PROBLEM SET NO. 3 - ISAAC VIVIANO

I affirm that I have adhered to the Honor Code on this assignment.

Isaac Viviano

1. Solutions:

- **Problem 1 - Interchanging integrals and derivatives:**

Theorem. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a continuous. If $\frac{\partial f}{\partial y}$ exists and and is continuous on all of $[a,b] \times [c,d]$, then one has*

$$\frac{\mathrm{d}}{\mathrm{d}y} \int_a^b f(x,y)\ \mathrm{d}x = \int_a^b \frac{\partial f}{\partial y}(x,y)\ \mathrm{d}x\ .$$

Proof. Let $\epsilon > 0$ be given. Since $\frac{\partial f}{\partial y}$ is continuous, $\frac{\partial f}{\partial y} : [c,d] \to \mathbb{R}$ is uniformly continuous. Pick $\delta > 0$ such that for all $x \in [a,b]$ and all $y_0, y_1 \in [c,d]$,

$$|y_0 - y_1| < \delta \implies \left| \frac{\partial f}{\partial y}(x,y_0) - \frac{\partial f}{\partial y}(x,y_1) \right|$$

Fix $x \in [a,b]$. At y_0,

$$\begin{aligned} \frac{d}{dy} \int_a^b f(x,y) &:= \frac{d}{dy} F(y_0) \\ &= \lim_{y \to y_0} \frac{F(y) - F(y_0)}{y - y_0} \end{aligned}$$

Now,

$$\begin{aligned} \left| \frac{F(y) - F(y_0)}{y - y_0} - \int_a^b \frac{\partial f}{\partial y}(x,y_0)\ dx \right| &= \left| \frac{\int_a^b f(x,y)\ dx - \int_a^b f(x,y_0)}{y - y_0}\ dx - \int_a^b \frac{\partial f}{\partial y}(x,y_0)\ dx \right| \\ &= \left| \int_a^b \frac{f(x,y) - f(x,y_0)}{y - y_0}\ dx - \int_a^b \frac{\partial f}{\partial y}(x,y_0)\ dx \right| \\ &= \left| \int_a^b \frac{f(x,y) - f(x,y_0)}{y - y_0} - \frac{\partial f}{\partial y}(x,y_0)\ dx \right| \\ &\leqslant \int_a^b \left| \frac{f(x,y) - f(x,y_0)}{y - y_0} - \frac{\partial f}{\partial y}(x,y_0) \right|\ dx \end{aligned}$$

By the (MVT), pick $y_1 \in [y, y_0]$ with

$$\frac{f(x,y) - f(x,y_0)}{y - y_0} = \frac{\partial f}{\partial y}(x,y_1)$$

Since $|y_0 - y| < \delta$ and $y_1 \in [y, y_0]$, $|y_0 - y_1| \leqslant |y_0 - y| < \delta$

$$\begin{aligned}
\int_a^b \left| \frac{f(x,y) - f(x,y_0)}{y - y_0} - \frac{\partial f}{\partial y}(x, y_0) \right| \, dx &= \int_a^b \left| \frac{\partial f}{\partial y}(x, y_1) - \frac{\partial f}{\partial x}(x, y_0) \right| \, dx \\
&\leqslant \int_a^b \frac{\epsilon}{b-a} \, dx \\
&= \epsilon
\end{aligned}$$

$\square$

- **Problem 2 - Properties of convolutions:**
  (c) **Smoothening property**, i.e.
  (c-i)

Proposition 1.1. *If $f \in \mathcal{R}_1(\mathbb{R}; \mathbb{C})$ and $g \in \mathcal{C}_1(\mathbb{R}; \mathbb{C})$, then $f \star g \in \mathcal{C}_1(\mathbb{R}; \mathbb{C})$.*

