PROBLEM SET NO. 10 (DUE ON WEDNESDAY, MAY 08 AT 1:30 PM ET (BEGINNING OF CLASS))

I affirm that I have adhered to the Honor Code in this assignment.
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1. Problems:

- **Problem 1:**
  (a) As a first step, define

Proposition 1.1. *The function K defined by*

$$K(x) := \int_{-1}^{1} (1 - |y|) \cdot \mathrm{e}^{2\pi i x y} \; \mathrm{d}y \;, \; \textit{for } x \in \mathbb{R} \;, \tag{1}$$

may be written as

$$K(x) = \left(\frac{\sin(\pi x)}{\pi x} \right)^2 \;. \tag{2}$$

Proof.

$$\begin{aligned}
K(x) &= \int_{-1}^{1} (1-|y|)\cdot e^{2\pi ixy}\ dy \\
&= \int_{-1}^{0} (1-|y|)\cdot e^{2\pi ixy}\ dy + \int_{0}^{1} (1-|y|)\cdot e^{2\pi ixy}\ dy \\
&= \int_{-1}^{0} (1+y)\cdot e^{2\pi ixy}\ dy + \int_{0}^{1} (1-y)\cdot e^{2\pi ixy}\ dy \\
&\overset{\text{IBP}}{=} (1+y)\frac{e^{2\pi ixy}}{2\pi ix}\bigg|_{y=-1}^{0} - \int_{-1}^{0} \frac{e^{2\pi ixy}}{2\pi ix}\ dx \\
&\quad + (1-y)\frac{e^{2\pi ixy}}{2\pi ix}\bigg|_{y=0}^{-1} + \int_{0}^{1} \frac{e^{2\pi ixy}}{2\pi ix}\ dx \\
&= \frac{1}{2\pi ix} - \frac{e^{2\pi ixy}}{(2\pi ix)^2}\bigg|_{y=-1}^{0} - \frac{1}{2\pi ix} + \frac{e^{2\pi ixy}}{(2\pi ixy)^2}\bigg|_{y=0}^{1} \\
&= -\frac{1-e^{-2\pi ix}}{-(2\pi x)^2} + \frac{e^{2\pi ix}-1}{-(2\pi x)^2} \\
&= \frac{2-e^{2\pi ix}-e^{-2\pi ix}}{(2\pi x)^2} \\
&= \frac{2-\Re\mathrm{e}(e^{2\pi ix})}{(2\pi x)^2} \\
&= \frac{2-\cos 2\pi x}{(2\pi x)^2} \\
&= \frac{\sin^2 \pi x}{(2\pi x)^2} \\
&= \left(\frac{\sin \pi x}{2\pi x}\right)^2
\end{aligned}$$

□

(b)

Theorem 1.1 (Continuous Analogue of Fejer's Theorem). *Let $f \in \mathcal{L}^1 \cap \mathcal{C}_\infty(\mathbb{R};\mathbb{C}))$, then one has*

$$f(t) = \lim_{\lambda\to+\infty} \int_{-\lambda}^{\lambda} \left(1-\frac{|\nu|}{\lambda}\right)\cdot \widehat{f}(\nu)\mathrm{e}^{2\pi i\nu t}\ \mathrm{d}\nu\ ,\ \textit{for all } t \in \mathbb{R}. \tag{3}$$

Proof. Applying a change of variables, we find a more useful expression of K_λ:

$$K_\lambda(x) = \lambda K(\lambda x) = \int_{-1}^{1} (1-|y|)e^{2\pi i \lambda x y}\, dy = \int_{-\lambda}^{\lambda} \left(1 - \frac{|\nu|}{\lambda}\right) e^{2\pi i \nu x}\, d\nu$$

Compute:

$$\begin{aligned}
f \star K_\lambda &= \int_{-\infty}^{\infty} f(s) \cdot K_\lambda(t-s)\, ds \\
&= \int_{-\infty}^{\infty} f(s) \int_{-\lambda}^{\lambda} \left(1 - \frac{|\nu|}{\lambda}\right) e^{2\pi i \nu (t-s)}\, d\nu\, ds \\
&\overset{\text{Fubini}}{=} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\nu|}{\lambda}\right) e^{2\pi i \nu t} \left(\underbrace{\int_{-\infty}^{\infty} f(s)\, e^{-2\pi i \nu s}\, ds}_{=\widehat{f}(\nu)} \right) d\nu \\
&= \int_{-\lambda}^{\lambda} \left(1 - \frac{|\nu|}{\lambda}\right) \widehat{f}(\nu)\, e^{2\pi i \nu t}\, d\nu
\end{aligned}$$

Since

$$\left| f(s) \left(1 - \frac{|\nu|}{\lambda}\right) e^{2\pi i \nu t - s} \right| = |f(s)| \cdot \underbrace{\left(1 - \frac{|\nu|}{\lambda}\right)}_{\leqslant 1} \cdot \underbrace{\left| e^{2\pi i \nu t - s} \right|}_{=1} \leqslant |f(s)|$$

we can apply Fubini because f is a Schwartz function in s and the λ domain is compact.

