AMS361 - Differential Equations

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Contents

Introduction

0.0.1 Lecture Information

Class: AMS 361: Differential Equations
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Chapter 1

Introduction to Linear Equations

1.1 Characteristics of Differential Equations

- linear vs nonlinear - an equation is linear if the dependent variable or its derivative is linear
- order of ordinary differential equations - the order of an ordinary differential equation is the number of the highest derivative in the equation
- homogeneous vs inhomogeneous - an equation is homogeneous if it can be arranged so its dependent variable terms are equal to 0

num. DV	num. IV	1	2+
1		ordinary differential equation	partial differential equation
2		system of oridnary differential equations	system of partial differential equations

Chapter 2

First Order Equations

2.1 Separable First Order Equations

Form:

$$y' = \frac{f(x)}{g(y)} \tag{1}$$

Solution:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \tag{1}$$

$$g(y)dy = f(x)dx \tag{2}$$

$$\int g(y)dy = \int f(x)dx \tag{3}$$

$$\text{integrate and solve for } y \tag{4}$$

2.2 Linear First Order Equations

Form:

$$y' + P(x)y + Q(x) = 0 \tag{1}$$

Solution Derivation:

$$y' + P(x)y + Q(x) = 0 \tag{1}$$

$$v(x)[y' + P(x)y + Q(x) = 0] \text{ - multiply by some factor } v(x) \tag{2}$$

$$v(x)y' + v(x)P(x)y + v(x)Q(x) = 0 \tag{3}$$

$$\text{let } v'(x) = v(x)P(x) \text{ - this way, } \frac{d[v(x)y(x)]}{dx} = v(x)y' + v(x)P(x)y \tag{4}$$

$$\frac{dv}{dx} = v(x)P(x) \tag{5}$$

$$\int \frac{dv}{v(x)} = \int P(x)dx \tag{6}$$

$$ln(v(x)) = \int P(x)dx + c \tag{7}$$

$$v(x) = e^{\int P(x)dx + c} = Ae^{\int P(x)dx} \tag{8}$$

$v(x)y' + v(x)P(x)y + v(x)Q(x) = 0$
plug in $v(x)$ and solve as a first order linear differential equation

Solution:

$y' + P(x)y + Q(x) = 0$
let $v(x) = Ae^{\int P(x)dx}$
$v(x)[y' + P(x)y + Q(x) = 0]$
$v(x)y' + v(x)P(x)y + v(x)Q(x) = 0$
substitute and solve as a first order linear differential equation

2.3 Polynomial Functions

Form:

$y' = F(ax + by + c)$

Solution Derivation:

$y' = F(ax + by + c)$
let $u = ax + by + c$
$u' = a + by'$
$\frac{dy}{dx} = (\frac{du}{dx} - a)\frac{1}{b}$ - solve for y’
$y' = F(u) = \frac{u'-a}{b}$

Solution:

$y' = F(ax + by + c)$
let $u = ax + by + c$
$y' = F(u) = \frac{u'-a}{b}$
solve for u as a separable first order linear differential equation
substitute and solve for y

2.4 Homogeneous Differential Equations

Form:

$y' = F(\frac{y}{x})$

Solution Derivation:

$y' = F(\frac{y}{x})$
let $u = \frac{y}{x}$
$ux = y$
$u'x + u = y'$

$y' = F(\frac{y}{x}) = F(u) = u'x + u$
$F(u) - u = \frac{du}{dx}x$
$\frac{dx}{x} = \frac{du}{F(u)-u}$
$\int \frac{dx}{x} = \int \frac{du}{F(u)-u}$
$ln(x) = \int \frac{du}{F(u)-u} + c$
$x = Ae^{\int \frac{du}{F(u)-u}}$
solve for u, then substitute and solve back for y

Solution:

$y' = F(\frac{y}{x})$
let $u = \frac{y}{x}$
$x = Ae^{\int \frac{du}{F(u)-u}}$
solve for u, then substitute and solve back for y

2.5 Bernoulli Differential Equations

Form:

$y' + P(x)y = Q(x)y^n$

Solution:

$y' + P(x)y = Q(x)y^n$
$y'y^{-n} + P(x)y^{1-n} = Q(x)$
let $u = y^{1-n}$
$u' = (1-n)y^{-n}y'$
$\frac{1}{1-n}u' + P(x)u = Q(x)$ solve for u as a separable first order linear equation

2.6 Ricatti Differential Equation

Form:

