1. Algebraic Number Theory

Definition 1.0.1. Let K be a global field. A place of K is an equivalence class of non-trivial absolute values on K. The set of places of K is denoted by V_K. Let $v \in V_K$ be a place. Denote by K_v the completion of K with respect to any absolute value representing v. An absolute value on K is called non-Archimedean if it satisfies the ultrametric inequality, and it is called Archimedean otherwise.

Lemma 1.0.2. An absolute value $|-|$ is Archimedean if and only if $|n| \to \infty$ as $n \to \infty$ for integers n.

Lemma 1.0.3. Any absolute value on a global function field is non-Archimedean.

Lemma 1.0.4. Let K be a number field. Then K has finitely many Archimedean places. Let r_1 denote the number of distinct embeddings $K \to \mathbb{R}$. Let $2r_2$ denote the number of distinct embeddings $K \to \mathbb{C}$ with image not contained in $\mathbb{R}$. Then the number of Archimedean places of K is equal to $r_1 + r_2$.

Remark 1.0.5. Complex conjugation.

Example 1.0.6. There is one Archimedean place ∞ on $\mathbb{Q}$, represented by the ordinary absolute value, and the completion $\mathbb{Q}_\infty$ is denoted by $\mathbb{R}$. The non-Archimedean places are in bijective correspondence with the prime numbers. The completion $\mathbb{Q}_p$ is the field of p-adic numbers.

Definition 1.0.7. Let $K \to L$ be a homomorphism of global fields. Any non-trivial absolute value on L restricts to a non-trivial absolute value on K. So we have a canonical map $V_L \to V_K$. This map is surjective with finite fibers. For $w \in V_L$, we denote its restriction by $w|_K$. For $v \in V_K$, we write $w \mid v$ if $w|_K = v$. If $w \mid v$ the the inclusion $K \to L$ extends naturally to a homomorphism of completions $K_v \to L_w$.

Lemma 1.0.8. Let v be an Archimedean place of K. There is a unique absolute value $|-|$ representing v such that $|n| = n$ for all integer $n \geq 1$. By the Gelfand–Mazur theorem, the completion K_v is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$, with $|-|$ coinciding with the restriction of the standard absolute value. We say that v is real or complex, respectively. We say the normalized absolute value at v is $|-|$ (real) or $|-|^2$ (complex).

Definition 1.0.9. Let v be a non-Archimedean place on K. Let $|-|$ be an absolute value representing v. Then $-\log|K^\times|$ is a non-trivial discrete subgroup of $\mathbb{R}$. So there exists a unique $\alpha \in \mathbb{R}_{>0}$ such that $-\alpha \log|K^\times| = \mathbb{Z}$. The valuation

$$\mathrm{ord}_v = -\alpha \log|\cdot| : K \to \mathbb{Z} \cup \{\infty\}$$

is intrinsic to v and does not depend on $|\cdot|$. The valuation ord_v extends in a unique way to a discrete valuation on the completion K_v. The subring

$$\mathcal{O}_v = \{x \in K_v \mid \mathrm{ord}_v(x) \geq 0\}$$

is a complete discrete valuation ring with maximal ideal

$$\mathfrak{m}_v = \{x \in K_v \mid \mathrm{ord}_v(x) > 0\}.$$

Its residue field k_v is finite. The normalized v-adic absolute value is

$$\|x\|_v = |k_v|^{-\mathrm{ord}_v(x)}.$$

The restriction of $\|\cdot\|_v$ to K is a canonical representation of the place v.

Lemma 1.0.10. Now we have a canonical normalized absolute value $\|\cdot\|_v$ representing each place v of a global field K. Then for $x \in K^\times$, we have $\|x\|_v = 1$ for almost all (i.e. all but finitely many) places v, and

$$\prod_{v \in V_K} \|x\|_v = 1.$$

Lemma 1.0.11 (Artin–Whaples)**.** Let K be a field. Let V be a set of places of K. Suppose

(1) For each place $v \in V$ there exists an absolute value $\|\cdot\|_v$ representing v such that for $x \in K^\times$ we have $\|x\|_v = 1$ for almost all v, and $\prod_{v \in V} \|x\|_v = 1$.
(2) The completion K_v is locally compact for at least one place $v \in V$.

Then K is a global field and $V = V_K$.

Definition 1.0.12. Let S be a non-empty finite set of places containing the Archimedena palces of K. The ring of S-integers in K is

$$\mathcal{O}_{K,S} = \{x \in K \mid \|x\|_v \leq 1, v \in V_K \backslash S\}.$$

This is a Dedekind domain. For $v \notin S$, we define

$$\mathfrak{p}_v = \{x \in \mathcal{O}_{K,S} \mid \|x\|_v < 1\}.$$

This is a prime ideal in $\mathcal{O}_{K,S}$, and $\mathcal{O}_v$ is canonically identified with the $\mathfrak{p}_v$-adic completion of $\mathcal{O}_{K,S}$, with $\mathfrak{m}_v = \mathfrak{p}_v \mathcal{O}_v$, and $\mathfrak{p}_v = \mathcal{O}_{K,S} \cap \mathfrak{m}_v$, and $k_v = \mathcal{O}_{K,S}/\mathfrak{p}_v$.

