Math 357 / Spring 2024 (C. Marx) - Midterm Exam (**take-home**)

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I affirm that I have adhered to the Honor Code in this exam. Isaac Viviano

1 Problems

Problem 1: Rate of convergence in Fejér's theorem

(a)

Proposition 1.1. *If $f \in \mathcal{C}_1(\mathbb{R};\mathbb{C})$ is α-Hölder continuous, then for every fixed $0 < \delta < \frac{1}{2}$, we have the bound*

$$\|F_n \star f - f\|_\infty \leq C\delta^\alpha + \frac{\|f\|_\infty}{2n\delta^2} \text{ , for all } n \in \mathbb{N}. \tag{1}$$

Proof. Fix $0 < \delta < \frac{1}{2}$ and let $f \in \mathcal{C}_1(\mathbb{R};\mathbb{C})$ be α-Hölder continuous. Pick $C > 0$ such that for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

For any $t \in \mathbb{R}$,

$$\begin{aligned}
|(F_n \star f)(t) - f(t)| &= \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} F_n(s) f(t-s)\, ds - f(t) \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} F_n(s)\, ds}_{=1,\ F_n \text{ is a PAI}} \right| \\
&= \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} F_n(s)[f(t-s) - f(t)]\, ds \right| \\
&\leq \left| \int_{(-\delta,\delta)} F_n(s)[f(t-s) - f(t)]\, ds \right| \\
&\qquad + \left| \int_{[-\frac{1}{2},\frac{1}{2}]-(-\delta,\delta)} F_n(s)[f(t-s) - f(t)\, ds \right| \\
&:= I_1 + I_2
\end{aligned}$$

We use the α-Hölder condition to estimate I_1. For all $s \in (-\delta, \delta)$, $|t - s - t| = |s| < \delta$, so $|s|^\alpha < \delta^\alpha$ by the monotonicity of log:

$$\begin{aligned}
&|s| < \delta \\
\iff &\log |s| < \log \delta \\
\iff &\alpha \log |s| < \alpha \log \delta \\
\iff &\log |s|^\alpha < \log \delta^\alpha \\
\iff &|s|^\alpha < \delta^\alpha
\end{aligned}$$

Therefore,

$$
\begin{aligned}
I_1 &\le \int_{-\delta}^{\delta} |F_n(s)| \cdot \underbrace{|f(t-s) - f(s)|}_{\le C \cdot \delta^\alpha} \ ds && (2)\\
&\le C \cdot \delta^\alpha \int_{-\delta}^{\delta} |F_n(s)| \ ds && (3)\\
&\le C\delta^\alpha \int_{-\frac{1}{2}}^{\frac{1}{2}} \underbrace{|F_n(s)|}_{=F_n(s)} \ ds && (4)\\
&\le C\delta^\alpha \int_{-\frac{1}{2}}^{\frac{1}{2}} F_n(s) \ ds && (5)\\
&= C\delta^\alpha && (6)
\end{aligned}
$$

For (5), we use that F_n is positive and real. For (6), we use that F_n is a periodic approximate identity (PAI-1 from Proposition 3.5.1 of [CM]). For I_2,

