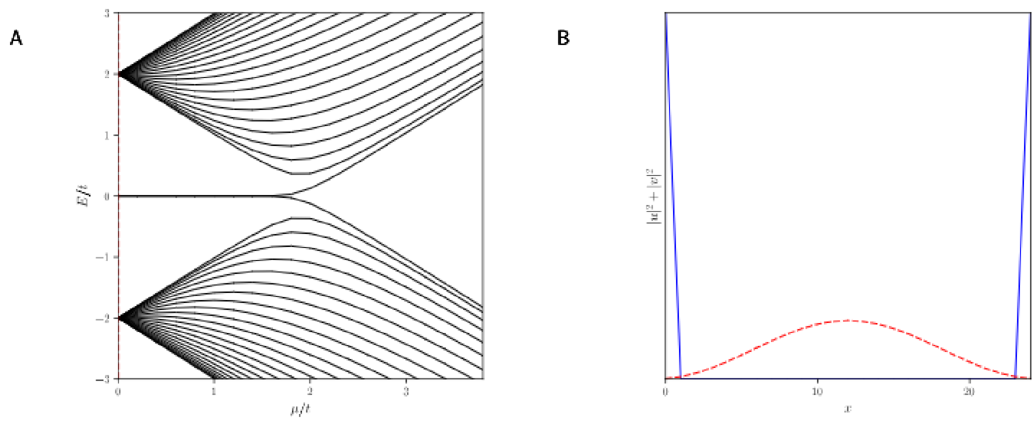
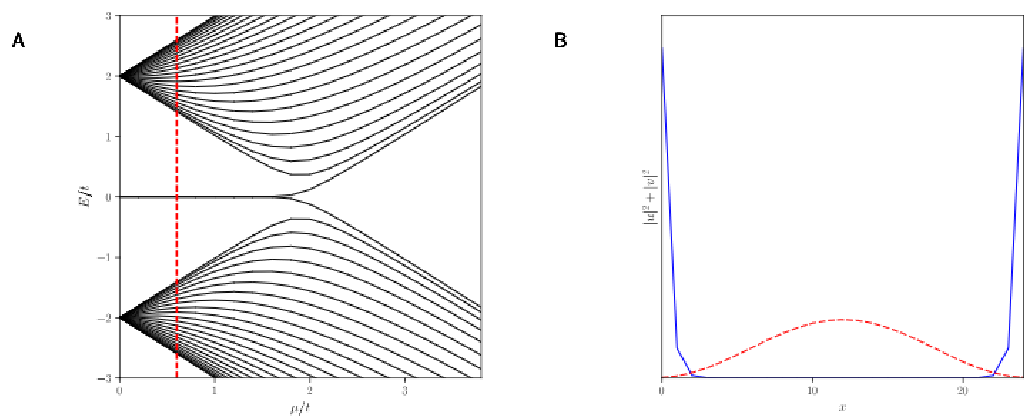
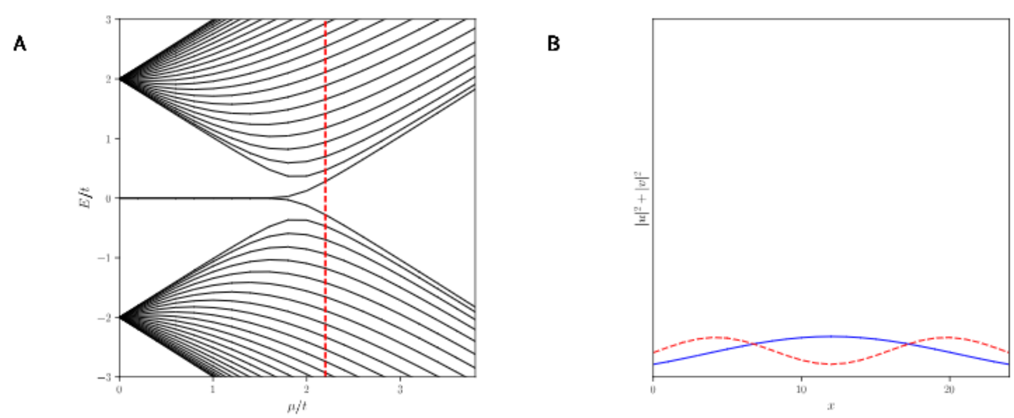
5. Majoranas in topological superconductors-b

Topological protection of Majorana modes

Let's start from the situation with unpaired Majorana modes ($\Delta = t, \mu = 0$) and then increase μ. Then let's plot the energy spectrum of a chain with N=25 sites as a function of μ, and also keep track how do the two lowest energy states of our system look like, when we change μ:



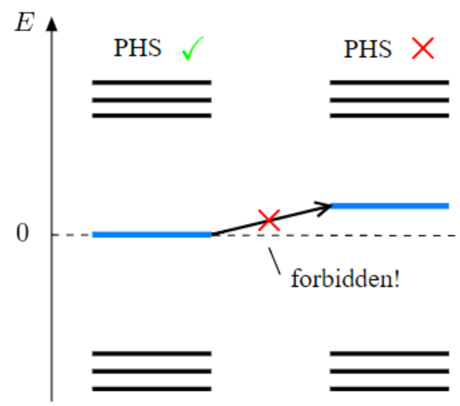




The left panel shows the spectrum, where we see two states at zero energy that split. On the right panel the blue line is the wave function of the state corresponding to the pair of Majorana modes, while the red dashed line is the wave function of the first excited state.

As you can see, the zero energy eigenvalues corresponding to the two unpaired Majorana zero modes persist for a long time, and they only split in energy when $\mu \cong 2t$. Another thing that we observe is that the wave function of the Majoranas stays zero in the middle of our wire. As we increase μ, the wave function of the Majoranas becomes less localized near the edges of the wire, but the coupling between two ends only appears later. You would observe a similar behavior if you varied μ in the negative direction starting from $\mu = 0$. The Majoranas persist until $\mu \cong -2t$, where the bulk gap closes.

