PROBLEM SET NO. 9 - ISAAC VIVIANO

I affirm that I have adhered to the Honor Code in this assignment.
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1. Problems:

- **Problem 1 - $\mathcal{L}^1$-Functions & Schwartz Functions:**

  (a)

Proposition 1.1. *If $f : \mathbb{R} \to \mathbb{C}$ is an $\mathcal{L}^1$-function, then the improper Riemann integral $\int_{-\infty}^{+\infty} f(t)\ \mathrm{d}t$ always exists.*

Proof. Suppose $f \in \mathcal{L}^1$. That is,

$$\int_{-\infty}^{\infty} |f(t)|\ dt < \infty$$

Since the integral converges, we have both

$$\int_{0}^{\infty} |f(t)|\ dt < \infty$$
$$\int_{-\infty}^{0} |f(t)|\ dt < \infty$$

Note that sequence convergence always implies Cauchy sequence. Let $\epsilon > 0$ be given and pick $N_-, N_+ \in \mathbb{N}$ such that

$$\left|\int_0^R |f(t)|\ dt - \int_0^S |f(t)|\ dt\right| < \epsilon, \quad \text{for all } R > S \geqslant N_+$$
$$\left|\int_{-R}^0 |f(t)|\ dt - \int_{-S}^0 |f(t)|\ dt\right| < \epsilon, \quad \text{for all } R > S \geqslant N_-$$

For $R > S \geqslant N_+$, we have

$$\begin{aligned}
\left|\int_0^R f(t)\ dt - \int_0^S f(t)\ dt\right| &= \left|\int_S^R f(t)\ dt\right| \\
&\leqslant \int_S^R |f(t)|\ dt \\
&= \underbrace{\int_0^R |f(t)|\ dt - \int_0^S |f(t)|\ dt}_{\geqslant 0} \\
&= \left|\int_0^R |f(t)|\ dt - \int_0^S |f(t)|\ dt\right| \\
&< \epsilon
\end{aligned}$$

and similarly for $R > S \geqslant N_-$, we have

$$\begin{aligned}
\left|\int_{-R}^{0} f(t)\ dt - \int_{-S}^{0} f(t)\ dt\right| &= \left|\int_{-R}^{-S} f(t)\ dt\right| \\
&\leqslant \int_{-R}^{-S} |f(t)|\ dt \\
&= \underbrace{\int_{-R}^{0} |f(t)|\ dt - \int_{-S}^{0} |f(t)|\ dt}_{\geqslant 0} \\
&= \left|\int_{-R}^{0} |f(t)|\ dt - \int_{-S}^{0} |f(t)|\ dt\right| \\
&< \epsilon
\end{aligned}$$

This shows that both $\int_0^R f(t)\ dt$ and $\int_{-R}^0 f(t)\ dt$ are Cauchy sequences in $\mathbb{R}$. Since $\mathbb{R}$ is complete, Cauchy implies convergence. Thus, the improper Riemann integral of f converges. $\square$

(b)

Proposition 1.2. *If $f \in \mathcal{S}(\mathbb{R};\mathbb{C})$, then $f \in \mathcal{L}^1$.*

Proof. Let $f \in \mathcal{S}(\mathbb{R};\mathbb{C})$ and let

$$C_{m,n} := \sup_{x\in\mathbb{R}} |x^m f^{(n)}(x)|$$

Let $1 < R \in \mathbb{R}$ be given. Estimate

$$\begin{aligned}
\int_{-R}^{R} |f(x)|\ dx &= \underbrace{\int_{-1}^{1} |f(x)|\ dx}_{:=C\in\mathbb{R}} + \int_{[-R,R]\setminus(-1,1)} |f(x)|\ dx \\
&= C + \int_{[-R,R]\setminus(-1,1)} \frac{1}{x^2} \underbrace{|x^2 f(x)|}_{\leqslant C_{2,0}}\ dx \\
&\leqslant C + C_{2,0} \int_{[-R,R]\setminus(-1,1)} \frac{1}{x^2}\ dx \\
&= C + 2C_{2,0} \int_{1}^{R} \frac{1}{x^2}\ dx \\
&= C + 2C_{2,0} \left(\frac{-1}{x}\Big|_1^R\right) \\
&= C + 2C_{2,0} \left(1 - \frac{1}{R}\right)
\end{aligned}$$

