# PROBLEM SET NO. 2 - ISAAC VIVIANO

## 1. Problems:

- **Problem 1 - The vibrating string & harmonics:**

(b) Suppose $y(x,t) = f(x) \cdot g(t)$ is a solution to the 1-dimensional wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

Note that

$$\begin{aligned}
\frac{\partial y}{\partial t} &= \frac{\partial}{\partial t}(f(x) \cdot g(t)) = f(x) \cdot \frac{\partial}{\partial t} g(t) = f(x) g'((t) \\
\frac{\partial^2 y}{\partial t^2} &= f(x) g''(t) \\
\frac{\partial^2 y}{\partial x^2} &= f''(x) g(t)
\end{aligned}$$

So,

$$f(x) g''(t) = c^2 f''(x) g(t)$$

On $f(x) \neq 0$ and $g(t) \neq 0$, this may be written

$$\frac{g''(t)}{g(t)} = c^2 \frac{f''(x)}{f(x)} \tag{2}$$

Note that if $f(x) = 0$ or $g(t) = 0$, $y(x,t) = 0$, so $y$ satisfies (1). So, any separated solution $y$ to (1) must only satisfy (2).
Suppose $y(x,t) = f(x) g(t)$ is a solution to (1). Fix $t_0 \in \mathbb{R}$ with $g(t_0) \neq 0$ and let

$$-k = \frac{g''(t_0)}{g(t_0)}$$

Then, for all $x \in \mathbb{R}$ with $f(x) \neq 0$,

$$-k = c^2 \frac{f''(x)}{f(x)} \tag{3}$$

So, $f$ is a solution to (2). Similarly, fixing an $x$ shows that $g$ is a solution to

$$-k = \frac{g''(t)}{g(t)} \tag{4}$$

(c) We now apply the boundary condition $y(0,t) = y(l,t) = 0$. We have $y(x,t) = f(x) \cdot g(t)$. If $f(0) = 0$ and $f(l) = 0$, then for all $t$,

$$\begin{aligned}
y(0,t) &= f(0) \cdot g(t) = 0 \cdot g(t) = 0 \\
y(l,t) &= f(l) \cdot g(t) = 0 \cdot g(t) = 0
\end{aligned}$$

so the boundary conditions are satisfied.

Suppose $f(0) \neq 0$. We need that for all $t$, $y(0,t) = 0$:

$$0 = y(0,t) = f(0) \cdot g(t) \implies g(t) = 0$$

But this is only possible if $g$ is identically 0. This gives $f$ identically 0, not a solution we are interested in. The same argument shows that $f(l) = 0$ for any separated solution to the initial value problem. Thus, $y(x,t) = f(x) \cdot g(t)$ is a solution to the initial value problem (1) if and only if $f(0) = f(l) = 0$.

(d) Solve:

$$f'' + kf = 0$$

From Set 1, the general solution is

$$f(x) = A\cos\sqrt{k}x + B\sin\sqrt{k}x$$

Suppose there is a nonzero solution $f$ that satisfies the initial value problem $f(0) = f(l) = 0$. Then,

$$\begin{aligned} 0 = f(0) &= A\cos\sqrt{k}0 + B\sin\sqrt{k}0 \\ &= A \end{aligned}$$

So, $A = 0$. The second boundary condition gives

$$\begin{aligned} 0 = f(l) &= B\sin(\sqrt{k}l) \\ \sin(\sqrt{k}l) &= 0 \end{aligned}$$

This occurs when $\sqrt{k} \cdot l = \pi \cdot n$ for some integer $n$. Solving for $k$, we see that possible values are

$$k = \left(\frac{n\pi}{l}\right)^2$$

Let $k_n = \left(\frac{n\pi}{l}\right)^2$. Then, for all $n$, the function $f_n$ defined by

$$f_n(x) = \sin\sqrt{k}x$$

is a solution to the initial value problem:

$$\begin{aligned} f_n'(x) &= \sqrt{k}\cos\sqrt{k}x \\ f_n''(x) &= -k\sin\sqrt{k}x \\ f_n(0) &= 0 \\ f_n(l) &= \sin\sqrt{k}l = \sin\left(\frac{n\pi}{l} \cdot l\right) = 0 \\ f''(x) + kf(x) &= -k\sin\sqrt{k}x + k\sin\sqrt{k}x = 0 \end{aligned}$$