In fact, the Majoranas only split when the higher-energy states in the bulk, originally separated by an energy gap of $2t$, come very close to zero energy. So our investigation shows that the Majorana modes are protected as long as the bulk energy gap is finite.



How can we understand this? Recall that we are dealing with a particle-hole symmetric Hamiltonian. Hence, the spectrum has to be symmetric around zero energy. When $\mu = 0$, we have two zero energy levels, corresponding to the Majorana modes which are localized far away from each other and separated by a gapped medium. Trying to move these levels from zero energy individually is impossible, as it would violate particle-hole symmetry as shown in the right figure.

The only possibility to move the energy levels from zero is to couple the two unpaired Majorana modes to each other. However, because of the spatial separation between Majoranas and of the presence of an energy gap, this coupling is impossible. The only way to split the Majorana modes in energy is to first close the bulk energy gap, and that is exactly what happens at large values of μ (to be precise, it happens at $\mu = 2t$).

So we have just learned the following: Isolated zero end-modes at each end in the Kitaev chain are protected by symmetry between positive and negative energy, and by the absence of zero-energy excitations in the bulk of the wire, but not by fine-tuning of the chain parameters.

As you see, our conclusion sounds a lot like what we learned about topology just before. We have come to these conclusions by studying a Kitaev chain on an open geometry with boundaries, and by focusing on the presence or absence of edge states localized at the boundaries of the chain.

Realization of Kitaev model

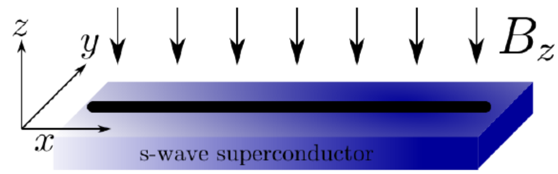
We are now all set to make Majoranas in a real system. We are going to invent a way to make Majoranas in a real system. The way we approach this problem is by considering the Kitaev chain a 'skeleton', and 'dressing' it with real physics phenomena until it becomes real.

Here is our 'skeleton', the Kitaev model Hamiltonian written in momentum space:

$$H_{\text{Kitaev}} = (-2t\cos k - \mu)\tau_z + 2\Delta\tau_y \sin k$$

The model seems OK for a start, because it has some superconducting pairing Δ and some normal dispersion given by terms proportional to μ and t.

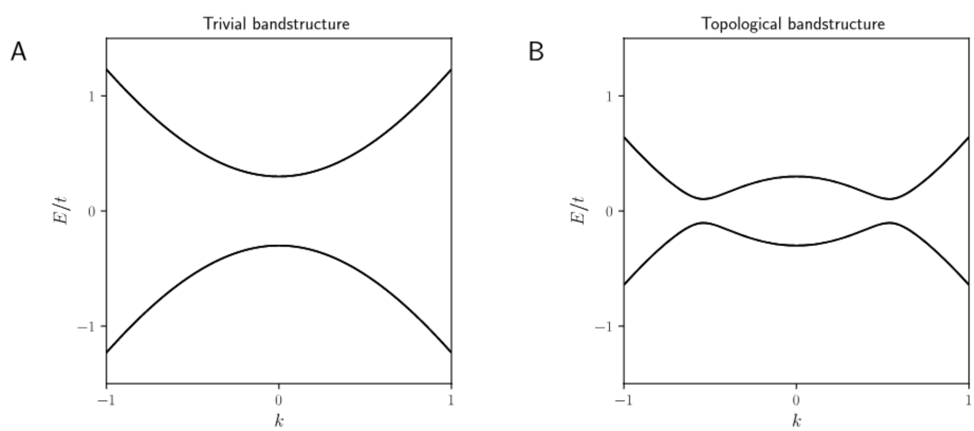
Before we proceed further, let's understand the relation between these parameters. First of all, we want to make a controllable system, so that we can tweak its parameters. That means that we need a semiconductor. In semiconductors the electron density is very low, so that the chemical potential is near the bottom of the band. This makes it easier to define μ with respect to the bottom of the band:



$$\mu \to \mu - 2t$$

Now the transition between trivial and non-trivial states happens when $\mu = 0$. Of course semiconductors are never additionally superconducting. We just paste a superconductor and semiconductor together into a hybrid structure, and let the superconductor induce superconductivity in the semiconductor.

The next thing we should consider is that μ will always stay small compared to the bandwidth, so $\mu \ll 2t$. The same holds for superconducting pairing: $\Delta \ll t$. This is because superconductivity is a very weak effect compared to the kinetic energy of electrons. These two inequalities combined mean that we can expand the $\cos k$ term and only work with the continuum limit of the Kitaev model:



$$H = \left(k^2/2m - \mu\right)\tau_z + 2\Delta\tau_y k$$

The effective electron mass m is just the coefficient of the expansion. Let's take a look at the band structure in this regime, both in the topological regime and in the trivial regime, as shown in the above figure.

The need for spin

Still, there is one obvious thing missing from the model, namely electron spin. This model works with some hypothetical spinless fermions, that do not really exist. So to make the model physical, we need to remember that every single particle has spin, and the Hamiltonian has some action in spin space, described by the Pauli matrices σ.

The simplest thing which we can do is to just add the spin as an extra degeneracy that is to multiply every term in the Hamiltonian by σ. Obviously this doesn't change the spectrum, and a zero energy solution stays a zero energy solution. However, the problem about adding spin is that the whole point of a Kitaev chain is to create unpaired Majorana modes. If we add an extra spin degeneracy to these Majoranas, the edge of our chain will host two Majoranas.