2. Algebraic Groups

Definition 2.0.1. Recall that a morphism of algebraic groups $(G, m) \to (G', m')$ is a morphism of k-schemes $G \to G'$ that is compatible with m and m', i.e. the following diagram

$$\begin{tikzcd} G \times G \arrow[r] \arrow[d] & G \arrow[d] \\ G' \times G' \arrow[r] & G' \end{tikzcd}$$

commutes. In particular, there is no requirements on e and inv. However, the compatibiliy for e and inv is automatic.

Lemma 2.0.2. Let $\phi : (G, m_G) \to (H, m_H)$ be a morphism of algebraic groups. Then $\phi \circ e_G = e_H$ and $\phi \circ \mathrm{inv}_G = \mathrm{inv}_H \circ \phi$.

Remark 2.0.3. In particular, the neutral element and the inverse map are uniquely determined by the multiplication map.

Lemma 2.0.4. Let (G, m) be an algebraic group over k. Then the map $G \to \mathrm{Spec}(k)$ is separated.

Proof. We have a pullback diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\Delta"] & \mathrm{Spec}(k) \arrow[d, "e"] \\ G \times G \arrow[r] & G \end{tikzcd}$$

where the morphism on the bottom is

$$m \circ (\mathrm{id} \times \mathrm{inv}) : G \times G \to G.$$

Then $\Delta : G \to G \times_k G$ is separated because the map $\mathrm{Spec}(k) \to G$ is a closed immersion. $\square$

Definition 2.0.5. An algebraic group (G, m) is commutative if $m \circ t = m$ where $t : G \times G \to G \times G$ is the transposition map.

Lemma 2.0.6. An algebraic group G is commutative if and only if $G(R)$ is commutative for every k-algebra R.

Lemma 2.0.7. Let X be a scheme over k of finite type. If X is reduced and k is perfect, then X is geometrically reduced.

Lemma 2.0.8. Let G be an algebraic group over k. Then G is geometrically reduced if and only if G is smooth.

Definition 2.0.9. Let G be an algebraic group over k. The connected component containing e is called the identity (or neutral) component of G, denoted by G°.

Lemma 2.0.10. Let X be a scheme locally of finite type over k.

(1) There exists an étale k-scheme $\pi_0(X)$ and a morphism $q_X : X \to \pi_0(X)$ with the following universal property: for every morphism $f : X \to Y$ of k-schemes where Y is an étale k-scheme, there exists a unique morphism $g : \pi_0(X) \to Y$ such that $f = g \circ q_X$.
(2) The morphism q_X is faithfully flat and the fibers of q_X are the connected components of X.

(3) For every morphism $f : X \to Y$ of k-schemes locally of finite type, there exists a unique morphism $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ such that $q_Y \circ f = \pi_0(f) \circ q_X$. The construction $X \mapsto \pi_0(X)$ is functorial.
(4) Let K/k be a field extension. Then we have an isomorphism of K-schemes
$$\pi_0(X \otimes_k K) \to \pi_0(X) \otimes_k K$$
functorial in X.
(5) The canonical morphism
$$\pi_0(X \times Y) \to \pi_0(X) \times \pi_0(Y)$$
is an isomorphism.
(6) X is geometrically connected over k if and only if $\pi_0(X) \simeq \mathrm{Spec}(k)$.

The scheme $\pi_0(X)$ is called the scheme of connected components of k.

Example 2.0.11. Let $X = \mathrm{Spec}(k[T])$. The points of X are the prime ideals of $k[T]$. These are the zero ideal and all principal ideals generated by irreducible polynomials. Hence every finite extension which is generated over k by a single element (in particular every finite separable extension) occurs as a residue class field of a point of X.

Example 2.0.12. Let K/k be a non-trivial extension. Then $X = \mathrm{Spec}(K)$ does not admit a k-morphism $\mathrm{Spec}(k) \to X$.

3. Homological Algebra

Definition 3.0.1. An Abelian category $\mathcal{C}$ is called a Grothendieck Abelian category, if the following conditions are satisfied:

(1) All (small?) coproducts exist in $\mathcal{C}$.
(2) Filtered colimits are exact.
(3) The category $\mathcal{C}$ admits a generator, i.e. there exists an object G of $\mathcal{C}$ such that the functor $\mathrm{Hom}_{\mathcal{C}}(G, -)$ reflects isomorphisms (i.e. conservative).

