Ch1 The basics

1.1 Classical motion of electrons in electrical and magnetic fields

The Hall effect; $m\dot{\vec{v}} = -e\vec{v} \times \vec{B}$
Transform into two differential equations

$$m\ddot{x} = -eB\dot{y} \qquad m\ddot{y} = eB\dot{x}$$

General solution:

$$x(t) = X - R\sin(\omega_c t + \Phi) \qquad y(t) = Y - R\cos(\omega_c t + \Phi)$$

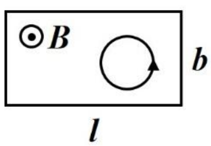
Cyclotron (angular) frequency

$$\omega_c = eB/m$$

1.1.1 Drude model

Apply an electric field $m\dot{\vec{v}} = -e(\vec{E} + \vec{v} \times \vec{B})$

When equilibrium: $\dot{\vec{v}} = 0$, that is $E_y = v_x B_z$



Assuming the electric and magnetic fields are perpendicular to each other, i.e. $\vec{E} = (E_x, 0,0)$ and $\vec{B} = (0,0, B_z)$, the solution to this equation is a similar 'cyclotron motion'; a superposition of a circular orbit with cyclotron frequency ω_c and a linear motion with velocity E/B.

General solution with the boundary condition $E_y/E_x = l/b$, is following, where l and b are the length and width of the system: $\vec{v} = -\frac{e}{m\omega_c}\begin{pmatrix} \frac{l}{b}E_x \\ -E_x \end{pmatrix} + v_0 \begin{pmatrix} \cos \omega_c \tau \\ \sin \omega_c \tau \end{pmatrix}$

Add friction with a damping term: $m\dot{\vec{v}} = -e(\vec{E} + \vec{v} \times \vec{B}) - m\vec{v}/\tau$

Matrix solution by making $\dot{\vec{v}} = 0$ in equilibrium: $\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} m/e\tau & -B \\ +B & m/e\tau \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$

1.1.2 Resistivity and Conductivity

Motion equ in matrix notation: $\begin{pmatrix} 1 & -\omega_c \tau \\ \omega_c \tau & 1 \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \frac{e^2 n\tau}{m} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$

Here, the conductivity σ is not a single number anymore due to the presence of the magnetic field.

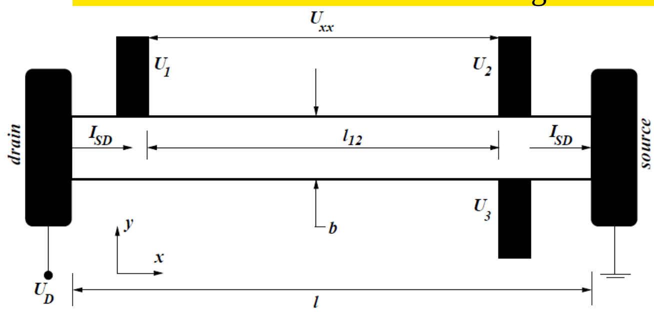
Conductivity tensor: $\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix} = \frac{\sigma_0}{1+\omega_c^2\tau^2}\begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix}$ with $\sigma_0 = \frac{e^2 n\tau}{m}$

Resistivity tensor: $\rho = \frac{1}{\sigma} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix} = \frac{1}{\sigma_0}\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix} = \begin{pmatrix} \sigma_0^{-1} & B/ne \\ -B/ne & \sigma_0^{-1} \end{pmatrix}$

Hall coefficient $R_H = -\frac{\omega_c\tau}{\sigma_0 B} = \frac{1}{ne}$

Consistent with that in semiconductor physics. Consistently, we can see that the Hall coefficient depends only on microscopic picture of the material: only the charge and density of the carriers. It does not depend on the scattering time, and insensitive to the friction process that happens in practical materials.

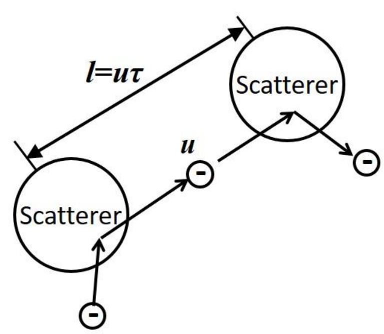
1.1.3 The classical Hall effect for low magnetic fields



The resistivity tensor is determined by applying a constant current along the *x*-direction through a rectangular sample, and measuring the voltage drop across the longitudinal ($U_{xx} = U_1 - U_2$) and transverse ($U_{xy} = U_2 - U_3$) directions as functions of the applied perpendicular magnetic field $\vec{B} = B\vec{e}_z$.
$U_{xx} = E_x l_{12}$ and $U_{xy} = E_y b$

Drift velocity v_d is simply the vector average over the velocities of all *N* electrons, $v_d = \frac{1}{N}\sum_{i=1}^{N} v_i$

In this regime of low magnetic field, the mobility of the electrons can be determined. The distance that an electron travels between scattering events is called the free path. It is straightforward to show that the average or mean free path for an electron is simply $l = \mu\tau$

