PROBLEM SET NO. 4 - ISAAC VIVIVANO

I affirm that I have adhered to the Honor Code on this assignment.
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1. Solutions:

- **Problem 1 - Riemann-Lebesgue Lemma:**
  (a) **Proof of Riemann-Lebesgue for continuous functions:**

Lemma 1.1. *For all $f \in \mathcal{T}$, if $N-1$ is the degree of f, for all $n \geqslant N$,*

$$|\widehat{f}_n| = 0$$

Proof. Let

$$f(t) = \sum_{k=-N}^{N} c_k e^{2\pi i k t}$$

and let $\epsilon > 0$ be given.

$$\begin{aligned}
\hat{f}_n &= \int_0^1 \left(\sum_{k=-N}^{N} c_k e_k(t) \right) \overline{e_n(t)} \ dt \\
&= \int_0^1 \sum_{k=-N}^{N} (c_k e_k(t) \overline{e_n(t)}) \ dt \\
&= \sum_{k=-N}^{N} c_k \int_0^1 e_k(t) \overline{e_n(t)} \ dt \\
&= \sum_{k=-N}^{N} c_k \cdot \delta_{n,k} \\
&= \begin{cases} c_n & \text{if } |n| \leqslant N \\ 0 & \text{if } |n| > N \end{cases}
\end{aligned}$$

Therefore, for all $n \geqslant N+1$ and $n \leqslant -N-1$,

$$|\hat{f}_n| = |0| = 0$$

□

Proposition 1.1. *For all $f \in \mathcal{C}(\mathbb{R};\mathbb{C})$,*

$$|\widehat{f}_n| \to 0 \ , \ as \ |n| \to \infty \ .$$

Proof. Let $f \in \mathcal{C}_1(\mathbb{R};\mathbb{C})$. Let $\epsilon > 0$ be given. With the Weierstrass approximation theorem, pick $p \in \mathcal{T}$ such that

$$\|f - p\|_\infty < \epsilon$$

From Set 2,

$$\left| [\widehat{f - p}]_n \right| \leqslant \|f - p\|_\infty < \epsilon$$

And,

$$\begin{aligned}
\widehat{[f-p]}_n &= \int_0^1 (f(t)-p(t))\overline{e_n(t)}\ dt \\
&= \int_0^1 f(t)\overline{e_n(t)}\ dt - \int_0^1 p(t)\overline{e_n(t)}\ dt \\
&= \hat{f}_n - \hat{p}_n
\end{aligned}$$

Let $N-1$ be the degree of p. Then, for all $|n| \geqslant N$,

$$\begin{aligned}
\left|\hat{f}_n\right| &= |\hat{f}_n - \hat{p}_n + \hat{p}_n| \\
&\leqslant \left|\hat{f}_n - \hat{p}_n\right| + \underbrace{|\hat{p}_n|}_{=0} \\
&= \left|\widehat{[f-p]}_n\right| \\
&< \epsilon
\end{aligned}$$

