1. Algebraic Groups

Definition 1.0.1. Some terminology.

(1) k is a field with algebraic closure k^a and separable closure k^s.
(2) Algebraic scheme means a scheme of finite type over k.
(3) Algebraic variety means a separated, geometrically reduced algebraic scheme over k.
(4) Let X be an algebraic scheme. We denote by $|X|$ the set of closed points of X. For $x \in X$, we use the usual notation $k(x)$ and $\mathfrak{m}_x$ for the residue field and the maximal ideal of the local ring $\mathcal{O}_{X,x}$. Note that $k \subset k(x)$.
(5) We set $* = \mathrm{Spec}(k)$.

Definition 1.0.2. An algebraic group over k is a group object in the category of algebraic schemes over k. In other words, an algebraic group over k is a tuple (G, m, e, inv) where

(1) G is an algebraic scheme over k.
(2) $m : G \times G \to G$ is the "multiplication".
(3) $e : * \to G$ is the "identity".
(4) $\mathrm{inv} : G \to G$ is the "inverse".

These datum are required to make the following diagrams commute.

(1) Associativity.

$$\begin{tikzcd}
G \times G \times G \arrow[r] \arrow[d] & G \times G \arrow[d] \\
G \times G \arrow[r] & G
\end{tikzcd}$$

(2) Identity.

$$\begin{tikzcd}
* \times G \arrow[r] \arrow[dr] & G \times G \arrow[d, "m"] & G \times * \arrow[l] \arrow[dl] \\
& G &
\end{tikzcd}$$

(3) Inverse.

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & G \times G \arrow[d] & G \arrow[l] \arrow[d] \\
* \arrow[r] & G & * \arrow[l]
\end{tikzcd}$$

When G is a variety, we say G is a group variety. When G is affine, we say G is an affine algebraic group. A morphism of algebraic groups $\phi : (G, m) \to (G', m')$ is a morphism $\phi : G \to G'$ of k-schemes such that the following diagram

$$\begin{tikzcd}
G \times G \arrow[r] \arrow[d] & G \arrow[d] \\
G' \times G' \arrow[r] & G'
\end{tikzcd}$$

commutes.

Remark 1.0.3. By removing the inverse morphism and related conditions, we obtain the notion of "algebraic monoid", i.e. an algebraic scheme equipped with a associative multiplication and a unit.

Definition 1.0.4. Let G be an algebraic group over k. Let R be a k-algebra. Set

$$G(R) = \mathrm{Hom}_k(\mathrm{Spec}(k), G).$$

It carries an induced group structure.

Lemma 1.0.5. If $\phi : G \to G'$ is a morphism of algebraic groups, then

$$\begin{tikzcd}
G \arrow[rr] & & G' \\
& * \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

commutes.

Definition 1.0.6. We say an algebraic group G is trivial if $e : * \to G$ is an isomorphism. We say a morphism $\phi : G \to G'$ is trivial if it can be factored as

$$G \to * \to G'.$$

Definition 1.0.7. Let (G, m_G) be an algebraic group over k. An algebraic subgroup of G over k is an algebraic group (H, m_H) such that H is a k-subscheme of G and the inclusion $H \to G$ is a morphism of algebraic groups. The algebraic subgroup H is called an subgroup variety if the underlying scheme is a variety.

Remark 1.0.8. An algebraic subgroup of a group variety may not be a subgroup variety.

Lemma 1.0.9. Let (G, m, e, inv) be an algebraic group. Let H be a k-subscheme of G. Suppose m and inv both factor through H. Then H is an algebraic subgroup of G.

Lemma 1.0.10. Let k'/k be an extension of fields.

(1) Let G be an algebraic group over k. Then $G_{k'} = G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ is an algebraic group over k'.
(2) Let $\phi : G \to G'$ be a morphism of algebraic groups over k. Then the base-change $G_{k'} \to G'_{k'}$ is a morphism of algebraic groups over k'.
(3) Let G be an algebraic group over k. Let H be an algebrai subgroup of G. Then $H_{k'}$ is an algebraic subgroup of $G_{k'}$ over k'.

Definition 1.0.11. Let X be an algebraic scheme over k. It defines a functor

$$\widetilde{X} : \mathrm{Alg}_k^{\mathrm{fg}} \to \mathrm{Set}, \quad R \mapsto X(R) = \mathrm{Hom}_k(\mathrm{Spec}(R), X).$$

Remark 1.0.12. Set theoretic issues. Note that the category $\mathrm{Alg}_k^{\mathrm{fg}}$ is essentially small.