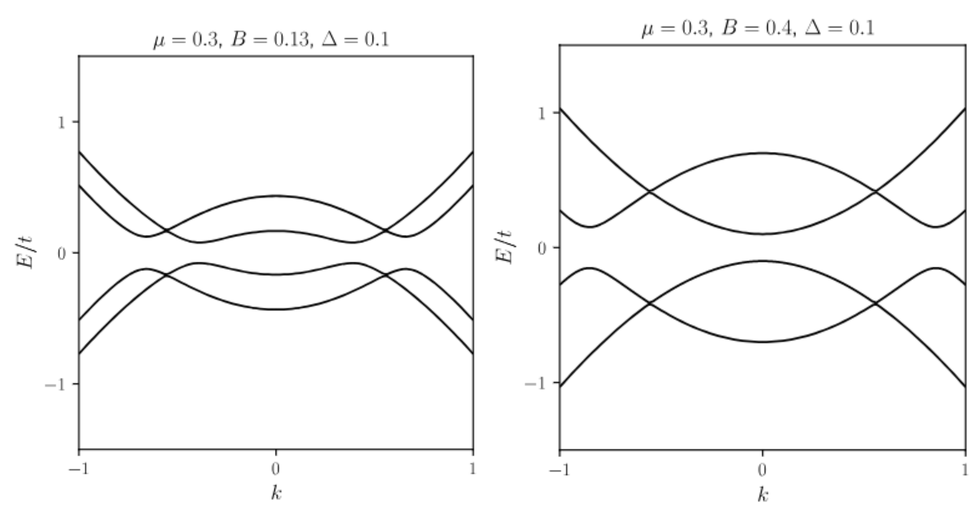
Let's add spin such that the Kitaev chain corresponding to one spin species is topologically trivial, and the Kitaev chain corresponding to the other spin species non-trivial. We know that the chemical potential μ controls whether a Kitaev chain is topological or trivial, so if say spin up has $\mu > 0$ and spin down $\mu < 0$, we're back in business.

We achieve this by adding Zeeman coupling of the spin to an external magnetic field:

$$H = \left(\frac{k^2}{2m} - \mu - B\sigma_z\right)\tau_z + 2\Delta\tau_y k$$

Whenever the Zeeman energy $|B|$ is larger than μ we have one Majorana fermion at the end of the chain.

Let's look at what happens with the dispersion as we increase the magnetic field from zero to a value larger than μ.



Realistic superconducting pairing

The next part for us to worry about is the superconductor.

Something that you probably saw in the Kitaev chain Hamiltonian is that the superconducting pairing Δ has a peculiar form. It pairs electrons from neighboring sites, and not those from the same site. In momentum space this means that the superconducting pairing is proportional to Δk.

Of course, in a Kitaev chain the superconducting pairing cannot couple two electrons from the same site since there is just one particle per site! Real world superconductors are different. Most of them, and specifically all the common superconductors like Al, Nb, Pb, Sn have *s*-wave pairing. This means that the pairing has no momentum dependence, and is local in real space. The Kitaev chain pairing is proportional to the first power of momentum and so it is a *p*-wave pairing.

All the *s*-wave superconductors are spin-singlet:

$$H_{\text{pair}} = \Delta(c_\uparrow c_\downarrow - c_\downarrow c_\uparrow) + \text{h.c.}$$

This means that now we need to modify the pairing, but before that we'll need to do one other important thing.

Important basis change

When you see Bogoliubov-de-Gennes Hamiltonians in the above content, you will find them written in two different bases. One variant is the one which we introduced before:

$$H_{\text{BdG}} = \begin{pmatrix} H & \Delta \\ -\Delta^* & -H^* \end{pmatrix}$$

It has the particle-hole symmetry $H_{\text{BdG}} = -\tau_x H_{\text{BdG}}^* \tau_x$. In this basis, the *s*-wave pairing is proportional to σ_y. However for systems with complicated spin and orbital structure, there is a different basis which makes the work much easier.

If we have a time-reversal symmetry operator $\mathcal{T} = UK$, we can apply the unitary transformation U to the holes, so that in the new basis we get the BdG Hamiltonian

$$H_{\text{BdG}} = \begin{pmatrix} H & \Delta' \\ \Delta'^\dagger & -\mathcal{T}H\mathcal{T}^{-1} \end{pmatrix} \text{ with } \Delta' = \Delta U^\dagger$$

1. Because in this new basis the *s*-wave pairing is a unit matrix regardless of system we consider.
2. Because it's easy to get the Hamiltonian of holes. We take the Hamiltonian for electrons, and change the signs of all terms that respect time-reversal symmetry, but not for those that break it, such as the term proportional to the magnetic field B. So if the electrons have a Hamiltonian $H(B)$, the Hamiltonian of the holes just becomes $-H(-B)$.

There is one disadvantage. The particle-hole symmetry now becomes more complicated. For our system with only one orbital and spin, it is $\mathcal{P} = \sigma_y \tau_y K$. But, the advantages are worth it.

s-wave superconductor with magnetic field

Let's look at how our chain looks once we change the superconducting coupling to be s-wave. The Zeeman field (or anything of magnetic origin) changes sign under time-reversal symmetry. This means that the Zeeman field has the same form for electrons and for holes in the new basis, and the full Hamiltonian is now:

$$H_{\mathrm{BdG}} = \left(k^2/2m - \mu\right)\tau_z + B\sigma_z + \Delta\tau_x$$

