THE HODGE–TATE PERIOD MAP

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Contents

1. Introduction

1.1. **Notations.** Throughout this paper, $0 \leq \epsilon < 1/2$ is a number such there exists an element in $\mathbb{Z}_p^{\mathrm{cycl}}$ of valuation ϵ, and any such element will be denoted by $p^\epsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$.

2. Technical Tools

2.1. Canonical Subgroups.

Definition 2.1.1. Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $A \to \mathrm{Spec}(R)$ be an Abelian scheme. Let $m \geq 1$ be an integer. We say that $A \to \mathrm{Spec}(R)$ satisfies the weak $O(m, \epsilon)$ condition if $\mathrm{Ha}(A_1/\mathrm{Spec}(R_1))^{(p^m-1)/(p-1)}$ divides p^ϵ as elements in $R_1 = R/p$. Here it seems to mean the following: there exists a section $u \in H^0(\mathrm{Spec}(R_1), \omega_{A_1/R_1}^{\otimes(-p^m+1)/(p-1)})$ such that $u \cdot \mathrm{Ha}(A_1/R_1) = p^\epsilon$.

Lemma 2.1.2. Let S be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let G be a finite locally free commutative group scheme over S. Let $C_1 \subset G \otimes_S S/p$ be a finite locally free subgroup. Assume that for $H = (G \otimes_S S/p)/C_1$, multiplication by p^ϵ on the Lie complex $\ell_H^\vee$ is homotopical to zero. Then there exists a finite locally free subgroup $C \subset G$ over S such that $C \otimes_S S/p^{1-\epsilon} = C_1 \otimes_{S/p} S/p^{1-\epsilon}$.

Lemma 2.1.3. Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $A \to \mathrm{Spec}(R)$ be an Abelian scheme satisfying weak $O(m, \epsilon)$. Then there is a unique closed subgroup $C_m \subset A[p^m]$ such that $C_m = \ker(F^m) \bmod p^{1-\epsilon}$.

Definition 2.1.4 (scholze/III/2.7)**.** Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. We say that an Abelian scheme $A \to \mathrm{Spec}(R)$ has a weak canonical subgroup of level m if $A \to \mathrm{Spec}(R)$ satisfies weak $O(m,\epsilon)$ for some $\epsilon < 1/2$. In that case, we call $C_m \subset A[p^m]$ in Lemma 2.1.3 the weak canonical subgroup of level m.

TODO: strong $O(m,\epsilon)$ If moreover $\mathrm{Ha}(A_1/\mathrm{Spec}(R_1))^{p^m}$ divides p^ϵ, then we say that C_m is a strong canonical subgroup. Here, $R_1 = R/p$ and $A_1 \to \mathrm{Spec}(R_1)$ is the reduction of $A \to \mathrm{Spec}(R)$.

Speculation 2.1.5. Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $A \to \mathrm{Spec}(R)$ be an Abelian scheme satisfying strong $O(m,\epsilon)$, with strong canonical subgroup $C \subset A[p]$ of level 1. Then $(A/C)_{1-\epsilon} \simeq A^{(p)}_{1-\epsilon}$, and in particular

$$\mathrm{Ha}((A/C)_{1-\epsilon}/R_{1-\epsilon}) = \mathrm{Ha}(A_{1-\epsilon}/R_{1-\epsilon})^p.$$

Lemma 2.1.6 (scholze/III/2.8)**.** Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let A and B be Abelian schemes over R.

(1) If A has a canonical subgroup $C_m \subset A[p^m]$ of level m, then it has a canonical subgroup $C_{m'} \subset A[p^{m'}]$ of every level $m' \leq m$, and $C_{m'} \subset C_m$.
(2) Let $f : A \to B$ be a map of Abelian schemes. Assume that both A and B have canonical subgroups $C_m \subset A[p^m]$ and $D_m \subset B[p^m]$ of level m. Then C_m maps into D_m under f.
(3) Assume that A has a canonical subgroup $C_m \subset A[p^m]$ of level m, and let $\overline{x}$ be a geometric point of $\mathrm{Spec}(R[p^{-1}])$. Then $C_m(\overline{x}) \simeq (\mathbb{Z}/p^m\mathbb{Z})^g$, where g is dimension of the Abelian variety over $\overline{x}$.

2.2. Hartog's Extension Principle.

Lemma 2.2.1 ([SGA2, Lemma III.3.1, Proposition III.3.3])**.** Let X be a locally Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $n \geq 1$ be an integer. Then the following are equivalent:

(1) For any open subscheme V of X, the map
$$H^i(V,\mathcal{F}) \to H^i(V\backslash Z,\mathcal{F})$$
is bijective for $i \leq n-2$ and injective for $i = n-1$.
(2) For any open subscheme V of X, the local cohomology
$$H^i_{V\cap Z}(V,\mathcal{F}) = 0$$
for all $i \leq n-1$.
(3) For any $x \in Z$ the depth of $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module is at least n.

Lemma 2.2.2 (Serre's criterion)**.** A Noetherian ring R is normal if and only if $R_\mathfrak{p}$ is regular for every $\mathfrak{p}$ of height ≤ 1 and $R_\mathfrak{p}$ has depth ≥ 2 for every $\mathfrak{p}$ of height ≥ 2.

Lemma 2.2.3 (gortz-wedhorn-1/6.45)**.** Let X be a locally Noetherian normal scheme. Let U be an open subscheme of X with codimension ≥ 2. Then the map $H^0(X,\mathcal{O}_X) \to H^0(U,\mathcal{O}_X)$ is an isomorphism.

Proof. We may assume that $X = \mathrm{Spec}(A)$ where A is normal integral domain. For every non-empty open V of X, the ring $\Gamma(V,\mathcal{O}_X)$ may be considered as a subring of the function field $K(X) = \mathrm{Frac}(A)$ such that the restriction maps are given by inclusions of rings. Let Z be an irreducible closed subset of X of codimension 1. Then U intersects Z non-trivially, so it contains the generic point η of Z. In other words, the subring $\Gamma(U,\mathcal{O}_X)$ of the function field $K(X)$ is contained in the stalk $\mathcal{O}_{X,\eta}$. But $A = \Gamma(X,\mathcal{O}_X)$ is the intersection of all the stalks $\mathcal{O}_{X,\eta}$, where η is a prime ideal of height 1; in other words, where η is the generic point of an irreducible closed subset of codimension 1. ☐

Lemma 2.2.4 (scholze/III/2.9; zhu/3.10.4)**.** Let R be a normal ring, i.e. the localization $R_\mathfrak{p}$ is an integrally closed domain for every prime ideal $\mathfrak{p}$ of R. Assume R is Noetherian. Let $Z \subset \mathrm{Spec}(R)$ be a closed subscheme of codimension at least 2, i.e. every $\mathfrak{p} \in Z$ has height at least 2. Then for $U = \mathrm{Spec}(R)\backslash Z$,

$$H^0(\mathrm{Spec}(R),\mathcal{O}_{\mathrm{Spec}(R)}) \simeq H^0(U,\mathcal{O}_{\mathrm{Spec}(R)}).$$

Proof. Consider $n = 2$ and $\mathcal{F} = \mathcal{O}_X$ in Lemma 2.2.1. Serre's criterion, cf. Lemma 2.2.2, guarantees the third condition in Lemma 2.2.1. The first assertion gives the desired result. ☐

