observe that the action of G on $\mathcal{F}_n$ is transitive and the stabilizer of the "standard flag"

$$\left(\mathrm{Span}_k\left\{\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right\},\ \mathrm{Span}_k\left(\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}\right),\ \dots,\ \mathrm{Span}_k\left(\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix},\begin{bmatrix}0\\1\\\vdots\\\vdots\\0\end{bmatrix},\dots,\begin{bmatrix}0\\\vdots\\0\\1\\0\end{bmatrix}\right)\right)$$

is B.

We conclude that $B \leq GL_n$ is a parabolic subgroup.

Prop Let G be a linear algebraic group.

① G is a parabolic subgroup of itself.

② If G is commutative and connected, then ~~the~~ the only parabolic subgroup is G itself.

③ Any conjugate of a parabolic subgroup is again a parabolic subgroup.

pf. ① $G/G = pt$, which is complete.

② Suppose G is commutative. Then every closed subgroup H of G is normal. So every quotient G/H is affine. Therefore, G/P is complete iff $G/P = pt$ iff $G = P$. ∎

③ $\forall g \in G$, we have an isomorphism

$$\begin{aligned} G/P &\xrightarrow{\ \sim\ } G/gPg^{-1} \\ xP &\longmapsto xg^{-1}(gPg^{-1}) \end{aligned}$$

□

Prop Suppose $P \subseteq Q \subseteq G$ are closed subgroups.

① If P is parabolic in G, then so is Q.

② If P is parabolic in Q and Q is parabolic in G, then P is parabolic in G.

pf. ① Use $G/P \twoheadrightarrow G/Q$.

② Consider the following diagram, for any alg. variety X:

$$\begin{tikzcd} Q/P \times G \times X \arrow[r, "\alpha"] \arrow[d, "\mathrm{proj}_{2,3}"'] & G/P \times X \arrow[d, "\tilde{\pi}_Q \times 1"] \arrow[dr, "\mathrm{proj}_2"] & \\ G \times X \arrow[r, "\pi_Q \times 1"] & G/Q \times X \arrow[r, "\mathrm{proj}_2"] & X \end{tikzcd}$$

where $\alpha(qP, g, x) = (gqP, x)$ and all other maps are the obvious projections. Let

$$V \subseteq G/P \times X$$

be closed. We aim to show that $\mathrm{proj}_2(V)$ is closed in X.

Note that:

- $\alpha^{-1}(V)$ is closed in $Q/P \times G \times X$ [α is continuous]
- $\mathrm{proj}_{2,3}(\alpha^{-1}(V))$ is closed in $G \times X$ [Q/P is complete]
- $Q \curvearrowright \mathrm{proj}_{2,3}(\alpha^{-1}(V))$ [check directly]

$$\Longrightarrow \underbrace{(\pi_Q \times 1)(\mathrm{proj}_{2,3}(\alpha^{-1}(V)))}_{= (\pi_Q \times 1)(V)} \text{ is closed in } G/Q \times X$$

- $\mathrm{proj}_2((\pi_Q \times 1)(V))$ is closed in X [G/Q is complete]

Since $\mathrm{proj}_2((\pi_Q \times 1))(V) = \mathrm{proj}_2(V)$, we are done. $\square$

Def A group G is <u>solvable</u> if in the chain of subgroups

$$G = G^{(0)}, \quad G^{(1)} = [G,G], \quad G^{(2)} = [G^{(1)}, G^{(1)}],$$

$$\ldots \quad G^{(i)} = [G^{(i-1)}, G^{(i-1)}], \ldots$$

we have $G^{(n)} = \{1\}$ for some n.

<u>Note</u>: Here $[G,G] = \langle x y x^{-1} y^{-1} : x, y \in G \rangle$ is the commutator subgroup.

Ex We saw in an earlier lecture that

$$U_n = \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \in GL_n \right\} \text{ is solvable.}$$

One can similarly show that

$$B_n = \left\{ \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \in GL_n \right\} \text{ is solvable}$$

(<u>Hint</u>: $[B_n, B_n] = U_n$).