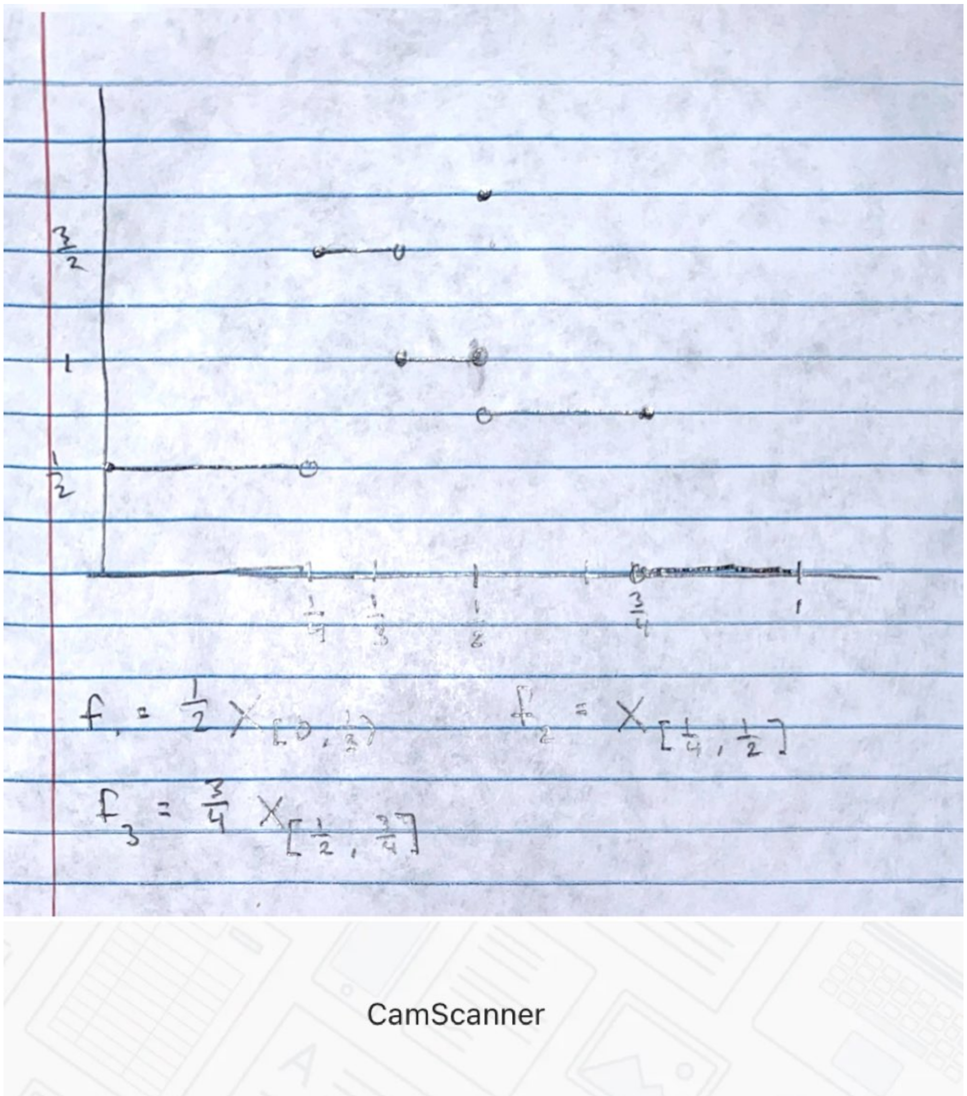
So,

$$|\hat{f}_n| \to 0, \text{ as } |n| \to \infty$$

$\square$

(b) **Proof of Riemann-Lebesgue for uniform limits of step functions:**
(b-i) Graph of $f = f_1 + f_2 + f_3$ defined by

$$f_1 = \frac{1}{2}\chi_{[0,\frac{1}{3})} \tag{1}$$
$$f_2 = \chi_{[\frac{1}{4},\frac{1}{2}]} \tag{2}$$
$$f_3 = \frac{3}{4}\chi_{[\frac{1}{2},\frac{3}{4}]} \tag{3}$$



(b-ii)

Lemma 1.2. *For all 1-periodic step-functions f,*

$$\left|\hat{f}_n\right| \to 0, \ \ as \ |n| \to \infty$$

Proof. Let $c < d \in \mathbb{R}$ with $d - c < 1$. Note that the Fourier coefficients of periodic extensions $\chi_{[c,d]}, \chi_{(c,d]}, \chi_{[c,d)}$ are the same since changing the value of a Riemann integrable function at finitely many points does not change its integral. Define f to be the 1-periodic extension of $\lambda\chi_{[c,d]}$ for some $\lambda \in \mathbb{C}$.

