Ivan Lin
AMS361
Dr. Yuefan Deng
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Homework 2

Problem 1

2.1) $xy' - y = x^7$

$$y' = \frac{x^7 + y}{x} = x^6 + \frac{y}{x}$$

let $u = \frac{y}{x} \Rightarrow y = ux, \; y' = u'x + u$

$$y' = x^6 + \frac{y}{x} = u'x + u$$

$$x^6 + \cancel{u} = u'x + \cancel{u}$$

$$x^5 = u'$$

$$\frac{du}{dx} = x^5$$

$$\int du = \int x^5 \, dx$$

$$u = \frac{1}{6}x^6 + C = \frac{y}{x}$$

$$\boxed{y = \frac{1}{6}x^7 + xC}$$

Problem 2

2.2) $x y' - y = y^3 \cos(7x)$

$$y' = \frac{y + y^3 \cos(7x)}{x}$$

$$\frac{y'}{y^3} = \frac{1}{y^2 x} + \frac{\cos(7x)}{x}$$

$$u = \frac{1}{y^2} = y^{-2},\ u' = -2y^{-3} y' = -\frac{2y'}{y^3}$$

$$-\frac{1}{2} u' = \frac{u}{x} + \frac{\cos(7x)}{x}$$

$$\rho(x)\left(u' + \frac{2u}{x} = \frac{-2\cos(7x)}{x}\right)$$

$$\rho(x) = e^{\int \frac{2}{x} dx} = e^{2\ln x + C} = x^2 \cdot e^C$$

$$e^C x^2 u' + x e^C 2u = -2e^C x \cos(7x)$$

$$(e^C x^2 u)' = -2e^C x \cos(7x)$$

$e^C = A$

$$\hookrightarrow \frac{d}{dx}(A x^2 u)' = -2A x \cos(7x)$$

$$\cancel{A} x^2 u = \int -2\cancel{A} x \cos(7x)\, dx$$

$$u = \frac{1}{x^2}\left(\frac{1}{7} x \sin(7x) + \frac{1}{49}\cos(7x) + C\right)$$

$$u = \frac{1}{7x}\sin(7x) + \frac{1}{49x^2}\cos 7x + \frac{A}{x^2}$$

$$y^{-2} = \frac{1}{7x}\sin(7x) + \frac{1}{49x^2}\cos(7x) + \frac{A}{x^2}$$

$$\boxed{y = \left(\frac{1}{7x}\sin(7x) + \frac{1}{49x^2}\cos(7x) + \frac{A}{x^2}\right)}$$

