Calculus problem from Farrington, "The Measurement and Interpretation of Age-Specific Vaccine Efficacy" (1992)

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Let $R_0 \in [0, 1] \subset \mathbb{R}$ and $\rho, \lambda \in \mathbb{R}_{\geq 0}$. Consider the following expression:

$$F(a) = \int_0^a g(u)h(a-u)du, \tag{1}$$

where

$$G(a) = 1 - (1 - R_0)\, e^{-\rho a} \tag{2}$$

$$G'(a) = g(a) = \rho(1 - R_0)\, e^{-\rho a} \tag{3}$$

and

$$h(b) = e^{-\lambda b}. \tag{4}$$

Question What is $F'(a) = \frac{dF}{da}$?

Proposed solutions

1. Using the fundamental theorem of calculus,

$$F'(a) = g(a)h(a-a) = g(a)h(0)$$

2. $F(a)$, on the other hand, is a convolution integral; therefore,

$$F(a) = \int_0^a g(u)h(a-u)du = \int_0^a g(a-u)h(u)du.$$

   Using this in combination with the fundamental theorem of calculus gives

$$F'(a) = g(a-a)h(a) = g(0)h(a).$$

   Usually, however, $g(0)h(a) \neq g(a)h(0)$.

3. First evaluating $F(a)$ and then taking the derivative with respect to a gives yet another result.