# PROBLEM SET NO. 6 - ISAAC VIVIANO

I affirm that I have adhered to the Honor Code on this assignment. Isaac Viviano

## 1. Problems:

- **Problem 1 - Uniform Convergence of Fourier Series:**

  (b)

**Lemma 1.1.** *Let $\epsilon > 0$ be given. Then, for each $t \in A$, there exists $N_t \in \mathbb{N}$ and $\delta_t > 0$ so that for all $s \in \mathbb{R}$ with $|s-t| < \delta_t$, one has*

$$\left|\widehat{[g^{(\pm)}_{\epsilon;s}]}_m\right| < \epsilon, \text{ for all } |m| \geqslant N_t$$

*for the function defined by*

$$g^{(\pm)}_{\epsilon;t_0}(s) := \begin{cases} \frac{f(t_0-s)-f(t_0)}{\sin(\pi s)}e^{\pm\pi i s} & , \text{ if } \epsilon \leqslant |s| \leqslant \frac{1}{2} \\ 0 & , \text{ if } s \in (-\epsilon, \epsilon) \end{cases}$$

*Proof.* Let $\epsilon \leqslant |s| \leqslant \frac{1}{2}$ and let $\eta > 0$ be given. By the uniform continuity of $f$, pick $\delta$ such that if $|s_0 - s_1| < \delta$, $|f(s_0) - f(s_1)| < \frac{\epsilon}{4}$. For such $s_0, s_2$ and an arbitrary $\epsilon \leqslant |s| \leqslant \frac{1}{2}$,

$$\begin{aligned}
\left|g^{(\pm)}_{\epsilon;s_0}(s) - g^{(\pm)}_{\epsilon;s_1}(s)\right| &= \left|\frac{f(s_0-s)-f(s_0)}{\sin(\pi s)}e^{\pm\pi i s} - \frac{f(s_1-s)-f(s_1)}{\sin(\pi s)}e^{\pm\pi i s}\right| \\
&= \left|\frac{(f(s_0-s)-f(s_1-s)) + (f(s_1)-f(s_0))}{\sin(\pi s)}e^{\pm\pi i s}\right| \\
&\leqslant (\underbrace{|f(s_0-s)-f(s_1-s)|}_{<\frac{\epsilon}{4}} + \underbrace{|f(s_1)-f(s_0)|}_{<\frac{\epsilon}{4}}) \cdot \underbrace{\frac{\|e^{\pm\pi i s}\|_\infty}{\|\sin(\pi s)\|_\infty}}_{=1} \\
&< \frac{\epsilon}{2}
\end{aligned}$$

Since this bound is independent of $s$, and

$$g^{(\pm)}_{\epsilon;s_0}(s) - g^{(\pm)}_{\epsilon;s_1}(s)$$

for all $|s| \leqslant \epsilon$, we get

$$\left\|g^{(\pm)}_{\epsilon;s_0}(s) - g^{(\pm)}_{\epsilon;s_1}(s)\right\|_\infty \leqslant \frac{\epsilon}{2} \tag{1}$$

Let $t_0 \in A$. By the Riemann-Lebesgue Lemma, there exists $N_{t_0} \in \mathbb{N}$ such that for all $n \geqslant N_{t_0}$,

$$\left|\widehat{[g^{(\pm)}_{\epsilon;t_0}]}_n\right| < \frac{\epsilon}{2}$$

Let $\delta_{t_0} = \delta$ from above. If $|s_0 - t_0| < \delta_{t_0}$, then (1) implies

$$\begin{aligned}
\left|\widehat{[g^{(\pm)}_{\epsilon;s_0}]}_n\right| &= \left|\widehat{[g^{(\pm)}_{\epsilon;s_0}]}_n - \widehat{[g^{(\pm)}_{\epsilon;t_0}]}_n + \widehat{[g^{(\pm)}_{\epsilon;t_0}]}_n\right| && (2)\\
&\leqslant \left|\widehat{[g^{(\pm)}_{\epsilon;s_0}]}_n - \widehat{[g^{(\pm)}_{\epsilon;t_0}]}_n\right| + \left|\widehat{[g^{(\pm)}_{\epsilon;t_0}]}_n\right| && (3)\\
&= \left|\widehat{[g^{(\pm)}_{\epsilon;s_0} - g^{(\pm)}_{\epsilon;t_0}]}_n\right| + \left|\widehat{[g^{(\pm)}_{\epsilon;t_0}]}_n\right| && (4)\\
&< \left\|g^{(\pm)}_{\epsilon;s_0}(s) - g^{(\pm)}_{\epsilon;s_1}(s)\right\|_\infty + \frac{\epsilon}{2} && (5)\\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} && (6)\\
&= \epsilon && (7)
\end{aligned}$$

