Colorful Helly

Key Lemma

Let K be a simplicial complex on the vertex set V and let $V = \cup_{i=1}^{p} V_i$ be the color classes with $|V_i| \geq t$. Assume that K contains all possible colorful subsets but no V_i is a face.
If $r^* = r^*(K)$ is bounded, then $p < S(m, t)$ for some $m = m(r^*)$.

- i.e.,"Too Many" color classes are forbidden.
- Here, $S(m, t)$ is the Stirling number of the second kind.

Def. The Stirling number of the second kind $S(m, t)$ is the number of ways to partition the set $[m]$ into t non-empty subsets.

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Theorem 1

If the Radon number $r = r(X, \mathcal{C})$ is bounded, then $(X, \mathcal{C})$ satisfies the colorful Helly property with $p = p(r)$.

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Theorem 1

If the Radon number $r = r(X, \mathcal{C})$ is bounded, then $(X, \mathcal{C})$ satisfies the colorful Helly property with $p = p(r)$.

- Choose any finite families $\mathcal{F}_1, \ldots, \mathcal{F}_p$, where $p = \sup\{S(m, t) : m = m(r^*)\}$.
- Apply Key Lemma to the nerve complex $K = N(\cup_{i=1}^{p} \mathcal{F}_i)$ with $t = h(X, \mathcal{C})$.
- **Contrapositive.** Since $p \geq S(m, t)$, there exists some $V_i \in K$.

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Sketch of Key Lemma.

Assume $t = 2$ and $r_2^*(K) = 4$. Note that $S(r, 2) = 2^{r-1} - 1$ and so, $S(4, 2) = 7$.
We want to make a contradiction when $p \geq 7$.

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We want to make a contradiction when $p \geq 7$. First, consider

$\mathcal{P}_1 = \{1\} \cup \{2, 3, 4\}$ $\mathcal{P}_2 = \{2\} \cup \{1, 3, 4\}$ $\mathcal{P}_3 = \{3\} \cup \{1, 2, 4\}$ $\mathcal{P}_4 = \{4\} \cup \{1, 2, 3\}$
$\mathcal{P}_5 = \{1, 2\} \cup \{3, 4\}$ $\mathcal{P}_6 = \{1, 3\} \cup \{2, 4\}$ $\mathcal{P}_7 = \{1, 4\} \cup \{2, 3\}$.

elements \ types	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	$\mathcal{P}_5$	$\mathcal{P}_6$	$\mathcal{P}_7$
1	1st	2nd	2nd	2nd	1st	1st	1st
2	2nd	1st	2nd	2nd	1st	2nd	2nd
3	2nd	2nd	1st	2nd	2nd	1st	2nd
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For each i, let $V_i = \{v_{i1}, v_{i2}\}$. Choose colorful subsets $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ as follows.

faces \ colors	V_1	V_2	V_3	V_4	V_5	V_6	V_7
σ_1	v_{11}	v_{22}	v_{32}	v_{42}	v_{51}	v_{61}	v_{71}
σ_2	v_{12}	v_{21}	v_{32}	v_{42}	v_{51}	v_{62}	v_{72}
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In this table, each σ_j is formed by the vertices in the j-th row.

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In this table, each σ_j is formed by the vertices in the j-th row.

- ▷ Since K contains all possible colorful subsets, $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are facets of K.
- ▷ Since $r_2 = 4$, there is a partition $\{1, 2, 3, 4\} = I_1 \cup I_2$ such that $(\cap_{j \in I_1} \sigma_j) \cup (\cap_{j \in I_2} \sigma_j) \in K$. — $(*)$

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Suppose that $I_1 = \{1\}$ and $I_2 = \{2, 3, 4\}$ as an example. Then by $(*)$,

$$\sigma_1 \cup (\sigma_2 \cap \sigma_3 \cap \sigma_4) \in K.$$

Observe that

$$V_1 = \{v_{11}\} \cup \{v_{12}\} \subseteq \sigma_1 \cup (\sigma_2 \cap \sigma_3 \cap \sigma_4) \in K.$$

This is impossible because V_1 is not a face of K.

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Indeed, we can check that no partition types are possible.

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- Suppose $\mathcal{P}_i = I_1 \cup I_2$ satisfies $(\cap_{j \in I_1} \sigma_j) \cup (\cap_{j \in I_2} \sigma_j) \in K$,

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- then $V_i = \{v_{i1}\} \cup \{v_{i2}\} \subseteq (\cap_{j \in I_1} \sigma_j) \cup (\cap_{j \in I_2} \sigma_j) \in K$.

This contradicts to the assumptions. □