Luby’s Algorithm for Maximal Independent Set

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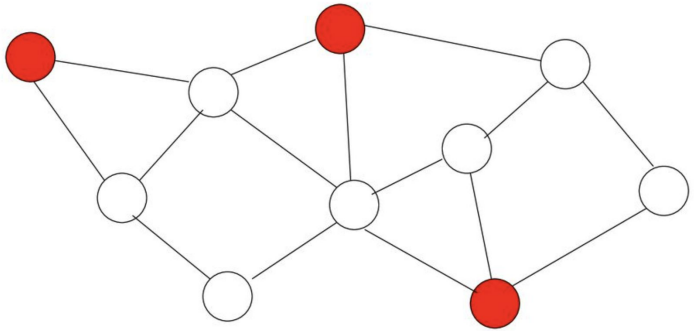
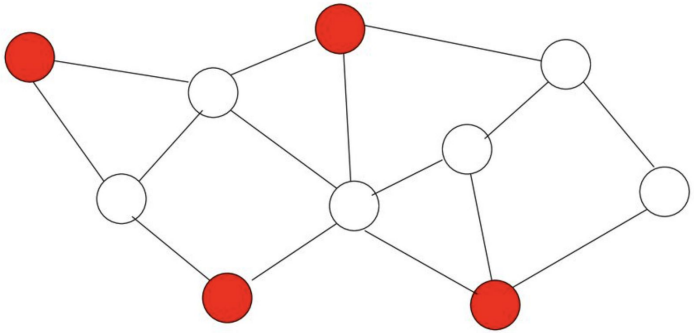
What is Maximal Independent Set (MIS)?

Independent Set (IS): Any set of nodes that are not adjacent



Maximal Independent Set: An independent set that is no subset of any other independent set



A General Algorithm for Computing MIS

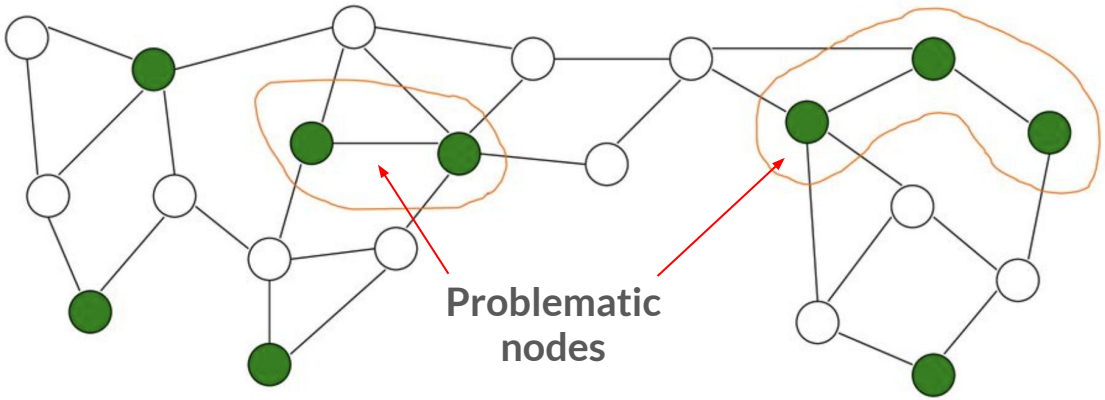
Algorithm 1: A high level description of the algorithm

Input: Graph $G = (V, E)$
Output: A maximal independent set I

1 $G' \leftarrow (V', E') \leftarrow (V, E)$;
2 $I \leftarrow \emptyset$;
3 **while** G' *is not empty* **do**
4 Select an independent set $I' \subseteq V'$ in G';
5 $I \leftarrow I \cup I'$;
6 $Y \leftarrow I' \cup N(I')$;
  // $N(I')$ is the set of neighbors of vertices in I'
7 $G' \leftarrow$ induced subgraph on $V' \setminus Y$;
8 **end**
9 **return** I;

Observation: The Select Step

- The **number of iterations** depends on the **choice of independent set** in each iteration.
- The **larger the independent set** at each iteration the **faster the algorithm**.



