3. Revisit the integer quantum Hall effect (b)

Spectral Flow in the Presence of Disorder

In the presence of disorder, in polar coordinates, the Hamiltonian takes the form

$$H_{\Phi=0} = \frac{1}{2m}\left[-\hbar^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial \phi}\right) + \left(-\frac{i\hbar}{r}\frac{\partial}{\partial \phi} + \frac{eBr}{2}\right)^2\right] + V(r,\phi)$$

where $V(r,\phi)$ is the random potential capturing the effects of disorder. Note that this depends on ϕ, so angular momentum is no longer a good quantum number in this system. Adding the flux through the center changes the Hamiltonian to

$$H_{\Phi} = \frac{1}{2m}\left[-\hbar^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial \phi}\right) + \left(-\frac{i\hbar}{r}\frac{\partial}{\partial \phi} + \frac{eBr}{2} + \frac{e\Phi}{2\pi r}\right)^2\right] + V(r,\phi)$$

Importantly, **the flux Φ affects only the extended states but does not change the localized states**. To see this, we attempt to undo the flux by a gauge transformation,

$$\psi(r,\phi) \to e^{-i(e/\hbar)\int dr \cdot A}\psi(r,\phi) = e^{-ie\Phi\phi/2\pi\hbar}\psi(r,\phi)$$

Again the spectrum of the Hamiltonian is left unchanged when Φ is an integer multiple of Φ_0. At this time, the Hamiltonian of the system is gauge equivalent to a Hamiltonian of a system without flux. This is because an integer multiple of the flux quantum can be removed by the gauge transformation acting on the wavefunction of the system. For the non-integer fluxes, this transformation alters the boundary condition $\psi(r,2\pi) = \psi(r,0)$.

There are always at least two extended states: one near the inner ring and one near the outer ring. An extended state localized at one radius is transformed into an extended state at the next available radius. The presence of disorder means that there are fewer extended states, but this doesn't change the overall conclusion: if all extended states in a given Landau level are filled, then the net effect of dialing the flux from $\Phi = 0$ to $\Phi = \Phi_0$ is to transport one electron from the inner to the outer edge. If n Landau levels are filled, we again get the result $\rho_{xy} = \frac{\mathcal{E}}{I_r} = \frac{2\pi\hbar}{e^2}\frac{1}{n}$.

The percolation transition

We know that eigenstates in a disordered Landau level are given by orbits along equipotential lines. The question is, if there is always an orbit in each Landau level that connects the two edges of our Corbino disk. If this is the case, this state mediates the sensitivity to the flux insertion in the disordered case. Luckily the answer to this question is an affirmative yes!
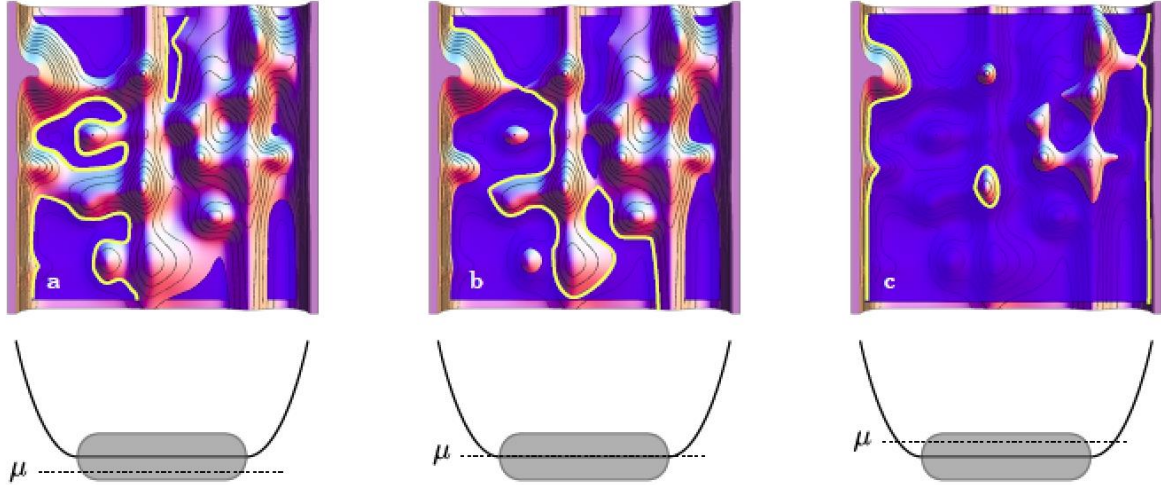
In the above figure, energy landscape for a disordered Landau level with a confining potential is shown. Eigenstates are given by equipotential orbits. At low chemical potential μ as shown here, all orbits are "lakes" and hence all states are localized.