By problem 2, K_λ is an approximate δ function and thus an $\mathbb{R}AI$ by Set 9, Problem 2 (c). Thus, the convolution

$$f \star K_\lambda \to f, \text{ uniformly}$$

by the Approximate Identities on $\mathbb{R}$ Theorem. So,

$$f(t) = \lim_{\lambda \to \infty} f \star K_\lambda$$

$\square$

Theorem 1.2 (Uniqueness Theorem for the Fourier Transform). *Suppose f, g are $\mathcal{L}^1 \cap \mathcal{C}_\infty(\mathbb{R}; \mathbb{C})$, then one has*

$$\widehat{f} = \widehat{g} \;\Rightarrow f = g\ .$$

Proof. if $\widehat{f} = \widehat{g}$,

$$\begin{aligned}
(f-g)(t) &= \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\nu|}{\lambda}\right) [\widehat{f-g}](\nu)\ e^{2\pi i\nu t}\ d\nu \\
&= \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\nu|}{\lambda}\right) [\widehat{f} - \widehat{g}](\nu)\ e^{2\pi i\nu t}\ d\nu \\
&= \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\nu|}{\lambda}\right) \cdot 0 \cdot\ e^{2\pi i\nu t}\ d\nu \\
&= \lim_{\lambda\to\infty} 0 \\
&= 0
\end{aligned}$$

$\square$

- **Problem 3 - Convolutions & Products:**

Proposition 1.2. *For all functions* $f, g \in \mathcal{S}(\mathbb{R};\mathbb{C})$,

$$\mathcal{F}[f\cdot g] = \widehat{f} \star \widehat{g}\ , \tag{4}$$

$$\mathcal{F}[f\star g] = \widehat{f} \cdot \widehat{g}\ . \tag{5}$$

Proof: (5) $\Longrightarrow$ *(4).* Suppose (5) and let $f, g \in \mathcal{S}(\mathbb{R};\mathbb{C})$. Note that $f\cdot g, f\star g \in \mathcal{S}(\mathbb{R};\mathbb{C})$. Define:

$$h^{(-)}(x) = h(-x)$$

for each $h \in \mathcal{S}(\mathbb{R};\mathbb{C})$. Note that $h^{(-)} \in \mathcal{S}(\mathbb{R};\mathbb{C})$.

$$\begin{aligned}
\widehat{h}(\nu) &= \int_{-\infty}^{\infty} h(t)\ e^{-2\pi i\nu t}\ dt \\
&= -\int_{\infty}^{-\infty} h(-t)\ e^{2\pi i\nu t}\ dt \\
&= \int_{-\infty}^{\infty} h^{(-)}(t)\ e^{2\pi i\nu t}\ dt \\
&= \widecheck{h^{(-)}}(\nu)
\end{aligned}$$

Also, note that

$$(f\cdot g)^{(-)} = f^{(-)} \cdot g^{(-)}$$

Compute:

$$\begin{aligned}\widehat{f} \star \widehat{g} &= \overbrace{f^{(-)}} \star \overbrace{g^{(-)}} \\ &= \mathcal{F}^{-1}(\mathcal{F}(\overbrace{f^{(-)}} \star \overbrace{g^{(-)}})) \\ &= \mathcal{F}^{-1}(f^{(-)} \cdot g^{(-)}) \\ &= \mathcal{F}^{-1}((f \cdot g)^{(-)}) \\ &= \mathcal{F}(f \cdot g)\end{aligned}$$

□

Proof: (5).

