PROBLEM SET NO. 7

I affirm that I have adhered to the Honor Code in this assignment. Isaac Viviano blank

1. Problems:

- **Problem 1 - Equidistribution Theorem for Riemann integrable functions:**

(a)

Lemma 1.1. *Given an interval $I \subseteq (0,1)$, let f_I denote the periodic version of the indicator function for I given in ([CM]; 7.66). For each $\epsilon > 0$ sufficiently small, there exist functions $f_{\pm}^{(\epsilon)} \in \mathcal{C}_1(\mathbb{R};\mathbb{C})$ satisfying*

$$f_{-}^{(\epsilon)}(t) \leqslant f_I(t) \leqslant f_{+}^{(\epsilon)}(t) \text{ , for all } t \in \mathbb{R} \text{ ,} \tag{1}$$

so that

$$\left| \int_0^1 \left(f_{\pm}^{(\epsilon)}(t) - f_I(t) \right) \, \mathrm{d}t \right| \leqslant \epsilon \ . \tag{2}$$

Proof. Let $0 < \epsilon < \min\{a, 1-b, \frac{b-a}{2}\}$ and define:

$$f_{-}^{(\epsilon)}(t) := \begin{cases} 0 & \text{if } x \in [0,a] \\ \frac{x-a}{\epsilon} & \text{if } x \in [a, a+\epsilon] \\ 1 & \text{if } x \in [a+\epsilon, b-\epsilon] \\ \frac{b-x}{\epsilon} & \text{if } x \in [b-\epsilon, b] \\ 0 & \text{if } x \in [b,1] \end{cases}$$

$$f_{+}^{(\epsilon)}(t) := \begin{cases} 0 & \text{if } x \in [0,a-\epsilon] \\ \frac{x-a+\epsilon}{\epsilon} & \text{if } x \in [a-\epsilon, a] \\ 1 & \text{if } x \in [a, b] \\ \frac{b-x+\epsilon}{\epsilon} & \text{if } x \in [b, b+\epsilon] \\ 0 & \text{if } x \in [b,1] \end{cases}$$

We see that $f_{+}^{(\epsilon)}$ and $f_{-}^{(\epsilon)}$ are the functions depicted in the picture. Using geometric evaluation, the trapezoidal areas may be found

$$\begin{aligned} \int_0^1 f_{+}^{(\epsilon)}(x) - f(x)\ dx &= \int_0^1 f_{+}^{(\epsilon)}(x)\ dx - \int_0^1 f(x)\ dx \\ &= (b-a) + \epsilon - (b-a) \\ &= \epsilon \\ \int_0^1 f_{-}^{(\epsilon)}(x) - f(x)\ dx &= \int_0^1 f_{-}^{(\epsilon)}(x)\ dx - \int_0^1 f(x)\ dx \\ &= (b-a-\epsilon) - (b-a) \\ &= -\epsilon \end{aligned}$$

showing (2). $\square$

(b)

Theorem 1.2. *Given $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, let R_α be the associated rotation map. For an arbitrary, fixed interval $I \subseteq [0,1)$, let*

$$\mathcal{N}_n(x;I) := \#\{k \in \{0,\ldots,n-1\} | R_\alpha^k(x) \in I\}$$

Then,

$$\frac{\mathcal{N}_n(x;I)}{n} \to |I|$$

uniformly, as $n \to \infty$.

Proof. We formulate Theorem 1.2 as a special case of Theorem 1.3. Let f_I be the 1-periodic extension of

$$\chi_I(x) = \begin{cases} 1, \text{if } x \in I \\ 0, \text{if } x \notin I \end{cases}$$