When filling in more water (raising μ) we switch at some point from "lakes" to "island". Right at the point where this happens, the shoreline has to connect through the whole sample. This is the consequence after extended state in the middle of the sample. Above the center of the Landau level we are left with "islands" where all states in the bulk are localized.

We can now summarize our discussion of disorder effects: The disorder allows the chemical potential to smoothly change between the Landau level's. Therefore, there is an extensive window where the chemical potential lies in the range of the gap and hence we find

$$\sigma_{xy} = -v\frac{e^2}{h} \quad v \in \mathbf{Z}$$

The arguments above involving gauge transformations start to give a hint of the topological nature of the quantum Hall effect - *independent of the system-specific details*, the addition of a flux quantum through the Corbino ring transfers an integer charge across the system. This suggests that the effect should be understandable in terms of some kind of topological index lurking behind the mapping of a parameter space (presently, the amount of flux through the annulus) into the Hilbert space of the problem.

The role of topology

In this section, we describe a set-up in which we can see the deep connections between topology and the Hall conductivity. However, we will consider the Hall system on a spatial torus $\mathbf{T}^2$. This can be viewed as a rectangle with opposite edges identified. We'll take the lengths of the sides to be L_x and L_y.

We consider wavefunctions over the torus and ask: what periodicity requirements should we put on the wavefunction? The first guess is that we should insist that wavefunctions obey $\psi(x, y) = \psi(x + L_x, y) = \psi(x, y + L_y)$. But this turns out to be too restrictive when there is a magnetic flux through the torus. Instead, one has to work in patches; on the overlap between two different patches, wavefunctions must be related by a gauge transformation.

Operationally, we introduce the magnetic translation operators, which are defined via the canonical momentum operator $(i\hbar\nabla - e\mathbf{A})$,

$$T(\mathbf{d}) = e^{-i\mathbf{d}\cdot\mathbf{p}/\hbar} = e^{-i\mathbf{d}\cdot(i\nabla + e\mathbf{A}/\hbar)}$$

These operators translate a state $\psi(x, y)$ by position vector $\mathbf{d}$. The appropriate boundary conditions will be that when a state is translated around a cycle of the torus, it comes back to itself. So $T_x \psi(x, y) = \psi(x, y)$ and $T_y \psi(x, y) = \psi(x, y)$ where $T_x = T(\mathbf{d} = (L_x, 0))$ and $T_y = T(\mathbf{d} = (0, L_y))$.

We'll choose Landau gauge $A_x = 0$ and $A_y = Bx$. With this choice, translations in the x direction are the same as those in the absence of a magnetic field, while translations in the y direction pick up an extra phase. If we take a state $\psi(x, y)$, translated around a cycle of the torus, becomes

$$T_x \psi(x, y) = \psi(x + L_x, y) = \psi(x, y)$$

$$T_y \psi(x, y) = e^{-ieBL_y x/\hbar}\psi(x, y + L_y) = \psi(x, y)$$

The two wavefunctions agree only up to a gauge transformation.

