MAT137

Yue Cheng

October 22, 2023

1 Limit and Continuity

Let $a, L \in \mathbb{R}$.
Let f be a function defined at least on an interval centered at a, except maybe at a.

Theorem 1 ($\lim_{x \to a} f(x) = L$)

$$\forall \varepsilon > 0, \exists \delta > 0 \ s.t.$$
$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$

Theorem 2 ($\lim_{x \to a} f(x) \neq L$)

$$\exists \varepsilon > 0 \ s.t. \ \forall \delta > 0$$
$$\exists x \in \mathbb{R} \ s.t. \ 0 < |x - a| < \delta \wedge |f(x) - L| \geq \varepsilon$$

Theorem 3 ($\lim_{x \to a} f(x)$ **DNE**)

$$\forall L \in \mathbb{R}, \exists \varepsilon > 0 \ s.t. \ \forall \delta > 0$$
$$\exists x \in \mathbb{R} \ s.t. \ 0 < |x - a| < \delta \wedge |f(x) - L| \geq \varepsilon$$

The formal definitons can be transferred to imply infinity, by removing absolute values.

1.1 Laws and Theorems

Let $a, L \in \mathbb{R}$.
Let f, g, h be functions defined at least on an interval centered at a, except maybe at a.

Theorem 4 (Limit Laws)

$$\begin{array}{rl} \text{IF} & \lim_{x\to a} f(x) = L \\ & \lim_{x\to a} g(x) = M \\ \text{THEN} & \lim_{x\to a}[f(x) + g(x)] = L + M \\ & \lim_{x\to a}[f(x)g(x)] = LM \\ & \lim_{x\to a}[\frac{f(x)}{g(x)}] = \frac{L}{M} \ (M \neq 0) \end{array}$$

WARNING: Limit Laws only apply if the initial limits exist.

Theorem 5 (The Squeeze Theorem)

$$\begin{array}{rl} \text{IF} & \exists p > 0 \ s.t. \\ & 0 < |x-a| < p \implies f(x) \leq h(x) \leq g(x) \\ \text{AND} & \lim_{x\to a} f(x) = L \\ & \lim_{x\to a} g(x) = L \\ \text{THEN} & \lim_{x\to a} h(x) = L \end{array}$$

1.2 Continuity

Let $a, L \in \mathbb{R}$.
Let f be a function defined at least on an interval centered at a, except maybe at a.
WARNING: If the function oscillates near a, then it is NOT continuous at a.

Theorem 6 (Continuity)
We say f is continuous at a when

$$\lim_{x \to a} f(x) = f(a)$$

This means:

1. $\lim_{x \to a} f(x)$ exists and is a number
2. $f(a)$ is defined
3. The limit and the value is equal

Theorem 7 (Alternative Definition of Continuity)
We say f is continuous at a when

$$\forall \varepsilon > 0, \exists \delta > 0 \ s.t.$$

$$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$$

$\lim_{x \to a} f(x) = L$ means that:

$$\left\{ \begin{array}{l} x \text{ close to } a \\ x \neq a \end{array} \right\} \implies f(x) \text{ close to } L$$

f is continuous at a means that:

$$\left\{ \ x \text{ close to } a \ \right\} \implies f(x) \text{ close to } f(a)$$

f is continuous on the interval $[a, b]$ means that:

1. $\forall c \in (a, b)$, f is continuous at c
2. $\lim_{x \to a^+} f(x) = f(a)$
3. $\lim_{x \to b^-} f(x) = f(b)$

Theorem 8 (The Main Continuity Theorem)
Any function we can construct with sum, product, quotient, and **composition** of polynomials, roots, trigonometric functions, exponentials, logarithms, and absolute values is continuous **on its domain**.

WARNING: There is NO limit laws for composition, but the composition of continuous functions is still continuous.

1.3 About Composition of Functions

Theorem 9 (RIGHT)

$$\begin{array}{rll} \text{IF} & (1) & f \text{ is continuous at } a \\ & (2) & g \text{ is continuous at } f(a) \\ \text{THEN} & (3) & g \circ f \text{ is continuous at } a \end{array}$$

$$\begin{array}{ll} (1) & \text{means } x \text{ close to } a \implies f(x) \text{ close to } f(a) \\ (2) & \text{means } y \text{ close to } f(a) \implies g(y) \text{ close to } g(f(a)) \end{array}$$

They can be concatenated.

Theorem 10 (WRONG)

$$\begin{array}{rll} \text{IF} & (1) & \lim\limits_{x \to a} f(x) = L \\ & (2) & \lim\limits_{y \to L} g(y) = M \\ \text{THEN} & (3) & \lim\limits_{x \to a} g(f(x)) = M \end{array}$$

$$\begin{array}{ll} (1) & \text{means} \left\{ \begin{array}{l} x \text{ close to } a \\ x \neq a \end{array} \right\} \implies f(x) \text{ close to } L \\ (2) & \text{means} \left\{ \begin{array}{l} y \text{ close to } L \\ y \neq L \end{array} \right\} \implies g(y) \text{ close to } M \end{array}$$

They can NOT be concatenated, but can be fixed in two ways, either on f or g.