where for (5), we use the infinity-norm bound for Fourier coefficients from Set 2, Problem 5, part (a). $\square$

- **Problem 2 - rational & irrational rotations:** For $\alpha \in (0,1)$, we consider the map

$$R_\alpha : [0,1] \to [0,1]\ ,\ R_\alpha(x) = (x + \alpha) \mod 1\ . \qquad (8)$$

(a)

**Proposition 1.1.** *$R_\alpha$ is periodic if and only if $\alpha \in \mathbb{Q}$.*

*Proof.* ( $\Longleftarrow$ ) Suppose $\alpha \in \mathbb{Q}$. Let $\alpha = \frac{p}{q}$ with $p, q \in \mathbb{N}$ and $\gcd(p,q) = 1$. For each $x \in [0,1)$,

$$\begin{aligned}
R^q_\alpha(x) &= x + q \cdot \alpha \mod 1\\
&= x + p \mod 1\\
&= x
\end{aligned}$$

So, $R^q_\alpha = \mathrm{id}$. Thus, $R_\alpha$ is periodic.
( $\Longrightarrow$ ) Suppose $R^q_\alpha = \mathrm{id}$ for some $q \in \mathbb{N}$. Then for all $x \in [0,1)$,

$$x = R^q_\alpha(x) = x + q \cdot \alpha \mod 1$$

Therefore, $q \cdot \alpha \equiv 0 \mod 1$, which implies $q\alpha = p \in \mathbb{Z}$. Since $q$ is nonzero, we may divide to get

$$\alpha = \frac{p}{q} \in \mathbb{Q}$$

$\square$

(b)

**Proposition 1.2.** *For* irrational $\alpha$*,* $R_\alpha$ *never returns to any point twice, i.e., for all* $x \in [0,1]$*, we have*

$$R_\alpha^n(x) \neq R_\alpha^m(x) \text{ , whenever } n \neq m \text{ .}$$

*Proof.* We argue by contrapositive. Suppose $R_\alpha^n(x) = R_\alpha^m(x)$ for some $x \in [0,1)$ and $n < m \in \mathbb{N}$. We have

$$x + n\alpha \equiv x + m\alpha \mod 1$$

So,

$$(m-n)\alpha \equiv 0 \mod 1$$

Since $0 \neq m - n \in \mathbb{N}$, we have $\alpha \in \mathbb{Q}$ as before. □

- **Problem 3 - "Irrational Rotations are Dense"**

Fix an open interval $I \subseteq [0,1]$. Denote the length of $I$ by $0 < \epsilon \leqslant 1$.

(a) Let $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$ by the Archimedean property of $\mathbb{R}$. Divide the half open unit interval into $N$ half open intervals:

$$I_i = \left[\frac{i}{N}, \frac{i+1}{N}\right), 0 \leqslant i < N$$

Note that

$$\bigcup_{i=0}^{N-1} I_i = [0,1)$$

so any collection of $N+1$ points in $[0,1)$ must have at least two elements in $I_i$ for some $0 \leqslant i < N$ by the Pigeonhole Principle. One such collection is the finite orbit:

$$\mathcal{O}_\alpha(0,N) = \{R_\alpha^k(0) : 0 \leqslant k \leqslant N\}$$

So, we may pick $n_1 < n_2 \leqslant N$ and $i < N$ with

$$R_\alpha^{n_1}(0), R_\alpha^{n_2}(0) \in I_i$$

Note that

$$0 \leqslant |R_\alpha^{n_1}(0) - R_\alpha^{n_2}(0)| < \frac{i+1}{N} - \frac{i}{N} = \frac{1}{N} < \epsilon$$

