Ivan Lin
AMS361
Dr. Yuefan Deng
5/21/17

Homework 7

Problem 1

7.1) $\begin{cases} x' = x - 2y, & x(0) = 6 \\ y' = 3x - 4y, & y(0) = 1 \end{cases}$

① Substitution

$y' + 4y = 3x \qquad y'' + 4y' = 3x - 6y \qquad \lambda^2 + 3\lambda + 2 = 0$

$y'' + 4y' = 3x' \qquad y'' + 4y' = y' + 4y - 6y \qquad (\lambda+2)(\lambda+1) = 0$

$3x' = 3x - 6y \qquad y'' + 3y' + 2y = 0 \qquad \lambda = -1, -2$

$y = c_1 e^{-t} + c_2 e^{-2t}$

$y' = -c_1 e^{-t} - 2c_2 e^{-2t}$

$-c_1 e^{-t} - 2c_2 e^{-2t} = 3x - 4(c_1 e^{-t} + c_2 e^{-2t})$

$3c_1 e^{-t} + 2c_2 e^{-2t} = 3x$

$c_1 e^{-t} + \frac{2}{3} c_2 e^{-2t} = x$

$x = c_1 e^{-t} + \frac{2}{3} c_2 e^{-2t}$

$y = c_1 e^{-t} + c_2 e^{-2t}$

$6 = c_1 e^{-0} + \frac{2}{3} c_2 e^{-2\cdot 0} \qquad 1 = c_1 e^{-0} + c_2 e^{-2\cdot 0}$

$6 = c_1 + \frac{2}{3} c_2 \qquad 1 = c_1 + c_2$

$5 = -\frac{1}{3} c_2, \quad c_2 = -15, \quad c_1 = 16$

$x = 16e^{-t} - 10e^{-2t}, \quad y = 16e^{-t} - 15e^{-2t}$

7.1) $\begin{cases} x' = x - 2y, & x(0) = 6 \\ y' = 3x - 4y, & y(0) = 1 \end{cases}$

② Operator

let $D = d/dt$

$\begin{cases} Dx = x - 2y \\ Dy = 3x - 4y \end{cases}$

$\begin{cases} (D-1)x + 2y = 0 & \cdot 3 \\ (D+4)y - 3x = 0 & \cdot (D-1) \end{cases}$

$\begin{cases} 3(D-1)x + 6y = 0 \\ (D+4)y(D-1) - 3(D-1)x = 0 \end{cases}$

$(D+4)(D-1)y + 6y = 0$

$D^2y + 3Dy + 2y = 0$

$\lambda^2 + 3\lambda + 2 = 0 \quad \lambda = -1, -2$

Given $\lambda = -1, -2$, follow the same process as ① to obtain

GS: $x = c_1 e^{-t} + \frac{2}{3} c_2 e^{-t}$

$y = c_1 e^{-t} + c_2 e^{-t}$

PS: $x = 16e^{-t} - 10e^{-2t}$

$y = 16e^{-t} - 15e^{-2t}$

③ Eigen

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{vmatrix}$

$0 = (1-\lambda)(-4-\lambda) - 3(-2)$

$0 = \lambda^2 + 3\lambda + 2 \quad 0 = (\lambda+1)(\lambda+2)$

$$\lambda = -1, -2$$

case $\lambda = -1$

$$\begin{pmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_1 = v_2 \qquad \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

case $\lambda = -2$

$$\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad 3v_1 = 2v_2 \qquad \vec{v} = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$v_1 = \frac{2}{3} v_2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix}$$

$x(0) = 6, \ y(0) = 1$

$$\begin{pmatrix} 6 \\ 1 \end{pmatrix} = c_1 e^0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2 \cdot 0} \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 6 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} \frac{2}{3} c_2 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + \frac{2}{3} c_2 \\ c_1 + c_2 \end{pmatrix} \qquad 5 = c_1 - \frac{1}{3} c_2$$

$$c_2 = -15, \ c_1 = 16$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = 16 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 15 e^{-2t} \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$$

