Helly-Type Theorems for Abstract Convexity

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Radon's Lemma. In $\mathbb{R}^d$, any $(d+2)$-point set P can be partitioned into two parts P_1 and P_2 such that

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Here, $conv(S)$ denotes the smallest convex set containing S.

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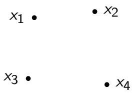

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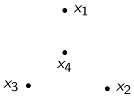

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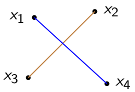

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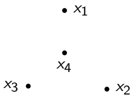

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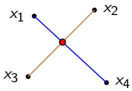

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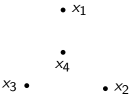

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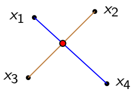

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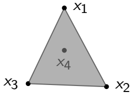

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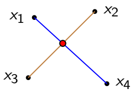

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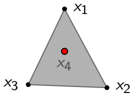

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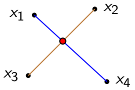

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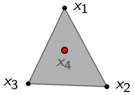

$I_1 = \{1, 4\}$ and $I_2 = \{2, 3\}$ $\qquad$ $I_1 = \{1, 2, 3\}$ and $I_2 = \{4\}$

Partitioning the index set is more useful :

Radon's Lemma. For any $d+2$ points $x_1, \ldots, x_{d+2}$ in $\mathbb{R}^d$, the index set $[d+2]$ can be partitioned into two parts I_1 and I_2 such that

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Definition

The Radon number of $\mathbb{R}^d$ is the smallest integer m satisfying the following.
For any m points $x_1, \ldots, x_m$ in $\mathbb{R}^d$, there is a partition $[m] = I_1 \cup I_2$ such that

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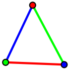
- ▷ The Radon number of $\mathbb{R}^d$ is at most $d+2$.
- ▷ Indeed, the Radon number of $\mathbb{R}^d$ is exactly $d+2$.

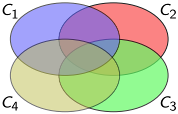


$3 <$ (the Radon number of plane)

Helly Number

Helly's Theorem. Let $C_1, \ldots, C_n$ be convex sets in $\mathbb{R}^d$. If every $d+1$ convex sets of them are intersecting, then all the convex sets are intersecting.

Example) In plane, let $\mathcal{F} = \{C_1, C_2, C_3, C_4\}$.



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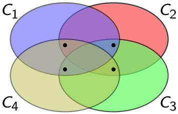

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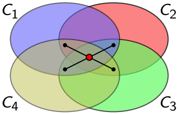

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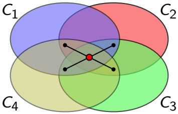

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The Helly number of $\mathbb{R}^d$ is the smallest integer k satisfying the following.
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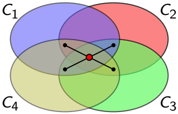

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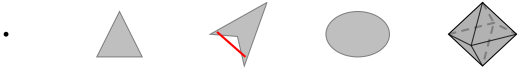
- The Helly number of $\mathbb{R}^d$ is $d+1$.
- The Helly number of $\mathbb{R}^d$ is less than the Radon number($=d+2$) of $\mathbb{R}^d$

Abstract Convexity

Convex set?



A set $C \subseteq \mathbb{R}^d$ is a standard convex set if
for any two points $x, y \in C$, the segment $\{tx + (1-t)y : t \in [0,1]\}$ is contained in C.

A Combinatorial Property:
The intersection of two standard convex sets is also a standard convex set.

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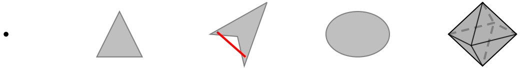
Convex set? 

