Homework -4 (Due on Apr 15, 2019)

1. Consider a band crossing described by the 2x2 matrix Hamiltonian

$$\mathcal{H}(q) = \begin{pmatrix} m & q_x^2 - q_y^2 - 2iq_xq_y \\ q_x^2 - q_y^2 + 2iq_xq_y & -m \end{pmatrix}$$

where m is a constant parameter.

(a) Find the spectrum and show that $\mathcal{H}(q)$ describes a quadratic crossing between two bands as m goes through zero.
(b) If the Chern numbers of the two bands are zero for $m > 0$ what are they when $m < 0$?

2. The effect of the external magnetic field on the tight binding electrons can be taken into account by means of Peierls substitution,

$$t_{ij} \to t_{ij}e^{i\phi_{ij}}, \ \ \phi_{ij} = \frac{e}{\hbar}\int_{\mathrm{R}_i}^{\mathrm{R}_j} \mathbf{A} \cdot dl$$

where $\mathbf{A}$ is the vector potential of the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ and the integral is taken along the straight-line connecting points R_i and R_j. Consider tight binding electrons with nearest neighbor hopping t and on-site energy $E_0 = 0$ on a 2D square lattice with uniform magnetic field B applied perpendicular to the lattice plane.

(a) Consider the field strength

$$B = \frac{p}{q}\frac{\Phi_0}{a^2}$$

with p, q integer, $\Phi_0 = h/e$ the flux quantum and a the lattice constant. Show that in this case the smallest primitive unit cell contains q lattice sites. Construct explicitly an example of such unit cell. What does this imply for the band structure?

Hint: It is easiest to work in the gauge $\mathbf{A} = B(0, x, 0)$. Also note that $\oint_C \mathbf{A} \cdot dl = \Phi$, the flux enclosed by contour C.

(b) Calculate the energy spectrum for a special case of $p = 1, q = 2$, i.e. half x quantum per elementary plaquette. Show that in this case the spectrum has two Dirac points per Brillouin zone. Sketch the first BZ and mark the position of the Dirac points.