We may write:

$$\frac{1}{n}\mathcal{N}_n(x,I) = \frac{1}{n}\sum_{n=0}^{n-1} f_I(x+k\alpha)$$

and

$$|I| = \int_0^1 f_I(x)\ dx = \widehat{[f_I]}_0$$

Now, we prove Theorem 1.3 for the 1-periodic indicator function f_I. Let $s_\pm^{(\epsilon)} = f_\pm^{(\epsilon)}$ and $f = f_I$. Estimate:

$$\begin{aligned}
\left|\frac{1}{n}\sum_{k=0}^{n-1} f(x+k\alpha) - \hat{f}_0\right| &= \left|\frac{1}{n}\sum_{k=0}^{n-1} f(x+k\alpha) - \frac{1}{n}\sum_{k=0}^{n-1} s_-^{(\epsilon)}(x+k\alpha) + \frac{1}{n}\sum_{k=0}^{n-1} s_-^{(\epsilon)}(x+k\alpha) - \hat{f}_0\right| \quad (3)\\
&\leqslant \left|\frac{1}{n}\sum_{k=0}^{n-1} f(x+k\alpha) - \frac{1}{n}\sum_{k=0}^{n-1} s_-^{(\epsilon)}(x+k\alpha)\right| + \left|\frac{1}{n}\sum_{k=0}^{n-1} s_-^{(\epsilon)}(x+k\alpha) - \hat{f}_0\right| \quad (4)\\
&\leqslant \left|\frac{1}{n}\sum_{k=0}^{n-1} f(x+k\alpha) - s_-^{(\epsilon)}(x+k\alpha)\right| \quad (5)\\
&+ \left|\frac{1}{n}\sum_{k=0}^{n-1} s_-^{(\epsilon)}(x+k\alpha) - \widehat{[s_-^{(\epsilon)}]}_0\right| \quad (6)\\
&+ \left|\widehat{[s_-^{(\epsilon)}]}_0 - \hat{f}_0\right| \quad (7)
\end{aligned}$$

Note that (6) converges to 0 uniformly by the version of Theorem 1.3 proven in class for continuous functions. For (7), Lemma 1.1 gives

$$\left|\widehat{[s_-^{(\epsilon)}]}_0 - \hat{f}_0\right| = \left|\int_0^1 s_-^{(\epsilon)}(t) - f(t)\ dt\right| < \epsilon$$

We estimate (5):

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} f(x+k\alpha) - s_-^{(\epsilon)}(x+k\alpha)\right| = \frac{1}{n}\sum_{k=0}^{n-1} \underbrace{f(x+k\alpha)}_{\leqslant\ s_+^{(\epsilon)}(x+k\alpha)} - s_-^{(\epsilon)}(x+k\alpha) \tag{8}$$

$$\leqslant \frac{1}{n}\sum_{k=0}^{n-1} s_+^{(\epsilon)}(x+k\alpha) - s_-^{(\epsilon)}(x+k\alpha) \tag{9}$$

$$= \left|\frac{1}{n}\sum_{k=0}^{n-1} s_+^{(\epsilon)}(x+k\alpha) - s_-^{(\epsilon)}(x+k\alpha)\right| \tag{10}$$

$$\leqslant \left|\frac{1}{n}\sum_{k=0}^{n-1} s_+^{(\epsilon)}(x+k\alpha) - \widehat{[s_-^{(\epsilon)}]}_0\right| \tag{11}$$

$$+ \left|\widehat{[s_+^{(\epsilon)}]}_0 - \widehat{[s_-^{(\epsilon)}]}_0\right| \tag{12}$$

$$+ \left|\frac{1}{n}\sum_{k=0}^{n-1} s_+^{(\epsilon)}(x+k\alpha) - \widehat{[s_-^{(\epsilon)}]}_0\right| \tag{13}$$

Again, (11) and (13) go to zero uniformly in x by the version of Theorem 1.3 proven in class for continuous functions. For (12) gives,

$$\begin{aligned}
\left|\widehat{[s_+^{(\epsilon)}]}_0 - \widehat{[s_-^{(\epsilon)}]}_0\right| &= \left|\int_0^1 s_+^{(\epsilon)}(t) - s_-^{(\epsilon)}(t)\ dt\right| \leqslant \epsilon \\
&= \left|\int_0^1 s_+^{(\epsilon)}(t) - f(t)\ dt + \int_0^1 f(t) - s_-^{(\epsilon)}(t)\ dt\right| \\
&= \left|\int_0^1 s_+^{(\epsilon)}(t) - f(t)\ dt\right| + \left|\int_0^1 f(t) - s_-^{(\epsilon)}(t)\ dt\right| \\
&\leqslant \epsilon + \epsilon
\end{aligned}$$

so, we have proven Theorem 1.3 for 1-periodic indicator functions. □

(c)

Theorem 1.3. *Fix $\alpha \in \mathbb{R} - \mathbb{Q}$ and let $f \in \mathcal{R}_1(\mathbb{R};\mathbb{C})$ be arbitrary. The, we have the Cesaro means of the values of f evaluated along the $\alpha-$finite rotational orbits converge uniformly in x with*

$$\frac{1}{n}\sum_{k=0}^{n-1} f(x+k\alpha) \to \int_0^1 f(t)\ dt = \hat{f}_0$$

as $n \to \infty$.