Lemma 2.2.5 (scholze/III/2.10; zhu/3.10.5)**.** Let R be a topologically finitely generated, flat, and p-adically complete $\mathbb{Z}_p$-algebra, such that $\overline{R} = R/p$ is normal. Fix $f \in R$ such that its reduction $\overline{f} \in \overline{R}$ is not a zero-divisor. Let $0 < \epsilon \le 1$. Set $S = (R \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u \rangle/(u \cdot f - p^\epsilon)$. Then S is p-adically complete and flat over $\mathbb{Z}_p^{\mathrm{cycl}}$. Fix a closed subscheme $Y \subset \mathrm{Spec}(\overline{R})$ of codimension ≥ 2. Let Z be the inverse image of Y in $\mathrm{Spf}(S)$. Then for $U = |\mathrm{Spf}(S)| \backslash Z$,

$$S = H^0(\mathrm{Spf}(S), \mathcal{O}_{\mathrm{Spf}(S)}) \simeq H^0(U, \mathcal{O}_{\mathrm{Spf}(S)}).$$

Proof. We first show that the map

$$S \simeq H^0(\mathrm{Spf}(S), \mathcal{O}_{\mathrm{Spf}(S)}) \to H^0(U, \mathcal{O}_{\mathrm{Spf}(S)})$$

is injective. Since S is p-adically separated and $H^0(U, \mathcal{O}_{\mathrm{Spf}(S)})$ is flat over $\mathbb{Z}_p^{\mathrm{cycl}}$, it suffices to show that

$$S_\epsilon \simeq H^0(\mathrm{Spec}(S_\epsilon), \mathcal{O}_{\mathrm{Spec}(S_\epsilon)}) \to H^0(U_\epsilon, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$$

is injective, where $S_\epsilon = S/p^\epsilon$, Z_ϵ is the inverse image of Y in $\mathrm{Spec}(S_\epsilon)$, and $U_\epsilon = \mathrm{Spec}(S_\epsilon) \backslash Z_\epsilon$. Note that

$$S_\epsilon = (R \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u \rangle/(uf, p^\epsilon) = R_\epsilon[u]/(uf_\epsilon)$$

where $R_\epsilon = R \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)$ and $f_\epsilon \in R_\epsilon$ is the image of $f \in R$.

Let $W \subset \mathrm{Spec}(S_\epsilon)$ be the preimage of $V = V(\overline{f}) \subset \mathrm{Spec}(\overline{R})$. Then $W = V \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathbb{A}^1_{\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon}$ is affine. The map $S_\epsilon \to R_\epsilon$ sending u to zero induces a section $\mathrm{Spec}(R_\epsilon) \to \mathrm{Spec}(S_\epsilon)$. We have a decomposition $\mathrm{Spec}(S_\epsilon) = N \cup W$, where $N = \mathrm{Spec}(R_\epsilon[u]/(u)) \simeq \mathrm{Spec}(R_\epsilon)$ is the image of the section $\mathrm{Spec}(R_\epsilon) \to \mathrm{Spec}(S_\epsilon)$. Take $V_\epsilon = V \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)$. Then $W = V_\epsilon \times_{\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)} \mathbb{A}^1_{\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon}$, and $N \cap W = V_\epsilon$.

We then have the following interpretations:

1. Each section in $\Gamma(\mathrm{Spec}(S_\epsilon), \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$ is a pair (f_1, f_2) such that $f_1 \in \Gamma(N, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$ and $f_2 \in \Gamma(W, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$ such that $f_1 = f_2$ on $N \cap W = V_\epsilon$.
2. Each section in $H^0(U_\epsilon, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$ is a pair (f_1, f_2) such that $f_1 \in H^0(U_\epsilon \cap N, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$, and $f_2 \in H^0(U_\epsilon \cap W, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$, such that $f_1 = f_2$ on $U_\epsilon \cap N \cap W$.

The (classical) Hartog's extension principle, i.e. Lemma 2.2.4 applied to $Y \subset \mathrm{Spec}(\overline{R})$, shows that

$$\Gamma(\mathrm{Spec}(\overline{R}) \backslash Y) \simeq \Gamma(\mathrm{Spec}(\overline{R})).$$

Under base-change this gives

$$\Gamma(U_\epsilon \cap N) \simeq \Gamma(\mathrm{Spec}(\overline{Y}) \backslash Y) \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon \simeq \Gamma(\mathrm{Spec}(\overline{R})) \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon \simeq \Gamma(N).$$

Thus injectivity reduces to show that

$$\Gamma(V) \otimes_{\mathbb{F}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)[u]\Gamma(W) \to \Gamma(U_\epsilon \cap W) = \Gamma(V \backslash Y) \otimes_{\mathbb{F}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)[u]$$

is injective. It suffices to show that $\Gamma(V) \to \Gamma(V \backslash Y)$ is injective, where both V and $V \backslash Y$ are $\mathbb{F}_p$-schemes. We have $\mathrm{depth}(\mathcal{O}_{V,y}) = \mathrm{depth}(\overline{R}_y) - 1$ for all $y \in V$, cf. [SGA-2/III/2.5]. Thus $\mathrm{depth}(\mathcal{O}_{V,y}) \ge 1$ for every $x \in V \cap Y$ by Serre's criterion, i.e. Lemma 2.2.2. Then the desired injectivity follows from Lemma 2.2.1.

We then show the surjectivity.

TODO □

2.3. **Tate's Normalized Traces.**

Lemma 2.3.1 (scholze/III/2.21; zhu/3.12.2)**.** Let R be a p-adically complete flat $\mathbb{Z}_p$-algebra. Let $Y_1, \dots, Y_n \in R$. Let $P_1, \dots, P_n \in R\langle X_1, \dots, X_n \rangle$ be topologically nilpotent elements, or equivalently, each P_i has topologically nilpotent coefficients in R. Let

$$S = R\langle X_1, \dots, X_n \rangle/(X_1^p - Y_1 - P_1, \dots, X_n - Y_n - P_n).$$

Then

1. The ring S is a finite free R-module of rank p^n, with a basis given by $X_1^{i_1} \cdots X_n^{i_n}$ with $0 \le i_1, \dots, i_n \le p - 1$.
2. Let I be the ideal of R generated by p together with all the coefficients of all P_i. Then the trace map $\mathrm{tr}_{S/R} : S \to R$ sends S to I^n, i.e. $\mathrm{tr}_{S/R}(S) \subset I^n$.

Lemma 2.3.2 (scholze/III/2.22; zhu/3.12.4)**.** Let R be a p-adically complete flat $\mathbb{Z}_p$-algebra topologically of finite type, formally smooth of dimension n over $\mathbb{Z}_p$. Let $f \in R$ such that its reduction $\overline{f} \in \overline{R} = R/p$ is not a zero-divisor. Let $0 \le \epsilon < 1/2$. Let

$$S_\epsilon = (R \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u_\epsilon \rangle/(u_\epsilon \cdot f - p^\epsilon).$$

Suppose $\varphi : S_\epsilon \to S_{\epsilon/p}$ is a map of $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra such that modulo $p^{1-\epsilon}$ it is given by the relative Frobenius. In other words, $\varphi \mod p^{1-\epsilon}$ is the map

$$R_{1-\epsilon}[u_\epsilon]/(f \cdot u_\epsilon - p^\epsilon) \to R_{1-\epsilon}[u_{\epsilon/p}]/(f \cdot u_{\epsilon/p} - p^{\epsilon/p}),$$

where $R_{1-\epsilon} = \overline{R} \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^{1-\epsilon})$, which sends u_ϵ to $u_{\epsilon/p}^p$, and restricts to $\mathrm{Fr}_{\overline{R}} \otimes \mathrm{id}$ on $R_{1-\epsilon}$. Then

(1) The map
$$\varphi[1/p] : S_\epsilon[1/p] \to S_{\epsilon/p}[1/p]$$
is finite and flat of degree p^n.

(2) The trace map
$$\mathrm{tr} = \mathrm{tr}_{S_{\epsilon/p}[1/p]/S_\epsilon[1/p]} : S_{\epsilon/p}[1/p] \to S_\epsilon[1/p]$$
sends $S_{\epsilon/p}$ into $p^{n-(2n+1)\epsilon} S_\epsilon$. Here $S_{\epsilon/p}[1/p]$ is viewed as an $S_\epsilon[1/p]$-algebra via $\varphi[1/p]$.