$$
\begin{aligned}
I_2 &\le \int_{[-\frac{1}{2},\frac{1}{2}]-(-\delta,\delta)} |F_n(s)| \cdot |[f(t-s) - f(t)]| \ ds \\
&\le \int_{[-\frac{1}{2},\frac{1}{2}]-(-\delta,\delta)} |F_n(s)| \cdot (\underbrace{|f(t-s)|}_{\le \|f\|_\infty} + \underbrace{|f(t)|}_{\le \|f\|_\infty}) \ ds \quad (\triangle \text{ inequality}) \\
&\le 2\|f\|_\infty \int_{[-\frac{1}{2},\frac{1}{2}]-(-\delta,\delta)} \underbrace{|F_n(s)|}_{=F_n(s)} \ ds \\
&= 2\|f\|_\infty \int_{[-\frac{1}{2},\frac{1}{2}]-(-\delta,\delta)} \frac{1}{n} \left(\underbrace{\sin(\pi n s)}_{\le 1} \right)^2 \left(\frac{1}{\underbrace{\sin(\pi s)}_{\ge 2s}} \right)^2 \ ds \\
&\le 2\|f\|_\infty \int_{[-\frac{1}{2},\frac{1}{2}]-(-\delta,\delta)} \underbrace{\frac{1}{4ns^2}}_{\text{even}} \ ds \\
&\le \frac{\|f\|_\infty}{n} \int_{(\delta,\frac{1}{2}]} \frac{1}{s^2} \ ds \\
&\le \frac{\|f\|_\infty}{n\delta^2} \cdot \left(\frac{1}{2} - \delta \right) \quad (\text{ML estimate}) \\
&< \frac{\|f\|_\infty}{2n\delta^2}
\end{aligned}
$$

Since the bounds on I_1 and I_2 have no dependence on t, we have shown the intended bound:

$$
\|F_n \star f - f\|_\infty \le C\delta^\alpha + \frac{\|f\|_\infty}{2n\delta^2}
$$

□

(b)

Proposition 1.2. *For every α-Hölder continuous function $f \in \mathcal{C}_1(\mathbb{R};\mathbb{C})$, the rate of convergence for approximating f by Fejér polynomials $F_n \star f$ is of the form*

$$\|F_n \star f - f\|_\infty \leq C_f \cdot n^{-\gamma} \ , \tag{7}$$

where the constants $C_f > 0$ and $\gamma = \gamma(\alpha) > 0$ are given by

$$C_f := C + 2\|f\|_\infty \tag{8}$$
$$\gamma := \frac{\alpha}{2+\alpha} \tag{9}$$

Proof. Let $\delta = \frac{1}{2} n^{-\frac{1}{2+\alpha}}$ and apply the bound (1) from (a).

$$\begin{aligned}
\|F_n \star f - f\|_\infty &\leq C\delta^\alpha + \frac{\|f\|_\infty}{2n\delta^2} \\
&= \frac{C}{\underbrace{2^\alpha}_{\geq 1} n^{\frac{\alpha}{2+\alpha}}} + \frac{\|f\|_\infty}{2n\left(\frac{1}{2}n^{-\frac{1}{2+\alpha}}\right)^2} \\
&\leq \frac{C}{n^{\frac{\alpha}{2+\alpha}}} + \frac{2\|f\|_\infty}{n^{1-\frac{2}{2+\alpha}}} \\
&\leq \frac{C}{n^{\frac{\alpha}{2+\alpha}}} + \frac{2\|f\|_\infty}{n^{\frac{\alpha}{2+\alpha}}} \\
&= \frac{C + 2\|f\|_\infty}{n^{\frac{\alpha}{2+\alpha}}} \\
&:= \frac{C_f}{n^\gamma}
\end{aligned}$$

□

Problem 2 - Queen Dido's problem:

(a)

Lemma 1.1. *For every function $f \in \mathcal{C}_1^1(\mathbb{R};\mathbb{C})$ which satisfies*

$$\widehat{f_0} = \int_0^1 f(t)\ \mathrm{d}t = 0 \ ,$$

one has the equality

$$\int_0^1 |f(t)|^2\ \mathrm{d}t \leq \frac{1}{4\pi^2} \int_0^1 |f'(t)|^2\ \mathrm{d}t \ . \tag{10}$$

