PROBLEM SET NO. 5 - ISAAC VIVIANO

I affirm that I have adhered to the Honor Code in this assignment. Isaac Viviano

1. Solutions:

Problem 1:

(b)

Proposition 1.1.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \ . \tag{1}$$

Proof. From integration by parts, we have

$$\int x^2 \cos x \ dx = 2x \cos x + (x^2 - 2) \sin x + C$$

Let $f : [0,1] \to \mathbb{R}$ be the 1-periodic extension of $f(t) = (\pi t)^2$. As $f \in \mathcal{PC}_1(\mathbb{R};\mathbb{R})$, its real Fourier series converges pointwise:

$$\frac{1}{2}(f(t+) + f(t-)) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi nt) + \sum_{n=1}^{\infty} b_n \sin(2\pi nt), \text{ for all } t \in \mathbb{R}$$

where

$$a_n := 2\int_0^1 f(t)\cos(2\pi int) \ dt$$
$$b_n := 2\int_0^1 f(t)\sin(2\pi int) \ dt$$

Compute:

$$
\begin{aligned}
(n > 0) \quad a_n &= 2\int_0^1 f(t)\cos(2\pi nt)\ dt \\
&= 2\int_0^1 (\pi t)^2 \cos(2\pi nt)\ dt \quad \text{(substitute } u = 2\pi nt\text{)} \\
&= \frac{1}{4\pi n^3}\int_0^{2\pi n} u^2 \cos u\ du \\
&= \frac{1}{4\pi n^3}(2u\cos u + (u^2-2)\sin u)\Big|_0^{2\pi n} \\
&= \frac{1}{4\pi n^3}(4\pi n \cdot \underbrace{\cos(2\pi n)}_{=1} + ((2\pi n)^2 - 2)\cdot \underbrace{\sin(2\pi n)}_{=0} - (2\cdot 0\cos 0 - 2\sin 0)) \\
&= \frac{1}{4\pi n^3}\cdot 4\pi n \\
&= \frac{1}{n^2} \\
(n = 0) \quad a_0 &= 2\int_0^1 (\pi t)^2 \cos(2\pi 0 t)\ dt \\
&= 2\int_0^1 \pi^2 t^2\ dt \\
&= \frac{2\pi^2 t^3}{3}\Big|_0^1 \\
&= \frac{2\pi^2}{3}
\end{aligned}
$$

Thus,

$$\frac{1}{2}(f(t+) + f(t-)) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos(2\pi nt) + \sum_{n=1}^{\infty} b_n \sin(2\pi nt)$$

Note that at 0,

$$
\begin{aligned}
f(t+) &= 0 \\
f(t-) &= \pi^2 \\
\frac{1}{2}(f(t+) + f(t-)) &= \frac{\pi^2}{2}
\end{aligned}
$$

So,

$$\begin{aligned}\frac{\pi^2}{2} &= \frac{1}{2}(f(0+)+f(0-)) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{1}{n^2} \underbrace{\cos(2\pi n0)}_{=1} + \sum_{n=1}^{\infty} b_n \underbrace{\sin(2\pi n0)}_{=0} \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{1}{n^2}\end{aligned}$$

So,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{2} - \frac{\pi^2}{3} = \frac{\pi^2}{6}$$

$\square$

Proposition 1.2.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12} \tag{2}$$

Proof. Let $f : [\frac{-1}{2}, \frac{1}{2}] \to \mathbb{R}$ be the 1-periodic extension of $f(t) = (\pi t)^2$. Since $f \in \mathcal{C}_1^1(\mathbb{R};\mathbb{R})$, its real Fourier series converges pointwise:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi nt), \text{ for all } t \in \mathbb{R}$$