$y' = A_0(x) + A_1(x)y + A_2(x)y^2$, given $y_0(x)$

Solution:

$y_0(x) = A_0(x) + A_1(y) + A_2(x)y^2$, given $y_0(x)$
$y_0' = A_0(x) + A_1(x)y_0 + A_2(x)y_0^2$
the general solution will be of the form $y = y_0(x) + \frac{1}{z(x)}$
$y^2 = y_0(x)^2 + 2y_0(x)\frac{1}{z(x)} + \frac{1}{z(x)^2}$
$y' = y_0'(x) - \frac{1}{z(x)^2}z(x)'$

$y' = y_0'(x) - \frac{1}{z^2}z' = A_0(x) + A_1(x)(y_0 + \frac{1}{z}) + A_2(x)(y_0^2 + 2y_0\frac{1}{z} + \frac{1}{z^2}$
$y_0' = A_0(x) + A_1(x)y_0 + A_2(x)y_0^2$ - subtract $\quad -\frac{1}{z^2}z' = A_1(x)\frac{1}{z} +$
$A_2(x)\frac{2y_0}{z} + \frac{A_2(x)}{z^2}$
$z' = -A_1(x)z(x) - A_2(x)2y_0(x)z(x) - A_2(x)$
$z' = (-A_1 - 2A_2y_0)z(x) - A_2(x)$
solve as a first order linear differential equation

2.7 Exact Differential Equations

Form:

$M(x,y)dx + N(x,y)dy = 0$
the equation is exact if $\frac{M(x,y)}{\delta y} = \frac{N(x,y)}{\delta x}$

Solution:

if the equation is not exact, multiply it by a factor ρ

$\rho[M(x,y)dx + N(x,y)dy = 0]$
$\rho(x)M(x,y) = \rho(x)N(x,y)$ or $\rho(y)M(x,y) = \rho(y)N(x,y)$
$\frac{d(\rho M(x,y))}{dy} = \frac{d(\rho N(x,y))}{dx}$
$\rho(x)M_y = \rho'(x)N + \rho(x)N_x$ or $\rho'(y)M + \rho(y)M_y = \rho(y)N_x$
$\rho(x)[M_y - N_x] = \rho'(x)N$ or $\rho(y)[M_y - N_x] = -\rho'(y)M$
$\rho(x)\frac{M_y - N_x}{N} = \frac{d\rho}{dx}$ or $\rho(y)\frac{M_y - N_x}{M} = -\frac{d\rho}{dy}$
$\int \frac{M_y - N_x}{N}dx = \int \frac{d\rho}{\rho(x)}$ or $\int \frac{M_y - N_x}{M} = \int -\frac{d\rho(y)}{dy}$
$\rho = e^{\int \frac{M_y - N_x}{N}dx}$ or $\rho = e^{\int \frac{N_x - M_y}{M}dy}$

if the equation is exact:

$F_1 = \int M(x,y)dx$
$F_1(x,y) + g(y)$
$F_2 = \int N(x,y)dy$
$F_2(x,y) + h(x)$
combine F_1 and F_2 to yield $f(x)$

Chapter 3

Mathematical Models

3.1 Newton's Equation of Cooling

Newton's Law of Cooling formula is used to calculate the temperature of an object as it loses heat as a function of its original body temperature, the environmental temperature, and time.

Form:

$\frac{dT}{dt} = k(A - T)$ 1

$T(t = 0) = T_0$ 2

Variables:

T = temperature of the body 1

A = temperature of the environment 2

k = thermal conductivity 3

t = time 4

3.2 Toricelli's Draining Equation

Toricelli's law relates the speed of fluid exiting from an opening in a container with the height of the fluid above the opening.

Form:

$\frac{dV}{dt} = \frac{A(y)dy}{dt} = -k\sqrt{g}$ 1

$y(t = 0) = y_0$ 2

Variables:

V = volume of fluid in the container 1

A = area of the upper surface of the fluid in the container 2

y = height of fluid over the opening 3
k = constant based on viscosity of water, other variables 4
g = constant of accelereation due to gravity 5

3.3 Population Differential Equation

This is the logistic model of the population growth model equation.

Form:

$\frac{dP}{dt} = kP(M - P) = kMP - kP^2$ 1
$P(t = 0) = P_0$ 2

Variables:

P = population 1
M = carrying capacity 2
k = rate of growth 3
t = time 4

3.4 Rocket Differential Equation

This model is used to find the velocity of a rocket fired in into the atmosphere, taking into account air resistance. This equation can be modified to suit any scenario that involves applying external forces to a moving mass.