$$\begin{aligned}\left|\hat{f}_n\right| &= \left|\int_0^1 f(t)\overline{e_n(t)}\ dt\right| \\ &\leqslant \int_0^1 |f(t)| \cdot \left|\overline{e_n(t)\ dt}\right|\ dt \\ &= \int_0^1 |f(t)|\ dt\end{aligned}$$

$$\begin{aligned}\hat{f}_n &= \int_0^1 f(t)\overline{e_n(t)}\ dt \\ &= \int_c^{c+1} f(t)\overline{e_n(t)}\ dt \\ &= \int_c^d \lambda e^{-2\pi int}\ dt \\ &= \frac{1}{2\pi in}\lambda e^{-2\pi int}\bigg|_c^d \\ &= \frac{\lambda}{2\pi in}(e^{-2\pi ind} - e^{-2\pi inc}) \\ \left|\hat{f}_n\right| &= \left|\frac{\lambda}{2\pi in}(e^{-2\pi ind} - e^{-2\pi inc})\right| \\ &= \frac{|\lambda|}{2\pi|n|}\left|e^{-2\pi ind} - e^{-2\pi inc}\right| \\ &\leqslant \frac{|\lambda|}{2\pi|n|}(\left|e^{-2\pi ind}\right| + \left|e^{-2\pi inc}\right|) \\ &= \frac{|\lambda|}{2\pi|n|}\cdot 2 \\ &\leqslant \frac{|\lambda|}{|n|}\end{aligned}$$

So, $|\hat{f}_n| \to 0$ as $|n| \to \infty$.
By the linearity of integration, we have that the Fourier coefficients of any 1-periodic step-function decay as desired. □

Proposition 1.2. *For all functions f which may be realized as a uniform limit of 1-periodic step functions,*

$$\left|\hat{f}_n\right| \to 0, \ \ as\ |n| \to \infty$$

Proof. Suppose $f \in \mathcal{R}_1(\mathbb{R};\mathbb{C})$ is the uniform limit of 1-periodic step-functions g_k. Let $\epsilon > 0$ be given and pick N such that for all $k \geqslant N$,

$$\|f - g_k\|_\infty < \frac{\epsilon}{2}$$

and for all $n \geqslant N$,

$$\left|\widehat{[g_N]}_n\right| < \frac{\epsilon}{2}$$

If $k \geqslant N$,

$$\begin{aligned} |\hat{f}_n - \widehat{[g_k]}_n| &= |\widehat{[f - g_k]}_n \\ &\leqslant \|f - g_k\|_\infty \\ &< \frac{\epsilon}{2} \end{aligned}$$

So, for $n \geqslant N$,

$$\begin{aligned} |\hat{f}_n| &= \left|\hat{f}_n - \widehat{[g_N]}_n + \widehat{[g_N]}_n\right| \\ &\leqslant \left|\hat{f}_n - \widehat{[g_N]}_n\right| + \left|\widehat{[g_N]}_n\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

□

(c)

Proposition 1.3. *Every function $f \in \mathcal{PC}_1(\mathbb{R}; \mathbb{C})$ can be realized as the uniform limit of step functions.*

Proof. Let $f : [0, 1] \to \mathbb{C}$ be piecewise continuous with 1 jump discontinuity x_0. Fix $n \in \mathbb{N}$. As f is uniform continuous, pick $\delta > 0$ such that for all $x, y \in [a, b]$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{1}{n}$$

Let

$$m := \left\lfloor \frac{x_0 - a}{\delta} \right\rfloor \tag{4}$$

$$l := \left\lfloor \frac{b - x_0}{\delta} \right\rfloor \tag{5}$$

so,

$$a + m\delta \leqslant x_0 < a + (m + 1)\delta$$

$$x_0 + l\delta \leqslant b < x_0 + (l + 1)\delta$$

Define

$$
\begin{aligned}
f_n(x) := \sum_{i=0}^{m-1} & f(a+i\delta)\chi_{[a+i\delta, a+(i+1)\delta)} + f\left(a+m\delta\right)\chi_{[a+m\delta, b)} \\
& + f(x_0)\chi_{\{x_0\}} + f(x_0)\chi_{(x_0, x_0+\delta)} \\
& + \sum_{i=1}^{l-1} f(a+i\delta)\chi_{[x_0+i\delta, x_0+(i+1)\delta)} + f(a+l\delta)\chi_{[x_0+l\delta, b]}
\end{aligned}
$$

