Root Systems

Let V be a vector space over $\mathbb{R}$, equipped with a positive definite symmetric bilinear form:

$$(,) : V \times V \longrightarrow \mathbb{R}$$

[This means that:
- $(v,w) = (w,v) \quad \forall v,w \in V$
- $(\lambda_1 v_1 + \lambda_2 v_2 , w) = \lambda_1 (v_1, w) + \lambda_2 (v_2, w)$ $\quad \forall v_1, v_2, w \in V \ , \ \lambda_1, \lambda_2 \in \mathbb{R}$
- $(v,v) > 0$ for all nonzero $v \in V$.]

<u>Note</u>: $\exists$ basis of V for which $(,)$ is just the dot product:

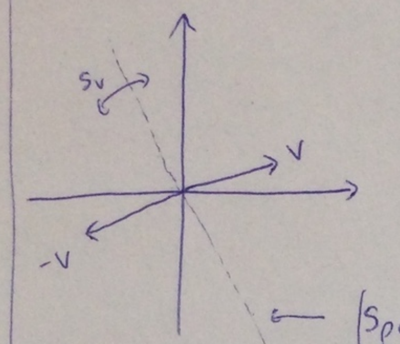
$$x \cdot y = \sum_{i=1}^{n} x_i y_i$$

$\boxed{\text{Def}}$ Let $v \in V$ be a nonzero vector. The <u>reflection</u> across v is defined as

$$S_v : V \longrightarrow V$$

$$w \longmapsto w - 2 \frac{(v,w)}{(v,v)} v$$

$\boxed{\text{Ex}}$ $V = \mathbb{R}^2$ with the dot product, and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



$$S_v = \frac{1}{5}\begin{bmatrix} -3 & -4 \\ 4 & 3 \end{bmatrix}$$

$\leftarrow [\mathrm{Span}(v)]^{\perp} = \{ w \in V : w \cdot v = 0 \}$

Let $s_v : V \longrightarrow V$ be the reflection across v. Then:

- $s_v(\lambda v) = -\lambda v \quad \forall \lambda \in \mathbb{R}$.
- $s_{\lambda v} = s_v \quad \forall \lambda \in \mathbb{R}$
- $s_v(w) = w$ iff $(v, w) = 0$
- $s_v^2 = id$
- $(s_v(w), s_v(w')) = (w, w') \quad \forall w, w' \in V$.

$\boxed{\text{Def}}$ A <u>root system</u> in V is a finite set Φ of nonzero vectors in V s.t.:

(1) The roots span V

(2) $\forall \alpha \in \Phi$, $\mathrm{Span}_{\mathbb{R}}(\alpha) \cap \Phi = \{\alpha, -\alpha\}$

(3) $\forall \alpha \in \Phi$, $s_\alpha(\Phi) = \Phi$

(4) $\forall \alpha, \beta \in \Phi$, $\dfrac{(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}[\frac{1}{2}]$

Remarks:

- (1) means that $\mathrm{Span}_{\mathbb{R}}(\{\alpha : \alpha \in \Phi\}) = V$.
- (2) means that if $\alpha \in \Phi$, then $-\alpha$ is also in Φ, but no other multiple of α is in Φ.
- (3) means that Φ is sent to itself under any reflection s_α for $\alpha \in \Phi$.
- (4) means that $\forall \alpha, \beta \in \Phi$, the projection of β onto $\mathrm{Span}_{\mathbb{R}}(\alpha)$ is a half-integer multiple of α.

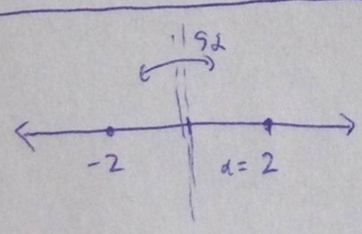
Def Let $\Phi \subseteq V$ be a root system. The associated reflection group is

$$W = W_\Phi = \langle s_\alpha : \alpha \in \Phi \rangle \subseteq GL(V)$$

It's called a reflection group since it's generated by the reflections s_α..