Monte Carlo Algorithm A

Algorithm 2: Select Step

```
1 for i ∈ V do                                          // ALGVERTEX(i), in parallel
2     π(i) ← a number randomly chosen from {1, ..., n^4}
3 I ← V
4 for (i, j) ∈ E do                                     // ALGEDGE(i, j), in parallel
5     if π(i) ≥ π(j) then
6         I ← I \ {i}
7     else
8         I ← I \ {j}
```

Monte Carlo Algorithm B

- If $d(i) > 0$, then $\text{coin}(i) = 1$ with probability $\frac{1}{2d(i)}$, and $\text{coin}(i) = 0$ otherwise.
- If $d(i) = 0$, then $\text{coin}(i) = 1$ with probability 1.

Monte Carlo Algorithm B

Algorithm 3: Algorithm B: Select Step

```
1  for i ∈ V' do                                              // in parallel
2  |  compute d(i)
3  end
4  X ← ∅ ;
5  for i ∈ V' do                                   // Choice Step, in parallel
6  |  Randomly choose a value for coin(i) ;
7  |  if coin(i) = 1 then X ← X ∪ {i} ;
8  end
9  I' ← X ;
10 for (i, j) ∈ E' do                                         // in parallel
11 |  if i ∈ X and j ∈ X then
12 |  |  if d(i) ≤ d(j) then I' ← I' \ {i} ;
13 |  |  else I' ← I' \ {j} ;
14 |  end
15 end
```

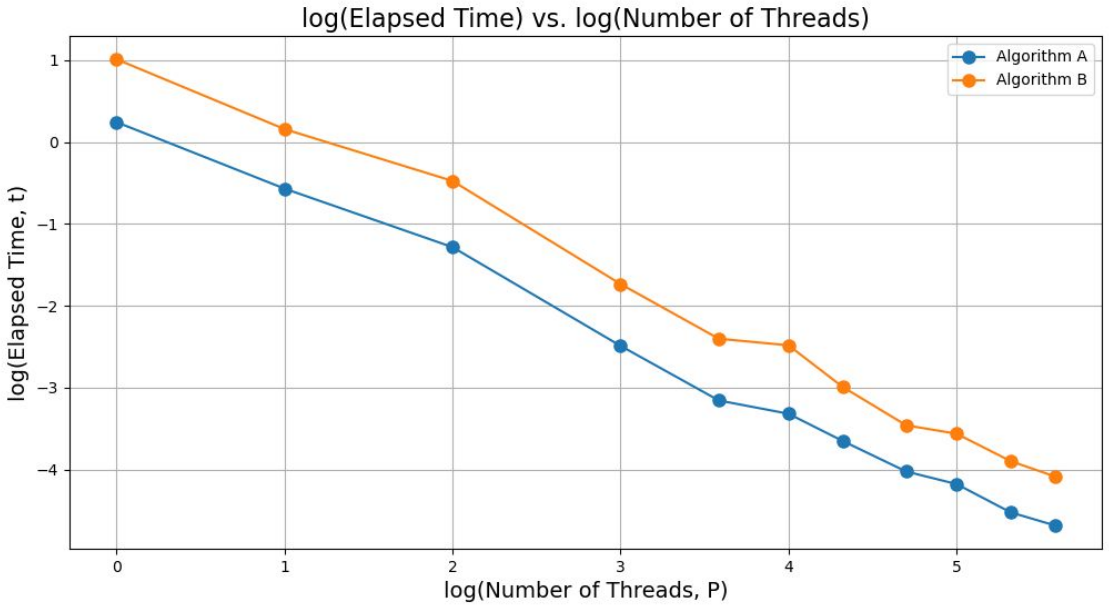
Distributed Algorithm with YGM

	Overall Runtime	Algorithm-A (Least Priority)	Algorithm-B (Least Degree)
Theoretical Result	$O(\log \lvert V \rvert \cdot \log d)$ (w.h.p.)	$O(\log d)$	$O(\log d)$
My Implementation	$O(\log \lvert V \rvert \cdot d)$ (w.h.p.)	$O(d)$	$O(d)$