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An (abstract) convexity space is a pair $(X, \mathcal{C})$ such that

- $\mathcal{C}$ is a collection of subsets of X.
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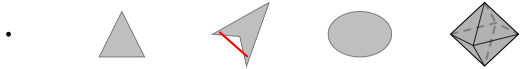

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- ▷ Every member of $\mathcal{C}$ is called an (abstract) convex set.
- ▷ "Convex hull" $conv(S)$ denotes the "inclusion-minimal" convex set containing S. In other words,

$$conv(S) := \bigcap_{C \in \mathcal{C}, S \subset C} C$$

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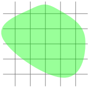

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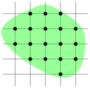

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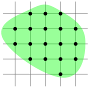

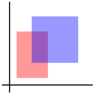

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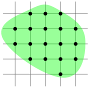

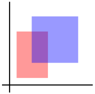

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Radon Number

The Radon number $r(X, \mathcal{C})$ is the smallest integer m satisfying the following. For any m points $x_1, \ldots, x_m$ in X, there is a partition $[m] = I_1 \cup I_2$ such that

$$conv(\{x_i : i \in I_1\}) \cap conv(\{x_i : i \in I_2\}) \neq \emptyset.$$

▷ $r(\mathbb{R}^d, \mathcal{S}) = d + 2$. (Radon, 1921)

▷ $r(\mathbb{Z}^d, \mathcal{L}) \leq d \cdot (2^d - 1) + 3$. (Sierksma, 1977)

▷ $r(\mathbb{R}^d, \mathcal{B}) = \min\{r : \binom{r}{\lfloor r/2 \rfloor} > 2d\}$. (Eckhoff, 1969)

The Helly Property with k

The Helly number $h(X, \mathcal{C})$ is the smallest integer k satisfying the following.
Let $C_1, \ldots, C_n$ be convex sets in $\mathcal{C}$. If every k convex sets of them are intersecting, then all the convex sets are intersecting.

- $h(\mathbb{R}^d, \mathcal{S}) = d + 1$. (Radon, 1921)
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What do we want to do? Let $(X, \mathcal{C})$ be a convexity space.

If the Radon number $r(X, \mathcal{C})$ is bounded, then

- $(X, \mathcal{C})$ satisfies the Helly property. (Revi, 1951)

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What do we want to do? Let $(X, \mathcal{C})$ be a convexity space.

If the Radon number $r(X, \mathcal{C})$ is bounded, then

- $(X, \mathcal{C})$ satisfies the Helly property. (Revi, 1951)
- $(X, \mathcal{C})$ satisfies "the colorful Helly property". (**Theorem 1**)
- $(X, \mathcal{C})$ satisfies "the weak fractional Helly property". (**Theorem 2**)

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Colorful Helly

Colorful Helly Theorem. (Lovász, 1974)
Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be families of convex sets in $\mathbb{R}^d$. If every colorful tuple is intersecting, then some $\mathcal{F}_i$ of the families is intersecting.

- Here, a colorful tuple denotes a family $\{C_1, \ldots, C_{d+1}\}$, where $C_1 \in \mathcal{F}_1, \ldots, C_{d+1} \in \mathcal{F}_{d+1}$.

For example)

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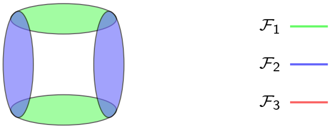

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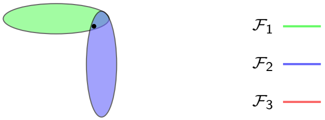

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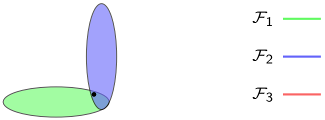

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Colorful Helly Theorem. (Lovász, 1974)
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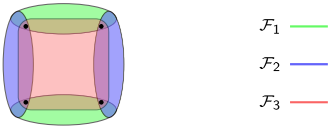

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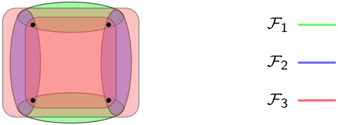

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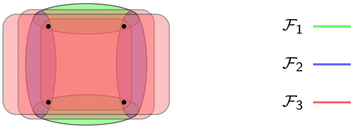

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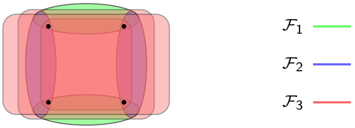

Definition. (Colorful Helly Property with k)

$(X, \mathcal{C})$ satisfies the colorful Helly property if there exists k satisfying the following.
Let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be families of convex sets in $\mathcal{C}$. If every colorful tuple is intersecting, then some $\mathcal{F}_i$ of the families is intersecting.

- ▷ $(\mathbb{R}^d, \mathcal{S})$ satisfies the colorful Helly property with $d+1$.
- ▷ The colorful Helly property $\Rightarrow$ the Helly property by setting $\mathcal{F}_1 = \cdots = \mathcal{F}_k$.