Examples

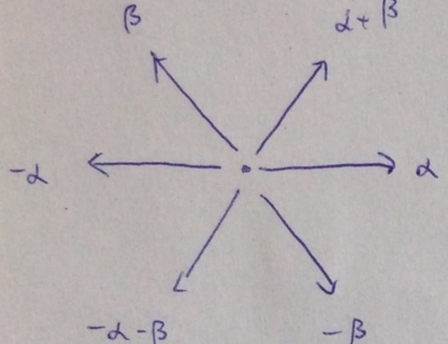
① The A_1 root system: $V = \mathbb{R}$, $\Phi = \{2, -2\}$



$$s_\alpha = -id$$

$$W = \mathbb{Z}/2\mathbb{Z}$$

② **The A_2 root system:** $V = \mathbb{R}^2$



$$\Phi = \{ \pm\alpha,\ \pm\beta,\ \pm(\alpha+\beta) \}$$

where $\alpha = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$.

The reflection group is

$$W = S_3$$

$$s_\alpha \longmapsto (12)$$

$$s_\beta \longmapsto (23)$$

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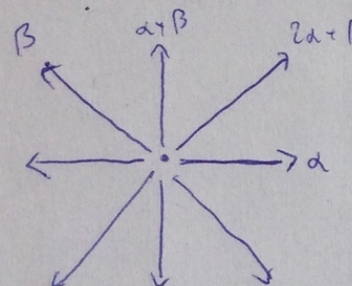
③ **The B_2 root system:** $V = \mathbb{R}^2$



$$\Phi = \{ \pm\alpha,\ \pm\beta,\ \pm(\alpha+\beta),\ \pm(2\alpha+\beta) \}$$

where $\alpha = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

The reflection group is D_4, the dihedral group of order 8.

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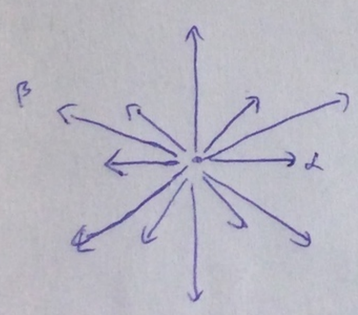
④ **The G_2 root system:** $V = \mathbb{R}^2$

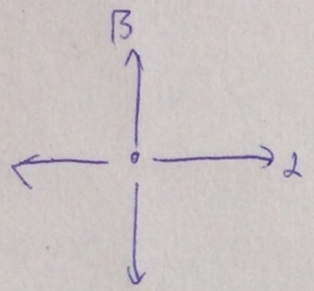


$$\Phi = \{ \pm\alpha,\ \pm\beta,\ \pm(\alpha+\beta),\ \pm(2\alpha+\beta),\ \pm(3\alpha+\beta),$$
$$\pm(2\alpha+2\beta) \}$$

where $\alpha = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\beta = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$

$$W = D_6$$

(5) <u>The $A_1 \times A_1$ root system</u>: $V = \mathbb{R}^2$, $\Phi = \{\pm\alpha, \pm\beta\}$



where $\alpha = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$$W = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

(<u>Rmk</u>: Examples (2) - (5) cover all (isomorphism classes) of 2-dimensional root systems. We will see that Examples (2) - (4) are irreducible while example (5) is reducible.)

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(6) <u>The A_n root system</u>:

Let $\{e_i\}_{i=1}^{n+1}$ be the standard basis of $\mathbb{R}^{n+1}$. Set

$$V = \{ x = \sum_{i=1}^{n+1} x_i e_i \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i = 0 \} \ (\cong \mathbb{R}^n)$$

