MAT137

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December 9, 2023

1 Limit and Continuity

Let $a, L \in \mathbb{R}$.
Let f be a function defined at least on an interval centered at a, except maybe at a.

Theorem 1 ($\lim_{x \to a} f(x) = L$)

$$\forall \varepsilon > 0, \exists \delta > 0 \ s.t.$$
$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$

Theorem 2 ($\lim_{x \to a} f(x) \neq L$)

$$\exists \varepsilon > 0 \ s.t. \ \forall \delta > 0$$
$$\exists x \in \mathbb{R} \ s.t. \ 0 < |x - a| < \delta \wedge |f(x) - L| \geq \varepsilon$$

Theorem 3 ($\lim_{x \to a} f(x)$ **DNE**)

$$\forall L \in \mathbb{R}, \exists \varepsilon > 0 \ s.t. \ \forall \delta > 0$$
$$\exists x \in \mathbb{R} \ s.t. \ 0 < |x - a| < \delta \wedge |f(x) - L| \geq \varepsilon$$

The formal definitons can be transferred to imply infinity, by removing absolute values.

1.1 Laws and Theorems

Let $a, L \in \mathbb{R}$.
Let f, g, h be functions defined at least on an interval centered at a, except maybe at a.

Theorem 4 (Limit Laws)

$$\begin{array}{rl} \text{IF} & \lim_{x\to a} f(x) = L \wedge \lim_{x\to a} g(x) = M \\ \text{THEN} & \lim_{x\to a} [f(x) + g(x)] = L + M \\ & \lim_{x\to a} [f(x)g(x)] = LM \\ & \lim_{x\to a} [\frac{f(x)}{g(x)}] = \frac{L}{M} \ (M \neq 0) \end{array}$$

WARNING: Limit Laws only apply if the initial limits exist.

Theorem 5 (A New Theorem about Product)

$$\begin{array}{rl} \text{IF} & \lim_{x\to a} f(x) = 0 \\ & \exists M > 0 \ s.t. \ \forall x \neq a, |g(x)| \leq M \ (g \text{ is bounded}) \\ \text{THEN} & \lim_{x\to a} [f(x)g(x)] = 0 \end{array}$$

Theorem 6 (The Squeeze Theorem)

$$\begin{array}{rl} \text{IF} & \exists p > 0 \ s.t. \ 0 < |x - a| < p \implies f(x) \leq h(x) \leq g(x) \\ \text{AND} & \lim_{x\to a} f(x) = L \wedge \lim_{x\to a} g(x) = L \\ \text{THEN} & \lim_{x\to a} h(x) = L \end{array}$$

Theorem 7 (The New Squeeze Theorem)

$$\begin{array}{rl} \text{IF} & \exists p > 0 \ s.t. \ 0 < |x - a| < p \implies f(x) \leq g(x) \\ & \lim_{x\to a} f(x) = \infty \\ \text{THEN} & \lim_{x\to a} g(x) = \infty \end{array}$$

1.2 Continuity

Let $a, L \in \mathbb{R}$.
Let f be a function defined at least on an interval centered at a, except maybe at a.
WARNING: If the function oscillates near a, then it is NOT continuous at a.

Theorem 8 (Continuity)
We say f is continuous at a when

$$\lim_{x \to a} f(x) = f(a)$$

This means:

1. $\lim_{x \to a} f(x)$ exists and is a number
2. $f(a)$ is defined
3. The limit and the value is equal

Theorem 9 (Alternative Definition of Continuity)
We say f is continuous at a when

$$\forall \varepsilon > 0, \exists \delta > 0 \ s.t.$$

$$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$$

$\lim_{x \to a} f(x) = L$ means that:

$$\left\{ \begin{array}{l} x \text{ close to } a \\ x \neq a \end{array} \right\} \implies f(x) \text{ close to } L$$

f is continuous at a means that:

$$\left\{ \ x \text{ close to } a \ \right\} \implies f(x) \text{ close to } f(a)$$

f is continuous on the interval $[a, b]$ means that:

1. $\forall c \in (a, b)$, f is continuous at c
2. $\lim_{x \to a^+} f(x) = f(a)$
3. $\lim_{x \to b^-} f(x) = f(b)$

Theorem 10 (The Main Continuity Theorem)
Any function we can construct with sum, product, quotient, and **composition** of polynomials, roots, trigonometric functions, exponentials, logarithms, and absolute values is continuous **on its domain**.

