Mathematics Exam

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Exercise 7

Evaluate the integral

$$I = \int_0^\infty \frac{x \ln(x)}{1 + x^3}\, dx. \tag{1}$$

Integral I cannot be solved through algebraic manipulations nor by integration by parts, indeed the integrand lacks an elementary antiderivative. An alternative approach involves leveraging special functions within the complex domain.

The Beta function

The Beta function can be expressed in several ways. Among the different representations the following is the one most similarly structured to (1)

$$B(x, y) = \int_0^\infty \frac{s^{x-1}}{(1 + s)^{x+y}}\, dx. \tag{2}$$

Relationship with the Beta function

Now, consider the function

$$J(a) = \int_0^\infty \frac{x^a}{1 + x^3}\, dx. \tag{3}$$

Notice that $J'(1) = I$, indeed

$$\begin{aligned} J'(a) &= \frac{\partial}{\partial a} \int_0^\infty \frac{x^a}{1 + x^3}\, dx \\ &= \int_0^\infty \frac{\partial}{\partial a} \frac{x^a}{1 + x^3}\, dx \\ &= \int_0^\infty \frac{x^a \ln(x)}{1 + x^3}\, dx. \end{aligned}$$

Where the derivative of an integral is the integral of the derivative under specific conditions, following a special case of the Leibniz integral rule [1]. The conditions to be met are the following:

- The integrand $f(x, a)$ and its partial derivative $\frac{\partial}{\partial a} f(x, a)$ are continuous functions of two variables when x is in the range of integration and a is in some interval around a_0.

  The integrand $f(x, a)$ and its partial derivative $\frac{\partial}{\partial a} f(x, a)$ are continuous for $x \neq -1$ and $x \neq 0$. Thus, this condition is met since they are both continuous within the ranges $x \in (0, \infty)$ and $a \in [1 - \epsilon, 1 + \epsilon]$ for small $\epsilon > 0$.

- For a in some interval around a_0 there are upper bounds $|f(x, a)| \leq A(x)$ and $|\frac{\partial}{\partial a} f(x, a)| \leq B(x)$, both bounds being independentof a, such that $\int_b^c A(x)\, dx$ and $\int_b^c B(x)\, dx$ exist.

  The second condition is met for $-1 < a < 2$.

The derivative of $J(a)$ at $a = 1$ is equal to the original integral I (1). This is useful because, after making some simple adjustments, function (3) can be expressed in terms of function (2).

At first consider the change of variables $x^3 = y$. Therefore

$$\partial x = \frac{y^{\frac{-2}{3}}}{3}\partial y.$$

The bounds remain unchanged and the integral becomes

$$J(a) = \int_0^\infty \frac{y^{\frac{a}{3}}}{1+y}\frac{y^{\frac{-2}{3}}}{3}\,dy$$

$$= \int_0^\infty \frac{y^{\frac{a-2}{3}}}{3(1+y)}\,dy$$

$$= \frac{1}{3}\int_0^\infty \frac{y^{\frac{a+1}{3}-1}}{(1+y)}\,dy.$$

The last expression is related to the Beta function (2) with arguments $x = \frac{(a+1)}{3}$ and $y = 1 - \frac{a+1}{3}$

$$J(a) = \frac{1}{3}\int_0^\infty \frac{y^{\frac{a+1}{3}-1}}{(1+y)}\,dy = \frac{1}{3}B(\frac{a+1}{3}, 1 - \frac{a+1}{3}). \tag{4}$$

Beta reflection formula

The last expression in (4) is a specific case of the Beta function and can be easily evaluated using the Euler reflection formula[2]:

$$B(x, 1-x) = \frac{\Gamma(x)\Gamma(1-x)}{\Gamma(1)}$$

$$= \Gamma(x)\Gamma(1-x)$$

$$= \frac{\pi}{sin(\pi x)}.$$

Now,

$$J(a) = \frac{1}{3}\frac{\pi}{sin(\frac{\pi}{3}(a+1))}. \tag{5}$$

Therefore,

$$J'(a) = \frac{\partial}{\partial a}\left(\frac{1}{3}\frac{\pi}{sin\big(\frac{\pi}{3}(a+1)\big)}\right)$$

$$= -\frac{\pi^2 cos\big(\frac{\pi}{3}(a+1)\big)}{9sin^2\big(\frac{\pi}{3}(a+1)\big)}.$$

Evaluating $J'(a)$ in $a = 1$ yields the result

$$J'(1) = -\frac{\pi^2 cos(\frac{2\pi}{3})}{9sin^2(\frac{2\pi}{3})} = -\frac{\pi^2(-\frac{1}{2})}{9\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2\pi^2}{27}, \tag{6}$$

which is equal to the original integral I.

References

[1] Gottfried Wilhelm Leibniz. A new method for maxima and minima, as well as tangents, which is not obstructed by fractional or irrational quantities, and a singular kind of calculus for them. *Acta Eruditorum*, 1684.

[2] Leonhard Euler. On transcendental progressions, that is, those whose general terms cannot be given algebraically, commentarii academiae scientiarum petropolitanae. *Commentarii academiae scientiarum Petropolitanae*, 1738.