Lemma 1.0.13. The functor $X \mapsto \widetilde{X}$ is fully faithful.

Definition 1.0.14. We say a functor $F : \mathrm{Alg}_k^{\mathrm{fg}} \to \mathrm{Set}$ is representable if there exists an algebraic k-scheme X such that $F \simeq \widetilde{X}$.

Lemma 1.0.15. Let X be an algebraic scheme over k. Suppose the functor $\widetilde{X} : \mathrm{Alg}_k^{\mathrm{fg}} \to \mathrm{Set}$ factors through $\mathrm{Grp} \to \mathrm{Set}$. Then X carries a structure of an algebraic group over k.

Lemma 1.0.16. An algebraic group over k is a functor $\mathrm{Alg}_k^{\mathrm{fg}} \to \mathrm{Grp}$ whose underlying functor to Set is representable by an algebraic scheme over k.

Remark 1.0.17. Let G be an algebraic group over k. Let H be an algebraic subscheme of G over k. Suppose $H(R) \subset G(R)$ is a subgroup for every $R \in \mathrm{Alg}_k^{\mathrm{fg}}$. Then H has a structure of an algebraic subgroup of G over k.

Example 1.0.18. For every $R \in \mathrm{Alg}_k^{\mathrm{fg}}$, define

$$\mathrm{SL}_n(R) = \{g \in M_n(R) \mid \det(g) = 1\}.$$

Then we have a functor $\mathrm{SL}_n : \mathrm{Alg}_k^{\mathrm{fg}} \to \mathrm{Grp}$. In order to obtain an algebraic group over k, we need to show that

$$\mathrm{SL}_n : \mathrm{Alg}_k^{\mathrm{fg}} \to \mathrm{Set}$$

is representable. Actually, it is represented by the affine scheme

$$\mathrm{Spec}(k[T_{ij} \mid 1 \le i, j \le n]/(\det(T_{ij}) = 1)).$$

Therefore, we obtain an affine algebraic group SL_n over k.

Example 1.0.19. For every $R \in \mathrm{Alg}_k^{\mathrm{fg}}$, define

$$\mathrm{GL}_n(R) = \{g \in M_n(R) \mid \det(g) \in R^\times\}.$$

Then we have a functor $\mathrm{GL}_n : \mathrm{Alg}_k^{\mathrm{fg}} \to \mathrm{Grp}$. It is represented by

$$\mathrm{Spec}\Big(k[T_{ij} \mid 1 \le i, j \le n][S]/(\det(T_{ij}) \cdot S - 1)\Big).$$

Therefore, we obtain an affine algebraic group GL_n over k.

Example 1.0.20. The functor $R \mapsto (R, +)$ is represented by the affine scheme $\mathrm{Spec}(k[T])$, and hence is an algebraic group, denoted by $\mathbb{G}_a$.

Example 1.0.21. The functor $R \mapsto (R^\times, \times)$ is represented by the affine scheme $\mathrm{Spec}(k[T,S]/(TS-1))$, and hence is an algebraic group, denoted by $\mathbb{G}_m$. ((TODO: $k[T, T^{-1}]$))

2. The Koszul Complex

Definition 2.0.1. Let A be a ring. Let $f_1, \ldots, f_r$ be elements of A. The Koszul complex $K_\bullet(f_1, \ldots, f_r)$ is defined by

$$K_p(f_1, \ldots, f_r) = \wedge^p A^{\oplus r}$$

for all $p \geq 0$. Let $e_1, \ldots, e_r$ be the standard basis of $A^{\oplus r}$. Then

$$\{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid i_1 < \cdots < i_p\}$$

form a basis for $K_p(f_1, \ldots, f_r)$. The differential

$$\partial_p : K_p(f_1, \ldots, f_r) \to K_{p-1}(f_1, \ldots, f_r)$$

is defined to be the contraction by $f_1 e_1^* + \cdots f_r e_r^*$, where $(e_i^*)_i$ is the basis of $\mathrm{Hom}_A(A^{\oplus r}, A)$ dual to $(e_i)_i$. In other words, we have

$$\partial(e_{i_1} \wedge \cdots \wedge e_{i_p}) = (f_1 e_1^* + \cdots + f_r e_r^*) \lrcorner (e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{k=1}^{p} (-1)^{k-1} f_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_k} \wedge \cdots \wedge e_{i_p}.$$

Lemma 2.0.2. We have $\partial_p \circ \partial_{p+1} = 0$.