WARNING: There is NO limit laws for composition, but the composition of continuous functions is still continuous.

1.3 About Composition of Functions

Theorem 11 (RIGHT)

$$\begin{array}{rll} \text{IF} & (1) & f \text{ is continuous at } a \\ & (2) & g \text{ is continuous at } f(a) \\ \text{THEN} & (3) & g \circ f \text{ is continuous at } a \end{array}$$

$$\begin{array}{ll} (1) & \text{means } x \text{ close to } a \implies f(x) \text{ close to } f(a) \\ (2) & \text{means } y \text{ close to } f(a) \implies g(y) \text{ close to } g(f(a)) \end{array}$$

They can be concatenated.

Theorem 12 (WRONG)

$$\begin{array}{rll} \text{IF} & (1) & \lim_{x\to a} f(x) = L \\ & (2) & \lim_{y\to L} g(y) = M \\ \text{THEN} & (3) & \lim_{x\to a} g(f(x)) = M \end{array}$$

$$\begin{array}{ll} (1) & \text{means } \left\{ \begin{array}{l} x \text{ close to } a \\ x \neq a \end{array} \right\} \implies f(x) \text{ close to } L \\ (2) & \text{means } \left\{ \begin{array}{l} y \text{ close to } L \\ y \neq L \end{array} \right\} \implies g(y) \text{ close to } M \end{array}$$

They can NOT be concatenated, but can be fixed in two ways, either on f or g.

Theorem 13 (RIGHT)

$$\begin{array}{rll} \text{IF} & (1) & \lim_{x\to a} f(x) = L \\ & & \exists p > 0 \ s.t. \ 0 < |x-a| < p \implies f(x) \neq L \\ & (2) & \lim_{y\to L} g(y) = M \\ \text{THEN} & (3) & \lim_{x\to a} g(f(x)) = M \end{array}$$

This adds a hypothesis on f.

Theorem 14 (RIGHT)

$$\begin{array}{rll} \text{IF} & (1) & \lim_{x\to a} f(x) = L \\ & (2) & g \text{ is continuous at } L \\ \text{THEN} & (3) & \lim_{x\to a} g(f(x)) = M \end{array}$$

This adds a hypothesis on g.

1.4 Discontinuity

Theorem 15 (Removable Discontinuity)
$\lim_{x \to a} f(x)$ exists, but is not equal to $f(a)$.

Intuitively, removable discontinuity looks like a hole and can be fixed.

Theorem 16 (Non-removable Discontinuity)
$\lim_{x \to a} f(x)$ does not exist.

1.5 EVT and IVT

Theorem 17 (Maximum)

$$\exists c \in I \ s.t.\ \forall x \in I, f(x) \leq f(c)$$

Theorem 18 (The Extreme Value Theorem)

$$\begin{array}{rl} \text{IF} & f \text{ is continuous on } [a,b] \\ \text{THEN} & f \text{ has a maximum and minimum on } [a,b] \end{array}$$

Theorem 19 (The Intermediate Value Theorem)

$$\begin{array}{rl} \text{IF} & f(a) < M < f(b) \\ & f \text{ is continuous on } [a,b] \\ \text{THEN} & \exists c \in (a,b)\ s.t.\ f(c) = M \end{array}$$

This means that f takes all the values between $f(a)$ and $f(b)$.

2 Derivatives

Theorem 20 (Derivative)

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \frac{dy}{dx}$$

f is differentiable at a when this limit exists.

Theorem 21 (Tangent Line)
1. through the point $(a, f(a))$
2. with slope $f'(a)$

$$y = f(a) + f'(a)(x - a)$$

Theorem 22
Given $f(x)$, we say $g(x)$ is a tangent line of $f(x)$ at $x = a$, when

$$\exists I \ni a \ s.t.$$

$$\forall x \in I, f(x) = g(x) \implies x = a$$

Theorem 23 (Vertical Tangent Line)
f is continuous at a

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \pm\infty$$

WARNING: The vertical tangent line is NOT vertical asymptote.

2.1 Differentiation

If f, g are both differentiable:

Theorem 24 (Differentiation Rules)

$$\begin{aligned} &\frac{d}{dx}[c] = 0 && \frac{d}{dx}[x^c] = cx^{c-1} \\ &(f+g)' = f' + g' && (cf)' = cf' \\ &(f \cdot g)' = f'g + fg' && (\frac{f}{g})' = \frac{f'g - fg'}{g^2} \end{aligned}$$

Proof (The Product Rule)

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a} \tag{1}$$

$$= \lim_{x \to a} \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x - a} \tag{2}$$

We can prove the Power Rule by induction.