$$\mathcal{F}[f \star g](\nu) = \int_{-\infty}^{\infty} (f \star g)(t)\ e^{-2\pi i\nu t}\ dt \tag{6}$$

$$= \int_{-\infty}^{\infty} e^{-2\pi i\nu t}\left[\int_{-\infty}^{\infty} f(s) \cdot g(t-s)\ ds\right]\ dt \tag{7}$$

$$\overset{\text{Fubini}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s) \cdot g(t-s)\ e^{-2\pi i\nu t}\ dt\ ds \tag{8}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s) \cdot g(u)\ e^{-2\pi i\nu(u+s)}\ du\ ds \quad \text{(change variables: } u = t-s\text{)} \tag{9}$$

$$= \int_{-\infty}^{\infty} f(s)\ e^{-2\pi i\nu s}\underbrace{\left[\int_{-\infty}^{\infty} g(u)\ e^{-2\pi i\nu u}\ du\right]}_{=\widehat{g}(\nu)}\ ds \tag{10}$$

$$= \widehat{g}(\nu) \cdot \underbrace{\int_{-\infty}^{\infty} f(s)\ e^{-2\pi i\nu s}\ ds}_{=\widehat{f}(\nu)} \tag{11}$$

$$= (\widehat{f} \cdot \widehat{g})(\nu) \tag{12}$$

Note that (8) we can apply Fubini's theorem for integration on $\mathbb{R}^2$, since the integrand satisfies an $\mathcal{L}^1$ bound:

$$\left|e^{-2\pi i\nu t} f(s) \cdot g(t-s)\right| = |f(s)| \cdot |g(t-s)|$$

because f is Schwartz in s and g is Schwartz in t. □

- **Problem 4 - Uncertainty principle:**

(a)

Proposition 1.3. *The class of functions which minimize the uncertainty product is given by*

$$f(t) = A\mathrm{e}^{-\pi|A|^4 \frac{t^2}{2}}$$

for parameter $A \in \mathbb{C}$. These functions satisfy

$$w_t \cdot w_\nu = \frac{1}{4\pi}$$

Proof. The first inequality in the proof of the Uncertainty principle applies Cauchy Schwartz:

$$\|t \cdot f(t)\|_{\mathcal{L}^2(dt)} \cdot \|\frac{1}{2\pi i} \cdot \frac{df}{dt}(t)\|_{\mathcal{L}^2(dt)} \geqslant \left| \left\langle t \cdot f(t), \frac{1}{2\pi i} \frac{df}{dt}(t) \right\rangle_{\mathcal{L}^2} \right|$$

Here, equality holds if and only if the inner product functions are linearly dependent. That is, there exists a $\lambda \in \mathbb{C}$ such that

$$t \cdot f(t) = -\lambda f'(t)$$

By the general solution to the decay equation, we see that

$$f(t) = A \cdot e^{-\int \frac{t}{\lambda} dt} = A \cdot e^{-\frac{t^2}{2\lambda}}$$

for some $A \in \mathbb{C}$.

The second inequality in the Uncertainty principle proof reduced the modulus of a complex number by looking at the modulus of the real part. Equality holds here if and only if the imaginary part is 0:

$$\Im\mathrm{m}(\langle tf(t), f'(t)\rangle) = 0 \tag{13}$$

Computing the LHS,

$$\begin{aligned}
\Im\mathrm{m}(\langle tf(t), f'(t)\rangle) &= \frac{1}{2i}(\langle tf(t), f'(t)\rangle - \overline{\langle tf(t), f'(t)\rangle}) \\
&= \frac{1}{2i}(\langle tf(t), f'(t)\rangle - \langle f'(t), tf(t)\rangle) \\
&= \frac{1}{2i}(\langle f(t), tf'(t)\rangle + \langle f(t), tf'(t) + f(t)\rangle) \\
&= \frac{1}{2i}(\langle f(t), f(t) + 2tf'(t\rangle) \\
&= \frac{1}{2i}(\langle f(t), f(t)\rangle + \langle f(t), 2tf'(t)\rangle) \\
&= \frac{1}{2i}\left(\|f\|^2_{\mathcal{L}^2} + \int_{-\infty}^{\infty} \overline{f(t)} \cdot 2tf'(t)\ dt\right) \\
&= \frac{1}{2i}\left(1 - \int_{-\infty}^{\infty} \overline{f(t)} \cdot \frac{2t^2}{\lambda} f(t)\ dt\right) \\
&= \frac{1}{2i}\left(1 - \frac{2}{\lambda}\int_{-\infty}^{\infty} \underbrace{t^2|f(t)|^2}_{\geqslant 0}\ dt\right)
\end{aligned}$$

we see that (13) implies

$$\lambda = 2\int_{-\infty}^{\infty} t^2|f(t)|^2\ dt$$

so, $\lambda \in \mathbb{R}$ with $\lambda > 0$.

