Ch3 Revisit the integer quantum Hall effect (a)

In this section we revisit the integer quantum Hall effect. This phenomenon can be understood without taking into account the interactions between electrons. This means that we will assume that the quantum states for a single particle in a magnetic field will remain the quantum states when there are many particles present. The only way that one particle knows about the presence of others is through the Pauli exclusion principle: they occupy space.

3.1 Conductivity in filled Landau levels

The experimental data for the Hall resistivity shows a number of plateaux labelled by an integer υ. Meanwhile, the energy spectrum forms Landau levels, also labelled by an integer. Each Landau level can accommodate a large, but finite number of electrons.

It's tempting to think that these integers are the same: $\rho_{xy} = 2\pi\hbar/e^2\nu$ with $\nu \in \mathbf{Z}$ and when precisely ν Landau levels are filled. But, from our classical calculation in the Drude model, we have the expectation that the Hall conductivity $\rho_{xy} = \frac{B}{ne}$. Comparing these two expressions, we see that the density needed to get the resistivity of the ν^{th} plateau is $n = \frac{B}{\Phi_0}\nu$ with $\Phi_0 = 2\pi\hbar/e$. This is indeed the density of electrons required to fill ν Landau levels.

Further, when ν Landau levels are filled, there is a gap in the energy spectrum: to occupy the next state costs an energy $\hbar\omega_B$ where $\omega_B = eB/m$ is the cyclotron frequency. As long as we're at temperature $k_BT \ll \hbar\omega_B$, these states will remain empty. When we turn on a small electric field, there's nowhere for the electrons to move: they're stuck in place like in an insulator. This means that the scattering time $\tau \to \infty$ and we have $\rho_{xx} = m/(ne^2\tau) = 0$ as expected.

Conductivity in semiclassical picture

The above calculation involved a curious mixture of quantum mechanics and the classical Drude mode. Here we'll describe how *to compute the conductivity for a single free particle*.

We know that the velocity of the particle is given by $m\dot{\boldsymbol{x}} = \boldsymbol{p} + e\boldsymbol{A}$ where p_i is the canonical momentum. The current is $\boldsymbol{I} = -e\dot{\boldsymbol{x}}$, which means that, in the quantum mechanical picture, the total current is given by

$$\mathbf{I} = -\frac{e}{m}\sum_{\text{filled states}} \langle\psi|-i\hbar\nabla + e\mathbf{A}|\psi\rangle$$

It's best to do these kind of calculations in Landau gauge, $\mathbf{A} = xB\hat{y}$. We introduce an electric field E in the x-direction so the Hamiltonian with application of electrical field is given by $H = \frac{1}{2m}\left(p_x^2 + \left(p_y + eBx\right)^2\right) - eEx$ and the shifted states by $\psi(x,y) = \psi_{n,k}(x - mE/eB^2, y)$. With the ν Landau levels filled, the current in the x-direction is

$$I_x = -\frac{e}{m}\sum_{n=1}^{v}\sum_{k}\left\langle\psi_{n,k}\right|-i\hbar\frac{\partial}{\partial x}\left|\psi_{n,k}\right\rangle = 0$$

This vanishes because it's computing the momentum expectation value of harmonic oscillator eigenstates. Meanwhile, the current in the *y*-direction is

$$I_y = -\frac{e}{m}\sum_{n=1}^{v}\sum_{k}\left\langle\psi_{n,k}\right|-i\hbar\frac{\partial}{\partial y}+exB\left|\psi_{n,k}\right\rangle = -\frac{e}{m}\sum_{n=1}^{v}\sum_{k}\left\langle\psi_{n,k}\right|\hbar k+eBx\left|\psi_{n,k}\right\rangle$$

The second term above is computing the position expectation value $\langle x\rangle$ of the eigenstates. But we know from $\psi_{n,k}(x,y)\sim e^{iky}H_n(x+kl_B^2)e^{-(x+kl_B^2)/2l_B^2}$ and $\psi(x,y)=\psi_{n,k}(x-mE/eB^2,y)$ that these harmonic oscillator states are shifted from the origin, so that $\left\langle\psi_{n,k}\right|x\left|\psi_{n,k}\right\rangle = -\hbar k/eB + mE/eB^2$, then

$$\left\langle\psi_{n,k}\right|eBx\left|\psi_{n,k}\right\rangle = -\hbar k + mE/B$$

$$\left\langle\psi_{n,k}\right|\hbar k+eBx\left|\psi_{n,k}\right\rangle = mE/B$$

The first of these terms cancels the explicit $\hbar k$ term in the expression for I_y. We're left with

$$I_y = -ev\sum_{k}\frac{E}{B}$$

The sum over k just gives the number of electrons which we computed in the degeneracy to be $N = AB/\Phi_0$. We divide through by the area to get the current density **J** instead of the current **I**: $j_y = -\left(\frac{evE}{B}\frac{AB}{\Phi_0}\right)/A = -\frac{evE}{\Phi_0}$. The result of this is that

$$\mathbf{E} = \begin{pmatrix}E\\0\end{pmatrix} \;\rightarrow\; \mathbf{J} = \begin{pmatrix}0\\-evE/\Phi_0\end{pmatrix}$$

