PROBLEM SET NO. 8 - ISAAC VIVIANO

I affirm that I have adhered to the Honor Code in this assignment. Isaac Viviano blank

1. Problems:

- **Problem 1 - Existence of Diophantine Numbers:**

Proposition 1.1. *Consider the cohomological equation*

$$h(x+\alpha) - h(x) = f(x) - \hat{f}_0 \tag{1}$$

Let $\phi : (0, \infty) \to (0, \infty)$ be a strictly increasing function which satisfies

$$\sum_{n=1}^{\infty} \frac{1}{\phi(n)} < +\infty$$

Then, there exists a 1-periodic, continuous function $f \in \mathcal{C}_1(\mathbb{R}; \mathbb{R})$ with $\hat{f}_0 = 0$ and

$$\hat{f}_{\pm n} = \frac{1}{\phi(n)}, \text{ for all } n \in \mathbb{N}$$

and an irrational $\alpha \in \mathbb{R}$ so that the cohomological equation (1) has no continuous solution h.

Proof. Since ϕ maps into $(0, \infty)$,

$$\left\| \frac{1}{\phi(n)} e_{\pm n} \right\|_{\infty} = \left\| \frac{1}{\phi(n)} \right\|_{\infty} = \frac{1}{\phi(n)}$$

Since $\frac{1}{\phi(n)}$ is summable, the Weierstrass M-test implies that the Fourier series

$$\sum_{n=-\infty}^{\infty} \frac{1}{\phi(|n|)} e_n$$

converges uniformly. Defining

$$f := \sum_{n=-\infty}^{\infty} \frac{1}{\phi(|n|)} e_n$$

we see that

$$\hat{f}_{\pm n} = \frac{1}{\phi(n)}$$

Define two mutually recursive sequences $\{n_k\}_{k \in \mathbb{N}}$ and $\{b_k\}_{k \in \mathbb{N}}$:

$$\begin{aligned} b_1 &:= 1 \\ n_k &:= 10^{b_k} \\ b_{k+1} &:= \min \left(\{ l \in \mathbb{N} : \frac{4\pi}{10^l} \leqslant \frac{1}{\phi(n_k)} \} \backslash \{ b_1, \ldots, b_{k-1} \} \right) \end{aligned}$$

For each $k \in \mathbb{N}$, let

$$\begin{aligned} x_{b_k} &:= 1 \\ x_{b_k+1}, \ldots, x_{b_{k+1}-1} &:= 0 \end{aligned}$$

