THE EISENSTEIN SERIES

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Contents

1. Automorphic Forms

1.1. Setup.
Let F be a number field. Let $\mathbb{A}$ be the ring of adèles of F.

Let G be a connected algebraic group over F. Denote by $X^*(G) = \mathrm{Hom}_F(G, \mathbb{G}_m)$ the group of characters of G. It is a commutative group and the group law is written additively. Define $\mathfrak{a}_G = \mathrm{Hom}(X^*(G), \mathbb{R})$ to be the set of group maps from $X^*(G)$ to $\mathbb{R}$, and set $\mathfrak{a}_G^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}_G, \mathbb{R})$ be the linear dual of $\mathfrak{a}_G$.

Remark 1.1.1. Another approach is to first define $\mathfrak{a}_G^* = X^*(G) \otimes_{\mathbb{Z}} \mathbb{R}$ and then set $\mathfrak{a}_G = (\mathfrak{a}_G^*)^*$.

The Harish-Chandra map $H_G : G(\mathbb{A}) \to \mathfrak{a}_G$ is defined by $x \in G(\mathbb{A}) \mapsto (\chi \in X^*(G) \mapsto \log|\chi(x)|)$, where $|-|$ is the idèle norm. Denote by $G(\mathbb{A})^1$ the kernel of H_G.

Lemma 1.1.2. The map H_G is surjective.

Remark 1.1.3. In the case that F is a function field, the Harish-Chandra map is not surjective.

Now suppose G is reductive. Let Z_G be the center of G. Let $G_{\mathbb{Q}}$ be the restriction of scalars from F to $\mathbb{Q}$ of G. Let A_G be the neutral component of the group of $\mathbb{R}$-points of the maximal split torus of the center of $G_{\mathbb{Q}}$. Then A_G is a connected Lie subgroup of $Z_\infty = Z_G(F \otimes_{\mathbb{Q}} \mathbb{R}) \subset Z_G(\mathbb{A})$.

Remark 1.1.4. Another source: A_G is contained in $Z_G(F_\infty)$, where F_∞ is the Archimedean part of $\mathbb{A}$.

The restrction of H_G induces an isomorphism $A_G \to \mathfrak{a}_G$.

Now fix a minimal parabolic subgroup P_0 of G with a Levi decomposition $P_0 = M_0 \ltimes U_0$. A parabolic subgroup P of G is called standard if $P_0 \subset P$. A standard parabolic subgroup P of G admits a unique Levi decompositioin $P = M \ltimes U$ with $M_0 \subset M$.

For two standard parabolic subgroups P and Q of G, denote by $W(P, Q)$ the finite set of cosets $w \in G(F)/M_P(F)$ such that $wM_Pw^{-1} = M_Q$. Any $w \in W(P, Q)$ induces a linear isomorphism $\mathfrak{a}_P \to \mathfrak{a}_Q$, which in turn uniquely determines w.

1.2. Siegel Sets.

Lemma 1.2.1. There exists a maximal compact subgroup $K = \prod_v K_v$ of $G(\mathbb{A})$ such that
1. Each K_v is a maximal compact subgroup of $G(k_v)$.
2. $G(\mathbb{A}) = P_0(\mathbb{A})K$.
3. $P(\mathbb{A}) \cap K = (M(\mathbb{A}) \cap K) \ltimes (U(\mathbb{A}) \cap K)$ for every standard parabolic subgroup $P = MU$ of G.

(4) $M(\mathbb{A})\cap K$ is a maximal compact subgroup of $M(\mathbb{A})$ for every standard parabolic subgroup $P = MU$ of G.

Remark 1.2.2. The group K is said to be in good position with respect to (P_0, M_0) in some literature.