Form:

$m\frac{dv}{dt} = -mg - kv$ 1
$v(t = 0) = v_0$ 2

Variables:

v = velocity of the rocket 1
m = mass of the rocket 2
g = acceleration due to gravity 3
k = drag/air resistant coefficient 4
t = time 5

3.5 Finance Differential Equation

This equation is used to model interest.

Form:

$\frac{dz}{dt} = zr - w$
$z(t = 0) = z_0$

Variables:

z = initial investment
r = interest rate
w = payments
t = time

3.6 Swimmer's Problem

This equation is used to model a swimmer trying to cross a river with a current.

Form:

$v_w(x) = \frac{dy}{dx} = \frac{v_c}{v_s}(1 - (\frac{x}{a})^2)$

Variables:

v_c = velocity of the current (perpendicular to swimmer)
v_s = velocity of the swimmer (perpendicular to current)
y = distance from the shortest path across from the start position(parallel to current)
x = position along shortest path between shores (perpendicular to current)
a = half the distance of the distance between shores

3.7 Plane Landing Equation

This equation is used model the landing of a plane through perpendicular winds, assuming the plane continually adjusts its angle to face the destination

Form:

$y = \frac{x}{2}[(\frac{x}{a})^{-\alpha} - (\frac{x}{a})^{\alpha}]$
$\frac{dy}{dx} = \frac{y}{x} - \frac{W}{v_0}\sqrt{1 + \frac{y^2}{x^2}}$
$\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = (\frac{x}{a})^{-\alpha}$
$\alpha = \frac{W}{v_0}$

Variables:

a = distance at which descent begins
w = wind (perpendicular to plane's route)

v_0 = plane's velocity 3

Chapter 4

Higher Order Differential Equations

4.1 General Overview of Higher Order Differentials

Higher order differential equations of the second degree follow the form:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

[1]

$f(x)$	$a_i(x)$	constant	variable
0		characteristic equation method	cauchy-euler
$\neq 0$		method of undetermined coefficients or variation of parameters	variation of parameters

4.2 Homogeneous Linear Differential Equations

This is the characteristic equation method, where one side consisting of all dependent variable terms and their derivatives is converted to a simple polynomial equation in terms of λ

Form:

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y^{(1)} + a_0 y = 0$
make substitution $y = e^{\lambda x}$ -¿ this is the only way a derivative will not go to zero due to falling order or produce a coefficient with x
$a_n \lambda^n e^{\lambda x} + a_{n-1} \lambda^{n-1} e^{\lambda x} + ... + a_1 \lambda e^{\lambda x} + a_0 + e^{\lambda x} = 0$
$e^{\lambda x}(a_n \lambda^n + a_{n-1} \lambda^{n-1} + ... + a_1 \lambda + a_0) = 0$
$e^{\lambda x} \neq 0$
$a_n \lambda^n + a_{n-1} \lambda^{n-1} + ... + a_1 \lambda + a_0 = 0$
solve for λ and substitute to solve for y
$y_{GS} = c_0 y_0 + c_1 y_1 + ... + c_{n-1} y_{n-1} + c_n y_n$

4.2.1 Addressing Duplicates

When solving the characteristic equation, there is a possibility that there can be duplicate values for λ and y. In the case that there is more than one solution of $y_i = e^{cx}$, multiply duplicate solutions by the independent variable, in this case x, to produce a solution that is linearly independent from the others. The reasoning behind this is that $\frac{d(xe^{cx})}{dx} = e^{cx} + xe^c x$, so the extra terms get absorbed into lower derivative values.

4.3 Euler-Cauchy Linear Differential Equations

Form:

$x^n y^{(n)} + x^{n-1} y^{(n-1)} + ... + xy^(1) + y = 0$
let $y = x^\lambda$ -¿ when multiplied with the coefficients, will produce terms all of the same order
$x^n (x^\lambda)^{(n)} + x^{n-1}(x^\lambda)^{(n-1)} + ... + (x^\lambda) y^(1) + (x^\lambda) = 0$
$x^n[(\lambda)(\lambda - 1)...(\lambda - n)]x^{(\lambda - n)} + ... + x\lambda x^{\lambda - 1} + x^\lambda = 0$
$x^n[(\lambda)(\lambda - 1)...(\lambda - n) + ... + \lambda + 1] = 0$
$x^\lambda \neq 0$
$(\lambda)(\lambda - 1)...(\lambda - n) + ... + \lambda + 1 = 0$
solve for λ and substitute to solve for y

4.3.1 Addressing Duplicates

In the case of duplicate solutions for $y_i = x^\lambda$, multiply solutions by $ln(x)$ until all solutions are of the form $y_i = ln(x)^c x^\lambda$ and linearly independent.