Define

$$\alpha := \sum_{k=0}^{\infty} \frac{x_k}{10^k} \tag{2}$$

We have for each k,

$$\left|\frac{x_k}{10^k}\right| \leqslant \left|\frac{1}{10^k}\right|$$

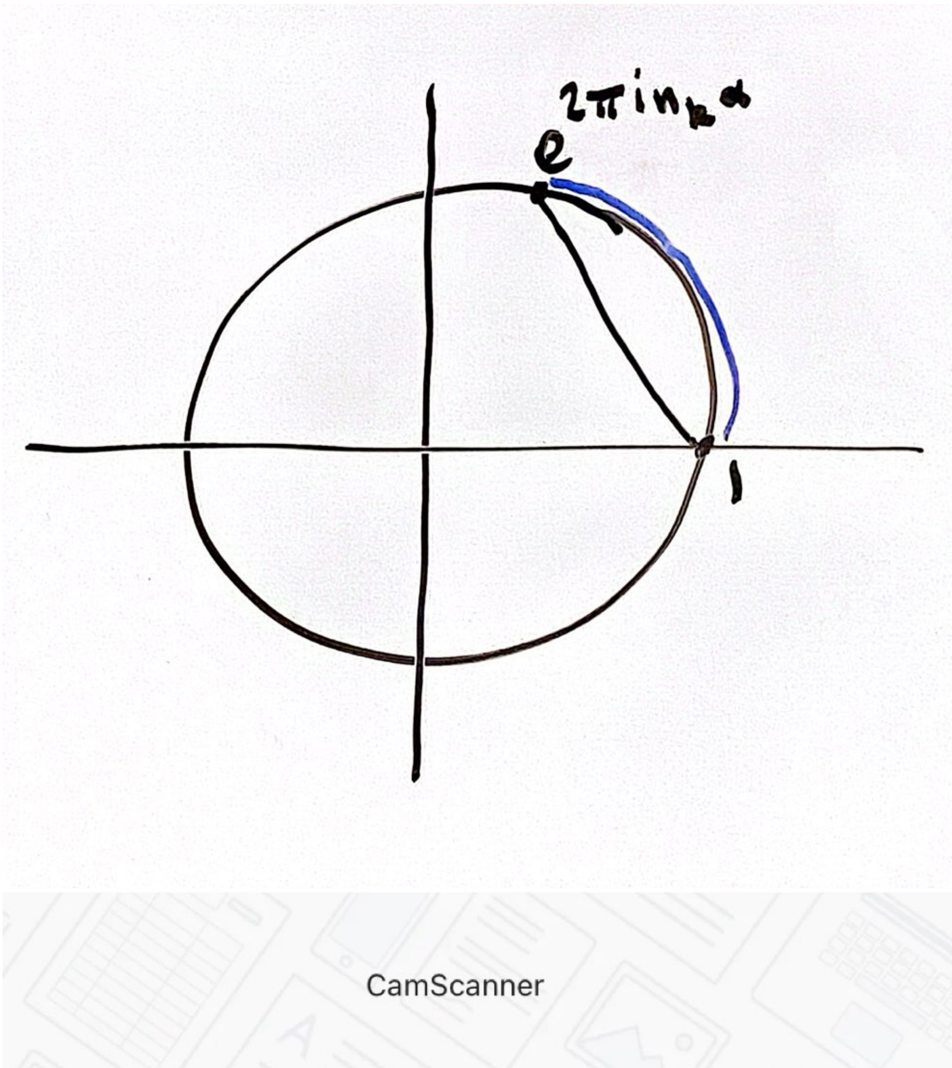
Since $\frac{1}{10^k}$ is a geometric series with radius less than 1, it converges absolutely. Thus, the series in (2) converges to α by the comparison test. Note also that we may write the decimal expansion of α as

$$\alpha = x_0.x_1x_2\ldots$$

Noting that all of the b_k's are unique, we see that α is irrational since it never terminates or repeats.

We estimate

$$\begin{aligned} n_k\alpha \mod 1 &= 10^{b_k}\sum_{l=0}^{\infty}\frac{x_l}{10^l} \mod 1 \\ &= \left(\underbrace{\sum_{l=0}^{\infty}10^{b_k-l}x_l}_{\in\mathbb{N}} - \underbrace{\sum_{l=b_k+1}^{\infty}\frac{x_l}{10^{l-b_k}}}_{<1}\right) \mod 1 \\ &= \sum_{l=b_k+1}^{\infty}\frac{x_l}{10^{l-b_k}} \\ &= \sum_{l=b_k+1}^{b_{k+1}}\frac{x_l}{10^{l-b_k}} + \sum_{l=b_{k+1}+1}^{\infty}\frac{x_l}{10^{l-b_k}} \\ &\leqslant \sum_{l=b_k+1}^{b_{k+1}}\frac{x_l}{10^{l-b_k}} + \frac{1}{10^{b_k+1}} \\ &= \frac{2}{10^{b_{k+1}}} \end{aligned}$$