Colorful Helly

Colorful Helly Property with k

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Theorem 1

If the Radon number $r = r(X, \mathcal{C})$ is bounded, then $(X, \mathcal{C})$ satisfies the colorful Helly property with $p = p(r)$.

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- Tool : Nerve complex.
- Translate the Radon number to the language of a simplicial complex.
- By using "Duplication", we may assume $\mathcal{F}_1, \ldots, \mathcal{F}_k$ are pairwise disjoint.

Nerve Complex

Let $\mathcal{F} = \{C_1, \ldots, C_n\}$ be a finite family of sets. The nerve complex of $\mathcal{F}$ is defined as

$$N(\mathcal{F}) := \{\sigma \in 2^{[n]} : \cap_{i \in \sigma} C_i \neq \emptyset\}.$$

In other words, for an index set σ

$$\cap_{i \in \sigma} C_i \neq \emptyset \quad \Leftrightarrow \quad \sigma \in N(\mathcal{F}).$$

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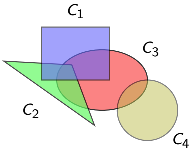

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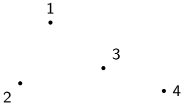
Nerve Complex $\rightsquigarrow$



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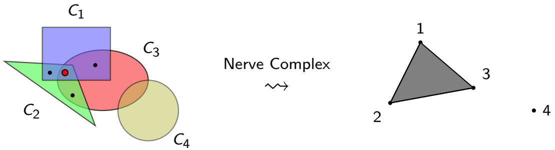

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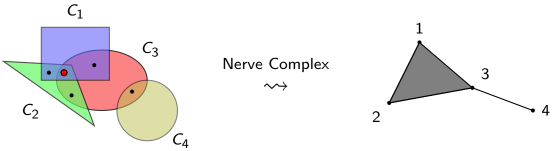

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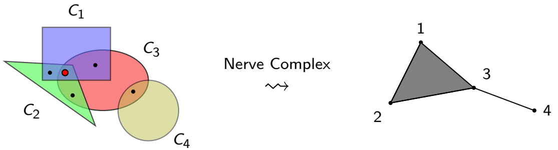

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- ▷ A nerve complex is an (abstract) simplicial complex.
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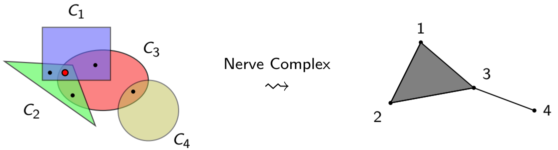

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$$\tau \in K \text{ and } \sigma \subseteq \tau \quad \Rightarrow \quad \sigma \in K.$$
- $v \in V$ is called a vertex. e.g.) $1, 2, 3, \dots$
- $\sigma \in K$ is called a face. e.g.) $\{1\}, \{1,2\}, \{1,2,3\}, \dots$
- If σ is inclusion-maximal in K, then σ is called a facet. e.g.) $\{1,2,3\}, \{3,4\}$

Colorful Helly

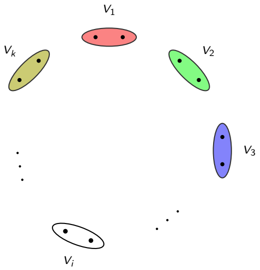
Colorful Helly Property with k

Let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be families of convex sets in $\mathcal{C}$. If every colorful tuple is intersecting, then there exists an intersecting family $\mathcal{F}_i$ among the families.

Let $K = N(\cup_{i=1}^{k} \mathcal{F}_i)$ denote the nerve complex. Consider

$$\mathcal{F}_i \quad \leftrightarrow \quad V_i.$$

For example,



Each V_i is a "color class".

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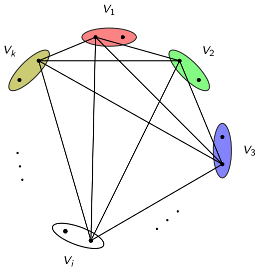

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A "colorful subset" S is a set of vertices such that $|S \cap V_i| = 1$ for all $1 \leq i \leq k$.