2.4. Riemann's Hebbarkeitssatz.

Definition 2.4.1 (scholze/II/3.8)**.** Let p be a prime. Let K be a perfectoid field (of any characteristic). Let t be a non-zero element of K with $|p| \le |t| < 1$. A triple $(\mathcal{X}, \mathcal{Z}, \mathcal{U})$, where $\mathcal{X}$ is an affinoid perfectoid space over K, $\mathcal{Z}$ is a closed subset of $\mathcal{X}$, and $\mathcal{U}$ is a quasi-compact open subset of $\mathcal{X}\backslash\mathcal{Z}$, is said to be good, if

$$H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}^+/t)^a \simeq H^0(\mathcal{X}\backslash\mathcal{Z}, \mathcal{O}_\mathcal{X}^+/t)^a \hookrightarrow H^0(\mathcal{U}, \mathcal{O}_\mathcal{U}^+/t)^a.$$

Remark 2.4.2. This notion is independent of the choice of t, and is compatible with tilting.

Situation 2.4.3. Let $K = \mathbb{F}_p((t^{1/p^\infty}))$. Let R_0 be a reduced Tate K-algebra topologically of finite type. Let $\mathcal{X}_0 = \mathrm{Spa}(R_0, R_0^\circ)$ be the associated affinoid adic space of finite type over K. Let R be the completed perfection of R_0, which is a p-finite perfectoid K-algebra. Let $\mathcal{X} = \mathrm{Spa}(R, R^+)$ with $R^+ = R^\circ$, the associated p-finite affinoid perfectoid space over K. Let I_0 be an ideal of R_0. Let $I = I_0 R \subset R$. Let $\mathcal{Z}_0 = V(I_0) \subset \mathcal{X}_0$. Let $\mathcal{Z} = V(I) \subset \mathcal{X}$. Let $\mathcal{U}_0$ be a quasi-compact open subset of $\mathcal{X}_0\backslash\mathcal{Z}_0$ with preimage $\mathcal{U} \subset \mathcal{X}\backslash\mathcal{Z}$.

Lemma 2.4.4 (scholze/II/3.10)**.** Assume Situation 2.4.3. Suppose $(\mathcal{X}, \mathcal{Z}, \mathcal{U})$ is good. Suppose that R_0 is normal, and that $V(I_0) \subset \mathrm{Spec}(R_0)$ is of codimension ≥ 2. Let R_0' be a finite normal R_0-algebra which is étale outside $V(I_0)$, and such that no irreducible component of $\mathrm{Spec}(R_0')$ maps into $V(I_0)$. Let $I_0' = I_0 R_0'$, and $\mathcal{U}_0' \subset \mathcal{X}_0'$ the preimage of $\mathcal{U}_0$. Let R', I', $\mathcal{X}'$, $\mathcal{Z}'$, $\mathcal{U}'$ be the associated perfectoid objects.

(1) There is a perfect trace pairing
$$\mathrm{tr}_{R_0'/R_0} : R_0' \otimes_{R_0} R_0' \to R_0.$$

(2) The trace pairing induces a trace pairing
$$\mathrm{tr}_{R'^\circ/R^\circ} : R'^\circ \otimes_{R^\circ} R'^\circ \to R^\circ.$$
which is almost perfect.

(3) For all open subsets $\mathcal{V} \subset \mathcal{X}$ with preimage $\mathcal{V}' \subset \mathcal{X}'$, the trace pairing induces an isomorphism
$$H^0(\mathcal{V}', \mathcal{O}_{\mathcal{X}'}^+/t)^a \simeq \mathrm{Hom}_{R^\circ/t}(R'^\circ/t, H^0(\mathcal{V}, \mathcal{O}_\mathcal{X}^+/t))^a.$$

(4) The triple $(\mathcal{X}', \mathcal{Z}', \mathcal{U}')$ is good.

(5) If $\mathcal{X}' \to \mathcal{X}$ is surjective, then the map
$$H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}^+/t) \to H^0(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}^+/t) \cap H^0(\mathcal{U}, \mathcal{O}_\mathcal{X}^+/t)$$
is an almost isomorphism.

Lemma 2.4.5 (scholze/II/3.11)**.** Suppose we have a filtered inductive system $(R_0^{(i)})_{i\in I}$ as in the previous lemma, giving rise to $\mathcal{X}^{(i)}$, $\mathcal{Z}^{(i)}$, $\mathcal{U}^{(i)}$. Assume that all transition maps $\mathcal{X}^{(i)} \to \mathcal{X}^{(j)}$ are surjective. Let $\widetilde{\mathcal{X}}$ be the inverse limit of the $\mathcal{X}^{(i)}$ in the category of perfectoid spaces over K, with preimage $\widetilde{\mathcal{Z}} \subset \widetilde{\mathcal{X}}$ of $\mathcal{Z}$, and $\widetilde{\mathcal{U}} \subset \widetilde{\mathcal{X}}\backslash\widehat{\mathcal{U}}$ of $\mathcal{U}$. Then the triple $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{Z}}, \widetilde{\mathcal{U}})$ is good.

2.5. The Hodge–Tate Filtration.

Lemma 2.5.1. Let C be an algebraically closed and complete extension of $\mathbb{Q}_p$. Let $A \to \mathrm{Spec}(C)$ be an Abelian variety. Then A has its Hodge–Tate filtration

$$0 \to \mathrm{Lie}(A)(1) \to T_p(A) \otimes_{\mathbb{Z}_p} C \to (\mathrm{Lie}(A^\vee))^* \to 0.$$

TODO

3. Siegel Modular Varieties

Let p be a fixed prime. Let $g \geq 1$ be an integer.

Definition 3.0.1. The symplectic similitude group GSp_{2g} is the reductive group scheme over $\mathbb{Z}$ whose points in a commutative ring R are given by

$$\mathrm{GSp}_{2g}(R) = \{x \in \mathrm{GL}_{2g}(V); \exists \nu(x) \in R^\times, x^t \Omega x = \nu(x)\Omega\}$$

where $\Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ is the standard symplectic matrix of order $2g$.

In the following discussion, we write $G = \mathrm{GSp}_{2g}$. Let $K_p = G(\mathbb{Z}_p)$. Let K^p be a compact open subgroup of $G(\mathbb{A}^{\infty,p})$ that is contained in

$$\Gamma(N)^{(p)} = \{g \in G(\mathbb{A}^{\infty,p}); g \equiv 1 \bmod N\}$$

for some integer $N \geq 3$ not divisible by p.

Definition 3.0.2. Let $m \geq 1$ be an integer.

$$\Gamma_0(p^m) = \left\{g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \bmod p^m \right\}$$

$$\Gamma_{\mathrm{s}}(p^m) = \left\{g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \bmod p^m, \nu(g) \equiv 1 \bmod p^m \right\}$$

$$\Gamma_1(p^m) = \left\{g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \bmod p^m \right\}$$

$$\Gamma(p^m) = \left\{g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod p^m \right\}$$

Let X be the scheme over $\mathrm{Spec}(\mathbb{Z}_{(p)})$ classifying principally polarized projective Abelian schemes of relative dimension g with level K^p structures. Let X^* be the minimal compactification of X.

TODO: Add references.

For each $U \in \{\Gamma(p^m), \Gamma_{\mathrm{s}}(p^m), \Gamma_0(p^m)\}$, we have a scheme $X_{U,\mathbb{Q}}$ over $\mathbb{Q}$ with certain moduli interpretations.