Proof. We have

$$\begin{aligned}
\partial^2(e_{i_1} \wedge \cdots \wedge e_{i_{p+1}}) &= \sum_{k=1}^{p+1} (-1)^{k-1} f_{i_k} \partial(e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_k} \wedge \cdots \wedge e_{i_{p+1}}) \\
&= \sum_{k=1}^{p+1} (-1)^{k-1} f_{i_k} \sum_{j=1}^{k-1} (-1)^{j-1} f_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_j} \wedge \cdots \wedge \widehat{e}_{i_k} \wedge \cdots \wedge e_{i_{p+1}} \\
&\quad + \sum_{k=1}^{p+1} (-1)^{k-1} f_{i_k} \sum_{j=k+1}^{p+1} (-1)^{j-2} f_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_k} \wedge \cdots \wedge \widehat{e}_{i_j} \wedge \cdots \wedge e_{i_{p+1}} \\
&= 0.
\end{aligned}$$

□

Definition 2.0.3. Let M be an A-module. We define

$$\begin{aligned}
K_\bullet(f_1, \ldots, f_r; M) &= K_\bullet(f_1, \ldots, f_r) \otimes_A M \\
K^\bullet(f_1, \ldots, f_r; M) &= \mathrm{Hom}_A(K_\bullet(f_1, \ldots, f_r), M) \\
H_p(f_1, \ldots, f_r; M) &= H_p(K_\bullet(f_1, \ldots, f_r; M)) \\
H^p(f_1, \ldots, f_r; M) &= H^p(K_\bullet(f_1, \ldots, f_r; M))
\end{aligned}$$

Remark 2.0.4. An element g in $K^p(f_1, \ldots, f_r; M)$ is completely determined by its value on the basis $e_{i_1} \wedge \cdots \wedge e_{i_p}$.

Lemma 2.0.5. Let $g \in K^p(f_1, \ldots, f_r; M)$. We have

$$\begin{aligned}
(dg)(e_{i_1} \wedge \cdots \wedge e_{i_{p+1}}) &= g(\partial(e_{i_1} \wedge \cdots \wedge e_{i_{p+1}})) \\
&= \sum_{k=1}^{p+1} (-1)^{k-1} f_{i_k} g(e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_k} \wedge \cdots \wedge e_{i_{p+1}}).
\end{aligned}$$

Lemma 2.0.6. We have

$$K^p(f_1,\dots,f_r;M) = \mathrm{Hom}_A(K_p(f_1,\dots,f_r),M) \simeq \wedge^p(A^{\oplus r})^* \otimes_A M.$$

Hence every $g \in K^p(f_1,\dots,f_r;M)$ can be written as

$$g = \sum_{i_1<\cdots<i_p} e^*_{i_1}\wedge\cdots\wedge e^*_{i_p}\otimes g_{i_1\cdots i_p}$$

where

$$g_{i_1\dots i_p} = g(e_{i_1}\wedge\cdots\wedge e_{i_p}).$$

Lemma 2.0.7. Let $g \in K^p(f_1,\dots,f_r;M)$. Then

$$dg = (f_1e_1^* + \cdots f_re_r^*)\wedge g.$$

Proof. We have

$$\begin{aligned}(f_1e_1^* + \cdots f_re_r^*)\wedge g &= (f_1e_1^* + \cdots f_re_r^*)\wedge\Big(\sum_{i_1<\cdots<i_p} e^*_{i_1}\wedge\cdots\wedge e^*_{i_p}\otimes g_{i_1\dots i_p}\Big)\\ &= \sum_{i_1<\cdots<i_{p+1}} e^*_{i_1}\wedge\cdots\wedge e^*_{i_{p+1}}\otimes\sum_{k=1}^{p+1}(-1)^{k-1}f_{i_k}g_{i_1\dots\widehat{i}_k\dots i_{p+1}}.\end{aligned}$$

$\square$

Definition 2.0.8. Let $\star_p : K_p(f_1,\dots,f_r;M)\to K^{r-p}(f_1,\dots,f_r;M)$ be the A-module homomorphism defined by

$$\star_p(e_{i_1}\wedge\cdots\wedge e_{i_p}) = \mathrm{sign}(i_1\dots i_pj_1\dots j_{r-p})e^*_{j_1}\wedge\cdots\wedge e^*_{j_{r-p}},$$

where $\{j_1,\dots,j_{r-p}\} = \{1,\dots,r\}\backslash\{i_1,\dots,i_p\}$, and $\mathrm{sign}(\bullet)$ is the sign of permutation. Note that the right hand side is well-defined, i.e. it does not depend on the order of $j_1,\dots,j_{r-p}$.