Theorem 25 (Differentiable implies Continuous)
If f is differentiable at c, then f is continuous at c.

We can find functions that are continuous but not differentiable.

The $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ may not exist (the function is not differentiable) for several reasons:

1. because the side limits are different (CORNER)
2. because the limit is $\pm\infty$ (VERTICAL TANGENT)

Theorem 26
If $f(x)$ is continuous at a

$$\lim_{x \to a} f'(x) = \pm\infty \implies \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \pm\infty$$

WARNING: The first method is to use differentiation rules, while the second is to use the definition of derivative.

2.2 Higher-order Derivatives

Theorem 27 (Lagrange Notation)
The n-th derivative of f is

$$f^{(n)}$$

Theorem 28 (Leibnitz Notation)
The n-th derivative of y with respect to x is

$$\frac{d^n y}{dx^n}$$

2.3 The Chain Rule

Theorem 29

$$\begin{array}{rl} \text{IF} & g \text{ is differentiable at } a \\ & f \text{ is differentiable at } g(a) \\ \text{THEN} & f \circ g \text{ is differentiable at } a \end{array}$$

$$(f \circ g)'(a) = f'(g(a))g'(a)$$

PROOF (WRONG)

$$(f \circ g)'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \tag{1}$$

$$= \lim_{x \to a} [\frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}] \tag{2}$$

WARNING: For this to work, we must ensure that:

$$\exists p > 0 \ s.t. \ 0 < |x - a| < p \implies g(x) \neq g(a)$$

PROOF (RIGHT) Define

$$u = g(x)$$

$$y = f(u) = f(g(x))$$

Then

$$(f \circ g)'(a) = f'(g(a))g'(a)$$

can be written as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

2.4 Implicit Differentiation

Consider an equation involving two variables x, y that defines y implicitly:

$$F(x, y) = 0$$

To find the derivative of y with respect to x, denoted as $\frac{dy}{dx}$, we differentiate each side of the equation with respect to x, treating y as a function of x.

$$\frac{dF}{dx} = \frac{dF}{dx}\Big|_{\text{terms involving } x} + \frac{dF}{dy} \cdot \frac{dy}{dx}\Big|_{\text{terms involving } y} = 0$$

3 Inverse Function

3.1 Notation

A function f consists of:
Name: f.
Domain: set of inputs.
Codomain: set of **potential** outputs.
Range: set of **actual** outputs. Can be calculated from other information.
A rule that matches each input to exactly one output.

$$f : A \to B$$

f is the name, A is the domain, B is the codomain.

3.2 Convention

The domain is the largest subset of $\mathbb{R}$ possible, the codomain is always $\mathbb{R}$.

3.3 Computer Science

The notation is not universal.

Math	**CS**
domain	domain
codomain	range
range	image

Table 1: Comparison of terms in Math and Computer Science

3.4 Inverse

Let $f : A \to B$ be a function.
The inverse of f is another function

$$f^{-1} : B \to A$$

defined by

$$\forall x \in A, \forall y \in B, x = f^{-1}(y) \iff y = f(x)$$

3.5 Injective and Surjective

Let $f : A \to B$ be a function.

Theorem 30 (Surjective)
f is surjective or onto when: Range $f = B$.

Theorem 31 (Injective)
f is injective or one-to-one when:

$$\forall x_1, x_2 \in A, x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

Or, equivalently:

$$\forall x_1, x_2 \in A, f(x_1) = f(x_2) \implies x_1 = x_2$$

f has an inverse $\iff$ f is surjective and injective.

Theorem 32 (Inverse)
Let f be a one-to-one function. Let Domain $f = A$, Range $f = C$.
The inverse f^{-1} is another function satisfying: Domain $f^{-1} = C$, Range $f^{-1} = A$.

$$\forall x \in A, f^{-1}(f(x)) = x$$

$$\forall y \in C, f(f^{-1}(y)) = y$$

3.6 Derivative of Inverse

Theorem 33
Let f be a function defined on an interval I.

$$\begin{array}{rl} \text{IF} & f \text{ has an inverse} \\ & f \text{ is differentiable} \\ & \forall x \in I, f'(x) \neq 0 \\ \text{THEN} & f^{-1} \text{ is differentiable} \end{array}$$

Assume that f, f^{-1} are differentiable.