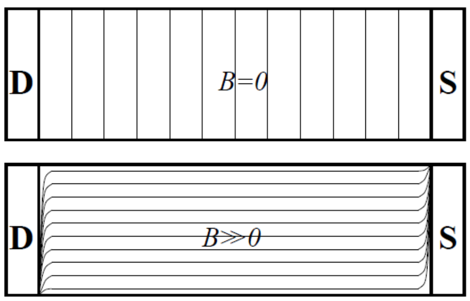


1.1.4 The classical Hall effect for high magnetic fields

Meaning of $\sigma_{xx} \propto \rho_{xx}$

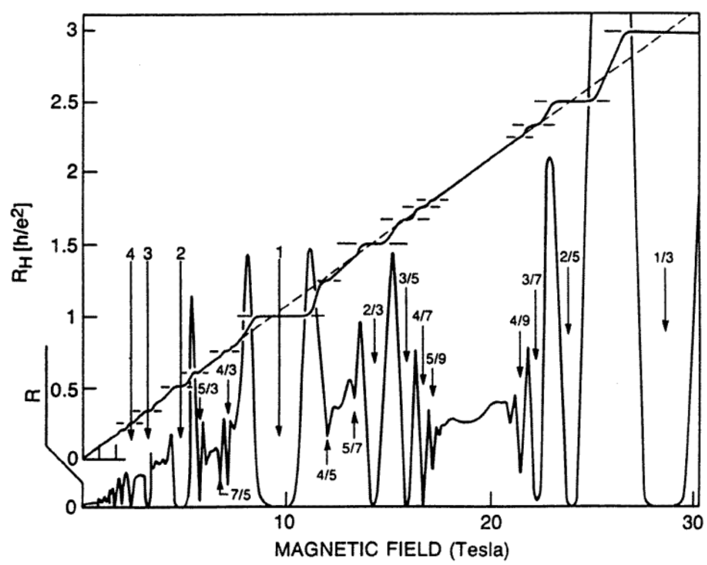
In the large field limit, we have $\omega_c\tau \gg 1$. We examine the longitudinal conductivity and the result is somewhat surprising: $\sigma_{xx} \to \frac{\sigma_0}{{\omega_c}^2\tau^2} = \frac{e^2n^2}{B}\frac{1}{\sigma_0} \propto \rho_{xx}$

This indicates that in the presence of a strong magnetic field, Ohm's law of the form $U = R \cdot I$ is no longer valid. The results can be understood by considering the equipotential lines in the sample. For the case without any external magnetic field or negligibly small magnetic field, the current density is evenly distributed across the sample, and $j_y = 0$. When a strong perpendicular magnetic field is applied, by solving Eq.$\begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix}\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \frac{e^2n\tau}{m}\begin{pmatrix} E_x \\ E_y \end{pmatrix}$, it can be seen that $j_y \neq 0$ due to a potential difference is build up in the y-direction. The off-diagonal elements for large applied magnetic fields are $-\sigma_{xy} = \rho_{xy}^{-1}(= en/B)$ and for small magnetic fields reduce to $-\sigma_{xy} = \sigma_0^2\rho_{xy}$.



In strong magnetic fields, the Hall voltage is identical to the source-drain voltage, i.e. $U_{SD} = U_x = U_y = U_{Hall}$. The equipotential lines indicate that the upper right corner of the sample is at the same potential as the drain contact, and the lower left corner is at the same potential as the source contact.

1.2 The integer quantum Hall effect



$$\rho_{xy} = \frac{1}{\nu}\frac{2\pi\hbar}{e^2}, \nu \in \mathbf{Z}$$

The center of each of these plateaus occurs when the magnetic field takes the value $B = \frac{2\pi\hbar \mathrm{n}}{\nu e} = \frac{n}{\nu}\Phi_0$ where *n* is the electron density and $\Phi_0 = \frac{2\pi\hbar}{e}$ is known as the flux quantum.

$$\sigma_{xx} = \frac{\rho_{xx}}{{\rho_{xx}}^2 + {\rho_{xy}}^2}, \sigma_{xy} = \frac{-\rho_{xy}}{{\rho_{xx}}^2 + {\rho_{xy}}^2}$$

Meaning of $\rho_{xx} \to 0$, $\sigma_{xx} \to 0$

1.2.1 Landau levels

The Lagrangian for a particle of charge $-e$ and mass m moving in a background magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ is $L = \frac{1}{2}m\dot{\mathbf{x}}^2 - e\dot{\mathbf{x}} \cdot \mathbf{A}$

Gauge transformation $\mathbf{A} \to \mathbf{A} + \nabla\alpha$, the Lagrangian changes by a total derivative: $L \to L - e\dot{\alpha}$.;

canonical and mechanical momentum; commutation relations;

Hamiltonian $H = \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^{\mathbf{2}}$

Harmonic oscillator; ground state; raising and lowering operators

Energy $E_n = \hbar\omega_B\left(n + \frac{1}{2}\right), n \in \mathbf{N}$

1.2.2 Landau gauge

$$\mathbf{A} = xB\hat{\mathbf{y}} = (0, xB, 0)$$

Hamiltonian

$$H = \frac{1}{2m}\left(p_x^2 + \left(p_y + eBx\right)^2\right) = \cdots$$

$$\psi_{n,k}(x,y) \sim e^{iky} H_n(x + kl_B^2) e^{-(x+kl_B^2)/2l_B^2}$$

1.2.3 Degeneracy

$$\mathcal{N} = \frac{L_y}{2\pi}\int_{-L_x/l_B^2}^{0} dk = \frac{L_x L_y}{2\pi l_B^2} = \frac{eBA}{2\pi\hbar}$$

1.2.4 Turning on an electric field

$$E_{n,k} = \hbar\omega_B\left(n + \frac{1}{2}\right) + eE\left(kl_B^2 - \frac{eE}{m\omega_B^2}\right) + \frac{m}{2}\frac{E^2}{B^2}$$

1.2.5 How does Landau quantization look like?

Refer to “Full momentum- and energy-resolved spectral function of a 2D electronic system”, Science 358, 901–906 (2017).