Lemma 2.0.9. We have $\star_{p-1}\partial_p = (-1)^{p-1}d_{r-p}\star_p$

Proof. We have

$$\begin{aligned}\star_{p-1}\partial_p(e_{i_1}\wedge\cdots\wedge e_{i_p}) &= \star_{p-1}\left(\sum_{k=1}^{p}(-1)^{k-1}f_{i_k}e_{i_1}\wedge\cdots\wedge\widehat{e}_{i_k}\wedge\cdots\wedge e_{i_p}\right)\\ &= \sum_{k=1}^{p}(-1)^{k-1}f_{i_k}\star_{p-1}(e_{i_1}\wedge\cdots\wedge\widehat{e}_{i_k}\wedge\cdots\wedge e_{i_p})\\ &= \sum_{k=1}^{p}(-1)^{k-1}f_{i_k}\mathrm{sign}(i_1\dots\widehat{i}_k\dots i_pi_kj_1\dots j_{r-p})e^*_{i_k}\wedge e^*_{j_1}\wedge\cdots\wedge e^*_{j_{r-p}}\\ &= (-1)^{p-1}\sum_{k=1}^{p}f_{i_k}\mathrm{sign}(i_1\dots i_pj_1\dots j_{r-p})e^*_{i_k}\wedge e^*_{j_1}\wedge\cdots\wedge e^*_{j_{r-p}}\\ &= (-1)^{p-1}d_{r-p}\star_p.\end{aligned}$$

Note that we used the fact that $i_1<\cdots<i_p$. $\square$

Lemma 2.0.10. We have

$$H_p(f_1,\dots,f_r;M)\simeq H^{r-p}(f_1,\dots,f_r;M)$$

for every p.

Proof. We have a chain map

$$\begin{array}{ccccccccc}\cdots & \longrightarrow & K_{p+1} & \xrightarrow{\partial_{p+1}} & K_p & \xrightarrow{\partial_p} & K_{p-1} & \longrightarrow & \cdots\\ & & \downarrow{\scriptstyle\star_{p+1}} & & \downarrow{\scriptstyle\star_p} & & \downarrow{\scriptstyle\star_{p-1}} & & \\ \cdots & \longrightarrow & K^{r-p-1} & \longrightarrow & K^{r-p} & \xrightarrow[(-1)^{p-1}d_{r-p}]{} & K^{r-p+1} & \longrightarrow & \cdots\end{array}$$

It is an isomorphism of complexes as every $\star_p$ is an isomorphism of A-modules. Therefore

$$H_p(f_1,\dots,f_r;M) \simeq H^{r-p}(f_1,\dots,f_r;M)$$

for every p. $\square$

Lemma 2.0.11. Suppose $f_1,\dots,f_r$ generate the unit ideal of A. Then

$$H_p(f_1,\dots,f_r;M) = H^p(f_1,\dots,f_r;M) = 0$$

for all p and every A-module M.

Proof. It suffices to show $H^p(f_1,\dots,f_r;M)=0$. Since $f_1,\dots,f_r$ generate the unit ideal of A, we can choose elements $a_1,\dots,a_r$ in A such that

$$a_1f_1+\cdots+a_rf_r = 1.$$

For every n, define the A-module homomorphism

$$\phi_n : K^n(f_1,\dots,f_r;M) \to K^{n-1}(f_1,\dots,f_r;M)$$

to be the contraction by $a_1e_1+\cdots a_re_r$. In other words,

$$\phi_n(e^*_{i_1}\wedge\cdots\wedge e^*_{i_n}) = \sum_{k=1}^{n}(-1)^{k-1}a_{i_k}e^*_{i_1}\wedge\cdots\wedge\widehat{e^*_{i_k}}\wedge\cdots\wedge e^*_{i_n}.$$

For every $g\in K^p(f_1,\dots,f_r;M)$, we have

$$\begin{aligned}(d\circ\phi+\phi\circ d)(g) &= \sum_{k=1}^{r} f_ke^*_k\wedge\phi(g) + \sum_{l=1}^{r} a_le_l \lrcorner\, d(g)\\ &= \sum_{k,l=1}^{r} f_ka_le^*_k\wedge(e_l\lrcorner\, g) + \sum_{k,l=1}^{r} a_lf_ke_l\lrcorner\,(e^*_k\wedge g)\\ &= \sum_{k,l=1}^{r} f_ka_l\delta_{kl}g\\ &= g.\end{aligned}$$