(e) Solving the time differential equation,

$$g'' + kc^2 g = 0$$

we get the general solution

$$g(x) = A\cos c\sqrt{k}t + B\sin c\sqrt{k}t$$

which may be rewritten

$$g(x) = B\sin(c\sqrt{k}t + \phi)$$

The general solution to the wave equation is the product of the these two solutions for the same $k$ value. So,

$$y = C\sin\left(\frac{cn\pi t}{l} + \phi\right)\sin\frac{n\pi t}{l}$$

where $C \in \mathbb{R}$ is a constant that represents the product of the temporal and spacial amplitudes.

– **Problem 2 - Bernoulli solutions & dividing the string:**
This will restrict to the Bernoulli solutions with a node at $x = \frac{l}{N}$. We need Bernoulli solutions with

$$f_n\left(\frac{l}{N}\right) = 0$$

Solving

$$0 = \sin\left(\frac{n\pi x}{\frac{l}{N}}\right) = \sin\left(\frac{nN\pi x}{l}\right)$$

we get $f_k$ where $k = nN$ for some $n \in \mathbb{N}$. These are the Bernoulli solutions where $k$ is divisible by $N$.

– **Problem 4 - Cesàro means:**
Let $a_n$ be a sequence in $\mathbb{C}$ with $a_n \to a \in \mathbb{C}$. Let

$$C_n = \frac{1}{n}\sum_{k=1}^{n} a_k$$

Let $\epsilon > 0$ be given and pick $M \in \mathbb{N}$ such that for all $n \geqslant M$,

$$|a_n - a| < \frac{\epsilon}{2}$$

Choose $N \geqslant M$ such that

$$\frac{1}{N}\sum_{k=1}^{M} |a_k - a| < \frac{\epsilon}{2}$$

Then, if $n \geqslant N$,

$$
\begin{aligned}
|C_n - a| &= \left| \frac{1}{n} \sum_{k=1}^{n} a_k - a \right| \\
&= \left| \frac{1}{n} \sum_{k=1}^{n} a_k - \frac{1}{n} \sum_{k=1}^{n} a \right| \\
&= \left| \frac{1}{n} \sum_{k=1}^{n} (a_k - a) \right| \\
&\leqslant \frac{1}{n} \sum_{k=1}^{n} |a_k - a| \quad (\Delta \text{ inequality in } \mathbb{C}) \\
&= \frac{1}{n} \sum_{k=1}^{M} |a_k - a| + \frac{1}{n} \sum_{k=M+1}^{n} |a_k - a| \\
&\leqslant \frac{1}{N} \sum_{k=1}^{M} |a_k - a| + \frac{1}{n} \sum_{k=M+1}^{n} |a_k - a| \\
&< \frac{\epsilon}{2} + \frac{(n-N)}{n} \cdot \frac{\epsilon}{2} \\
&\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
$$

So, $C_n \to a$.

– **Problem 5 - Fourier coefficients & derivatives:**

(a) Note that

$$\left| e^{-2\pi i n t} \right| = 1$$

for all $t \in \mathbb{R}$, and for a 1-periodic function $f$,

$$\|f\|_\infty = \sup\{f(x) : x \in [0, 1]\}$$

Then,

$$\begin{aligned}
|\hat{f}_n| &= \left|\int_0^1 e^{-2\pi int} f(t)\ dt\right| \\
&\leqslant \int_0^1 \left|e^{-2\pi int} f(t)\right| dt \quad (\Delta \text{ inequality}) \\
&= \int_0^1 \left|e^{-2\pi int}\right| \cdot |f(t)|\ dt \\
&= \int_0^1 |f(t)|\ dt \\
&\leqslant 1 \cdot \sup_{x\in[0,1]} f(x) \quad (\text{ML estimate}) \\
&= \|f\|_\infty
\end{aligned}$$