$$\begin{aligned} f(f^{-1}(y)) &= y \\ \frac{d}{dy} f(f^{-1}(y)) &= \frac{d}{dy} y \\ f'(f^{-1}(y)) \cdot (f^{-1})'(y) &= 1 \\ f'(x) \cdot (f^{-1})'(y) &= 1 \\ (f^{-1})'(y) &= \frac{1}{f'(x)} \end{aligned}$$

3.7 Exponentials

$f(x) = a^x$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x (a^h - 1)}{h} \\ &= a^x \lim_{h \to 0} \frac{a^h - 1}{h} \end{aligned}$$

Call $L_a = \lim_{h \to 0} \dfrac{a^h - 1}{h}$, then $\dfrac{d}{dx} a^x = L_a a^x$.

Theorem 34 (e)
Define e is the only number such that $L_e = 1$.

$$\frac{d}{dx} e^x = e^x$$

Theorem 35

$$a^c = e^{c \ln a}$$

3.8 Definition

Theorem 36 (e^x)
Define $E(x) = e^x$ as the only function that satisfies

$$\left\{ \begin{array}{l} E'(x) = E(x) \\ E(0) = 1 \end{array} \right\}$$

Theorem 37 ($\ln x$)
Define

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

3.9 Derivative of Logarithm

$$\forall x > 0, e^{\ln x} = x$$

$$\frac{d}{dx} e^{\ln x} = \frac{d}{dx} x$$

$$x \cdot \frac{d}{dx} \ln x = 1$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

3.10 Derivative of Exponential

$$a^x = e^{x \ln a}$$

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \cdot \frac{x}{dx} x \ln a = a^x \cdot \ln a$$

3.11 Change Base

$$a^x = b^{x \log_b a}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

3.12 Natural Logarithm

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = 1$$

4 Inverse of Trignometry

Theorem 38 (arcsin)
arcsin is the inverse function of the restriction of sin to $[-\pi/2, \pi/2]$.

$$x = \arcsin y \iff y = \sin x$$

$$\text{for } -\frac{\pi}{x} \leq x \leq \frac{\pi}{2}, -1 \leq y \leq 1$$

Theorem 39 (arctan)
arctan is the inverse function of the restriction of tan to $(-\pi/2, \pi/2)$.

$$x = \arctan y \iff y = \tan x$$

$$\text{for } -\frac{\pi}{x} < x < \frac{\pi}{2}, y \in \mathbb{R}$$

Theorem 40 (arccos)
arccos is the inverse function of the restriction of cos to $[0, \pi]$.

$$x = \arccos y \iff y = \cos x$$

$$\text{for } 0 \leq x \leq \pi, -1 \leq y \leq 1$$

Theorem 41

$$\frac{d}{dx}[\sin(\arcsin x)] = \frac{d}{dx}[x]$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

5 MVT

5.1 Local Extremum

Let f be a function with domain I. Let $c \in I$.
We say f has a maximum at c when

$$\forall x \in I, f(x) \leq f(c)$$

We say f has a local maximum at c when

$$\exists \delta > 0 \ s.t. \ |x - c| < \delta \implies f(x) \leq f(c)$$

Theorem 42 (Local Extreme Value Theorem)
Let f be a function with domain on an interval I, let $c \in I$.

IF	f has an local extremum at c
	c is an interior point to I
THEN	$f'(c)$ is 0 or DNE.

Proof WTS: The limit is 0 or DNE.
Assume this limit exists, and it is 0.

Theorem 43 (Critical Point)
c is called a **critical point** of f when
c is an interior point of the domain of f, and
$f'(c) = 0$ or DNE.

Theorem 44 (Rolle's Theorem)
Let $a < b$, let f be a function defined on $[a, b]$.

IF	f is continuous on $[a, b]$
	f is differentiable on (a, b)
	$f(a) = f(b)$
THEN	$\exists c \in (a, b) \ s.t. \ f'(c) = 0$

5.2 Zero

To find zeroes:
1. Use the IVT to prove it has at least n.
2. Use Rolle's Theorem to prove it has at most n.

PROOF To use Rolle's Theorem:
1. Assume f is continuous and differentiable everywhere.
2. Conclusion 1: between any two zeroes of f, there must be at least one zero of f'.
3. Conclusion 2: (# of zeroes of f') $\geq$ (# of zeroes of f) -1.