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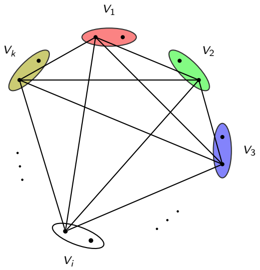

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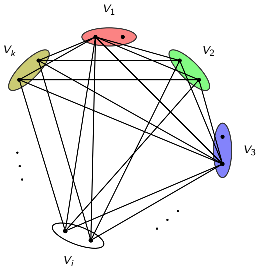

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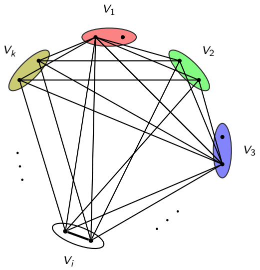

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Nerve Complex

Definition

The Radon number $r^*(K)$ of a simplicial complex K is the smallest integer m with the following. For any m facets $\sigma_1, \ldots, \sigma_m$ of K, there exists a partition $[m] = I_1 \cup I_2$ such that

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Compare to

The Radon number $r(X, \mathcal{C})$ is the smallest integer m with the following. For any m points $x_1, \ldots, x_m$ in X, there is a partition $[m] = I_1 \cup I_2$ such that

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Let $(X, \mathcal{C})$ be a convexity space. For any finite family $\mathcal{F}$ of convex sets in $\mathcal{C}$,

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Reduced Problem

If $r^*(K)$ is bounded, then the **Claim** holds?

Colorful Helly

Key Lemma

Let K be a simplicial complex on the vertex set V and let $V = \cup_{i=1}^{p} V_i$ be the color classes with $|V_i| \geq t$. Assume that K contains all possible colorful subsets but no V_i is a face.
If $r^* = r^*(K)$ is bounded, then $p < S(m, t)$ for some $m = m(r^*)$.

- ▷ i.e., "Too Many" color classes are forbidden.
- ▷ Here, $S(m, t)$ is the Stirling number of the second kind.

Def. The Stirling number of the second kind $S(m, t)$ is the number of ways to partition the set $[m]$ into t non-empty subsets.

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Theorem 1

If the Radon number $r = r(X, \mathcal{C})$ is bounded, then $(X, \mathcal{C})$ satisfies the colorful Helly property with $p = p(r)$.

- Choose any finite families $\mathcal{F}_1, \ldots, \mathcal{F}_p$, where $p = \sup\{S(m, t) : m = m(r^*)\}$.
- Apply Key Lemma to the nerve complex $K = N(\cup_{i=1}^{p} \mathcal{F}_i)$ with $t = h(X, \mathcal{C})$.
- **Contrapositive.** Since $p \geq S(m, t)$, there exists some $V_i \in K$.

Fractional Helly

Fractional Helly Theorem. (Katchalski and Liu, 1979)

For every $\alpha \in (0,1]$, there exists $\beta \in (0,1]$ satisfying the following. Let $C_1, \ldots, C_n$ be convex sets in $\mathbb{R}^d$.
If

$$\left(\text{\# of intersecting } (d+1)\text{-tuples of the } C_i\text{'s}\right) \geq \alpha \binom{n}{d+1},$$

then

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Example) In plane, $n = 4$ and $\binom{n}{d+1} = \binom{4}{3} = 4$.

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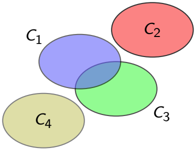

$\alpha = 0$ and $\beta = \frac{1}{2}$

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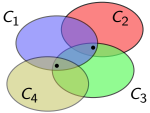

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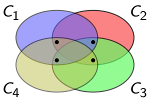

$\alpha = 1$ and $\beta = 1$

- ▷ Here, $\beta = \beta(\alpha)$ should not depend on n.
- ▷ If $\beta(1) = 1$, then we obtain the Helly property.

Fractional Helly

Definition. (Fractional Helly property with k)

$(X, \mathcal{C})$ satisfies the fractional Helly property if there exists k with the following.
For every $\alpha \in (0,1]$, there exists $\beta \in (0,1]$ such that for any n convex sets $C_1, \ldots, C_n$ in $\mathcal{C}$, if the number of intersecting k-tuples of the C_i's is at least $\alpha\binom{n}{k}$, then at least βn convex sets among the C_i's are intersecting.