Proof. We proved Theorem 1.3 for 1-periodic indicator functions in part (b). We begin by extension of Theorem 1.3 from 1-periodic indicator functions to 1-periodic real-valued step functions. We consider an arbitrary step function f defined by

$$f = \sum_{I \in A} a_I f_I$$

where A is a finite collection of intervals and $a_I \in \mathbb{R}$ for all $I \in A$, and f_I denotes the 1-periodic extension of the indicator χ_I. For any $x \in \mathbb{R}$,

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} f(x + k\alpha) - \hat{f}_0 \right| = \left| \frac{1}{n} \sum_{k=0}^{n-1} \sum_{I \in A} a_I f_I(x + k\alpha) - \left[\widehat{\sum_{I \in A} a_I f_I} \right]_0 \right| \tag{14}$$

$$= \left| \sum_{I \in A} a_I \frac{1}{n} \sum_{k=0}^{n-1} f_I(x + k\alpha) - \sum_{I \in A} a_I [\widehat{f_I}]_0 \right| \tag{15}$$

$$= \left| \sum_{I \in A} a_I \left(\frac{1}{n} \sum_{k=0}^{n-1} f_I(x + k\alpha) - [\widehat{f_I}]_0 \right) \right| \tag{16}$$

where each term of the finite sum in (16) converges to 0 uniformly in x. Therefore, the Equidistribution Theorem holds for all real-valued 1-periodic step functions.

We next extend from 1-periodic step functions to real-valued Riemann integrable functions. Let $f \in \mathcal{R}_1(\mathbb{R}; \mathbb{R})$ and $\epsilon > 0$ be arbitrary.

First, we show there exist 1-periodic step functions $s_{\pm}^{(\epsilon)}$ such that $s_{-}^{(\epsilon)} \leqslant f \leqslant s_{+}^{(\epsilon)}$ and

$$\left| \int_0^1 \left(s_{\pm}^{(\epsilon)}(t) - f(t) \right) \, dt \right| \leqslant \epsilon \tag{17}$$

We first observe that for any partition of $[0, 1]$, the upper and lower Riemann sums may be written as the integrals of step functions. By the definition of Riemann integrable, pick a partition $x_0 < \cdots < x_n$ of $[0, 1]$ such that

$$\sum_{i=0}^{n-1} (M_i - m_i)(x_{i+1} - x_i) < \epsilon$$

where

$$M_i := \sup_{x \in [x_i, x_{i+1})} f(x),\ 0 \leqslant i < n$$
$$m_i := \inf_{x \in [x_i, x_{i+1})} f(x),\ 0 \leqslant i < n$$

Note: use a closed interval for the final $i = n-1$ case. Define two step functions

$$s_+^{(\epsilon)}(x) := \sum_{i=0}^{n-1} M_i \chi_{[x_i, x_{i+1})}(x) \geqslant f(x)$$
$$s_-^{(\epsilon)}(x) := \sum_{i=0}^{n-1} m_i \chi_{[x_i, x_{i+1})}(x) \leqslant f(x)$$

Observe that these step functions integrate to the Riemann sums:

$$\int_0^1 s_+^{(\epsilon)}(x)\ dx = \sum_{i=0}^{n-1} M_i (x_{i+1} - x_i)$$
$$\int_0^1 s_-^{(\epsilon)}(x)\ dx = \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i)$$