Note that since f is even, $b_n = 0$ for all n. Compute:

$$
\begin{aligned}
\text{for } n > 0, a_n &= 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)\cos(2\pi n t)\ dt \\
&= 4\int_{0}^{\frac{1}{2}} f(t)\cos(2\pi n t)\ dt \quad (f, \cos\ \text{ even}) \\
&= 4\int_{0}^{\frac{1}{2}} (\pi t)^2\cos(2\pi n t)\ dt \\
&= \frac{1}{2\pi n^3}(2u\cos u + (u^2-2)\sin u)\Big|_0^{\pi n} \\
&= \frac{1}{2\pi n^3}(2\pi n\cdot \underbrace{\cos(\pi n)}_{=(-1)^n} + ((\pi n)^2-2)\cdot \underbrace{\sin(\pi n)}_{=0} - (2\cdot 0\cdot\cos 0 - 2\sin 0)) \\
&= \frac{1}{2\pi n^3}\cdot 2\pi n(-1)^n \\
&= \frac{(-1)^n}{n^2} \\
\text{for } n = 0, a_0 &= 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)\cos(2\pi 0 t)\ dt \\
&= 4\int_{0}^{\frac{1}{2}} (\pi t)^2\ dt \\
&= \frac{4\pi^2 t^3}{3}\Big|_0^{\frac{1}{2}} \\
&= \frac{\pi^2}{6}
\end{aligned}
$$

Thus,

$$f(t) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(2\pi n t)$$

So,

$$
\begin{aligned}
0 &= f(0) \\
&= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(2\pi n 0) \\
&= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}
\end{aligned}
$$

So,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

$\square$

Problem 2:

Theorem 1.1 (Cauchy-Schwarz inequality). *Let $(V, \langle . \,, \, \rangle)$ be a $\mathbb{C}$-IP space. Then, for each $x, y \in V$, one has*

$$|\langle x, y\rangle| \leqslant \|x\| \cdot \|y\| \ , \tag{3}$$

with equality if and only if x and y are linearly dependent.

Proof. We show that the inequality of (3) holds if x and y are linearly independent. Then, we show that equality holds if and only if x and y are linearly dependent.

Let $x, y \in V$ be two linearly independent vectors. Since $y \neq 0$, $\langle y, y\rangle \neq 0$. Define

$$\lambda := \frac{\langle y, x\rangle}{\|y\|^2}$$
$$\bar{\lambda} := \frac{\langle x, y\rangle}{\|y\|^2}$$

Using repeated applications of linearity in the second entry and conjugate symmetry, compute:

$$\begin{aligned}
\|x - \lambda y\|^2 &= \langle x - \lambda y, x - \lambda y\rangle \\
&= \langle x - \lambda y, x\rangle - \lambda\langle x - \lambda y, y\rangle \\
&= \overline{\langle x, x - \lambda y\rangle} - \lambda\overline{\langle y, x - \lambda y\rangle} \\
&= \overline{\langle x, x\rangle - \lambda\langle x, y\rangle} - \lambda(\overline{\langle y, x\rangle - \lambda\langle y, y\rangle}) \\
&= \overline{\langle x, x\rangle} - \bar{\lambda} \cdot \overline{\langle x, y\rangle} - \lambda\overline{\langle y, x\rangle} + \lambda \cdot \bar{\lambda} \cdot \overline{\langle y, y\rangle} \\
&= \langle x, x\rangle - \bar{\lambda} \cdot \overline{\langle x, y\rangle} - \lambda\overline{\langle y, x\rangle} + |\lambda|^2\langle y, y\rangle \\
&= \|x\|^2 - \frac{\langle x, y\rangle\overline{\langle x, y\rangle}}{\|y\|^2} - \frac{\langle y, x\rangle\overline{\langle y, x\rangle}}{\|y\|^2} + \frac{|\langle x, y\rangle|^2}{\|y\|^4}\|y\|^2 \\
&= \|x\|^2 - 2\frac{|\langle x, y\rangle|^2}{\|y\|^2} + \frac{|\langle x, y\rangle|^2}{\|y\|^2} \\
&= \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}
\end{aligned}$$

Since $0 \leqslant \|x - \lambda y\|^2$,

$$\frac{|\langle x, y\rangle|^2}{\|y\|^2} \leqslant \|x\|^2 \iff |\langle x, y\rangle|^2 \leqslant \|x\|^2\|y\|^2$$