Noting that we can associate $e^{2\pi i n_k \alpha}$ with the rotation angle $n_k\alpha$. As shown in Figure 1, we estimate $|1 - e^{2\pi i n_k \alpha}|$ by the rotation map, since the straight-line distance between the points 1 and $e^{2\pi i n_k \alpha}|$ will always be less than the distance around the circle (of circumference 2π). Thus,

$$
\begin{aligned}
|e^{2\pi i n_k \alpha} - 1| &\leqslant 2\pi(n_k\alpha - 1 \mod 1) \\
&= 2\pi(n_k\alpha \mod 1) \\
&\leqslant 2\pi \cdot \frac{2}{10^{b_{k+1}}} \\
&\leqslant \frac{1}{\phi(n_k)}
\end{aligned}
$$

If h were a continuous solution to (1), we have

$$\hat{h}_n := \begin{cases} \frac{\hat{f}_n}{e^{2\pi i n\alpha}-1} & \text{if } n \neq 0 \\ \text{const} & \text{if } n = 0 \end{cases}$$

In particular, we have

$$|\hat{h}_{n_k}| = \frac{\left|\hat{f}_{n_k}\right|}{|1 - e^{2\pi i n_k \alpha}|} \geqslant 1, \text{ for all } k \in \mathbb{N} \tag{3}$$

The Riemann Lebesgue lemma implies that $\hat{h}_n \to 0$ as $n \to \infty$. But any subsequence of a convergent subsequence converges to the same value. (3) implies that $\hat{h}_{n_k} \not\to 0$. This contradicts the Riemann-Lebesgue lemma, showing that there is no continuous solution h to (1) for the α chosen.

□

- **Problem 2 - Pythagoras' Tuning Problem and Approximate Return Times:**

  (b)

  Theorem 1.1. *For a fixed irrational $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, let $\frac{p_n}{q_n}$ be the sequence of convergents in the continued fraction expansion of α. Given $M \in \mathbb{N}$, let $n_M \in \mathbb{N}$ be the unique natural number such that*

$$q_{n_M} \leqslant M < q_{n_M+1}$$

  Then, one has

$$\|\alpha \cdot q_{n_M}\| = \min_{1 \leqslant k \leqslant M} \|\alpha \cdot k\| \tag{4}$$

  in particular, for all $n \in \mathbb{N}$,

$$\|\alpha \cdot q_n\| = \min_{1 \leqslant k < q_{n+1}} \|\alpha \cdot k\|$$

  Proof. Show that n_M exists and is unique:
  Let $M \in \mathbb{N}$ be given. Let $A = \{k \in \mathbb{N}_0 : q_k \leqslant M\}$
  Note that $A \subseteq \mathbb{N}$ is nonempty (it contains $q_0 = 1$) and bounded (by M). Thus, it has a maximum:

$$n_M := \max A$$

  We have $n_M \in A$ and thus,

$$q_{n_M} \leqslant M$$

  That n_M is the maximum of A implies $n_M + 1 \notin A$. So, $q_{n_M+1} \nleqslant M \iff M < q_{n_M}$.
  If $k < n_M$,

$$q_{k+1} \leqslant q_{n_M} \leqslant M$$

so,

$$q_{k+1} \not> M$$

If $k > n_M$,

$$q_k \geqslant q_{n_M+1} > M$$

so,

$$q_k \not\leqslant M$$

So, we have shown the existence and uniqueness of n_M.
From Corollary 8.1.1, q_{n_M} and q_{n_M+1} are approximate return times and for all $q_{n_M} < k < q_{n_M+1}$, k is not an approximate return time. If (4) were false, we would have

$$\|\alpha \cdot q_{n_M}\| > \|\alpha \cdot k_0\| = \min_{1 \leqslant k \leqslant M} \|\alpha \cdot k\|$$

for some $1 \leqslant k_0 \leqslant M$ with $k_0 \neq q_{n_M}$. Consider two cases. If $k_0 < q_{n_M}$, this contradicts that q_{n_M} is BA-2. Otherwise, $k_0 > q_{n_M}$. So,