Proof. Suppose $f \in \mathcal{R}_1(\mathbb{R}; \mathbb{C})$ and $g \in \mathcal{C}_1(\mathbb{R}; \mathbb{C})$. Let

$$h(t) := (f \star g)(t) = \int_0^1 f(s) g(t-s) \, ds$$

Note that f is Riemann integrable and thus bounded, and g is periodic and continuous and thus uniformly continuous.
Let $\epsilon > 0$ be given and fix $t_0 \in \mathbb{R}$. Since g is uniformly continuous, pick $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|x - y| < \delta$, $|g(x) - g(y)| < \frac{\epsilon}{\|f\|_\infty}$. Let $t \in \mathbb{R}$ with $|t - t_0| < \delta$. Then for all $s \in [0, 1]$,

$$\begin{aligned}
|(t-s) - (t_0 - s)| &= |t - s - t_0 + s| \\
&= |t - t_0| \\
&< \delta
\end{aligned}$$

So,

$$\begin{aligned}
|h(t) - h(t_0)| &= \left| \int_0^1 f(s)g(t-s)\ ds - \int_0^1 f(s)g(t_0 - s)\ ds \right| \\
&= \left| \int_0^1 f(s)g(t-s) - f(s)g(t_0-s)\ ds \right| \\
&= \left| \int_0^1 f(s)(g(t-s) - g(t_0-s))\ ds \right| \\
&\leqslant \int_0^1 |f(s)| \cdot |g(t-s) - g(t_0-s)|\ ds \\
&< \int_0^1 |f(s)| \cdot \frac{\epsilon}{\|f\|_\infty}\ ds \\
&= \frac{\epsilon}{\|f\|_\infty} \int_0^1 |f(s)|\ ds \\
&\leqslant \frac{\epsilon}{\|f\|_\infty} \cdot \|f\|_\infty \quad \text{(ML estimate)} \\
&= \epsilon
\end{aligned}$$

Therefore, h is continuous.
For all t, we have

$$\begin{aligned}
h(t) &= \int_0^1 f(s)g(t-s)\ ds \\
&= \int_0^1 f(s)g(t-s+1)\ ds \quad (g \text{ is 1-periodic}) \\
&= \int_0^1 f(s)g((t+1)-s)\ ds \\
&= h(t+1)
\end{aligned}$$

showing that h is also 1-periodic. □

(c-ii)

Proposition 1.2. *If $f \in \mathcal{C}_1(\mathbb{R};\mathbb{C})$ and, for some $k \in \mathbb{N}$, one has $g \in \mathcal{C}_1^k(\mathbb{R};\mathbb{C})$, then $f \star g \in \mathcal{C}_1^k(\mathbb{R};\mathbb{C})$.*

Proof. Let $f \in \mathcal{C}_1(\mathbb{R};\mathbb{C})$ and $g \in \mathcal{C}_1^k(\mathbb{R};\mathbb{C})$. Let $h = f \star g$. We have

Note that $f(s)g(t-s)$ is k-times continuously differentiable with respect to t. By problem (1),

$$\begin{aligned}\frac{d}{dt}h(t) &= \int_0^1 \frac{\partial}{\partial t}(f(s)g(t-s))\ ds \\ &= \int_0^1 f(s)g'(t-s)\ ds\end{aligned}$$

We may do this k times iteratively, showing

$$h^{(k)}(t) = \int_0^1 f(s)g^{(k)}(t-s)\ ds$$

Since f and $g^{(k)}$ are continuous and 1-periodic, $h^{(k)}$ is continuous and 1-periodic. Therefore,

$$f \star g = h \in \mathcal{C}_1^k(\mathbb{R};\mathbb{C})$$

□

(d) **A-priori bound for convolutions**

Proposition 1.3. *If $f, g \in \mathcal{R}_1(\mathbb{R};\mathbb{C})$, then $f \star g$ is bounded with*

$$\|f \star g\|_\infty \leqslant \min\{\|f\|_\infty \cdot \|g\|_1\ ;\ \|g\|_\infty \cdot \|f\|_1\} \tag{1}$$

Proof. It suffices to show that

$$\|f \star g\|_\infty \leqslant \|g\|_\infty \cdot \|f\|_1$$

Then, the symmetry property in part (a) gives

$$\|f \star g\|_\infty = \|g \star f\|_\infty \leqslant \|f\|_\infty \cdot \|g\|_1$$

