Homework -3 (Due on Apr 1, 2020)

In integer quantum Hall effect, we assume that all electrons behave identically and that they do not interact. We thus have to solve the Schrödinger equation for one single electron in the presence of a magnetic field. The Hamiltonian can be written as

$$\hat{H} = \frac{1}{2m}[\hat{\mathbf{p}} + e\mathbf{A}]^2 + V(z)$$

where we ignore the driving potential $V(y)$ for simplicity. $\mathbf{A}$ is the vector potential. It is related to the B-field as B = $\nabla \times \mathbf{A}$. We choose the Landau gauge $\mathbf{A} = (-yB, 0,0)$ and use the fact that $\nabla \cdot \mathbf{A} = \hat{\mathbf{p}} \cdot \mathbf{A} = 0$. We further assume that the electrons are vertically trapped in an infinite potential well of width Δ_z.

(a) Prove that the Landau gauge fulfills $\mathbf{B} = \nabla \times \mathbf{A}$.

(b) Prove that $\hat{\mathbf{p}} \cdot (\mathbf{A}\Psi) = \Psi(\hat{\mathbf{p}} \cdot \mathbf{A}) + \mathbf{A} \cdot (\hat{\mathbf{p}}\Psi) = \mathbf{A} \cdot (\hat{\mathbf{p}}\Psi)$

(c) Separate the time and *z*-dependence from the *x*, *y*-dependence using $\Psi(x, y, z, t) = \psi_{x,y}(x, y)\psi_z(z)\phi(t)$. What is the differential equation for $\psi_{x,y}$ and the solutions for ψ_z? How is the total energy of the electron determined?

(d) To solve for $\psi_{x,y}$, assume a plane wave dependence along the *x*-direction, i.e. $\psi_{x,y}(x, y) = \psi_y(y)\exp(ik_x x)$. Make the substitution $\psi_y(y) \to \psi_y(\tilde{y})$ where $\tilde{y} = \frac{y}{r_c} - r_c k_x$, $r_c = \sqrt{\frac{\hbar}{eB}}$ (cyclotron orbit radius) to bring the equation for ψ_y to the form of a harmonic oscillator. What are the expressions for the energy levels and the angular frequency ω (Landau frequency)?

(e) Determine the solution for the total wavefunction $\Psi(x, y, z, t)$ and the total energy E.

(f) The system that we are considering is a two-dimensional system with two-dimensional density of states. For zero temperature, show that the number of states n_e per unit area ($n_e = \frac{dN}{dA}$) is $n_e = \frac{mE_F}{2\pi\hbar^2}$. Remember: the topmost filled energy level is the Fermi energy E_F. n_e corresponds to the areal density of participating electrons.

(g) Ignore the energy contributions from $\psi_z(z)$. The number N_f of filled energy levels (energy levels with energies smaller than the Fermi energy E_F) is now $N_f = \frac{E_F}{\hbar\omega}$, with ω being the Landau frequency. N_f is refered to as the filling factor and obviously, it can have only discrete values. Use the above expression to eliminate E_F in the previous question and substitute n_e into the expression for the Hall resistance. You can see that the discrete values of N_f impose a discrete behavior of ρ_{Hall}.

2. Reading: TKNN paper, DOI: 10.1103/PhysRevLett.49.405