ISAAC VIVIANO - PROBLEM SET NO. 1

1. Solutions:

- **Problem 1:**
  (a) Suppose that f is a solution to (1) and let $g(x) = f(k^{\frac{-1}{2}} \cdot x)$. Note that (1) implies $f = \mp kf$. We have

$$g'(x) = \frac{d}{dx} f(k^{\frac{-1}{2}} \cdot x) = k^{\frac{-1}{2}} f'(k^{\frac{-1}{2}} \cdot x)$$
$$g''(x) = \frac{d}{dx} g'(x) = \frac{d}{dx}\left(k^{\frac{-1}{2}} f'(k^{\frac{-1}{2}} \cdot x)\right) = k^{-1} f''\left(k^{\frac{-1}{2}} \cdot x\right)$$

so,

$$\begin{aligned} g''(x) \pm g(x) &= k^{-1} f''(k^{\frac{-1}{2}} x) \pm f(k^{\frac{-1}{2}} x) = k^{-1} \cdot \mp kf(k^{\frac{-1}{2}} x) \pm f(k^{\frac{-1}{2}} x) \\ &= \mp f(k^{\frac{-1}{2}} x) \pm f(k^{\frac{-1}{2}} x) = 0 \end{aligned}$$

Therefore, g is a solution to (3).
Suppose that $g(x) = f(k^{\frac{-1}{2}} x)$ and g is a solution to (3). So, $g = \mp g''$, giving

$$f(k^{\frac{-1}{2}} x) = \mp k^{-1} f''(k^{\frac{-1}{2}} x)$$

Substituting $y = k^{\frac{-1}{2}} x$,

$$f(y) = \mp k^{-1} f''(y) \implies f = \mp k^{-1} f''$$

Then,

$$f'' \pm kf = f'' \pm k \cdot \mp k^{-1} f'' = 0$$

Therefore, f is a solution of (1).

  (b) Let g be a solution of (3) with $g(0) = 0$ and $g'(0) = 0$. Then,

$$\begin{aligned} \frac{d}{dx}\{(g'(x))^2 \pm (g(x))^2\} &= \frac{d}{dx}(g'(x))^2 \pm \frac{d}{dx}(g(x))^2 \\ &= 2g'(x) \cdot g''(x) \pm 2g(x) \cdot g'(x) \\ &= 2g'(x) \cdot \mp g(x) \pm 2g(x) \cdot g'(x) \\ &= \mp 2g'(x)g(x) \pm 2g(x)g'(x) = 0 \end{aligned}$$

By Proposition 1.1 of the DE handout,

$$(g'(x))^2 \pm (g(x))^2 = C$$

for some constant $C \in \mathbb{R}$. Applying the initial conditions:

$$C = (g'(0))^2 \pm (g(0))^2 = 0^2 \pm 0^2 = 0$$

Therefore,

$$(g(x))^2 = \mp (g'(x))^2$$

Since the square of a real number must be positive, $(g(x))^2$ is positive, and

$$(g(x))^2 = (g'(x))^2$$

Taking square roots, we get

$$|g(x)| = |g'(x)|$$

On the set $g(x) \neq 0$, me may divide by $|g(x)|$:

$$1 = \frac{|g'(x)|}{|g(x)|} = \left|\frac{g'(x)}{g(x)}\right|$$

The set O of x for which $g(x) \neq 0$ is open in $\mathbb{R}$: its compliment is $C = g^{-1}(\{0\})$. If x_n is a sequence in C and $x_n \to x$, then $g(x_n) \to g(x)$ by the continuity of g. Additionally, $g(x_n) = 0$ for all n. Therefore, $g(x_n) \to 0 = g(x)$, so $x \in C$. So, C contains all of its limit points and is thus closed. Therefore, O is open and may be written as a countable union of disjoint intervals. Since g and g' are continuous, $\frac{g'(x)}{g(x)}$ is also continuous. So for a particular g, on any one of these intervals (a, b),