However, these equations are not consistent for any choice of $\mathbf{B}$. This follows by comparing what happens if we translate around the x-cycle, followed by the y-cycle, or if we do these in the opposite order. We have

$$T_y T_x = e^{-ieBL_x L_y x/\hbar} T_x T_y$$

Since both are required to give us back the same state, we must have

$$\frac{eBL_x L_y}{\hbar} = \frac{L_x L_y}{l_B^2} \in 2\pi\mathbf{Z}$$

This is the Dirac quantization condition. In other words, an integer number flux quanta has to pierce the surface of the torus (we can only put quantized magnetic monopoles inside the torus).

Adding Flux

Let's consider a general, multi-particle Hamiltonian H_0 where the subscript 0 means that this is the unperturbed Hamiltonian before we turn on an electric field. At this point, H_0 could be that of many non-interacting particles each. Later, we'll apply the Kubo formula both to Hamiltonians which describe particles moving in the continuum and to Hamiltonians that describe particles moving on a lattice.

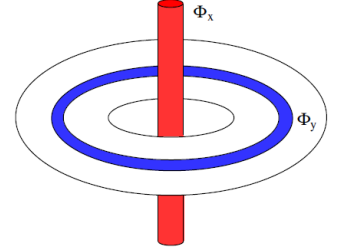We use the Hamiltonian of the form $H = H_0 + \Delta H$ with

$$\Delta H = -\boldsymbol{J} \cdot \boldsymbol{A}$$

where $\boldsymbol{J}$ is the quantum operator in more general definitions rather than the current operator.

Now we're going to perturb the system. We do this by threading two fluxes, Φ_x and Φ_y through the x and y-cycles of the torus respectively. This means that the gauge potential becomes

$$A_x = \frac{\Phi_x}{L_x} \text{ and } A_y = \frac{\Phi_y}{L_y} + Bx$$

which corresponds to slowly turning on the fluxes through the opening of the torus. This is the same kind of set-up that we discussed in Section *The Role of Gauge Invariance*; the only difference is that now the geometry allows us to introduce two fluxes instead of one. If we increase either Φ_i from zero to Φ_0, then the spectrum of the quantum system must be invariant. However, the system can undergo spectral flow.

The addition of the fluxes adds an extra term to the Hamiltonian of the form,

$$\Delta H = -\sum_{i=x,y} \frac{J_i \Phi_i}{L_i}$$

We want to see how this affects the ground state of the system which we will denote as $|\psi_0\rangle$. We'll assume that the ground state is nondegenerate and that there is a gap to the first excited state. Then, to first order in perturbation theory, the ground state becomes

$$|\psi_0\rangle' = |\psi_0\rangle + \sum_{n \neq \psi_0} \frac{\langle n|\Delta H|\psi_0\rangle}{E_n - E_0} |n\rangle$$

Considering infinitesimal changes of Φ_i, we can write this as

$$\left|\frac{\partial \psi_0}{\partial \Phi_i}\right\rangle = -\frac{1}{L_i} \sum_{n \neq \psi_0} \frac{\langle n|J_i|\psi_0\rangle}{E_n - E_0} |n\rangle$$

But the right-hand-side is exactly the kind of expression that appeared in the Kubo formula. This means that, including the correct factors of the spatial area, we can write the Hall conductivity as

$$\sigma_{xy} = i\hbar L_x L_y \sum_{n \neq \psi_0} \frac{\langle \psi_0 | J_y | n \rangle \langle n | J_x | \psi_0 \rangle - \langle \psi_0 | J_x | n \rangle \langle n | J_y | \psi_0 \rangle}{(E_n - E_0)^2}$$

$$= i\hbar \left[\left\langle \frac{\partial \psi_0}{\partial \Phi_y} \middle| \frac{\partial \psi_0}{\partial \Phi_x} \right\rangle - \left\langle \frac{\partial \psi_0}{\partial \Phi_x} \middle| \frac{\partial \psi_0}{\partial \Phi_y} \right\rangle \right]$$

$$= i\hbar \left[\frac{\partial}{\partial \Phi_y} \left\langle \psi_0 \middle| \frac{\partial \psi_0}{\partial \Phi_x} \right\rangle - \frac{\partial}{\partial \Phi_x} \left\langle \psi_0 \middle| \frac{\partial \psi_0}{\partial \Phi_y} \right\rangle \right]$$

As we now explain, this final way of writing the Hall conductivity provides a novel perspective on the integer quantum Hall effect.