Additionally, we have $s_-^{(\epsilon)} \leqslant f \leqslant s_+^{(\epsilon)}$. For all $x \in [0, 1]$, we have $x_i \leqslant x < x_{i+1}$ for some i or $x_{n-1} \leqslant x \leqslant x_n$. In the first case,

$$s_-^{(\epsilon)}(x) = m_i \leqslant f(x) \leqslant M_i = s_+^{(\epsilon)}(x)$$

and in the second case,

$$s_-^{(\epsilon)}(x) = m_{n-1} \leqslant f(x) \leqslant M_{n-1} = s_+^{(\epsilon)}(x)$$

For the desired bound in (17):

$$0 \leqslant \int_0^1 s_+^{(\epsilon)}(t) - \underbrace{f(t)}_{\geqslant s_-^{(\epsilon)}(t)} \; dt \leqslant \int_0^1 s_+^{(\epsilon)}(t)\; dt - \int_0^1 s_-^{(\epsilon)}(t)\; dt$$

$$= \sum_{i=0}^{n-1} M_i(x_{i+1} - x_i) - \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i)$$

$$= \sum_{i=0}^{n-1} (M_i - m_i)(x_{i+1} - x_i)$$

$$< \epsilon$$

$$0 \leqslant \int_0^1 \underbrace{f(t)}_{\leqslant s_+^{(\epsilon)}(t)} - s_-^{(\epsilon)}(t)\; dt \leqslant \int_0^1 s_+^{(\epsilon)}(t)\; dt - \int_0^1 s_-^{(\epsilon)}(t)\; dt < \epsilon$$

Giving

$$\left| \int_0^1 s_\pm^{(\epsilon)}(t) - f(t)\; dt \right| < \epsilon$$

Observe that this is the same exact set up as part (b), since we have Theorem 1.3 for $s_\pm^{(\epsilon)}$. Thus, we have proven Theorem 1.3 for real-valued Riemann integrable functions f.

Extension to complex valued functions: Let $f \in \mathcal{R}_1(\mathbb{R}; \mathbb{C})$ be defined by

$$f = h + ig$$

for $h, g \in \mathcal{R}_1(\mathbb{R}; \mathbb{R})$.

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} f(x + k\alpha) - \hat{f}_0 \right| = \left| \frac{1}{n} \sum_{k=0}^{n-1} (h(x + k\alpha) + ig(x + k\alpha)) - [\widehat{h - ig}]_0 \right| \quad (18)$$

$$= \left| \frac{1}{n} \sum_{k=0}^{n-1} h(x + k\alpha) - \hat{h}_0 + \frac{1}{n} \sum_{k=0}^{n-1} ig(x + k\alpha) - i\hat{g}_0 \right| \quad (19)$$

$$\leqslant \left| \frac{1}{n} \sum_{k=0}^{n-1} h(x + k\alpha) - \hat{h}_0 \right| + \left| \frac{1}{n} \sum_{k=0}^{n-1} g(x + k\alpha) - \hat{g}_0 \right| \quad (20)$$

where both terms of (20) converge to 0 uniformly in x as $n \to \infty$ by the Equidistribution Theorem for real-valued Riemann integrable functions. □

- **Problem 2 - Rate of Convergence and Diophantine Conditions:**

(c)

Proposition 1.1. *There exists an irrational number α such that $\alpha \notin \mathcal{DC}$.*

Proof. Let $\phi : [0, \infty) \to [0, \infty)$ be defined by

$$\phi(x) := e^x$$

Define

$$f_n(t) := e^{-|n|}\; e^{2\pi int},\; t \in \mathbb{R}$$

Note that

$$\|f_n\|_\infty = e^{-|n|}$$

At $t = 0$, $f_n(t) = e^{-|n|}$, for all $n \in \mathbb{Z}\backslash\{0\}$. So,

$$\sum_{0\neq n=-\infty}^{\infty} \|f_n\|_\infty = \sum_{n=1}^{\infty} \|f_{-n}\|_\infty + \sum_{n=1}^{\infty} \|f_n\|_\infty = \sum_{n=1}^{\infty} e^{-n} + \sum_{n=1}^{\infty} e^{-n}$$

That e^{-n} is summable follows from the comparison test. Since log is monotonic,

$$\begin{aligned} e^{-n} &\leqslant \frac{1}{n^2} \\ \iff n^2 &\leqslant\; e^{-n} \\ \iff n^2 &\leqslant e^n \\ \iff \log n^2 &\leqslant \log e^n \\ \iff 2\log n &\leqslant n \end{aligned}$$