$$\|\alpha \cdot k_0\| = \min_{1 \leqslant k \leqslant M} \|\alpha \cdot k\| = \min_{1 \leqslant k \leqslant k_0} \|\alpha \cdot k\|$$

which implies that k_0 is an approximate return time. Since the sequence of denominators q_n is monotonic and $q_{n_M} < k_0 < q_{n_M+1}$, we have contradicted Corollary 8.1.1, so (4) holds.
Fix $n \in \mathbb{N}$ and let $M = q_{n+1} - 1$. Note that taking $n_M = n$, we have

$$q_{n_M} \leqslant q_{n_M+1} - 1 = M$$

since the n_M satisfying this is unique for a given M, we have

$$\|\alpha \cdot q_{n_M}\| = \min_{1 \leqslant k \leqslant M} \|\alpha \cdot k\|$$

which we may rewrite as

$$\|\alpha \cdot q_n\| = \min_{1 \leqslant k < q_{n+1}} \|\alpha \cdot k\|$$

□

- **Problem 3 - $\mathcal{C}^\infty$-Bump Functions:**

  (a)

Proposition 1.2. *The function:*

$$h : \mathbb{R} \to \mathbb{R} \ , \ h(x) = \begin{cases} \mathrm{e}^{-1/x^2} & , \textit{ if } x > 0 \ , \\ 0 & , \textit{ if } x \leqslant 0 \ . \end{cases} \tag{5}$$

satisfies $h \in \mathcal{C}^\infty$.

Proof.

$$h : \mathbb{R} \to \mathbb{R},\ h(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that h is $\mathcal{C}^\infty$.
Let $P(n)$ be that

$$h^{(n)}(x) = p_n\left(\frac{1}{x}\right) \cdot e^{-\frac{1}{x^2}},\ \text{for all } x > 0$$

with $h^{(n)}(0) = 0$ for some polynomial p_n.
Base case: $n = 0$. Let $p_0 = 1$. We get

$$h^{(0)}(x) = 1 \cdot h(x) = h(x),\ \text{for all } x > 0$$

For $x = 0$, the definition of h gives

$$h^{(0)}(0) = h(0) = 0$$

Inductive Step: Suppose $P(n)$ holds for some $n \geqslant 0$ for a polynomial $p_n = \sum_{k=0}^{n_k} a_k x^k$.

For $x > 0$,

$$\begin{aligned}
h^{(n+1)}(x) &= \frac{d}{dx} h^{(n)}(x) \\
&= \frac{d}{dx}\left(p_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}\right) \\
&= \frac{d}{dx}\left(\sum_{k=0}^{n_k} a_k x^{-k}\ e^{-\frac{1}{x^2}}\right) \\
&= \sum_{k=0}^{n_k} a_k \frac{d}{dx}\left(x^{-k}\ e^{-\frac{1}{x^2}}\right) \\
&= \sum_{k=0}^{n_k} a_k \left(-kx^{-k-1}\ e^{-\frac{1}{x^2}} + x^{-k} e^{-\frac{1}{x^2}} \cdot \frac{d}{dx}\left(\frac{-1}{x^2}\right)\right) \\
&= \sum_{k=0}^{n_k} a_k \left(-kx^{-k-1}\ e^{-\frac{1}{x^2}} + x^{-k} e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3}\right) \\
&= \sum_{k=0}^{n_k} a_k \left(-kx^{-k-1}\ e^{-\frac{1}{x^2}} + x^{-k} e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3}\right) \\
&= \sum_{k=0}^{n_k} a_k \left(-k\left(\frac{1}{x}\right)^{k+1}\ e^{-\frac{1}{x^2}} + 2\left(\frac{1}{x}\right)^{k+3} e^{-\frac{1}{x^2}}\right) \\
&= e^{-\frac{1}{x^2}} \sum_{k=0}^{n_k} a_k \left(-k\left(\frac{1}{x}\right)^{k+1} + 2\left(\frac{1}{x}\right)^{k+3}\right) \\
&:= e^{-\frac{1}{x^2}} p_{n+1}\left(\frac{1}{x}\right)
\end{aligned}$$