Hall Conductivity and the Chern Number

The fluxes Φ_i appear as parameters in the perturbed Hamiltonian. These parameters should be periodic: the space of the flux parameters is itself a torus, $\mathbf{T}_\Phi^2$. We'll introduce dimensionless angular variables, θ_i, to parameterize this torus,

$$\theta_i = \frac{2\pi \Phi_i}{\Phi_0} \quad \text{with} \quad \theta_i \in [0, 2\pi)$$

It is natural to consider the Berry phase that arises as the parameters are varied. This is described by the Berry connection which, in this case, lives over $\mathbf{T}_\Phi^2$. It is

$$\mathcal{A}_i(\Phi) = -i \langle \psi_0 | \frac{\partial}{\partial \theta_i} | \psi_0 \rangle$$

The field strength, or curvature, associated to the Berry connection is given by

$$\mathcal{F}_{xy} = \frac{\partial \mathcal{A}_x}{\partial \theta_y} - \frac{\partial \mathcal{A}_y}{\partial \theta_x} = -i \left[\frac{\partial}{\partial \theta_y} \left\langle \psi_0 \middle| \frac{\partial \psi_0}{\partial \theta_x} \right\rangle - \frac{\partial}{\partial \theta_x} \left\langle \psi_0 \middle| \frac{\partial \psi_0}{\partial \theta_y} \right\rangle \right]$$

This is precisely our expression for the Hall conductivity! We learn that, for the torus with fluxes, we can write

$$\sigma_{xy} = i\hbar \left[\frac{\partial}{\partial \Phi_y} \langle \psi_0 | \frac{\partial \psi_0}{\partial \Phi_x} \rangle - \frac{\partial}{\partial \Phi_x} \langle \psi_0 | \frac{\partial \psi_0}{\partial \Phi_y} \rangle \right] = i\hbar \frac{e^2}{\hbar^2} \left[\frac{\partial}{\partial \theta_y} \langle \psi_0 | \frac{\partial \psi_0}{\partial \theta_x} \rangle - \frac{\partial}{\partial \theta_x} \langle \psi_0 | \frac{\partial \psi_0}{\partial \theta_y} \rangle \right]$$

$$= i \frac{e^2}{\hbar} \frac{\mathcal{F}_{xy}}{-i} = -\frac{e^2}{\hbar} \mathcal{F}_{xy}$$

This is a nice formula. But, so far, it doesn't explain why σ_{xy} is quantized. However, suppose that we average over all fluxes. In this case we integrate over the torus $\mathbf{T}_\Phi^2$ of parameters to get

$$\sigma_{xy} = -\frac{e^2}{\hbar} \int_{\mathbf{T}_\Phi^2} \frac{d^2\theta}{(2\pi)^2} \mathcal{F}_{xy}$$

The integral of the curvature over $\mathbf{T}_\Phi^2$, is a number known as the first Chern number

$$C = \frac{1}{2\pi} \int_{\mathbf{T}_\Phi^2} d^2\theta \, \mathcal{F}_{xy}$$

Importantly, this is always an integer: $C \in \mathbf{Z}$. The net result is that if we average over the fluxes, the Hall conductivity is necessarily quantized as

$$\sigma_{xy} = -\frac{e^2}{2\pi\hbar} C$$

This, of course, is the integer quantum Hall effect. The relationship between the Hall conductivity and the Chern number is usually referred to as the TKNN invariant (named after Thouless, Kohomoto, Nightingale and den Nijs).

TKNN invariants

We saw in the previous section that there is a deep relationship between the Hall conductivity and a certain topological quantity called the Chern number that is related to the Berry phase.

The advantage of looking at the particle on a lattice is that the momentum of a particle lies on a torus $\mathbf{T}^2$, known as the Brillouin zone. Indeed, it will play a very similar role to the parameter space $\mathbf{T}_\Phi^2$ that we met in the previous section.