Table 2: Comparison of Theoretical vs. Implemented Runtime Complexities

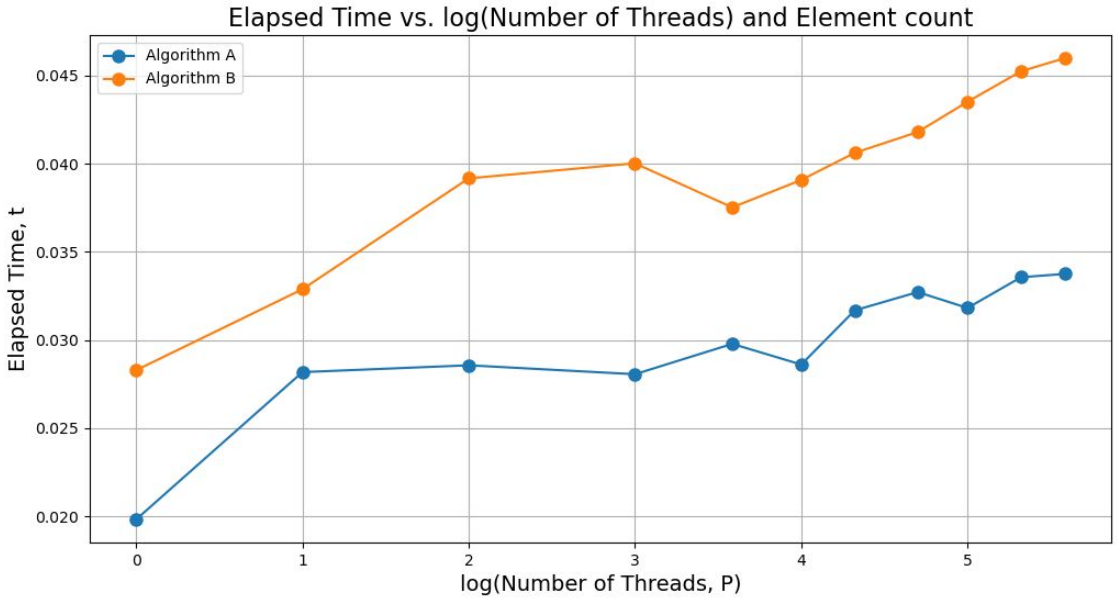
- Theoretical Result: **O(log d)-time reduction to find the neighbour with the least priority (Algorithm A) / degree (Algorithm B).**
- My Implementation: **O(d)-time to find the neighbour with the least priority (Algorithm A) / degree (Algorithm B).**

Strong Scaling Results



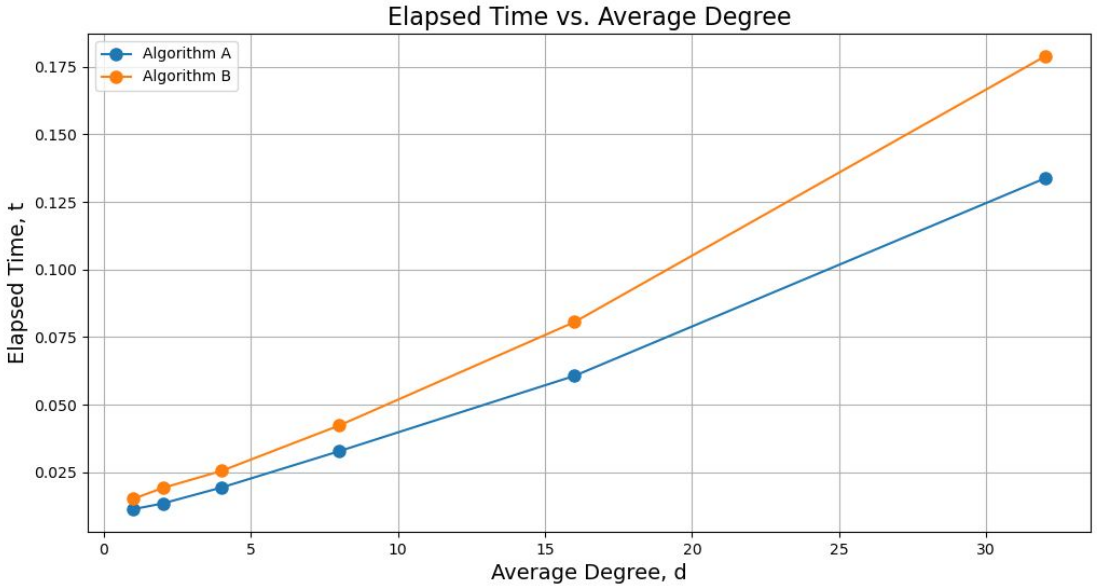
Number of Vertices: 80,000 | Number of Edges: 640,000 | Processor Count: 1 to 48