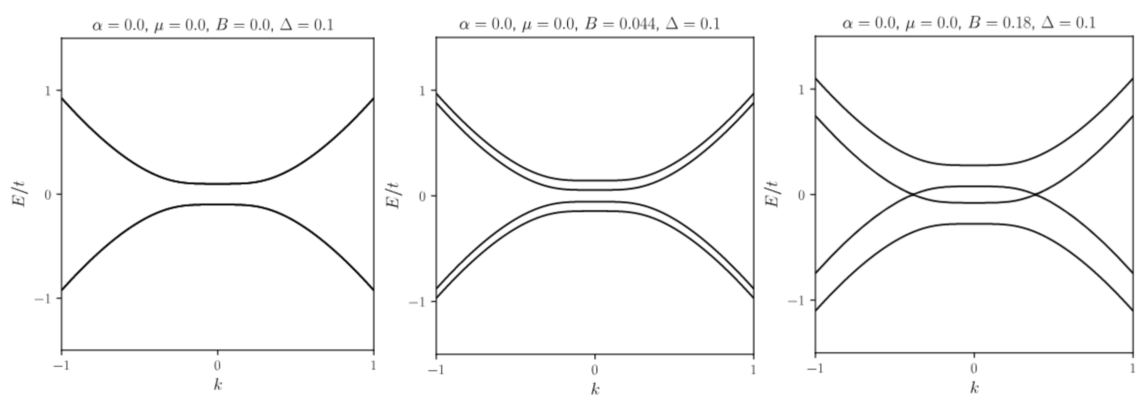
This Hamiltonian is easy to diagonalize since every term only has either a τ matrix or a σ matrix. At $k = 0$ it has 4 levels with energies $E = \pm B \pm \sqrt{\mu^2 + \Delta^2}$. We can use this expression to track the crossings. We also know that when $B = 0$ the system is trivial due to spin degeneracy. Together this means that we expect the system to be non-trivial (and will have a negative Pfaffian invariant) when

$$B^2 > \Delta^2 + \mu^2$$

Problem with singlets

A singlet superconductor has an important property: Since electrons are created in singlets, the total spin of every excitation is conserved. Zeeman field conserves the spin in z-direction, so together every single state of our system has to have a definite spin, including the Majoranas.

And that is a big problem. Majoranas are their own particle-hole partners, and that means that they cannot have any spin (energy, charge, or any other observable property at all). Let's look at the band structure.



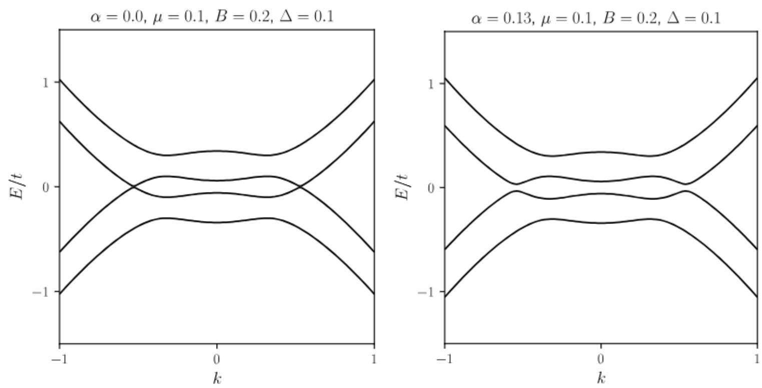
1.1.1 How to open the gap?

We know that there is no gap because of conservation of one of the spin projections, so we need to break the spin conservation. If we don't want to create an inhomogeneous magnetic field, we have to use a different term that couples to spin. That term is spin-orbit interaction. In its simplest form this interaction appears in our wire as

$$H_{\mathrm{so}} = \lambda_{SO}\sigma_y k$$

It is like a Zeeman field pointing in y-direction with a strength proportional to the particle momentum. Note that this term is invariant under time reversal symmetry (both σ_y and k change sign). So now we have our final Hamiltonian:

$$H_{\text{wire}} = \left(\frac{k^2}{2m} + \lambda_{SO}\sigma_y k - \mu\right)\tau_z + B\sigma_z + \Delta\tau_x$$



At $k = 0$, spin-orbit coupling vanishes, so it has no effect on the system being topologically trivial or non-trivial. Let's now check if this can open the gap at a finite momentum, as shown in the right figure. Indeed it does open a gap.

An important remark: You might now think that since spin-orbit interaction depends on spin, it makes the magnetic field unnecessary. This is not true: Since spin-orbit interaction preserves time-reversal symmetry, in the absence of a magnetic field the energy spectrum of the model would have a Kramers degeneracy, as you learned in the last chapter. To get one unpaired Majorana mode per edge and not two edges, we need to break Kramers degeneracy and therefore break time-reversal symmetry. So the combination of both Zeeman field and spin-orbit coupling is needed.

Let's now rest for a moment and reflect on what we have done.

We started from a toy model, which has a very special feature. Despite the model being very simple, it has four independent parameters already in our simplest formulation. Let's enumerate the parameters once again:

1. The chemical potential μ, which sets the overall electron density in the wire.
2. The induced superconducting gap Δ, which is required to make particle-hole symmetry play a role.
3. The spin-orbit coupling α, which breaks spin conservation.
4. The Zeeman field **B**, which breaks Kramers degeneracy.

Let's summarize the entire process to obtain a Majorana wire:

We present a realization of a 1D semiconductor nanowire coupled to an s-wave superconductor, thus, with an induced superconductivity. Rashba spin–orbit coupling, leading to an effective magnetic field $B_{SO} \propto k \times E$ (where k is the momentum along the wire and E is the electric field perpendicular to the wire), separates electrons with opposite spins in momentum space. Applying a magnetic field perpendicular to B_{SO} will mix the two spin bands, forming two pseudo-spin bands, Zeeman gapped by $2E_z$ at $k = 0$. Inducing superconductivity modifies the Zeeman gap at $k = 0$ and opens up a gap at the Fermi momentum k_F. The overall gap $\mathrm{E_g}$ is the smaller of these two gaps.