Let $\mathfrak{X}$ be the formal scheme over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$ defined as the p-completion of $X_{\mathbb{Z}_p^{\mathrm{cycl}}} = X \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} \mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$.

Remark 3.0.3. The moduli interpretations can be described as follows.

(1) $\mathrm{Sh}_{K^p G(\mathbb{Z}_p), \mathbb{Z}_{(p)}}$ represents the following problem $S \mapsto \{(A, \lambda, \eta)\}/\sim$ where
   - A is a projective Abelian scheme over S of relative dimension g.
   - λ is a principal polarization of A.
   - η is a level K^p structure on A.

(2) $\mathrm{Sh}_{K^p\Gamma(p^m), \mathbb{Q}}$ represents the following problem $S \mapsto \{(A, \lambda, \eta, \eta_p)\}/\sim$ where
   - $(A, \lambda, \eta) \in \mathrm{Sh}_{K^p G(\mathbb{Z}_p), \mathbb{Z}_{(p)}}(S)$.
   - η_p is a level $\Gamma(p^m)$ structure on A.

(3) $\mathrm{Sh}_{K^p\Gamma_0(p^m), \mathbb{Q}}$ represents the following problem $S \mapsto \{(A, \lambda, \eta, D)\}/\sim$ where
   - $(A, \lambda, \eta) \in \mathrm{Sh}_{K^p G(\mathbb{Z}_p), \mathbb{Z}_{(p)}}(S)$.
   - D is a totally isotropic subgroup of $A[p^m]$.

(4) $\mathrm{Sh}_{K^p\Gamma_{\mathrm{s}}(p^m), \mathbb{Q}}$ represents the following problem $S \mapsto \{(A, \lambda, \eta, D, t)\}/\sim$ where
   - $(A, \lambda, \eta, D) \in \mathrm{Sh}_{K^p\Gamma_0(p^m), \mathbb{Q}}(S)$.

- $t : \mu_{p^m} \to \mathbb{Z}/p^m\mathbb{Z}$ is an isomorphism.

The universal Abelian scheme $A \to X$ gives a line bundle $\omega = \omega_{A/S} = \bigwedge^g \Omega^1_{A/X}$. The sheaf ω extends to the minimal compactification X^*. The Hasse invariant defines a section $\mathrm{Ha} \in H^0(X_{\mathbb{F}_p}, \omega^{\otimes(p-1)})$. The section Ha extends to $\mathrm{Ha} \in H^0(X^*_{\mathbb{F}_p}, \omega^{\otimes(p-1)})$. For $g = 1$, this follows from direct inspection. For $g \geq 2$, this follows from the (classical) Hartog's extension principle. TODO: Clearify this paragraph.

Let $\mathfrak{A} \to \mathfrak{X}$ be the universal formal Abelian scheme.

Speculation 3.0.4. Minimal compactification is compatible with p-completion.

4. The Anti-Canonical Towers

Lemma 4.0.1. Let S be a p-adically complete $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. There is a bijection

$$\mathrm{Hom}_{\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})}(\mathrm{Spf}(S), \mathfrak{X}^*) \simeq \mathrm{Hom}_{\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})}(\mathrm{Spec}(S), X^*_{\mathbb{Z}_p^{\mathrm{cycl}}}).$$

Speculation 4.0.2. Let Y be a scheme over $\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$. Let $\mathfrak{Y}$ be the formal scheme over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$ obtained as the p-completion of Y. Let S be a p-adically complete $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Then there is a bijection

$$\mathrm{Hom}_{\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})}(\mathrm{Spf}(S), \mathfrak{Y}) \simeq \mathrm{Hom}_{\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})}(\mathrm{Spec}(S), Y).$$

Definition 4.0.3. Let $0 \leq \epsilon < 1$ such that there exists an element $p^\epsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$ with p-adic valuation ϵ. Let $\mathcal{M}_\epsilon$ be the functor sending a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra S to the set of pairs $(f, [u])$, where

- f is a map $\mathrm{Spf}(S) \to \mathfrak{X}^*$; it's equivalent to a map $\mathrm{Spec}(S) \to X^*_{\mathbb{Z}_p^{\mathrm{cycl}}}$ by Lemma 4.0.1.
- Let $\overline{f} : \mathrm{Spec}(S/p) \to X^*_{\mathbb{F}_p}$ be the reduction of $\mathrm{Spec}(S) \to X^*_{\mathbb{Z}_p^{\mathrm{cycl}}}$. Recall that we have the Hasse section $\mathrm{Ha} \in H^0(X^*_{\mathbb{F}_p}, \omega^{\otimes(p-1)})$. It pullbacks to $\overline{f}^*\mathrm{Ha} \in H^0(\mathrm{Spec}(S/p), \overline{f}^*\omega^{\otimes(p-1)})$. Then $[u]$ is an equivalence class of sections $u \in H^0(\mathrm{Spec}(S/p), \overline{f}^*\omega^{\otimes(1-p)})$ satisfying $u \cdot \overline{f}^*\mathrm{Ha} = p^\epsilon \in S_1$ under the equivalence relation that $u \sim u'$ if and only if there exists some $h \in S$ such that $u' = u(1 + p^{1-\epsilon}h)$.

Lemma 4.0.4. Then the functor $\mathcal{M}_\epsilon$ is representable by a formal scheme flat over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$. Moreover, locally it is ...

Definition 4.0.5. Let $\mathfrak{X}(\epsilon) \to \mathfrak{X}$ be the pullback of $\mathfrak{X}^*(\epsilon) \to \mathfrak{X}^*$ along $\mathfrak{X} \to \mathfrak{X}^*$. Let $\mathfrak{A}(\epsilon) \to \mathfrak{X}(\epsilon)$ be the pullback of $\mathfrak{A} \to \mathfrak{X}$ along $\mathfrak{X}(\epsilon) \to \mathfrak{X}$.

Let $\mathcal{X}$ be the generic fiber of the adic space associated to the formal scheme $\mathfrak{X}$. Let $\mathcal{X}(\epsilon)$ be the generic fiber of the adic space associated to $\mathfrak{X}(\epsilon)$. Then $\mathcal{X}$ admits an open embedding to the X^{ad}, the adic space associated to the scheme $X_{\mathbb{Q}_p^{\mathrm{cycl}}}$. Let $\mathcal{X}_{\Gamma_s(p^m)}$ be the inverse image of $\mathcal{X}$ under the map $X^{\mathrm{ad}}_{\Gamma_s(p^m)} \to X^{\mathrm{ad}}$.

Remark 4.0.6. TODO: moduli interpretation of $\mathfrak{X}(\epsilon)$. Should be almost identical to $\mathcal{M}_\epsilon$.

Definition 4.0.7. For a formal scheme $\mathfrak{Y}$ over $\mathbb{Z}_p^{\mathrm{cycl}}$ and $a \in \mathbb{Z}_p^{\mathrm{cycl}}$, we write $\mathfrak{Y}/a$ for $\mathfrak{Y} \times_{\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})} \mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/a)$.

Definition 4.0.8. For a formal scheme $\mathfrak{Y}$ over $\mathbb{Z}_p^{\mathrm{cycl}}/p$, we write $\mathfrak{Y}^{(p)}$ for the pullback of $\mathfrak{Y}$ along the (absolute) Frobenius $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p) \to \mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$.

Lemma 4.0.9. We have a natural isomorphism

$$(\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \simeq \mathfrak{X}^*(\epsilon)/p$$

of formal schemes over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$. Furthermore, by pullback we get the following commutative diagram

$$\begin{tikzcd}
(\mathfrak{A}(p^{-1}\epsilon)/p)^{(p)} \arrow[r] \arrow[d] & (\mathfrak{X}(p^{-1}\epsilon)/p)^{(p)} \arrow[r] \arrow[d] & (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \arrow[d] \\
\mathfrak{A}(\epsilon)/p \arrow[r] & \mathfrak{X}(\epsilon)/p \arrow[r] & \mathfrak{X}^*(\epsilon)/p
\end{tikzcd}$$

where each vertical map is an isomorphism.