- ▷ $(\mathbb{R}^d, \mathcal{S})$ satisfies the f. Helly property with $d+1$ and $\beta = \frac{\alpha}{d+1}$. (Katchalski and Liu, 1979)
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Definition. (Fractional Helly property with k)

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"WEAK" ?

Even though we cannot find $\beta \in (0,1]$ for every $\alpha \in (0,1]$, but if such β exists for some α, then we say that $(X, \mathcal{C})$ satisfies the weak fractional Helly property.

- In $(\mathbb{R}^d, \mathcal{B})$, in order for $\beta = 1 - \sqrt{d(1-\alpha)} > 0$, it needs to be $\alpha > 1 - \frac{1}{d}$.

Weak Fractional Helly

“Weak” Fractional Helly property with k

For every $\alpha \in (1-\delta, 1]$, there exists $\beta \in (0,1]$ such that for any n convex sets $C_1, \dots, C_n$ in $\mathcal{C}$, if the number of intersecting k-tuples of the C_i's is at least $\alpha\binom{n}{k}$, then at least βn convex sets among the C_i's are intersecting.

Theorem 2

If the Radon number $r = r(X, \mathcal{C})$ is bounded, then $(X, \mathcal{C})$ satisfies the weak fractional Helly property with h.

- Here, h is the Helly number of $(X, \mathcal{C})$.

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- $\beta = 1 - ph(1-\alpha)^{\frac{1}{h}}$, where p is the Stirling number of the second kind in **Theorem 1**.
- $\alpha > 1 - \frac{1}{(ph)^h}$ not to vanish β. i.e., $\delta = \frac{1}{(ph)^h}$.

Weak Fractional Helly

Sketch of Theorem 2.

Theorem (M. Kim, 2017)

If $(X, \mathcal{C})$ satisfies the colorful Helly property with k, then $(X, \mathcal{C})$ also satisfies the weak fractional Helly property with k and $\beta = 1 - k(1-\alpha)^{\frac{1}{k}}$.

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If the Radon number $r = r(X, \mathcal{C})$ is bounded, then $(X, \mathcal{C})$ satisfies the weak fractional Helly property with h and $\beta = 1 - ph(1-\alpha)^{\frac{1}{h}}$, where $h = h(X, \mathcal{C})$.

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Let $(X, \mathcal{C})$ be a convexity space. If the Radon number r_2 is bounded, then

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Tverberg numbers $r_k(X, \mathcal{C})$ of a convexity space

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Compare to

Tverberg numbers $r_k^*(K)$ of a simplicial complex

For any m facets $\sigma_1, \ldots, \sigma_m$ of K, there exists a partition $[m] = I_1 \cup \cdots \cup I_k$ such that

$$(\cap_{i \in I_1} \sigma_i) \cup \cdots \cup (\cap_{i \in I_k} \sigma_i) \in K.$$

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Key Lemma

Let K be a simplicial complex. Then there is a convexity space $(X, \mathcal{C})$ such that

$$r_k(X, \mathcal{C}) = r_k^*(K) \text{ for every } k \geq 2.$$

- Such $(X, \mathcal{C})$ is called the upper convexity space of K.

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We will construct a simplicial complex K such that

$$r_2^*(K) = 4 \quad \text{but} \quad r_3^*(K) > 3(k-1)+1 = 7.$$

Let $N = \{1,2,3,4,5,6,7\}$ and let

$$V_1 = \{S \subset N : |S| = 1\},\ V_2 = \{S \subset N : |S| = 2\} \text{ and } V_3 = \{S \subset N : |S| = 3\}.$$

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Step 2. The facets of K are of the following forms: for any distinct $x, y, z, w \in N$,

- $A_x = \{S \in V : x \in S\}$
- $B_{xy,zw} = \big\{\{x,y\},\{z,w\}\big\} \cup \{S \in V_3 : S \cap \{x,y,z,w\} \neq \emptyset\}$
- $C_{xy} = \big\{\{x,y\}\big\} \cup V_3$.

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Step 3. The simplcial complex K is defined as
$$K := \big\{\sigma \subseteq A_x : x \in N\big\} \cup \big\{\sigma \subseteq B_{xy,zw} : x,y,z,w \in N\big\} \cup \big\{\sigma \subseteq C_{xy} : x,y \in N\big\}.$$

Claim 1: $r_3^*(K) > 7$.