Three parameters are of significance: the spin–orbit energy $\Delta_{SO}= k_{SO}^2/2m$, with $\pm k_{SO} = \pm\hbar/\lambda_{SO}$; the Zeeman gap $2E_z = g\mu_B B$, where g is the Landé g-factor, μ_B is the Bohr magnetron and B is the external magnetic field; and the induced superconducting gap in the nanowire $2\Delta_{ind}$(the Al superconducting gap is $2\Delta_{Al}$).

The spectrum for constant μ, λ_{SO}, Δ, and B, is revealed by squaring H_{wire} twice, which yields

$$E_{\pm}^2 = B^2 + \Delta^2 + \xi_k^2 + (\lambda_{SO}k)^2 \pm 2\sqrt{B^2\Delta^2 + B^2\xi_k^2 + (\lambda_{SO}k)^2\xi_k^2}$$

where $\xi_k = \frac{k^2}{2m} - \mu$. A linear vanishing and reopening gap when μ, Δ, and B vary indicates a topological phase transition. The $k = 0$ gap, $E_{g,0}$, is the key to the emergence of the Majorana states. Examining E_- at $k = 0$ we notice that

$$E_{g,0} = E(k = 0) = \left|B - \sqrt{\Delta^2 + \mu^2}\right|$$

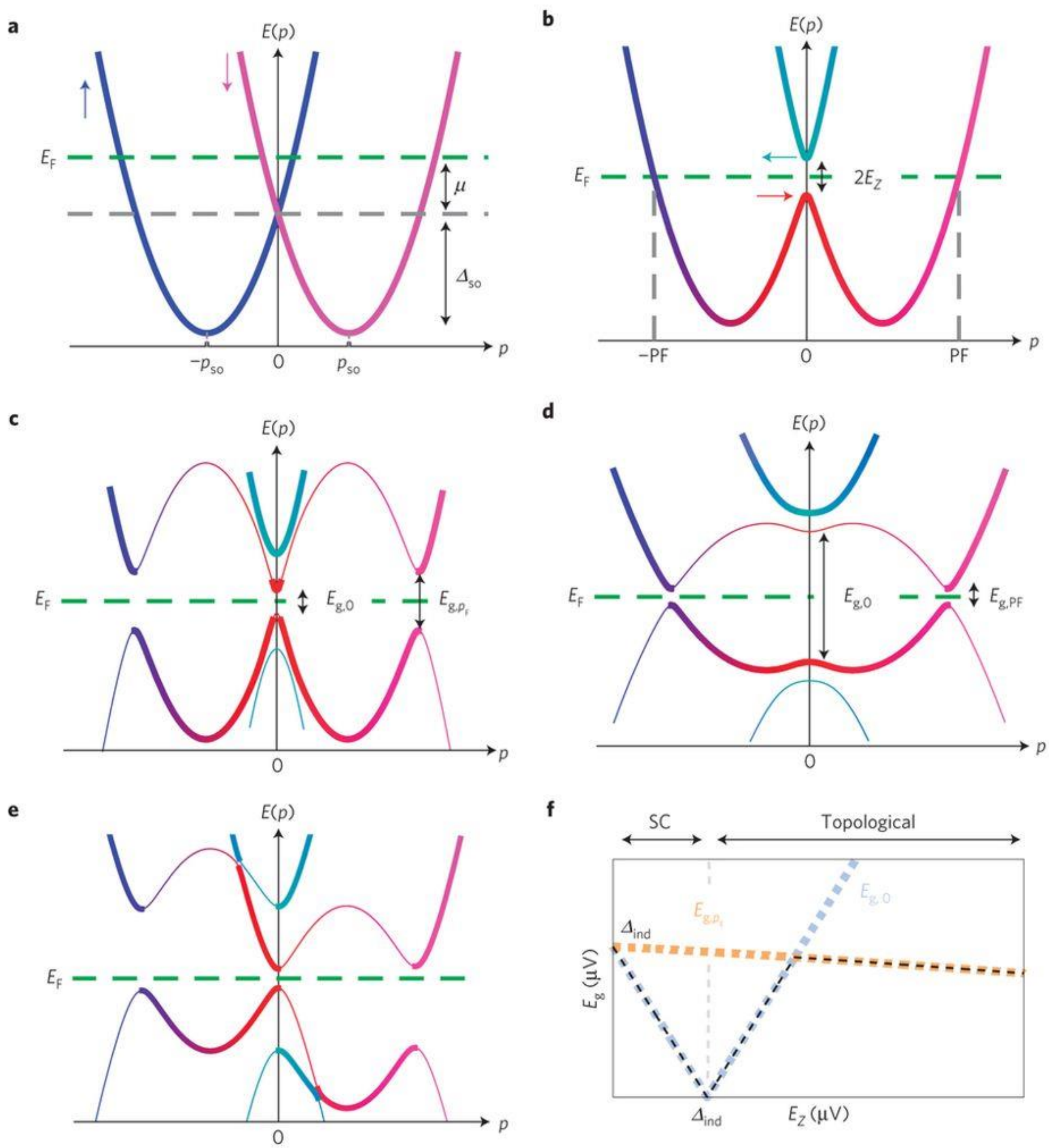
- For $B^2 > \Delta^2 + \mu^2$, $E_{g,0}$ is a B- dominated (or strong interaction induced) gap; the wire is in its topological phase, with Majorana states at the wire's ends;
- When $B^2 < \Delta^2 + \mu^2$ the gap is pairing dominated, with no end states;
- When $B^2 = \Delta^2 + \mu^2$ a quantum phase transition occurs.

The gap $E_{g,k_F} = E(k = k_F)$ near $k = \pm\sqrt{2\mu m}$ remains a finite pairing gap throughout, since Δ always stays finite.