Also, note the normalization condition:

$$\begin{aligned}
1 &= \|f\|_{\mathcal{L}^2}^2 \\
&= \int_{-\infty}^{\infty} |f(t)|^2\ dt \\
&= |A|^2 \int_{-\infty}^{\infty} e^{-\frac{t^2}{\lambda}}\ dt \\
&= |A|^2\sqrt{\pi\lambda}\underbrace{\int_{-\infty}^{\infty} e^{-\pi u^2}\ du}_{=1} \quad \text{(change variables: } u = \frac{t}{\sqrt{\pi\lambda}}\text{)} \\
&= |A|^2\sqrt{\pi\lambda}
\end{aligned}$$

Where the real change of variables is allowed since we showed $\lambda > 0$. So, $|A|^2 = \frac{1}{\sqrt{\pi\lambda}}$. Since $\lambda > 0$ we can write:

$$\lambda = \frac{1}{\pi|A|^4}$$

Therefore, we have that the class of functions which minimize uncertainty is

$$f(t) = Ae^{-\pi|A|^4\frac{t^2}{2}} \tag{14}$$

where $A \in \mathbb{C}$ is a parameter.

Via explicit computation, we can verify that f defined in (14) minimizes uncertainty. First compute the Fourier Transform with Gaussian invariance:

$$\widehat{f}(\nu) = \frac{1}{|A|^2}e^{\frac{-\pi\nu^2}{|A|^4}}$$

Noting that the standard deviation of a normal random variable with distribution:

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}$$

is σ, we see

$$\begin{aligned}
w_t &= \frac{1}{\sqrt{2\pi}|A|^2} \\
w_\nu &= \frac{|A|^2}{\sqrt{2\pi}}
\end{aligned}$$

for f and thus,

$$w_t \cdot w_\nu = \frac{1}{4\pi}$$

$\square$

(b) For $f \in \mathcal{S}(\mathbb{R};\mathbb{C})$, define

$$T_{\text{avg}} := \int_{-\infty}^{+\infty} |f(t)|^2 \; t \; \mathrm{d}t \tag{15}$$

$$\nu_{\text{avg}} := \int_{-\infty}^{+\infty} |\widehat{f}(\nu)|^2 \; \nu \; \mathrm{d}\nu \tag{16}$$

$$w_t := \left\{ \int_{-\infty}^{+\infty} |t - T_{\text{avg}}|^2 \;\; |f(t)|^2 \; \mathrm{d}t \right\}^{1/2} = \|f(t) \cdot (t - T_{\text{avg}})\|_{\mathcal{L}^2(\mathrm{d}t)} \tag{17}$$

$$w_\nu := \left\{ \int_{-\infty}^{+\infty} |\nu - \nu_{\text{avg}}|^2 \;\; |\widehat{f}(\nu)|^2 \; \mathrm{d}\nu \right\}^{1/2} = \|\widehat{f}(\nu) \cdot (\nu - \nu_{\text{avg}})\|_{\mathcal{L}^2(\mathrm{d}\nu)} \tag{18}$$

We have the following version of the Uncertainty Principle from class:

Theorem 1.3 (Uncertainty Principle for functions centered in time and frequency)**.** *For all $f \in \mathcal{S}(\mathbb{R};\mathbb{C})$ with $T_{\text{avg}} = \nu_{\text{avg}} = 0$,*

$$w_t \cdot w_\nu \geqslant \frac{pi}{4}$$

We first extend to functions which may not be centered in the time domain before extending to all Schwartz functions.

Lemma 1.4 (Uncertainty Principle for functions centered in frequency)**.** *For all $f \in \mathcal{S}(\mathbb{R};\mathbb{C})$ with $\nu_{\text{avg}} = 0$,*

$$w_t \cdot w_\nu \geqslant \frac{pi}{4}$$

Proof. Let $f \in \mathcal{S}(\mathbb{R};\mathbb{C})$ with $\nu_{\text{avg}} = 0$.

Define $g : \mathbb{R} \to \mathbb{C}$ with

$$g(t) = f(t + T_{\text{avg}})$$

Note that

$$\begin{aligned}
|\widehat{g}(\nu)| &= \left|\int_{-\infty}^{\infty} g(t)\ e^{-2\pi i\nu t}\ dt\right| \\
&= \left|\int_{-\infty}^{\infty} f(t+T_{\text{avg}})\ e^{-2\pi i\nu t}\ dt\right| \\
&= \left|\int_{-\infty}^{\infty} f(\tau)\ e^{-2\pi i\nu(\tau - T_{\text{avg}})}\ d\tau\right| \quad \text{(change variables: } \tau = t + T_{\text{avg}}) \\
&= \left| e^{2\pi i\nu T_{\text{avg}}} \underbrace{\int_{-\infty}^{\infty} f(\tau)\ e^{-2\pi i\nu\tau}\ d\tau}_{=\widehat{f}(\nu)} \right| \\
&= \left|e^{2\pi i\nu T_{\text{avg}}}\right| \cdot \left|\widehat{f}(\nu)\right| \\
&= \left|\widehat{f}(\nu)\right|
\end{aligned}$$

