# Mathematics Exam

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## Exercise 7

Evaluate the integral

$$I = \int_0^\infty \frac{x\ln(x)}{1+x^3}\,dx. \tag{1}$$

Integral $I$ cannot be solved through algebraic manipulations nor by integration by parts, indeed the integrand lacks an elementary antiderivative. An alternative approach involves leveraging special functions within the complex domain.

### The Beta function

The Beta function can be expressed in several ways. Among the different representations the following is the one most similarly structured to (1)

$$B(x,y) = \int_0^\infty \frac{s^{x-1}}{(1+s)^{x+y}}\,dx. \tag{2}$$

### Relationship with the Beta function

Now, consider the function

$$J(a) = \int_0^\infty \frac{x^a}{1+x^3}\,dx. \tag{3}$$

Notice that $J'(1) = I$, indeed

$$\begin{aligned}
J'(a) &= \frac{\partial}{\partial a}\int_0^\infty \frac{x^a}{1+x^3}\,dx \\
&= \int_0^\infty \frac{\partial}{\partial a}\frac{x^a}{1+x^3}\,dx \\
&= \int_0^\infty \frac{x^a\ln(x)}{1+x^3}\,dx.
\end{aligned}$$

Where the second passage is possible, under specific conditions, following a special case of the Leibniz integral rule [1]. The conditions to be met are the following.

- The integral should converge: For $x \to 0$ the integral behaves as $x^a$ which converges if and only if $a > -1$. While for $x \to \infty$ the leading term is $x^{a-3}$, which converges if $a < 2$. Thus, the function is integrable for $-1 < a < 2$.
- The derivative should be locally integrable: Under the previous conditions the function $\frac{x^a\ln(x)}{1+x^3}\,dx.$ is locally integrable.

The derivative of $J(a)$ at $a = 1$ is equal to the original integral $I$ (1). This is useful because, after making some simple adjustments, function (3) can be expressed in terms of function (2).

At first consider the change of variables $x^3 = y$. Therefore

$$\partial x = \frac{y^{\frac{-2}{3}}}{3}\partial y.$$

The bounds remain unchanged and the integral becomes

$$J(a) = \int_0^\infty \frac{y^{\frac{a}{3}}}{1+y}\frac{y^{\frac{-2}{3}}}{3}\,dy$$

$$= \int_0^\infty \frac{y^{\frac{a-2}{3}}}{3(1+y)}\,dy$$

$$= \frac{1}{3}\int_0^\infty \frac{y^{\frac{a+1}{3}-1}}{(1+y)}\,dy.$$

The last expression is related to the Beta function (2) with arguments $x = \frac{(a+1)}{3}$ and $y = 1 - \frac{a+1}{3}$

$$J(a) = \frac{1}{3}\int_0^\infty \frac{y^{\frac{a+1}{3}-1}}{(1+y)}\,dy = \frac{1}{3}B(\frac{a+1}{3}, 1 - \frac{a+1}{3}). \tag{4}$$

## Beta reflection formula

The last expression in (4) is a specific case of the Beta function and can be easily evaluated using the Euler reflection formula[2]:

$$B(x, 1-x) = \frac{\Gamma(x)\Gamma(1-x)}{\Gamma(1)}$$

$$= \Gamma(x)\Gamma(1-x)$$

$$= \frac{\pi}{sin(\pi x)}.$$

Now,

$$J(a) = \frac{1}{3}\frac{\pi}{sin(\frac{\pi}{3}(a+1))}. \tag{5}$$

Therefore,

$$J'(a) = \frac{\partial}{\partial a}\left(\frac{1}{3}\frac{\pi}{sin\big(\frac{\pi}{3}(a+1)\big)}\right)$$

$$= -\frac{\pi^2 cos\big(\frac{\pi}{3}(a+1)\big)}{9sin^2\big(\frac{\pi}{3}(a+1)\big)}.$$

Evaluating $J'(a)$ in $a = 1$ yields the result

$$J'(1) = -\frac{\pi^2 cos(\frac{2\pi}{3})}{9sin^2(\frac{2\pi}{3})} = -\frac{\pi^2(-\frac{1}{2})}{9\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2\pi^2}{27}, \tag{6}$$

which is equal to the original integral $I$.

## References


[1] Gottfried Wilhelm Leibniz. A new method for maxima and minima, as well as tangents, which is not obstructed by fractional or irrational quantities, and a singular kind of calculus for them. *Acta Eruditorum*, 1684.

[2] Leonhard Euler. On transcendental progressions, that is, those whose general terms cannot be given algebraically, commentarii academiae scientiarum petropolitanae. *Commentarii academiae scientiarum Petropolitanae*, 1738.