$$\frac{g'(x)}{g(x)} = 1$$

or

$$\frac{g'(x)}{g(x)} = -1$$

and on the endpoints, we have $g(a) = g(b) = 0$. For the positive 1 case, the general solution of g is given by $g(x) = Ae^x$ for $A \in \mathbb{R}$. The constraint $g(a) = 0$ gives

$$0 = g(a) = Ae^a$$

since $e^a \neq 0$, $A = 0$ and $g(x)$ is uniformly 0 on (a, b). For the negative 1 case, the general solution of g on (a, b) is given by $g(x) = Ae^{-x}$. Again, the constraint $g(a) = 0$ gives

$$0 = g(a) = Ae^{-a}$$

since $e^{-a} \neq 0$, $A = 0$ and $g(x)$ is uniformly 0 on (a, b). Thus, g is 0 on O. Since g is 0 on the compliment of O by definition, g is 0 on $\mathbb{R}$.

(c) Let $h(x) = g(x) - (\alpha \cos x + \beta \sin x)$. Suppose g satisfies (3) and $g(0) = \alpha$, $g'(0) = \beta$. Then, h has the initial conditions:

$$\begin{aligned}
h(0) &= g(0) - (\alpha \cos 0 + \beta \sin 0) = \alpha - \alpha = 0 \\
h'(x) &= g'(x) - (-\alpha \sin x + \beta \cos x) \\
h'(0) &= g'(0) - (-\alpha \sin 0 + \beta \cos 0) = \beta - \beta = 0 \\
h''(x) &= g''(x) - (-\alpha \cos x - \beta \sin x) = g''(x) + (\alpha \cos x + \beta \sin x)
\end{aligned}$$

The function h also satisfies (3), since

$$\begin{aligned}
h''(x) + h(x) &= g''(x) + (\alpha\cos x - \beta\sin x) + (g(x) - \alpha\cos x + \beta\sin x) \\
&= g''(x) - g(x) + \alpha\cos x - \alpha\cos x - \beta\sin x + \beta\sin x \\
&= g''(x) - g(x) = 0
\end{aligned}$$

since g is a solution to (3). Using (b), $h(x) = 0$ for all x, so

$$g(x) = \alpha\cos x + \beta\sin x$$

(d) Let

$$h(x) = g(x) - \left(\frac{\alpha+\beta}{2}e^{x} + \frac{\alpha-\beta}{2}e^{-x}\right)$$

and suppose g satisfies (3) and $g(0) = \alpha, g'(0) = \beta$. Then,

$$\begin{aligned}
h(0) &= g(0) - \left(\frac{\alpha+\beta}{2}e^{0} + \frac{\alpha-\beta}{2}e^{0}\right) = \alpha - \left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right) \\
&= \alpha - \frac{\alpha+\alpha}{2} = 0 \\
h'(x) &= g'(x) - \left(\frac{\alpha+\beta}{2}e^{x} + \frac{\beta-\alpha}{2}e^{-x}\right) \\
h'(0) &= g'(0) - \left(\frac{\alpha+\beta}{2}e^{0} + \frac{\beta-\alpha}{2}e^{0}\right) = \beta - \left(\frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{2}\right) \\
&= \beta - \frac{\beta+\beta}{2} = 0 \\
h''(x) &= g''(x) - \left(\frac{\alpha+\beta}{2}e^{x} + \frac{\alpha-\beta}{2}e^{-x}\right)
\end{aligned}$$

The function h also satisfies (3), since

$$\begin{aligned}
h''(x) - h(x) &= g''(x) - \left(\frac{\alpha+\beta}{2}e^{x} + \frac{\alpha-\beta}{2}e^{-x}\right) - \left(g(x) - \left(\frac{\alpha+\beta}{2}e^{x} + \frac{\alpha-\beta}{2}e^{-x}\right)\right) \\
&= g''(x) - g(x) - \left(\frac{\alpha+\beta}{2}e^{x} + \frac{\alpha-\beta}{2}e^{-x}\right) + \left(\frac{\alpha+\beta}{2}e^{x} + \frac{\alpha-\beta}{2}e^{-x}\right) \\
&= g''(x) - g(x) = 0
\end{aligned}$$

since g is a solution of (3).