Let

$$\Phi = \{ e_i - e_j \ : \ i,j = 1, \dots, n+1 \} \setminus \{0\} \subseteq V$$

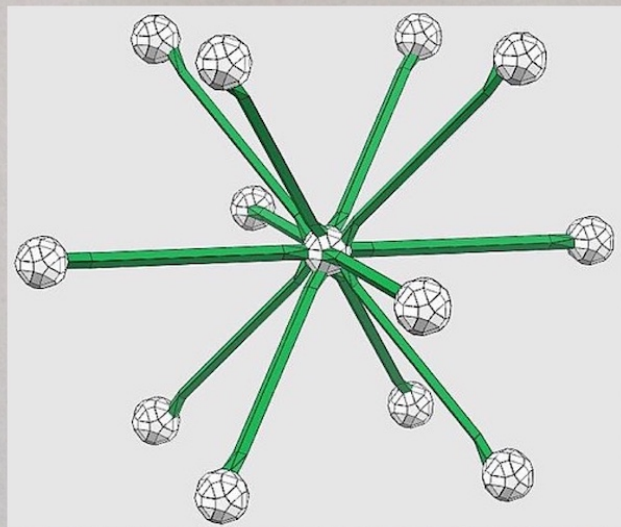
This is a root system, and the reflection group is

$$W = S_{n+1}$$

$$s_{e_i - e_{i+1}} \longleftrightarrow \text{the two cycle } (i, i+1)$$

for $i = 1, \dots, n$.

Picture of the A_3 root system:



$\boxed{\text{Def}}$ A root system Φ is <u>irreducible</u> if it cannot be written as a disjoint union $\Phi = \Phi_1 \sqcup \Phi_2$ of two root systems Φ_1 and Φ_2 s.t. $(\alpha, \beta) = 0 \quad \forall \alpha \in \Phi_1, \beta \in \Phi_2$.

(Technically, Φ_i is a root system of some subspace V_i of V for $i = 1, 2$ and $V = V_1 \overset{\perp}{\oplus} V_2$)

$\boxed{\text{Def}}$ A subset Φ^+ of a root system Φ is said to form a subset of <u>positive roots</u> if

- For any $\alpha \in \Phi$, exactly one of $\{\alpha, -\alpha\}$ lies in Φ^+.
- For any $\alpha, \beta \in \Phi^+$ distinct, if $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi^+$.

An element of Φ^+ is called a <u>simple root</u> if it cannot be written as a sum of two elements of Φ^+.

Rmk. Let $H \subseteq V$ be a hyperplane such that $H \cap \Phi = \emptyset$ (this is a generic condition). Then half of the roots in Φ lie on one side of H and half on the other.
Either of these halves forms a set of positive roots.

$\boxed{\text{Ex}}$ Consider the A_2 root system. Then

$$\Phi^+ = \{\alpha, \beta, \alpha+\beta\}$$

is a set of positive roots and $\{\alpha, \beta\} \subseteq \Phi^+$ is a set of simple roots. This choice is not canonical. For example,

$$\{\alpha, \alpha+\beta, -\beta\}$$

is another set of positive roots with $\{\alpha+\beta, -\beta\}$ being the simple roots.

$\boxed{\text{Lemma}}$ Let α, β be distinct roots and θ the angle between them. Then:

① θ is a multiple of $\frac{\pi}{6}$ or $\frac{\pi}{4}$. (integer multiple)

② If α and β are simple roots in a set of positive roots, then

$$\theta \in \left\{ \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6} \right\}.$$

Idea: We have these two observations:

(I) The angle θ satisfies
$$\cos\theta = \frac{(\alpha, \beta)}{\sqrt{(\alpha,\alpha)}\sqrt{(\beta,\beta)}}.$$

(II) The root system axioms require that

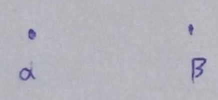
$$2\,\frac{(\alpha,\beta)}{(\alpha,\alpha)} \quad \text{and} \quad 2\,\frac{(\alpha,\beta)}{(\beta,\beta)} \quad \text{are integers.}$$

These facts imply that $2\cos\theta \in \{0, \pm 1, \pm\sqrt{2}, \pm\sqrt{3}\}$; hence we obtain ①. For ②, one argues that θ must be at least $\pi/2$ and less than π, which limits the possibilities to those listed.