Proof. Let S be a (discrete? flat) $(\mathbb{Z}_p^{\mathrm{cycl}}/p)$-algebra. Then

$$(\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)}(S) = (\mathfrak{X}^*(p^{-1}\epsilon)/p)(\mathrm{Fr}_*S),$$

where Fr_*S is the the $(\mathbb{Z}_p^{\mathrm{cycl}}/p)$-algebra obtained from S by precomposing with $\mathrm{Fr} : \mathbb{Z}_p^{\mathrm{cycl}}/p \to \mathbb{Z}_p^{\mathrm{cycl}}/p$. Each map $\mathrm{Spf}(\mathrm{Fr}_*S) \to \mathfrak{X}^*(p^{-1}\epsilon)/p$ is equivalent to a pair $(f, [u])$, where

- $f : \mathrm{Spec}(\mathrm{Fr}_*S) \to X^*_{\mathbb{Z}_p^{\mathrm{cycl}}}$ is a map over $\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$.
- $u \in H^0(\mathrm{Spec}(\mathrm{Fr}_*S), f^*\omega^{\otimes(1-p)})$ is a section such that $u \cdot f^*\mathrm{Ha} = p^{p^{-1}\epsilon} \in \mathrm{Fr}_*S$. Note that $(\mathrm{Fr}_*S)/p = \mathrm{Fr}_*S$ since S is defined over $\mathbb{Z}_p^{\mathrm{cycl}}/p$.

Recall that $X^*_{\mathbb{Z}_p^{\mathrm{cycl}}} = X^*_{\mathbb{Z}_p} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$, and thus $(f, [u])$ is equivalent (TODO: should be more precise) to the following datum

- $f : \mathrm{Spec}(\mathrm{Fr}_*S) \to X^*_{\mathbb{Z}_p}$ is a map over $\mathrm{Spec}(\mathbb{Z}_p)$.
- (TODO: Check the reduction of u) $u \in H^0(\mathrm{Spec}(\mathrm{Fr}_*S), f^*\omega^{\otimes(1-p)})$ is a section such that $u \cdot f^*\mathrm{Ha} = p^{p^{-1}\epsilon} \in \mathrm{Fr}_*S$.

Note that the Frobenius on $\mathbb{Z}_p/p = \mathbb{F}_p$ is simply the identity, and thus the map $\mathrm{Spec}(\mathrm{Fr}_*S) \to \mathrm{Spec}(\mathbb{Z}_p)$ is identical to $\mathrm{Spec}(S) \to \mathrm{Spec}(\mathbb{Z}_p)$. But under this identification the element $p^{p^{-1}\epsilon} \in \mathrm{Fr}_*S$ corresponds to $p^\epsilon \in S$. Then $f : \mathrm{Spec}(\mathrm{Fr}_*S) \to X^*_{\mathbb{Z}_p}$ can be reinterpreted as a map $g : \mathrm{Spec}(S) \to X^*_{\mathbb{Z}_p}$ over $\mathrm{Spec}(\mathbb{Z}_p)$. We write $v = u$ for clarity. The section v then satisfies $v \cdot g^*\mathrm{Ha} = p^\epsilon \in S$. The pair $(g, [v])$ then corresponds to a map $\mathrm{Spf}(S) \to \mathfrak{X}^*(\epsilon)/p$ over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$. □

Lemma 4.0.10. The Frobenius map $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p) \to \mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$ induces the following commutative diagram

$$\begin{tikzcd}
\mathfrak{A}(p^{-1}\epsilon)/p \arrow[r] \arrow[d] & \mathfrak{X}(p^{-1}\epsilon)/p \arrow[r] \arrow[d] & \mathfrak{X}^*(p^{-1}\epsilon)/p \arrow[d] \\
(\mathfrak{A}(p^{-1}\epsilon)/p)^{(p)} \arrow[r] & (\mathfrak{X}(p^{-1}\epsilon)/p)^{(p)} \arrow[r] & (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)}
\end{tikzcd}$$

Proof. This follows from the universal property of pullback. □

Remark 4.0.11. TODO: Explain the moduli interpretation of

$$\mathfrak{X}^*(p^{-1}\epsilon)/p \to (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \simeq \mathfrak{X}^*(\epsilon)/p.$$

Speculation 4.0.12. This is written explicitly in [Zhu23, p. 40]. But I cannot find a reference. Let S be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $A \to \mathrm{Spec}(S)$ be an Abelian scheme. Then ω_{A_1/S_1} (or even $\omega_{A/S}$) is the trivial line bundle, where $S_1 = S/p$.

Speculation 4.0.13. Let S be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $f : \mathrm{Spf}(S) \to \mathfrak{X}$ be a map over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$. Let $A \to \mathrm{Spec}(S)$ be the corresponding Abelian scheme. Suppose $A \to \mathrm{Spec}(S)$ satisfies strong $O(1,\epsilon)$. Let C be the strong canonical subgroup of $A \to \mathrm{Spec}(S)$ of level 1. Then $B = A/C$ satisfies weak $O(1,\epsilon)$.

Speculation 4.0.14. Let S be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $A \to \mathrm{Spec}(S)$ be an Abelian scheme. The set of points $s \in \mathrm{Spec}(S)$ such that A_s is ordinary forms a dense subset of $\mathrm{Spec}(S)$.

Lemma 4.0.15. There is a unique commutative diagram

$$\begin{tikzcd}
\mathfrak{A}(p^{-1}\epsilon) \arrow[r] \arrow[d] & \mathfrak{X}(p^{-1}\epsilon) \arrow[r] \arrow[d] & \mathfrak{X}^*(p^{-1}\epsilon) \arrow[d] \\
\mathfrak{A}(\epsilon) \arrow[r] & \mathfrak{X}(\epsilon) \arrow[r] & \mathfrak{X}^*(\epsilon)
\end{tikzcd}$$

that is identified with the following commutative diagram from Lemma 4.0.9 and Lemma 4.0.10, after modulo $p^{1-\epsilon}$.

$$\begin{tikzcd}
\mathfrak{A}(p^{-1}\epsilon)/p \arrow[r] \arrow[d] & \mathfrak{X}(p^{-1}\epsilon)/p \arrow[r] \arrow[d] & \mathfrak{X}^*(p^{-1}\epsilon)/p \arrow[d] \\
(\mathfrak{A}(p^{-1}\epsilon)/p)^{(p)} \arrow[r] \arrow[d] & (\mathfrak{X}(p^{-1}\epsilon)/p)^{(p)} \arrow[r] \arrow[d] & (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \arrow[d] \\
\mathfrak{A}(\epsilon)/p \arrow[r] & \mathfrak{X}(\epsilon)/p \arrow[r] & \mathfrak{X}^*(\epsilon)/p
\end{tikzcd}$$

Proof. TODO: The map $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon)$ comes from the moduli interpretation, the weak canonical subgroup, and the Hasse invariant. Then $\mathfrak{A}(p^{-1}\epsilon) \to \mathfrak{A}(\epsilon)$ is obtained by base-change. The extension to $\mathfrak{X}^*(p^{-1}\epsilon) \to \mathfrak{X}^*(\epsilon)$ is done using Hartog's extension principle.

We first construct the map $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon)$. Let S be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $(f, [u])$ be a pair where

- $f : \mathrm{Spf}(S) \to \mathfrak{X}$ is a map of formal schemes over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$; its equivalent to a map $f : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$.
- $u \in H^0(\mathrm{Spec}(S/p), \overline{f}^* \omega^{\otimes(1-p)})$ is a section such that $u \cdot \overline{f}^* \mathrm{Ha} = p^{p^{-1}\epsilon} \in S/p$.