Since $\sqrt{}$ is monotonic, we have

$$|\langle x, y\rangle| \leqslant \|x\|\|y\|$$

Let $x, y \in V$ be two linearly dependent vectors with $x = \lambda y$. Then,

$$\begin{aligned}
|\langle x, y\rangle| &= |\langle x, \lambda x\rangle| \\
&= |\lambda\langle x, x\rangle| \\
&= |\lambda| \cdot |\underbrace{\|x\|}_{\geqslant 0}\ \|x\|| \\
&= |\lambda| \cdot \|x\|\ \|x\| \\
&= \|x\|\ \|\lambda x\| \\
&= \|x\|\ \|y\|
\end{aligned}$$

Suppose $|\langle x, y\rangle| = \|x\| \cdot \|y\|$. If $y = 0$, then, x and y are trivially linearly dependent. Otherwise, let $\lambda := \frac{\langle y, x\rangle}{\|y\|^2}$, and continue the prior computation:

$$\begin{aligned}
\|x - \lambda y\|^2 &= \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2} \\
&= \|x\|^2 - \frac{(\|x\|\|y\|)^2}{\|y\|^2} \\
&= \|x\|^2 - \|x\|^2 \\
&= 0
\end{aligned}$$

By the positive definiteness of the induced norm (Problem 3),

$$x - \lambda y = 0 \iff x = \lambda y$$

showing that x and y are linearly dependent. □

Problem 3:

Proposition 1.3. *Let $(V, \langle . \, , \, . \rangle)$ be a $\mathbb{C}$-IP space. Then,*

$$\|x\| := \sqrt{\langle x, x\rangle}\ , \text{ for } x \in V. \tag{4}$$

defines a norm on V.

Proof. We show that $\| \, . \, \|$ satisfies the three norm axioms:

(N-1) Positive definiteness:

By the positive definiteness of inner products, we have

$$\begin{aligned} & 0 = x \\ \iff & 0 = \langle x, x\rangle \\ \iff & 0 = \sqrt{\langle x, x\rangle} = \|x\| \end{aligned}$$

(N-2) Positive homogeneity: For every $x \in V$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned} \|\lambda x\| &= \sqrt{\langle \lambda x, \lambda x\rangle} \\ &= \sqrt{\lambda\langle \lambda x, x\rangle} \\ &= \sqrt{\lambda\overline{\langle x, \lambda x\rangle}} \\ &= \sqrt{\lambda(\overline{\lambda\langle x, x\rangle})} \\ &= \sqrt{\lambda \cdot \bar{\lambda} \cdot \langle x, x\rangle} \\ &= \sqrt{|\lambda|^2\langle x, x\rangle} \\ &= |\lambda|\ \|x\| \end{aligned}$$

(N-3) Triangle inequality: For every $x, y \in V$, one has

□

Problem 4:

(a) For $n \in \mathbb{N}$, let $f_n := \sqrt{g_n}$ where $g_n : [0,1] \to \mathbb{R}$ is given by

$$g_n(x) = \begin{cases} 4n^2 x & , 0 \leqslant x \leqslant \frac{1}{2n} , \\ 4n^2(\frac{1}{n} - x) & , \frac{1}{2n} \leqslant x \leqslant \frac{1}{n} , \\ 0 & , \frac{1}{n} \leqslant x \leqslant 1 , \end{cases}$$

Proposition 1.4. *$f_n \to 0$ point-wise as $n \to \infty$*

Proof. Fix $x \in [0,1]$ and consider two cases. If $x = 0$, then, for all $n \in \mathbb{N}$,

$$f_n(x) = \sqrt{g_n(x)} = \sqrt{4n^2 x} = 0$$

So, $f_n(0) \to 0$. If $x > 0$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < x$. For all $n \geqslant N$,