Lemma 3.0.2. Let $\mathcal{C}$ be a Grothendieck Abelian category. A functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ is representable if and only if it commutes with small limits, i.e. $F(\mathrm{colim}_i X_i) = \lim_i F(X_i)$ for any diagram $X : \mathcal{I} \to \mathcal{C}$.

Lemma 3.0.3. A Grothendieck Abelian category has enough injectives.

Definition 3.0.4. Let $\mathcal{A}$ be an additive category. The category of complexes in $\mathcal{A}$ is denoted by $C(\mathcal{A})$. The shift functor $T : C(\mathcal{A}) \to C(\mathcal{A})$ changes differential by a sign
$$d^j_{TX^\bullet} = -d^{j+1}_{X^\bullet}.$$
The category $C(\mathcal{A})$ is additive. If $\mathcal{A}$ is Abelian, then $C(\mathcal{A})$ is also Abelian. We can form finite products and finite coproducts in $C(\mathcal{A})$ component wise. More generally, if $\mathcal{A}$ admits limits (resp. colimits) indexed by some category, then the same is true for $C(\mathcal{A})$ and these limits (resp. colimits) are formed componentwise. If $\mathcal{A}$ is a Grothendieck Abelian category, then $C(\mathcal{A})$ is a Grothendieck Abelian category.

Definition 3.0.5. A complex $X^\bullet$ is called bounded (resp. bounded below, resp. bounded above) if $X^j = 0$ for $|j| >> 0$ (resp. $j << 0$, resp. $j >> 0$). We denote by $C^b(\mathcal{A})$, $C^+(\mathcal{A})$, $C^-(\mathcal{A})$ the full subcategories of $C(\mathcal{A})$ that are bounded, bounded below, and bounded above.

Let $M \subset \mathbb{Z}$ be a subset. A complex $X^\bullet$ is said to be concentrated in degree M if $X^j = 0$ for all $j \notin M$. The full subcategory is denoted by $C^M(\mathcal{A})$.

Definition 3.0.6. Let $\mathcal{A}$ be an Abelian category. We consider the following functors $C(\mathcal{A}) \to \mathcal{A}$:
$$\begin{aligned}
\pi_n &: X^\bullet \mapsto X^n \\
Z^n &: X^\bullet \mapsto \ker(d^n : X^n \to X^{n+1}) \\
B^n &: X^\bullet \mapsto \mathrm{im}(d^{n-1} : X^{n-1} \to X^n) \\
H^n &: X^\bullet \mapsto H^n(X^\bullet).
\end{aligned}$$

Definition 3.0.7. Let $\mathcal{A}$ be an Abelian category. The truncation functors are defined as

$$\tau^{\leq n}(X^\bullet) : \cdots \to X^{n-2} \to X^{n-1} \to \ker(d^n) \to 0 \to \cdots$$
$$\tau^{\geq n}(X^\bullet) : \cdots \to 0 \to \operatorname{coker}(d^{n-1}) \to X^{n+1} \to X^{n+2} \to \cdots$$

We have

$$H^i(\tau^{\leq n}(X)) = \begin{cases} H^i(X) & i \leq n \\ 0 & i > n \end{cases}$$

and

$$H^i(\tau^{\geq n}(X)) = \begin{cases} 0 & i < n \\ H^i(X) & i \geq n \end{cases}$$

Definition 3.0.8. The homotopy category of $C(\mathcal{A})$ is denoted by $K(\mathcal{A})$, i.e. the morphisms are equivalence classes of chain maps up to chain homotopy.

Definition 3.0.9. Let $f : X \to Y$ be a chain map of complexes. The mapping cone C_f of f the complex defined by

$$C_f^n = X^{n+1} \oplus Y^n$$

with differential given by

$$d^n_{C_f} = \begin{bmatrix} -d_X & 0 \\ f & d_Y \end{bmatrix}.$$

Remark 3.0.10. Let $u : X \to Y$ be a morphism in $C(\mathcal{A})$. The inclusions

$$Y^i \to X^{i+1} \oplus Y^i = C_u^i$$

defines a chain map $\iota : Y \to C_u$. The composition

$$X \to Y \to C_u$$

is homotopic to zero via the homotopy h defined by the inclusions

$$h^i : X^i \to X^i \oplus Y^{i-1} = C_u^{i-1}.$$

The mapping cone C_u has the following universal property of a "homotopy cokernel". If $v : Y \to Z$ be another morphism and $\widetilde{h} : v \circ u \simeq 0$ is a homotopy, then there exists a unique morphism of complexes $w : C_u \to Z$ such that $v = w \circ \iota$ and $\widehat{h}^i = w^{i-1} \circ h^i$. Indeed, w is given by

$$w^n = \widetilde{h}^{n+1} + v^n.$$