The map $f : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ gives an Abelian scheme $A \to \mathrm{Spec}(S)$ with principal polarization and level K^p structure. We claim that $A \to \mathrm{Spec}(S)$ satisfies strong $O(1, \epsilon)$, i.e. $\mathrm{Ha}(A_1/\mathrm{Spec}(S_1))^p$ divides p^ϵ. This follows from

$$p^{p^{-1}\epsilon} = u \cdot \overline{f}^* \mathrm{Ha} = u \cdot \mathrm{Ha}(A_1/\mathrm{Spec}(S_1)).$$

Let $C \subset A[p]$ be the strong canonical subgroup of level 1. We get an Abelian scheme $A/C \to \mathrm{Spec}(S)$ equipped with induced polarization and level structure TODO: Clarify; use totally isotropic, which corresponds to a map $g : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$.

Then we declare that the pair $(f, [u])$ gets mapped to the pair $(g, [u^p])$.

By Speculation 4.0.13, the quotient $A/C \to \mathrm{Spec}(S)$ satisfies weak $O(1, \epsilon)$, i.e. there exists a section $v \in H^0(\mathrm{Spec}(S/p), \overline{g}^* \omega^{\otimes(1-p)})$ such that $v \cdot \overline{g}^* \mathrm{Ha} = p^\epsilon$. Then we declar that the pair $(f, [u])$ gets mapped to the pair $(g, [v])$. We need to check that $[v]$ is well-defined. It suffices to show that $\overline{g}^* \mathrm{Ha} = \mathrm{Ha}((A/C)_1/S_1)$ is not a zero-divisor. Otherwise, for every geometric point x of $\mathrm{Spec}(S)$, the Abelian scheme $(A/C)_x$ is not ordinary. This contradicts Speculation 4.0.14. Therefore we obtain a well-defined map $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon)$.

We have

$$p^\epsilon = u^p \cdot \mathrm{Ha}(A_1/\mathrm{Spec}(S_1))^p = u^p \cdot \mathrm{Ha}(A_1^{(p)}/\mathrm{Spec}(S_1)).$$

Modulo $p^{1-\epsilon}$,

$$p^\epsilon = u^p \cdot \mathrm{Ha}(A_{1-\epsilon}^{(p)}/\mathrm{Spec}(S_{1-\epsilon})) = u^p \cdot \mathrm{Ha}(B_{1-\epsilon}/\mathrm{Spec}(S_{1-\epsilon})).$$

Thus there is $v \in H^0(\mathrm{Spec}(S_1), \overline{g}^* \omega^{\otimes(1-p)})$ such that $v = u^p \bmod p^{1-\epsilon}$ and $v \cdot \mathrm{Ha}(B_1/\mathrm{Spec}(S_1)) = p^\epsilon \bmod p^{1-\epsilon}$.

We then declare that the pair $(f, [u])$ gets mapped to $(g, [v])$. This is well-defined since equivalent u gives the same u^p. ∎

Construction 4.0.16. Let $m \geq 1$.

We first construct a map $\mathfrak{X}(p^{-m}\epsilon) \to \mathfrak{X}$. Let S be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $\mathrm{Spf}(S) \to \mathfrak{X}(p^{-m}\epsilon)$ be a map over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$. It corresponds to a pair $(f, [u])$ where

- $f : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ is a map over $\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$.
- $u \in H^0(\mathrm{Spec}(S/p), \overline{f}^* \mathrm{Ha})$ is a section such that $u \cdot f = p^{p^{-m}\epsilon}$ in S/p.

The map $f : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ gives an Abelian scheme $A \to \mathrm{Spec}(S)$. The section u shows that $A \to \mathrm{Spec}(S)$ satisifies strong $O(m, \epsilon)$, and thus has a strong canonical subgroup $C_m \subset A[p^m]$ of level m. The Abelian scheme $A/C_m \to \mathrm{Spec}(S)$ has induced principal polarization and level structure, and thus corresponds to a map $\mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$, which gives a map $\mathrm{Spf}(S) \to \mathfrak{X}$ over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$.

Passing to the adic fiber (i.e. the generic fiber of the associated adic space), we get a map $\mathcal{X}(p^{-m}\epsilon) \to \mathcal{X}$ of adic spaces. Now we construct a factorization $\mathcal{X}(p^{-m}\epsilon) \to \mathcal{X}_{\Gamma_s(p^m)} \to \mathcal{X}$, where the map $\mathcal{X}_{\Gamma_s(p^m)} \to \mathcal{X}$ is given by the moduli interpretation "$(A, D) \mapsto A/D$".

TODO: construct the factorization

Lemma 4.0.17. For each $m \geq 1$, the $\mathcal{X}(p^{-m}\epsilon) \to \mathcal{X}_{\Gamma_s(p^m)}$ extends uniquely to $\mathcal{X}^*(p^{-m}\epsilon) \to \mathcal{X}^*_{\Gamma_s(p^m)}$, and both maps are open immersions of adic spaces. Moreover, the following diagram

$$\begin{tikzcd} \mathcal{X}^*(p^{-m-1}\epsilon) \arrow[r] \arrow[d] & \mathcal{X}^*_{\Gamma_s(p^{m+1})} \arrow[d] \\ \mathcal{X}^*(p^{-m}\epsilon) \arrow[r] & \mathcal{X}^*_{\Gamma_s(p^m)} \end{tikzcd}$$

is a pullback diagram for all $m \geq 1$, where the vertical map on the left is induced from the map $\mathfrak{X}(p^{-m-1}\epsilon) \to \mathfrak{X}(p^{-m}\epsilon)$, cf. Lemma 4.0.15.

Proof. TODO: write down the proof

- Extension to minimal compactification:
- Open immersion:
  - The map $\theta : X_{\mathbb{Q}_p^{\mathrm{cycl}}} \to X_{\mathbb{Q}_p^{\mathrm{cycl}}}$ defined by $A \mapsto A/A[p^m]$ is an isomorphism.
  - The following diagram

    $$\begin{tikzcd} \mathcal{X}(p^{-m}\epsilon) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ X^{\mathrm{ad}} \arrow[r, "\theta"] & X^{\mathrm{ad}} \end{tikzcd}$$

    commutes.
  - So the composition $\mathcal{X}^*(p^{-m}\epsilon) \to \mathcal{X}_{\Gamma_s(p^m)} \to \mathcal{X}$ is an open immersion.
  - The map $\mathcal{X}_{\Gamma_s(p^m)} \to \mathcal{X}$ is finite étale.
  - Thus the map $\mathcal{X}(p^{-m}\epsilon) \to \mathcal{X}_{\Gamma_s(p^m)}$ is an open immersion.
  - Then pass to minimal compactification.
- Pullback diagram:
  - First show that

    $$\begin{tikzcd} \mathcal{X}(p^{-m-1}\epsilon) \arrow[r] \arrow[d] & \mathcal{X}_{\Gamma_s(p^{m+1})} \arrow[d] \\ \mathcal{X}(p^{-m}\epsilon) \arrow[r] & \mathcal{X}_{\Gamma_s(p^m)} \end{tikzcd}$$

    is pullback diagram.
  - Commutativity of the diagram:
  - It is a pullback since both vertical maps are finite étale of degree $p^{g(g+1)/2}$.
  - Then pass to minimal compactification.

□

Definition 4.0.18. Let $\mathcal{X}_{\Gamma_s(p)}(\epsilon)$ be the pullback of $\mathcal{X}(\epsilon)$ along $\mathcal{X}_{\Gamma_s(p)} \to \mathcal{X}$.