Theorem 45
If f is continuous and differentiable on an interval

$$(\#\text{ of zeroes of } f) \leq (\#\text{ of zeroes of } f') + 1$$

5.3 MVT

Theorem 46 (The Mean Value Theorem)
Let $a < b$, let f be a function defined on $[a, b]$.

$$\begin{array}{rl} \text{IF} & f \text{ is continuous on } [a,b] \\ & f \text{ is differentiable on } (a,b) \\ \text{THEN} & \exists c \in (a,b) \ s.t. \ f'(c) = \dfrac{f(b)-f(a)}{b-a} \end{array}$$

PROOF Use Rolle's Theorem on $h(x) = f(x) - f(a) - m(x - a)$ where

$$m = \frac{f(b) - f(a)}{b - a}$$

5.4 Zero Derivative

Theorem 47 (Constant)
Let $a < b$, let f be a function defined on $[a, b]$.

$$\begin{array}{rl} \text{IF} & \forall x \in (a,b), f'(x) = 0 \\ & f \text{ is continuous on } [a,b] \\ \text{THEN} & f \text{ is constant on } [a,b] \end{array}$$

Corollary 1 *If f, g have the same derivative on an open interval I, then $f - g$ is constant on I.*

PROOF $f - g$ has zero derivative.

5.5 Monotonicity

Theorem 48 (Increasing)
Let f be a function defined on an interval I.
f is increasing on I when

$$\forall x_1, x_2 \in I, x_1 < x_2 \implies f(x_1) < f(x_2)$$

f is non-decreasing on I when

$$\forall x_1, x_2 \in I, x_1 < x_2 \implies f(x_1) \leq f(x_2)$$

Theorem 49
Let $a < b$, let f be a function defined on (a, b).

IF $\forall x \in (a, b), f'(x) > 0$
THEN f is increasing on (a, b)

AND

IF $\forall x \in (a, b), f'(x) > 0$
f is continuous on $[a, b]$
THEN f is increasing on $[a, b]$

6 Applications

Indeterminate Forms

We say that $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, 0 \cdot \infty, \infty - \infty, 0^0, \infty^0, 1^{\pm\infty}$ is a (limit) indeterminate form.

Theorem 50 (L'Hopital's Theorem)
Let f, g be functions, let $a \in \mathbb{R}$.

IF f, g are differentiable as $x \to a$
g, g' are never 0 as $x \to a$
The limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$
The limit $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists or is $\pm\infty$
THEN $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

6.1 Concavity

Theorem 51
Let f be a differentiable function defined on an interval I.
We say that f is concave-up on I when f' is increasing on I.
We say that f is concave-down on I when f' is decreasing on I.
Let c be an interior point to I.
We say that f has an inflection point at c when f changes concavity at c.

Theorem 52
Let I be an open interval.
Let f be a twice-differentiable function defined on I.
IF $\forall x \in I, f''(x) > 0$, THEN f is concave-up on I.
IF $\forall x \in I, f''(x) < 0$, THEN f is concave-down on I.

Theorem 53
Let I be an open interval. Let $c \in I$
Let f be a twice-differentiable function defined on I.
IF f has an inflection point at c, THEN $f''(c) = 0$ or DNE.

6.2 Asymptotes

Let L be a line and C be a curve in the plane.

L is an asymptote for C when they become arbitrarily close as we move away from the origin in one direction.

Theorem 54 (Vertical Asymptotes)
The vertical line $x = a$ is an asymptote of f when

$$\lim_{x \to a^{\pm}} f(x) = \pm\infty$$

Theorem 55 (Horizontal Asymptotes)
The vertical line $y = L$ is an asymptote of f when

$$\lim_{x \to \pm\infty} f(x) = L$$

Theorem 56 (Slant Asymptotes)
The vertical line $y = mx + b$ is an asymptote of f when

$$\lim_{x \to \pm\infty} [f(x) - (mx + b)] = 0$$

7 Notes

Theorem 57 (Floor)
We difined the floor of x, denoted by $\lfloor x \rfloor$, as the largest integer smaller than or equal to x.

$$\sqrt{x^2} = |x|$$

If f is differentiable everywhere, then either its derivative is continuous, or the derivative has Oscillation Discontinuity Point.

Derivative is continuous $\implies$ Differentiable $\implies$ Continuous

To prove a claim is not true from definition, write down the nagetion of the definition first.

The term **Extremum** measn maximum or minimum.

Global extremum means extremum.

The plurals are extrema/maxima/minima.

Endpoints do not count as local extrema.