Proof. Let $f \in \mathcal{C}_1^1(\mathbb{R};\mathbb{C})$ satisfy

$$\hat{f}_0 = \int_0^1 f(t)\ dt = 0$$

Note that the Fourier differentiation mantra (Theorem 5.3 of [CM]) states:

$$\hat{f}_n = \frac{\hat{f'}_n}{2\pi i n}, \text{ for all } |n| \in \mathbb{N}$$

Using Plancherel's Identity (Theorem 6.9 of [CM]) for f and f', we see

$$\begin{aligned}
\int_0^1 |f(t)|^2 \, dt &= \|f\|_2^2 \\
&= \sum_{n=-\infty}^{\infty} \left|\hat{f}_n\right|^2 \\
&= \sum_{n=1}^{\infty} \left|\hat{f}_{-n}\right|^2 + \hat{f}_0 + \sum_{n=1}^{\infty} \left|\hat{f}_n\right| \\
&= \sum_{n=1}^{\infty} \left|\frac{\hat{f'}_{-n}}{2\pi i n}\right|^2 + 0 + \sum_{n=1}^{\infty} \left|\frac{\hat{f'}_n}{2\pi i n}^2\right| \\
&\leq \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{\left|\hat{f'}_{-n}\right|^2}{n} + \frac{1}{4\pi^2} \left|\hat{f'}_0\right|^2 + \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{\left|\hat{f'}_n\right|^2}{n} \\
&\leq \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \left|\hat{f'}_{-n}\right|^2 + \frac{1}{4\pi^2} \left|\hat{f'}_0\right|^2 + \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \left|\hat{f'}_n\right|^2 \quad \text{(comparison test)} \\
&= \frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} \left|\hat{f'}_n\right|^2 \\
&= \frac{1}{4\pi^2} \|f'\|_2^2 \\
&= \frac{1}{4\pi^2} \int_0^1 |f'(t)|^2 \, dt
\end{aligned}$$

where the comparison test refers to (Theorem 2.7.4 and Theorem 2.3.4 of Abott) □

(b)

Lemma 1.2. *for every $\alpha, \beta \in \mathbb{C}$, one has*

$$|\alpha\beta| \leq \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) \ . \tag{11}$$

Proof.

$$|\alpha\beta| = |\alpha| \cdot |\beta| = \sqrt{|\alpha|^2 \cdot |\beta|^2}$$

So, the claim of Lemma 1.2 reduces to showing that in general, the geometric mean of two non-negative real numbers is bound by their arithmetic mean:

$$\sqrt{ab} \leq \frac{a+b}{2} \tag{12}$$

Consider the squares:

$$0 \leq \left(\frac{a}{2} - \frac{b}{2}\right)^2 = \frac{a^2}{4} - \frac{ab}{2} + \frac{b^2}{4}$$
$$\iff ab \leq \frac{a^2}{4} + \frac{ab}{2} + \frac{b^2}{4} = \left(\frac{1}{2}(a+b)\right)^2$$

showing (11) for all non-negative real numbers. □

(c)

Theorem. *<u>For all</u> closed, simple $\mathcal{C}^1$-curves $\gamma = (x, y)$ with fixed perimeter $L = 1$ and parametrized by component functions which satisfy $x, y \in \mathcal{C}^1_1(\mathbb{R}; \mathbb{R})$, one has the inequality*

$$A(\gamma) \leq \frac{1}{4\pi} \int_0^1 \left(|x'(t)|^2 + |y'(t)|^2\right) \, \mathrm{d}t = \frac{1}{4\pi} \; . \tag{13}$$

Proof.