Lemma 4.0.19. The following diagram

$$\begin{tikzcd} \mathcal{X}(p^{-1}\epsilon) \arrow[r] \arrow[d] & \mathcal{X}_{\Gamma_s(p)}(\epsilon) \arrow[d] \\ \mathcal{X}(\epsilon) \arrow[r, "\mathrm{id}"] & \mathcal{X}(\epsilon) \end{tikzcd}$$

commutes. Moreover, the map $\mathcal{X}(p^{-1}\epsilon) \to \mathcal{X}_{\Gamma_s(p)}(\epsilon)$ is an open immersion, and the image of $\mathcal{X}(p^{-1}\epsilon)$ in $\mathcal{X}_{\Gamma_s(p)}(\epsilon)$ is both open and closed.

Definition 4.0.20. Let $\mathcal{X}_{\Gamma_s(p)}(\epsilon)_a$ be the open and closed subset of $\mathcal{X}_{\Gamma_s(p)}(\epsilon)$ "parametrizing those $D \subset \mathcal{A}(\epsilon)[p]$ with $D \cap C = \{0\}$". Let $\mathcal{X}_{\Gamma_s(p)}(\epsilon)_a$ be the image of $\mathcal{X}(p^{-1}\epsilon)$ in $\mathcal{X}_{\Gamma_s(p)}(\epsilon)$. Let $\mathcal{X}^*_{\Gamma_s(p)}(\epsilon)_a$ be the image of $\mathcal{X}^*(p^{-1}\epsilon)$ in $\mathcal{X}^*_{\Gamma_s(p)}(\epsilon)$. Let $\mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a$ be the pullback of $\mathcal{X}^*_{\Gamma_s(p)}(\epsilon)_a$ along $\mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon) \to \mathcal{X}^*_{\Gamma_s(p)}(\epsilon)$.

Remark 4.0.21. $\mathcal{X}^*_{\Gamma_s(p)}(\epsilon)_a$ is both open and closed in $\mathcal{X}^*_{\Gamma_s(p)}(\epsilon)$.

Lemma 4.0.22. Then for m sufficiently large, $\mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a$ is affinoid.

Proof. TODO: Write down the proof

- There exists an integer $m \geq 0$ such that $H^i(X^*, \omega^{\otimes p^m(p-1)}) = 0$ for all $i \geq 1$, since ω is an ample line bundle on X^*.
- We can find a lift $s \in H^0(X^*, \omega^{\otimes p^m(p-1)})$ lifting $\mathrm{Ha}^{p^m} \in H^0(X^*_{\mathbb{F}_p}, \omega^{\otimes p^m(p-1)})$.
- The condition $|\mathrm{Ha}| \geq |p|^{p^{-m}\epsilon}$ is equivalent to $|s| \geq |p|^{\epsilon}$.
- The condition defines an affinoid space $\mathcal{X}^*(p^{-m}\epsilon) \simeq \mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a$.

□

Lemma 4.0.23. There exists a unique perfectoid space $\mathcal{X}^*_{\Gamma_s(p^\infty)}(\epsilon)_a$ such that

$$\mathcal{X}^*_{\Gamma_s(p^\infty)}(\epsilon)_a \sim \lim_m \mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a.$$

Lemma 4.0.24. The space $\mathcal{X}^*_{\Gamma_s(p^\infty)}(\epsilon)_a$ is affinoid.

5. The Topological Hodge–Tate Map

6. The Perfectoid Hodge–Tate Map

Appendix A. Review of Abelian Schemes

Definition A.0.1. Let S be a scheme. A geometric point of S is a map $\mathrm{Spec}(k) \to S$ where k is an algebraically closed field.

Definition A.0.2. Let S be a scheme. An Abelian scheme over S is a proper smooth group scheme over S that is geometrically connected.

Definition A.0.3. Let $A \to S$ be an Abelian scheme where S is over $\mathrm{Spec}(\mathbb{F}_p)$. We have Frobenius map $F : A \to A^{(p)}$ and Verschiebung map $V : A^{(p)} \to A$ with the composition $V \circ F = [p]$.

The map $V : A^{(p)} \to A$ induces a map $\omega_{A/S} \to \omega_{A^{(p)}/S} \simeq \omega_{A/S}^{\otimes p}$. This gives a canonical section of $\omega_{A/S}^{\otimes(p-1)}$, called the Hasse invariant of A/S, denoted $\mathrm{Ha}(A/S) \in \Gamma(S, \omega_{A/S}^{\otimes(p-1)})$.

Appendix B. Review of Deformation Theory

Definition B.0.1 ([Ill71, pp. II.1.2.1, II.1.2.3])**.** Let $A \to B$ be a map of rings. The simplicial A-algebra $P_A(B)$ is defined by $P_A(B)_0 = A[B]$ and $P_A(B)_n = A[P_A(B)_{n-1}]$ for $n \geq 1$. The standard resolution of B over A is the argumentation $P_A(B) \to B$ where B is viewed as a constant simplicial A-algebra. The cotangent complex of B over A is the simplicial B-module $L_{B/A} = \Omega^1_{P_A(B)/A} \otimes_{P_A(B)} B$.

Remark B.0.2. This definition works in a general topos.

Definition B.0.3 ([Ill72, p. VII.1.1.1])**.** Let S be a scheme. Let S_{zar} be the small Zariski site over S. Let S_{fpqc} be the big fqpc site over S. The natural inclusion $S_{\mathrm{zar}} \to S_{\mathrm{fpqc}}$ induces a geometric map $(\epsilon^*, \epsilon_*) : \mathrm{Sh}(S_{\mathrm{zar}}) \rightleftarrows \mathrm{Sh}(S_{\mathrm{fpqc}})$.

Definition B.0.4. Let $f : X \to Y$ be a map of schemes. The cotangent complex is $L_{X/Y}$.

Definition B.0.5. Let S be a scheme. Let G be a group scheme over S that is flat and locally of finite presentation. Let $e : S \to G$ be the unit. The co-Lie complex is $\ell_G = Le^* L_{G/S}$, and the Lie complex is $\ell_G^\vee = R\underline{\mathrm{Hom}}(\ell_G, \mathcal{O}_S)$. Define $\underline{\ell}_G = L\epsilon^* \ell_G$.

Lemma B.0.6 ([Ill72, Theorem VII.4.2.5])**.** Let $f : S \to T$ be a map of schemes. Let $i : S \to S'$ be a T-extension by a quasi-coherent module I. Let A be a "schéma en anneaux" over T that is, as a scheme over T, tor-independent (c.f. [SGA6, Definition III.1.5]) with both S and S'. Let F (resp. G') be "schéma en A-modules" that are flat and locally of finite presentation over S (resp. S'). Let G be a "schéma en A-module" over S induced by G'. Let $u : F \to G$ be a morphism of "schémas en A-modules". Let K be the complex fitting into the distinguished triangle $K \to \ell_F^\vee \to \ell_G^\vee \to K[1]$. It is an object in $D(A \otimes_{\mathbb{Z}}^L \mathcal{O})$. Then there is an obstruction $\omega(u, G') \in \mathrm{Ext}_A^2(F, K \otimes_{\mathcal{O}}^L \epsilon^* I)$ which is zero if and only if there exists a pair (F', u') where F' is a deformation of F as "un schéma en A-modules" flat over S' and a map $u' : F' \to G'$ extending u.

Lemma B.0.7. Let S be a scheme. Let $i : S \to S'$ be an extension by a quasi-coherent module I. Suppose S and S' are both tor-indepeendent with $\mathrm{Spec}(\mathbb{Z})$. Let F (resp. G') be commutative group schemes over S (resp. S') that are flat and locally of finite presentation. Let G be a commutative group scheme over S induced by G'. Let $u : F \to G$ be a morphism of group schemes over S. Let K be the cone of the map $\ell_F^\vee \to \ell_G^\vee$. There is an obstruction $\omega(u, G') \in \mathrm{Ext}^1(F, K \otimes^L I)$ which vanishes if and only if there exists a pair (F', u') where F' is a deformation of F as a commutative group scheme that is flat over S', and $u' : F' \to G'$ is a map extending u.