Since $\sqrt{}$ is also a monotonic function, if $n \geqslant 16$,

$$2\log n = 4\log\sqrt{n} \leqslant \sqrt{n}\sqrt{n} = n$$

So, we see that

$$e^{-n} \leqslant \frac{1}{n^2}$$

for all $n \geqslant 16$. Since $e^{-n} = |e^{-n}|$, and $\frac{1}{n^2}$ is a summable p-series we see that e^{-n} is absolutely summable and thus summable. So,

$$\sum_{0\neq n=-\infty}^{\infty} f_n$$

converges uniformly in x to a function

$$f := \sum_{0\neq n=-\infty}^{\infty} f_n$$

We use induction to show that $f \in \mathcal{C}^\infty$. The base case was just shown. Suppose that for some $k \geqslant 0$, f is k times continuously differentiable with derivative

$$\frac{d^k}{dx^k} f = f^{(k)} = \sum_{0 \neq n = -\infty}^{\infty} f_n^{(k)}$$

Note that this is equivalent to $f \in \mathcal{C}^k$. We have that

$$\sum_{0 \neq n = -\infty}^{\infty} f_n^{(k)}$$

converges point-wise everywhere per the inductive hypothesis.
We have

$$f_n^{(k)}(t) = (2\pi i|n|)^k e^{-|n|}\ e^{2\pi i n t}$$

For any closed interval I,

$$\|f^{(k)}\|_{\infty;I} \leqslant \|f^{(k)}\|_\infty = (2\pi|n|)^k e^{-|n|}$$

$$\sum_{0 \neq n = -\infty}^{\infty} \|f_n^{(k)}\|_{\infty;I} = \sum_{n=1}^{\infty} \|f_{-n}^{(k)}\|_{\infty;I} + \sum_{n=1}^{\infty} \|f_n^{(k)}\|_{\infty;I}$$

Again, we use the comparison test to show that f' is norm summable:

$$\begin{aligned}
&(2\pi|n|)^k e^{-|n|} \leqslant \frac{1}{|n|^2} \\
&\iff (2\pi)^k |n|^{k+2} \leqslant e^{|n|} \\
&\iff k \log 2\pi + (k+2)\log|n| \leqslant |n|
\end{aligned}$$

Note that $\log 2\pi < \log|n|$ if $|n| \geqslant 2\pi$. Pick an integer $N > 2\pi, (4k + 2)^4$. Then, if $|n| \geqslant N$,

$$\begin{aligned}
k \log 2\pi + (k+2)\log|n| &\leqslant (2k+2)\log|n| \\
&= (4k+4)\log\sqrt{|n|} \\
&\leqslant \sqrt{|n|}\log\sqrt{|n|} \\
&\leqslant \sqrt{|n|}\sqrt{|n|} \\
&= |n|
\end{aligned}$$

So, we see that

$$\|f_n^{(k)}\|_{\infty;I} \leqslant |n|^2$$

for all $n \geqslant N$. Thus, the comparison test implits that the double sided sequence of functions $f_n^{(k)}$ is norm summable under the supremum norm.

We have verified both hypotheses of Theorem 1.4. Thus,

$$f^{(k)} := \sum_{0\neq n=-\infty}^{\infty} f_n^{(k)}$$

is a $\mathcal{C}^1$ function with

$$f^{(k+1)} = \sum_{0\neq n=-\infty}^{\infty} f_n^{(k+1)}$$

Additionally, $f \in \mathcal{C}^{k+1}$.
By (PMI), we have $f \in \mathcal{C}^k$ for all $k \in \mathbb{N}_0$ and thus,

$$f \in \mathcal{C}^\infty = \bigcap_{k=0}^{\infty} \mathcal{C}^k$$

Consider the cohomological equation:

$$f(x) - \hat{f}_0 = h(x+\alpha) - h(x) \tag{21}$$

We showed that $\{e^{-n}\}_{n\in\mathbb{N}}$ is summable. Thus, Propostion 7.5.2 of [CM] implies that there exists a continuous function and an irrational α such that the cohomological equation (21) has no continuous solution h. We also note from the convergent Fourier series of f, that it has the same Fourier coefficients as the function given by Proposition 7.5.2. Thus, the uniqueness property of continuous functions implies that for the f and α chosen, there is no continuous solution h to (21).