Thus, the distributions $|\widehat{g}|^2$ and $\left|\widehat{f}\right|^2$ have the same mean and standard deviation. We have the expected value of t for distribution $|g|^2$:

$$\begin{aligned}
T'_{\text{avg}} &= \int_{-\infty}^{\infty} t \cdot |g(t)|^2\ dt \\
&= \int_{-\infty}^{\infty} t \cdot |f(t+T_{\text{avg}})|^2\ dt \\
&= \int_{-\infty}^{\infty} (\tau - T_{\text{avg}})|f(\tau)|^2\ d\tau \quad \text{(change variables: } \tau = t + T_{\text{avg}}) \\
&= \underbrace{\int_{-\infty}^{\infty} \tau|f(\tau)|^2\ d\tau}_{=T_{\text{avg}}} - T_{\text{avg}} \underbrace{\int_{-\infty}^{\infty} |f(\tau)|^2\ d\tau}_{=1} \\
&= T_{\text{avg}} - T_{\text{avg}} \\
&= 0
\end{aligned}$$

and the variance:

$$\begin{aligned}
\underbrace{w_t^2}_{\text{for } f} &= \int_{-\infty}^{\infty} |t - T_{\text{avg}}|^2 |f(t)|^2 \, dt \\
&= \int_{-\infty}^{\infty} |\tau|^2 |f(\tau + T_{\text{avg}})|^2 \, d\tau \quad (\text{change variables: } \tau = t - T_{\text{avg}}) \\
&= \int_{-\infty}^{\infty} |\tau|^2 |g(\tau)|^2 \, d\tau \\
&= \underbrace{w_t^2}_{\text{for } g}
\end{aligned}$$

Since $g \in \mathcal{S}(\mathbb{R};\mathbb{C})$ with $T'_{\text{avg}} = 0$, it satisfies the Uncertainty Principle. So, for f, we have

$$w_t \cdot w_\nu \geqslant \frac{\pi}{4}$$

□

Theorem 1.5 (Uncertainty Principle)**.** *For all* $f \in \mathcal{S}(\mathbb{R};\mathbb{C})$*,*

$$w_t \cdot w_\nu \geqslant \frac{pi}{4}$$

Proof. Let $f \in \mathcal{S}(\mathbb{R};\mathbb{C})$. By the inversion property, let $h : \mathbb{R} \to \mathbb{C}$ such that

$$\widehat{h}(\nu) = \widehat{f}(\nu + \nu_{\text{avg}})$$

By the above computation, $|\widehat{h}|^2$ is centered at 0 with the same standard deviation w_ν as $|\widehat{f}|^2$. So, by Lemma 1.4, we have the uncertainty principle for h.

Apply the inversion property:

$$\begin{aligned}
|h(\nu)| &= \left| \int_{-\infty}^{\infty} \widehat{h}(\nu) \; e^{2\pi i \nu t} \right| \, d\nu \\
&= \left| \int_{-\infty}^{\infty} \widehat{f}(\nu + \nu_{\text{avg}}) e^{2\pi i \nu t} \, d\nu \right| \\
&= \left| \int_{-\infty}^{\infty} \widehat{f}(s) e^{2\pi i t (s - \nu_{\text{avg}})} \, ds \right| \quad (\text{change variables: } s = \nu + \nu_{\text{avg}}) \\
&= \left| e^{-2\pi i t \nu_{\text{avg}}} \underbrace{\int_{-\infty}^{\infty} \widehat{f}(s) \; e^{2\pi i t s} \, ds}_{= f(t)} \right| \\
&= \left| e^{2\pi i \alpha \nu} \; f(t) \right| \\
&= |f(t)|
\end{aligned}$$

Thus, h has the same intensity as f for all $t \in \mathbb{R}$. So, $|h|^2$ and $|f|^2$ have the same expected value and standard deviation in the time domain. Since h and f have the same width in both the time and frequency domain, the uncertainty principle extends to f:

$$w_t \cdot w_\nu \geqslant \frac{\pi}{4}$$

$\square$