At $x = 0$,

$$h^{(n+1)}(0) = \lim_{x\to 0} \frac{h^{(n)}(x) - h^{(n)}(0)}{x - 0} = \lim_{x\to 0} \frac{h^{(n)}(x)}{x}$$

Since

$$\begin{aligned}
&\lim_{x\to 0} h^{(n)}(x) \to 0 \\
&\lim_{x\to 0} x \to 0
\end{aligned}$$

by the inductive hypothesis, the rule de l'Hopital says that this is indeed the limit:

$$h^{(n+1)}(0) = \lim_{x\to 0} \frac{\frac{d}{dx} h^{(n)}(x)}{\frac{d}{dx} x} = \lim_{x\to 0} \frac{h^{(n+1)}(x)}{1} = \lim_{x\to 0} h^{(n+1)}(x)$$

Note that $h^{(n)}$ differentiable for all $x > 0$ was shown above, and this is what we need to compute the limit.

On set 7, problem 4, we showed that

$$x^m e^{-x^2} \to 0 \text{ as } x \to \infty, \text{ for all } m \in \mathbb{N}_0$$

Consider an arbitrary term $b_k x^k$ of p_{n+1}. Let $\epsilon > 0$ be given and pick x_0 such that for all $x > x_0$,

$$|x^k e^{-x^2}| < \frac{\epsilon}{b_k}$$

If $0 < x < \frac{1}{x_0}$, then,

$$\frac{1}{x} > x_0$$

and

$$\left| b_k \left(\frac{1}{x}\right)^k e^{-\frac{1}{x^2}} \right| < \epsilon$$

Thus, we see that each term of $p_{n+1}(x)e^{-\frac{1}{x^2}} \to 0$ as $x \to 0$ and thus

$$h^{(n+1)}(x) = p_{n+1}(x)e^{-\frac{1}{x^2}} \to 0, \text{ as } x \to 0$$

This concludes the inductive step.

By (PMI), we have $P(n)$ for all $n \in \mathbb{N}_0$.

$\square$

(b)

Proposition 1.3. *The function*

$$g : \mathbb{R} \to \mathbb{R}\ ,\ g(x) := \frac{h(x)}{h(x) + h(1-x)}\ . \tag{6}$$

satisfies $g \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R})$, $0 \leqslant g \leqslant 1$, *and*

$$g(x) = 0\ ,\ \textit{if } x \leqslant 0\ ,\ \textit{and } g(x) = 1\ ,\ \textit{if } x \leqslant 1. \tag{7}$$

Proof. Recall the product rule:

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := h(x) + h(1-x)$$

so we have $g(x) = \frac{h(x)}{f(x)}$.

Let $P(n)$ be that $g \in \mathcal{C}^n$.

Base case:

Note that $h(x)$ is 0 if and only if $x = 0$. Also, h, is non-negative. Thus, $h(x) = -h(1-x)$ is false for all x and f is nonzero.

Since f is nonzero and continuous, the quotient h is continuous, by the algebra of continuous functions. This shows $P(0)$.
Inductive Step: Suppose $P(n)$ for some $n-1 \geqslant 0$.
We have $g^{(k)}$ is continuous for all $k \leqslant n-1$. Note that $f^{(k)}$ and $h^{(k)}$ are also continuous for all k by part (a).

Write $h = g \cdot f$ and apply the product rule:

$$\begin{aligned} h^{(n)} &= (f \cdot g)^{(n)} \\ &= \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)} \\ &= f \cdot g^{(n)} + \sum_{k=0}^{n-1} \binom{n}{k} f^{(n-k)} g^{(k)} \\ \iff \quad g^{(n)} &= \frac{h^{(n)} - \sum_{k=0}^{n-1} \binom{n}{k} f^{(n-k)} g^{(k)}}{f} \end{aligned}$$

where we can divide by f in the last step since it is nonzero. Thus, the algebra of continuous functions implies that $g^{(n)}$ is continuous. So, $g \in \mathcal{C}^n$.