If α were Diophantine for some $r > 0$, then Proposition 7.5.1 of [CM] would imply that for all $g \in \mathcal{C}_1^l(\mathbb{R};\mathbb{C})$ with $l > r+1$, the cohomological equation (21) would have a continuous solution. That $f \in \mathcal{C}_1^\infty(\mathbb{R};\mathbb{C}) \subseteq \mathcal{C}_1^l(\mathbb{R};\mathbb{C})$ with no continuous solution to (21) implies that $\alpha \notin \mathcal{DC}(r)$ for any $r > 0$. In particular, we have found $\alpha \notin \mathcal{DC}$. $\square$

- **Problem 3 - Interchanging Derivatives and Infinite Sums:**

  (b)

Theorem 1.4 ("Interchanging Derivatives and Series")**.** *Given and open interval $I \subseteq \mathbb{R}$ and $\mathcal{C}^1$-functions $f_n : I \to \mathbb{R}$, $n \in \mathbb{N}$. Suppose that* **both** *of the following conditions hold:*
(i) *The series $\sum_{n=1}^{\infty} f_n(x)$ converges for at least one point $x \in I$.*
(ii) *For all closed sub-intervalls $[a,b] \subseteq I$, we have that*

$$0 \leqslant \sum_{n=1}^{\infty} \|f_n'\|_{\infty;[a,b]} < +\infty \ ;$$

here, we denote as usual

$$\|f'\|_{\infty;[a,b]} := \sup_{x\in[a,b]} |f'(x)| \tag{22}$$

Then, $f(x) = \sum_{n=1}^{\infty} f_n(x)$ *converges uniformly on all compact subsets of* I *(and, thus, (absolutely) for all* $x \in I$*) and one has that*

$$\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{\mathrm{d}f_n}{\mathrm{d}x} \ .$$

Proof. Let f_n be a sequence satisfying (i) and (ii) of 1.4 on an open interval I.
Note that any compact subset of $\mathbb{R}$ has a minimum and maximum by (EVT). For any compact subset K of I,

$$K \subseteq [\min K, \max K] := [a,b]$$

Also,

$$\|f_n'\|_{\infty;K} \leqslant \|f_n'\|_{\infty'[a,b]} =: M_n$$

By the second hypothesis (1.4), the series

$$\sum_{n=1}^{\infty} M_n < \infty$$

So, the series

$$\sum_{n=1}^{\infty} f_n'$$

converges uniformly to $g \in \mathcal{B}(I,\mathbb{C})$ by the Weierstrass M-Test.
Letting

$$\{g_n\}_{n\in\mathbb{N}_0} := \left\{\sum_{k=1}^{n} f_n\right\}_{n\in\mathbb{N}_0}$$

we see that, the sequence

$$\{g_n'\}_{n\in\mathbb{N}_0} = \left\{\frac{d}{dx}\sum_{k=1}^{n} f_n\right\}_{n\in\mathbb{N}_0} = \left\{\sum_{k=1}^{n} f_n'\right\}_{n\in\mathbb{N}_0}$$

converges uniformly to g on all compact subsets of I.
By the first hypothesis, we have

$$g_n(x) = \sum_{k=1}^{n} f_n(x) \text{ converges for some } x \in I \tag{23}$$

Thus, the sequence g_n satisfies both conditions of the "interchanging limits and derivatives" theorem. So, $g_n \to f$ uniformly on compact subsets of I for some function $f : I \to \mathbb{R}$ satisfying $f' = g$. We see that $f = \sum_{n=1}^{\infty} f_n$ from (23). And,

$$\frac{d}{dx}\sum_{n=1}^{\infty} f_n = f' = g = \sum_{n=1}^{\infty} f_n' = \sum_{n=1}^{\infty} \frac{df_n}{dx}$$

□

- **Problem 4 - Schwartz functions:**

(a)

Proposition 1.2. *$\mathcal{S}(\mathbb{R};\mathbb{C})$ is a vector subspace of $\mathcal{C}^\infty$.*