Proof:

$$\begin{aligned}\|f \star g\|_\infty &= \left\| \int_0^1 f(s)g(t-s)\ ds \right\|_\infty \\ &= \sup_{t \in \mathbb{R}} \left\{ \left| \int_0^1 f(s)g(t-s)\ ds \right| \right\} \\ &\leqslant \sup_{t \in \mathbb{R}} \left\{ \int_0^1 |f(s)| \cdot |g(t-s)|\ ds \right\} \\ &\leqslant \sup_{t \in \mathbb{R}} \left\{ \|g\|_\infty \int_0^1 |f(s)|\ ds \right\} \\ &= \|g\|_\infty \int_0^1 |f(s)|\ ds \\ &= \|g\|_\infty \cdot \|f\|_1\end{aligned}$$

□

(e) **Continuity w.r.t. uniform convergence**

Proposition 1.4. *if $f \in \mathcal{R}_1(\mathbb{R};\mathbb{C})$ and (g_n) is a sequence in $\mathcal{R}_1(\mathbb{R};\mathbb{C})$ such that $g_n \to g$ in the L^1-sense, then*

$$f \star g_n \to f \star g \text{ , uniformly .} \tag{2}$$

Proof. Suppose $g_n \to g$ with respect to the L1 norm and let $\epsilon > 0$ be given. Pick $N \in \mathbb{N}$ such that for all $n \geqslant N$,

$$\|g_n - g\|_1 < \frac{\epsilon}{\|f\|_\infty}$$

Then,

$$\begin{aligned}
\|f \star g_n - f \star g\|_\infty &= \|f \star (g_n - g)\|_\infty \\
&\leqslant \min\{\|f\|_\infty \cdot \|g_n - g\|_1, \|g_n - g\|_\infty \cdot \|f\|_1\} \\
&\leqslant \|f\|_\infty \cdot \|g_n - g\|_1 \\
&< \|f\|_\infty \cdot \frac{\epsilon}{\|f\|_\infty} \\
&= \epsilon
\end{aligned}$$

So, $f \star g_n$ converges to $f \star g$ uniformly. $\square$

- **Problem 3 - "the Dirichlet kernel is not a periodic approximate identity:"** Recall the Dirichlet kernel from class:

$$D_n(x) = \frac{\sin(\pi(2n+1)x)}{\sin(\pi x)} \text{ , } x \in [-\tfrac{1}{2}, \tfrac{1}{2}], \text{ for } n \in \mathbb{N}.$$

In the following you will show that

$$\frac{4}{\pi^2}\log(n) \leqslant \int_{-1/2}^{1/2} |D_n(x)| \, \mathrm{d}x \leqslant 3 + \log(n) \text{ ,} \tag{3}$$

whence (D_n) is *not* an approximate identity because it fails the property (PAI-2) from our definition. To prove (3), follow the outline below.

(a)

Proposition 1.5. *If $f : [1, +\infty] \to \mathbb{R}$ is non-negative and monotone decreasing then*

$$\int_1^{N+1} f(x) \, \mathrm{d}x \leqslant \sum_{n=1}^{N} f(n) \leqslant f(1) + \int_1^N f(x) \, \mathrm{d}x \text{ .}$$

Proof. Note that for $a, b \geqslant 1$,

$$\sup\{f(x) : x \in [a,b]\} = f(a)$$

and

$$\inf\{f(x) : x \in [a,b]\} = f(b)$$

because f is monotone decreasing.
Thus, for any partition $s_1, \dots, s_{N+1}$ of $[1, N+1]$, our upper and lower bounds are given by

$$\begin{aligned} M_i &= f(s_i) \\ m_i &= f(s_{i+1}) \end{aligned}$$

Consider the partition given by $s_i = i$ for all $1 \leqslant i \leqslant N+1$. By definition,

$$\begin{aligned} \int_1^{N+1} f(x)\ dx &\leqslant \sum_{i=1}^{N} M_i(s_{i+1} - s_i) \\ &= \sum_{i=1}^{N} f(s_i)(i+1-i) \\ &= \sum_{i=1}^{N} f(i) \end{aligned}$$

and

$$\begin{aligned} f(1) + \int_1^{N} f(x)\ dx &\geqslant f(1) + \sum_{i=1}^{N-1} m_i(s_{i+1} - s_i) \\ &= f(1) + \sum_{i=1}^{N-1} f(s_{i+1})(i+1-i) \\ &= f(1) + \sum_{i=1}^{N-1} f(i+1) \\ &= f(1) + \sum_{i=2}^{N} f(i) \quad \text{(change index)} \\ &= \sum_{i=1}^{N} f(i) \end{aligned}$$