Since $\mathbf{J}=\sigma\mathbf{E}$, we have

$$\begin{pmatrix}0\\-evE/\Phi_0\end{pmatrix} = \begin{pmatrix}\sigma_{xx} & \sigma_{xy}\\-\sigma_{xy} & \sigma_{xx}\end{pmatrix}\begin{pmatrix}E\\0\end{pmatrix} = \begin{pmatrix}\sigma_{xx}E\\-\sigma_{xy}E\end{pmatrix}$$

Therefore

$$\sigma_{xx}=0,\;\; \sigma_{xy}=\frac{ev}{\Phi_0} \quad\rightarrow\quad \rho_{xx}=0,\;\; \rho_{xy}=-\frac{\Phi_0}{ev}=-\frac{2\pi\hbar}{e^2v}$$

This is exactly the conductivity seen on the quantum Hall plateaux.

3.2 Edge modes

Chiral edge states

Electrons near the edge doesn't have enough space to complete a circle. It will hit the boundary before the circle can be completed. And then, the electron will be reflected by the boundary and start another circle and hit the boundary again. If we zoom out, we find that the electrons near the edge can move along the edge in one direction (the direction is determined by the B field). This edge states are known as the chiral edge states and they are responsive for the quantized Hall conductivity.

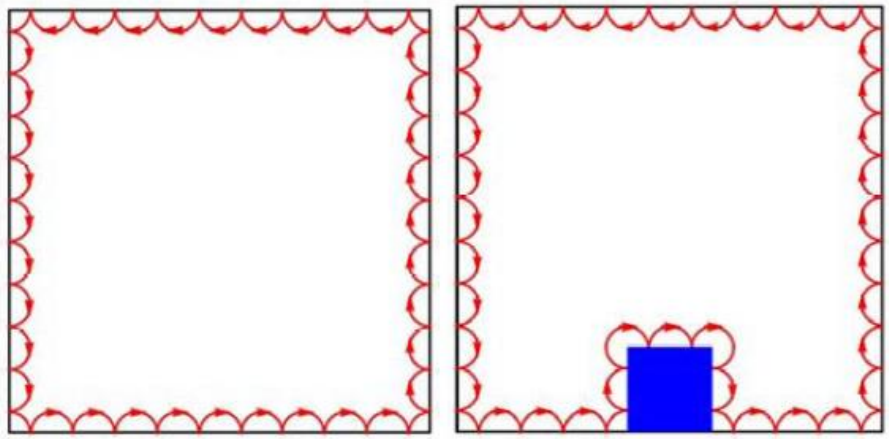


Fig. 3. the quantum Hall edge states. The figure on the right shows that impurity doesn't Interrupt the chiral edge state.

The dispersion relation of the chiral edge states

We know that the group velocity is defined as

$$v = \frac{d\omega}{dk} = \frac{\hbar d\omega}{\hbar dk} = \frac{d\epsilon}{\hbar dk}$$

For an edge state, the electron can only moving in one direction. Here, we consider a 1D chiral edge state moving to the right. Because electrons can only moving to the right, the group velocity must be positive, $v > 0$. Therefore, $\frac{d\epsilon}{dk} > 0$. If we only focus on the state with energy close to the Fermi energy, $\epsilon \sim E_F$, we can expand the dispersion relation around the Fermi wave-vector k_F.

$$\epsilon(k) = E_F + v_F\hbar(k - k_F) + \cdots$$

Here, $E_F = \epsilon(k_F)$ and $v_F = \frac{d\epsilon}{\hbar dk}\Big|_{k=k_F}$ is the Fermi velocity. For right moving edge states, $v_F > 0$ and for left moving ones, $v_F < 0$. Similarly, if we consider the chiral edge states moving to the left, we find that $\frac{d\epsilon}{dk} < 0$and thus ϵ is a monotonically decreasing function of k.

Hall conductivity

We consider a Hall bar here. The circular motion of the electrons tells us that the top and bottom edge must have the opposite Fermi velocity (one moving to the left and the other moving to the

right). Here, we assume that the top edge has negative Fermi velocity (left moving) and the bottom edge has positive Fermi velocity (right moving).

Case I: $E_y = 0$. Here, the top and bottom edges have the same electric potential. So we expect the number of electrons near the top edge equals to the number of electrons near the bottom edge. This means that we have the same number of electrons moving to the right and left. Thus, the total current along the x direction is 0.

Case II: $E_y \neq 0$. Here, the top and bottom edges have different electric potentials. Let's consider here the situation where the top edge has electric potential +V/2, while the bottom edge has -V/2. Because we have a positive potential at the top edge, it will attract more electrons to the top edge. For similar reasons, the bottom edge will have less number of electrons. Therefore, the number of electrons moving to the left will be higher than the number moving to the right. This means that we will have a current flowing to the right.

