THE HODGE–TATE PERIOD MAP

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Contents

1. Introduction

1.1. **Notations.** Throughout this paper, $0 \leq \epsilon < 1/2$ is a number such there exists an element in $\mathbb{Z}_p^{\mathrm{cycl}}$ of valuation ϵ, and any such element will be denoted by $p^\epsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$.

2. Technical tools

2.1. Canonical subgroups.

Definition 2.1.1. Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $A \to \mathrm{Spec}(R)$ be an Abelian scheme. Let $m \geq 1$ be an integer. We say that $A \to \mathrm{Spec}(R)$ satisfies the weak $O(m, \epsilon)$ condition if $\mathrm{Ha}(A_1/\mathrm{Spec}(R_1))^{(p^m-1)/(p-1)}$ divides p^ϵ ((as elements in $R_1 = R/p$?? seems wrong)).

((todo: understand the precise meaning of "divide"; Here it seems to mean the following: there exists a section $u \in H^0(\mathrm{Spec}(R_1), \omega_{A_1/R_1}^{\otimes(-p^m+1)/(p-1)})$ such that $u \cdot \mathrm{Ha}(A_1/R_1) = p^\epsilon$.))

Lemma 2.1.2. Let S be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let G be a finite locally free commutative group scheme over S. Let $C_1 \subset G \otimes_S S/p$ be a finite locally free subgroup. Assume that for $H = (G \otimes_S S/p)/C_1$, multiplication by p^ϵ on the Lie complex $\ell_H^\vee$ is homotopical to zero. Then there exists a finite locally free subgroup $C \subset G$ over S such that $C \otimes_S S/p^{1-\epsilon} = C_1 \otimes_{S/p} S/p^{1-\epsilon}$.

Lemma 2.1.3. Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $A \to \mathrm{Spec}(R)$ be an Abelian scheme satisfying weak $O(m, \epsilon)$. Then there is a unique closed subgroup $C_m \subset A[p^m]$ such that $C_m = \ker(F^m)$ mod $p^{1-\epsilon}$.

Definition 2.1.4. Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. We say that an Abelian scheme $A \to \mathrm{Spec}(R)$ has a weak canonical subgroup of level m if $A \to \mathrm{Spec}(R)$ satisfies weak $O(m, \epsilon)$ for some $\epsilon < 1/2$. In that case, we call $C_m \subset A[p^m]$ in Lemma 2.1.3 the weak canonical subgroup of level m.

((todo: strong $O(m, \epsilon)$) If moreover $\mathrm{Ha}(A_1/\mathrm{Spec}(R_1))^{p^m}$ divides p^ϵ, then we say that C_m is a strong canonical subgroup. Here, $R_1 = R/p$ and $A_1 \to \mathrm{Spec}(R_1)$ is the reduction of $A \to \mathrm{Spec}(R)$.

Speculation 2.1.5. ((todo: check: Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $A \to \mathrm{Spec}(R)$ be an Abelian scheme satisfying strong $O(m, \epsilon)$, with strong canonical subgroup $C \subset A[p]$ of level 1. Then $(A/C)_{1-\epsilon} \simeq A_{1-\epsilon}^{(p)}$, and in particular

$$\mathrm{Ha}((A/C)_{1-\epsilon}/R_{1-\epsilon}) = \mathrm{Ha}(A_{1-\epsilon}/R_{1-\epsilon})^p.$$

))

Lemma 2.1.6. Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let A and B be Abelian schemes over R.

(1) If A has a canonical subgroup $C_m \subset A[p^m]$ of level m, then it has a canonical subgroup $C_{m'} \subset A[p^{m'}]$ of every level $m' \le m$, and $C_{m'} \subset C_m$.
(2) Let $f : A \to B$ be a map of Abelian schemes. Assume that both A and B have canonical subgroups $C_m \subset A[p^m]$ and $D_m \subset B[p^m]$ of level m. Then C_m maps into D_m under f.
(3) Assume that A has a canonical subgroup $C_m \subset A[p^m]$ of level m, and let $\overline{x}$ be a geometric point of $\mathrm{Spec}(R[p^{-1}])$. Then $C_m(\overline{x}) \simeq (\mathbb{Z}/p^m\mathbb{Z})^g$, where g is dimension of the Abelian variety over $\overline{x}$.

2.2. **Hartog's extension principle.**

Lemma 2.2.1 ([GR68, Lemma III.3.1, Proposition III.3.3]). Let X be a locally Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $n \ge 1$ be an integer. Then the following are equivalent:

(1) For any open subscheme V of X, the map

$$H^i(V, \mathcal{F}) \to H^i(V \backslash Z, \mathcal{F})$$

is bijective for $i \le n - 2$ and injective for $i = n - 1$.
(2) For any open subscheme V of X, the local cohomology

$$H_{V \cap Z}^i(V, \mathcal{F}) = 0$$

for all $i \le n - 1$.
(3) For any $x \in Z$ the depth of $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module is at least n.

Lemma 2.2.2 (Serre's criterion). A Noetherian ring R is normal if and only if $R_{\mathfrak{p}}$ is regular for every $\mathfrak{p}$ of height ≤ 1 and $R_{\mathfrak{p}}$ has depth ≥ 2 for every $\mathfrak{p}$ of height ≥ 2.

Lemma 2.2.3. Let X be a locally Noetherian normal scheme. Let U be an open subscheme of X with codimension ≥ 2. Then the map $H^0(X, \mathcal{O}_X) \to H^0(U, \mathcal{O}_X)$ is an isomorphism.

Proof. We may assume that $X = \mathrm{Spec}(A)$ where A is normal integral domain. For every non-empty open V of X, the ring $\Gamma(V, \mathcal{O}_X)$ may be considered as a subring of the function field $K(X) = \mathrm{Frac}(A)$ such that the restriction maps are given by inclusions of rings. Let Z be an irreducible closed subset of X of codimension 1. Then U intersects Z non-trivially, so it contains the generic point η of Z. In other words, the subring $\Gamma(U, \mathcal{O}_X)$ of the function field $K(X)$ is contained in the stalk $\mathcal{O}_{X,\eta}$. But $A = \Gamma(X, \mathcal{O}_X)$ is the intersection of all the stalks $\mathcal{O}_{X,\eta}$, where η is a prime ideal of height 1; in other words, where η is the generic point of an irreducible closed subset of codimension 1. $\square$

Lemma 2.2.4. Let R be a normal ring, i.e. the localization $R_{\mathfrak{p}}$ is an integrally closed domain for every prime ideal $\mathfrak{p}$ of R. Assume R is Noetherian. Let $Z \subset \mathrm{Spec}(R)$ be a closed subscheme of codimension at least 2, i.e. every $\mathfrak{p} \in Z$ has height at least 2. Then for $U = \mathrm{Spec}(R) \backslash Z$,

$$H^0(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}) \simeq H^0(U, \mathcal{O}_{\mathrm{Spec}(R)}).$$

Proof. Consider $n = 2$ and $\mathcal{F} = \mathcal{O}_X$ in Lemma 2.2.1. Serre's criterion, cf. Lemma 2.2.2, guarantees the third condition in Lemma 2.2.1. The first assertion gives the desired result. $\square$

Lemma 2.2.5. Let R be a topologically finitely generated, flat, and p-adically complete $\mathbb{Z}_p$-algebra, such that $\overline{R} = R/p$ is normal. Fix $f \in R$ such that its reduction $\overline{f} \in \overline{R}$ is not a zero-divisor. Let $0 < \epsilon \leq 1$. Set $S = (R \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u\rangle/(u \cdot f - p^\epsilon)$. Then S is p-adically complete and flat over $\mathbb{Z}_p^{\mathrm{cycl}}$. Fix a closed subscheme $Y \subset \mathrm{Spec}(\overline{R})$ of codimension ≥ 2. Let Z be the inverse image of Y in $\mathrm{Spf}(S)$. Then for $U = |\mathrm{Spf}(S)| \backslash Z$,

$$S = H^0(\mathrm{Spf}(S), \mathcal{O}_{\mathrm{Spf}(S)}) \simeq H^0(U, \mathcal{O}_{\mathrm{Spf}(S)}).$$

Proof. We first show that the map

$$S \simeq H^0(\mathrm{Spf}(S), \mathcal{O}_{\mathrm{Spf}(S)}) \to H^0(U, \mathcal{O}_{\mathrm{Spf}(S)})$$

is injective. Since S is p-adically separated and $H^0(U, \mathcal{O}_{\mathrm{Spf}(S)})$ is flat over $\mathbb{Z}_p^{\mathrm{cycl}}$, it suffices to show that

$$S_\epsilon \simeq H^0(\mathrm{Spec}(S_\epsilon), \mathcal{O}_{\mathrm{Spec}(S_\epsilon)}) \to H^0(U_\epsilon, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$$

is injective, where $S_\epsilon = S/p^\epsilon$, Z_ϵ is the inverse image of Y in $\mathrm{Spec}(S_\epsilon)$, and $U_\epsilon = \mathrm{Spec}(S_\epsilon) \backslash Z_\epsilon$. Note that

$$S_\epsilon = (R \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u\rangle/(uf, p^\epsilon) = R_\epsilon[u]/(uf_\epsilon)$$

where $R_\epsilon = R \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)$ and $f_\epsilon \in R_\epsilon$ is the image of $f \in R$.

Let $W \subset \mathrm{Spec}(S_\epsilon)$ be the preimage of $V = V(\overline{f}) \subset \mathrm{Spec}(\overline{R})$. Then $W = V \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathbb{A}^1_{\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon}$ is affine. The map $S_\epsilon \to R_\epsilon$ sending u to zero induces a section $\mathrm{Spec}(R_\epsilon) \to \mathrm{Spec}(S_\epsilon)$. We have a decomposition $\mathrm{Spec}(S_\epsilon) = N \cup W$, where $N = \mathrm{Spec}(R_\epsilon[u]/(u)) \simeq \mathrm{Spec}(R_\epsilon)$ is the image of the section $\mathrm{Spec}(R_\epsilon) \to \mathrm{Spec}(S_\epsilon)$. Take $V_\epsilon = V \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)$. Then $W = V_\epsilon \times_{\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)} \mathbb{A}^1_{\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon}$, and $N \cap W = V_\epsilon$.

We then have the following interpretations:

(1) Each section in $\Gamma(\mathrm{Spec}(S_\epsilon), \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$ is a pair (f_1, f_2) such that $f_1 \in \Gamma(N, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$ and $f_2 \in \Gamma(W, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$ such that $f_1 = f_2$ on $N \cap W = V_\epsilon$.
(2) Each section in $H^0(U_\epsilon, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$ is a pair (f_1, f_2) such that $f_1 \in H^0(U_\epsilon \cap N, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$, and $f_2 \in H^0(U_\epsilon \cap W, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$, such that $f_1 = f_2$ on $U_\epsilon \cap N \cap W$.

The (classical) Hartog's extension principle, i.e. Lemma 2.2.4 applied to $Y \subset \mathrm{Spec}(\overline{R})$, shows that

$$\Gamma(\mathrm{Spec}(\overline{R}) \backslash Y) \simeq \Gamma(\mathrm{Spec}(\overline{R})).$$

Under base-change this gives

$$\Gamma(U_\epsilon \cap N) \simeq \Gamma(\mathrm{Spec}(\overline{Y}) \backslash Y) \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon \simeq \Gamma(\mathrm{Spec}(\overline{R})) \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon \simeq \Gamma(N).$$

Thus injectivity reduces to show that

$$\Gamma(V) \otimes_{\mathbb{F}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)[u]\Gamma(W) \to \Gamma(U_\epsilon \cap W) = \Gamma(V \backslash Y) \otimes_{\mathbb{F}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)[u]$$

is injective. It suffices to show that $\Gamma(V) \to \Gamma(V \backslash Y)$ is injective, where both V and $V \backslash Y$ are $\mathbb{F}_p$-schemes. We have $\mathrm{depth}(\mathcal{O}_{V,y}) = \mathrm{depth}(\overline{R}_y) - 1$ for all $y \in V$, cf. [SGA-2/III/2.5]. Thus $\mathrm{depth}(\mathcal{O}_{V,y}) \geq 1$ for every $x \in V \cap Y$ by Serre's criterion, i.e. Lemma 2.2.2. Then the desired injectivity follows from Lemma 2.2.1.

We then show the surjectivity.