Proposition

Let $N = I_1 \cup I_2 \cup I_3$ be a partition. If $r_3^*(K) \leq 7$, then

$$\{I_1, I_2, I_3\} \subseteq \tau \text{ for some facet } \tau \in K.$$

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However, no facet of K contains three disjoint subsets I_1, I_2, I_3.

- $A_x = \{S \in V : x \in S\} \quad \Rightarrow \quad x \in I_1 \cap I_2 \cap I_3$.
- $B_{xy,zw} = \{\{x,y\},\{z,w\}\} \cup \{S \in V_3 : S \cap \{x,y,z,w\} \neq \emptyset\}$
  $\Rightarrow \quad I_1 = \{x,y\}, I_2 = \{z,w\}$ but $I_3 \cap \{x,y,z,w\} \neq \emptyset$.
- $C_{xy} = \{\{x,y\}\} \cup V_3 \quad \Rightarrow \quad 8 \leq |I_1| + |I_2| + |I_3| = 7$.

Claim 2: $r_2^*(K) = 4$.

Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ be distinct facets of K.

- For any $i \neq j$, $\sigma_i \cap \sigma_j \subseteq C_{xy}$ for some x, y.
- If $\sigma_i \cap \sigma_j$ has no sets of size 2, then $\sigma_i \cap \sigma_j \subseteq V_3$. Then, **WLOG**, $(\sigma_1 \cap \sigma_2) \cup (\sigma_3 \cap \sigma_4) \subseteq C_{xy}$ for some x, y.
- By the above, **WMA** every pair of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ have a common set of size 2.
- If $\sigma_i = C_{xy}$, then $\{x, y\} \in \sigma_j$ for all $j \neq i$. Hence $(\sigma_1 \cap \sigma_2) \cup (\sigma_3 \cap \sigma_4) \subseteq C_{xy}$.
- **WMA** the facets σ_1, σ_2, σ_3, σ_4 are of the types A_x or $B_{xy,zw}$.

The remaining cases are the following:

- If $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (A_x, A_y, A_z, A_w)$, then $(\sigma_1 \cap \sigma_2) \cup (\sigma_3 \cap \sigma_4) \subseteq B_{xy,zw}$
- If $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (A_x, A_y, A_z, B_{xy,zw})$, then $(\sigma_1 \cap \sigma_2 \cap \sigma_3) \cup \sigma_4 \subseteq B_{xy,zw}$
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- If $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (A_x, B_{xy,zw}, B_{xy',zw}, B_{xy'',zw})$, then $(\sigma_1 \cap \sigma_2) \cup (\sigma_3 \cap \sigma_4) \subseteq B_{xy,zw}$
- If $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (B_{xy,zw}, B_{xy,z'w'}, B_{xy,z''w''}, B_{xy,z''',w'''})$, then $(\sigma_1 \cap \sigma_2) \cup (\sigma_3 \cap \sigma_4) \subseteq C_{xy}$.

By checking all possible cases, $r_2^*(K) \leq 4$.

Eckhoff's Conjecture

- Hence, our K satisfies $r_2^*(K) = 4$ but $r_3^*(K) > 7$.
- This method can be extended for arbitrary $k \geq 3$.

Theorem 3

For each $k \geq 3$, there exists a simplicial complex K such that

$$r_2^*(K) = 4 \text{ but } r_k^*(K) > 3(k-1) + 1.$$

- Because of the upper convexity space of K,

Counterexample to Eckhoff's Conjecture

For each $k \geq 3$, there exists a convexity space $(X, \mathcal{C})$ such that

$$r_2(X, \mathcal{C}) = 4 \text{ but } r_k(X, \mathcal{C}) > 3(k-1) + 1.$$

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$(X, \mathcal{C})$: abstract convexity space. If $r = r(X, \mathcal{C})$ is bounded, then

- ▷ $(X, \mathcal{C})$ satisfies the Helly property with $h < r$.
- ▷ $(X, \mathcal{C})$ satisfies the colorful Helly property with $p = p(r)$.
- ▷ $(X, \mathcal{C})$ satisfies the weak fractional Helly property with h.
- ▷ For each $k \geq 3$, an upper convexity space satisfies $r = 4$ and $r_k > 3(k-1)+1$. This is a counterexample to Eckhoff's conjecture.

Any Questions? Thank You!