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Homework 10-b

Problem B

Let a_n be the number of sequences of blocks that can be placed consecutively in a line of length n if each block has length either 1 or 2. (Empty spots are not allowed.)

(a). Write a recurrence for a_n. (You do not have to solve the recurrence.)

$a_n = a_{n-1} + a_{n-2}$

Reasoning: Since blocks are of size 1 or 2, a sequence of size n can be formed by:

- adding a size 1 block to a sequence of size $n - 1$
- adding a size 2 block to a sequence of size $n - 2$
- Note: adding 2 size 1 blocks would fall under the case of item 1

(b). Write a complete set of initial conditions.

$a_1 = 1$ (1 size 1 block)
$a_2 = 2$ (1 size 2 block or 2 size 1 blocks)

(c). Calculate a_6 using your recursion from (a) and initial conditions from (b).

$a_1 = 1$
$a_2 = 2$
$a_3 = a_2 + a_1 = 1 + 2 = 3$
$a_4 = a_3 + a_2 = 3 + 2 = 5$
$a_5 = a_4 + a_3 = 5 + 3 = 8$
$a_6 = a_5 + a_4 = 8 + 5 = 13$

(d). Now assume that no 2 blocks of length 1 can be placed consecutively. Write a recurrence for a_n. (You do not have to solve the recurrence.)

$a_n = a_{n-2} + a_{n-3}$

Reasoning:

A sequence can be made to length n by adding a block of size of 2 to a sequence of size $n-2$ or adding a block of size 1 to a sequence of size $n-1$ if it ends in a size 2 block. The latter case is only possible by adding a size 2 block to a sequence of length $n-3$.

(e). Write a complete set of initial conditions for your recurrence of part (d).

$a_1 = 1$ (1 size 1 block)
$a_2 = 1$ (1 size 2 block)
$a_3 = 2$ (size 1 block + size 2 block, size 2 block + size 1 block)

<u>Problem C</u>

Solve the following recurrence relation: $a_n = 4a_{\frac{n}{2}} + 3n$, $a_1 = 1$
(You may assume that $n = 2^m$, for some $m = 0, 1, 2, ...$)

$a_n = 4a_{\frac{n}{2}} + 3n$, $a_1 = 1$
$c = 4$, $c \neq k$, $f(n) = 3n$
when $c \neq k, f(n) = dn, a_n = An^{log_k c} + (\frac{kd}{k-c})n$
$a_n = n^{log_2 4} + \frac{2*3}{2-4}n$
$a_n = n^2 - 3n$

<u>Problem D</u>

Consider the recurrence relation: $a_n = 2a_{n-1} + 1, a_1 = 1$
Prove that the solution to this recurrence is $a_n = 2^n - 1$

proof by induction
base case: $a_1 = 1$
induction hypothesis: we will prove for all n that $a_n = 2a_{n-1} + 1 = a_n = 2^n - 1$
induction step: we will assume the relation is true for $n-1$
$a_{n-1} = 2a_{n-2} + 1 = 2^{n-1} - 1$
$2a_{n-1} = 4a_{n-2} + 2 = 2^n - 2$ - multiply by 2
$2a_{n-1} = 2(a_{n-2} + 1) = 2^n - 2$ - factor out
$2a_{n-1} = 2a_{n-1} = 2^n - 2$ - substitute
$2a_{n-1} = 2a_{n-1} + 1 = 2^n - 1$ - add 1
we have proven the base case and the induction step true, therefore by the law of induction, we have proven the proposition true