((todo: ...)) $\square$

2.3. **Tate's normalized traces.**

Lemma 2.3.1. Let R be a p-adically complete flat $\mathbb{Z}_p$-algebra. Let $Y_1, \ldots, Y_n \in R$. Let $P_1, \ldots, P_n \in R\langle X_1, \ldots, X_n \rangle$ be topologically nilpotent elements, or equivalently, each P_i has topologically nilpotent coefficients in R. Let

$$S = R\langle X_1, \ldots, X_n \rangle / (X_1^p - Y_1 - P_1, \ldots, X_n - Y_n - P_n).$$

Then

(1) The ring S is a finite free R-module of rank p^n, with a basis given by $X_1^{i_1} \cdots X_n^{i_n}$ with $0 \leq i_1, \ldots, i_n \leq p - 1$.

(2) Let I be the ideal of R generated by p together with all the coefficients of all P_i. Then the trace map $\operatorname{tr}_{S/R} : S \to R$ sends S to I^n, i.e. $\operatorname{tr}_{S/R}(S) \subset I^n$.

Lemma 2.3.2. Let R be a p-adically complete flat $\mathbb{Z}_p$-algebra topologically of finite type, formally smooth of dimension n over $\mathbb{Z}_p$. Let $f \in R$ such that its reduction $\overline{f} \in \overline{R} = R/p$ is not a zero-divisor. Let $0 \leq \epsilon < 1/2$. Let

$$S_\epsilon = (R \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u_\epsilon \rangle / (u_\epsilon \cdot f - p^\epsilon).$$

Suppose $\varphi : S_\epsilon \to S_{\epsilon/p}$ is a map of $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra such that modulo $p^{1-\epsilon}$ it is given by the relative Frobenius. In other words, $\varphi \mod p^{1-\epsilon}$ is the map

$$R_{1-\epsilon}[u_\epsilon]/(f \cdot u_\epsilon - p^\epsilon) \to R_{1-\epsilon}[u_{\epsilon/p}]/(f \cdot u_{\epsilon/p} - p^{\epsilon/p}),$$

where $R_{1-\epsilon} = \overline{R} \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^{1-\epsilon})$, which sends u_ϵ to $u_{\epsilon/p}^p$, and restricts to $\operatorname{Fr}_{\overline{R}} \otimes \operatorname{id}$ on $R_{1-\epsilon}$. Then

(1) The map

$$\varphi[1/p] : S_\epsilon[1/p] \to S_{\epsilon/p}[1/p]$$

is finite and flat of degree p^n.

(2) The trace map

$$\operatorname{tr} = \operatorname{tr}_{S_{\epsilon/p}[1/p]/S_\epsilon[1/p]} : S_{\epsilon/p}[1/p] \to S_\epsilon[1/p]$$

sends $S_{\epsilon/p}$ into $p^{n-(2n+1)\epsilon} S_\epsilon$. Here $S_{\epsilon/p}[1/p]$ is viewed as an $S_\epsilon[1/p]$-algebra via $\varphi[1/p]$.

2.4. **Riemann's Hebbarkeitssatz.**

Definition 2.4.1. Let p be a prime. Let K be a perfectoid field (of any characteristic). Let t be a non-zero element of K with $|p| \leq |t| < 1$. A triple $(\mathcal{X}, \mathcal{Z}, \mathcal{U})$, where $\mathcal{X}$ is an affinoid perfectoid space over K, $\mathcal{Z}$ is a closed subset of $\mathcal{X}$, and $\mathcal{U}$ is a quasi-compact open subset of $\mathcal{X} \backslash \mathcal{Z}$, is said to be good, if

$$H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^+/t)^a \simeq H^0(\mathcal{X} \backslash \mathcal{Z}, \mathcal{O}_{\mathcal{X}}^+/t)^a \hookrightarrow H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}}^+/t)^a.$$

Remark 2.4.2. This notion is independent of the choice of t, and is compatible with tilting.

Situation 2.4.3. Let $K = \mathbb{F}_p((t^{1/p^\infty}))$. Let R_0 be a reduced Tate K-algebra topologically of finite type. Let $\mathcal{X}_0 = \operatorname{Spa}(R_0, R_0^\circ)$ be the associated affinoid adic space of finite type over K. Let R be the completed perfection of R_0, which is a p-finite perfectoid K-algebra. Let $\mathcal{X} = \operatorname{Spa}(R, R^+)$ with $R^+ = R^\circ$, the associated p-finite affinoid perfectoid space over K. Let I_0 be an ideal of R_0. Let $I = I_0 R \subset R$. Let $\mathcal{Z}_0 = V(I_0) \subset \mathcal{X}_0$. Let $\mathcal{Z} = V(I) \subset \mathcal{X}$. Let $\mathcal{U}_0$ be a quasi-compact open subset of $\mathcal{X}_0 \backslash \mathcal{Z}_0$ with preimage $\mathcal{U} \subset \mathcal{X} \backslash \mathcal{Z}$.

Lemma 2.4.4. Assume Situation 2.4.3. Suppose $(\mathcal{X}, \mathcal{Z}, \mathcal{U})$ is good. Suppose that R_0 is normal, and that $V(I_0) \subset \operatorname{Spec}(R_0)$ is of codimension ≥ 2. Let R_0' be a finite normal R_0-algebra which is étale outside $V(I_0)$, and such that no irreducible component of $\operatorname{Spec}(R_0')$ maps into $V(I_0)$. Let $I_0' = I_0 R_0'$, and $\mathcal{U}_0' \subset \mathcal{X}_0'$ the preimage of $\mathcal{U}_0$. Let R', I', $\mathcal{X}'$, $\mathcal{Z}'$, $\mathcal{U}'$ be the associated perfectoid objects.

(1) There is a perfect trace pairing

$$\operatorname{tr}_{R_0'/R_0} : R_0' \otimes_{R_0} R_0' \to R_0.$$

(2) The trace pairing induces a trace pairing

$$\operatorname{tr}_{R'^\circ/R^\circ} : R'^\circ \otimes_{R^\circ} R'^\circ \to R^\circ.$$

which is almost perfect.

(3) For all open subsets $\mathcal{V} \subset \mathcal{X}$ with preimage $\mathcal{V}' \subset \mathcal{X}'$, the trace pairing induces an isomorphism

$$H^0(\mathcal{V}', \mathcal{O}_{\mathcal{X}'}^+/t)^a \simeq \operatorname{Hom}_{R^\circ/t}(R'^\circ/t, H^0(\mathcal{V}, \mathcal{O}_{\mathcal{X}}^+/t))^a.$$

(4) The triple $(\mathcal{X}', \mathcal{Z}', \mathcal{U}')$ is good.

(5) If $\mathcal{X}' \to \mathcal{X}$ is surjective, then the map

$$H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^+/t) \to H^0(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}^+/t) \cap H^0(\mathcal{U}, \mathcal{O}_{\mathcal{X}}^+/t)$$

is an almost isomorphism.

Lemma 2.4.5. Suppose we have a filtered inductive system $(R_0^{(i)})_{i \in I}$ as in the previous lemma, giving rise to $\mathcal{X}^{(i)}$, $\mathcal{Z}^{(i)}$, $\mathcal{U}^{(i)}$. Assume that all transition maps $\mathcal{X}^{(i)} \to \mathcal{X}^{(j)}$ are surjective. Let $\widetilde{\mathcal{X}}$ be the inverse limit of the $\mathcal{X}^{(i)}$ in the category of perfectoid spaces over K, with preimage $\widetilde{\mathcal{Z}} \subset \widetilde{\mathcal{X}}$ of $\mathcal{Z}$, and $\widehat{\mathcal{U}} \subset \widetilde{\mathcal{X}} \backslash \widehat{\mathcal{U}}$ of $\mathcal{U}$. Then the triple $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{Z}}, \widehat{\mathcal{U}})$ is good.

2.5. **The Hodge–Tate filtration.**

Lemma 2.5.1. Let C be an algebraically closed and complete extension of $\mathbb{Q}_p$. Let $A \to \mathrm{Spec}(C)$ be an Abelian variety. Then A has its Hodge–Tate filtration

$$0 \to \mathrm{Lie}(A)(1) \to T_p(A) \otimes_{\mathbb{Z}_p} C \to (\mathrm{Lie}(A^\vee))^* \to 0.$$

((todo: ...))

3. Siegel Modular Varieties

Let p be a fixed prime. Let $g \geq 1$ be an integer.

Definition 3.0.1. The symplectic similitude group GSp_{2g} is the reductive group scheme over $\mathbb{Z}$ whose points in a commutative ring R are given by

$$\mathrm{GSp}_{2g}(R) = \{x \in \mathrm{GL}_{2g}(V); \exists \nu(x) \in R^\times, x^t \Omega x = \nu(x)\Omega\}$$

where $\Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ is the standard symplectic matrix of order $2g$.

In the following discussion, we write $G = \mathrm{GSp}_{2g}$. Let $K_p = G(\mathbb{Z}_p)$. Let K^p be a compact open subgroup of $G(\mathbb{A}^{\infty,p})$ that is contained in

$$\Gamma(N)^{(p)} = \{g \in G(\mathbb{A}^{\infty,p}); g \equiv 1 \bmod N\}$$

for some integer $N \geq 3$ not divisible by p.

Definition 3.0.2. Let $m \geq 1$ be an integer.

$$\Gamma_0(p^m) = \left\{ g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \bmod p^m \right\}$$

$$\Gamma_s(p^m) = \left\{ g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \bmod p^m, \nu(g) \equiv 1 \bmod p^m \right\}$$

$$\Gamma_1(p^m) = \left\{ g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \bmod p^m \right\}$$

$$\Gamma(p^m) = \left\{ g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod p^m \right\}$$

Let X be the scheme over $\mathrm{Spec}(\mathbb{Z}_{(p)})$ classifying principally polarized projective Abelian schemes of relative dimension g with level K^p structures. Let X^* be the minimal compactification of X.

((todo: add references))

For each $U \in \{\Gamma(p^m), \Gamma_s(p^m), \Gamma_0(p^m)\}$, we have a scheme $X_{U,\mathbb{Q}}$ over $\mathbb{Q}$ with certain moduli interpretations.

Let $\mathfrak{X}$ be the formal scheme over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$ defined as the p-completion of $X_{\mathbb{Z}_p^{\mathrm{cycl}}} = X \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} \mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$.

Remark 3.0.3. ((todo: understand the moduli interpretation as stated in [Zhu23]:

The moduli interpretations can be described as follows.

(1) $\mathrm{Sh}_{K^p G(\mathbb{Z}_p), \mathbb{Z}_{(p)}}$ represents the following problem $S \mapsto \{(A, \lambda, \eta)\}/\sim$ where

- A is a projective Abelian scheme over S of relative dimension g.
- λ is a principal polarization of A.
- η is a level K^p structure on A.

(2) $\mathrm{Sh}_{K^p\Gamma(p^m),\mathbb{Q}}$ represents the following problem $S \mapsto \{(A, \lambda, \eta, \eta_p)\}/\sim$ where
- $(A, \lambda, \eta) \in \mathrm{Sh}_{K^p G(\mathbb{Z}_p), \mathbb{Z}_{(p)}}(S)$.
- η_p is a level $\Gamma(p^m)$ structure on A.

(3) $\mathrm{Sh}_{K^p\Gamma_0(p^m),\mathbb{Q}}$ represents the following problem $S \mapsto \{(A, \lambda, \eta, D)\}/\sim$ where
- $(A, \lambda, \eta) \in \mathrm{Sh}_{K^p G(\mathbb{Z}_p), \mathbb{Z}_{(p)}}(S)$.
- D is a totally isotropic subgroup of $A[p^m]$.

(4) $\mathrm{Sh}_{K^p\Gamma_s(p^m),\mathbb{Q}}$ represents the following problem $S \mapsto \{(A, \lambda, \eta, D, t)\}/\sim$ where
- $(A, \lambda, \eta, D) \in \mathrm{Sh}_{K^p\Gamma_0(p^m),\mathbb{Q}}(S)$.
- $t : \mu_{p^m} \to \mathbb{Z}/p^m\mathbb{Z}$ is an isomorphism.

))

The universal Abelian scheme $A \to X$ gives a line bundle $\omega = \omega_{A/S} = \bigwedge^g \Omega^1_{A/X}$. The sheaf ω extends to the minimal compactification X^*. The Hasse invariant defines a section $\mathrm{Ha} \in H^0(X_{\mathbb{F}_p}, \omega^{\otimes(p-1)})$. The section Ha extends to $\mathrm{Ha} \in H^0(X^*_{\mathbb{F}_p}, \omega^{\otimes(p-1)})$. For $g = 1$, this follows from direct inspection. For $g \geq 2$, this follows from the (classical) Hartog's extension principle. ((todo: clearify this paragraph))

Let $\mathfrak{A} \to \mathfrak{X}$ be the universal formal Abelian scheme.

Speculation 3.0.4. ((todo: check: Minimal compactification is compatible with p-completion.))