$$\begin{aligned}
A(\gamma) &= \frac{1}{2} \int_0^1 x(t)y'(t) - x'(t)y(t) \; dt & (14)\\
&\leq \frac{1}{2} \int_0^1 |x(t)y'(t)| + |x'(t)y(t)| \; dt & (15)\\
&= \frac{1}{2} \int_0^1 |\sqrt{2\pi}x(t) \cdot \frac{1}{\sqrt{2\pi}} y'(t)| + |\sqrt{2\pi}x'(t) \cdot \frac{1}{\sqrt{2\pi}} y(t)| \; dt & (16)\\
&\leq \frac{1}{2} \int_0^1 \frac{1}{2} \left(|\sqrt{2\pi}x(t)|^2 + \left|\frac{1}{\sqrt{2\pi}} y'(t)\right|^2 \right) + \frac{1}{2} \left(|\sqrt{2\pi}y(t)| + \left|\frac{1}{\sqrt{2\pi}} x'(t)\right| \right) \; dt & (17)\\
&= \frac{1}{4} \cdot 2\pi \int_0^1 |x(t)|^2 \; dt + \frac{1}{4} \cdot \frac{1}{2\pi} \int_0^1 |y'(t)|^2 + |x'(t)|^2 \; dt + \frac{1}{4} \cdot 2\pi \int_0^1 |y(t)|^2 \; dt & (18)\\
&\leq \frac{\pi}{2} \cdot \frac{1}{4\pi^2} \int_0^1 |x'(t)|^2 \; dt + \frac{1}{4} \cdot \frac{1}{2\pi} \int_0^1 |x'(t)|^2 + |y'(t)|^2 \; dt + \frac{\pi}{2} \cdot \frac{1}{4\pi^2} \int_0^1 |y'(t)|^2 \; dt & (19)\\
&= \frac{1}{4\pi} \int_0^1 |x'(t)|^2 + |y'(t)|^2 \; dt & (20)\\
&= \frac{1}{4\pi} & (21)
\end{aligned}$$

(17) follows from Lemma 1.2 and (19) uses Lemma 1.1. □

Problem 3 - Minimal Period and the Fourier Coefficients

Proposition 1.3. *Let $f \in \mathcal{R}_T(\mathbb{R}; \mathbb{C})$. If $0 < T' < T$ is another period of f, then*

$$\hat{f}_n = 0, \;\; \textit{for all } n \in \mathbb{Z} \textit{ for which } n\frac{T'}{T} \notin \mathbb{Z} \tag{22}$$

Proof. Let $0 < T' < T$ be another period of f and let $n \in \mathbb{Z}$ such that $n\frac{T'}{T} \notin \mathbb{Z}$. Note that $e^{2\pi i k} = 1$ precisely when $k \in \mathbb{Z}$. So, $e^{2\pi i n \frac{T'}{T}} \neq 1$. We have

$$\begin{aligned}
\hat{f}_n &= \frac{1}{T}\int_0^T f(t)\ e^{-2\pi i n \frac{t}{T}}\ dt \\
&= \frac{1}{T}\int_{T'}^{T+T'} \underbrace{f(s-T')}_{=f(s)}\ e^{-2\pi i n \frac{s-T'}{T}}\ ds \quad \text{(change variables } t = s - T'\text{)} \\
&= e^{2\pi i n \frac{T'}{T}} \frac{1}{T}\int_0^T f(s)\ e^{-2\pi i n \frac{s}{T}}\ ds \\
&= e^{2\pi i n \frac{T'}{T}} \hat{f}_n
\end{aligned}$$

Thus, we have

$$\hat{f}_n(e^{2\pi i n \frac{T'}{T}} - 1) = 0$$

Since $e^{2\pi i n \frac{T'}{T}} \neq 1$, $\hat{f}_n = 0$. □

Proposition 1.4. *Suppose $f \in \mathcal{C}_T(\mathbb{R};\mathbb{C})$. If (22) holds for some positive $T' < T$, then T' is another period of f.*

Proof. Let $f \in \mathcal{C}_T(\mathbb{R};\mathbb{C})$. Suppose

$$\hat{f}_n = 0, \text{ for all } n \in \mathbb{Z}, \text{ for which } n\frac{T'}{T} \notin \mathbb{Z}$$

for some $T' > 0$. Let $g : \mathbb{R} \to \mathbb{C}$ be defined by

$$g(t) = f(t + T'), \text{ for all } t \in \mathbb{R}$$

Note that $g \in \mathcal{C}_T(\mathbb{R};\mathbb{C})$. Compute

$$\begin{aligned}
\hat{g}_n &= \frac{1}{T}\int_0^T g(t)\ e^{-2\pi i n \frac{t}{T}}\ dt \\
&= \frac{1}{T}\int_0^T f(t+T')\ e^{-2\pi i n \frac{t}{T}}\ dt \\
&= \frac{1}{T}\int_{T'}^{T+T'} f(s)\ e^{-2\pi i n \frac{s-T'}{T}}\ ds \quad \text{(change variables } s = t + T'\text{)} \\
&= e^{2\pi i n \frac{T'}{T}} \frac{1}{T}\int_0^T f(s)\ d^{-2\pi i n \frac{s}{T}}\ ds \\
&= e^{2\pi i n \frac{T'}{T}} \hat{f}_n
\end{aligned}$$