Proof. Clealy,

$$\mathcal{S}(\mathbb{R};\mathbb{C}) \subseteq \mathcal{C}^\infty$$

Let $f, g \in \mathcal{S}(\mathbb{R};\mathbb{C})$ and $a, b \in \mathbb{C}$ and $m, n \in \mathbb{N}_0$. Let $C_{m,n}$ and $D_{m,n}$ be the Schwartz constants of f and g, respectively. We see:

$$\begin{aligned}
\sup_{x\in\mathbb{R}} \left|x^m(af^{(n)}(x) + bg^{(n)}(x))\right| &= \sup_{x\in\mathbb{R}} \left|ax^m f^{(n)}(x) + bx^m g^{(n)}(x))\right| && (24)\\
&\leqslant \sup_{x\in\mathbb{R}} a\left|x^m(f^{(n)}(x)\right| + b\sup_{x\in\mathbb{R}}\left|g^{(n)}(x))\right| && (25)\\
& && (26)\\
&= aC_{m,n} + bD_{m,n} && (27)\\
&< \infty && (28)
\end{aligned}$$

so, $af + bg \in \mathcal{S}(\mathbb{R};\mathbb{C})$. Note that (25) uses the triangle inequality and positive homogeneity for the supremum norm.

Clearly, $0 \in \mathcal{S}(\mathbb{R};\mathbb{C})$ with $C_{m,n} = 0$ for all $m, n \in \mathbb{N}_0$. Therefore, $\mathcal{S}(\mathbb{R};\mathbb{C})$ is a vector subspace of $\mathcal{C}^\infty(\mathbb{R};\mathbb{C})$.

□

Proposition 1.3. *$\mathcal{S}(\mathbb{R};\mathbb{C})$ is closed under multiplication:*

$$f, g \in \mathcal{S}(\mathbb{R};\mathbb{C}) \Longrightarrow f \cdot g \in \mathcal{S}(\mathbb{R};\mathbb{C}) \ . \qquad (29)$$

Proof. Repeated applications of the product rule show:

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

$$\begin{aligned}
\sup_{x\in\mathbb{R}} \left|x^m (f\cdot g)^{(n)}(x)\right| &= \sup_{x\in\mathbb{R}} \left| x^m \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) \right| && (30)\\
&\leqslant \sum_{k=0}^{n} \binom{n}{k} \sup_{x\in\mathbb{R}} \left|x^m f^{(k)}(x) g^{(n-k)}(x)\right| && (31)\\
&\leqslant \sum_{k=0}^{n} \binom{n}{k} \sup_{x\in\mathbb{R}} \left|x^m f^{(k)}(x)\right| \cdot \sup_{x\in\mathbb{R}} \left|g^{(n-k)}(x)\right| && (32)\\
&= \sum_{k=0}^{n} \binom{n}{k} C_{m,k} \cdot D_{0,n-k} && (33)\\
&< \infty && (34)
\end{aligned}$$

where (33) is finite since it is a finite sum. □

Proposition 1.4. *If $f \in \mathcal{S}(\mathbb{R};\mathbb{C})$, then also all its derivatives satisfy*

$$f^{(n)} \in \mathcal{S}(\mathbb{R};\mathbb{C}) \text{ , for all } n \in \mathbb{N} \text{ .} \tag{35}$$

Proof. Let $f \in \mathcal{S}(\mathbb{R};\mathbb{C})$ have Schwartz constants $C_{m,n}$. For a fixed $k \in \mathbb{N}$, and any $n, m \in \mathbb{N}_0$, let $[f^{(k)}]^{(n)}$ denote the n-th derivative of the k-th derivative of f. We have

$$\sup_{x\in\mathbb{R}} \left|x^m [f^{(k)}]^{(n)}\right| = \sup_{x\in\mathbb{R}} \left|x^m f^{(k+n)}(x)\right| = C_{m,n+k} < \infty$$

showing that $f^{(k)} \in \mathcal{S}(\mathbb{R};\mathbb{C})$. □

(b)

Proposition 1.5. *The Gaussian,*

$$f : \mathbb{R} \to \mathbb{R} \text{ , } f(x) = \mathrm{e}^{-\pi x^2} \tag{36}$$

is a Schwartz function.