4. The anti-canonical towers

4.1. The Frobenius tower of formal models.

Lemma 4.1.1. Let S be a p-adically complete $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. There is a bijection
$$\mathrm{Hom}_{\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})}(\mathrm{Spf}(S), \mathfrak{X}^*) \simeq \mathrm{Hom}_{\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})}(\mathrm{Spec}(S), X^*_{\mathbb{Z}_p^{\mathrm{cycl}}}).$$

Speculation 4.1.2. ((todo: check: Let Y be a scheme over $\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$. Let $\mathfrak{Y}$ be the formal scheme over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$ obtained as the p-completion of Y. Let S be a p-adically complete $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Then there is a bijection
$$\mathrm{Hom}_{\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})}(\mathrm{Spf}(S), \mathfrak{Y}) \simeq \mathrm{Hom}_{\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})}(\mathrm{Spec}(S), Y).$$
))

Definition 4.1.3. Let $\mathcal{N}_\epsilon$ be the functor sending a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra S to the set of pairs $(f, [u])$, where

- f is a map $\mathrm{Spf}(S) \to \mathfrak{X}^*$; it's equivalent to a map $\mathrm{Spec}(S) \to X^*_{\mathbb{Z}_p^{\mathrm{cycl}}}$ by Lemma 4.1.1.
- Let $\overline{f} : \mathrm{Spec}(S/p) \to X^*_{\mathbb{F}_p}$ be the reduction of $\mathrm{Spec}(S) \to X^*_{\mathbb{Z}_p^{\mathrm{cycl}}}$. Recall that we have the Hasse section $\mathrm{Ha} \in H^0(X^*_{\mathbb{F}_p}, \omega^{\otimes(p-1)})$. It pullbacks to $\overline{f}^*\mathrm{Ha} \in H^0(\mathrm{Spec}(S/p), \overline{f}^*\omega^{\otimes(p-1)})$. Then $[u]$ is an equivalence class of sections $u \in H^0(\mathrm{Spec}(S), f^*\omega^{\otimes(1-p)})$ satisfying $u \cdot \overline{f}^*\mathrm{Ha} = p^\epsilon \in S/p$ under the equivalence relation that $u \sim u'$ if and only if there exists some $h \in S$ such that $u' = u(1 + p^{1-\epsilon}h)$.

Speculation 4.1.4. ((todo: check: Let S be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $A \to \mathrm{Spec}(S)$ be an Abelian scheme. Then ω_{A_1/S_1} (or even $\omega_{A/S}$) is the trivial line bundle, where $S_1 = S/p$.)) ((todo: I don't think this could be true; the correct version is the following. The line bundle $\omega_{A/S}$ is locally free of rank 1. Then there exists an affine covering $\mathrm{Spec}(S) = \bigcup_i \mathrm{Spec}(S_i)$ where every S_i is p-adically complete an flat over $\mathbb{Z}_p^{\mathrm{cycl}}$, such that $\omega_{A/S}$ restricts to the trivial line bundle on every $\mathrm{Spec}(S_i)$.))

Lemma 4.1.5. Then the functor $\mathcal{N}_\epsilon$ is representable by a formal scheme flat over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$. ((todo: moreover, locally it is ...))

((todo: examine this: here it seems that R needs to be chosen such that ω is trivial on R, cf. Speculation 4.1.4; such R covers X^*)) For $\mathrm{Spf}(R) \subset \mathfrak{X}^*_{\mathbb{Z}_p}$, we have
$$\mathcal{N}_\epsilon \times_{\mathfrak{X}^*} \mathrm{Spf}(R \,\widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}}) = \mathrm{Spf}((R \,\widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u \rangle/(u\widetilde{\mathrm{Ha}} - p^\epsilon))$$

where $\widetilde{\mathrm{Ha}} \in H^0(\mathrm{Spec}(R), \omega^{\otimes(p-1)})$ is a lift of $\mathrm{Ha} \in H^0(\mathrm{Spec}(R/p), \omega^{\otimes(p-1)})$.

((remark: it seems that the explicit local construction of $\mathfrak{X}^*(\epsilon)$ is used only when we need to apply Lemma 2.2.5; it should be sufficient to check on a specific cover for the unique extension problem, so picking a cover doesn't seem to cause trouble; need to inspect the situation more closely))

Definition 4.1.6. Let $\mathfrak{X}(\epsilon) \to \mathfrak{X}$ be the pullback of $\mathfrak{X}^*(\epsilon) \to \mathfrak{X}^*$ along $\mathfrak{X} \to \mathfrak{X}^*$. Let $\mathfrak{A}(\epsilon) \to \mathfrak{X}(\epsilon)$ be the pullback of $\mathfrak{A} \to \mathfrak{X}$ along $\mathfrak{X}(\epsilon) \to \mathfrak{X}$.

Let $\mathcal{X}$ be the generic fiber of the adic space associated to the formal scheme $\mathfrak{X}$. Let $\mathcal{X}(\epsilon)$ be the generic fiber of the adic space associated to $\mathfrak{X}(\epsilon)$. Then $\mathcal{X}$ admits an open embedding to the X^{ad}, the adic space associated to the scheme $X_{\mathbb{Q}_p^{\mathrm{cycl}}}$. Let $\mathcal{X}_{\Gamma_\mathrm{s}(p^m)}$ be the inverse image of $\mathcal{X}$ under the map $X_{\Gamma_\mathrm{s}(p^m)}^{\mathrm{ad}} \to X^{\mathrm{ad}}$.

Remark 4.1.7. ((todo: moduli interpretation of $\mathfrak{X}(\epsilon)$. Should be almost identical to $\mathcal{M}_\epsilon$.))

Definition 4.1.8. For a formal scheme $\mathfrak{Y}$ over $\mathbb{Z}_p^{\mathrm{cycl}}$ and $a \in \mathbb{Z}_p^{\mathrm{cycl}}$, we write $\mathfrak{Y}/a$ for $\mathfrak{Y} \times_{\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})} \mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/a)$.

Definition 4.1.9. For a formal scheme $\mathfrak{Y}$ over $\mathbb{Z}_p^{\mathrm{cycl}}/p$, we write $\mathfrak{Y}^{(p)}$ for the pullback of $\mathfrak{Y}$ along the (absolute) Frobenius $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p) \to \mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$.

Lemma 4.1.10. We have a natural isomorphism
$$(\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \simeq \mathfrak{X}^*(\epsilon)/p$$
of formal schemes over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$. Furthermore, by pullback we get the following commutative diagram

$$
\begin{array}{ccccc}
(\mathfrak{A}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{A}(\epsilon)/p & \longrightarrow & \mathfrak{X}(\epsilon)/p & \longrightarrow & \mathfrak{X}^*(\epsilon)/p
\end{array}
$$

where each vertical map is an isomorphism.

Proof. Let S be a ((discrete? flat)) $(\mathbb{Z}_p^{\mathrm{cycl}}/p)$-algebra. Then
$$(\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)}(S) = (\mathfrak{X}^*(p^{-1}\epsilon)/p)(\mathrm{Fr}_*S),$$
where Fr_*S is the the $(\mathbb{Z}_p^{\mathrm{cycl}}/p)$-algebra obtained from S by precomposing with $\mathrm{Fr} : \mathbb{Z}_p^{\mathrm{cycl}}/p \to \mathbb{Z}_p^{\mathrm{cycl}}/p$. Each map $\mathrm{Spf}(\mathrm{Fr}_*S) \to \mathfrak{X}^*(p^{-1}\epsilon)/p$ is equivalent to a pair $(f, [u])$, where

- $f : \mathrm{Spec}(\mathrm{Fr}_*S) \to X^*_{\mathbb{Z}_p^{\mathrm{cycl}}}$ is a map over $\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$.
- $u \in H^0(\mathrm{Spec}(\mathrm{Fr}_*S), f^*\omega^{\otimes(1-p)})$ is a section such that $u \cdot f^*\mathrm{Ha} = p^{p^{-1}\epsilon} \in \mathrm{Fr}_*S$. Note that $(\mathrm{Fr}_*S)/p = \mathrm{Fr}_*S$ since S is defined over $\mathbb{Z}_p^{\mathrm{cycl}}/p$.

Recall that $X^*_{\mathbb{Z}_p^{\mathrm{cycl}}} = X^*_{\mathbb{Z}_p} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$, and thus $(f, [u])$ is equivalent ((todo: should be more precise)) to the following datum

- $f : \mathrm{Spec}(\mathrm{Fr}_*S) \to X^*_{\mathbb{Z}_p}$ is a map over $\mathrm{Spec}(\mathbb{Z}_p)$.
- ((todo: Check the reduction of u)) $u \in H^0(\mathrm{Spec}(\mathrm{Fr}_*S), f^*\omega^{\otimes(1-p)})$ is a section such that $u \cdot f^*\mathrm{Ha} = p^{p^{-1}\epsilon} \in \mathrm{Fr}_*S$.

Note that the Frobenius on $\mathbb{Z}_p/p = \mathbb{F}_p$ is simply the identity, and thus the map $\mathrm{Spec}(\mathrm{Fr}_*S) \to \mathrm{Spec}(\mathbb{Z}_p)$ is identical to $\mathrm{Spec}(S) \to \mathrm{Spec}(\mathbb{Z}_p)$. But under this identification the element $p^{p^{-1}\epsilon} \in \mathrm{Fr}_*S$ corresponds to $p^\epsilon \in S$. Then $f : \mathrm{Spec}(\mathrm{Fr}_*S) \to X^*_{\mathbb{Z}_p}$ can be reinterpreted as a map $g : \mathrm{Spec}(S) \to X^*_{\mathbb{Z}_p}$ over $\mathrm{Spec}(\mathbb{Z}_p)$. We write $v = u$ for clarity. The section v then satisfies $v \cdot g^*\mathrm{Ha} = p^\epsilon \in S$. The pair $(g, [v])$ then corresponds to a map $\mathrm{Spf}(S) \to \mathfrak{X}^*(\epsilon)/p$ over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$. $\qquad\square$

Lemma 4.1.11. The Frobenius map $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p) \to \mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$ induces the following commutative diagram

$$
\begin{array}{ccccc}
\mathfrak{A}(p^{-1}\epsilon)/p & \longrightarrow & \mathfrak{X}(p^{-1}\epsilon)/p & \longrightarrow & \mathfrak{X}^*(p^{-1}\epsilon)/p \\
\downarrow & & \downarrow & & \downarrow \\
(\mathfrak{A}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)}
\end{array}
$$

Proof. This follows from the universal property of pullback. $\qquad\square$

Remark 4.1.12. ((todo: Explain the moduli interpretation of

$$\mathfrak{X}^*(p^{-1}\epsilon)/p \to (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \simeq \mathfrak{X}^*(\epsilon)/p.$$

))

Speculation 4.1.13. ((todo: check: Let S be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $f : \mathrm{Spf}(S) \to \mathfrak{X}$ be a map over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$. Let $A \to \mathrm{Spec}(S)$ be the corresponding Abelian scheme. Suppose $A \to \mathrm{Spec}(S)$ satisfies strong $O(1,\epsilon)$. Let C be the strong canonical subgroup of $A \to \mathrm{Spec}(S)$ of level 1. Then $B = A/C$ satisfies weak $O(1,\epsilon)$.))

Speculation 4.1.14. ((todo: cf. [Wed99]))

Lemma 4.1.15. There is a unique commutative diagram

$$\begin{array}{ccccc}
\mathfrak{A}(p^{-1}\epsilon) & \longrightarrow & \mathfrak{X}(p^{-1}\epsilon) & \longrightarrow & \mathfrak{X}^*(p^{-1}\epsilon) \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{A}(\epsilon) & \longrightarrow & \mathfrak{X}(\epsilon) & \longrightarrow & \mathfrak{X}^*(\epsilon)
\end{array}$$

that is identified with the following commutative diagram from Lemma 4.1.10 and Lemma 4.1.11, after modulo $p^{1-\epsilon}$.

$$\begin{array}{ccccc}
\mathfrak{A}(p^{-1}\epsilon)/p & \longrightarrow & \mathfrak{X}(p^{-1}\epsilon)/p & \longrightarrow & \mathfrak{X}^*(p^{-1}\epsilon)/p \\
\downarrow & & \downarrow & & \downarrow \\
(\mathfrak{A}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{A}(\epsilon)/p & \longrightarrow & \mathfrak{X}(\epsilon)/p & \longrightarrow & \mathfrak{X}^*(\epsilon)/p
\end{array}$$

Proof. ((todo: finish the proof: The map $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon)$ comes from the moduli interpretation, the weak canonical subgroup, and the Hasse invariant. Then $\mathfrak{A}(p^{-1}\epsilon) \to \mathfrak{A}(\epsilon)$ is obtained by base-change. The extension to $\mathfrak{X}^*(p^{-1}\epsilon) \to \mathfrak{X}^*(\epsilon)$ is done using Hartog's extension principle.))