□

(b)

Proposition 1.6.

$$\frac{4}{\pi^2}\log(n) \leqslant \int_{-1/2}^{1/2} |D_n(x)|\ \mathrm{d}x \leqslant 3 + \log(n)\ , \tag{4}$$

Proof. First note that $|D_n(x)|$ is even:

$$\begin{aligned}|D_n(-x)| &= \left|\frac{\sin(\pi(2n+1)(-x))}{\sin(-\pi x)}\right| \\ &= \left|\frac{-\sin(\pi(2n+1)x)}{-\sin(\pi x)}\right| \\ &= \left|\frac{\sin(\pi(2n+1)x)}{\sin(\pi x)}\right| \\ &= |D_n(x)|\end{aligned}$$

Therefore,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |D_n(x)|\ dx = 2\int_0^{\frac{1}{2}} |D_n(x)|\ \ dx := 2I_n$$

Let

$$4a_n = \frac{2}{2n+1}$$

be the period of $f_n := |\sin(\pi(2n+1)x)|$. f_n has zeros when $\pi(2n+1)x$ is an integer multiple of π, that is, $(2n+1)x$ is an integer, or $x = 2ka_n$ for some $k \in \mathbb{N}$. It has maxima at $\pi(2n+1)x - \frac{\pi}{2}$ is an integer. These occur when $x = (2k+1)a_n$ for some $k \in \mathbb{N}$. The only extrema are these zeros and maxima. Since f_n is continuous, it is monotonic non-decreasing from a zero to a maximum and monotonic non-increasing from a maximum to a zero. These intervals of monotonicity are

$$\begin{aligned}&[0, a_n] \\ &[a_n, 2a_n] \\ &[2a_n, 3a_n] \\ &[3a_n, 4a_n]\end{aligned}$$

on its period interval $[0, 4a_n]$.

$$\begin{aligned}
I_n &= \int_0^{\frac{1}{2}} |D_n(x)|\ dx \\
&= \int_0^{\frac{1}{2}} \frac{|\sin(\pi(2n+1)x)|}{|\sin(\pi x)|}\ dx \\
&= \sum_{j=0}^{2n} \int_{ja_n}^{(j+1)a_n} \frac{|\sin(\pi(2n+1)x)|}{\sin \pi x}\ dx \\
&\geqslant \sum_{j=0}^{2n} \int_{ja_n}^{(j+1)a_n} \frac{|\sin(\pi(2n+1)x)|}{\pi x}\ dx \\
&\geqslant \sum_{j=0}^{2n} \frac{1}{\pi(j+1)a_n} \int_{ja_n}^{(j+1)a_n} |\sin(\pi(2n+1)x)|\ dx
\end{aligned}$$

From the previous observation,

$$\int_{ja_n}^{(j+1)a_n} |\sin(\pi(2n+1)x)\ dx = \int_0^{a_n} \sin(\pi(2n+1)x)\ dx$$

We may use the change of variables $u = \pi(2n+1)x$ to write

$$\begin{aligned}
\int_{ja_n}^{(j+1)a_n} \sin(\pi(2n+1)x) &= \int_0^{a_n} \sin(\pi(2n+1)x)\ dx \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{\pi(2n+1)} \sin u\ du \\
&= \left.\frac{-\cos u}{\pi(2n+1)}\right|_0^{\frac{\pi}{2}} \\
&= 0 - \frac{-1}{\pi(2n+1)} \\
&= \frac{2a_n}{\pi}
\end{aligned}$$