To compute this current, we need to count how many electrons got attracted to the top edge (and expel by the bottom edge). Let's start from the bottom edge, the energy of an electron is known as its kinetic energy + the potential energy due to the -V/2 electric potential

$$\epsilon = E_F + v_F\hbar(k - k_F) + eV/2$$

Here, E_F is the Fermi energy, v_F is the Fermi velocity, k_F is the Fermi wave vector before we apply the potential energy. After we applied the potential energy, the Fermi wave-vector is no longer v_F, because the dispersion relation is not different. We should find the new Fermi wave using the condition $\epsilon = E_F$

$$\epsilon = E_F + v_F\hbar(k - k_F) + eV/2 = E_F$$

$$\hbar v_F(k - k_F) = -eV/2$$

$$k = k_F - \frac{eV}{2\hbar v_F}$$

Before we apply the potential, electrons fills up all quantum states with $k < k_F$. After we apply the potential, electrons fill up all quantum states with $k < k_F - \frac{eV}{2\hbar v_F}$. If we compare these two cases, it is easy to notice that if we apply the potential, the quantum states with $k_F - \frac{eV}{2\hbar v_F} < k < k_F$ turns from occupied to unoccupied. In other words, some electrons are expelled from the bottom edge. This is exactly what we want to compute. The number of electrons expelled from the bottom edge by the electric potential is

$$N_b = \int_{k_F - eV/2}^{k_F} \frac{dk}{2\pi/L_x} = \frac{L_x}{2\pi}\left[k_F - \left(k_F - \frac{eV}{2\hbar v_F}\right)\right] = \frac{L_x}{2\pi}eV/2 = \frac{L_x}{2\pi}\frac{eV}{2\hbar v_F}$$

Because current $j = -env_F$, where n is the density of electrons, the change of current due to the change of electron number is $\delta j = -e\delta n v_F$. Here, the number of right moving electrons is reduced by N_b, so that the change of density $\delta n = -N_b/L_x/L_y$. So we have

$$\delta j_x^b = -e\left(\frac{-N_b}{L_x L_y}\right) v_F = e\frac{1}{2\pi}\frac{eV}{2v_F L_y \hbar} v_F = \frac{e^2}{h}\frac{V}{2L_y} = \frac{e^2}{h}\frac{E_y}{2}$$

Here, $E_y = V/L_y$ is the strength of the electric field in the y direction (perpendicular to the current).

Similarly, we can do the same calculation for the top edge and we will get that

$$\delta j_x^t = \frac{e^2}{h}\frac{E_y}{2}$$

If we combine the contributions from both edges together, we find

$$j_x = \delta j_x^b + \delta j_x^t = \frac{e^2}{h} E_y$$

Thus, we have

$$\sigma_{xy} = \frac{j_x}{E_y} = \frac{e^2}{h}$$

So, for one chiral edge state, we get $\sigma_{xy} = \frac{e^2}{h}$. If we have n chiral edge states, $\sigma_{xy} = n\frac{e^2}{h}$. Notice that n here is the number of edge states, which must be an integer. This is the reason why the Hall conductivity must take integer values.

It can be shown that each filled Landau level gives us one chiral edge states. Therefore, the number of chiral edge states (or say the Hall conductivity) is just the number of filled Landau levels.

There are a couple of aspects of the story which the simple description above does not capture. One of these is the role played by disorder, why the plateaux can persist over a certain magnetic field range. The other is the special importance of modes at the edge of the system. Here we describe some basic facts about edge modes.

3.3 Robustness of the Hall State

The calculations above show that if an integer number of Landau levels are filled, then the longitudinal and Hall resistivities are those observed on the plateaux. But it doesn't explain why these plateaux exist in the first place, nor why there are sharp jumps between different plateaux.

Yet the whole point of the quantum Hall effect is that the experiments reveal that the quantized values of the resistivity persist over a range of magnetic field. How is this possible?

The role of disorder

In a perfect two-dimensional sheet of electrons, all energy levels are confined to the exact integer Landau energies. In a realistic system, where it is not so easy to manufacture a perfect system, this picture is not quite correct. There are often impurities in the surrounding

semiconductor material that jut into the electron layer. These impurities introduce local variations in the energy levels, and many can even grab electrons and keep them bound in so-called localized states. Those states don't conduct electricity!

To see how this works, let's think about what disorder will likely do to the system. Our first expectation is that it will split the degenerate eigenstates that make up a Landau level, which is to alter the nearby potential energy and thereby change the mechanics of how electrons fill up Landau levels. This follows on general grounds from quantum perturbation theory: any generic perturbation, which doesn't preserve a symmetry, will break degeneracies. Such a fluctuation could be either a decrease or an increase in the energy of the level, resulting in a broadening of the Landau levels in the new density of states. The essence of this result is that there are an extra number of available electron energy states just above and below each Landau level that do not conduct electricity. We will further ask that the strength of disorder is small relative to the splitting of the Landau levels,

$$V \ll \hbar\omega_B$$

In practice, this means that the samples which exhibit the quantum Hall effect actually have to be very clean. We need disorder, but not too much disorder! The energy spectrum in the presence of this weak disorder is the expected to change the quantized Landau levels from the familiar picture in a well-aligned Landau levels, to the more broad spectrum.