By (PMI), we have $g \in \mathcal{C}^\infty$. $\square$

(c) Let $\chi_\delta : \mathbb{R} \to \mathbb{C}$ be defined by:

$$\chi_\delta(x) := \begin{cases} 0 & \text{if } t \notin (-1-\delta, 1+\delta) \\ 1 & \text{if } t \in [-1, 1] \\ g\left(\frac{1+\delta-x}{\delta}\right) & \text{if } t \in (1, 1+\delta) \\ g\left(\frac{x+1+\delta}{\delta}\right) & \text{if } t \in (-1-\delta, -1) \end{cases} \tag{8}$$

For all $t \in [-1, 1]$,

$$\chi_\delta(t) = 1$$

For all $t \notin (-1-\delta, 1-\delta)$,

$$\chi_\delta(t) = 0$$

If $1 < t < 1 + \delta$,

$$0 < \frac{1+\delta-t}{\delta} < 1$$

So,

$$0 < g(t) < 1$$

If $-1-\delta < t < -1$,

$$0 < \frac{x+1+\delta}{\delta} < 1$$

So,

$$0 < g(t) < 1$$

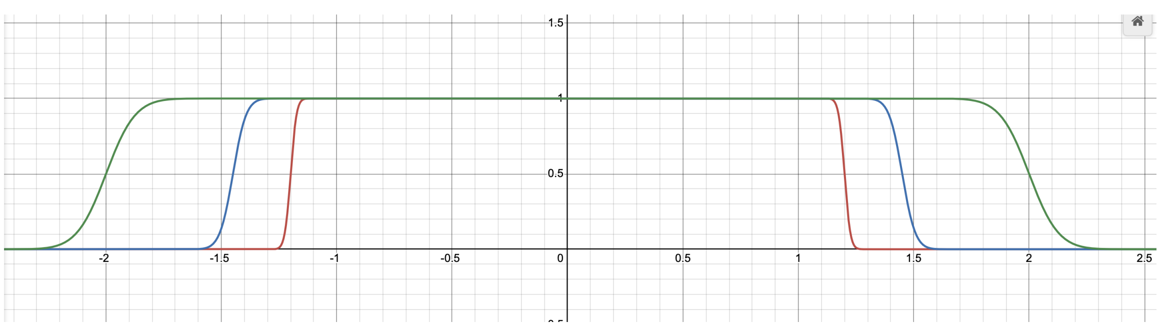
Clearly, χ_δ is $\mathcal{C}^\infty$ except possibly at $t = -1-\delta, -1, 1, 1+\delta$. Noting that χ_δ is even, we consider $t = 1, 1+\delta$.
At $t = 1$, for all $n > 0$, the left limit of $\chi_\delta^{(n)}$ is 0, since χ_δ is constant left of 1. The right limit is also 0, since we have

$$\frac{1+\delta-1}{\delta} = 1$$

and, b implies that $g^{(n)}$ is 0 at 1. For $n = 0$, χ_δ is uniformly 1 to the right of 1 (and at 1). Also, $g(1) = 1$.
At $t = 1+\delta$, for all $n > 0$, the left limit of $\chi_\delta^{(n)}$ is 0, since we have

$$\frac{1+\delta-1-\delta}{\delta} = 0$$

and, b implies that $g^{(n)}$ is 0 at 0. The right limit is 0, since χ_δ is constant right of $1+\delta$. For $n = 0$, χ_δ is uniformly 0 to the right of $1+\delta$. Also, $g(0) = 0$.
Here is a graph of χ_δ for $\delta = .4$, $\delta = .9$, and $\delta = 2$:



- **Problem 4:** This problem follows up on our discussion of the **Steinhaus three distance (aka as three gap) theorem** from class and its proof, which was the part of the reading assignment for Monday, 04/15.