The phase transition evident in $E_{g,0}$ allows Majorana states to form. These can be achieved in various ways since $E_{g,0}$ depends on μ, Δ, and B. As in 2D topological insulator edges, a MF bound state forms when B changes spatially and crosses Δ, e.g., at $y = 0$, or when Δ varies and crosses B. Here we emphasize a third possibility: varying the chemical potential μ. E.g., consider $B > \Delta$ so that for $\mu = 0$, we have a B-dominated gap $E_{g,0}$. But when $\mu > \sqrt{B^2 - \Delta^2}$, the gap $E_{g,0}$, is pairing dominated. Thus, we can form a Majorana state by tuning μ between these values.

For $2\Delta_{ind}> 0$ and $E_z = 0$, the wire is a trivial superconductor with a gapped spectrum. When E_z is increased to $E_z = \sqrt{\Delta_{ind}^2 + \mu^2}$, where μ is the chemical potential, the gap at $k = 0$ closes at the Fermi energy, and the wire enters the topological phase; with a topological gap reopening with a further increase in E_z. Continuously changing the parameters along the wire from its topological phase into another gapped phase must close the gap at the phase transition point, forming a Majorana state. Another quasi-particle state is formed at the other end of the topological segment.

The required condition for a topological superconductor is therefore $E_z > \sqrt{\Delta_{ind}^2 + \mu^2}$ (or equivalently $E_z > \Delta_{ind}$ and $|\mu| < \sqrt{E_z^2 - \Delta_{ind}^2}$. For Zeeman energies close to the topological phase transition, the smaller gap is at $k = 0$: $E_g = 2\left|E_z - \sqrt{\Delta_{ind}^2 + \mu^2}\right|$ (Fig. 1f). For larger Zeeman energies, the gap at k_F is the smaller of the two, which for $E_z \gg \Delta_{SO}, \Delta_{ind}, |\mu|$ is given by $E_g = 4\Delta_{ind}\sqrt{\Delta_{SO}/E_z}$ (Fig. 1f). In this regime the gap decreases with E_z, and even more so owing to the diminishing of the superconducting gap with field. The Majoranas, with energy ε pinned to the Fermi energy, should be robust for a range of system parameters that keep $|\mu| < \sqrt{E_z^2 - \Delta_{ind}^2}$. The immediate consequence of the Majorana is a sharp enhancement in the tunnelling density of states and a $2e^2/h$ differential conductance at zero applied bias and $T = 0$.



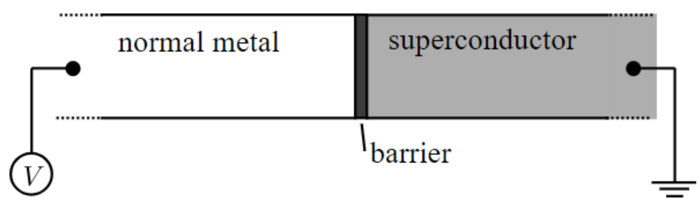
Heavy (light) lines show electron-like (hole-like) bands. Opposite spin directions are denoted in blue and magenta for the spin–orbit effective field direction; red and cyan for the spins in the perpendicular direction; relative mixture denotes intermediate spin directions. **a**, Split electronic spin bands due to spin–orbit coupling. The spin–orbit energy is denoted as Δ_{SO}; the chemical potential is μ, with respect to spin bands crossing at $k = 0$. **b**, With applied magnetic field $B \perp B_{SO}$, leading to a Zeeman gap $E_z = \frac{1}{2} g\mu_B B$ at $k = 0$. **c**, Bringing a superconductor into close proximity opens a superconducting gap at the crossing of particle and hole curves (light lines). **d**, The same as in **c** but for a larger E_z with the gap at k_F dominant. **e**, Field rotated to a direction of 30° with respect to B_{SO}, leading to shifts of the original spin–orbit bands. **f**, The evolution of the energy gap at $k = 0$ (dotted blue), at k_F (dotted yellow), and the overall energy gap (dashed black) with Zeeman energy, E_z, for $\mu = 0$.

We need to control every single parameter out of these 4 to create Majoranas (and there are even more). This is why the task of creating Majoranas is extremely challenging.

How to detect Majoranas

Andreev reflection off a Majorana zero mode

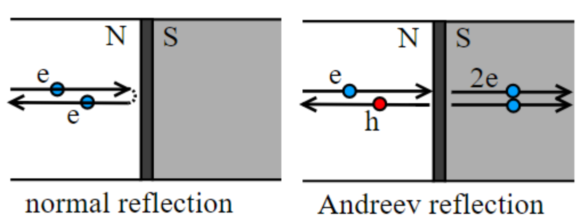
To understand how conductance through a Majorana works, we first have to learn how charge is transferred from a metallic lead to a superconductor. In general this transfer takes place via a mechanism known as Andreev reflection. Before we discuss the conductance signatures of a Majorana zero mode, it is useful to learn what Andreev reflection is.



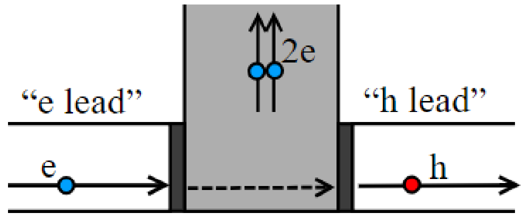
Let's consider the following very simple circuit with two electrodes as shown in the right figure. One electrode is a normal metal, the other a superconductor, and they are kept at a voltage difference *V*. At the interface between the normal metal and the superconductor (in short, NS interface) there is a barrier. We are particularly interested in the case when the voltage difference is very small compared to the energy gap in the superconductor, $eV < \Delta$, where e is the charge of the electron.

What happens when electrons arrive at the interface with superconductor? The superconductor has no states available up to an energy Δ around the Fermi level, and the voltage is not enough to provide for this energy difference. How can a current develop?