Problem 2

7.2) $\begin{cases} x' = 4x - 3y, & x(0) = 6 \\ y' = 3x + 4y. & y(0) = 1 \end{cases}$

Substitution

$3x = y' - 4y \qquad y'' - 4y' = 12x - 9y$

$3x' = y'' - 4y' \qquad y'' - 4y' = 4y' - 16y - 9y$

$x' = 4x - 3y \qquad y'' - 8y' + 25y = 0$

$3x' = 12x - 9y \qquad \lambda^2 - 8\lambda + 25 = 0$

$$\lambda = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(25)}}{2(1)} = 4 \pm 3i$$

$$y = C_1 e^{(4+3i)t} + C_2 e^{(4-3i)t}$$

$$y' = (4+3i)C_1 e^{(4+3i)t} + (4-3i)C_2 e^{(4-3i)t}$$

$$= \cancel{4}C_1 e^{(4+3i)t} + \cancel{4}C_2 e^{(4-3i)t} + 3x$$

$$\cancel{3}x = C_1 \cancel{3} i e^{(4+3i)t} - C_2 \cancel{3} i C_2 e^{(4-3i)t}$$

$1 = C_1 e^{(4+3i)0} + C_2 e^{(4-3i)0}, \quad 6 = C_1 i e^{(4+3i)0} - C_2 i e^{(4-3i)0}$

$1 = C_1 + C_2 \qquad 6 = C_1 - C_2$

$7 = 2C_1, \; C_1 = 7/2, \; C_2 = 5/2$

CS: $x = C_1 i e^{(4+3i)t} - C_2 i e^{(4-3it)}, \; y = C_1 e^{(4+3i)t} + C_2 e^{(4-3i)t}$

PS:

$$x = \frac{7}{2} i e^{(4+3i)t} - \frac{5}{2} i e^{(4-3i)t}, \quad y = \frac{7}{2} e^{(4+3i)t} + \frac{5}{2} e^{(4-3i)t}$$

Eigen

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det(x - \lambda) = 0 = \begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix}$$

$$(4-\lambda)^2 - 3(-3) = 16 - 8\lambda + \lambda^2 + 9 = \lambda^2 - 8\lambda + 25$$

$$\lambda = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(25)}}{2(1)}$$

$$= 4 \pm \tfrac{1}{2}\sqrt{-36} = 4 \pm 3i$$

case $\lambda = 4 + 3i$

$$\begin{pmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{pmatrix} \to \begin{pmatrix} -3i & -3 \\ 3 & -3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3v_1 - 3iv_2 = 0, \quad v_1 = iv_2 \qquad v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

Case $\lambda = 4 - 3i$

$$\begin{pmatrix} 3i & -3 \\ 3 & 3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3v_1 + 3iv_2 = 0, \quad v_1 = -iv_2, \quad v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{(4+3i)t} \begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 e^{(4-3i)t} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} x: 6 &= c_1 i e^{(4+3i)0} + c_2 i e^{(4-3i)0} = c_1 i + c_2 i \\ y: 1 &= c_1 e^{(4+3i)0} + c_2 e^{(4-3i)0} = c_1 + c_2 \end{aligned}\right\} \; c_1 = 7/2,\; c_2 = 9/2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{7}{2} e^{(4+3i)t} \begin{pmatrix} i \\ 1 \end{pmatrix} + \frac{5}{2} e^{(4-3i)t} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Problem 3

7.3) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

EIGEN

det $(A-\lambda)=0=\begin{vmatrix}1-\lambda & -3\\ 3 & 7-\lambda\end{vmatrix}$

$0=(1-\lambda)(7-\lambda)-3(-3)$

$0=\lambda^2-8\lambda+7+9$

$0=\lambda^2-8\lambda+16$ $\qquad \lambda=4$

$0=(\lambda-4)^2$

$\lambda=4)$ $\begin{pmatrix}1-\lambda & -3\\ 3 & 7-\lambda\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

