# Math 357 / Spring 2024 (C. Marx) - Final Exam (**problem-based portion**)

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# 1 Problems

## Problem 1

**Theorem 1.1** (Poisson Formula)**.** *For all Schwartz functions $f \in \mathcal{S}(\mathbb{R};\mathbb{C})$,*

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{n=-\infty}^{+\infty} \widehat{f}(n)\ . \tag{1}$$

(a)

**Lemma 1.2.** *For all Schwartz functions $g \in \mathcal{S}(\mathbb{R};\mathbb{C})$, the series*

$$\sum_{n=0}^{\infty} g(n) \tag{2}$$

*converges.*

*Proof.* Let $g \in \mathcal{S}(\mathbb{R};\mathbb{C})$. Pick $C \in \mathbb{R}$ such that

$$|g(t)| \leq \frac{C}{|t|^2},\ \text{for all } t \neq 0$$

In particular, we have

$$|g(n)| \leq \frac{C}{n^2},\ \text{for all } n \in \mathbb{N}$$

$\frac{C}{n^2}$ is a $p$-series with $p = 2 > 1$, so it converges. By the comparison test (Abott Theorem 2.7.4), $g(n)$ is absolutely summable. Therefore, the sum in (2) converges. $\square$

*Remark* 1.3. Lemma 1.2 immediately implies that both sides of (1) are summable. Let $f \in \mathcal{S}(\mathbb{R};\mathbb{C})$ and define

$$f^{(-)} : \mathbb{R} \to \mathbb{C},\ f^{(-)}(t) = f(-t)$$

Note that $f^{(-)} \in \mathcal{S}(\mathbb{R};\mathbb{C})$:

$$\left| t^m \cdot \frac{d^n f^{(-)}}{dt^n}(t) \right| = \left| t^m \cdot (-1)^n \frac{d^n f}{dt^n}(t) \right| < \infty$$

Thus, by Lemma 1.2, both of the series

$$\sum_{n=0}^{\infty} f(n),\ \sum_{n=1}^{\infty} f^{(-)}(n)$$

converge. Since

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=0}^{\infty} f(n) + \sum_{n=1}^{\infty} f^{(-)}(n)$$

we have the double sided series on the left side of (1) converges. For the right side, we note that $f \in \mathcal{S}(\mathbb{R};\mathbb{C})$ implies $\widehat{f} \in \mathcal{S}(\mathbb{R};\mathbb{C})$ (Fourier transform is a linear map on the class of Schwartz functions).

(b)

**Lemma 1.4.** *Let*

$$f_n : \mathbb{R} \to \mathbb{C},\ f_n(t) = f(t+n), \quad \textit{for all } t \in \mathbb{R}, n \in \mathbb{Z} \tag{3}$$

*and define the auxiliary function*

$$g : \mathbb{R} \to \mathbb{C}\ ,\ g(t) = \sum_{n=-\infty}^{+\infty} f_n(t)\ . \tag{4}$$

*Then,*

1. *The series of functions $f_n$ in (4) converges uniformly to $g$ on all compact subsets of $\mathbb{R}$.*
2. *$g$ is a 1-periodic function whose Fourier coefficients are given by*

$$\widehat{g}_n = \widehat{f}(n)\ ,\ \textit{for all } n \in \mathbb{Z}\ . \tag{5}$$

*Proof.* 1. Let $K \subseteq \mathbb{R}$ be compact. We will to show that

$$\sum_{n=-\infty}^{\infty} \|f_n\|_{\infty;K} = \sum_{n=0}^{\infty} \|f_n\|_{\infty;K} + \sum_{n=1}^{\infty} \|f_{-n}\|_{\infty;K} \tag{6}$$

converges. We first show the first term on the right-hand side is summable. Let $t_0 = \min(K)$ by (EVT). Let $N \in \mathbb{N}$ such that $N > |t_0| + 1$ by (A). Since $f$ is a Schwartz function, select $C \in \mathbb{R}$ with

$$|f(t)| \leq \frac{C}{|t|^2},\ \text{for all } t \neq 0$$

If $n \geq N$,

$$\begin{aligned}
\|f_n\|_{\infty;K} &= \sup_{t\in K} |f(t+n)| \\
&\leq \sup_{t\in K} \frac{C}{|\underbrace{t+n}_{>0}|^2} \\
&= \sup_{t\in K} \frac{C}{(t+n)^2} \\
&= \frac{C}{(t_0+n)^2} \\
&< \frac{C}{(n-N+1)^2}
\end{aligned}$$

Thus, we have $\|f_n\|_{\infty;K}$ for $n \geq N$ is bounded by the terms of a convergent $p$ series for $p = 2 > 1$. Thus,

$$\sum_{n=0}^{\infty} \|f_n\|_{\infty;K} = \sum_{n=0}^{N-1} \|f_n\|_{\infty;K} + \sum_{n=N}^{\infty} \|f_n\|_{\infty;K}$$

converges.