$$\frac{1}{n} \leqslant \frac{1}{N} < x$$

so,

$$|f_n(x)| = \left|\sqrt{g_n(x)}\right| = \left|\sqrt{0}\right| = 0$$

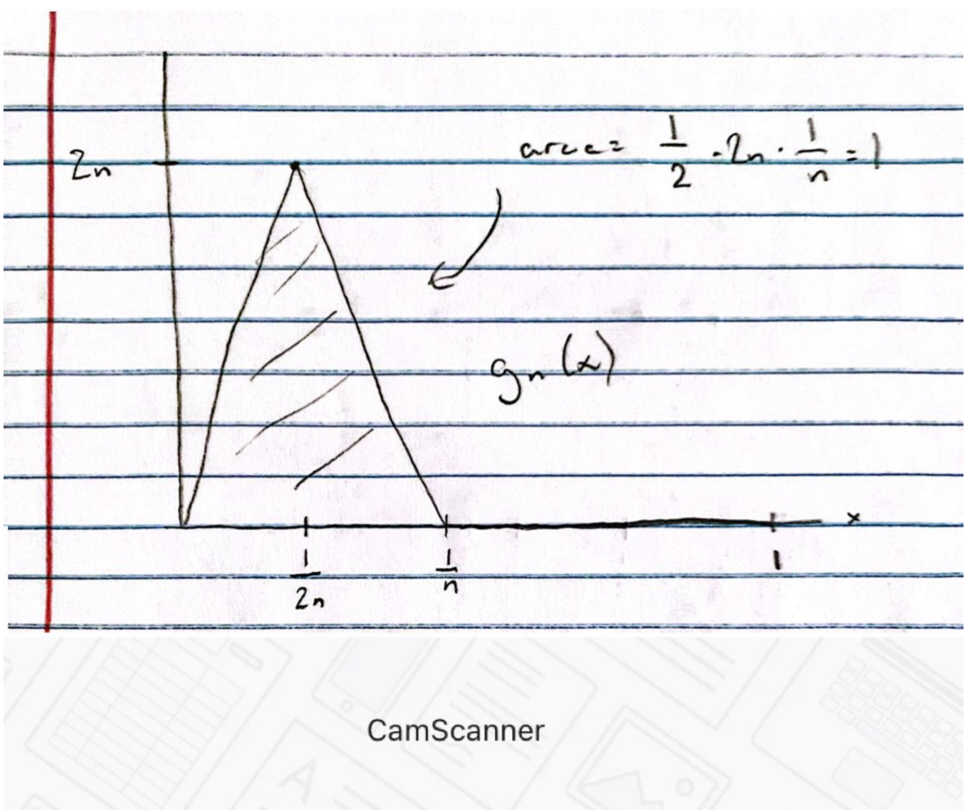
Therefore, $f_n(x) \to 0$ for all $x \in [0,1]$, so $f_n \to 0$ pointwise. □

Proposition 1.5. *For all $n \in \mathbb{N}$, $\|f_n\| = 1$*

Proof. f_n does not converge to 0 in the L^2 norm:

$$\begin{aligned}
\|f_n\|_2^2 &= \langle f_n, f_n \rangle_2 \\
&= \int_0^1 \overline{f_n(t)} f_n(t)\ dt \\
&= \int_0^1 f_n(t) f_n(t)\ dt \\
&= \int_0^1 \left(\sqrt{g_n(t)}\right)^2\ dt \\
&= \int_0^1 g_n(t)\ dt \\
&= 1
\end{aligned}$$

where the integral of g_n was computed geometrically from its graph:



Since $\|f_n\|_2^2 = 1$, $\|f_n\|_2 = 1$ for all n. $\square$

(b) Given $n \in \mathbb{N}$, write n in dyadic form,

$$n = 2^j + k \ , \tag{5}$$

for $j \in \mathbb{N}_0$ and $0 \leqslant k \leqslant 2^j - 1$; note that both j and k in this decomposition are unique! Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = f_{2^j+k}(x) := \begin{cases} 1 & , \text{ if } x \in [\frac{k}{2^j}, \frac{k+1}{2^j}] \ , \\ 0 & , \text{ otherwise} \ . \end{cases}$$

Proposition 1.6. *$\|f_n\|_2 \to 0$ as $n \to \infty$*

Proof.