Problem 3

2.3) $\begin{cases} y' - x^2 y = 7xy \\ y(1) = 1 \end{cases}$

$$\frac{y'}{y} - x^2 = 7x$$

$$\frac{y'}{y} = x^2 + 7x$$

$$\frac{dy}{dx} \cdot \frac{1}{y} = x^2 + 7x$$

$$\int \frac{dy}{y} = \int (x^2 + 7x)\, dx$$

$$\ln y = \frac{1}{3}x^3 + \frac{7}{2}x^2 + C$$

$$y = Ae^{\frac{1}{3}x^3 + \frac{7}{2}x^2}$$

$y(1) = 1$: $1 = Ae^{\frac{1}{3} \cdot 1^3 + \frac{7}{2} \cdot 1^2}$

$$1 = Ae^{\frac{23}{6}}$$

$$A = e^{-\frac{6}{23}}$$

$$y = e^{-\frac{6}{23}} e^{\frac{1}{3}x^3 + \frac{7}{2}x^2}$$

$$\boxed{y = e^{\frac{1}{3}x^3 + \frac{7}{2}x^2 - \frac{6}{23}}}$$

Problem 4

$$y' = (xy' + y)y^3$$
$$y' = xy'y^3 + y^4$$
$$y' - xy'y^3 = y^4$$
$$y'(1 - xy^3) = y^4$$
$$y' = \frac{y^4}{1 - xy^3}$$
$$\frac{1}{y'} = \frac{1 - xy^3}{y^4}$$
$$\frac{y^4}{y'} + xy^3 = 1$$
let $u = y^{-3}$, $u' = -3y^{-4}y'$, $\frac{1}{u'} = \frac{y^4}{-3y'}$
$$-\frac{3}{u'} + \frac{x}{u} = 1$$
$$-\frac{3}{u'} = 1 - \frac{x}{u}$$
$$\frac{dx}{du} = \frac{x}{3u} - \frac{1}{3} = \frac{x-u}{3u}$$
$$v = \frac{u}{x}, \quad \frac{1}{v} = \frac{x}{u}$$
$$u = vx, \; u' = v'x + v, \; \frac{1}{u'} = \frac{1}{v'x+v}$$
$$\frac{dx}{du} = \frac{1}{v'x+v} = \frac{1}{3v} - \frac{1}{3}$$
$$\frac{1}{v'x+v} = \frac{1-v}{3v}$$
$$u' = v'x + v = \frac{3v}{1-v}$$
$$v'x = \frac{3v}{1-v} - v = \frac{3v}{1-v} - \frac{v-v^2}{1-v}$$
$$x\frac{dv}{dx} = \frac{2v+v^2}{1-v}$$
$$\int \left(\frac{dx}{x} = \frac{1-v}{2v+v^2}dv \right)$$
$$\frac{1-v}{2v+v^2} = \frac{1/2}{2+v} + \frac{-3/2}{v}$$
$$C + \ln x = \frac{1}{2}\ln(2+v) - \frac{3}{2}\ln(v)$$
$$x = \frac{e^{1/2}(2+v)}{e^{3/2}\, v}$$
$$\frac{1}{e}x = \frac{2}{v} + 1$$
$$\frac{y^{-3}}{x} = \frac{u}{x} = v = \frac{2}{\frac{x}{e} - 1}$$
$$y^{-3} = \frac{2x}{\frac{x}{e} - 1}$$
$$y = \left(\frac{\frac{x}{e} - 1}{2x}\right)^{1/3} = \left(\frac{1}{2e} - \frac{1}{2x}\right)^{1/3}$$

Problem 5

$xy'' + 3y' = 7x^6$

$u = y'\ ,\ u' = y''$

$xu' + 3u = 7x^6$

$u' = \dfrac{7x^6 - 3u}{x}$ $\qquad a = \dfrac{u}{x}$

$ax = u$

$7x^{5} - 3a = xa' + a$ $\qquad xa' + a = u'$

$\to a' = \dfrac{7x^5 - 4a}{x}$ $\qquad b = \dfrac{a}{x}$

$7x^4 - 4b = xb' + b$ $\qquad bx = a$

$\to b' = \dfrac{7x^4 - 5b}{x}$ $\qquad xb' + b = a'$

pattern continues

$\vdots$

$e' = \dfrac{7x - 8e}{x}$ $\qquad f = \dfrac{e}{x}$

$7 - 8f = f + xf'$ $\qquad e = fx$

$e' = f + xf'$

$\dfrac{df}{dx} = f' = \dfrac{7 - 9f}{x}$

$\dfrac{df}{7 - 9f} = \dfrac{dx}{x}$

$\ln x = -\frac{1}{9}\ln(7 - 9f) + C$

$f = \dfrac{e}{x} = \dfrac{d}{x^2} = \dfrac{c}{x^3} = \dfrac{b}{x^4} = \dfrac{a}{x^5} = \dfrac{u}{x^6} = \dfrac{y'}{x^6}$

$-1/9$

$x = A(7 - 9f)^{-1/9}$

$x = A\left(7 - 9\dfrac{y'}{x^6}\right)$

$\dfrac{A}{x^9} = 7 - 9\dfrac{y'}{x^6}$

$\frac{7}{9}x^6 - \dfrac{A}{9x^3} = \dfrac{dy}{dx}$

$\int \frac{7}{9}x^6 - \dfrac{A}{9x^3}\,dx = \int dy$

$y = \frac{1}{9}x^7 - \dfrac{A}{2x^2} + B$