Consider two cases for n. If $n\frac{T'}{T} \in \mathbb{Z}$, then $e^{2\pi i n \frac{T'}{T}} = 1$, so

$$\hat{g}_n = \hat{f}_n$$

Otherwise the hypothesis that f satisfies (22) gives,

$$g_n = e^{2\pi i n \frac{T'}{T}} \hat{f}_n = e^{2\pi i n \frac{T'}{T}} \cdot 0 = 0 = \hat{f}_n$$

Thus,

$$\hat{f}_n = \hat{g}_n, \text{ for all } n \in \mathbb{Z}$$

and $f, g \in \mathcal{C}_T(\mathbb{R};\mathbb{C})$, so the uniqueness property (Theorem 3.5 of [CM]) gives $f = g$. In particular,

$$f(t) = g(t) = f(t + T'), \text{ for all } t \in \mathbb{R}$$

showing that f is T'-periodic. □

Problem 4 - Rotationally Invariant Functions

(a)

Proposition 1.5. *Let $\alpha \in \mathbb{R}$ be a fixed irrational number. If $f \in \mathcal{C}_1(\mathbb{R};\mathbb{C})$ satisfies*

$$f(x + \alpha) = f(x) \text{ , for all } x \in \mathbb{R} \text{ ,} \tag{23}$$

then f is a constant function.

Proof. Fix an irrational $\alpha \in \mathbb{R}$ and let $f \in \mathcal{C}_1(\mathbb{R};\mathbb{C})$ be α-rotationally invariant. Let $g : \mathbb{R} \to \mathbb{C}$ be defined by $g(t) := f(t + \alpha)$. By (23), we have

$$\hat{f}_n = \hat{g}_n, \text{, for all } n \in \mathbb{Z}$$

For each $n \in \mathbb{Z}$,

$$\begin{aligned}
\hat{f}_n &= \hat{g}_n \\
&= \int_0^1 f(t+\alpha) e^{-2\pi i n t} \, dt \\
&= \int_\alpha^{1+\alpha} f(s) e^{-2\pi i n (s-\alpha)} \, ds \quad \text{(change variables } s = t + \alpha) \\
&= e^{2\pi i n \alpha} \int_0^1 f(s) e^{-2\pi i n s} \, ds \\
&= e^{2\pi i n \alpha} \hat{f}_n
\end{aligned}$$

So,

$$\hat{f}_n (e^{2\pi i n \alpha} - 1) = 0, \text{ for all } n \in \mathbb{Z} \tag{24}$$

The irrationality of α implies that $n\alpha \notin \mathbb{Z}$ if $n \neq 0$. Note that $e^{2\pi i k} = 1$ precisely when $k \in \mathbb{Z}$. Therefore, $e^{2\pi i n \alpha} \neq 0$ for all $n \neq 0$. Then, (24) implies that $\hat{f}_n = 0$ for all $n \in \mathbb{Z}$ with $n \neq 0$. We now use the uniqueness property to show f is constant. Let $h : \mathbb{R} \to \mathbb{C}$ be defined by $h(t) = c \in \mathbb{C}$ for all $t \in \mathbb{R}$. Compute

$$\begin{aligned}
\hat{h}_0 &= \int_0^1 c \, dt = c \\
\hat{h}_n &= \int_0^1 c e^{-2\pi i n t} \, dt = c\delta_{-n,0} = 0, \text{ for all } n \neq 0
\end{aligned}$$

By choosing $c = \hat{f}_0$, we see that

$$\hat{f}_n = \hat{h}_n, \text{ for all } n \in \mathbb{Z}$$

Since $f, h \in \mathcal{C}_1(\mathbb{R};\mathbb{C})$, $f = h$ is constant (Theorem 3.5 of [CM]). □