There is a second effect of disorder: it turns many of the quantum states from extended to localized. Here, an extended state is spread throughout the whole system. In contrast, a localized state is restricted to lie in some region of space. We can easily see the existence of these localized

states in a semi-classical picture which holds if the potential, in addition to obeying (3.4), varies appreciably on distance scales much greater than the magnetic length l_B,

$$|\nabla V| \ll \frac{\hbar\omega_B}{l_B}$$

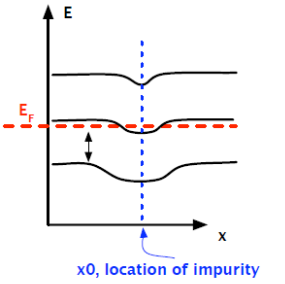
With this assumption, the cyclotron orbit of an electron takes place in a region of essentially constant potential. The center of the orbit, **X** then drifts along equipotentials. To see this, we can review the classical equations of motion:

$$x(t) = X - R\sin(\omega_c t + \phi) \quad y(t) = Y + R\cos(\omega_c t + \phi)$$

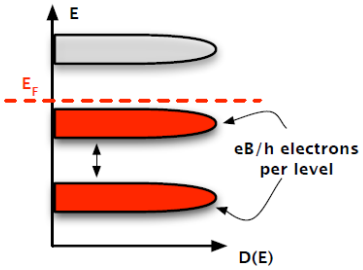
We think about the coordinate labelling the center of the orbit (X, Y) as quantum operator. This is also named as guiding center. We can rearrange to the following:

$$X = x(t) + R\sin(\omega_c t + \phi) = x - \frac{\dot{y}}{\omega_c} = x - \frac{\pi_y}{m\omega_c}$$

$$Y = y(t) - R\cos(\omega_c t + \phi) = y + \frac{\dot{x}}{\omega_c} = y + \frac{\pi_x}{m\omega_c}$$



(a) Impurities alter the local energy potential



(b) Broadened density of states

These quantum operators (X, Y) therefore describe the center of the orbit, with $\boldsymbol{\pi}$ the mechanical momentum. We notice the fact that, in the absence of a potential, $[X, H] = [Y, H] = 0$ (these are constants of motion in terms of the guiding center, therefore the operators commute with the Hamiltonian) together with the commutation relation $[X, Y] = il_B^2$.

When considering a current moving in a sample, one has to consider the particles in that sample to affect the charge carriers in the current. This is described by two kinds of potentials, confinement potential and an impurity potential. Since the current is confined to the sample, the confinement potential varies only along the y direction, ($V_{conf}(y)$) and does not effect the particles moving along the sample in the x-direction. The other potential that need to be considered is the impurity potential caused by the impurities in the sample, $V_{imp}(x, y)$, which effects the particles in both directions. If one wants to describe a particle moving in this system, one has to use the total potential in the Hamiltonian which is the sum of these potentials

$$V(\vec{r}) = V_{conf}(y) + V_{imp}(x, y)$$

When $V(\vec{r})$ is smooth on the length scale of l_B it does not generate Landau mixing, one can approximate the argument of the potential with the position of the guiding center, $V(\vec{r}) \cong V(\vec{R})$. Since $V(\vec{R})$ and $\vec{R}$ do not commute, one may consider the Heisenberg equation of motion

$$\frac{dA}{dt} = \frac{1}{i\hbar}[A, H]$$

which in this case can be written as

$$i\hbar\dot{X} = [X, H + V] = [X, V] = [X, Y]\frac{\partial V}{\partial Y} = il_B^2\frac{\partial V}{\partial Y}$$

$$i\hbar\dot{Y} = [Y, H + V] = [Y, V] = [Y, X]\frac{\partial V}{\partial X} = -il_B^2\frac{\partial V}{\partial X}$$

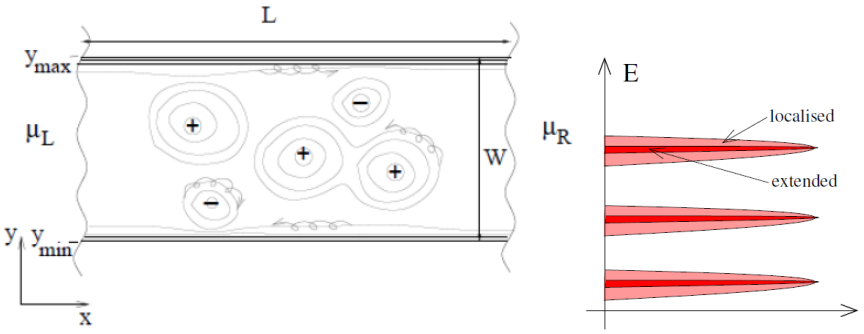
The guiding center component is moving along the equipotential lines since $\vec{R}$ is perpendicular to $V(\vec{R})$. This corresponds to the Hall drift,

$$\langle\dot{R}\rangle = \frac{-\nabla V \times \vec{B}}{eB^2} = \frac{\vec{E} \times \vec{B}}{B^2} = \vec{v}_D$$

This says that the center of mass drifts in a direction $[\dot{X}, \dot{Y}]$ which is perpendicular to ∇V; in other words, the motion is along equipotentials.