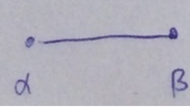
Dynkin diagram of a root system

Let Φ be a root system. Pick a set of positive roots Φ^+. Let $\Delta \subseteq \Phi^+$ be the simple roots. We associate a graph to Δ as follows:

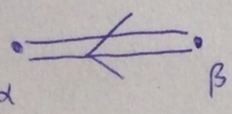
- Vertices are indexed by Δ.
- For $\alpha, \beta \in \Delta$ distinct, let θ be the angle between them.
  - If $\theta = \pi/2$, there is no edge between the vertices corresponding to α and β.



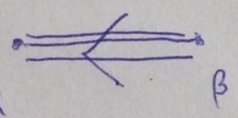
  - If $\theta = 2\pi/3$, there is a single undirected edge b/w the vertices corresponding to α and β



- If $\theta = \frac{3\pi}{4}$, there is a directed double edge, pointing toward the shorter vector

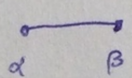


- If $\theta = \frac{5\pi}{6}$, there is a directed triple edge, pointing toward the shorter vector.

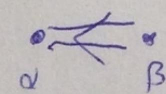


Ex ① A_1 $\circ \alpha$

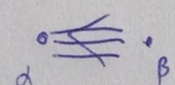
② A_2



③ B_2

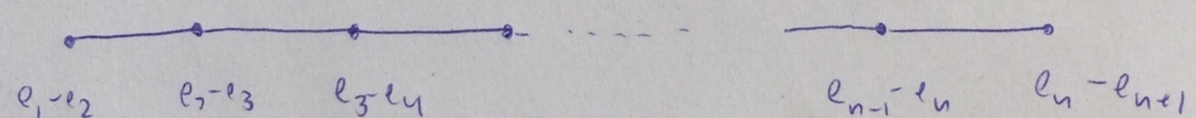


④ G_2



⑤ $A_1 \times A_1$ $\overset{\circ}{\alpha}$ $\overset{\circ}{\beta}$

⑥ A_n



Def Let $\Phi \subseteq V$ and $\Phi' \subseteq V'$ be irreducible root systems. An isomorphism between Φ and Φ' is an invertible linear map

$$f : V \longrightarrow V'$$

s.t. • $\exists \lambda \in \mathbb{R}_{>0}$ s.t. $(f(v), f(w)) = \lambda (v, w)$ $\forall v, w \in V$

• $f(\Phi) = \Phi'$.

Thm [Classification Theorem for root systems].

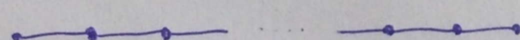
(1) The Dynkin diagram of a root system depends only on its isomorphism class (not on the choice of positive roots).

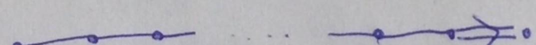
(2) The Dynkin diagram of Φ is connected iff Φ is irreducible.

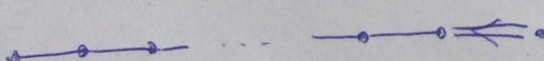
(3) The Dynkin diagram is a complete isomorphism invariant for root systems (i.e. Φ and Φ' have the same Dynkin diagram iff Φ and Φ' are isomorphic).

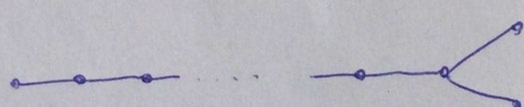
The possible Dynkin diagrams are:

Type

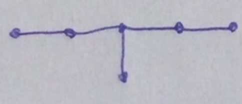
A_n  $(n \geq 1)$

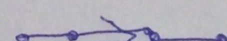
B_n  $(n \geq 2)$

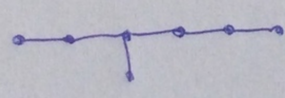
C_n  $(n \geq 3)$

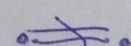
D_n  $(n \geq 4)$

n nodes in each case

E_6 

F_4 

E_7 

G_2 

E_8