We'll consider a particle moving on a rectangular lattice. The distance between lattice sites in the x-direction is a; the distance in the y-direction is b. The energy spectrum of this system forms bands. Within each band, states are labelled by lattice momentum which takes values in the Brillouin zone, parameterized by

$$-\frac{\pi}{a} < k_x < \frac{\pi}{a} \quad \text{and} \quad -\frac{\pi}{b} < k_y < \frac{\pi}{b}$$

The states with momenta at the edges of the Brillouin zone are identified. This means that the Brillouin zone is a torus $\mathbf{T}^2$. The wavefunctions in a given band can be written in Bloch form as

$$\psi_{\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} u_{\mathbf{k}}(\mathbf{x})$$

where $u_{\boldsymbol{k}}(\boldsymbol{x})$ is usually periodic on a unit cell so that $u_{\boldsymbol{k}}(\boldsymbol{x} + \mathbf{e}) = u_{\boldsymbol{k}}(\boldsymbol{x})$ with either $\mathbf{e} = (a, 0)$ or $\mathbf{e} = (0, b)$.

The results below are very general: they don't rely on any specific Hamiltonian, but rather apply to any system that satisfies a few simple criteria.

- First, we will assume that the single particle spectrum decomposes into bands, with each band parameterized by a momentum label $\mathbf{k}$ which lives on a torus $\mathbf{T}^2$. This is obviously true for simple lattice models. It is also true (under certain assumptions) for particles moving in a lattice in the presence of a magnetic field where the torus is a slightly different concept called a magnetic Brillouin zone. (In this case, the periodicity conditions on $u_{\boldsymbol{k}}$ are altered slightly but the formula we derive below still hold).
- Second, we'll assume that the electrons are non-interacting. This means that we get the multi-particle spectrum simply by filling up the single-particle spectrum, subject to the Pauli Exclusion Principle.
- Finally, we'll assume that there is a gap between bands and that the Fermi energy E_F lies in one of these gaps. This means that all bands below E_F are completely filled while all bands above E_F are empty. In band theory, such a situation describes an insulator.

Whenever these three criteria are obeyed, one can assign an integer-valued topological invariant $C \in \mathbf{Z}$ to each band. The topology arises from the way the phase of the states winds as we move around the Brillouin zone $\mathbf{T}^2$. This is captured by a Berry connection over $\mathbf{T}^2$, defined by

$$\mathcal{A}_i(\boldsymbol{k}) = -\langle u_{\boldsymbol{k}} | \frac{\partial}{\partial k^i} | u_{\boldsymbol{k}} \rangle$$

There is one slight conceptual difference from the type of Berry connection we met previously. In Section *Berry Phase*, the connections lived on the space of parameters of the Hamiltonian; here the connection lives on the space of states.

We can compute the field strength associated to $\mathcal{A}_i$. This is

$$\mathcal{F}_{xy} = \frac{\partial \mathcal{A}_x}{\partial k^y} - \frac{\partial \mathcal{A}_y}{\partial k^x} = -i \left\langle \frac{\partial u}{\partial k^y} \Big| \frac{\partial u}{\partial k^x} \right\rangle + i \left\langle \frac{\partial u}{\partial k^x} \Big| \frac{\partial u}{\partial k^y} \right\rangle$$

Once again, we can compute the first Chern number by integrating $\mathcal{F}$ over the Brillouin zone $\mathbf{T}^2$,

$$C = -\frac{1}{2\pi} \int_{\mathbf{T}^2_{\Phi}} d^2\theta \, \mathcal{F}_{xy}$$

In the present context, it is usually referred to as the TKNN invariant. As we've seen before, the Chern number is always an integer: $C \in \mathbf{Z}$. In this way, we can associate an integer C_α to each band α.

The Chern number once again has a beautiful physical manifestation: it is related to the Hall conductivity σ_{xy} of a non-interacting band insulator by

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar} \sum_\alpha C_\alpha$$

where the sum is over all filled bands α and C_α is the Chern class associated to that band. This is the famous TKNN formula.