Due to the potential $V(\vec{R})$ a landscape is formed with hills and valleys which the charge carriers are moving through. Now consider a random potential. We've drawn some contour lines of such a potential with '+' denoting the local maxima of the potential and '−' denoting the local minima. The particles move anti-clockwise around the maxima and clockwise around the minima depending on the charge. In both cases, the particles are trapped close to the extrema. They can't move throughout the sample. This means that the charge carriers are localized around this topological region and therefore may not contribute to the actual current in the sample. One therefore calls these charge carriers localized. One place that they're guaranteed to exist is on the edge of the sample. At the edge, the equipotential lines are not closed, but open because the confinement potential $V_{conf}(y)$ increases at the edges such that it confines the charge carriers to the sample. These states are then called extended states and contribute to a current in the sample.

The upshot of this is that the states at the far edge of a band — either of high or low energy — are localized. Only the states close to the



center of the band will be extended.

Suppose now one wants to measure the magneto conductivity and Hall conductivity as the Fermi energy is risen from fully below one Landau level to fully above it. At the beginning, localized electron states with lower energies that the integer Landau energy are being depopulated, resulting in no change to the number of conducting electrons in that level. During this time, both conductivities remain constant. Next, and for most of the transition, electrons are being transferred to the extended states, which do conduct electricity, so both conductivities of the sample increase proportionally with the increasing Fermi level. Toward the end, all the conduction electrons have been filled into this Landau level, but localized states are about to be filled, and then both conductivities again flat, this time at a higher value. In practice, the Fermi energy is effectively increased by decreasing the magnetic field, thereby lowering the capacity of the lower Landau levels. Thus, as we vary B, we expect to see plateaus at the quantized values.

Electrostatic potential with translation invariance in the x-direction

The assumptions made in the previous chapter has some constraints, since one assumed that the confinement potential was smooth on a scale of l_B, while confinement potential varies significantly on the scale of l_B at the edges (y_{min}; y_{max}) and therefore one needs to treat it accurately while solving the Schrödinger equation. One chooses to work within the Landau gauge, since it is translation invariant in x-direction. In this case the Hamiltonian becomes,

$$H = \frac{p_y^2}{2m} + \frac{1}{2}m\omega_c(y - y_0)^2 + V_{conf}(y)$$

where $y_0 = kl_B^2$ is the center of oscillation and k is the wave vector in the x-direction. Then one can expand the potential around this point,

$$V_{conf}(y) = V(y_0) + \frac{\partial V(y_0)}{\partial y}(y - y_0) + \mathcal{O}\left(\frac{\partial^2 V}{\partial y^2}\right) = V(y_0) - eE(y_0)(y - y_0) + \mathcal{O}\left(\frac{\partial^2 V}{\partial y^2}\right)$$

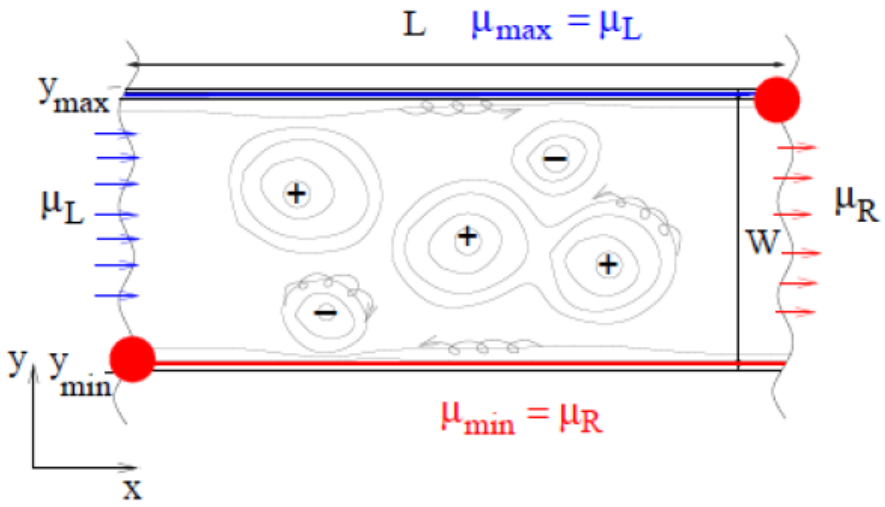
and writing the Hamiltonian in terms of the expanded confinement potential and excluding the second and higher terms one gets

$$H = \frac{p_y^2}{2m} + \frac{1}{2}m\omega_c(y - y_0')^2 + V_{conf}(y_0')$$

where the harmonic oscillator is now shifted $y_0 \to y_0' = y_0 + eE(y_0)/m\omega_c$. The energy levels given by this Hamiltonian are,

$$\epsilon_{n,y_0'} = \hbar\omega_c\left(n + \frac{1}{2}\right) + V(y_0')$$

which gives the same energy levels as before, but with an additional term that solves the problem of large variation at the edges. In summary, we can see that the edge current comes from $V_{conf}(y)$.



Here one can see the potential landscape and the left contact μL is in thermodynamic equilibrium with the upper edge μ_{max}, while the right contact μR is in thermodynamical equilibrium with lower edge μ_{min}. The red dots corresponds to the hotspots, where the chemical potential suddenly drops as it gets in contact with the other side. The Hall voltage is the voltage between these edges and the resistance between the contacts are the Hall resistance.