(b) Integration by parts:

$$\int u(t)v'(t)\ dt = u(t)v(t) - \int u'(t)v(t)\ dt \quad (\text{IBP})$$

Let $u(t) = f(t)$ and $v'(t) = e^{-2\pi int}$. By problem (3),

$$v(t) = \frac{e^{-2\pi int}}{-2\pi in}$$

is an antiderivative of $v$. Note that

$$\begin{aligned}
v(0) &= \frac{-1}{2\pi in} \\
v(1) &= -\frac{e^{-2\pi in}}{2\pi in}
\end{aligned}$$

(c) let $P(m)$ be that

$$\hat{f}_n = \left(\frac{1}{2\pi in}\right)^m \left(\hat{f_n}^{(m)} + \sum_{l=0}^{m-1} (2\pi in)^{m-1-l}(f^{(l)}(0) - e^{-2\pi in} f^{(l)}(1))\right)$$

**Base Case:** $m = 1$

$$\frac{1}{2\pi in}\left(\hat{f}'_n + \sum_{l=0}^{0}(2\pi in)^{0-l}(f^{(l)}(0) - e^{-2\pi in} f^{(l)}(1))\right) = \frac{1}{2\pi in}(\hat{f}'_n + f(0) - e^{-2\pi in} f(1))$$

which equals $\hat{f}_n$ by (b), so $P(1)$ is true.
**Inductive Step:** Assume $P(m)$ for some $1 \leqslant m < k$

Then,

$$\begin{aligned}
\hat{f_n}^{(m)} &= \frac{f^{(m)}(0) - e^{-2\pi in} f^{(m)}(1) + \hat{f_n}^{(m+1)}}{2\pi in} \quad \text{(part b)} \\
\hat{f_n} &= \left(\frac{1}{2\pi in}\right)^m \left(\hat{f_n}^{(m)} + \sum_{l=0}^{m-1} (2\pi in)^{m-1-l} (f^{(l)}(0) - e^{-2\pi in} f^{(l)}(1))\right) \quad \text{(ind. hyp.)} \\
&= \left(\frac{1}{2\pi in}\right)^m \left(\frac{f^{(m)}(0) - e^{-2\pi in} f^{(m)}(1) + \hat{f_n}^{(m+1)}}{2\pi in} + \sum_{l=0}^{m-1} (2\pi in)^{m-1-l} (f^{(l)}(0) - e^{-2\pi in} f^{(l)}(1))\right) \\
&= \left(\frac{1}{2\pi in}\right)^{m+1} \left(\hat{f_n}^{(m+1)} + f^{(m)}(0) - e^{-2\pi in} f^{(m)}(1) + \sum_{l=0}^{m-1} (2\pi in)^{m-l} (f^{(l)}(0) - e^{-2\pi in} f^{(l)}(1))\right) \\
&= \left(\frac{1}{2\pi in}\right)^{m+1} \left(\hat{f_n}^{(m+1)} + (2\pi in)^{m-m} (f^{(m)}(0) - e^{-2\pi in} f^{(m)}(1)) + \sum_{l=0}^{m-1} (2\pi in)^{m-l} (f^{(l)}(0) - e^{-2\pi in} f^{(l)}(1))\right) \\
&= \left(\frac{1}{2\pi in}\right)^{m+1} \left(\hat{f_n}^{(m+1)} + \sum_{l=0}^{m} (2\pi in)^{m-l} (f^{(l)}(0) - e^{2\pi in} f^{(l)}(1))\right) \\
&= \left(\frac{1}{2\pi in}\right)^{m+1} \left(\hat{f_n}^{(m+1)} + \sum_{l=0}^{(m+1)-1} (2\pi in)^{(m+1)-1-l} (f^{(l)}(0) - e^{2\pi in} f^{(l)}(1))\right)
\end{aligned}$$

So, $P(m+1)$.