$\begin{pmatrix}-3 & -3\\ 3 & 3\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $\qquad -v_1=v_2$ $\qquad \vec{v}=\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}$

let $x_2=(v_1 t+v_2)e^{\lambda t}$

det $(A-\lambda)^2 v_2=0$

$\begin{pmatrix}-3 & -3\\ 3 & 3\end{pmatrix}^2 v_2=\vec{0}\, v_2$

v_2 can be any vector, let $\vec{v}_2=\begin{pmatrix}c_3\\c_4\end{pmatrix}$

$x_2=\left(\begin{pmatrix}-1\\1\end{pmatrix}t+\begin{pmatrix}1\\0\end{pmatrix}\right)e^{4t}=0$

$\begin{pmatrix}x\\y\end{pmatrix}'=c_1e^{4t}\begin{pmatrix}-1\\1\end{pmatrix}+c_2e^{4t}\left[\begin{pmatrix}-1\\1\end{pmatrix}t+\begin{pmatrix}c_3\\c_4\end{pmatrix}\right]$

Substitution

$x' = x - 3y$

$y' = 3x + 7y$

$3x' = y'' - 7y'$

$3x - 9y = y'' - 7y'$

$y' - 7y - 9y = y'' - 7y'$

$y'' - 8y' + 16y = 0$

$\lambda^2 - 8\lambda + 16 = 0$

$(\lambda - 4)^2 = 0$

$\lambda = 4$

$y = c_1 e^{4t} + c_2 t e^{4t}$

$y' = 4c_1 e^{4t} + 4c_2 t e^{4t}$

$4c_1 e^{4t} + 4t c_2 e^{4t} = 3x + 7(c_1 e^{4t} + c_2 t e^{4t})$

$-3c_1 e^{4t} - 3c_2 t e^{4t} = 3x$

$x = -c_1 e^{4t} - c_2 t e^{4t}$

$y = c_1 e^{4t} + c_2 t e^{4t}$

Problem 4

7.4) $x'(t) = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix} x(t) + \begin{pmatrix} e^{-t} \\ -t^2 \end{pmatrix}$

Eigen

$x_c(t)$: $x' = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix} x$

$$\det(A - \lambda) = 0 = \begin{vmatrix} 2-\lambda & 4 \\ 1 & -1-\lambda \end{vmatrix}$$

$$0 = (2-\lambda)(-1-\lambda) - 1(4)$$

$$0 = -2 - \lambda + \lambda^2 - 4 = \lambda^2 - \lambda - 6$$

$$0 = (\lambda - 3)(\lambda + 2)$$

$$\lambda = 3, -2$$

$\lambda = 3$) $\begin{pmatrix} 2-\lambda & 4 \\ 1 & -1-\lambda \end{pmatrix} \to \begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix} \vec{v} = \vec{0}$, $v_1 = 4v_2$, $\vec{v} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\lambda = -2$) $\begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix} \vec{v} = 0$, $v_1 = -v_2$, $\vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

x_p: $x_T = \vec{a} e^{-t} + \vec{b} t^2 + ct + d$

$$x_T' = -\vec{a} e^{-t} + 2\vec{b} t + c$$

e^{-t}: $-\vec{a} = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix} \vec{a} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2a_1 + 4a_2 \\ a_1 - a_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3a_1 + 4a_2 + 1 \\ a_1 \end{pmatrix} \Rightarrow \vec{a} = \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}$$

t^2: $\vec{0} = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix} \vec{b} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2b_1 + 4b_2 \\ b_1 - b_2 - 1 \end{pmatrix}$