We first construct the map $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon)$. Let S be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $(f, [u])$ be a pair where

- $f : \mathrm{Spf}(S) \to \mathfrak{X}$ is a map of formal schemes over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$; its equivalent to a map $f : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$.
- $u \in H^0(\mathrm{Spec}(S), f^*\omega^{\otimes(1-p)})$ is a section such that $u \cdot \overline{f}^*\mathrm{Ha} = p^{p^{-1}\epsilon} \in S/p$.

The map $f : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ gives an Abelian scheme $A \to \mathrm{Spec}(S)$ ((todo: with principal polarization and level K^p structure)). We claim that $A \to \mathrm{Spec}(S)$ satisfies strong $O(1,\epsilon)$, i.e. $\mathrm{Ha}(A_1/\mathrm{Spec}(S_1))^p$ divides p^ϵ. This follows from

$$p^{p^{-1}\epsilon} = u \cdot \overline{f}^*\mathrm{Ha} = u \cdot \mathrm{Ha}(A_1/\mathrm{Spec}(S_1)).$$

Let $C \subset A[p]$ be the strong canonical subgroup of level 1. We get an Abelian scheme $A/C \to \mathrm{Spec}(S)$ ((todo: explain: equipped with induced polarization and level structure: use totally isotropic)), which corresponds to a map $g : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$. This gives a map $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}$. We will show next that it can be factored as $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon) \to \mathfrak{X}$.

((seems wrong: Then we declare that the pair $(f, [u])$ gets mapped to the pair $(g, [u^p])$.))

((seems wrong: By Speculation 4.1.13, the quotient $A/C \to \mathrm{Spec}(S)$ satisfies weak $O(1,\epsilon)$, i.e. there exists a section $v \in H^0(\mathrm{Spec}(S), \overline{g}^*\omega^{\otimes(1-p)})$ such that $v \cdot \overline{g}^*\mathrm{Ha} = p^\epsilon$. Then we declare that the pair $(f, [u])$ gets mapped to the pair $(g, [v])$. We need to check that $[v]$ is well-defined. ((wrong!)) It suffices to show that $\overline{g}^*\mathrm{Ha} = \mathrm{Ha}((A/C)_1/S_1)$ is not a zero-divisor. Otherwise, for every geometric point x of $\mathrm{Spec}(S)$, the Abelian scheme $(A/C)_x$ is not ordinary. This contradicts Speculation 4.1.14. Therefore we obtain a well-defined map $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon)$.))

Let $B = A/C$. We have

$$p^\epsilon = u^p \cdot \mathrm{Ha}(A_1/\mathrm{Spec}(S_1))^p = u^p \cdot \mathrm{Ha}(A_1^{(p)}/\mathrm{Spec}(S_1)).$$

Modulo $p^{1-\epsilon}$,

$$p^\epsilon = u^p \cdot \mathrm{Ha}(A_{1-\epsilon}^{(p)}/\mathrm{Spec}(S_{1-\epsilon})) = u^p \cdot \mathrm{Ha}(B_{1-\epsilon}/\mathrm{Spec}(S_{1-\epsilon})).$$

Thus there is $v \in H^0(\mathrm{Spec}(S), g^*\omega^{\otimes(1-p)})$ such that $v = u^p \bmod p^{1-\epsilon}$ and $v \cdot \mathrm{Ha}(B_1/\mathrm{Spec}(S_1)) = p^\epsilon \bmod p^{1-\epsilon}$. Hence

$$v \cdot \mathrm{Ha}(B_1/\mathrm{Spec}(S_1)) = p^\epsilon + p^{1-\epsilon}t = p^\epsilon(1 + p^{1-2\epsilon}t) \in S/p$$

for some $t \in S$.

((check: $1 + p^{1-2\epsilon}t$ is invertible in S)) Then

$$(1 + p^{1-2\epsilon}t)^{-1}v \cdot \mathrm{Ha}(B_1/\mathrm{Spec}(S_1)) = p^\epsilon \in S/p.$$

(This shows that B is weak $O(1, \epsilon)$.) We claim that the pair $(f, [u])$ gets mapped to the pair $(g, [(1 + p^{1-2\epsilon}t)^{-1}v])$.

- First check this map is well-defined.
  - Any choice $u' \in [u]$ leads to $u^p = (u')^p \bmod p^{1-\epsilon}$.
  - Now choose another lift $v + p^{1-\epsilon}v'$ of v.
-

Another attempt at constructing the factorization $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon) \to \mathfrak{X}$.

- We already know that B is weak $O(1, \epsilon)$, i.e. there exists a section $v \in H^0(\mathrm{Spec}(S/p), \overline{g}^*\omega^{\otimes(1-p)})$ such that
  $$v \cdot \mathrm{Ha}(B_1/S_1) = p^\epsilon \bmod p.$$
- Modulo $p^{1-\epsilon}$,
  $$p^\epsilon = v \cdot \mathrm{Ha}(B_{1-\epsilon}/S_{1-\epsilon}) = v \cdot \mathrm{Ha}(A_{1-\epsilon}^{(p)}/S_{1-\epsilon}) = u^p \cdot \mathrm{Ha}(A_{1-\epsilon}^{(p)}/S_{1-\epsilon}) \bmod p^{1-\epsilon}.$$
- Then
  $$(v - u^p) \cdot \mathrm{Ha}(A_{1-\epsilon}^{(p)}/S_{1-\epsilon}) = 0 \bmod p^{1-\epsilon}.$$
- Maybe $\mathrm{Ha}(A_{1-\epsilon}^{(p)}/S_{1-\epsilon})$ is not a zero-divisor in $S/p^{1-\epsilon}$.
- Then $v = u^p \bmod p^{1-\epsilon}$.

Another attempt at constructing the factorization $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon) \to \mathfrak{X}$.

- Let $\mathrm{Spf}(R) \subset \mathfrak{X}$ on which ω is trivial.
- Choose a lift $\widetilde{\mathrm{Ha}} \in H^0(\mathrm{Spec}(R), \omega^{\otimes(p-1)})$ of $\mathrm{Ha} \in H^0(\mathrm{Spec}(R/p), \omega^{\otimes(p-1)})$.
- We want

$$\mathrm{Spf}(R\langle u\rangle/(u \cdot \widetilde{\mathrm{Ha}} - p^\epsilon))$$
$$\mathrm{Spf}(R\langle u\rangle/(u \cdot \widetilde{\mathrm{Ha}} - p^{p^{-1}\epsilon})) \longrightarrow \mathrm{Spf}(R).$$

In other words,

$$R\langle u\rangle/(u \cdot \widetilde{\mathrm{Ha}} - p^\epsilon)$$
$$R\langle u\rangle/(u \cdot \widetilde{\mathrm{Ha}} - p^{p^{-1}\epsilon}) \longleftarrow R.$$

We need to show that $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon)$ extends to $\mathfrak{X}^*(p^{-1}\epsilon) \to \mathfrak{X}^*(\epsilon)$ ((cf. the remark in Lemma 4.1.5)).

- We'd like to apply Lemma 2.2.5 for the case $g \geq 2$.
- Let $\mathrm{Spf}(R) \subset \mathfrak{X}^*_{\mathbb{Z}_p}$ ((such that $\omega^{\otimes(p-1)}$ is trivial on $\mathrm{Spf}(R)$)). This gives an affine open $\mathrm{Spf}(R \,\widehat{\otimes}_{\mathbb{Z}_p}\, \mathbb{Z}_p^{\mathrm{cycl}})$ of $\mathfrak{X}^*$, and such affines cover $\mathfrak{X}^*$.
- Check that R is a topologically finitely generated flat p-adically complete $\mathbb{Z}_p$-algebra, and that R/p is normal.
- Check that $\mathrm{Ha} \in H^0(\mathrm{Spec}(R/p), \omega^{\otimes(p-1)}) \simeq R/p$ not a zero-divisor, where ω is the natural (ample) line bundle on $X^*_{\mathbb{F}_p}$.

- $\mathrm{Spec}(R/p)$ is an affine open in $X^*_{\mathbb{F}_p}$ as $\mathrm{Spec}(R)$ is an affine open of $X^*_{\mathbb{Z}_p}$.
  - We have inclusion of opens

  $$X^{\mathrm{ord}}_{\mathbb{F}_p} \subset X_{\mathbb{F}_p} \subset X^*_{\mathbb{F}_p}.$$

  The first inclusion is dense by Lemma A.2.2, and the second is dense by the property of minimal compactification ((todo: add reference)).
  - Thus the intersection

  $$\mathrm{Spec}(R/p) \cap X^{\mathrm{ord}}_{\mathbb{F}_p}$$

  is non-empty.
  - Therefore Ha is not a zero-divisor since it is non-zero at a point.
- We need a map

$$\mathfrak{X}^*(p^{-1}\epsilon) \times_{\mathfrak{X}^*} \mathrm{Spf}(R\,\widehat{\otimes}_{\mathbb{Z}_p}\,\mathbb{Z}^{\mathrm{cycl}}_p) \to \mathfrak{X}^*(\epsilon) \times_{\mathfrak{X}^*} \mathrm{Spf}(R\,\widehat{\otimes}_{\mathbb{Z}_p}\,\mathbb{Z}^{\mathrm{cycl}}_p)$$

- Choose a lift $\widetilde{\mathrm{Ha}} \in H^0(\mathrm{Spec}(R), \omega^{\otimes(p-1)}) \simeq R$ of $\mathrm{Ha} \in R/p$.
- Let $S_\epsilon = (R\,\widehat{\otimes}_{\mathbb{Z}_p}\,\mathbb{Z}^{\mathrm{cycl}}_p)\langle u\rangle/(u\cdot\widetilde{\mathrm{Ha}} - p^\epsilon)$. Let $S_{p^{-1}\epsilon} = (R\,\widehat{\otimes}_{\mathbb{Z}_p}\,\mathbb{Z}^{\mathrm{cycl}}_p)\langle u\rangle/(u\cdot\widetilde{\mathrm{Ha}} - p^{p^{-1}\epsilon})$. Then we need a map

$$
\begin{array}{ccc}
\mathrm{Spf}(S_{p^{-1}\epsilon}) & \dashrightarrow & \mathrm{Spf}(S_\epsilon) \\
\downarrow & & \downarrow \\
\mathfrak{X}^*(p^{-1}\epsilon) & \dashrightarrow & \mathfrak{X}^*(\epsilon)
\end{array}
$$

- Consider the pullback diagram

$$
\begin{array}{ccc}
U & \longrightarrow & \mathrm{Spec}(R/p) \\
\downarrow & & \downarrow \\
X_{\mathbb{F}_p} & \longrightarrow & X^*_{\mathbb{F}_p}.
\end{array}
$$

  Then U is an open in $\mathrm{Spec}(R/p)$. Let Y be the complement of U in $\mathrm{Spec}(R/p)$.
- Check that Y has codimension ≥ 2 in $\mathrm{Spec}(R/p)$. This follows from that $Y \cap X_{\mathbb{F}_p} = \emptyset$ and that boundary of $X^*_{\mathbb{F}_p}$ has codimension $g \geq 2$.
- Let Z be the preimage of Y in $\mathrm{Spf}(S_\epsilon)$. Then Lemma 2.2.5 shows that the natural map

$$H^0(\mathrm{Spf}(S_\epsilon), \mathcal{O}_{\mathrm{Spf}(S_\epsilon)}) \to H^0(\mathrm{Spf}(S_\epsilon)\backslash Z, \mathcal{O}_{\mathrm{Spf}(S_\epsilon)})$$

  is an isomorphism.
- Define $\mathfrak{U}$ by the pullback diagram

$$
\begin{array}{ccc}
\mathfrak{U} & \longrightarrow & \mathrm{Spf}(R\,\widehat{\otimes}_{\mathbb{Z}_p}\,\mathbb{Z}^{\mathrm{cycl}}_p) \\
\downarrow & & \downarrow \\
\mathfrak{X} & \longrightarrow & \mathfrak{X}^*.
\end{array}
$$

- Now we need to construct

$$
\begin{array}{ccc}
\mathfrak{X}(p^{-1}\epsilon) \times_{\mathfrak{X}} \mathfrak{U} & \longrightarrow & \mathrm{Spf}(S_{p^{-1}\epsilon}) \\
\downarrow & & \downarrow \\
\mathfrak{X}(\epsilon) \times_{\mathfrak{X}} \mathfrak{U} & \longrightarrow & \mathrm{Spf}(S_\epsilon).
\end{array}
$$

- We claim that $\mathfrak{X}(\epsilon) \times_{\mathfrak{X}} \mathfrak{U} = \mathrm{Spf}(S_\epsilon)\backslash Z$.