We may write

$$
f_n(x) = \begin{cases}
f(a+i\delta) & \exists i \leqslant m : x \in [a+i\delta, a+(i+1)\delta) \\
f(a+m\delta) & x \in [a+m\delta, x_0) \\
f(x_0) & x = x_0 \\
f(x_0) & x \in (x_0, x_0+\delta) \\
f(x_0+i\delta) & \exists i \leqslant l : x \in [x_0+i\delta, x_0+(i+1)\delta) \\
f(x_0+l\delta) & x \in [x_0+l\delta, b]
\end{cases}
$$

In case (1),

$$
\begin{aligned}
x &\geqslant a+i\delta \\
x &< a+(i+1)\delta \\
|x-(a+i\delta)| &= x-a-i\delta \\
&< a+(i+1)\delta - a - i\delta \\
&= \delta
\end{aligned}
$$

So,

$$
\begin{aligned}
|f(x)-f_n(x)| &= |f(x)-f(a+i\delta)| \\
&< \frac{1}{n}
\end{aligned}
$$

In case (2), noting that

$$
\begin{aligned}
x &\geqslant a+m\delta \\
x &< x_0 < a+(m+1)\delta
\end{aligned}
$$

we get the same bound

$$
|f(x)-f_n(x)| < \frac{1}{n}
$$

In case (3),

$$
|f(x)-f_n(x)| = |f(x_0)-f_n(x_0)| = 0
$$

In case (4),

$$
x \in (x_0, x_0+\delta) \implies |x-x_0| < \delta
$$

So,

$$|f(x) - f_n(x)| < \frac{1}{n}$$

Case (5) is analogous to (1):

$$\begin{aligned} x &\geqslant x_0 + i\delta \\ x &< x_0 + (i+1)\delta \\ |x - (x_0 + i\delta)| &= x - x_0 - i\delta \\ &< x_0 + (i+1)\delta - x_0 - i\delta \\ &= \delta \end{aligned}$$

giving

$$|f(x) - f_n(x)| < \frac{1}{n}$$

In case (6), noting that

$$\begin{aligned} x &\geqslant x_0 + l\delta \\ x &\leqslant b < x_0 + (l+1)\delta \end{aligned}$$

we get the same bound

$$|f(x) - f_n(x)| < \frac{1}{n}$$

Therefore, for all $x \in [0,1]$, $|f(x) - f_n(x)| < \frac{1}{n}$, so

$$\|f - f_n\|_\infty < \frac{1}{n}$$

Let $\epsilon > 0$ be given and pick $N \in \mathbb{N}$ with $\frac{1}{N} < \epsilon$. If $n \geqslant N$,

$$\|f - f_n\|_\infty < \frac{1}{n} \leqslant \frac{1}{N} < \epsilon$$

So, the sequence of step functions converges to f uniformly on $[0,1]$. By the (PMI), we see that there is a sequence of step functions uniformly converging to all piecewise continuous functions on $[0,1]$. Clearly, periodic extension maintains this property, so it holds for all $f \in \mathcal{PC}_1(\mathbb{R};\mathbb{C})$.

$\square$

- **Problem 2 - Pointwise Convergence of Fourier Series**
  (b) Denote

$$\alpha_{x_0} := \frac{1}{2}\left(f(x_0+) + f(x_0-)\right) \ . \tag{6}$$

Proposition 1.4. *For a function $f \in \mathcal{PC}^1_1(\mathbb{R};\mathbb{C})$, for all points of discontinuity x_0,*

$$\begin{aligned}|S_n[f](x_0) - \alpha_{x_0}| \leqslant &\left|\int_0^{1/2} D_n(y)\left(f(x_0+y) - f(x_0+)\right)\ \mathrm{d}y\right| \\ &+ \left|\int_0^{1/2} D_n(y)(f(x_0-y) - f(x_0-))\ \mathrm{d}y\right| =: I_n^{(+)} + I_n^{(-)}\ . \qquad (7)\end{aligned}$$

Proof. Let $f \in \mathcal{PC}^1_1(\mathbb{R};\mathbb{C})$ have a single point of discontinuity x_0 on its period interval $(0,1)$.