(e) Let $f_1 = \cos\sqrt{k}x$ and $f_2 = \sin\sqrt{k}x$. Then, f_1 and f_2 are solutions to the differential equation:

$$f'' + kf = 0 \tag{1}$$

since

$$\begin{aligned} f_1' &= -\sqrt{k}\sin\sqrt{k}x \\ f_1'' &= -k\cos\sqrt{k}x \\ f_2' &= \sqrt{k}\cos\sqrt{k}x \\ f_2'' &= -k\sin\sqrt{k}x \end{aligned}$$

so

$$\begin{aligned} f_1'' + kf_1 &= -k\cos\sqrt{k}x + k\cos\sqrt{k}x = 0 \\ f_2'' + kf_2 &= -k\sin\sqrt{k}x + k\sin\sqrt{k}x = 0 \end{aligned}$$

Additionally, f_1 and f_2 are linearly independent, so

$$f = Af_1 + Bf_2$$

is the general solution to (1).
Let $f_1 = e^{\sqrt{k}x}$ and $f_2 = e^{-\sqrt{k}x}$. Then, f_1 and f_2 are solutions to the differential equation:

$$f'' - kf = 0 \tag{2}$$

since

$$\begin{aligned} f_1' &= \sqrt{k}e^{\sqrt{k}x} \\ f_1'' &= ke^{\sqrt{k}x} \\ f_2' &= -\sqrt{k}e^{-\sqrt{k}x} \\ f_2'' &= ke^{-\sqrt{k}x} \end{aligned}$$

so

$$\begin{aligned} f_1'' + kf_1 &= ke^{\sqrt{k}x} - ke^{\sqrt{k}x} = 0 \\ f_2'' + kf_2 &= ke^{-\sqrt{k}x} - ke^{-\sqrt{k}x} = 0 \end{aligned}$$

Additionally, f_1 and f_2 are linearly independent, so

$$f = Af_1 + Bf_2$$

is the general solution to (2).

We may write

$$f'' \pm kf = 0 \tag{3}$$

as

$$|f''| = k|f|$$

Let f be any solution to (3). On $O = \{x : f(x) \neq 0\}$, this may be written

$$\left|\frac{f''}{f}\right| = k$$

Write O as a countable union of disjoint open intervals. On any such interval (a, b). If f'' changed sign f would as well. Therefore,

$$k = \left|\frac{f''}{f}\right| = \begin{cases} \frac{f''}{f} & \text{if } f > 0 \text{ and } f'' > 0 \\ -\frac{f''}{f} & \text{otherwise} \end{cases}$$

These cases show that f satisfies either (1) or (2) on (a, b). If f is not constant and

$$f = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$$

then, by the continuity of f, $a = -\infty$ and $b = \infty$ (otherwise, $f(a) = 0, f(b) = 0$). If f is not constant and

$$f = A\cos\sqrt{k}x + B\sin\sqrt{k}x$$

then by the differentiability of f, $a = -\infty$ and $b = \infty$. Otherwise, we have an interval $[b, b + \epsilon]$ where f is uniformly 0. Since f is differentiable at b, A and B would have to be 0. This argument shows that if f is a solution to (3), it must be of the form (11) on all of $\mathbb{R}$.

- **Problem 2:**

(a) Fix $a \in \mathbb{R}$. Let $\alpha = \sin a$, $\beta = \cos a$. We have

$$\begin{aligned} g(x) &= \sin(a + x) \\ g'(x) &= \cos(a + x) \\ g''(x) &= -\sin(a + x) \end{aligned}$$

So, $g(0) = \alpha$ and $g'(0) = \beta$. Additionally,

$$g''(x) + g(x) = -\sin(a + x) + \sin(a + x) = 0$$

so, g is a solution to the IVP (6). This implies that

$$g(x) = \alpha\cos x + \beta\sin x$$

by (c). Substituting the values of α and β, we arrive at the familiar formula:

$$\sin(a + x) = \sin a\cos x + \cos a\sin x$$

Now let $\alpha = \cos a$ and $\beta = -\sin a$. For $g(x) = \cos(a + x)$,

$$\begin{aligned} g(x) &= \cos(a + x) \\ g'(x) &= -\sin(a + x) \\ g''(x) &= -\cos(a + x) \end{aligned}$$