$\square$

4.2. The anti-canonical tower of level Γ_s.

Construction 4.2.1. Let $m \geq 1$.

We first construct a map $\mathfrak{X}(p^{-m}\epsilon) \to \mathfrak{X}$. Let S be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $\mathrm{Spf}(S) \to \mathfrak{X}(p^{-m}\epsilon)$ be a map over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$. It corresponds to a pair $(f, [u])$ where

- $f : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ is a map over $\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$.
- $u \in H^0(\mathrm{Spec}(S), f^*\omega^{\otimes(1-p)})$ is a section such that $u \cdot \overline{f}^*\mathrm{Ha} = p^{p^{-m}\epsilon}$ in S/p.

The map $f : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ gives an Abelian scheme $A \to \mathrm{Spec}(S)$. The section u shows that $A \to \mathrm{Spec}(S)$ satisifies strong $O(m, \epsilon)$, and thus has a strong canonical subgroup $C_m \subset A[p^m]$ of level m. The Abelian scheme $A/C_m \to \mathrm{Spec}(S)$ has induced principal polarization and level structure, and thus corresponds to a map $\mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$, which gives a map $\mathrm{Spf}(S) \to \mathfrak{X}$ over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$.

Passing to the adic fiber (i.e. the generic fiber of the associated adic space), we get a map $\mathcal{X}(p^{-m}\epsilon) \to \mathcal{X}$ of adic spaces. Now we construct a factorization $\mathcal{X}(p^{-m}\epsilon) \to \mathcal{X}_{\Gamma_\mathrm{s}(p^m)} \to \mathcal{X}$, where the map $\mathcal{X}_{\Gamma_\mathrm{s}(p^m)} \to \mathcal{X}$ is given by the moduli interpretation "$(A, D) \mapsto A/D$".

((todo: construct the factorization))

Lemma 4.2.2. For each $m \geq 1$, the $\mathcal{X}(p^{-m}\epsilon) \to \mathcal{X}_{\Gamma_\mathrm{s}(p^m)}$ extends uniquely to $\mathcal{X}^*(p^{-m}\epsilon) \to \mathcal{X}^*_{\Gamma_\mathrm{s}(p^m)}$, and both maps are open immersions of adic spaces. Moreover, the following diagram

$$
\begin{array}{ccc}
\mathcal{X}^*(p^{-m-1}\epsilon) & \longrightarrow & \mathcal{X}^*_{\Gamma_\mathrm{s}(p^{m+1})} \\
\downarrow & & \downarrow \\
\mathcal{X}^*(p^{-m}\epsilon) & \longrightarrow & \mathcal{X}^*_{\Gamma_\mathrm{s}(p^m)}
\end{array}
$$

is a pullback diagram for all $m \geq 1$, where the vertical map on the left is induced from the map $\mathfrak{X}(p^{-m-1}\epsilon) \to \mathfrak{X}(p^{-m}\epsilon)$, cf. Lemma 4.1.15.

Proof. ((todo: write down the proof))

- Extension to minimal compactification:
- Open immersion:
  - The map $\theta : X_{\mathbb{Q}_p^{\mathrm{cycl}}} \to X_{\mathbb{Q}_p^{\mathrm{cycl}}}$ defined by $A \mapsto A/A[p^m]$ is an isomorphism.
  - The following diagram

$$
\begin{array}{ccc}
\mathcal{X}(p^{-m}\epsilon) & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
X^{\mathrm{ad}} & \xrightarrow{\ \theta\ } & X^{\mathrm{ad}}
\end{array}
$$

  commutes.
  - So the composition $\mathcal{X}^*(p^{-m}\epsilon) \to \mathcal{X}_{\Gamma_\mathrm{s}(p^m)} \to \mathcal{X}$ is an open immersion.
  - The map $\mathcal{X}_{\Gamma_\mathrm{s}(p^m)} \to \mathcal{X}$ is finite étale.
  - Thus the map $\mathcal{X}(p^{-m}\epsilon) \to \mathcal{X}_{\Gamma_\mathrm{s}(p^m)}$ is an open immersion.
  - Then pass to minimal compactification as follows.
  - We'd like to apply Lemma 2.2.5.
- Pullback diagram:
  - First show that

$$
\begin{array}{ccc}
\mathcal{X}(p^{-m-1}\epsilon) & \longrightarrow & \mathcal{X}_{\Gamma_\mathrm{s}(p^{m+1})} \\
\downarrow & & \downarrow \\
\mathcal{X}(p^{-m}\epsilon) & \longrightarrow & \mathcal{X}_{\Gamma_\mathrm{s}(p^m)}
\end{array}
$$

  is pullback diagram.
  - Commutativity of the diagram:
  - It is a pullback since both vertical maps are finite étale of degree $p^{g(g+1)/2}$.
  - Then pass to minimal compactification.

$\square$

Definition 4.2.3. Let $\mathcal{X}_{\Gamma_s(p)}(\epsilon)$ be the pullback of $\mathcal{X}(\epsilon)$ along $\mathcal{X}_{\Gamma_s(p)} \to \mathcal{X}$.

Lemma 4.2.4. The following diagram

$$
\begin{array}{ccc}
\mathcal{X}(p^{-1}\epsilon) & \longrightarrow & \mathcal{X}_{\Gamma_s(p)}(\epsilon) \\
\downarrow & & \downarrow \\
\mathcal{X}(\epsilon) & \xrightarrow{\;\mathrm{id}\;} & \mathcal{X}(\epsilon)
\end{array}
$$

commutes. Moreover, the map $\mathcal{X}(p^{-1}\epsilon) \to \mathcal{X}_{\Gamma_s(p)}(\epsilon)$ is an open immersion, and the image of $\mathcal{X}(p^{-1}\epsilon)$ in $\mathcal{X}_{\Gamma_s(p)}(\epsilon)$ is both open and closed.

Definition 4.2.5. ((todo: how to make this statement precise? do we actually need this?: Let $\mathcal{X}_{\Gamma_s(p)}(\epsilon)_a$ be the open and closed subset of $\mathcal{X}_{\Gamma_s(p)}(\epsilon)$ "parametrizing those $D \subset \mathcal{A}(\epsilon)[p]$ with $D \cap C = \{0\}$".))

Let $\mathcal{X}_{\Gamma_s(p)}(\epsilon)_a$ be the image of $\mathcal{X}(p^{-1}\epsilon)$ in $\mathcal{X}_{\Gamma_s(p)}(\epsilon)$. Let $\mathcal{X}^*_{\Gamma_s(p)}(\epsilon)_a$ be the image of $\mathcal{X}^*(p^{-1}\epsilon)$ in $\mathcal{X}^*_{\Gamma_s(p)}(\epsilon)$. Let $\mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a$ be the pullback of $\mathcal{X}^*_{\Gamma_s(p)}(\epsilon)_a$ along $\mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon) \to \mathcal{X}^*_{\Gamma_s(p)}(\epsilon)$.

Remark 4.2.6. $\mathcal{X}^*_{\Gamma_s(p)}(\epsilon)_a$ is both open and closed in $\mathcal{X}^*_{\Gamma_s(p)}(\epsilon)$.

Lemma 4.2.7. For m sufficiently large, $\mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a$ is affinoid.

Proof. ((todo: write down the proof))

- There exists an integer $m \geq 0$ such that $H^i(X^*_{\mathbb{Z}_p}, \omega^{\otimes p^m(p-1)}) = 0$ for all $i \geq 1$, since ω is an ample line bundle on $X^*_{\mathbb{Z}_p}$.
- We can find a lift $s \in H^0(X^*_{\mathbb{Z}_p}, \omega^{\otimes p^m(p-1)})$ lifting $\mathrm{Ha}^{p^m} \in H^0(X^*_{\mathbb{F}_p}, \omega^{\otimes p^m(p-1)})$. ((todo: add a proof; should follows from vanishing of first cohomology; maybe use a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules, and then pass to a long exact sequence))
- The condition $|\mathrm{Ha}| \geq |p|^{p^{-m}\epsilon}$ is equivalent to $|s| \geq |p|^\epsilon$.
- The condition defines an affinoid space $\mathcal{X}^*(p^{-m}\epsilon) \simeq \mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a$.

$\square$

Definition 4.2.8. Fix an element $t \in (\mathbb{Z}_p^{\mathrm{cycl}})^\flat$ such that $|t| = |t^\sharp| = |p|$, such that t admits a $(p-1)$-th root. Then we get an identification

$$(\mathbb{Z}_p^{\mathrm{cycl}})^\flat = \mathbb{F}_p[[t^{1/(p-1)f^\infty}]].$$

Definition 4.2.9. Let $\mathfrak{X}'$ be the formal scheme over $\mathrm{Spf}(\mathbb{F}_p[[t^{1/(p-1)p^\infty}]])$ given by the t-adic completion of $X \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} \mathrm{Spec}(\mathbb{F}_p[[t^{1/(p-1)p^\infty}]])$. Let $\mathcal{X}'$ be the generic fiber of the adic space associated to $\mathfrak{X}'$. Define $\mathfrak{X}'^*$ and $\mathfrak{A}^*$ similarly, with generic fibers $\mathcal{X}'^*$ and $\mathcal{A}'$.

Lemma 4.2.10. There exists a unique perfectoid space $\mathcal{X}^*_{\Gamma_s(p^\infty)}(\epsilon)_a$ such that

$$\mathcal{X}^*_{\Gamma_s(p^\infty)}(\epsilon)_a \sim \lim_m \mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a.$$

Similar results hold for $\mathcal{X}_{\Gamma_s(p^\infty)}(\epsilon)_a$ and $\mathcal{A}_{\Gamma_s(p^\infty)}(\epsilon)_a$.

Proof. ((todo: use tilting))

- Define

$$\mathfrak{X}^*_{\Gamma_s(p^\infty)}(\epsilon)_a = \lim_m \mathfrak{X}(p^{-m}\epsilon),$$

  where the inverse limit is taken in the category of formal schemes over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$. Note that the transition maps are finite.
- Let $\mathrm{Spf}(R_{m_0}) \subset \mathfrak{X}(p^{-m_0}\epsilon)$ be affine. Let $\mathrm{Spf}(R_m) \subset \mathfrak{X}(p^{-m}\epsilon)$ be the preimage of $\mathrm{Spf}(R_{m_0})$ for $m \geq m_0$.
- We get an affine open $\mathrm{Spf}(R_\infty)$ of $\mathfrak{X}^*_{\Gamma_s(p^\infty)}(\epsilon)_a$, where R_∞ is the p-adic completion of $\mathrm{colim}_m R_m$. Then R_∞ is flat over $\mathbb{Z}_p^{\mathrm{cycl}}$.
- The transition map $R_m/p^{1-\epsilon} \to R_{m+1}/p^{1-\epsilon}$ agrees with the relative Frobenius. The absolute Frobenius then induces an isomorhpism

$$R_\infty/p^{(1-\epsilon)/p} = \mathrm{colim}_m R_{m+1}/p^{(1-p)/p} \simeq \mathrm{colim}_m R_m/p^{1-\epsilon} = R_\infty/p^{1-\epsilon}.$$

- Thus R_∞^a is a perfectoid $\mathbb{Z}_p^{\mathrm{cycl},a}$-algebra, cf. [Sch12, Definition 5.1.(ii)].
- Then $R_\infty[1/p]$ is a perfectoid $\mathbb{Q}_p^{\mathrm{cycl}}$-algebra, cf. [Sch12, Lemma 5.6].
- Then the generic fiber of $\mathfrak{X}_{\Gamma_s(p^\infty)}^*(\epsilon)_a$ is a perfectoid space $\mathcal{X}_{\Gamma_s(p^\infty)}^*(\epsilon)_a$ over $\mathbb{Q}_p^{\mathrm{cycl}}$, and

$$\mathcal{X}_{\Gamma_s(p^\infty)}^*(\epsilon)_a \sim \lim_m \mathcal{X}_{\Gamma_s(p^m)}^*(\epsilon)_a,$$

  cf. [SW13, Definition 2.4.1, Proposition 2.4.2].
- Uniqueness follows from [SW13, Proposition 2.4.5].

$\square$

Lemma 4.2.11. The tilt $\mathcal{X}_{\Gamma_s(p^\infty)}^*(\epsilon)_a^\flat$ identifies naturally with the open subset $\mathcal{X}'^{*\mathrm{perf}}(\epsilon) \subset \mathcal{X}'^{*\mathrm{perf}}$ where $|\mathrm{Ha}| \geq |t|^\epsilon$. The similar result holds for $\mathcal{A}$.