To understand this, let's look more closely at an electron arriving at the interface with the superconductor. There are two possible processes that can take place, normal reflection and Andreev reflection. In normal reflection, the electron is simply reflected at the interface with the superconductor. With normal reflection, there is no net charge transfer from the left electrode to the right electrode. Hence this process does not contribute any net current. Normal reflection obviously doesn't even require a superconductor and would take place also if the right electrode was normal.



Andreev reflection, instead, is unique to the NS interface. In Andreev reflection, an electron is converted to a hole by the superconductor, and a Cooper pair is created in the superconductor. You can see that a net charge of $2e$ is transferred from the left to the right electrode, and at low voltages Andreev reflection is the only process responsible for the electrical current. Above the superconducting gap, $eV > \Delta$ transmission of an incident electron into the superconductor also contributes to the current.

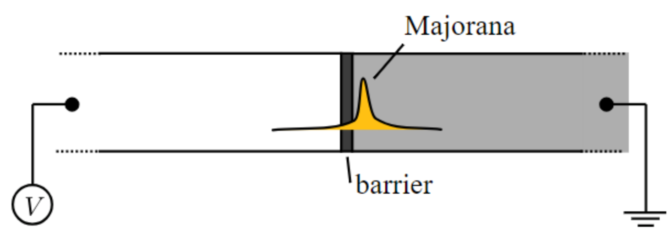


You can also think of Andreev reflection as a transmission problem. Because of the presence of the superconductor, both electrons and holes participate in the transfer of charge in the normal metal lead. Conceptually, you can imagine to separate the left lead into two leads, one only carrying electrons and one only carrying holes. These two leads are connected by the superconductor, which converts incoming electrons in the first lead into outgoing holes in the second lead, and vice versa.

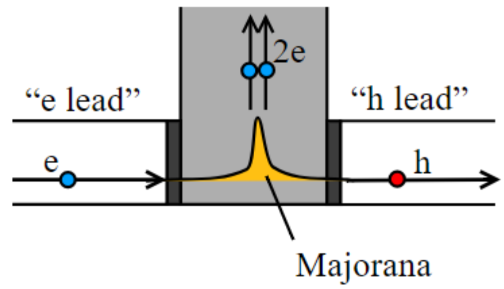
With this picture, you can understand that Andreev reflection is very similar to the problem of transmission through a double barrier. Let's call r_{eh} the amplitude for Andreev reflection. Its absolute value squared, $|r_{eh}|^2$, is the probability that an incoming electron from the normal metal is Andreev reflected as a hole. Once we know r_{eh}, we can compute the conductance $G(V)$, which relates the current I that develops as a response to a small voltage V, $G(V) = dI/dV$. The conductance is given by the following formula:

$$G(V) = 2G_0|r_{eh}|^2$$

We will not derive this equation, since it can be understood intuitively. The conductance is proportional to the probability $|r_{eh}|^2$ of Andreev reflection, since we know that at low voltages this is the only process that transfers electric charge from the left to the right electrode. The factor of 2 is due to each Andreev reflection transferring a charge of a Cooper pair, $2e$. Finally, $G_0 = e^2/h$ is the conductance quantum, the fundamental proportionality constant which relates currents to voltages.

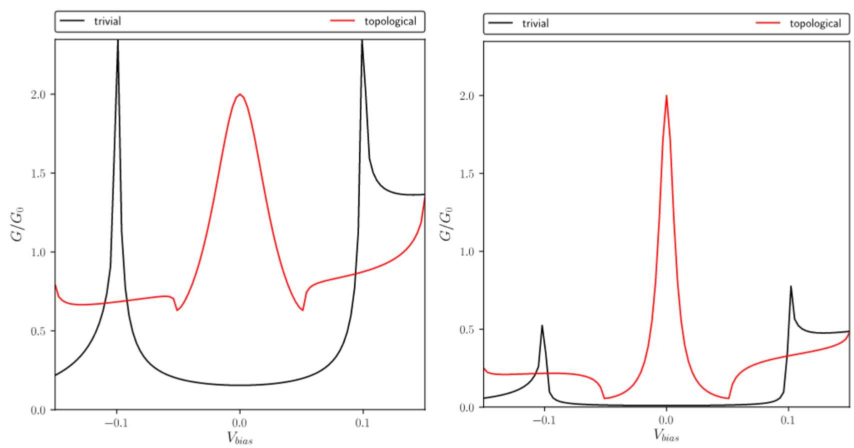


Now that we understand a conventional NS interface, let's see what happens if our superconductor is topological as shown in the right figure. You can imagine that the superconducting electrode is now a nanowire in the topological phase, like the one you have just studied. Because the superconductor is topological, there is a Majorana mode at the NS interface. Of course, there will be also a second Majorana mode, but we place it far enough from the NS interface, so that it does not have a role in the transport. Does the Majorana zero mode at the interface change the Andreev reflection properties?



Yes. Going back to the picture of Andreev reflection as a transmission process through a double barrier, the crucial difference is that the Majorana mode now appears as a bound state between the two barriers as shown in the right figure. In the double barrier problem in quantum mechanics, you can have resonant transmission in the presence of a bound state. This means that the probability $|r_{eh}|^2$ to pass through the barriers is dramatically enhanced if the energy of the incident electron matches the energy of the bound state. In our case, the energy of the incident electron is eV, and the energy of the bound state, the Majorana mode, is zero. So the presence of the Majorana mode leads to a resonant peak in the conductance of the NS interface at V=0.

Majorana resonance



Seeing the resonant peak is the most direct way we know to measure the presence of a Majorana zero mode. However, the presence of a resonance associated with Majorana modes is not uniquely topological, because tunneling into any low energy bound state produces resonance.

Is there anything which distinguishes the Majorana resonance from any other resonance?