Thus,

$$
\begin{aligned}
I_n &\geqslant \sum_{j=0}^{2n} \frac{1}{\pi(j+1)a_n} \cdot \frac{2a_n}{\pi} \\
&= \sum_{j=1}^{2n+1} \frac{2}{\pi^2 j} \\
&= \frac{2}{\pi^2} \sum_{j=1}^{2n+1} \frac{1}{j} \\
&\geqslant \frac{2}{\pi^2} \int_1^{2n+2} \frac{1}{x}\ dx \qquad \text{(a)} \\
&= \frac{2}{\pi^2} \ln x \Big|_1^{2n+2} \\
&= \frac{2}{\pi^2} \ln(2n+2) \\
&\geqslant \frac{2}{\pi^2} \ln n
\end{aligned}
$$

For the upper bound,

$$
\begin{aligned}
I_n &= \int_0^{\frac{1}{2}} \frac{|\sin(\pi(2n+1)x)|}{|\sin(\pi x)|}\ dx \\
&= \int_0^{a_n} \frac{|\sin(\pi(2n+1)x)|}{|\sin(\pi x)|}\ dx + \sum_{j=1}^{2n} \int_{ja_n}^{(j+1)a_n} \frac{|\sin(\pi(2n+1)x)|}{|\sin(\pi x)|}\ dx \\
&= \int_0^{a_n} \left|\frac{\sin(\pi(2n+1)x)}{\pi(2n+1)x}\right| \cdot \left|\frac{\pi(2n+1)x}{\pi x}\right| \cdot \left|\frac{\pi x}{\sin(\pi x)}\right|\ dx + \\
&\qquad \sum_{j=1}^{2n} \int_{ja_n}^{(j+1)a_n} \frac{|\sin(\pi(2n+1)x)|}{|\sin(\pi x)|}\ dx \\
&\leqslant \int_0^{a_n} 2n+1\ dx + \sum_{j=1}^{2n} \int_{ja_n}^{(j+1)a_n} \frac{|\sin(\pi(2n+1)x)|}{|\sin(\pi x)|}\ dx \\
&\leqslant \int_0^{\frac{1}{2(2n+1)}} 2n+1\ dx + \sum_{j=1}^{2n} \int_{ja_n}^{(j+1)a_n} \frac{1}{2x}\ dx \\
&= \frac{1}{2} + \sum_{j=1}^{2n} \frac{1}{2ja_n} \cdot a_n \quad \text{(ML-estimate)} \\
&= \frac{1}{2} + \frac{1}{2} \sum_{j=1}^{2n} \frac{1}{j} \\
&\leqslant \frac{1}{2} + \frac{1}{2}\left(1 + \int_1^{2n} \frac{1}{x}\ dx\right) \\
&= \frac{1}{2} + \frac{1}{2}\left(\ln x \Big|_1^{2n}\right) \\
&= \frac{1}{2} + \frac{1}{2}(\ln 2n)
\end{aligned}
$$

We have

$$
\begin{aligned}
I_n &\geqslant \frac{2}{\pi^2} \log n \\
I_n &\leqslant \frac{1}{2} + \frac{1}{2}(\ln 2n) \\
2I_n &\leqslant 1 + \ln(2n) \\
&= 1 + \ln 2 + \ln n \\
&\leqslant 3 + \ln n \\
2I_n &\geqslant \frac{4}{\pi^2} \log n
\end{aligned}
$$

showing the desired bounds. □

- **Problem 4 - Existence of Minimal Periods for Periodic Functions:**

Lemma 1.1. *Let $f : \mathbb{R} \to \mathbb{C}$ be periodic and let Π_f denote the set of all periods of f. Then,*
(1) *$\Pi_f \cup \{0\}$ is an additive subgroup of $\mathbb{R}$*
(2) *If $0 \notin cl\, \Pi_f$, then Π_f has no other limit point and f has a minimal period*

Proof. (1) Let $a, b \in \Pi_f \cup \{0\}$. We have that for all $x \in \mathbb{R}$,

$$f(x) = f(a + x)$$

and

$$f(x) = f(b + x) = f(b + (x - 2b)) = f(x - b)$$

Then,

$$\begin{aligned} f(x + a - b) &= f((x + a) - b) \\ &= f(x + a) \\ &= f(x) \end{aligned}$$

So, $a - b \in \Pi_f \cup \{0\}$, showing that $\Pi_f \cup \{0\}$ is an additive subgroup of $\mathbb{R}$.