Hence $(\phi_n)_n$ defines a chain homotopy from identity to zero. Therefore $H^p(f_1,\dots,f_r;M)=0$ for every p. $\square$

Remark 2.0.12. Here we used the identity

$$e^*_k\wedge(e_l\lrcorner\, g)+e_l\lrcorner\,(e^*_k\wedge g)=\delta_{kl}g.$$

It should be understood rather as

$$e^*_k\wedge(e_l\lrcorner\, g)+e_l\lrcorner\,(e^*_k\wedge g)=(e_l\lrcorner\, e^*_k)g.$$

Lemma 2.0.13. We have

$$H_0(f_1,\dots,f_r;M)\simeq M/(f_1,\dots,f_r)M.$$

Lemma 2.0.14. Suppose the homomorphism

$$M/(f_1,\dots,f_{k-1})M\to M/(f_1,\dots,f_{k-1})M$$

defined by multiplication by f_k is injective for every $1\le k\le r$. Then

(1) $H_p(f_1,\dots,f_r;M)=0$ for every $p\neq 0$.
(2) $H^p(f_1,\dots,f_r;M)=0$ for every $p\neq r$.

Proof. It suffices to show that $H_p(f_1,\dots,f_r;M)=0$ for $p\neq 0$. We shall proceed using induction on $r\ge 1$.

Let $r=1$. Then $K_\bullet(f_1;M)$ is the following complex

$$\begin{tikzcd} 0 \arrow[r] & K_1(f_1;M) \arrow[r] \arrow[d, equal] & K_0(f_1;M) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

where the morphism $M \to M$ is given by multiplication by f_1, which is injective by our assumption. Hence $H_1(f_1; M) = 0$.

Let $r \geq 2$. We can regard $K_\bullet(f_1, \ldots, f_{r-1}; M)$ as a sub-complex of $K_\bullet(f_1, \ldots, f_r; M)$. Moreover, we have isomorphisms

$$K_{p-1}(f_1, \ldots, f_{r-1}; M) \to K_p(f_1, \ldots, f_r; M)/K_p(f_1, \ldots, f_{r-1}; M)$$

defined by

$$e_{i_1} \wedge \cdots \wedge e_{i_{p-1}} \otimes m \mapsto e_{i_1} \wedge \cdots \wedge e_{i_{p-1}} \wedge e_r \otimes m$$

for $1 \leq i_1 < \cdots < i_{p-1} \leq r-1$ (note that this is already an isomorphism taking $M = A$). Note that these isomorphisms commute with ∂. Hence we obtain an exact sequence of complexes

$$0 \to K_\bullet(f_1, \ldots, f_{r-1}; M) \to K_\bullet(f_1, \ldots, f_r; M) \to K_{\bullet-1}(f_1, \ldots, f_{r-1}; M) \to 0.$$

We then obtain a long exact sequence

$$\cdots \to H_p(f_1, \ldots, f_{r-1}; M) \to H_p(f_1, \ldots, f_r; M) \to H_{p-1}(f_1, \ldots, f_{r-1}; M) \to \cdots .$$

By the induction hypothesis, we have

$$H_p(f_1, \ldots, f_{r-1}; M) = H_{p-1}(f_1, \ldots, f_{r-1}; M) = 0$$

for $p \geq 2$. Hence

$$H_p(f_1, \ldots, f_r; M) = 0$$

for $p \geq 2$. The last few terms of long exact sequence is

$$0 \to H_1(f_1, \ldots, f_r; M) \to H_0(f_1, \ldots, f_{r-1}; M) \to H_0(f_1, \ldots, f_{r-1}; M).$$

The last morphism can be identified with the multiplication by f_r on $M/(f_1, \ldots, f_{r-1})M$, which is assumed to be injective. Therefore

$$H_1(f_1, \ldots, f_r; M) = 0.$$

$\square$

3. Algebraic Number Theory

Remark 3.0.1. Main references.

gebraic Number Theory
ate, Class Field Theory
Rabinoff

Definition 3.0.2. A global field is a finite extension of $\mathbb{Q}$ or $\mathbb{F}_p((t))$.

Definition 3.0.3. Let K be a global field. A place of K is an equivalence class of non-trivial absolute values on K. The set of places of K is denoted by V_K.