Proof. ((todo: split the proof))

- We define $\mathfrak{X}'^*(\epsilon) \to \mathfrak{X}'^*$ in a way similar to $\mathfrak{X}^*(\epsilon) \to \mathfrak{X}^*$, parametrizing sections $u \in \omega^{\otimes(1-p)}$ such that $u \cdot \mathrm{Ha} = t^\epsilon$.
- We have the map

$$\mathfrak{X}'^*(p^{-1}\epsilon) \to \mathfrak{X}'^*(\epsilon)$$

  given by the raltive Frobenius.
- The inverse limit $\lim_m \mathfrak{X}'^*(p^{-m}\epsilon)$ is representable by a perfect flat formal scheme over $\mathbb{F}_p[[t^{1/(p-1)p^\infty}]]$ which is naturally the same as $\mathfrak{X}'^*(\epsilon)^{\mathrm{perf}}$.
- Its generic fiber is thus a perfectoid space over $\mathbb{F}_p((t^{1/(p-1)p^\infty}))$, that is identified with the open subset of $\mathcal{X}'^{*\mathrm{perf}}$ where $|\mathrm{Ha}| \geq |t|^\epsilon$.
- We have a canonical identification

$$\mathfrak{X}'^*(p^{-m}\epsilon)/t^{1-\epsilon} \simeq \mathfrak{X}^*(p^{-m}\epsilon)/p^{1-\epsilon}$$

  compatible with transition maps.
- For an open affine $\mathrm{Spf}(R_{m_0}) \subset \mathfrak{X}^*(p^{-m}\epsilon)$ with preimages $\mathrm{Spf}(R_m)$, we get affine opens $\mathrm{Spf}(S_m) \subset \mathfrak{X}'^*(p^{-m}\epsilon)$, with

$$S_m/t^{1-\epsilon} = R_m/p^{1-\epsilon}.$$

- Let R_∞ be the p-adic completion of $\mathrm{colim}_m R_m$. Let S_∞ be the t-adic completion of $\mathrm{colim}_m S_m$.
- Then $\mathrm{Spf}(R_\infty) \subset \mathfrak{X}_{\Gamma_s(p^\infty)}(\epsilon)_a$ and $\mathrm{Spf}(S_\infty) \subset \mathfrak{X}'^*(\epsilon)^{\mathrm{perf}}$ give corresponding open subsets, and

$$R_\infty/p^{1-\epsilon} = \mathrm{colim}_m R_m/p^{1-\epsilon} = \mathrm{colim}_m S_m/t^{1-\epsilon} = S_\infty/t^{1-\epsilon}.$$

- It follows that $R_\infty[1/p]$ and $S_\infty[1/t]$ are tilts by [Sch12, Theorem 5.2].

$\square$

Lemma 4.2.12. The space $\mathcal{X}_{\Gamma_s(p^\infty)}^*(\epsilon)_a$ is affinoid.

Proof. ((todo: use tilting))

- It suffices to check for the tilts.
- The open subset $\mathcal{X}'^*(\epsilon) \subset \mathcal{X}'^*$ given by $|\mathrm{Ha}| \geq |\epsilon|^\epsilon$ is affinoid.

$\square$

4.3. **Lifting to level Γ_1.**

Lemma 4.3.1. Let $\mathfrak{X}_{\Gamma_s(p^\infty)}(\epsilon)_a$ be the formal scheme over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$ defined as

$$\mathfrak{X}_{\Gamma_s(p^\infty)}(\epsilon)_a = \lim_m \mathfrak{X}(p^{-m}\epsilon).$$

Let $0 \leq m \leq m'$. Then

(1) The maps

$$1/p^{(m'-m)g(g+1)/2}\mathrm{tr} : \mathcal{O}_{\mathfrak{X}(p^{-m'}\epsilon)}[1/p] \to \mathcal{O}_{\mathfrak{X}(p^{-m}\epsilon)}[1/p]$$

are compatible for varying m', and thus induces a map

$$\overline{\mathrm{tr}}_m : \lim_{m'} \mathcal{O}_{\mathfrak{X}(p^{-m'}\epsilon)}[1/p] \to \mathcal{O}_{\mathfrak{X}(p^{-m}\epsilon)}[1/p].$$

(2) The image of $\overline{\mathrm{tr}}_m$ is contained in $p^{-C_m}\mathcal{O}_{\mathfrak{X}(p^{-m}\epsilon)}$ for some constant C_m, with $C_m \to 0$ as $m \to +\infty$. Thus $\overline{\mathrm{tr}}_m$ extends by continuity to a map

$$\overline{\mathrm{tr}}_m : \mathcal{O}_{\mathfrak{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\epsilon)_a}[1/p] \to \mathcal{O}_{\mathfrak{X}(p^{-m}\epsilon)}[1/p],$$

called Tate's normalized trace.

(3) For every $x \in \mathcal{O}_{\mathfrak{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\epsilon)_a}[1/p]$, we have

$$x = \lim_{m\to+\infty} \overline{\mathrm{tr}}_m(x).$$

Assume $g \geq 2$.

Lemma 4.3.2. Fix $m \geq 1$ sufficiently large such that $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$ is affinoid, cf. Lemma 4.2.7. Let $\mathcal{Y}^*_m \to \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$ be a finite morphism. Let $\mathcal{Y}_m \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$ be the pullback of $\mathcal{Y}^*_m \to \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$ along $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$. Assume that

- The map $\mathcal{Y}_m \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$ is étale.
- $\mathcal{Y}^*_m$ is normal.
- None of the irreducible components of $\mathcal{Y}^*_m$ is mapped into the boundary of $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$.

For $m' \geq m$, define $\mathcal{Y}^*_{m'} \to \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^{m'})}(\epsilon)_a$ to be the ((todo: normalization??)) pullback of $\mathcal{Y}^*_m \to \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$ along $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^{m'})}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$. Define $\mathcal{Y}_{m'} \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p^{m'})}(\epsilon)_a$ by pullback. Let $\mathcal{Y}_\infty$ be the pullback of $\mathcal{Y}_m$ to $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\epsilon)_a$, which exists as $\mathcal{Y}_m \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$ is finite étale.

Since every $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^{m'})}(\epsilon)_a$ is affinoid, each $\mathcal{Y}^*_{m'}$ is affinoid. We write $\mathcal{Y}^*_{m'} = \mathrm{Spa}(S_{m'}, S^+_{m'})$. ((todo: scholze says $S^+_{m'} = S^\circ_{m'}$??)) Then

(1) For all $m' \geq m$,

$$S^+_{m'} = H^0(\mathcal{Y}_{m'}, \mathcal{O}^+_{\mathcal{Y}_{m'}}).$$

(2) The map

$$\mathrm{colim}_{m'} S^+_{m'} \to H^0(\mathcal{Y}_\infty, \mathcal{O}^+_{\mathcal{Y}_\infty})$$

is injective with dense image. Moreover, there is a canonical continuous retraction

$$H^0(\mathcal{Y}_\infty, \mathcal{O}_{\mathcal{Y}_\infty}) \to S_{m'}.$$

(3) Assume that $S_\infty = H^0(\mathcal{Y}_\infty, \mathcal{O}_{\mathcal{Y}_\infty})$ is a perfectoid $\mathbb{Q}_p^{\mathrm{cycl}}$-algebra. Then

$$\mathcal{Y}^*_\infty = \mathrm{Spa}(S_\infty, S^+_\infty)$$

where $S^+_\infty = S^\circ_\infty$, is an affinoid perfectoid space over $\mathbb{Q}_p^{\mathrm{cycl}}$, and

$$\mathcal{Y}^*_\infty \sim \lim_{m'} \mathcal{Y}^*_{m'},$$

and S^+_∞ is the p-adic completion of $\mathrm{colim}_{m'} S^+_{m'}$.

Proof. ((todo: write a proof))

- For (1), we only need to prove for $m' = m$. Let $S = S_m$ and $R = H^0(\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a, \mathcal{O}_{\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a})$.
- Both R and S are normal and Noetherian. S is an R-module of finite type.
- Let $Z \subset \mathrm{Spec}(R)$ denote the boundary, which is of codimension ≥ 2. Let $Z' \subset \mathrm{Spec}(S)$ be the preimage, again of codimension ≥ 2, by the assumption on irreducible components.
- Then $S = H^0(\mathrm{Spec}(S)\backslash Z', \mathcal{O}_{\mathrm{Spec}(S)})$ and $R = H^0(\mathrm{Spec}(R)\backslash Z, \mathcal{O}_{\mathrm{Spec}(R)})$.
- We have a trace map $\mathrm{tr}_{S/R} : S \to R$ as the map ((which??)) is finite étale away from Z. The trace pairing induces an isomorphism

$$S \to \mathrm{Hom}_R(S, R).$$

- Explanation of the isomorphism: If $s_1 \in S$ is in the kernel, it lies in the kernel of the pairing away from the boundary, on which the pairing is perfect as the map is finite étale. Thus s_1 vanishes away from the boundary, and then s_1 is zero.
- For all open subsets $\mathcal{U} \subset \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$ with preimage $\mathcal{V} \subset \mathcal{Y}^*_m$, the trace pairing gives an isomorphism

$$H^0(\mathcal{V}, \mathcal{O}_{\mathcal{Y}^*_m}) \simeq \mathrm{Hom}_R(S, H^0(\mathcal{U}, \mathcal{O}_{\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a})),$$

cf. the argument in Riemann's Hebbarkeitssatz, Lemma 2.4.4.

- Then (1) follows from
$$H^0(\mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a, \mathcal{O}_{\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a}) \simeq H^0(\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a, \mathcal{O}_{\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a})$$
  which is a direct corollary of Lemma 4.2.7.
- For (2), use Tate's normalized traces and Lemma 4.3.1.
- (3) follows directly from (2).

$\square$

First assume $g \geq 2$.

Definition 4.3.3. Note that on the tower $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$, we have the tautological Abelian variety $\mathcal{A}^t_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$ (which are related to each other by pullback), as well as the Abelian varieties $\mathcal{A}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a = \mathcal{A}(p^{-m}\epsilon)$ over $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a \simeq \mathcal{X}(p^{-m}\epsilon)$. They are related by an isogeny
$$\mathcal{A}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a \to \mathcal{A}^t_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$$
whose kernel is the canonical subgroup $C_m \subset \mathcal{A}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a[p^m]$ of level m. We get an induced subgroup
$$D_m = \mathcal{A}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a[p^m]/C_m \subset \mathcal{A}^t_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a.$$
Let $D_{m\Gamma_{\mathrm{s}}(p^{m'})}$ be the pullback of D_m to $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^{m'})}(\epsilon)_a$ for $m' \geq m$. We have
$$D_{m\Gamma_{\mathrm{s}}(p^{m'})} = D_{m'}[p^{m'}].$$
Also, the D_m give the $\Gamma_{\mathrm{s}}(p^m)$ level structure. Let $D_{m\Gamma_{\mathrm{s}}(p^\infty)}$ be the pullback of D_m to $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\epsilon)_a$. Since $D_m \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$ is finite étale, $D_{m\Gamma_{\mathrm{s}}(p^\infty)}$ is a perfectoid space.

Lemma 4.3.4. The map
$$\mathcal{A}_{\Gamma_{\mathrm{s}}(p^\infty)}(\epsilon)_a \to D_{m\Gamma_{\mathrm{s}}(p^\infty)}$$
is an isomorphism of perfectoid spaces.

Definition 4.3.5. Let $D'_m \to \mathcal{X}'(\epsilon) \subset \mathcal{X}'$ be the quotient $\mathcal{A}'(\epsilon)[p^m]/C'_m$, where C'_m denotes the canonical subgroup on the ordinary locus in characteristic p. Note that all Abelian varieties over $\mathbb{F}_p((t^{1/(p-1)p^\infty}))$ parametrized by $\mathcal{X}'(\epsilon)$ are ordinary, as the Hasse invariant divides t^ϵ, and thus is invertible.

Lemma 4.3.6. The tilt of $D_{m\Gamma_{\mathrm{s}}(p^\infty)}$ identifies canonically with the perfection of D'_m.