(2) For contradiction, suppose $0 \notin \text{cl } \Pi_f$ and Π_f has a limit point. Let $a \in \text{cl } \Pi_f$ be a limit point of Π_f. Then, there exists a sequence

$$\{a_n\} \subseteq \Pi_f$$

with $a_n \to a$. Let $\epsilon > 0$ be given. Pick N such that for all $n, m \geqslant N$,

$$|a_n - a_m| < \epsilon$$

Fix $n \geqslant N$. We have $a_n - a_m, a_m - a_n \in \Pi_f$ by (1). Let

$$b_n = \max\{a_n - a_m, a_m - a_n\} = |a_n - a_m|$$

So, $0 \leqslant b_n < \epsilon$. Therefore, $b_n \in \Pi_f$ and $b_n \to 0$. This shows that 0 is a limit point of Π_f and thus $0 \in \text{cl } \Pi_f$.

Now suppose $0 \notin \text{cl } \Pi_f$. Let T be a period of f. If

$$[0, T] \cap \Pi_f$$

is infinite, it has a limit point. But, this would be a limit point of Π_f, which has no limit points. Thus, $[0, T] \cap \Pi_f$ is finite and nonempty. $[0, T] \cap \Pi_f$ must have a minimum, showing that f has a minimal positive period. □

Proposition 1.7. *If $f : \mathbb{R} \to \mathbb{C}$ is periodic and has atleast one point of continuity, then f has a minimal period.*

Proof. Suppose $f : \mathbb{R} \to \mathbb{C}$ is periodic, non-constant, and has at least point of continuity.

Let
$$P = \{|p| : f \text{ is } p\text{-periodic}\}$$
Note that P is nonempty since f is periodic and bounded below by zero, so P has an infimum $\alpha = \inf P$

Let x_0 be a point of continuity of f. Since f is not constant, pick $y \in \mathbb{R}$ with $f(x_0) \neq f(y)$. Pick $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x_0 - x| < \delta$,
$$|f(x_0) - f(x)| < |f(x_0) - f(y)|$$
Claim: $\frac{\delta}{2}$ is a lower bound for P. If $p \leqslant \frac{\delta}{2}$ is a period of f, then for all $n \in \mathbb{Z}$,
$$f(y + np) = f(y)$$
Taking
$$n = \left\lfloor \frac{1}{p}(x_0 - y) \right\rfloor$$
we have
$$\begin{aligned} y + np &= y + p \left\lfloor \frac{1}{p}(x_0 - y) \right\rfloor \\ &\leqslant y + p \cdot \frac{1}{p}(x_0 - y) \\ &= y + x_0 - y \\ &= x_0 \\ &< x_0 + \delta \end{aligned}$$
and
$$\begin{aligned} y + np &= y + p \left\lfloor \frac{1}{p}(x_0 - y) \right\rfloor \\ &\geqslant y + p \left(\frac{1}{p}(x_0 - y) - 1 \right) \\ &= y + x_0 - y - p \\ &= x_0 - p \\ &\geqslant x_0 - \frac{\delta}{2} \\ &> x_0 - \delta \end{aligned}$$
Therefore,
$$y + np \in (x_0 - \delta, x_0 + \delta)$$
since
$$y + np < x_0 + \delta$$

and

$$y + np > x_0 - \delta$$

But $f(y+np) \in (x_0 - \delta, x_0 + \delta)$ implies that

$$|f(x_0) - f(y+np)| < |f(x_0) - f(y)|$$

In particular,

$$|f(x_0) - f(y+np)| \neq |f(x_0) - f(y)|$$

Thus,

$$f(y+np) \neq f(y)$$

contradicting that p is a period. By contradiction, for all $p \in P$, $\frac{\delta}{2} < p$.

Since α is greater than or equal to any lower bound for P,

$$\alpha \geqslant \frac{\delta}{2} > 0$$

Suppose 0 were a limit point of Π_f. Then we would have a sequence p_n in Π_f with $\pi_n \to 0$. But $|p_n|$ is a sequence in P and $|p_n| \to 0$, contradicting that $\alpha \geqslant 0$. Therefore, 0 is a not a limit point of Π_f, and f is has a minimal positive period by Lemma 1.1.

□