Contents

Vector Spaces

1. What is a Vector Space?

A **vector space** is a set where you can add things together and multiply them by numbers ("scalars"), and the results stay within the set. The rules are:

- **Addition**: If you have two vectors $\vec{u}, \vec{v}$, then $\vec{u} + \vec{v}$ is also a vector in that space.
- **Scalar multiplication**: Multiply a vector $\vec{v}$ by a number a to get another vector: $a\vec{v}$.
- There are other properties (like distributivity, existence of a zero vector, etc.), but these are the core.

Example: The set of all ordered pairs of real numbers (x, y) is a vector space: $\mathbb{R}^2$.

2. Hierarchy of Spaces

- **Vector Space**: Most general setting for linear algebra.
- **Normed Space**: A vector space with a way to measure "length" (norm).
- **Inner Product Space**: A normed space where you can take "dot products" (measuring angles).
- **Euclidean Space**: The example you know best—$\mathbb{R}^n$ with the usual dot product, forming our usual geometry.

A simple mnemonic: "All Euclidean spaces are inner product spaces, all inner product spaces are normed spaces, all normed spaces are vector spaces—but not vice versa."

3. Norm and Inner Product

- **Norm**: Given a vector $\vec{v}$, the norm $\|\vec{v}\|$ is its length: For $\mathbb{R}^n$, $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$.
- **Inner Product** (a.k.a. "Dot Product"): Measures the "angle" and "length together" between two vectors: $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$

This leads to the **norm**: $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$.

4. Euclidean Spaces

- These are just $\mathbb{R}^n$, with our standard rules for addition, scaling, and dot product.
- **Geometry**: Distance between points, measuring angles, etc. are all linked to the inner product structure.

5. Subspaces

What is a Subspace?

A **subspace** is a "smaller" vector space inside a bigger one. For example, a line or a plane through the origin inside $\mathbb{R}^3$.

Three Key Criteria

A subset W of a vector space V is a **subspace** if:

1. **Zero Vector:** $\vec{0} \in W$.
2. **Closed Under Addition:** For any $\vec{u}, \vec{v} \in W$, the sum $\vec{u} + \vec{v} \in W$.
3. **Closed Under Scalar Multiplication:** For any $\vec{v} \in W$ and any scalar a, $a\vec{v} \in W$.

If you see a set that doesn't satisfy these, it's not a subspace.

Quick Examples

- **Entire vector space** itself: Always a subspace.
- **Line through origin:** E.g., all multiples of $(1, 2)$ inside $\mathbb{R}^2$.
- **Plane through origin:** E.g., all $(x, y, 0)$ inside $\mathbb{R}^3$.
- **Set of solutions to a homogeneous linear equation:** Like $ax + by = 0$, forms a subspace.

Not Subspaces (Counterexamples)

- **Line not through origin:** E.g., $(1, 2) + t(3, 4)$ for $t \in \mathbb{R}$, where $(1, 2)$ isn't the origin — this fails the zero vector criterion.
- **Subset not closed under addition/scalar multiplication:** E.g., only vectors of length 1.

Why Are Subspaces Important?

- They often describe **solution sets** to linear systems.
- They're the backbone of **span**, **basis**, and **dimension** concepts.

Vector Geometry

1. Angles Between Vectors

The angle θ between two vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^n$ uses their dot product:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$$

To find θ:

$$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$$

This formula answers: "How much do the vectors 'point in the same direction'?"

2. Vector & Scalar Projections

Scalar projection (also called "component") of $\vec{a}$ onto $\vec{b}$:

$$\text{comp}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

This is just a number; it tells "how much of $\vec{a}$ is in the direction of $\vec{b}$".

Vector projection puts the scalar projection in the direction of $\vec{b}$:

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\vec{b}$$

Think of it as the **"shadow"** of $\vec{a}$ on $\vec{b}$.

3. Orthogonal Projections

> **Orthogonalization** is the process of transforming a set of vectors or features into a new set where each pair is independent or "perpendicular" to each other. This is achieved by creating vectors with a dot product of zero, making them mutually orthogonal, and is often done using methods like the Gram-Schmidt process.

Suppose you want the projection onto a subspace (like a plane) that is defined by some vector $\vec{b}$. The vector projection $\text{proj}_{\vec{b}}\vec{a}$ is always **orthogonal** to the difference $\vec{a} - \text{proj}_{\vec{b}}\vec{a}$ (the part left over is perpendicular to $\vec{b}$).

In a nutshell:

- The **orthogonal projection** of $\vec{a}$ onto a line (or vector) is its vector projection.
- The part perpendicular to $\vec{b}$ is called the "orthogonal complement".

4. Normals

A **normal** to a surface (like a plane) is a vector perpendicular to that surface.

- For a plane defined by $Ax + By + Cz = D$, the vector (A, B, C) is normal to the plane.
- Normals are important for defining hyperplanes, reflecting vectors, optimization, and computer graphics.

5. Hyperplanes

A **hyperplane** in $\mathbb{R}^n$ is a flat subset with dimension $n - 1$.

- In 3D, a plane is a hyperplane.
- General equation: $a_1x_1 + a_2x_2 + ... + a_nx_n = d$.
- The normal vector is $(a_1, ..., a_n)$, showing what direction is "perpendicular" to the hyperplane.