We may also write

$$R_\alpha^{n_1}(0) - R_\alpha^{n_2}(0) = n_1\alpha - n_2\alpha \mod 1$$

Pick $k_0 \in \mathbb{Z}$ such that

$$n_1\alpha - n_2\alpha - k_0 = n_1\alpha - n_2\alpha \mod 1$$

Then,

$$\begin{aligned} d_c(n_1\alpha, n_2\alpha) &= \min\{|\alpha(n_1 - n_2) - k|, k \in \mathbb{Z}\} \\ &\leqslant |\alpha n_1 - \alpha n_2 - k_0| \\ &= |R_\alpha^{n_1}(0) - R_\alpha^{n_2}(0)| \\ &< \frac{1}{N} \\ &< \epsilon \end{aligned}$$

(b)

**Proposition 1.3.** *Let $M := n_2 - n_2$. Then, for all $x \in [0, 1]$, there exists $k \in \mathbb{N}$ such that*

$$R_\alpha^{kM}(x) \in I \ . \tag{9}$$

*Proof.* Suppose not: Let $I = (a, b)$ and let $k_0 \in \mathbb{N}$ be the first $k$ such that

$$R_\alpha^{k_0 M}(x) \leqslant a$$

and

$$R_\alpha^{(k_0-1)M}(x) \geqslant b$$

Note that

$$R_\alpha^m(x) = x + n\alpha \mod 1 = x + R_\alpha^m(0) \mod 1$$

So,

$$k_0(n_2 - n_1)\alpha + x \mod 1 \leqslant a \tag{10}$$

and

$$(k_0 - 1)(n_2 - n_1)\alpha + x \mod 1 \geqslant b \tag{11}$$

Subtracting (11) from (10) gives

$$(n_1 - n_2)\alpha \mod 1 \geqslant b - a = \epsilon$$

which contradicts (9). ☐

(c)

**Proposition 1.4.** *For every $x \in [0, 1]$, $R_\alpha^n(x) \in I$ for infinitely many $n \in \mathbb{N}$.*

*Proof.* Let $M \in \mathbb{N}$ and $x \in [0, 1]$ be given. Divide $I$ into $M$ subintervals:

$$I_i = \left(a + \frac{i\epsilon}{M}, a + \frac{(i+1)\epsilon}{M}\right), \text{ for } 0 \leqslant i < M$$

and

$$I_M = \left(b - \frac{\epsilon}{M}, b\right)$$

By Kronecker's Theorem, for each $0 \leqslant i < M$, there exists $n_i \in \mathbb{N}$ such that

$$R_\alpha^{n_i}(x) \in I_i \subseteq I$$

Note that $n_i \neq n_j$ for all $i \neq j$, since $I_i \cap I_j = \varnothing$. We have shown that

$$\#(\mathcal{O}_\alpha(x) \cap I) \geqslant M$$

Since $M$ was arbitrary, $R_\alpha^n(x) \in I$ for infinitely many $n$. □

- **Problem 4:**

**Proposition 1.5.** *Let $a, b, c \in \mathbb{N}$ with $b, c \neq 1$. Suppose there exists a prime $p$ which only contributes to the prime factorization of precisely one of $a$ or $b$ or $c$. Then, $\log_a(b)$ and $\log_a(c)$ are independent over $\mathbb{Q}$: there are no $n_1, n_2, n_3 \in \mathbb{Z}$ such that*

$$n_1 \log_a(b) + n_2 \log_a(c) = n_3$$

*besides $n_1 = n_2 = n_3 = 0$*

*Proof.* Suppose there exist $n_1, n_2, n_3$ nonzero satisfying

$$n_1 \log_a b + n_2 \log_a c = n_3$$

Using log properties, we write

$$a^{n_3} = b^{n_1} c^{n_2}$$

We see that

$$\{q : q \text{ is a prime factor of } a\} = \{q : q \text{ is a prime factor of } c\} \cup \{q : q \text{ is a prime factor of } b\}\}$$

so there is no prime $p$ contributing to the prime factorization of precisely one of $a$, $b$, or $c$. □