Thus,

$$\lim_{R\to\infty}\int_{-R}^{R}|f(x)|\,dx \leqslant \sup_{R>1}\left(C+2C_{2,0}\left(1-\frac{1}{R}\right)\right) = C+2C_{2,0}<\infty$$

This shows that the Cauchy principal value and thus also the improper Riemann integral of $|f|$ is finite. So, $f\in\mathcal{L}^1$. □

- **Problem 2 - Approximate Identities on $\mathbb{R}$:**

(b)

Theorem 1.1 (Approximate Identities on $\mathbb{R}$)**.** *For $k\in\mathbb{N}_0\cup\{\infty\}$, let $\{\phi_\lambda,\lambda>0\}$ be a $\mathcal{C}^k$-$\mathbb{R}AI$. Then, for every function $f\in\mathcal{C}_\infty(\mathbb{R};\mathbb{C})$, one has*

$$f\star\phi_\lambda\to f,\ \textit{uniformly as}\ \lambda\to\infty$$

Before proving Theorem 1.1, we prove the following lemma that extends the regularity of continuous functions that vanish at ∞.

Lemma 1.2. *If f is continuous and f vanishes at ∞:*

$$f\in\mathcal{C}_\infty(\mathbb{R};\mathbb{C})$$

then f is uniformly continuous.

Proof. Let $f\in\mathcal{C}_\infty$, and let $\epsilon>0$ be given. Pick $R>0$ such that for all $|x|\geqslant R$,

$$|f(x)|<\frac{\epsilon}{3}$$

Since continuous functions are uniformly continuous on compact sets, pick $\delta>0$ such that for all $x,y\in[-R,R]$,

$$|f(x)-f(y)|<\frac{\epsilon}{3}$$

For all $x,y\in\mathbb{R}$ with $|x-y|<\delta$, we consider three cases without loss of generality. If $x,y\in[-R,R]$, we have

$$|f(x)-f(y)|<\frac{\epsilon}{3}<\epsilon$$

If $x\in[-R,R]$ and $y>R$,

$$\begin{aligned}|f(x)-f(y)| &\leqslant |f(x)-f(R)|+|f(R)-f(y)|\\ &<\frac{\epsilon}{3}+|f(R)|+|f(y)|\\ &<\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}\\ &=\epsilon\end{aligned}$$

If $x, y \notin [-R, R]$,

$$|f(x) - f(y)| \leqslant |f(x)| + |f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} \leqslant \epsilon$$

Thus, f is uniformly continuous. $\square$

Proof of Theorem 1.1. Let $f \in \mathcal{C}_\infty(\mathbb{R}; \mathbb{C})$. Let $\epsilon > 0$ be given. Since f vanishes at ∞, it is uniformly continuous on $\mathbb{R}$. Pick $\delta > 0$ such that for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$.

Since f vanishes at ∞, pick $R > 0$ such that

$$\|f\|_{\infty; \mathbb{R}\setminus(-R,R)} \leqslant \epsilon$$

We also have that that the continuous f is bounded on $[-R, R]$ by (EVT). Thus, f is bounded with

$$\|f\|_\infty = \max\{\|f\|_{\infty;[-R,R]}, \underbrace{\|f\|_{\infty;\mathbb{R}\setminus(-R,R)}}_{\leqslant \epsilon}\} < \infty$$

Pick Λ such that for all $\lambda > \Lambda$,

$$\int_{\mathbb{R}\setminus(-\delta,\delta)} |\phi_\lambda(t)| \, dt < \epsilon$$

Let C be the constant given by $\mathbb{R}AI$-2.

