NOTES

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Contents

1. Survey

1.1. Lawrence–Venkatesh.

Remark 1.1. The reference is [LV20].

Lemma 1.2 (Mordell Conjecture)**.** Let K be a number field. The set of K-rational points on a smooth projective K-curve of genus ≥ 2 is finite.

Remark 1.3. This is first proved by Faltings, see [Fal83]. Lawrence–Venkatesh gives a new proof based on a closer study of the variation of p-adic Galois representations in a family. Both proofs reduce the problem to the finiteness results for varieties with good reduction, cf. [Par68].

1.2. Betts–Stix.

Remark 1.4. Let K be a number field. Let Y be a smooth projective (geometrically connected) curve over K of genus ≥ 2. We have the fundamental exact sequence

$$1 \to \pi_1^{\text{ét}}(Y_{\overline{K}}) \to \pi_1^{\text{ét}}(Y) \to G_K \to 1$$

on étale fundamental groups (at appropriate basepoints), where G_K is the absolute Galois group of K.

Definition 1.5 (Gortz–Wedhorn, Definition II.26.1, Defintion I.15.14)**.** Let K be a field. A curve over the field K is a K-scheme of finite type that is equi-dimension of dimension 1.

Lemma 1.6 (Gortz–Wedhorn, Theorem I.15.18)**.** Let K be a field. A separated curve over K is quasi-projective over K.

Definition 1.7. Let K be a field. Let C be a proper seperated curve over K. The (arithmetic) genus of C is defined as

$$g(C) = 1 - \chi(\mathcal{O}_C) = 1 - \dim_k H^0(C, \mathcal{O}_C) + \dim_k H^1(C, \mathcal{O}_C).$$

Here χ is the Euler characteristic.

Definition 1.8. A number field E is said to be

(1) totally real, if the image of every homomorphism $E \to \mathbb{C}$ is contained in $\mathbb{R}$;
(2) totally imaginary, if the image is never contained in $\mathbb{R}$.

Definition 1.9. A number field E is CM if the following equivalent conditions are satisfied.

(1) E is a totally imaginary quadratic extension of a totally real numer field.
(2) There exists a non-trivial automorphism ι_E of E such that $\rho \circ \iota_E = \iota_E \circ \rho$ for all homomorphism $\rho : E \to \mathbb{C}$.
(3) $E = F[\alpha]$ with F totally real, $\alpha^2 \in F$, and $\rho(\alpha^2) < 0$ for all homomorphism $\rho : F \to \mathbb{C}$.

References

[Fal83] Gerd Faltings. "Endlichkeitssätze für abelsche Varietäten über Zahlkörpern". In: *Inventiones mathematicae* 73 (1983), pp. 349–366.

[LV20] Brian Lawrence and Akshay Venkatesh. "Diophantine problems and p-adic period mappings". In: *Inventiones mathematicae* 221 (2020), pp. 893–999.

[Par68] Aleksei Nikolaevich Paršin. "Algebraic curves over function fields. I". In: *Mathematics of the USSR-Izvestiya* 2.5 (1968), p. 1145.

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