Thm Let G be a connected linear algebraic group. Then G is solvable iff G contains no proper parabolic subgroups.

Cor GL_n and SL_n are not solvable.

(We have already seen proper parabolic subgroups of SL_n, GL_n).

proof of Theorem. ($\Rightarrow$) We proceed by induction on $\dim G$.

If $\dim G = 1$, then G is abelian, so G has no proper parabolics (as we have already observed).

Now suppose $\dim(G) > 1$ and assume G is solvable. Let $P \subseteq G$ be a parabolic subgroup. We aim to show that $G = P$.

To this end, observe first that

$$P \cap [G,G] \text{ is a parabolic subgroup of } [G,G]$$

$$\left(\underline{\text{Reason}}: \quad [G,G]/_{P \cap [G,G]} \simeq P.[G,G]/_{P} \xhookrightarrow{\text{closed}} G/_P \right)$$

Since $[G,G]$ is solvable and $\dim([G,G]) < \dim(G)$, by the induction hypothesis we get $[G,G] = P \cap [G,G]$, i.e. $[G,G] \subseteq P$.

Since $G/_{[G,G]}$ is abelian, the subgroup $P/_{[G,G]}$ is normal in $G/_{[G,G]}$. Hence the quotient

$$\frac{G/_{[G,G]}}{P/_{[G,G]}} \quad \text{is affine.}$$

This quotient is isomorphic to $G/_P$, which is complete as a variety. Hence $G/_P = pt \Rightarrow G = P$.

($\Leftarrow$) Suppose G contains no proper parabolics. We first show:

$\boxed{\text{Claim}}$ For any representation V of G, there is a basis of V in which G acts by upper-triangular matrices.

(Note: This is known as the Lie-Kolchin Theorem)

pf of Claim. We proceed by induction on $\dim V$. If $\dim V = 1$, there is nothing to show. So we assume $\dim V > 1$.

Let $X \subseteq \mathbb{P}(V)$ be a closed orbit (such an orbit exists by 2.3.3). Then X is a projective variety with a transitive G-action, so the stabilizer of any point is a parabolic. By our hypothesis, this stabilizer is all of G. Thus $X = \{x\}$ is a point. This point $x \in \mathbb{P}(V)$ corresponds to a line $\mathrm{Span}_k(v)$ of a nonzero vector $v \in V$ with the property that

$$g.v \in \mathrm{Span}_k(v) \quad \forall g \in G.$$

Then $W = V/_{\mathrm{Span}_k(v)}$ is a representation of G of dimension $\dim V - 1$. By the induction hypothesis, we can pick a basis $\{w_1, \dots, w_{n-1}\}$ of W in which G acts by upper-triangular matrices.

Pick lifts $\{v_1, \dots, v_{n-1}\} \subseteq V$ of the w's. Then, in the basis $\{v, v_1, v_2, \dots, v_{n-1}\}$ of V, G acts by upper triangular matrices. $\square$ (Claim).

Now, G embeds into $GL(V)$ for some V. Using the claim, we find a basis for V for which G embeds into $B_n = \left\{ \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \right\}$. The latter is solvable, hence so is G. $\square$

$\boxed{\text{Thm}}$ (Borel's fixed point theorem). Let G be a connected solvable linear algebraic group and X a complete G-variety. There exists a point in X that is fixed by all elements in G.

pf. By 2.3.3, there is a closed orbit $O \subseteq X$. Let $x \in O$, and let $H = \mathrm{Stab}_G(x)$. Since $O \simeq G/H$ is complete, H is parabolic. Since G is solvable, we have $G = H$, so $O = G/G = \{x\}$. Thus x is fixed by all elements of G. $\square$

$\boxed{\text{Def}}$ A <u>Borel subgroup</u> of a linear algebraic group G is a closed connected solvable group that is maximal for these properties

Lemma
1. Any conjugate of a Borel subgroup is again a Borel subgroup.
2. A Borel subgroup is parabolic.
3. A closed subgroup of G is parabolic iff it contains a Borel subgroup.
4. Any two Borel subgroups are conjugate.