Therefore, by (PMI), $P(m)$ for all $1 \leqslant m \leqslant k$. In particular,

$$\hat{f_n} = \left(\frac{1}{2\pi in}\right)^k \left(\hat{f_n}^{(k)} + \sum_{l=0}^{k-1} (2\pi in)^{k-1-l} (f^{(l)}(0) - e^{-2\pi in} f^{(l)}(1))\right)$$

Now,

$$\begin{aligned}
&\left|\hat{f_n}^{(k)} + \sum_{l=0}^{k-1} (2\pi in)^{k-1-l} (f^{(l)}(0) - e^{-2\pi in} f^{(l)}(1))\right| \\
&\leqslant \left|\hat{f_n}^{(k)}\right| + \left|\sum_{l=0}^{k-1} (2\pi in)^{k-1-l} (f^{(l)}(0) - e^{-2\pi in} f^{(l)}(1))\right| \\
&\leqslant \left\|f^{(k)}\right\| + \left|\sum_{l=0}^{k-1} (2\pi in)^{k-1-l} (f^{(l)}(0) - e^{-2\pi in} f^{(l)}(1))\right| \\
:&= C \in \mathbb{R}
\end{aligned}$$

Then,

$$\left|\hat{f_n}\right| \leqslant \left|\left(\frac{1}{2\pi in}\right)^k \cdot C\right| = \frac{C}{|n|^k}$$

– **Problem 6 - Real Fourier series and the sawtooth function:**

(b) Let $\phi$ be the $2\pi$-periodic function described by

$$\phi(\theta) = \frac{\pi - \theta}{2}$$

and $\phi(0) = \phi(2\pi) = 0$. For all $n \neq 0$,

$$\begin{aligned}
\int_0^{2\pi} e^{-ni\theta} d\theta &= -\left.\frac{e^{-ni\theta}}{-ni}\right|_0^{2\pi} = 0 \\
\int_0^{2\pi} \frac{\theta}{2} e^{-ni\theta}\, d\theta &= \left.\frac{-\theta e^{-ni\theta}}{2ni}\right|_0^{2\pi} - \int_0^{2\pi} \frac{1}{2} e^{-ni\theta}\, d\theta \\
&= \frac{-2\pi}{2ni} = -\frac{\pi}{ni} \\
\hat{f}_n &= \int_0^{2\pi} \phi(\theta) e^{-ni\theta}\, d\theta \\
&= \int_0^{2\pi} \frac{\pi - \theta}{2} e^{-ni\theta}\, d\theta \\
&= \int_0^{2\pi} \frac{\pi}{2} e^{-ni\theta} d\theta - \int_0^{2\pi} \frac{\theta}{2} e^{-ni\theta}\, d\theta \\
&= \frac{\pi}{ni}
\end{aligned}$$

And for $n = 0$,

$$\begin{aligned}
\hat{f}_0 &= \int_0^{2\pi} \phi(\theta) e^{-ni\theta}\, d\theta = \int_0^{2\pi} \frac{\pi - \theta}{2} d\theta \\
&= \left.\frac{\pi}{2}\theta - \frac{\theta^2}{4}\right|_0^{2\pi} \\
&= \pi^2 - \frac{(2\pi)^2}{4} = 0
\end{aligned}$$

So, the Fourier series is

$$
\begin{aligned}
f(t) &= \sum_{n=-1}^{-\infty} \hat{f}_n e^{int} + \hat{f}_0 + \sum_{n=1}^{\infty} \hat{f}_n e^{int} \\
&= \sum_{n=-1}^{-\infty} \hat{f}_n e^{int} + 0 + \sum_{n=1}^{\infty} \hat{f}_n e^{int} \\
&= \sum_{n=1}^{\infty} \hat{f_{-n}} e^{-int} + \sum_{n=1}^{\infty} \hat{f}_n e^{int} \\
&= \sum_{n=1}^{\infty} \frac{\pi}{-ni}(\cos(-nt) + i\sin(-nt)) + \sum_{n=1}^{\infty} \frac{\pi}{ni}(\cos nt + i\sin nt) \\
&= \sum_{n=1}^{\infty} \frac{\pi}{ni}(-\cos nt + i\sin nt) + \sum_{n=1}^{\infty} \frac{\pi}{ni}(\cos nt + i\sin nt) \\
&= \sum_{n=1}^{\infty} \frac{\pi}{ni}(-\cos nt + \cos nt + 2i\sin nt) \\
&= \sum_{n=1}^{\infty} \frac{2\pi}{n}\sin nt
\end{aligned}
$$