Weak Scaling Results



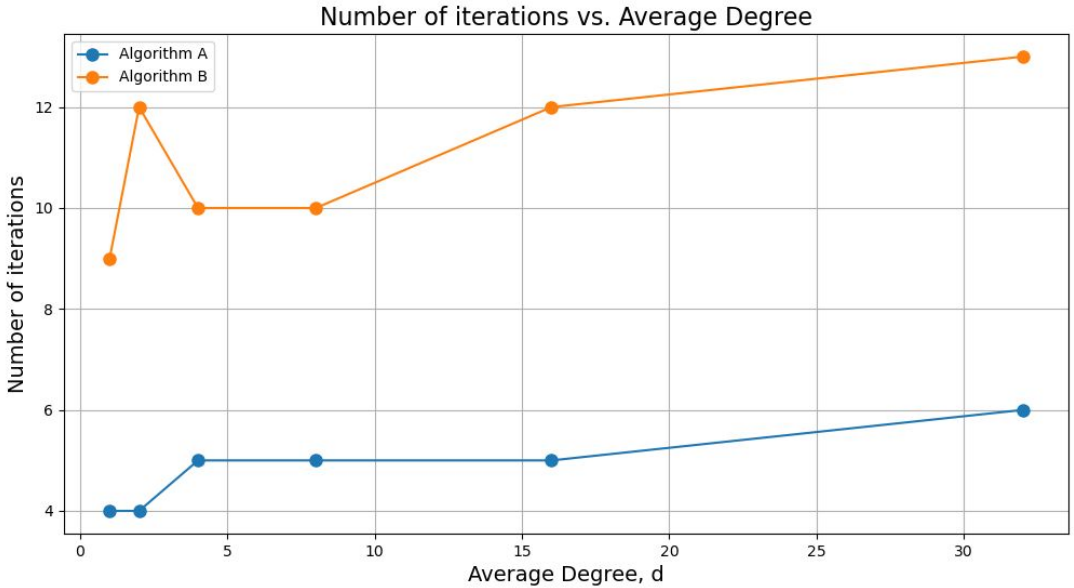
Number of Vertices: 1,250 to 10,000 | Number of Edges: 60,000 to 480,000 | Processor Count: 1 to 48

Degree Experiment - Time vs Degree



Number of Vertices: 40,000 | Number of Edges: 40,000 to 2,560,000 | Processor Count: 32

Degree Experiment - Iterations vs Degree



Number of Vertices: 40,000 | Number of Edges: 40,000 to 2,560,000 | Processor Count: 32

Removing randomization from parallel Algorithms

De-randomization strategy: Monte Carlo -> Deterministic Algorithm

1. **Isolate a random choice step** in the algorithm
2. Construct **a small set of independent random variables**
3. Run the algorithm over **all choices from this set**.

Algorithm	PRAM Type	Processors	Time
A	CRCW	$O(m)$	$EO(\log n)$
B	EREW	$O(m)$	$EO((\log n)^2)$
C	EREW	$O(m)$	$EO((\log n)^2)$
D	EREW	$O(n^2m)$	$O((\log n)^2)$

This is very powerful in parallel setting:

- All candidate choices can be evaluated from a bounded space simultaneously across processors -> guarantee success