$$\begin{aligned}
S_n[f](x_0) &= \sum_{k=-n}^{n}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)\overline{e_k(t)}\ dt\right) e_k(x_0)\ dt \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) \sum_{k=-n}^{n} \overline{e_k(t)} e_k(x_0)\ dt \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) \sum_{k=-n}^{n} e_k(x_0 - t)\ dt \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) D_n(x_0 - t)\ dt \\
&= (f \star D_n)(x_0) \\
&= (D_n \star f)(x_0) \quad \text{(commutativity of convolutions)} \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} D_n(t) f(x_0 - t)\ dt \\
&= \underbrace{\int_{-\frac{1}{2}}^{0} D_n(t) f(x_0 - t)\ dt}_{\text{change variables: } u=-t} + \int_0^{\frac{1}{2}} D_n(t) f(x_0 - t)\ dt \\
&= -\int_{\frac{1}{2}}^{0} D_n(-u) f(x_0 + u)\ du + \int_0^{\frac{1}{2}} D_n(t) f(x_0 - t)\ dt \\
&= \int_0^{\frac{1}{2}} \underbrace{D_n(u)}_{D_n \text{ is even}}\ f(x_0 + u)\ du + \int_0^{\frac{1}{2}} D_n(t) f(x_0 - t)\ dt \\
&= \int_0^{\frac{1}{2}} D_n(t)(f(x_0 + t) + f(x_0 - t))\ dt
\end{aligned}$$

Write:

$$\begin{aligned}\alpha_{x_0} &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \alpha_{x_0} D_n(t)\ dt \\ &= 2\int_0^{\frac{1}{2}} \alpha_{x_0} D_n(t)\ dt\end{aligned}$$

since D_n is even.
So,

$$\begin{aligned}|S_n[f](x_0) - \alpha_{x_0}| &= \left|\int_0^{\frac{1}{2}} D_n(t)(f(x_0+t) + f(x_0-t))\ dt - \alpha_{x_0}\right| \\ &= \left|\int_0^{\frac{1}{2}} D_n(t)(f(x_0+t) + f(x_0-t))\ dt - 2\int_0^{\frac{1}{2}} \alpha_{x_0} D_n(t)\ dt\right| \\ &= \left|\int_0^{\frac{1}{2}} D_n(t)(f(x_0+t) + f(x_0-t) - \alpha_{x_0})\ dt\right| \\ &= \left|\int_0^{\frac{1}{2}} D_n(t)(f(x_0+t) - f(x_0+) + f(x_0-t) - f(x_0-))\ dt\right| \\ &\leqslant \left|\int_0^{\frac{1}{2}} D_n(t)(f(x_0+t) - f(x_0+))\ dt\right| \\ &\qquad + \left|\int_0^{\frac{1}{2}} D_n(t)(f(x_0-t) - f(x_0-))\ dt\right| =: I_n^{(+)} + I_n^{(-)}\end{aligned}$$

□

(c)

Proposition 1.5.

$$I_n^{(+)} \to 0\ ,\ as\ n \to \infty\ . \tag{8}$$

Proof. Let $\epsilon > 0$ be given. We have that both directional derivatives of f exist at x_0. Pick a $\frac{1}{2} > \delta > 0$ such that for all $x \in (x_0, x_0+\delta)$,

$$\frac{f(x) - f(x_0)}{x - x_0} < \epsilon$$

Compute:

$$
\begin{aligned}
I_n^{(+)} &= \left|\int_0^{\frac{1}{2}} D_n(t)(f(x_0+t)-f(x_0+))\ dt\right| \\
&= \left|\int_0^{\delta} D_n(t)(f(x_0+t)-f(x_0+))\ dt + \int_{\epsilon}^{\frac{1}{2}} D_n(t)(f(x_0+t)-f(x_0+))\ dt\right| \\
&\leqslant \left|\int_0^{\delta} D_n(t)(f(x_0+t)-f(x_0+))\ dt\right| + \left|\int_{\delta}^{\frac{1}{2}} D_n(t)(f(x_0+t)-f(x_0+))\ dt\right| \\
&\quad := I_1 + I_2 \\
I_1 &\leqslant \int_0^{\delta} |D_n(t)|\cdot|f(x_0+t)-f(x_0+)|\ \ dt \\
&= \int_0^{\delta} \left|\frac{\sin(\pi(2n+1)t)}{\underbrace{\sin(\pi t)}_{\geqslant 2t}}\right|\cdot|f(x_0+t)-f(x_0+)|\ dt \\
&\leqslant \int_0^{\delta} \underbrace{|\sin(\pi(2n+1)t)|}_{\leqslant 1}\cdot\frac{|f(x_0+t)-f(x_0+)|}{2|t|}\ \ dt \\
&\leqslant \frac{1}{2}\int_0^{\delta} \underbrace{\frac{|f(x_0+t)-f(x_0+)|}{|t|}}_{<\epsilon}\ \ dt \\
&< \frac{1}{2}\delta\epsilon < \epsilon
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sin(\pi(2n+1)t) &= \Im\mathrm{m}(e^{\pi(2n+1)t}) \\
&= \frac{1}{2i}(e^{\pi(2n+1)t} - \overline{e^{\pi(2n+1)t}} \\
&= \frac{1}{2i}(e^{\pi(2n+1)t} - e^{-\pi(2n+1)t}) \\
&= \frac{1}{2i}(e^{2\pi int}e^{\pi it} - e^{-2\pi int}e^{-\pi it})
\end{aligned}
$$

Compute:

$$
\begin{aligned}
I_2 &= \left| \int_\delta^{\frac{1}{2}} D_n(t)(f(x_0+t) - f(x_0+))\ dt \right| \\
&= \left| \int_\delta^{\frac{1}{2}} \frac{\sin(\pi(2n+1)t)}{\sin \pi t}(f(x_0+t) - f(x_0+))\ dt \right| \\
&= \left| \int_\delta^{\frac{1}{2}} \frac{1}{2i}(e^{2\pi int}e^{\pi it} - e^{-2\pi int}e^{-\pi it})\frac{f(x_0+t) - f(x_0+)}{\sin \pi t}\ dt \right| \\
&\leqslant \left| \int_\delta^{\frac{1}{2}} \frac{1}{2i}(e^{2\pi int}e^{\pi it})\frac{f(x_0+t) - f(x_0+)}{\sin \pi t}\ dt\ \right| \\
&\qquad + \left| \int_\delta^{\frac{1}{2}} \frac{1}{2i}(e^{-2\pi int}e^{-\pi it})\frac{f(x_0+t) - f(x_0+)}{\sin \pi t}\ dt \right|
\end{aligned}
$$

Let

$$
g^{(\pm)}(t) = \begin{cases} \frac{f(x_0+t)-f(x_0+)}{\sin(\pi t)} e^{\pm \pi it} & \text{if } t \in (\delta, \frac{1}{2}] \\ 0 & \text{if } t \notin (\delta, \frac{1}{2}] \end{cases}
$$

and let $g_1^{(\pm)}$ be the 1-periodic extensions of $g^{(\pm)}$. On its period interval $g_1^{(\pm)}$ is discontinuous only at $\epsilon, \frac{1}{2}$. Therefore, $g_1^{(\pm)} \in \mathcal{PC}_1(\mathbb{R};\mathbb{C})$. By problem 1,

$$
\widehat{[g_1^{(+)}]_n} \to 0
$$
$$
\widehat{[g_1^{(-)}]_n} \to 0
$$

as $|n| \to \infty$. Thus,

$$
\begin{aligned}
I_2 &= \left| \int_\delta^{\frac{1}{2}} \frac{1}{2i} e^{2\pi int} g_1^{(+)}(t)\ dt \right| + \left| \int_\delta^{\frac{1}{2}} \frac{1}{2i} e^{2\pi int} g_1^{(-)}(t)\ dt \right| \\
&= \left| \frac{1}{2i} \int_0^1 e^{2\pi int} g_1^{(+)}(t)\ dt \right| + \left| \frac{1}{2i} \int_0^1 e^{2\pi int} g_1^{(-)}(t)\ dt \right| \\
&= \frac{1}{2} \left| \widehat{[g_1^{(+)}]_n} \right| + \frac{1}{2} \left| \widehat{[g_1^{(-)}]_n} \right| \\
&\to 0 \text{ as } |n| \to \infty
\end{aligned}
$$

Therefore, $I_n^{(+)} \to 0$ as $|n| \to \infty$ as desired.

$\square$