So, $g(0) = \alpha$ and $g'(0) = \beta$ again. We still have

$$g''(x) + g(x) = -\cos(a + x) + \cos(a + x) = 0$$

so, g is a solution to the IVP (6). This implies that

$$g(x) = \alpha \cos x + \beta \sin x$$

by (c). Substituting the values of α and β, we arrive at the familiar formula:

$$\cos(a + x) = \cos a \cos x - \sin a \sin x$$

(b) Let $z = a + ib$, $w = c + ib$.

$$\begin{aligned}
e^{z+w} &= e^{a+ib+c+ib} = e^{a+c+i(b+d)} = e^{a+c}e^{i(b+d)} \\
&= e^{a+c}(\cos(b + d) + i \sin(b + d)) \\
&= e^{a+c}(\cos b \cos d - \sin b \sin d + i(\sin b \cos d + \sin d \cos b)) \\
&= e^{a+c}(\cos b \cos d + i \sin b \cos d + i \sin d \cos b - \sin b \sin d) \\
&= e^{a+c}(\cos b \cos d + i \cos b \sin d + i \sin b \cos d + i^2 \sin b \sin d) \\
&= e^{a+c}(\cos b + i \sin b)(\cos d + i \sin d) \\
&= e^{a}(\cos b + i \sin b)e^{c}(\cos d + i \sin d) \\
&= e^{a}e^{-b}e^{c}e^{id} \\
&= e^{z}e^{w}
\end{aligned}$$

- **Problem 3: Complex numbers basics**

(a) (i) Let $z = a + ib$. Then, Re $z = a$ and

$$\frac{1}{2}(z + \bar{z}) = \frac{1}{2}(a + ib + a - ib) = \frac{1}{2}2a = a$$

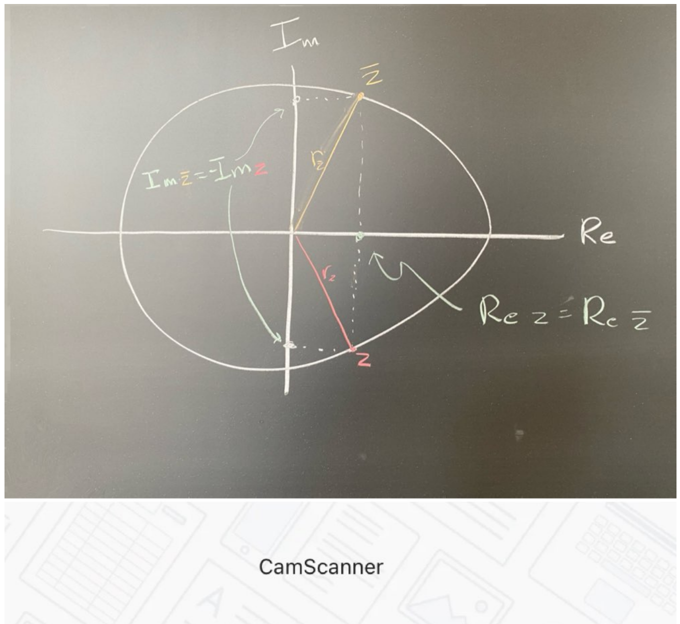
(ii) Let $z = a + ib$. Then, Im $z = b$ and

$$\frac{1}{2i}(z - \bar{z}) = \frac{1}{2i}(a + ib - (a - ib)) = \frac{1}{2i}(a - a + ib + ib) = \frac{1}{2i}(2ib) = b$$

(iii) Let $z = a + ib$. Then, $|z|^2 = a^2 + b^2$ and

$$z \cdot \bar{z} = (a + ib)(a - ib) = a^2 - i^2b^2 = a^2 + b^2$$

(iv) Let $z = a + ib$. Then, $|z| = \sqrt{a^2 + b^2}$ and $|\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$. Since z and $\bar{z}$ have the same real coordinate and opposite imaginary coordinates, they both lie on a circle centered at the origin:



(b) (i)

$$\frac{i^2}{i^3-4i+6} = \frac{-1}{-i-4i+6} = \frac{-1}{6-5i}\cdot\frac{6+5i}{6+5i} = \frac{-6-5i}{36-25i^2} = \frac{-6-5i}{61}$$

so, the real part is $\frac{-6}{61}$ and the [[Imaginary Part]] is $\frac{-5}{61}$

(ii)

$$e^{4(2+\sqrt{2}i)t} = e^{8t}e^{4\sqrt{2}it} = e^{8t}(\cos 4\sqrt{2}t + i\sin 4\sqrt{2}t)$$

so, the [[Real Part]] is $e^{8t}\cos 4\sqrt{2}t$ and the [[Imaginary Part]] is $e^{8t}\sin 4\sqrt{2}t$

(c) (i)

$$|(2-i)^2\cdot(4+6i)| = |2-i|^2|4+6i| = (4+1)\sqrt{16+36} = 5\sqrt{52}$$

(ii)

$$\left|\left(\frac{i+2}{i-2}\right)^{57}\right| = \left(\frac{|i+2|}{|i-2|}\right)^{57} = \left(\frac{1+4}{1+4}\right)^{57} = 1$$

(iii)

$$|(2+3i)e^{2+i}| = |2+3i||e^{2+i}| = \sqrt{4+9}e^2 = e^2\sqrt{13}$$

- **Problem 4:**

(a) Let $P(n)$ be

$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$$

Base case: $n = 0$

$$\sum_{k=0}^{0} z^k = z^0 = 1$$

and

$$\frac{1 - z^{0+1}}{1 - z} = \frac{1 - z}{1 - z} = 1$$

since $z \neq 1$. Therefore, $P(0)$ holds.
Inductive Step: Let $n \geqslant 0$ be given and assume $P(n)$ holds. Then,

$$\begin{aligned}\sum_{k=0}^{n+1} z^k = \sum_{k=0}^{n} z^k + z^{n+1} = \frac{1 - z^{n+1}}{1 - z} + z^{n+1} = \frac{1 - z^{n+1} + z^{n+1}(1 - z)}{1 - z} \\ = \frac{1 - z^{n+1} + z^{n+1} - z^{n+2}}{1 - z} = \frac{1 - z^{n+2}}{1 - z}\end{aligned}$$

Therefore, $P(n + 1)$ holds.
So by (PMI), $P(n)$ for all $n \in \mathbb{N}$.

(b) Goal:

$$\sum_{k=1}^{n} \sin kt = \frac{\sin \frac{1}{2}(n + 1)t \cdot \sin \frac{1}{2}nt}{\sin \frac{1}{2}t}$$

Note that

$$\sin \theta = \operatorname{Im}(e^{i\theta})$$

So,

$$\sum_{k=1}^{n} \sin kt = \sum_{k=1}^{n} \operatorname{Im}(e^{ikt}) = \operatorname{Im}\left(\sum_{k=1}^{n} e^{ikt}\right) = \operatorname{Im}\left(\sum_{k=1}^{n} e^{ikt} + 1\right)$$

and,

$$\sum_{k=1}^{n} e^{ikt} + 1 = \sum_{k=0}^{n} (e^{it})^k = \frac{1-(e^{it})^{n+1}}{1-e^{it}} \quad \text{(by part a)}$$

$$= \frac{e^{it\frac{n+1}{2}}\left(e^{-it\frac{n+1}{2}} - e^{it\frac{n+1}{2}}\right)}{1-e^{it}}$$

$$= \frac{e^{it\frac{n+1}{2}}\left(-2i\sin t\frac{n+1}{2}\right)}{1-e^{it}} \cdot \frac{1-e^{-it}}{1-e^{-it}}$$

$$= \frac{-2i\sin t\frac{n+1}{2}(e^{it\frac{n+1}{2}} - e^{it\frac{n-1}{2}})}{1-e^{-it}-e^{it}+1}$$