<u>pf.</u> It is enough to prove these in the case that G is connected.

1. Clear from the definition of a Borel subgroup.

2. By induction on $\dim(G)$.

If $\dim(G) = 1$, then G is abelian, hence soluble. So its only Borel subgroup is G itself, which is parabolic.

Suppose $\dim(G) > 1$. If G is soluble then its only Borel is G itself, so is parabolic. If G is not soluble, then let P be a proper parabolic subgroup of G. Let B be a Borel subgroup of G. Consider the action

$$B \curvearrowright G/P \qquad b \cdot gP = (bg)P$$

This is an action of a connected soluble group on a complete variety, so by Borel's fixed point theorem, there is a fixed point $gP \in G/P$.

Hence $bgP = gP \quad \forall\, b \in B \quad \Rightarrow \quad g^{-1}Bg \subseteq P$.

WLOG, assume $B \subseteq P$. Observe that B must be a Borel subgroup of P. By the induction hypothesis, B is parabolic in P. Since P is parabolic in G, it follows from an earlier result that B is parabolic in G.

③ ($\Rightarrow$) We already demonstrated this while proving ② (using Borel's fixed point theorem).

($\Leftarrow$) Follows from the fact that B is parabolic and any subgroup containing a parabolic is itself parabolic.

④ Let B and B' be Borel subgroups. Then $B \curvearrowright G/B'$ has a fixed point. Hence $\exists\, g \in G$ s.t. $g^{-1}Bg \subseteq B'$.

Similarly $\exists\, h \in G$ s.t. $h^{-1}B'h \subseteq B$ (from $B' \curvearrowright G/B$).

Since $\dim B = \dim B'$ it follows that these are equalities. $\square$

$\boxed{\text{Cor}}$ Let $f : G \longrightarrow H$ be a surjective map of linear algebraic groups. Then

$$P \subseteq G \text{ parabolic} \implies f(P) \subseteq H \text{ is parabolic}$$

$$B \subseteq G \text{ Borel} \implies f(B) \subseteq H \text{ Borel}.$$

Ex You will show in exercise 6.2.11 (1) that

(a) $\left\{ \begin{bmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{bmatrix} : \det \neq 0 \right\} \subseteq GL_n$ is a Borel subgroup.

(b) The Borel subgroups of GL_n are the subgroups that appear as stabilizers of a complete flag

$$\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subset \cdots \subset V_{n-1} \subset V_n = k^n.$$

Summary of §6.3 and §6.4

Theorem [Lie - Kolchin] Let G be a connected solvable linear algebraic group. If $G \subseteq GL_n$, then $\exists x \in GL_n$ s.t.

$$x G x^{-1} \subseteq \left\{ \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \right\} = B_n .$$

pf. Similar to the argument that solvable $\iff$ no proper parabolics. $\square$

Prop Let G be a connected solvable linear algebraic group

① The commutator subgroup $[G,G]$ is a closed connected unipotent subgroup of G. (also normal)

② The set G_u of unipotent elements of G is a closed connected normal unipotent subgroup of G.

③ Let T be a maximal torus of G. The multiplication map $T \times G_u \longrightarrow G$ is an isomorphism of varieties.

proof (idea) Embed $G \hookrightarrow B_n \subseteq GL_n$ and reduce to showing these for B_n. $\square$

Note: The multiplication map $T \times G_u \longrightarrow G$ in part ③ is generally not an isomorphism of groups. In many cases $G = G_u \rtimes T$.

$\boxed{\text{Thm}}$ Let G be a linear algebraic group.

① Any maximal torus lies in a Borel subgroup.

② Any two maximal tori are conjugate.

③ If B is a Borel subgroup of G, then

$$N_G(B) = B.$$

One consequence of ③: Let $\mathcal{B} = \{\text{Borel subgroups of } G\}$. We have a bijection

$$G/B \longrightarrow \mathcal{B}$$
$$gB \longmapsto gBg^{-1}$$