Let's just look at what happens if we compare conductance of an NS interface in the cases when S is trivial and non-trivial, and see how the conductance changes as we alter the tunnel barrier strength. We see a very robust and persistent characteristic: the peak height of the Majorana resonance is quantized to the value of $2G_0$, independent of the strength of the voltage barrier. From the formula above, this means that if a Majorana is present we have $|r_{eh}|^2 = 1$, that is we have perfect Andreev reflection.

Reflection matrix of a normal metal-superconductor interface

Quantum-mechanically, we can describe transport through the NS interface as a scattering problem. An incoming wave function Ψ_{in} propagates in the left electrode, until it is reflected back at the interface with the superconductor, turning into an outgoing wave function Ψ_{out}. Because of the presence of the superconductor, both the incoming and outgoing states can be electrons Ψ_e or holes Ψ_h. At zero energy, they are related to each other by particle-hole symmetry:

$$\Psi_e(E) = \mathcal{P}\Psi_h(-E)$$

The reflection enforces a linear relation between incoming and ingoing waves:

$$\Psi_{\text{out}} = r(V)\Psi_{\text{in}}$$

$$r(V) = \begin{pmatrix} r_{ee} & r_{eh} \\ r_{he} & r_{hh} \end{pmatrix}$$

The matrix r is known as the reflection matrix. Its complex elements are the amplitudes of normal and Andreev reflection of an incoming electron (r_{ee} and r_{eh}) and normal and Andreev reflection of an incoming hole (r_{hh} and r_{he}). (For brevity we don't write out explicitly that each of those depends on *V*.)

If there is more than one incoming electron state (in our case there are two due to spin), all 4 elements of r become matrices themselves. They then describe scattering between all the possible incoming and outgoing states.

Because for $eV \ll \Delta$ there are no propagating waves in the superconductor, the reflection process which relates Ψ_{out} and Ψ_{in} is unitary, $r^\dagger r = 1$. This implies that

$$|r_{ee}|^2 + |r_{eh}|^2 = |r_{he}|^2 + |r_{hh}|^2 = 1$$

This is the mathematical way of saying that an electron (or hole) arriving at the interface has no alternatives other than being normal-reflected or Andreev-reflected. Can we add any other constraint to r, that might help to distinguish any characteristic of the Majorana mode? Just like we did before, let's try to study r using symmetry and topology. Our circuit involves a superconductor, so we must have particle-hole symmetry in the problem. In order to derive r explicitly we could start directly from the Bogoliubov-de Gennes Hamiltonian of the NS system, and solve it for an energy *V*. This is a lot of work, which we won't do, but knowing this fact we can understand the consequences of particle-hole symmetry for r.

First, particle-hole symmetry exchanges electrons and hole components of the wave function, so it involves a Pauli matrix τ_x acting on Ψ_{in} and Ψ_{out}. Second, it is an anti-unitary symmetry, so it involves complex conjugation. Third, it changes the sign of the energy so it sends *V* into −*V*. Hence we arrive at the following symmetry for the reflection matrix:

$$\tau_x r^*(-V)\tau_x = r(V)$$

Together with unitarity, particle-hole symmetry imposes that the conductance is symmetric around zero voltage, *G*(*V*)=*G*(−*V*). In the most interesting point, *V*=0 we have:

$$\tau_x r_0^* \tau_x = r_0$$

where we defined $r_0 = r(V = 0)$. So much for the impact of symmetry on r.

What about topology?

Topological invariant of the reflection matrix

The Majorana zero mode is the consequence of a topological phase in the topological superconductor, and its presence is dictated by the bulk-boundary correspondence. Can we find any consequence of this fact in r_0? It turns out that reflection matrices r with particle-hole symmetry are also topological in their own way. Their topological invariant is

$$Q = \det r_0$$

Again, we will not derive this equation, but rather convince ourselves this expression is correct.

First of all, the determinant of a unitary matrix such as r_0 is always a complex number with unit norm, so $|\det r_0| = 1$. Second, because of particle-hole symmetry, the determinant is real: $\det r_0 = \det \tau_x r_0^* \tau_x = \det r_0^* = (\det r_0)^*$. Hence, $\det r_0 = \pm 1$.

This is quite promising! Two possible discrete values, just like the Pfaffian invariant of the Kitaev chain. Because it is just dictated by unitarity and particle-hole symmetry, the determinant of r_0 cannot change from +1 to −1 under a change of the properties of the NS interface. For instance, you can vary the height of the potential barrier at the interface, but this cannot affect the determinant of r_0. This corresponds to the following two cases, making $\det r_0$ a topological index: (i) Reflection from the trivial phase with perfect normal reflection $|r_{ee}| = 1$ and zero Andreev reflection $r_{eh} = 0$, corresponding to $\det r_0 = 1$, and (ii) reflection from the topological phase with perfect Andreev reflection $|r_{eh}| = 1$ and zero normal reflection $r_{ee} = 0$.

The only way to make the determinant change sign is to close the bulk gap in the superconducting electrode. If the gap goes to zero, then it is not true that an incoming electron coming from the normal metal can only be normal-reflected or Andreev-reflected. It can also just enter the superconducting electrode as an electron. Hence the reflection matrix no longer contains all the possible processes taking place at the interface, and it won't be unitary anymore. This allows the determinant to change sign. We conclude that $Q = \det r$ is a good topological invariant.

Explicitly, we have that

$$Q = |r_{ee}|^2 - |r_{eh}|^2 = \pm 1$$

We already saw that unitarity requires that $|r_{ee}|^2 + |r_{eh}|^2 = 1$. There are only two possibilities for both conditions to be true: either $|r_{ee}| = 1$ (perfect normal reflection) or $|r_{eh}| = 1$ (perfect Andreev reflection). The situation cannot change without a phase transition. Thus the quantized conductance of the Majorana mode is topologically robust in this case, and in fact survives past the tunneling limit.