If $\lambda > \Lambda$,

$$
\begin{aligned}
|(\phi_\lambda \star f)(t) - f(t)| &= \left| \int_{-\infty}^{\infty} \phi_\lambda(s) \cdot f(t-s)\ ds - f(t) \right| && (1)\\
&= \left| \int_{-\infty}^{\infty} \phi_\lambda(s) \cdot f(t-s)\ ds - f(t) \int_{-\infty}^{\infty} \phi_\lambda(s)\ ds \right| && (2)\\
&= \left| \int_{-\infty}^{\infty} \phi_\lambda(s)(f(t-s) - f(t))\ ds \right| && (3)\\
&= \left| \int_{\mathbb{R}\setminus(-\delta,\delta)} \phi_\lambda(s)(f(t-s) - f(t))\ ds + \int_{-\delta}^{\delta} \phi_\lambda(s)(f(t-s) - f(t))\ ds \right| && (4)\\
&\leqslant \int_{\mathbb{R}\setminus(-\delta,\delta)} |\phi_\lambda(s)(f(t-s) - f(t))|\ ds + \int_{-\delta}^{\delta} |\phi_\lambda(s)(f(t-s) - f(t))|\ ds && (5)\\
&\leqslant \int_{\mathbb{R}\setminus(-\delta,\delta)} |\phi_\lambda(s)| \cdot \underbrace{|(f(t-s) - f(t))|}_{\leqslant 2\|f\|_\infty}\ ds + \int_{-\delta}^{\delta} |\phi_\lambda(s)| \cdot \underbrace{|(f(t-s) - f(t))|}_{<\epsilon}\ ds && (6)\\
&\leqslant 2\|f\|_\infty \underbrace{\int_{\mathbb{R}\setminus(-\delta,\delta)} |\phi_\lambda(s)|\ ds}_{<\epsilon} + \epsilon \int_{-\delta}^{\delta} |\phi_\lambda(s)|\ ds && (7)\\
&< 2\epsilon\|f\|_\infty + \epsilon \underbrace{\int_{-\infty}^{\infty} |\phi_\lambda|\ ds}_{\leqslant C,\text{ by } \mathbb{R}AI-2} && (8)\\
&\leqslant 2\epsilon\|f\|_\infty + \epsilon C && (9)
\end{aligned}
$$

where the bound on $|f(t-s) - f(t)|$ in (6) comes from the uniform continuity of f and $|s| < \delta$. Since this bound in (9) a constant (independent of t) multiple of ϵ,

$$f \star \phi_\lambda = \phi_\lambda \star f \to 0, \quad \text{uniformly in } t$$

for all $f \in \mathcal{C}_\infty(\mathbb{R}; \mathbb{C})$. $\square$

(c)

Proposition 1.3. *For a $\mathcal{L}^1$ function ϕ satisfying*

$$\int_{-\infty}^{\infty} \phi(x)\ \mathrm{d}x = 1 \tag{10}$$

the family of approximate delta functions

$$\phi_\epsilon := \epsilon^{-1}\phi(\epsilon^{-1}x)\ ,\ 0 < \epsilon < 1\ ,$$

forms an approximate identity on $\mathbb{R}$.

Proof. Let $0 < \phi \in \mathcal{L}^1 \cap \mathcal{C}^k(\mathbb{R};\mathbb{C})$ satisfy

$$\int_{-\infty}^{\infty} \phi(x)\ dx = 1$$

Note that ϕ real valued and non-negative implies $|\phi| = \phi$.

$\mathbb{R}AI - 1$:

$$\begin{aligned}\int_{-\infty}^{\infty} \phi_\epsilon(x)\ dx &= \int_{-\infty}^{\infty} \frac{1}{\epsilon}\phi\left(\frac{1}{\epsilon}x\right)\ dx \\ &= \int_{-\infty}^{\infty} \phi(t)\ dt \quad (\text{change variables } t = \frac{x}{\epsilon}) \\ &= 1\end{aligned}$$

$\mathbb{R}AI - 2$: Since $|\phi| = \phi$,

$$\phi_\epsilon(x) = \frac{1}{\epsilon}\phi\left(\frac{1}{\epsilon}x\right) = \left|\frac{1}{\epsilon}\phi\left(\frac{1}{\epsilon}x\right)\right| = |\phi_\epsilon|$$

Thus, $\mathbb{R}AI - 2$ follows from $\mathbb{R}AI - 1$ by Remark 9.10 (2) of [CM].