Proof. Let $P(k)$ be that the k-th derivative of f is of the form

$$f^{(k)}(x) = p_k(x) e^{-\pi x^2}$$

for some polynomial p_k.

Base case: $k = 0$ Taking $p_0(x) = 1$, we see that $P(0)$ holds.

Inductive Step: Suppose for some $k \geqslant 0$, $P(k)$ holds for a polynomial $p_k(x) = \sum_{i=0}^{n_k} a_i x^i$.

We compute the $(k+1)$-st derivative of f:

$$\begin{aligned}
\frac{d}{dx}\left(p_k(x)e^{-\pi x^2}\right) &= \frac{d}{dx}\sum_{i=0}^{n_k} a_i x^i e^{-\pi x^2} \\
&= \sum_{i=0}^{n_k} \frac{d}{dx}\left(a_i x^i e^{-\pi x^2}\right) \\
&= \sum_{i=0}^{n_k} i a_i x^i e^{-\pi x^2} - 2a_i x^{i+1} e^{-\pi x^2} \\
&= \left(\sum_{i=0}^{n_k} i a_i x^i - 2a_i x^{i+1}\right) e^{-\pi x^2} \\
&:= p_{k+1}(x)e^{-\pi x^2}
\end{aligned}$$

showing $P(k+1)$.

Thus, by (PMI), $P(k)$ holds for all $k \in \mathbb{N}_0$.

Fix $n, m \in \mathbb{N}_0$ and examine the n-th derivative of f:

$$f^{(n)}(x) = p_n(x)e^{-\pi x^2}$$

with p_n degree k.

Pick $x_0 \in \mathbb{R}$ such that for for all $x \notin [-x_0, x_0]$,

$$|p_n(x)| \leqslant |x^{k+1}| \tag{37}$$

and $x_0 \geqslant m + k + 1$. Note that we can do (37) since for all $|x| \geqslant 1$,

$$\begin{aligned}
\left|\sum_{i=0}^{N} a_i x^i\right| &\leqslant \sum_{i=0}^{N} |a_i| \cdot |x|^i \\
&\leqslant \sum_{i=0}^{N} |a_i| \cdot |x|^N \\
&= |x|^N \sum_{i=0}^{N} |a_i|
\end{aligned}$$

So, if we also require $|x| \geqslant \sum_{i=0}^{N} |a_i|$, (37) holds.
We have:

$$|x^m f^{(n)}(x)| = |x^m p_n(x)e^{-\pi x^2}| \leqslant |x|^{m+k+1}e^{-\pi x^2}$$

Since log is monotonic, we have

$$\begin{aligned} &|x|^{m+k+1}e^{-\pi x^2} \leqslant 1 \\ \iff& |x|^{m+k+1} \leqslant e^{\pi x^2} \\ \iff& \log|x|^{m+k+1} \leqslant \log e^{\pi x^2} \\ \iff& (m+k+1)\log|x| \leqslant \pi x^2 \end{aligned}$$

For a fixed $x \notin [-x_0, x_0]$, $|x| \geqslant m+k+1$ and $\log|x| \leqslant |x|$. So we have

$$(m+k+1)\log|x| \leqslant (m+k+1)\cdot|x| \tag{38}$$

$$\leqslant (m+k+1)^2 \tag{39}$$

$$\leqslant |x|^2 \tag{40}$$

$$\leqslant \pi x^2 \tag{41}$$

where (40) is true because $y^2 \leqslant z^2 \iff y \leqslant z$ for all $y, z \in \mathbb{R}$. So, we have indeed have that

$$|x|^{m+k+1}e^{-\pi x^2} \leqslant 1$$

for all $x \notin [-x_0, x_0]$.
Then,

$$\sup_{x\notin[-x_0,x_0]} \left|x^m f^{(n)}(x)\right| \leqslant \sup_{x\notin[-x_0,x_0]} \left|x^{m+k+1}e^{-\pi x^2}\right| \leqslant 1$$

We also have the continuous function $x^m f^{(n)}(x)$ is bounded on the compact domain $[-x_0, x_0]$ by the (EVT). Therefore, there exists

$$C_{m,n} := \sup_{x\in\mathbb{R}} \left|x^m f^{(n)}(x)\right|$$

□