$0 = 2b_1 + 4b_2$ $\qquad$ $0 = 6b_2 + 2$

$0 = 2b_1 - 2b_2 - 2$ $\qquad$ $b_2 = -1/3$ $\quad b_1 = 2/3$

$$\vec{b} = \begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}$$

$t: \; \vec{b} = A\vec{c} \Rightarrow \begin{pmatrix} 2c_1+4c_2 \\ c_1-c_2 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 1/3 \end{pmatrix}$

$$\begin{aligned} 2c_1+4c_2 &= -2/3 \\ 2c_1-2c_2 &= 2/3 \end{aligned} \quad 6c_2 = -4/3 \quad c_1 = 1/9 \quad c_2 = -2/9 \quad \vec{c} = \begin{pmatrix} -2/9 \\ 4/9 \end{pmatrix}$$

$k: \; \vec{c} = A\vec{d} \Rightarrow \begin{pmatrix} 2d_1+4d_2 \\ d_1-d_2 \end{pmatrix} = \begin{pmatrix} -2/9 \\ 4/9 \end{pmatrix}$

$$\begin{aligned} 2d_1+4d_2 &= -2/9 \\ 2d_1-2d_2 &= 8/9 \end{aligned} \quad 6d_2 = -10/9 \quad d_1 = 7/27 \quad d_2 = -5/27 \quad \vec{d} = \begin{pmatrix} 7/27 \\ -5/27 \end{pmatrix}$$

$$\boxed{\vec{X}(t) = c_1 e^{3t}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/4 \end{pmatrix} e^{-t} + \begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix} t^2 + \begin{pmatrix} -2/9 \\ 4/9 \end{pmatrix} t + \begin{pmatrix} 7/27 \\ -5/27 \end{pmatrix}}$$

Substitution

$$\begin{aligned} x' &= 2x+4y+e^{-t} \\ y' &= x-y-t^2 \end{aligned} \Rightarrow \begin{aligned} x &= y'+y+t^2 \\ x' &= y''+y'+2t \end{aligned}$$

$$y''+y'+2t = 2y'+2y+2t^2+4y+e^{-t}$$

$$y'' - y' - 6y = 2t^2 - 2t + e^{-t}$$

y_c: $\lambda^2 - \lambda - 6 = 0$

$(\lambda - 3)(\lambda + 2) = 0 \quad \lambda = 3, -2$

$y_c = c_1 e^{3t} + c_2 e^{-2t}$

y_p: $y_T = ae^{-t} + bt^2 + ct + d$

$(ae^{-t} + 2b) - (-ae^{-t} + 2bt + c) -$

$6(ae^{-t} + bt^2 + ct + d) = 2t^2 - 2t + e^{-t}$

$a + a - 6a = 1$

$-4a = 1$

$a = -1/4$

$-6b\cancel{t^2} = 2\cancel{t^2}$

$b = -1/3$

$-2b\cancel{t} - 6c\cancel{t} = -2\cancel{t}$

$2/3 - 6c = -2$

$-6c = -8/3$

$c = 4/9$

$2b - c - 6d = 0$

$-2/3 - 4/9 - 6d = 0$

$-10/9 = 6d$

$d = -5/27$

$y = c_1 e^{3t} + c_2 e^{-2t}$

$-\frac{1}{4}e^{-t} - \frac{t^2}{3}$

$+ \frac{4}{9}t - \frac{5}{27}$

$$y' = 3c_1e^{3t} - 2c_2e^{-2t} + \frac{1}{4}e^{-t} - \frac{2}{3}t + \frac{4}{9}$$

$$y' = x - y - t^2$$

$$y' + y + t^2 = x$$

$$4c_1e^{3t} - c_2e^{-2t} + \cancel{0} \underbrace{- \frac{t^2}{3} + t^2}_{\frac{2}{3}t^2} - \frac{2}{9}t + \frac{7}{27}$$

$$= x$$

$$\boxed{\begin{aligned} x &= 4c_1e^{3t} - c_2e^{-2t} + \frac{2}{3}t^2 - \frac{2}{9}t + \frac{7}{27} \\ y &= c_1e^{3t} + c_2e^{-2t} - \frac{1}{4}e^{-t} - \frac{t^2}{3} \\ &\quad + \frac{4}{9}t - \frac{5}{27} \end{aligned}}$$