In complete analogy, we can show the second series converges. Let $t_0 = \max(K)$ by (EVT). Let $N \in \mathbb{N}$ such that $N > |t_0| + 1$ by (A). Since $f$ is a Schwartz function, select $C \in \mathbb{R}$ with

$$|f(t)| \leq \frac{C}{|t|^2}, \text{ for all } t \neq 0$$

If $n \geq N$,

$$\begin{aligned}
\|f_{-n}\|_{\infty;K} &= \sup_{t\in K} |f(t-n)| \\
&\leq \sup_{t\in K} \frac{C}{|\underbrace{t-n}_{<0}|^2} \\
&= \sup_{t\in K} \frac{C}{(n-t)^2} \\
&= \frac{C}{(n-t_0)^2} \\
&< \frac{C}{(n-N+1)^2}
\end{aligned}$$

Thus, we have $\|f_{-n}\|_{\infty;K}$ for $n \geq N$ is bounded by the terms of a convergent $p$ series for $p = 2 > 1$. Thus,

$$\sum_{n=1}^{\infty} \|f_{n-}\|_{\infty;K} = \sum_{n=1}^{N-1} \|f_{n-}\|_{\infty;K} + \sum_{n=N}^{\infty} \|f_{-n}\|_{\infty;K}$$

converges. So, both terms of (6) converge. Therefore, the series

$$\sum_{n=-\infty}^{\infty} f_n$$

converges uniformly to $g$ by the Weierstrass $M$-test.

2. 1-periodic:
   For each $t \in \mathbb{R}$:

$$\begin{aligned}
g(t+1) &= \sum_{n=-\infty}^{\infty} f(t+1+n) && (7)\\
&= \sum_{n=1}^{\infty} f(t+1-n) + \sum_{n=0}^{\infty} f(t+1+n) && (8)\\
&= \sum_{n=0}^{\infty} f(t-n) + \sum_{n=1}^{\infty} f(t+n) \quad \text{(reindex)} && (9)\\
&= \sum_{n=-\infty}^{\infty} f(t+n) && (10)\\
&= g(t) && (11)
\end{aligned}$$

We justify the reindexing of the infinite sum in (9) by:

$$
\begin{aligned}
\sum_{n=1}^{\infty} f(t+1-n) &= \lim_{n\to\infty} \sum_{k=1}^{n} f(t+1-k) \\
&= \lim_{n\to\infty} \sum_{j=0}^{n-1} f(t-j) \quad \text{(reindex: } j = k-1\text{)} \\
&= \sum_{n=0}^{\infty} f(t-n)
\end{aligned}
$$

and similarly for the second term of (9).

Recall the following theorem from analysis about interchanging limits and integrals:

**Theorem 1.5** (Abott Theorem 7.4.4)**.** *If the sequence of integrable functions $f_n \to f$ uniformly on $[a,b] \subseteq \mathbb{R}$, then $f$ is integrable with*

$$
\lim_{n\to\infty} \int_a^b f_n(x)\ dx = \int_a^b f(x)\ dx
$$

We can easily extend this to exchanging series and integrals by the linearity of the integral:

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} \int_a^b f_n(x)\ dx &= \lim_{N\to\infty} \sum_{n=0}^{N} \int_a^b f_n(x) dx + \lim_{N\to\infty} \sum_{n=1}^{N} \int_a^b f_{-n}(x) dx \\
&= \lim_{N\to\infty} \int_a^b \left( \sum_{n=0}^{N} f_n(x) \right)\ dx + \lim_{N\to\infty} \int_a^b \left( \sum_{n=1}^{N} f_{-n}(x) \right)\ dx \\
&= \int_a^b \left( \lim_{N\to\infty} \sum_{n=0}^{n} f_n(x) \right) dx + \int_a^b \left( \lim_{N\to\infty} \sum_{n=1}^{n} f_{-n}(x) \right) dx \\
&= \int_a^b \sum_{n=0}^{\infty} f_n(x)\ dx + \int_a^b \sum_{n=1}^{\infty} f_{-n}(x)\ dx \\
&= \int_a^b \sum_{n=-\infty}^{\infty} f_n(x)\ dx
\end{aligned}
$$

where the swap is allowed when the series convergence is uniform.