  (b)

Proposition 1.4. *For irrational α, it is impossible that every gap has length either g_m or $2g_m$.*

Proof. Suppose that every gap has length either g_m or $2g_m$. We first show that g_m must be rational.

$$\begin{aligned}\sum_{k=1}^{n} g_k &= 1 + b_1 - b_n + \underbrace{\sum_{k=2}^{n} b_k - b_{k-1}}_{\text{telescopes}} \\ &= 1 + b_1 - b_n - b_1 + b_n \\ &= 1\end{aligned}$$

Note also that each g_k is an integer multiple of g_m. In particular, there exists $k \in \mathbb{N}$ such that

$$1 = \sum_{k=1}^{n} g_k = k g_m$$

So,

$$g_m = \frac{1}{k} \in \mathbb{Q}$$

Consider two cases: Case 1: b_n is the first iterate corresponding to the fractional part of 1α. Then, $b_n = \alpha$. Since $n > 1$, we have some $k < n$ such that

$$b_n + \alpha = b_k (\mod 1)$$

Note that since $b_k < b_n$,

$$b_n + \alpha - 1 = b_k \iff b_n - b_k = 1 - \alpha$$

For the points in between b_k and b_n, we count the number of each distance:

$$\begin{aligned} l &:= \#\{i : k \leqslant i < n,\ b_{i+1} - b_i = g_m\} \\ j &:= \#\{i : k \leqslant i < n,\ b_{i+1} - b_i = 2g_m\}\end{aligned}$$

As in part b, a telescoping sum argument shows that the total distance between b_k and b_n is given by

$$1 - \alpha = b_n - b_k = l g_m + 2 j g_m$$

Thus,

$$1 - \alpha = (\underbrace{l + 2j}_{\in \mathbb{Q}}) g_m$$

and so $\alpha \in \mathbb{Q}$ by the closure of $\mathbb{Q}$ under multiplication and addition. Case 2: b_n is not the first iterate. Then, $b_n > \alpha$, so there exists k with $b_k = b_n - \alpha$ and $n > k$. We use the same argument as in Case 1, calculating the total distance between b_k and b_n as an integer multiple of g_m to show that $\alpha \in \mathbb{Q}$. $\square$

(c)

Theorem 1.2 (Steinhaus three gap theorem - musical scale version)**.** *Let $\alpha > 0$ be a fixed irrational generator. Consider a musical scale of length $q \in \mathbb{N}$ generated by α, given by the set of q distinct points (=pitches) of the form*

$$\mathcal{S}_q(k) := \{R_\alpha^j(0)\ : k \leqslant j \leqslant k + q - 1\}\ ,\ \textit{where}\ k \in \mathbb{Z}\ . \tag{9}$$

Then, one has:

(i) *The set $\mathcal{S}_n(k)$ will give rise to two or three distinct, consecutive (pitch) distances.*

(ii) *Scales with two consecutive (pitch) distances arise if and only if q corresponds to an optimal scale size of the first kind. Here, recall from class that optimal scales of the first kind are precisely those for which the "*discarded*" $(q + 1)$st pitch in the tuning process giving rise to the scale in (9), i.e., the iterate*

$$R_\alpha^j(0)\ ,\ \textit{for}\ j = k + q\ , \tag{10}$$

is a neighboring iterate for the scale's origin at $R_\alpha^k(0)$.