$\mathbb{R}AI - 3$: Since $\phi \in \mathcal{L}^1$, pick $R > 0$ such that

$$\int_{\mathbb{R}\setminus(-R,R)} \phi(x)\ dx < \epsilon$$

If $\epsilon < 1$, we have

$$\begin{aligned}\int_{\mathbb{R}\setminus(-R,R)} \phi_\epsilon(x)\ dx &= \int_{-\infty}^{-R} \frac{1}{\epsilon}\phi\left(\frac{1}{\epsilon}x\right) dx + \int_{R}^{\infty} \frac{1}{\epsilon}\phi\left(\frac{1}{\epsilon}x\right)\ dx \qquad (11)\\ &= \int_{-\infty}^{-\frac{R}{\epsilon}} \phi(t)\ dt + \int_{\frac{R}{\epsilon}}^{\infty} \phi(t)\ dt \quad (\text{change variables } t = \frac{x}{\epsilon}) \qquad (12)\\ &\leqslant \int_{-\infty}^{R} \phi(t)\ dt + \int_{R}^{\infty} \phi(t)\ dt \qquad (13)\\ &= \int_{\mathbb{R}\setminus(-R,R)} \phi(t)\ dt \qquad (14)\\ &< \epsilon \qquad (15)\end{aligned}$$

where the monotonicity of the integrals in (13) holds for the non-negative real-valued function ϕ.

□

(d)

Proposition 1.4. *The function* $f : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x) := \begin{cases} \sqrt{n} & x \in \left[n, n + \frac{1}{n^2}\right] \text{ for some } n \in \mathbb{N}_0 \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

satisfies $f \in \mathcal{L}^1$ *but* $f^2 \notin \mathcal{L}^1$.

Proof. We see

$$\int_{-\infty}^{0} f(x)\ dx = 0$$

For $n \in \mathbb{N}$, we see,

$$\int_0^n f(x)\ dx = \sum_{k=0}^{n-1} \frac{\sqrt{k}}{k^2} = \sum_{k=0}^{n-1} \frac{1}{k^{\frac{3}{2}}}$$

which converges as $n \to \infty$, since it is a p series with $p > 1$. Thus, $|f| = f$ has convergent improper Riemann integrals and $f \in \mathcal{L}^1$

However, for $n \in \mathbb{N}$,

$$\int_0^n (f(x))^2\ dx = \sum_{k=0}^{n-1} \frac{k}{k^2} = \sum_{k=0}^{n-1} \frac{1}{k}$$

which diverges as $n \to \infty$, since it is the harmonic series. Thus, $f^2 \notin \mathcal{L}^1$.

In particular, we have shown that $f, g \in \mathcal{L}^1$ is not sufficient for $f \star g$ to converge. □

- **Problem 3 - α-comma Meantone Temperaments:**

(b) For the mean-tone temperament, $g_\alpha = \frac{7}{12}$ ([CM] (8.41)), so

$$\begin{aligned} & g_\alpha = g - \alpha\epsilon_s \\ \iff & \alpha = \frac{g - g_\alpha}{\epsilon_s} = \frac{g - 7/12}{\epsilon_s} \end{aligned}$$

Compute the first couple continued fraction approximants:

$$
\begin{aligned}
\alpha &= .09090368\ldots \\
a_0 &= \lfloor \alpha \rfloor = 0 \\
x_1 &= \frac{1}{\alpha - a_0} = 11.0006548\ldots \\
a_1 &= \lfloor x_1 \rfloor = 11 \\
\alpha_1 &= a_0 + \frac{1}{a_1} = \frac{1}{11} \\
r_1 &= x_1 - a_1 = .00065477\ldots \\
x_2 &= \frac{1}{r_1} = 1527.25201\ldots \\
a_2 &= \lfloor x_2 \rfloor = 1527 \\
\alpha_2 &= a_0 + \frac{1}{a_1 + \frac{1}{a_2}} = \frac{1527}{16798}
\end{aligned}
$$

The first continued fraction approximant $\frac{1}{11}$ is a very good approximant because of the huge jump in denominators from 11 to 16798. Also, a scale of 16798 tones would be impractical.