Note that by Lemma 1.6, $g$ is $\mathcal{C}^\infty$. In particular $g$ is Riemann integrable, so the Fourier coefficients are well-defined. Using the uniform convergence of part 1, we apply Theorem 1.5 in (15) to compute the Fourier coefficients of $g$:

$$
\begin{aligned}
\hat{g}_n &= \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t)\ e^{-2\pi i n t}\ dt && (12)\\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \sum_{k=-\infty}^{\infty} f(t+k) \right)\ e^{-2\pi i n t}\ dt && (13)\\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \sum_{k=-\infty}^{\infty} f(t+k)\ e^{-2\pi i n t} \right)\ dt && (14)\\
&= \sum_{k=-\infty}^{\infty} \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t+k)\ e^{-2\pi i n t}\ dt \right) && (15)\\
&= \sum_{k=-\infty}^{\infty} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f(s)\ e^{-2\pi i n s} \cdot \underbrace{e^{2\pi i n k}}_{=1}\ ds && (16)\\
&= \lim_{N\to\infty} \sum_{k=0}^{N} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f(s)\ e^{-2\pi i n s}\ ds + \lim_{N\to\infty} \sum_{k=1}^{N} \int_{-k-\frac{1}{2}}^{-k+\frac{1}{2}} f(s)\ e^{-2\pi i n s}\ ds && (17)\\
&= \lim_{N\to\infty} \int_{-\frac{1}{2}}^{N+\frac{1}{2}} f(s)\ e^{-2\pi i n s}\ ds + \lim_{N\to\infty} \int_{-N-\frac{1}{2}}^{-\frac{1}{2}} f(s)\ e^{-2\pi i n s}\ ds && (18)\\
&= \int_{-\frac{1}{2}}^{\infty} f(s)\ e^{-2\pi i n s}\ ds + \int_{-\infty}^{-\frac{1}{2}} f(s)\ e^{-2\pi i n s}\ ds && (19)\\
&= \int_{-\infty}^{\infty} f(s)\ e^{-2\pi i n s}\ ds && (20)\\
&= \widehat{f}(n) && (21)
\end{aligned}
$$

where $e^{2\pi i n k} = 1$ in (16) since $nk \in \mathbb{Z}$ and $e^{2\pi i x} = 1 \iff x \in \mathbb{Z}$.

□

(c)

**Lemma 1.6.** *The function $g$ defined by (4) is $\mathcal{C}^\infty$.*

*Proof.* We verify the hypotheses of Set 7, problem 3b:

1. We need that the series $\sum_{n=-\infty}^{\infty} f_n(t)$ converges for at least one point $x \in \mathbb{R}$. This follows from the uniform convergence of Part 1.
2. We need that for all compact sets $K$, the series of $k$-th derivatives is infinity norm summable on $K$:

$$0 \leq \sum_{n=-\infty}^{\infty} \|f_n^{(k)}\|_{\infty;K} < \infty \tag{22}$$

Since $f$ is a Schwartz function, all of its derivatives are Schwartz functions (Set 7, problem 4a). By the chain rule,

$$f_n^{(k)}(t) = f^{(k)}(t+n)$$

So, for all $k \in \mathbb{N}$, (22) follows immediately from part 1 of Lemma 1.4.

Thus, the interchanging derivatives and series theorem implies that $g \in \mathcal{C}^\infty$. $\square$

(d) *Proof of Theorem 1.1:* By the Fourier series inversion property for $\mathcal{C}_1^2$ functions (Proposition 5.2.2 of [CM]), the Fourier series of $g$ converges uniformly to $g$. So for all $t \in \mathbb{R}$,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} f(t+n) = g(t)& \\ &= \sum_{n=-\infty}^{\infty} \hat{g}_n \; e^{2\pi int} \\ &= \sum_{n=-\infty}^{\infty} \widehat{f}(n) \; e^{-2\pi int}\end{aligned}$$

The desired equality of (1) follows from the $t = 0$ case.

$\square$

## Problem 2

**Proposition 1.1.** *For each Schwartz function $f \in \mathcal{S}(\mathbb{R};\mathbb{C})$,*

$$\sup_{t\in\mathbb{R}} |f(t)| \leq C \max\{\|f\|_{\mathcal{L}^2} \;,\; \|f'\|_{\mathcal{L}^2}\} \;. \tag{23}$$

*for some constant $C > 0$.*

*Proof.* First note that for all $a, b \in \mathbb{R}$,

$$(a+b)^2 \leq 4\max\{a,b\}^2 = 4\max\{a^2,b^2\} \tag{24}$$

Without loss of generality, assume $a \geq b$:

$$(a+b)^2 = a^2 + 2ab + b^2 \leq a^2 + 2a \cdot a + a^2 = 4a^2$$

If $b \geq a$, we get the other bound.