Theorem 11 (RIGHT)

$$\begin{array}{rll} \text{IF} & (1) & \lim\limits_{x \to a} f(x) = L \\ & & \exists p > 0 \ s.t. \ 0 < |x - a| < p \implies f(x) \neq L \\ & (2) & \lim\limits_{y \to L} g(y) = M \\ \text{THEN} & (3) & \lim\limits_{x \to a} g(f(x)) = M \end{array}$$

By adding a hypothesis on f, we ensure that g has the behavior of approaching.

Theorem 12 (RIGHT)

$$\begin{array}{rll} \text{IF} & (1) & \lim_{x\to a} f(x) = L \\ & (2) & g \text{ is continuous at } L \\ \text{THEN} & (3) & \lim_{x\to a} g(f(x)) = M \end{array}$$

By adding a hypothesis on g, we don't care if g has the behavior of approaching.

1.4 Discontinuity

Theorem 13 (Removable Discontinuity)
$\lim_{x \to a} f(x)$ exists, but is not equal to $f(a)$.

Intuitively, removable discontinuity looks like a hole and can be fixed.

Theorem 14 (Non-removable Discontinuity)
$\lim_{x \to a} f(x)$ does not exist.

1.5 EVT and IVT

Theorem 15 (Maximum)

$$\exists c \in I \text{ } s.t. \text{ } \forall x \in I, f(x) \leq f(c)$$

Theorem 16 (The Extreme Value Theorem)

$$\begin{array}{rl} \text{IF} & f \text{ is continuous on } [a,b] \\ \text{THEN} & f \text{ has a maximum and minimum on } [a,b] \end{array}$$

Theorem 17 (The Intermediate Value Theorem)

$$\begin{array}{rl} \text{IF} & f(a) < M < f(b) \\ & f \text{ is continuous on } [a,b] \\ \text{THEN} & \exists c \in (a,b) \text{ } s.t. \text{ } f(c) = M \end{array}$$

This means that f takes all the values between $f(a)$ and $f(b)$.

2 Derivatives

Theorem 18 (Derivative)

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \frac{dy}{dx}$$

f is differentiable at a when this limit exists.

Theorem 19 (Tangent Line)
1. through the point $(a, f(a))$
2. with slope $f'(a)$

$$y = f(a) + f'(a)(x - a)$$

Theorem 20 (Vertical Tangent Line)
f is continuous at a

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \pm\infty$$

WARNING: The vertical tangent line is NOT vertical asymptote.

2.1 Differentiation

If f, g are both differentiable:

Theorem 21 (Differentiation Rules)

$$\begin{aligned} &\frac{d}{dx}[c] = 0 && \frac{d}{dx}[x^c] = cx^{c-1} \\ &(f+g)' = f' + g' && (cf)' = cf' \\ &(f \cdot g)' = f'g + fg' && (\frac{f}{g})' = \frac{f'g - fg'}{g^2} \end{aligned}$$

Proof (The Product Rule)

$$\begin{aligned} h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} &= \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a} && (1) \\ &= \lim_{x \to a} \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x - a} && (2) \end{aligned}$$

We can prove the Power Rule by induction.

Theorem 22 (Differentiable implies Continuous)
If f is differentiable at c, then f is continuous at c.

We can find functions that are continuous but not differentiable.

The $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ may not exist (the function is not differentiable) for several reasons:

1. because the side limits are different (CORNER)
2. because the limit is $\pm\infty$ (VERTICAL TANGENT)

Theorem 23
If $f(x)$ is continuous at a

$$\lim_{x \to a} f'(x) = \pm\infty \implies \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \pm\infty$$

WARNING: The first method is to use differentiation rules, while the second is to use the definition of derivative.

2.2 Higher-order Derivatives

Theorem 24 (Lagrange Notation)
The n-th derivative of f is

$$f^{(n)}$$

Theorem 25 (Leibnitz Notation)
The n-th derivative of y with respect to x is

$$\frac{d^n y}{dx^n}$$

2.3 The Chain Rule

Theorem 26

$$\begin{array}{rl} \text{IF} & g \text{ is differentiable at } a \\ & f \text{ is differentiable at } g(a) \\ \text{THEN} & f \circ g \text{ is differentiable at } a \end{array}$$

$$(f \circ g)'(a) = f'(g(a))g'(a)$$

PROOF (WRONG)

$$(f \circ g)'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \tag{1}$$

$$= \lim_{x \to a} [\frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}] \tag{2}$$

WARNING: For this to work, we must ensure that:

$$\exists p > 0 \ s.t. \ 0 < |x - a| < p \implies g(x) \neq g(a)$$

PROOF (RIGHT) Define

$$u = g(x)$$

$$y = f(u) = f(g(x))$$

Then

$$(f \circ g)'(a) = f'(g(a))g'(a)$$

can be written as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

2.4 Implicit Differentiation

Consider an equation involving two variables x, y that defines y implicitly:

$$F(x, y) = 0$$

To find the derivative of y with respect to x, denoted as $\frac{dy}{dx}$, we differentiate each side of the equation with respect to x, treating y as a function of x.

$$\frac{dF}{dx} = \frac{dF}{dx}\bigg|_{\text{terms involving } x} + \frac{dF}{dy} \cdot \frac{dy}{dx}\bigg|_{\text{terms involving } y} = 0$$