Homework -6 (Due on June 3, 2020)

1. What will happen if we take a 100 site Kitaev chain in the topological phase and change the potential μ to a very large negative value for the last 50 sites? (Consider the band evolution and the position of the Majorana bound state)

2. Can we remove the last Majorana site in the Kitaev chain in the topological phase? Explain why.

3. The effective theory of the first experimentally discovered 2D topological insulator HgTe is given by the following Bloch Hamiltonian on the simple square lattice

$$\mathcal{H}(\mathbf{k}) = \lambda\sigma_z\left[s_x \sin(k_y) - s_y \sin(k_x)\right] + \sigma_x M_{\mathbf{k}}$$

Here $M_{\mathbf{k}} = \epsilon - 2t\left[\cos(k_x) + \cos(k_y)\right]$ and $\mathbf{s}, \boldsymbol{\sigma}$ represent the Pauli matrices in spin and orbital space, respectively. λ, ϵ, and t are model parameters. Hereafter, we take $\lambda = 1$ and measure ϵ and t in units of λ.

(a) Find the spectrum of $\mathcal{H}(\mathbf{k})$. Show that $\mathcal{H}(\mathbf{k})$ respects both time-reversal symmetry $\mathcal{T}$ and inversion symmetry $\mathcal{P}$. ((The inversion operation here involves exchange of the two orbitals on the same site implemented by σ_x).

(b) Use the Fu-Kane formula to classify the topological phases of $\mathcal{H}(\mathbf{k})$ when the two negative-energy bands are filled. It is most instructive to fix t to a positive value and sketch the phase diagram as a function of ϵ. Alternately you can give a phase diagram in the $\epsilon - t$ plane. Please label clearly all the phase transitions and assign the $\mathbb{Z}_2$ index to the gapped phases as appropriate.

4. Consider the Majorana chain with Hamiltonian

$$H = \sum_{j=1}^{N} -\mu c_j^\dagger c_j - \Delta c_{j+1}^\dagger c_j - \Delta c_j^\dagger c_{j+1} + \Delta c_{j+1}^\dagger c_j^\dagger + \Delta^* c_j c_{j+1}$$

Take Δ to be real. Apply the Jordan Wigner transformation and map it to a spin 1/2 chain

$$c_j^\dagger = \prod_{l<j} \sigma_l^z \left(\sigma_j^x + i\sigma_j^y\right)/2, \qquad c_j = \prod_{l<j} \sigma_l^z \left(\sigma_j^x - i\sigma_j^y\right)/2$$

(a) What does the total fermion parity symmetry $P_f = \prod_j \left(2c_j^\dagger c_j - 1\right)$ map to under the Jordan Wigner transformation?

(b) What does the $-\mu c_j^\dagger c_j$ term map to?

(c) In the middle of the chain $(1 \leq j \leq N)$, what does the $-\Delta c_{j+1}^\dagger c_j - \Delta c_j^\dagger c_{j+1} + \Delta c_{j+1}^\dagger c_j^\dagger + \Delta c_j c_{j+1}$ term map to?

(d) On an open chain, the total spin Hamiltonian is the Ising model with transverse field:

$$H_{Ising} = \sum_j \Delta\, \sigma_j^x \sigma_{j+1}^x - \mu\left(1 + \sigma_j^z\right)/2$$

What is the ground state when $\Delta = 0$, $\mu > 0$? Is the global $\mathbb{Z}_2$ symmetry spontaneously broken? Show this by calculating the connected correlation function of the order parameter (σ_x) in the symmetric ground state wave function.

(e) On an open chain, what are the ground states when $\mu = 0$, $\Delta < 0$? Do they break the global symmetry of the system?