$$= \frac{-2i\sin t\frac{n+1}{2}(e^{it\frac{n+1}{2}} - e^{it\frac{n-1}{2}})}{2-2\cos t}$$

$$= \frac{i\sin t\frac{n+1}{2}\left(e^{it\frac{n+1}{2}} - e^{it\frac{n-1}{2}}\right)}{\cos t - 1}$$

and

$$e^{it\frac{n+1}{2}} - e^{it\frac{n-1}{2}} = \cos t\frac{n+1}{2} + i\sin t\frac{n-1}{2} - \cos t\frac{n-1}{2} - i\sin t\frac{n-1}{2}$$

$$\begin{aligned}
\cos t\frac{n+1}{2} - \cos t\frac{n-1}{2} &= \cos\frac{tn}{2}\cos\frac{t}{2} - \sin\frac{tn}{2}\sin\frac{t}{2} - \cos\frac{tn}{2}\cos\frac{-t}{2} \\
&\quad + \sin\frac{tn}{2}\sin\frac{-t}{2} \\
&= \cos\frac{tn}{2}\cos\frac{t}{2} - \cos\frac{tn}{2}\cos\frac{t}{2} \\
&\quad - \sin\frac{tn}{2}\sin\frac{t}{2} - \sin\frac{tn}{2}\sin\frac{t}{2} \\
&= -2\sin\frac{tn}{2}\sin\frac{t}{2} \\
\cos t - 1 &= -2\sin^2\frac{t}{2}
\end{aligned}$$

So,

$$\operatorname{Im}\left(\sum_{k=1}^{n} e^{ikt}\right) = \frac{\sin t\frac{n+1}{2}\cdot -2\sin\frac{tn}{2}\sin\frac{t}{2}}{-2\sin^2\frac{t}{2}}$$

$$= \frac{\sin\frac{1}{2}t(n+1)\sin\frac{1}{2}tn}{\sin\frac{1}{2}t}$$

- **Problem 5:**
  (a)

Lemma 1.1. *For all $x, y \in \mathbb{R}$,*

$$x^2 \leqslant y^2 \implies |x| \leqslant |y|$$

Proof: Since $x^2 = |x|^2$,

$$0 \leqslant y^2 - x^2 = (|y| - |x|)(|y| + |x|)$$

so,

$$0 \leqslant |y| - |x| \implies |y| \geqslant |x|$$

Triangle Inequality:

$$|z + w|^2 = (z + w) \cdot (\overline{z + w})$$

Let $z = a + ib$ and $w = c + id$. We have the following:

$$\begin{aligned} z + w &= (a + c) + i(b + d) \\ \overline{z + w} &= (a + c) - i(b + d) \end{aligned}$$

So,

$$\begin{aligned} |z + w|^2 &= (a + c)^2 - i^2(b + d)^2 \\ &= a^2 + 2ac + c^2 + b^2 + 2bd + d^2 \\ &= a^2 + b^2 + c^2 + d^2 + 2ac + 2bd \end{aligned}$$

And,

$$\begin{aligned} (|z| + |w|)^2 &= |z|^2 + 2|z||w| + |w|^2 \\ &= a^2 + b^2 + c^2 + d^2 + 2|z||w| \end{aligned}$$

If $2|z||w| \geqslant 2ac + 2bd$, then we are done.

$$\begin{aligned} |z|^2|w|^2 &= (a^2 + b^2)(c^2 + d^2) \\ &= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \\ (ac + bd)^2 &= a^2c^2 + 2acbd + b^2d^2 \end{aligned}$$

Taking the difference, the desired inequality is:

$$a^2d^2 + b^2c^2 \geqslant 2adbc$$

which may be rewritten:

$$a^2d^2 - 2adbc + b^2c^2 \geqslant 0$$

and simplified to

$$(ad - bc)^2 \geqslant 0$$

This is true, since the square of any real number is nonnegative. We conclude that

$$|z + w|^2 \leqslant (|z| + |w|)^2$$

So by Lemma 1.1, (2.37) holds.