Definition 4.3.7. Let $X^{\mathrm{ord}*} \subset X^* \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} \mathrm{Spec}(\mathbb{F}_p)$ be the locus where the Hasse invariant is invertible. Then $X^{\mathrm{ord}*}$ is affine over $\mathbb{F}_p$. Let $X^{\mathrm{ord}} \subset X \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} \mathrm{Spec}(\mathbb{F}_p)$ be the preimage, which is the ordinary locus. Let $D^{\mathrm{ord}}_m \to X^{\mathrm{ord}}$ be the quotient of the p^m-torsion of the universal Abelian variety by the canonical subgroup. Let $X^{\mathrm{ord}}_{\Gamma_1(p^m)} \to X^{\mathrm{ord}}$ parametrize isomorphisms $D^{\mathrm{ord}}_m \simeq (\mathbb{Z}/p^m\mathbb{Z})^g$. Then $X^{\mathrm{ord}}_{\Gamma_1(p^m)} \to X^{\mathrm{ord}}$ is a finite map of schemes over $\mathbb{F}_p$. Define
$$X^{\mathrm{ord}*}_{\Gamma_1(p^m)} = \mathrm{Spec}(H^0(X^{\mathrm{ord}}_{\Gamma_1(p^m)}, \mathcal{O}_{X^{\mathrm{ord}}_{\Gamma_1(p^m)}})).$$
Then map $X^{\mathrm{ord}*}_{\Gamma_1(p^m)} \to X^{\mathrm{ord}*}$ is a finite map of affine schemes over $\mathbb{F}_p$, such that $X^{\mathrm{ord}}_{\Gamma_1(p^m)}$ is the preimage of X^{ord}. Also $X^{\mathrm{ord}*}_{\Gamma_1(p^m)}$ is normal.

Let $\mathcal{X}'^*_{\Gamma_1(p^m)}$ be the open locus of the adic space associated with $X^{\mathrm{ord}*}_{\Gamma_1(p^m)} \otimes \mathbb{F}_p((t^{1/(p-1)p^\infty}))$ where $|\mathrm{Ha}| \geq |t|^\epsilon$. Then
$$\mathcal{X}'^*_{\Gamma_1(p^m)}(\epsilon) \to \mathcal{X}'^*(\epsilon)$$
is finite, and étale away from the boundary. In particular, the base-change $\mathcal{X}'_{\Gamma_1(p^m)}(\epsilon) \to \mathcal{X}'(\epsilon) \subset \mathcal{X}'^*(\epsilon)$ is finite étale, parametrizing isomorphisms $D'_m \simeq (\mathbb{Z}/p^m\mathbb{Z})^g$. Let $\mathcal{Z}'^*(\epsilon) \subset \mathcal{X}'^*(\epsilon)$ denote the boundary, with pullback $\mathcal{Z}'^*_{\Gamma_1(p^m)}(\epsilon) \subset \mathcal{X}'^*_{\Gamma_1(p^m)}(\epsilon)$.

Lemma 4.3.8. The triple $(\mathcal{X}'^*(\epsilon)^{\mathrm{perf}}, \mathcal{Z}'^*(\epsilon)^{\mathrm{perf}}, \mathcal{X}'(\epsilon)^{\mathrm{perf}})$ is good.

Lemma 4.3.9. The triple $(\mathcal{X}'^*_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}}, \mathcal{Z}'^*_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}}, \mathcal{X}'_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}})$ is good.

Now fix $m \geq 1$, and consider $\mathcal{Y}^*_m = \mathcal{X}^*_{\Gamma_1(p^m)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^m)}(\epsilon)_a$.

Lemma 4.3.10. Th tilt of $\mathcal{Y}_\infty$ identifies with $\mathcal{X}'_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}}$.

Remark 4.3.11. Note that $\mathcal{Y}_m^* \backslash \partial \to \mathcal{X}_{\Gamma_s(p^m)}^*(\epsilon)_a \backslash \partial$ is finite étale. By pullbackwe get a perfectoid space $\mathcal{Y}_\infty^* \backslash \partial \to \mathcal{X}_{\Gamma_s(p^\infty)}^*(\epsilon)_a \backslash \partial$. Caution: $\mathcal{Y}_\infty^*$ is not defined yet.

Lemma 4.3.12. The tilt of $\mathcal{Y}_\infty^* \backslash \infty$ identifies with $\mathcal{X}_{\Gamma_1(p^m)}^{\prime*}(\epsilon)^{\mathrm{perf}} \backslash \partial$.

Lemma 4.3.13. The ring $S_\infty = H^0(\mathcal{Y}_\infty, \mathcal{O}_{\mathcal{Y}_\infty})$ is perfectoid, and the tilt of $\mathcal{Y}_\infty^* = \mathrm{Spa}(S_\infty, S_\infty^+)$ identifies with $\mathcal{X}_{\Gamma_1(p^m)}^{\prime*}(\epsilon)^{\mathrm{perf}}$.

Lemma 4.3.14. For any $m \geq 1$, there is a unique perfectoid spacae $\mathcal{X}_{\Gamma_1(p^m) \cap \Gamma_s(p^\infty)}^*(\epsilon)_a$ over $\mathbb{Q}_p^{\mathrm{cycl}}$ such that

$$\mathcal{X}_{\Gamma_1(p^m) \cap \Gamma_s(p^\infty)}^*(\epsilon)_a \sim \lim_{m'} \mathcal{X}_{\Gamma_1(p^m) \cap \Gamma_s(p^{m'})}^*(\epsilon)_a.$$

Moreover, $\mathcal{X}_{\Gamma_1(p^m) \cap \Gamma_s(p^\infty)}^*(\epsilon)_a$ and all $\mathcal{X}_{\Gamma_1(p^m) \cap \Gamma_s(p^{m'})}^*(\epsilon)_a$ for m' sufficiently large are affinoid, and

$$\mathrm{colim}_{m'} H^0(\mathcal{X}_{\Gamma_1(p^m) \cap \Gamma_s(p^{m'})}^*(\epsilon)_a, \mathcal{O}) \to H^0(\mathcal{X}_{\Gamma_1(p^m) \cap \Gamma_s(p^\infty)}^*(\epsilon)_a, \mathcal{O})$$

has dense image. Let $\mathcal{Z}_{\Gamma_1(p^m) \cap \Gamma_s(p^\infty)}^*(\epsilon)_a \subset \mathcal{X}_{\Gamma_1(p^m) \cap \Gamma_s(p^\infty)}^*(\epsilon)_a$ denote the boundar, and $\mathcal{X}_{\Gamma_1(p^m) \cap \Gamma_s(p^\infty)}(\epsilon)_a$ the preimage of $\mathcal{X}_{\Gamma_s(p)}(\epsilon)_a \subset \mathcal{X}_{\Gamma_s(p)}(\epsilon)_a$. Then the triple

$$(\mathcal{X}_{\Gamma_1(p^m) \cap \Gamma_s(p^\infty)}^*(\epsilon)_a, \mathcal{Z}_{\Gamma_1(p^m) \cap \Gamma_s(p^\infty)}(\epsilon)_a, \mathcal{X}_{\Gamma_1(p^m) \cap \Gamma_s(p^\infty)}(\epsilon)_a)$$

is good.

((todo: $g = 1$))

Lemma 4.3.15. There is a unique perfectoid space $\mathcal{X}_{\Gamma_1(p^\infty)}^*(\epsilon)_a$ over $\mathbb{Q}_p^{\mathrm{cycl}}$ such that

$$\mathcal{X}_{\Gamma_1(p^\infty)}^*(\epsilon)_a \sim \lim_m \mathcal{X}_{\Gamma_1(p^m)}^*(\epsilon)_a.$$

Moreover, $\mathcal{X}_{\Gamma_1(p^\infty)}^*(\epsilon)_a$ and all $\mathcal{X}_{\Gamma_1(p^m)}^*(\epsilon)_a$ for m sufficiently large are affinoid, and

$$\mathrm{colim}_m H^0(\mathcal{X}_{\Gamma_1(p^m)}^*(\epsilon)_a, \mathcal{O}) \to H^0(\mathcal{X}_{\Gamma_1(p^\infty)}^*(\epsilon)_a, \mathcal{O})$$

has dense image. Let $\mathcal{Z}_{\Gamma_1(p^\infty)}^*(\epsilon)_a \subset \mathcal{X}_{\Gamma_1(p^\infty)}^*(\epsilon)_a$ denote the boundar, and $\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a$ the preimage of $\mathcal{X}_{\Gamma_s(p)}(\epsilon)_a \subset \mathcal{X}_{\Gamma_s(p)}(\epsilon)_a$. Then the triple

$$(\mathcal{X}_{\Gamma_1(p^\infty)}^*(\epsilon)_a, \mathcal{Z}_{\Gamma_1(p^\infty)}(\epsilon)_a, \mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a)$$

is good.

4.4. **Lifting to level Γ.**

Lemma 4.4.1. For every $m \geq 1$, the map

$$\mathcal{X}_{\Gamma(p^m)}^*(\epsilon)_a \to \mathcal{X}_{\Gamma_1(p^m)}^*(\epsilon)_a$$

is finite étale.

Lemma 4.4.2. There is a unique perfectoid space $\mathcal{X}_{\Gamma(p^\infty)}^*(\epsilon)_a$ over $\mathbb{Q}_p^{\mathrm{cycl}}$ such that

$$\mathcal{X}_{\Gamma(p^m)}^*(\epsilon)_a \sim \lim_m \mathcal{X}_{\Gamma(p^m)}^*(\epsilon)_a.$$

5. The Hodge–Tate period map

5.1. **The map of popological spaces.** ((todo))

5.2. **The map of perfectoid spaces.** ((todo))

Appendix A. Review of Abelian schemes

A.1. Basics.

Definition A.1.1. Let S be a scheme. A geometric point of S is a map $\mathrm{Spec}(k) \to S$ where k is an algebraically closed field.

Definition A.1.2. Let S be a scheme. An Abelian scheme over S is a proper smooth group scheme over S that is geometrically connected.

Definition A.1.3. Let $A \to S$ be an Abelian scheme where S is over $\mathrm{Spec}(\mathbb{F}_p)$. We have Frobenius map $F : A \to A^{(p)}$ and Verschiebung map $V : A^{(p)} \to A$ with the composition $V \circ F = [p]$.

The map $V : A^{(p)} \to A$ induces a map $\omega_{A/S} \to \omega_{A^{(p)}/S} \simeq \omega_{A/S}^{\otimes p}$. This gives a canonical section of $\omega_{A/S}^{\otimes(p-1)}$, called the Hasse invariant of A/S, denoted $\mathrm{Ha}(A/S) \in \Gamma(S, \omega_{A/S}^{\otimes(p-1)})$.

A.2. Ordinary Abelian schemes.

Definition A.2.1. Let k be an algebraically closed field of characteristic $p > 0$. Let A be an Abelian scheme over $\mathrm{Spec}(k)$. We say that A is ordinary if one the following equivalent conditions are satisfied
   (1) $A[p](k)$ has cardinality p^g.
   (2) $A[p](k) \simeq \mathbb{Z}/p^g\mathbb{Z}$.
   (3) The maximal étale quotient of $A[p]$ has order p^g.
   (4) The p-divisible group $A[p^\infty]$ is isomorphic to $(\mu_{p^\infty}^g \times (\mathbb{Q}/\mathbb{Z}_p)^g$.
   (5) $\mathrm{Ha}(A/k)$ is non-zero
   (6) $\ker(V : A^{(p)} \to A)$ is an étale group scheme.

   ((todo: ordinariness over non algebraically closed fields))

Lemma A.2.2 ([Wed99, Theorem 1.6.2]). Let X be the moduli scheme of principally polarized abelian schemes with level K^p structure. It is a scheme over $\mathrm{Spec}(\mathbb{Z}_{(p)})$. Let $X_{\mathbb{F}_p}^{\mathrm{ord}}$ be the set of points $s \in X \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} \mathrm{Spec}(\mathbb{F}_p)$ such that the Abelian scheme X_s over $\mathrm{Spec}(\kappa(s))$ is ordinary. Then $X_{\mathbb{F}_p}^{\mathrm{ord}}$ is a (Zariski) dense subset of $X_{\mathbb{F}_p}$.

Appendix B. Review of deformation theory

Definition B.0.1 ([Ill71, II.1.2.1, II.1.2.3]). Let $A \to B$ be a map of rings. The simplicial A-algebra $P_A(B)$ is defined by $P_A(B)_0 = A[B]$ and $P_A(B)_n = A[P_A(B)_{n-1}]$ for $n \geq 1$. The standard resolution of B over A is the argumentation $P_A(B) \to B$ where B is viewed as a constant simplicial A-algebra. The cotangent complex of B over A is the simplicial B-module $L_{B/A} = \Omega^1_{P_A(B)/A} \otimes_{P_A(B)} B$.

Remark B.0.2. This definition works in a general topos.

Definition B.0.3 ([Ill72, VII.1.1.1]). Let S be a scheme. Let S_{zar} be the small Zariski site over S. Let S_{fpqc} be the big fqpc site over S. The natural inclusion $S_{\mathrm{zar}} \to S_{\mathrm{fpqc}}$ induces a geometric map $(\epsilon^*, \epsilon_*) : \mathrm{Sh}(S_{\mathrm{zar}}) \rightleftarrows \mathrm{Sh}(S_{\mathrm{fpqc}})$.

Definition B.0.4. Let $f : X \to Y$ be a map of schemes. The cotangent complex is $L_{X/Y}$.