Lemma 1.2. *$j \to \infty$ as $n \to \infty$*

Proof. Let $M > 0$ be given and pick N such that for all $n \geqslant N$,

$$n \geqslant 2^{M+1}$$

if

$$\frac{n}{2} = \frac{2^j + k}{2} \leqslant \frac{2^j + 2^j}{2} = 2^j$$

Therefore,

$$\begin{aligned} j &= \log_2(2^j) \\ &\geqslant \log_2\left(\frac{n}{2}\right) \quad (\log \text{ is monotonic}) \\ &\geqslant \log_2\left(\frac{2^{M+1}}{2}\right) \\ &= \log_2(2^M) \\ &= M \end{aligned}$$

□

for $n = 2^j + k$,

$$\begin{aligned} \|f_n\|_2^2 &= \int_0^1 \overline{f_n(t)} f_n(t) \ dt \\ &= \int_0^1 f_n(t) f_n(t) \ dt \\ &= \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} 1 \ dt \\ &= \frac{1}{2^j} \end{aligned}$$

Therefore, $f_n \to 0$ in the L^2 sense as $j \to \infty$. As $j \to \infty$ with n, $\|f_n\|_2 \to 0$

□

Proposition 1.7. *For all $x \in [0,1]$, $f_n(x)$ does not converge.*

Proof. Fix $x \in [0,1]$. To show that $f_n(x)$ does not converge, we show that it is not a Cauchy sequence. This follows from the statement: for all $N \in \mathbb{N}$, there exist $n, m \geqslant N$ such that

$$|f_n(x) - f_m(x)| = 1$$

Let $N \in \mathbb{N}$ be arbitrary and break into two cases: if $x = 0$, define

$$\begin{aligned} j_n &= N \\ k_n &= 0 \\ j_m &= N \\ k_m &= 1 \\ n &= 2^{j_n} + k_n = 2^N \geqslant N \\ m &= 2^{j_m} + k_m = 2^N + 1 \geqslant N \end{aligned}$$

We have

$$0 \in \left[0, \frac{1}{2^N}\right]$$

so, $f_n(0) = 1$, but

$$0 \notin \left[\frac{1}{2^N}, \frac{2}{2^N}\right]$$

so $f_m(0) = 0$. Thus,

$$|f_n(x) - f_m(x)| = 1$$

For the second case, we take $x > 0$. Using the Archimedean law consider an N satisfying $1 < 2^N x$.

Let $j_n = N$ and

$$K_n = \{k \in \mathbb{N} : k < 2^N x\}$$

Since $1 \in K_n$ and K_n is a set of integers bounded by $2^N x$, let $k_n = \max K_n$. $1 \in K_n$ implies that $k_n \geqslant 1 \geqslant 0$.

Suppose $k_n = 2^N$. Since $k_n \leqslant 2^N x$, $x = 1$. If $k \in K_n$, then $k \in \mathbb{N}$ and $k < 2^N$. So, $k \leqslant 2^N - 1$. This contradicts that k_n is an upper bound. Therefore, $k_n \neq 2^N$. Since $k \leqslant 2^N$, $k < 2^N$. Now, $k \in K_n$ implies $k \in \mathbb{N}$ so $k \leqslant 2^N - 1$. This shows that

$$n = 2^{j_n} + k$$

is in dyadic form with

$$n \geqslant 2^N \geqslant N$$

There $k \in K_n$ such that $k + 1 \geqslant 2^N x$. Otherwise, for all $l \in K_n$, $l + 1 < 2^N x$. But, then, $l + 1 \in K_n$, contradicting that K_n is bounded and nonempty. Thus, $k_n \geqslant k \geqslant 2^N x - 1$. So,

$$k_n \leqslant 2^{j_n} x \leqslant k_n + 1$$

which implies

$$x \in \left[\frac{k_n}{2^{j_n}}, \frac{k_n + 1}{2^{j_n}}\right] \iff f_n(x) = 1$$

Let $j_m = 2N$ and let $k_m = k_n - 1$. We still have $k_m \leqslant 2^N - 2 \leqslant 2^{2N} - 1 = 2^{j_m} - 1$. Thus,

$$m = 2^{j_m} + k_m$$

is in dyadic form with

$$m \geqslant 2^{2N} \geqslant N$$

We have

$$x \geqslant \frac{k_n}{2^N} = \frac{k_m + 1}{2^N} > \frac{k_m + 1}{2^N}$$

so,

$$x \notin \left[\frac{k_m}{2^{j_m}}, \frac{k_m + 1}{2^{j_m}}\right] \iff f_m(x) = 0$$

Therefore,

$$|f_n(x) - f_m(x)| = |1 - 0| = 1$$

$\square$