Let $P = MU$ be a standard parabolic subgroup of G. Then $G(\mathbb{A}) = P_0(\mathbb{A})K = P(\mathbb{A})K = U(\mathbb{A})M(\mathbb{A})K$. Write every $g \in G(\mathbb{A})$ as $g = umk$, with $u \in U(\mathbb{A})$, $m \in M(\mathbb{A})$, and $k \in K$. This gives a well-defined map $G(\mathbb{A}) \to M(\mathbb{A})/(M(\mathbb{A}) \cap K)$ by sending g to the equivalence class of m. The map $H_M : M(\mathbb{A}) \to \mathfrak{a}_M$ is a continuous map of groups, and thus H_M kills every compact subgroup of $M(\mathbb{A})$ because $\mathfrak{a}_M$ is a finite dimensional real vector space. In particular $M(\mathbb{A})\cap K$ is contained in $M(\mathbb{A})^1$. Therefore a well-defined map $m_P : G(\mathbb{A}) \to M(\mathbb{A})/M(\mathbb{A})^1$ is obtained.

Let $t = (t_\alpha)_{\alpha\in\Delta_0} \in \mathbb{R}^{\Delta_0}_{>0}$. Define $A_{M_0}(t)$ to be the subset of A_{M_0} of elements $x \in A_{M_0}$ such that $\alpha(x) > t_\alpha$ for all $\alpha \in \Delta_0$.

Lemma 1.2.3. There exists a compact subset $\omega \subset P_0(\mathbb{A})$ such that for sufficiently small t, it holds that $G(\mathbb{A}) = G(F)S$ where $S = \omega A_{M_0}(t)K$.

The set S is called a Siegel set.

1.3. **Automorphic Forms.** Write $\mathfrak{X} = G(F)\backslash G(\mathbb{A})$, and more generally,

$$\mathfrak{X}_P = U(\mathbb{A})P(F)\backslash G(\mathbb{A}) = U(\mathbb{A})M(F)\backslash G(\mathbb{A})$$

for every standard parabolic subgroup $P = M \ltimes U$ of G.

Fix a faithful representation $i' : G \to \mathrm{GL}_n$. Define $i : G \to \mathrm{SL}_{2n}$ by

$$g \mapsto \begin{bmatrix} i(g) & \\ & {}^t i(g)^{-1} \end{bmatrix}.$$

The height function $|-| : G(\mathbb{A}) \to \mathbb{R}_{>0}$ is defined by

$$g \mapsto \prod_v \sup_{1\le r,s\le 2n} |i(g)_{r,s}|_v,$$

where v runs through all places of F. A function $\phi : G(\mathbb{A}) \to \mathbb{C}$ has moderate growth if there exists $c \in \mathbb{R}_{>0}$ and $r \in \mathbb{R}$ such that $|\phi(g)| \le c|g|^r$ for all $g \in G(\mathbb{A})$.

Definition 1.3.1. TODO: Define $C^\infty(G(\mathbb{A}))$

Let $\mathfrak{Z}$ be the center of the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$.

Definition 1.3.2. An automorphic form on $\mathfrak{X}$ is a function $\varphi \in C^\infty(G(\mathbb{A}))$ such that

(1) φ is left $G(\mathbb{Q})$ invariant.
(2) φ is of moderate growth.
(3) φ is $\mathfrak{Z}$-finite.
(4) φ is K-finite.

2. The Eisenstein Series

Definition 2.0.1. Let $\xi, \lambda \in X_P$. Let $\phi \in \mathcal{A}(\mathfrak{X}_P)_\xi$, i.e. ϕ is an automorphic form on $\mathfrak{X}_P = P(F)U(F)\backslash G(\mathbb{A})$ such that $\phi(ag) = a^\xi \phi(g)$ for all $a \in A_M$. The Eisenstein series is $E(g, \phi, \lambda) = \sum_{\gamma\in P(F)\backslash G(F)} \phi_\lambda(\gamma g)$.

Lemma 2.0.2. There exists an open cone $\mathcal{C}$ in $\mathfrak{a}^*_P$ such that $E(g, \phi, \lambda)$ converges absolutely for $\lambda \in X_P$ with $\mathrm{Re}(\lambda) \in \mathcal{C}$, and locally uniformly in (g, λ). Moreover, the function $E(-, \phi, \lambda)$ has moderate growth.