Fix a $t \in \mathbb{R}$. By the Fourier Transform inversion property, we can write

$$f(t) = \int_{-\infty}^{\infty} \widehat{f}(\nu) \; e^{2\pi i\nu t} \; d\nu \tag{25}$$

$$= \int_{-\infty}^{\infty} \underbrace{\frac{1}{1+|\nu|}}_{=\overline{1/1+|\nu|}} \cdot (1+|\nu|)\widehat{f}(\nu) e^{2\pi i\nu t} \; d\nu \tag{26}$$

$$= \left\langle \frac{1}{1+|\nu|}, (1+|\nu|) \cdot \widehat{f}(\nu) \; e^{2\pi i\nu t} \right\rangle_{\mathcal{L}^2} \tag{27}$$

where we can take the $\mathcal{L}^2$ inner product of the Schwartz function $(1+|\nu|)\cdot\widehat{f}(\nu)\ e^{2\pi i\nu t}$ and the square integrable function $\frac{1}{1+|\nu|}$:

$$\begin{aligned}
\left\|\frac{1}{1+|\nu|}\right\|_{\mathcal{L}^2}^2 &= \int_{-\infty}^{\infty}\left|\frac{1}{1+|\nu|}\right|^2\ d\nu &(28)\\
&= \int_{-\infty}^{\infty}\frac{1}{(1+|\nu|)^2}\ d\nu &(29)\\
&= 2\int_{0}^{\infty}\frac{1}{(1+\nu)^2}\ d\nu &(30)\\
&= 2\lim_{R\to\infty}\left(\frac{-1}{1+\nu}\bigg|_0^R\right) &(31)\\
&= 2 &(32)
\end{aligned}$$

where (29) uses the even symmetry of the integrand to remove the absolute value. Applying the Cauchy Schwarz inequality to the inner product in (27), we have

$$\left|\left\langle\frac{1}{1+|\nu|},(1+|\nu|)\cdot\widehat{f}(\nu)\ e^{2\pi i\nu t}\right\rangle_{\mathcal{L}^2}\right| \le \left\|\frac{1}{1+|\nu|}\right\|_{\mathcal{L}^2}\cdot\left\|(1+|\nu|)\cdot\widehat{f}(\nu)\ e^{2\pi i\nu t}\right\|_{\mathcal{L}^2}$$

We compute the second norm factor:

$$\begin{aligned}
\left\|(1+|\nu|)\cdot\widehat{f}(\nu)\ e^{2\pi i\nu t}\right\|_{\mathcal{L}^2}^2 &= \int_{-\infty}^{\infty}\left|\underbrace{(1+|\nu|)}_{>0}\cdot\widehat{f}(\nu)\ e^{2\pi i\nu t}\right|^2\ d\nu\\
&= \int_{-\infty}^{\infty}\underbrace{(1+|\nu|)^2}_{\le 4\max\{1,|\nu|^2\}}\cdot\left|\widehat{f}(\nu)\right|^2\cdot\underbrace{\left|e^{2\pi i\nu t}\right|^2}_{=1}\ d\nu\\
&\le 4\max\left\{\int_{-\infty}^{\infty}\left|\widehat{f}(\nu)\right|^2\ d\nu,\int_{-\infty}^{\infty}\left|\nu\cdot\widehat{f}(\nu)\right|^2\ d\nu,\right\} \quad (24)
\end{aligned}$$

The first argument of the max is handled with Plancherel's identity for the Fourier Transform:

$$\int_{-\infty}^{\infty}\left|\widehat{f}(\nu)\right|^2\ d\nu = \int_{-\infty}^{\infty}|f(t)|^2\ dt = \|f\|_{\mathcal{L}^2}^2$$

For the second term, we must first apply the Fourier Differentiation Mantra:

$$\int_{-\infty}^{\infty}\underbrace{\left|\nu\cdot\widehat{f}(\nu)\right|}_{=\left|\frac{\widehat{f'}(\nu)}{2\pi i}\right|}{}^2\ d\nu = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widehat{f'}(\nu)\right|^2\ d\nu = \frac{1}{2\pi}\|f'\|_{\mathcal{L}^2}^2$$

where the second equality is Plancherel's identity for $f'$. We have shown

$$|f(t)|^2 \le 8\max\left\{\|f\|_{\mathcal{L}^2}^2,\frac{1}{2\pi}\|f'\|_{\mathcal{L}^2}^2\right\}$$

Since $\sqrt{\ }$ is monotonic, taking $C=\sqrt{8}$ gives the desired relation (23).

$\square$