Proof. Let's generalize the definition of a scale:

$$\mathcal{S}_n(k, x) := \{R_\alpha^j(x) : k \leqslant j \leqslant n + k - 1\}$$

Note that we wish to show that (i) and (ii) hold for all scales of type:

$$\begin{aligned} \mathcal{S}_n(k, 0) &= \{R_\alpha^j(0) : k \leqslant j \leqslant n + k - 1\} \\ &= \{\ \underbrace{R_\alpha^{i+k-1}(0)}_{=R_\alpha^i(R_\alpha^{k-1}(0))}\ : 1 \leqslant i \leqslant n\} \\ &= \mathcal{S}_n(1, R_\alpha^{k-1}(0)) \end{aligned}$$

We will show that (i) and (ii) hold for all scales of the form

$$\mathcal{S}_n(1, x) : x \in [0, 1)$$

which implies Theorem 1.2.

Consider the sequence b_m of Shiu's formula with

$$0 < b_1 < \cdots b_n < 1$$

and let $x \in [0, 1)$ be given. Let m be 1 if $b_1 + x \geqslant 1$. Otherwise, we take m to be the greatest integer such that $b_m + x < 1$:

$$m := \max\{k \in \mathbb{N} : b_k + x < 1\}$$

Observe that the sequence

$$\{a_k\}_{k=1}^n := \{b_k + x(\mod 1)\}_{k=1}^n$$

may be written in increasing order:

$$0 < a_{m+1} < \cdots < a_n < a_1 < \cdots < a_m < 1$$

since:

$$\begin{aligned}
k \leqslant m &\implies b_k \leqslant b_m \\
&\implies b_{k+x} \leqslant b_m + x < 1 \\
&\implies a_k = b_k + x \\
k > m &\implies b_m < b + k \\
&\implies b_k + k \nless 1 \\
&\implies a_k = b_k + x - 1 < a_1
\end{aligned}$$

So we see that the order of $a_1 \dots, a_m$ is the same and the order of $a_{m+1}, \dots a_n$ is the same, but with the larger terms first.

We compute the gaps of a_k:

$$G_1 = 1 + a_1 - a_n, \quad G_m = a_m - a_{m-1}, \quad m = 2, \dots, n$$

$$\begin{aligned}
G_1 &= 1 + a_1 - a_n \\
&= 1 + [(b_1 + x) \mod 1] - [(b_n + x) \mod 1] \\
&= 1 + [(b_1 + x) - (b_n + x) \mod 1] \\
&= 1 + [(\underbrace{b_1 - b_n}_{<1}) \mod 1] \\
&= 1 + b_1 - b_n \\
&= g_1 \\
G_m &= a_m - a_{m-1} \\
&= [(b_m + x) \mod 1] - [(b_{m-1} + x) \mod 1] \\
&= [(b_m + x) - (b_{m-1} + x)] \mod 1 \\
&= (\underbrace{b_m - b_{m-1}}_{<1}) \mod 1 \\
&= b_m - b_{m-1} \\
&= g_m
\end{aligned}$$

Thus, the sequence a_k has exactly the same gaps as b_m.
Note that

$$\{a_k\}_{k=1}^n = \mathcal{S}_n(1, x)$$

since for each $1 \leqslant k \leqslant n$, we can write

$$a_k = (j\alpha \mod 1) + x \mod 1 = j\alpha + x \mod 1 = R_\alpha^j(x)$$

for a unique j (uniqueness from Kronecker's Theorem). This concludes the argument of (i) for $\mathcal{S}_n(k)$.

We see that the gaps of the scale $\mathcal{S}_n(1, x)$ are independent of x. In particular, the gaps of $\mathcal{S}_n(k)$ are the same for all $k \in \mathbb{N}$. We consider the case of $\mathcal{S}_n(1)$. Notice that the elements of $\mathcal{S}_n(1)$ form exactly the sequence $\{b_m\}_{m=1}^n$. Also it is best scale of the first kind if and only if either the first or n-th iterate is closest to 0. This is equivalent to Shiu's definition of n being a node. So, the lemma gives (ii) for $\mathcal{S}_n(1)$ in particular. By the earlier comment, we see that it holds generally for $\mathcal{S}_n(k)$. $\square$