2.1. **Intertwining Operators.** Let $P = MU$ and $P' = M'U'$ be two standard parabolics in G. Suppose $w \in G(\mathbb{Q})$ is an element such that $wMw^{-1} = M'$. The element w induces isomorphism $X_P \to X_{P'}$ denoted by $\lambda \mapsto w\lambda$.

Definition 2.1.1. Let $\varphi \in \mathcal{A}(\mathfrak{X}_P)$ be an automorphic form. Let $\lambda \in X_P$. Write $(M(w, \lambda)\varphi)_{w\lambda}$ for the function

$$g \in G(\mathbb{A}) \mapsto \int_{(wUw^{-1}\cap U')(\mathbb{A})\backslash U'(\mathbb{A})} \varphi_\lambda(w^{-1}ug)du$$

whenever the integral converges.

Lemma 2.1.2. There exists an open cone $\mathcal{C} \subset \mathfrak{a}_P^*$ such that the integral converges absolutely and locally uniformly in g and λ when $\mathrm{Re}(\lambda) \in \mathcal{C}$. Moreover, for such λ, the map $(M(w,\lambda)\varphi)_{w\lambda}$ is an automorphic form on $\mathfrak{X}_{P'}$.

Proof. Reduce to the Eisenstein case. □

Lemma 2.1.3. Fix standard parabolic subgroups P and P' of G. Fix $\varphi \in \mathcal{A}(\mathfrak{X}_P)$.

(1) The partially defind map $\lambda \in X_P \mapsto M(w,\lambda)\varphi \in \mathcal{A}(\mathfrak{X}_{P'})$ has a meromorphic extension to all $\lambda \in \mathfrak{X}_P$.
(2) $E(M(w,\lambda)\varphi, w\lambda) = E(\varphi,\lambda)$ for all $\lambda \in X_P$.
(3) Let P'' be another standard parabolic subgroup of G. Suppose $w'M'(w')^{-1} = M''$. Then $M(w'w,\lambda) = M(w', w\lambda) \circ M(w,\lambda)$.

2.2. **Meromorphic Extension.**

Lemma 2.2.1. Let P be a standard parabolic subgroup of G. Let $\varphi \in \mathcal{A}_P$. Then the Eisenstein seris $E(-,\varphi,\lambda)$ admits a meromorphic extension $\lambda \mapsto E(-,\varphi,\lambda) \in \mathfrak{F}_{\mathrm{umg}}(\mathfrak{X})$ on X_P.

3. The Spectral Decomposition

TODO.

Appendix A. Review of algebraic groups

Remark A.0.1. Where are "connected" and "reductive" used?

Let k be a field of characteristic zero. Let G be a connected reductive algebraic group over k. Let T_0 be a maximal split torus in G. Let $M_0 = C_G(T_0)$ be the centralizer of T_0 in G.

Lemma A.0.2. The minimal parabolic subgroups of G are all conjugate by $G(k)$.

Let $\Phi(G,T_0)$ be the set of roots.

Lemma A.0.3. There is a bijection between the set of minimal parabolic subgroups of G containing M_0, and the set of root basis of $\Phi(G,T_0)$.

Let R_uG be the unipotent radical of G. A subgroup M of G is a Levi subgroup if $G = M \ltimes R_uG$.

Lemma A.0.4. The subgroup M_0 is a Levi subgroup of P_0.

A parabolic subgroup P of G is standard if $P_0 \subset P$. A Levi subgroup M of a parabolic subgroup P is standard if $M_0 \subset M$.

Lemma A.0.5. Every standard parabolic subgroup of G has a unique standard Levi subgroup.

Let G be a split reductive group over k. Then there exists a maximal torus T in G that splits, i.e. T is isomorphic to $\mathbb{G}_m^r$ for some $r \geq 1$. Then $X^*(G)$ is a free $\mathbb{Z}$-module of rank r.

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