Let's now see what kind of behavior we expect for the conductivity. Suppose that we've filled all the extended states in a given Landau level and consider what happens as we decrease *B* with fixed *n*. Each Landau level can accommodate fewer electrons, so the Fermi energy will increase. But rather than jumping up to the next Landau level, we now begin to populate the localized states. Since these states can't contribute to the current, the conductivity doesn't change. This leads to exactly the kind of plateaux that are observed, with constant conductivities over a range of magnetic field.

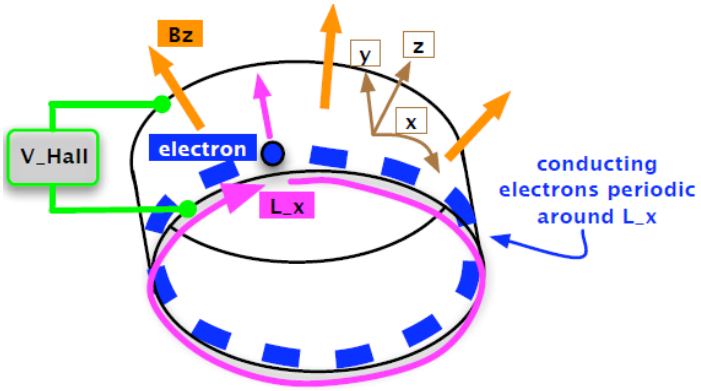
So the presence of disorder explains the presence of plateaux. But now we have to revisit our original argument of why the resistivities take the specific quantized values. These were computed assuming that all states in the Landau level contribute to the current. Now we know that many of these states are localized by impurities and don't transport charge. Surely we expect the value of the resistivity to be different. Right? Well, no. Remarkably, *current carried by the extended states increases to compensate for the lack of current transported by localized states*. This ensures that the resistivity remains quantized despite the presence of disorder.

The role of gauge invariance

One clue to the strangeness of the behavior is that the electrons appear not to interact with their host semiconductor material in any way, nor are their properties affected by different geometries of material. one can remove any interior part of a quantum Hall system, for instance, by drilling a hole and the system still behaves just as it did before. Indeed, if we visualize the electron wavefunctions classically, they are making circular orbits at the radius of the "size" of the wave-

function. Neighboring orbits have trajectories in opposite directions, and so the current they carry cancels out entirely. The exception is any electron traveling near the edge of the sample, where the mechanics are more complicated but the result is that not all the current is cancelled. These edge currents play a significant role in our understanding.

The explanation of IQHE demands that the physical, observable sheet of electrons obey invariance under transformations of the non-observable electromagnetic gauge. Imagine that the sample is folded under itself along the x axis, so that it forms a continuous loop. Let us now assign the coordinates x and y specific orientations so that the direction in which electron wavefunctions that are extended stretches around the loop (x) and the harmonic oscillator eigenstates are spaced across the ribbon (y). The magnetic field B still points through the face of ribbon everywhere (we can keep the same form for the Landau gauge, only now x is a cyclic coordinate).



We would like to know what happens when the loop is "threaded" with a unit of vector potential $\Delta \boldsymbol{A}$:

$$A \to By\hat{x} + \Delta A\hat{x}$$

In such case, we have a new Hamiltonian

$$H = \frac{1}{2m}\left[\vec{p} - e\left(\vec{A} + \Delta\vec{A}\right)\right]^2 - e\phi = \frac{1}{2m}\left[p_x^2 + p_y^2 - 2e(By + \Delta A)p_x + e^2(By + \Delta A)^2\right]$$

yielding the harmonic oscillator part

$$\frac{-\hbar^2}{2m}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{2}m\omega_c^2\left(y - y_0 + \frac{\Delta A}{B}\right)^2 Y = EY$$

and the same relationship between the harmonic oscillator centers and the extended state wavevectors $y_0 = p_x/eB$. Evidently, such a transformation will cause a shift in the center of each electron's wavefunction in the y direction, across the ribbon, by an amount

$$y_0 \to y_0 - \frac{\Delta A}{B}$$

The threading action also produces a phase shift in the wavefunctions along the extended direction x:

$$\psi(x) = e^{\frac{ip_x x}{\hbar}} = e^{\frac{iy_0 eBx}{\hbar}} \to e^{\frac{iy_0 eBx}{\hbar}} e^{\frac{-i\Delta Aex}{\hbar}}$$

This last result implies that in the direction around the loop, electrons may change their phase, and thereby change the location where they are most likely to be found. Yet the only action we performed was a transformation of the gauge, which is not supposed to have real, observable effects! The only way around this problem is to require that the electron remains unchanged -- any shift in vector potential most produce a complete translation around the loop, $x \to x + L_x$, so that the electron looks the same as it did before:

$$1 = e^{\frac{-i\Delta AeL_x}{\hbar}} \to \frac{\Delta AeL_x}{\hbar} = 2\pi$$

This results in a quantization of the vector potential

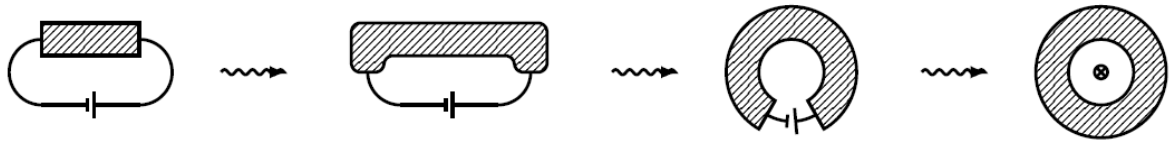
$$\Delta A = \frac{2\pi\hbar}{eL_x} = \frac{h}{eL_x}$$

or in terms of the wavefunction centers:

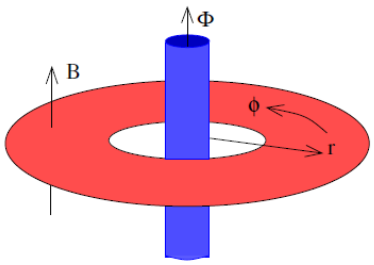
$$\frac{\Delta A}{B} = \frac{h}{eBL_x}$$

But if eB/h defines the number of conducting electrons, then we can show the above equation defines the quantum of vector potential as exactly the spacing between conducting electrons along y! Thus, for every quanta of magnetic flux added to the system, one electron per Landau level shifts over by one in the y direction. If, however, an electron is bound to an impurity in a localized state, we can show that its wavefunction also picks up a phase shift, but does not shift location. Hence, the IQHE is an exactly quantized charge pump, and is not disrupted by the presence of localized states, nor is it affected by a hole drilled in the ribbon. In this way, we can say the IQHE is of a topological nature.

If σ_{xy} is quantized, it should not depend on the details of the geometry. One is therefore allowed to smoothly deform a rectangular sample in the following way: where in the last step we replaced the applied voltage $V \to \partial_t \Phi$ (Faraday's law) with the electromotive force of a time-dependent flux through the opening of the "Corbino" ring. This operation is equivalent to use a periodic boundary condition to join the two ends, making a strip into a Corbino ring.



Instead of considering electrons moving in a rectangular sample, we'll instead consider electrons moving in the annulus. In this context, this is called a Corbino ring. The Hall conductivity is indeed quantized then it shouldn't depend on the geometry of the sample.



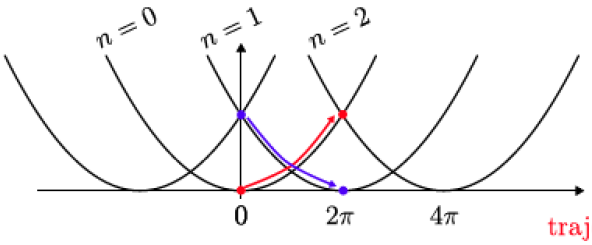
The nice thing about the ring geometry is that it provides us with an extra handle. In addition to the background magnetic field B which penetrates the sample, we can thread an additional flux Φ through the center of the ring. Inside the ring, this Φ is locally pure gauge.

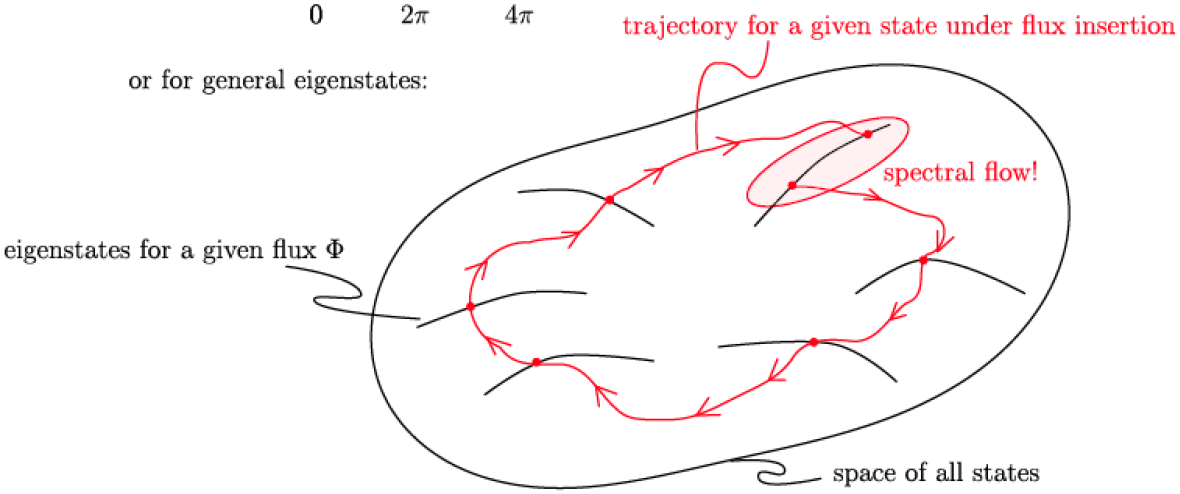
Let's first see what Φ has to do with the Hall conductivity. Suppose that we slowly increase Φ from 0 to $\Phi_0 = 2\pi\hbar/e$. Here "slowly" means that we take a time $T \gg 1/\omega_B$. This induces an emf around the ring, $\mathcal{E} = -\partial\Phi/\partial t = -\Phi_0/T$. Let's suppose that we can argue that n electrons are transferred from the inner circle to the outer circle in this time. This would result in a radial current $I_r = -ne/T$.

$$\rho_{xy} = \frac{\mathcal{E}}{I_r} = \frac{2\pi\hbar}{e^2}\frac{1}{n}$$

This is the result we want. Our task, therefore, is to argue that n electrons are indeed transferred across the ring as the flux is increased to Φ_0.