Definition B.0.5. Let S be a scheme. Let G be a group scheme over S that is flat and locally of finite presentation. Let $e : S \to G$ be the unit. The co-Lie complex is $\ell_G = Le^* L_{G/S}$, and the Lie complex is $\ell_G^\vee = R\underline{\mathrm{Hom}}(\ell_G, \mathcal{O}_S)$. Define $\underline{\ell}_G = Le^* \ell_G$.

Lemma B.0.6 ([Ill72, Theorem VII.4.2.5]). Let $f : S \to T$ be a map of schemes. Let $i : S \to S'$ be a T-extension by a quasi-coherent module I. Let A be a "schéma en anneaux" over T that is, as a scheme over T, tor-independent (c.f. [FJJ+71, Definition III.1.5]) with both S and S'. Let F (resp. G') be "schéma en A-modules" that are flat and locally of finite presentation over S (resp. S'). Let G be a "schéma en A-module" over S induced by G'. Let $u : F \to G$ be a morphism of "schémas en A-modules". Let K be the complex fitting into the distinguished triangle $K \to \ell_F^\vee \to \ell_G^\vee \to K[1]$. It is an object in $D(A \otimes_{\mathbb{Z}}^L \mathcal{O})$. Then there is an obstruction $\omega(u, G') \in \mathrm{Ext}_A^2(F, K \otimes_{\mathcal{O}}^L \epsilon^* I)$ which is zero if and only if there exists a pair (F', u') where F' is a deformation of F as "un schéma en A-modules" flat over S' and a map $u' : F' \to G'$ extending u.

Lemma B.0.7. Let S be a scheme. Let $i : S \to S'$ be an extension by a quasi-coherent module I. Suppose S and S' are both tor-indepeendent with $\mathrm{Spec}(\mathbb{Z})$. Let F (resp. G') be commutative group schemes over S (resp. S') that are flat and locally of finite presentation. Let G be a commutative group scheme over S induced by G'. Let $u : F \to G$ be a morphism of group schemes over S. Let K be the cone of the map $\ell_F^\vee \to \ell_G^\vee$. There is an obstruction $\omega(u, G') \in \mathrm{Ext}^1(F, K \otimes^L I)$ which vanishes if and only if there exists a pair (F', u') where F' is a deformation of F as a commutative group scheme that is flat over S', and $u' : F' \to G'$ is a map extending u.

Lemma B.0.8 ([Sch15, Theorem III.2.1]). Let A be a ring. Let G and H be commutative group schemes over A that are flat and of finite presentation, with a group map $u : H \to G$. Let $B \to A$ be a square-zero thickening with the argumentation ideal J. Let $\widetilde{G}$ be a lift of G to B. Let K be a cone of the map $\ell_H^\vee \to \ell_G^\vee$ of Lie complexes. Then there is an obstruction class $\omega \in \mathrm{Ext}^1(H, K \otimes^L J)$ which vanishes if and only if there exists a pair $(\widetilde{H}, \widetilde{u})$ where $\widetilde{H}$ is a flat commutative group scheme over B, and $\widetilde{u} : \widetilde{H} \to \widetilde{G}$ is a map lifting $u : H \to G$. Moreover, the obstruction class is functorial in J, in the following sense. If $B' \to A$ is another square-zero thickening with the argumentation ideal J', with a map $B \to B'$ over A, then $\omega' \in \mathrm{Ext}^1(H, K \otimes^L J')$ is the image of $\omega \in \mathrm{Ext}^1(H, K \otimes^L J)$ under the map $J \to J'$.

Appendix C. Review of Shimura varieties

C.1. Shimura datum and canonical models.

Definition C.1.1. A Shimura datum is a pair (G, X) where

- G is a reductive group over $\mathbb{Q}$;
- X is a $G(\mathbb{R})$-conjugacy class of maps $\mathbb{S} \to G_\mathbb{R}$;

satisfying the following properties

(1) For $h \in X$, only the characters $z/\overline{z}, 1, \overline{z}/z$ occur in the representation of $\mathbb{S}$ on $\mathrm{Lie}(G)$. In other words, the Hodge structure on $\mathrm{Lie}(G_\mathbb{R})$ defined by $\mathrm{Ad} \circ h$ is of type

$$\{(-1, 1), (0, 0), (1, -1)\}.$$

(2) $\mathrm{ad}(h(i))$ is a Cartan involution of G^{ad}, i.e. the real Lie group

$$\{g \in G^{\mathrm{ad}}(\mathbb{C}); \mathrm{ad}(h(i))\sigma(g) = g\}$$

is compact, where σ denotes the complex conjugation.

(3) G^{ad} has no factor defined over $\mathbb{Q}$ whose real points form a compact group. Equivalently, G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

Theorem C.1.2. Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. Let $\mathrm{Sh}_K(G, X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$.

(1) ((todo: add reference: Baily–Borel)) $\mathrm{Sh}_K(G, X)$ has a natural structure of an algebraic variety over $\mathbb{C}$.

(2) ((todo: add reference: Shimura, Deligne, Milne, ...)) $\mathrm{Sh}_K(G, X)$ has a model over a the reflex field $E(G, X)$.

Remark C.1.3. Let K denote a compact open subgroup of $G(\mathbb{A}_f)$. We get an inverse system of algebraic varieties (schemes) $(\mathrm{Sh}_K(G, X))_K$. There is an action ρ of $G(\mathbb{A}_f)$ on the system $(\mathrm{Sh}_K(G, X))_K$ defined by isomorphisms $\rho_K(g) : \mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{g^{-1}Kg}(G, X)$. For $k \in K$, $\rho_K(k)$ is the identity map. Therefore, for K' normal in K, there is an action of the finite group K/K' on $\mathrm{Sh}_{K'}(G, X)$, and the variety $\mathrm{Sh}_K(G, X)$ is the quotient of $\mathrm{Sh}_{K'}(G, X)$ by the action of K/K'.

C.2. **Siegel modular varieties.** Let (G, X) be a Siegel Shimura datum, i.e. the Shimura datum associated to a symplectic space. Then $G = \mathrm{GSp}_{2g}$, and the reflex field is $E(G, X) = \mathbb{Q}$ since G is split.

Lemma C.2.1. Let K^p be a compact open subgroup of $G(\mathbb{A}^\infty)$ contained in $\Gamma(N)^{(p)}$ for some integer $N \geq 3$ not divisible by p. Let $K_p = G(\mathbb{Z}_p)$. For compact open subgroup $U \subset K_p$, we have a smooth quasi-projective $\mathbb{Q}$-scheme $X_{K^p U, \mathbb{Q}}$ and a natural finite étale map $X_{K^p U, \mathbb{Q}} \to X_{K^p K_p, \mathbb{Q}}$ over $\mathbb{Q}$.

C.3. **PEL Shimura varieties.** For PEL type Shimura variety, see Milne.

This is [Kot92]; also, "PEL-type $\mathcal{O}$-lattice", cf. [Lan13, Definition 1.2.1.3]

Let p be a prime. Let B be a finite-dimensional simple $\mathbb{Q}$-algebra with center F. Let $\mathcal{O}_B$ be a $\mathbb{Z}_{(p)}$-order in B. Let $*$ be a positive involution on B that preserves $\mathcal{O}_B$. Let V be a non-degenerate skew-Hermitian B-module. Let G be the group of automorphisms of the skew-Hermitian B-module V. Let K^p be a compact open subgroup of $G(\mathbb{A}_f^p)$. Let $h : \mathbb{C} \to \operatorname{End}_B(V_\mathbb{R})$ be an $\mathbb{R}$-algebra homomorphism such that $(h(\bar{z}) = h(z)^*$, and that the symmetric bilinear form $(v, h(i)w)$ on $V_\mathbb{R}$ is positive definite. The map h determines a decomposition $V_\mathbb{C} = V_1 \oplus V_2$. Here V_1 is the subspace of $V_\mathbb{C}$ on which $h(z)$ acts by z. The field of definition of the isomorphism class of the complex representation V_1 of B is a number field E, with ring of integers $\mathcal{O}_E$.

Consider the following moduli problem over $\mathcal{O}_E \otimes_\mathbb{Z} \mathbb{Z}_{(p)}$.

Also, see [Lan13, Definition 1.4.1.4, Theorem 1.4.1.11, Remark 1.4.1.13].

Definition C.3.1 ([Lan13, Definition 1.2.1.3]). Let B be a finite-dimensional semi-simple algebra over $\mathbb{Q}$ with positive involution $*$ and center F, where positivity means $\operatorname{tr}_{B/\mathbb{Q}}(xx^*) > 0$ for all $x \neq 0$ in B. Let V be a finite B-module, equipped with a non-degenerate alternating bilinear form ψ, such that $\psi(bx, y) = \psi(x, b^*y)$ for all $x, y \in V$ and $b \in B$. Let $h : \mathbb{C} \to \operatorname{End}_B(V)_\mathbb{R}$ be a map over $\mathbb{R}$ such that complex conjugation on $\mathbb{C}$ corresponds by h to the adjunction in $\operatorname{End}_B(V)_\mathbb{R}$ with respect to the pairing ψ, and such that $(u, v) \mapsto \psi(u, h(i)v)$ is a positive definite symmetric pairing on $V_\mathbb{R}$. Let G be the reductive group over $\mathbb{Q}$ defined by

$$G(R) = \{g \in \operatorname{GL}_B(V \otimes_\mathbb{Q} R); \exists \mu(g) \in R^\times, \psi(gx, gy) = \mu(g)\psi(x, y)\},$$

where GL_B means B-equivariant linear maps. Let X be the $G(\mathbb{R})$-conjugacy class of $h^{-1} : \mathbb{C}^\times \to G_\mathbb{R}$. The pair (G, X) is called a PEL Shimura datum.

Definition C.3.2 ([Roz20, Section 1.2]). In order the define the integral model, we need to add the following new data. Let $\mathcal{O}_B$ be a $\mathbb{Z}_{(p)}$-order in B that is stable under the involution $*$ and becomes maximal after tensoring with $\mathbb{Z}_{(p)}$. We impose two more conditions which is omitted for now.

Let E be the reflex field.

Let $\mathcal{F}_{K^p}$ be the following category fibered in groupoids over the category of schemes over $\mathcal{O}_E \otimes_\mathbb{Z} \mathbb{Z}_{(p)}$:

- The objects over a scheme S are the tuples $(A, \lambda, \iota, \eta)$, where

APPENDIX D. REVIEW OF ARTIN'S CRITERION

Theorem D.0.1. Let S be a scheme of finite type over a field or an excellent Dedekind domain. Let X be a category fibered in groupoids over $\operatorname{Sch}_{/S}$. Then X is an algebraic stack locally of finite type over S if and only if the following conditions hold:

(1) X is a stack for the étale topology.
(2) X is locally of finite presentation.
(3) ...

APPENDIX E. REVIEW OF PERFECTOID SPACES

Definition E.0.1. Let $\mathfrak{Y}$ be a flat t-adic formal scheme over $\mathbb{F}_p[[t^{1/(p-1)p^\infty}]]$. Let $\Phi : \mathfrak{Y} \to \mathfrak{Y}$ be the relative Frobenius. Then the inverse limit $\lim_\Phi \mathfrak{Y}$ is representable by a perfect flat t-adic formal scheme $\mathfrak{Y}^{\operatorname{perf}}$ over $\mathbb{F}_p[[t^{1/(p-1)p^\infty}]]$, called the perfection of $\mathfrak{Y}$.

Locally,
$$(\operatorname{Spf}(S))^{\operatorname{perf}} = \operatorname{Spf}(S^{\operatorname{perf}})$$
where S^{perf} is the t-adic completion of $\lim_\Phi S$.

Definition E.0.2. Let $\mathcal{Y}$ be an adic space over $\mathbb{F}_p((t^{1/(p-1)p^\infty}))$. Let $\Phi : \mathcal{Y} \to \mathcal{Y}$ be the ralative Frobenius. Then there exists a unique perfectoid space $\mathcal{Y}^{\operatorname{perf}}$ over $\mathbb{F}_p((t^{1/(p-1)p^\infty}))$, called the perfection of $\mathcal{Y}$, such that
$$\mathcal{Y}^{\operatorname{perf}} \sim \lim_\Phi \mathcal{Y}.$$

Locally,
$$\operatorname{Spa}(S, S^+)^{\operatorname{perf}} = \operatorname{Spa}(S^{\operatorname{perf}}, S^{\operatorname{perf},+})$$
where $S^{\operatorname{perf},+}$ is the t-adic completion of $\lim_\Phi S^+$, and $S^{\operatorname{perf}} = S^{\operatorname{perf},+}[1/t]$.

((todo: the two constructions are compatible))

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