4.4 Inhomogeneous Differential Equations

4.4.1 Finding a Particular Solution

Theorem: The general solution to an inhomogeneous differential equation is the sum of the characteristic equation of the dependant terms and a particular solution to the equation.

$y_{GS}(x) = y_c(x) + y_p(x)$

For any extra terms not included in the characteristic equation, one can find a trial solution by setting a y value to all extra terms with variable coefficients. Polynomial terms must be reduced as well.

4.4.2 Reduction of Order

Given an higher order linear differential equation, $a_2xy'' + a_1xy' + a_0xy = 0$ and a particular solution $y_1(x)$, we can get a second solution.

$y = u(x)y_1$
$y' = u(x)y_1' + u'(x)y_1$
$y'' = u(x)y_1'' + 2u'(x)y_1' + u''(x)y_1$
$ln(u) = -\int a_1 + 2\frac{y_1'}{y+1}dx + c_0$
$u(x) = c_1 \int e^{-\int a_1 + 2\frac{y_1'}{y+1}dx} + c_2$
$y = u(x)y_1(x) = y_1[c_1 \int e^{-\int a_1 + 2\frac{y_1'}{y+1}dx} + c_2]$

4.4.3 Variational Principle Method

Given an equation $y'' + P(x)y' + Q(x)y = f(x)$

The solution of a higher order differential equation requires two solutions, $y_1(x)$, $y_2(x)$

Propose a particular solution:

$y_p(x) = u_1y_1 + u_2y_2$ $y_p'(x) = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$ assume
$u_1'y_1 + u_2'y_2 = 0$ $y_p'(x) = u_1y_1' + u_2y_2'$ $y_p''(x) = u_1'y_1' + u_1y_1'' + u_2'y_2' +$

u_2y_2'' plug into the original equation $u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' + P(x)(u_1y_1' + u_2y_2') + Q(x)(u_1y_1 + u_2y_2) = f(x)$ $(u_1'y_1' + u_2'y_2') + u_1(y_1'' + P(x)y_1' + Q(x)y_1) + u_2(y_2'' + P(x)y_2' + Q(x)y_2) = f(x)$ $(u_1'y_1' + u_2'y_2') + u_1x(0) + u_2x(0) = f(x)$ $u_1'y_1' + u_2'y_2' = f(x)$ $u_1'y_1 + u_2'y_2 = 0$ was an earlier assumption $u_1' = \frac{y_2f(x)}{y_1y_2' - y_2y_1'}$ - solve the system for u_1' $u_2' = \frac{y_1f(x)}{y_1y_2' - y_2y_1'}$ - solve the system for u_2' let $W(y_1, y_2) = y_1y_2' - y_2y_1' \neq 0$ $u_1 = -\int \frac{y_2f(x)}{W(y_1,y_2)}dx$, $u_2 = \int \frac{y_1f(x)}{W(y_1,y_2)}dx$ $y_p(x) = y_1u_1 + y_2u_2 = y_1[-\int \frac{y_2f(x)}{W(y_1,y_2)}dx] + y_2[\int \frac{y_1f(x)}{W(y_1,y_2)}dx]$ Wronskian: $W(y_1, y_2) = y_1y_2' - y_2y_1' \neq 0$

Chapter 5

Systems of Differential Equations

5.1 General Overview

Systems of differential equations are systems of equations involving differentials. Listed here are three of many possible ways of solving a system.

- Substitution Method
- Operator Method
- Eigen Method

5.2 Substitution Method

Given:
$x' = ax + by + c \; y' = dx + ey + f$

- Separate the variables in one of the equations.
- Take the derivative of both sides.
- Substitute into the other equation.
- Solve the equation as a second order linear differential equation.

5.3 Operator Method

Given:
$x' = ax + by + c \; y' = dx + ey + f$

- Let $D = \frac{d}{dt}$ and substitute into both equations
- Bring all terms to one side
- Factor out the dependent variables
- Solve like a regular systems of equations

5.4 Eigen Method

Just like any systems of equations, a systems of differential equations can be represented in matrix form.

$$\begin{pmatrix} x'_0 \\ x'_1 \\ . \\ . \\ . \\ x'_n \end{pmatrix} = \begin{pmatrix} a_0 + a_1 + ... + a_n \\ b_0 + a_1 + ... + a_n \\ ... \\ ... \\ ... \\ ... \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

To find the characteristic equation of the system, find the kernel of the primary coefficient matrix.