(b)

Proposition 1.6. *For each rational $\alpha \neq 0$, there exists a continuous function $f \in \mathcal{C}_1(\mathbb{R};\mathbb{C})$ which satisfies 23 but is not constant.*

Proof. Let $\alpha = \frac{p}{q}$ for $p, q \in \mathbb{N}$ and $p, q \neq 0$. Define f to be

$$f(t) = \begin{cases} t - \frac{n}{q} & t \in [\frac{n}{q}, \frac{n}{q} + \frac{1}{2q}), \text{ for some } n \in \mathbb{Z} \\ \frac{n+1}{q} - t & t \in [\frac{n}{q} + \frac{1}{2q}, \frac{n+1}{q}), \text{ for some } n \in \mathbb{Z} \end{cases}$$

For each $t \in \mathbb{R}$, if $t \in [\frac{n}{q}, \frac{n}{q} + \frac{1}{2q})$ for some $n \in \mathbb{Z}$,

$$t + \alpha = t + \frac{p}{q} \in \left[\frac{n+p}{q}, \frac{n+p}{q} + \frac{1}{2q}\right)$$

so,

$$\begin{aligned} f(t+\alpha) &= t + \frac{p}{q} - \frac{n+p}{q} \\ &= t - \frac{n}{q} \\ &= f(t) \end{aligned}$$

Otherwise $t \in [\frac{n}{q} + \frac{1}{2q}, \frac{n+1}{q})$ for some $n \in \mathbb{Z}$, so

$$t + \alpha = t + \frac{p}{q} \in \left[\frac{n+p}{q} + \frac{1}{2q}, \frac{n+p+1}{q}\right)$$

so,

$$\begin{aligned} f(t+\alpha) &= \frac{n+p+1}{q} - t - \frac{p}{q} \\ &= \frac{n+1}{q} - t \\ &= f(t) \end{aligned}$$

This shows that f is α-rotationally invariant.
For each $t \in \mathbb{R}$, if $t \in [\frac{n}{q}, \frac{n}{q} + \frac{1}{2q})$ for some $n \in \mathbb{Z}$,

$$t + 1 = t + \frac{q}{q} \in \left[\frac{n+q}{q}, \frac{n+q}{q} + \frac{1}{2q}\right)$$

so,

$$\begin{aligned} f(t+1) &= t + \frac{q}{q} - \frac{n+q}{q} \\ &= t - \frac{n}{q} \\ &= f(t) \end{aligned}$$

Otherwise $t \in [\frac{n}{q} + \frac{1}{2q}, \frac{n+1}{q})$ for some $n \in \mathbb{Z}$, so

$$t + 1 = t + \frac{q}{q} \in \left[\frac{n+q}{q} + \frac{1}{2q}, \frac{n+q+1}{q}\right)$$

so,

$$\begin{aligned} f(t+1) &= \frac{n+q+1}{q} - t - \frac{q}{q} \\ &= \frac{n+1}{q} - t \\ &= f(t) \end{aligned}$$

This shows that f is 1-periodic.

To verify that f is continuous, we consider the endpoints of each of its pieces.

At $t = \frac{n}{q}$: The left limit of f at t:

$$f(t-) = t - \frac{n}{q} = 0$$

The right limit and value of f at t:

$$f(t+) = \frac{n-1+1}{q} - t = \frac{n}{q} - \frac{n}{q} = 0$$

At $t = \frac{n}{q} + \frac{1}{2q}$: The left limit of f at t:

$$f(t-) = t - \frac{n}{q} = \frac{n}{q} + \frac{1}{2q} - \frac{n}{q} = \frac{1}{2q}$$

The right limit and value of f at t:

$$f(t+) = \frac{n+1}{q} - \frac{n}{q} - \frac{1}{2q} = \frac{1}{2q}$$

Therefore, f is continuous. Since f attains values 0 and $\frac{1}{2q}$, it is not constant. $\square$