Spectral Flow in Landau Levels





The key idea that we need is that of spectral flow. We try to understand this idea on the example of a particle on a ring threaded by a flux Φ:

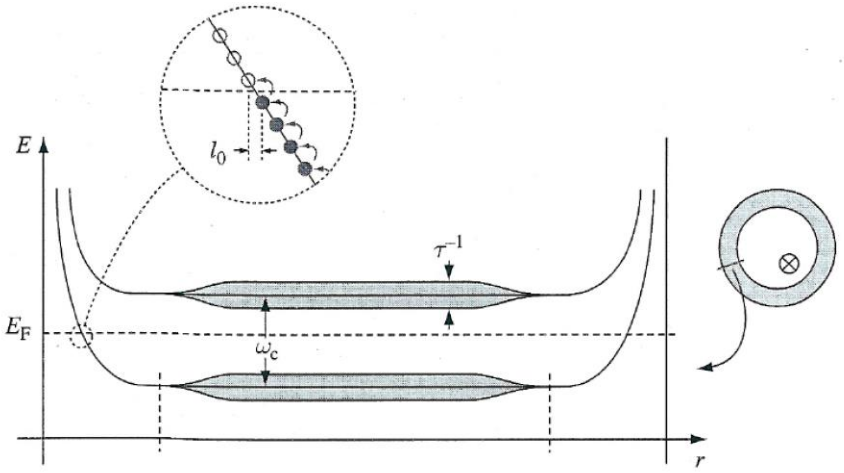
$$H = \frac{1}{2}\left(-\partial_\phi - eA\right)^2 \Phi \quad \rightarrow \quad \psi_n(\phi) = \frac{1}{\sqrt{2}} e^{in\phi} \quad \text{with} \quad \epsilon_n = \frac{1}{2}\left(n - \frac{\Phi}{\Phi_0}\right)^2$$

The spectrum of the Hamiltonian is the same whenever Φ is an integer multiple of Φ_0. However, if we start with a particular energy eigenstate when $\Phi = 0$, this will evolve into a different energy eigenstate with $\Phi = \Phi_0$. As the change is done suitably slowly, over a time $T \gg 1/\omega_B$, the adiabatic theorem ensures that the final energy eigenstate must lie in the same Landau level as the initial state.

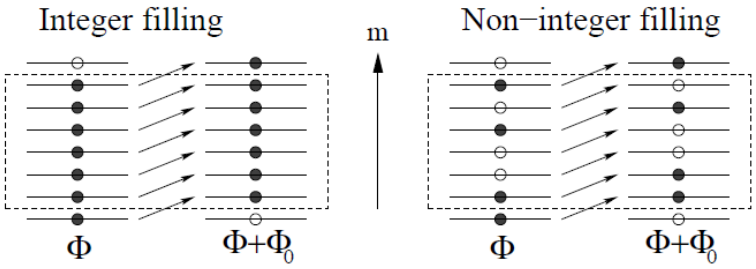
In short, after the insertion of a flux quantum, the Hamiltonian returns to itself. However, if we follow each state adiabatically, we see that the first excited state and the ground state exchanged their positions. This situation is called spectral flow: While the spectrum has to be the same for $\Phi = 0$ and $\Phi = \Phi_0$, the adiabatic evolution does not need to return the ground state to itself! This is illustrated in the figure above. While the example of a particle on a ring is particularly simple, the same situation can occur for a general setup where after the insertion of a flux Φ_0 the original ground state is adiabatically transferred to an excited state.

In other words, the non-invariance individual states upon completion of a round trip will go back to a gauge-equivalent Hamiltonian. Specifically, we shall see that upon sending a flux quantum through the ring, *n* states radially centered at the inner radius are pushed above E_F (*n* is the number of states below E_F). At the same time, *n* states at the outer radius sink below E_F. To regain thermal equilibrium, the system responds by sending *n* electrons from the inner to outer radius.

To substantiate this picture, let's consider a situation in which is E_F placed somewhere between the first and second Landau level. As a result, we obtain the level diagram below. The figure schematically shows the energies of the single-particle states as functions of the radial coordinate of the ring region. In the bulk of the annulus, disorder leads to a broadening of the Landau levels to energy bands of width τ^{-1}. At the outer/inner radius of the annulus, the confining potential



pushes the levels energetically up. Crucially, this implies the presence of as many Fermi energy states – "edge states" – as there are occupied Landau levels (Here, just one). At the inner edge of the system, one filled level has been pushed up the to the Fermi level, while at the outer edge, one empty level dives into the Fermi level. To repair this energy imbalance, the system wants to transfer one electron from inner edge to the outer edge.



At this point it is in place to review the assumptions that went into this argument:

1. $\hbar/t_0 \ll \hbar\omega_B$, i.e. we adiabatically inserted the flux. This is well justified as σ_{xy} describes linear response.
2. Spectral flow lead us to *an excited state,* i.e., the system was sensitive to the flux insertion!