$X_c(t) : det(A - \lambda) = 0 \begin{vmatrix} A_{0,0} - \lambda & A_{0,1} \\ A_{1,0} & A_{1,1} - \lambda \end{vmatrix} = 0$ solve for λ for each λ, plug it back into the matrix to find the eigenvectors $\begin{pmatrix} A_{0,0} - \lambda_0 & A_{0,1} \\ A_{1,0} & A_{1,1} - \lambda_0 \end{pmatrix} \vec{v_0} = 0$

$X_p(t)$: solve for the remaining terms as you would in the method of undetermined coefficients

Chapter 6

Laplace Transforms

Definition:
$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$

6.1 Common Laplace Transform Forms

Properties and Theorems of Laplace Transformations

Property/Theorem	t-space	S-space
Linearity	$af(t) + bf(t)$	$aF(s) + bF(s)$
Periodic Function	$f(t) = f(t+T)$	$\frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)dt$
Unit Step Function	$u(t-\alpha)$	$\frac{1}{s}e^{-\alpha s}$
Exponential Function	$e^{\alpha t}$	$\frac{1}{s-\alpha}$
Trig - cosine	$cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
Trig - sine	$sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
Polynomial Case 0	$f(t) = 1$	$\frac{1}{s}$
Polynomial Case 1	$f(t) = t$	$\frac{1}{s^2}$
Polynomial Case n	$f(t) = t^n$	$\frac{n!}{s^{n+1}}$
Deg. 1 Variable Coefficient	$tf(t)$	$-F'(s)$
Deg. n Variable Coefficient	$t^n f(t)$	$(-1)^n F^n(s)$
Derivative	$f'(t)$	$sF(s) - f(0)$
Derivative	$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$
Convolution	$f(t) \star g(t)$	$F(s)G(s)$

6.1.1 Convolution Theorem

$\star$ refers to the convolution operator, a way of combining two functions into a third. The formal definition is as follows:

$f(t) \star g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$

Convolution Theorem:

$\mathcal{L}[f(t) \star g(t)] = F(s)G(s)$ and $\mathcal{L}^{-1}[F(s)G(s)] = f(t) \star g(t)$

Proof of the Convolution Theorem:

let $u(t) = 1$ if $t \geq 0$ and $u(t) = 0$ otherwise
$[f(t) \star g(t)]u(t)$
$(\int_0^{\text{inf}} f(\tau)g(t-\tau)d\tau) * u(t-\tau)$
$\int_0^t f(\tau)g(t-\tau)u(t-\tau)d\tau + \int_0^t f(\tau)g(t-\tau)u(t-\tau)d\tau$
$\int_0^t f(\tau)g(t-\tau)d\tau$
$f(t) \star g(t) = \int_0^{\text{inf}} f(\tau)g(t-\tau)u(t-\tau)d\tau$
$\mathcal{L}[f(t) \star g(t) = \int_0^{\text{inf}} f(\tau)g(t-\tau)u(t-\tau)d\tau]$
$\int_0^{\text{inf}} \int_0^{\text{inf}} f(\tau)g(t-\tau)u(t-\tau)e^{-st}d\tau dt$
$\int_0^{\text{inf}} f(\tau)d\tau \int_0^{\text{inf}} g(t-\tau)u(t-\tau)e^{-st}dt$
let $t = t_1 + \tau$, $dt = dt_1$
$\int_0^{\text{inf}} f(\tau)d\tau \int_0^{\text{inf}} g(t_1)u(t_1)e^{-s(t_1+\tau)}dt_1$
$\int_0^{\text{inf}} f(\tau)e^{-s\tau}d\tau \int_0^{\text{inf}} g(t_1)u(t_1)e^{-st_1}dt_1$
let $F(s) = \int_0^{\text{inf}} f(\tau)e^{-s\tau}d\tau$, $G(s) = \int_0^{\text{inf}} g(t_1)u(t_1)e^{-st_1}dt_1$
$\mathcal{L}[f(t) \star g(t)] = F(s)G(s)$

Properties of Convolutions

- commutative
- associative
- distributive

6.2 Complex Inverses

More complex inverses can be solved three ways:

- Separate an inverse using a convolution, solve the components separately, and plug the values into the convolution

- Partial fraction decomposition into separate terms and solve individually