Lemma B.0.8 ([Sch15, Theorem III.2.1])**.** Let A be a ring. Let G and H be commutative group schemes over A that are flat and of finite presentation, with a group map $u : H \to G$. Let $B \to A$ be a square-zero thickening with the argumentation ideal J. Let $\widetilde{G}$ be a lift of G to B. Let K be a cone of the map $\ell_H^\vee \to \ell_G^\vee$ of Lie complexes. Then there is an obstruction class $\omega \in \mathrm{Ext}^1(H, K \otimes^L J)$ which vanishes if and only if there exists a pair $(\widetilde{H}, \widetilde{u})$ where $\widetilde{H}$ is a flat commutative group scheme over B, and $\widetilde{u} : \widetilde{H} \to \widetilde{G}$ is a map lifting $u : H \to G$. Moreover, the obstruction class is functorial in J, in the following sense. If $B' \to A$ is another square-zero thickening with the argumentation ideal J', with a map $B \to B'$ over A, then $\omega' \in \mathrm{Ext}^1(H, K \otimes^L J')$ is the image of $\omega \in \mathrm{Ext}^1(H, K \otimes^L J)$ under the map $J \to J'$.

Appendix C. Review of Shimura Varieties

C.1. Shimura Datum and Canonical Models.

Definition C.1.1. A Shimura datum is a pair (G, X) where

- G is a reductive group over $\mathbb{Q}$;
- X is a $G(\mathbb{R})$-conjugacy class of maps $\mathbb{S} \to G_{\mathbb{R}}$;

satisfying the following properties

(1) For $h \in X$, only the characters $z/\overline{z}$, 1, $\overline{z}/z$ occur in the representation of $\mathbb{S}$ on $\mathrm{Lie}(G)$. In other words, the Hodge structure on $\mathrm{Lie}(G_{\mathbb{R}})$ defined by $\mathrm{Ad} \circ h$ is of type
$$\{(-1, 1), (0, 0), (1, -1)\}.$$
(2) $\mathrm{ad}(h(i))$ is a Cartan involution of G^{ad}, i.e. the real Lie group
$$\{g \in G^{\mathrm{ad}}(\mathbb{C}); \mathrm{ad}(h(i))\sigma(g) = g\}$$
is compact, where σ denotes the complex conjugation.
(3) G^{ad} has no factor defined over $\mathbb{Q}$ whose real points form a compact group. Equivalently, G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

Theorem C.1.2. Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. Let $\mathrm{Sh}_K(G, X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$.

(1) (Baily–Borel) $\mathrm{Sh}_K(G, X)$ has a natural structure of an algebraic variety over $\mathbb{C}$.
(2) (Shimura, Deligne, Milne, ...) $\mathrm{Sh}_K(G, X)$ has a model over a the reflex field $E(G, X)$.

Remark C.1.3. Let K denote a compact open subgroup of $G(\mathbb{A}_f)$. We get an inverse system of algebraic varieties (schemes) $(\mathrm{Sh}_K(G, X))_K$. There is an action ρ of $G(\mathbb{A}_f)$ on the system $(\mathrm{Sh}_K(G, X))_K$ defined by isomorphisms $\rho_K(g) : \mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{g^{-1}Kg}(G, X)$. For $k \in K$, $\rho_K(k)$ is the identity map. Therefore, for K' normal in K, there is an action of the finite group K/K' on $\mathrm{Sh}_{K'}(G, X)$, and the variety $\mathrm{Sh}_K(G, X)$ is the quotient of $\mathrm{Sh}_{K'}(G, X)$ by the action of K/K'.

C.2. **Siegel Modular Varieties.** Let (G, X) be a Siegel Shimura datum, i.e. the Shimura datum associated to a symplectic space. Then $G = \mathrm{GSp}_{2g}$, and the reflex field is $E(G, X) = \mathbb{Q}$ since G is split.

Lemma C.2.1. Let K^p be a compact open subgroup of $G(\mathbb{A}^\infty)$ contained in $\Gamma(N)^{(p)}$ for some integer $N \geq 3$ not divisible by p. Let $K_p = G(\mathbb{Z}_p)$. For compact open subgroup $U \subset K_p$, we have a smooth quasi-projective $\mathbb{Q}$-scheme $X_{K^pU,\mathbb{Q}}$ and a natural finite étale map $X_{K^pU,\mathbb{Q}} \to X_{K^pK_p,\mathbb{Q}}$ over $\mathbb{Q}$.

C.3. **PEL Shimura Varieties.** For PEL type Shimura variety, see Milne.

This is [Kot92]; also, "PEL-type $\mathcal{O}$-lattice", cf. [Lan13, Definition 1.2.1.3]

Let p be a prime. Let B be a finite-dimensional simple $\mathbb{Q}$-algebra with center F. Let $\mathcal{O}_B$ be a $\mathbb{Z}_{(p)}$-order in B. Let $*$ be a positive involution on B that preserves $\mathcal{O}_B$. Let V be a non-degenerate skew-Hermitian B-module. Let G be the group of automorphisms of the skew-Hermitian B-module V. Let K^p be a compact open subgroup of $G(\mathbb{A}_f^p)$. Let $h : \mathbb{C} \to \mathrm{End}_B(V_\mathbb{R})$ be an $\mathbb{R}$-algebra homomorphism such that $(h(\overline{z}) = h(z)^*$, and that the symmetric bilinear form $(v, h(i)w)$ on $V_\mathbb{R}$ is positive definite. The map h determines a decomposition $V_\mathbb{C} = V_1 \oplus V_2$. Here V_1 is the subspace of $V_\mathbb{C}$ on which $h(z)$ acts by z. The field of definition of the isomorphism class of the complex representation V_1 of B is a number field E, with ring of integers $\mathcal{O}_E$.

Consider the following moduli problem over $\mathcal{O}_E \otimes_\mathbb{Z} \mathbb{Z}_{(p)}$.

Also, see [Lan13, Definition 1.4.1.4, Theorem 1.4.1.11, Remark 1.4.1.13].

Definition C.3.1 ([Lan13, Definition 1.2.1.3])**.** Let B be a finite-dimensional semi-simple algebra over $\mathbb{Q}$ with positive involution $*$ and center F, where positivity means $\mathrm{tr}_{B/\mathbb{Q}}(xx^*) > 0$ for all $x \neq 0$ in B. Let V be a finite B-module, equipped with a non-degenerate alternating bilinear form ψ, such that $\psi(bx, y) = \psi(x, b^*y)$ for all $x, y \in V$ and $b \in B$. Let $h : \mathbb{C} \to \mathrm{End}_B(V)_\mathbb{R}$ be a map over $\mathbb{R}$ such that complex conjugation on $\mathbb{C}$ corresponds by h to the adjunction in $\mathrm{End}_B(V)_\mathbb{R}$ with respect to the pairing ψ, and such that $(u, v) \mapsto \psi(u, h(i)v)$ is a positive definite symmetric pairing on $V_\mathbb{R}$. Let G be the reductive group over $\mathbb{Q}$ defined by

$$G(R) = \{g \in \mathrm{GL}_B(V \otimes_\mathbb{Q} R); \exists \mu(g) \in R^\times, \psi(gx, gy) = \mu(g)\psi(x, y)\},$$

where GL_B means B-equivariant linear maps. Let X be the $G(\mathbb{R})$-conjugacy class of $h^{-1} : \mathbb{C}^\times \to G_\mathbb{R}$. The pair (G, X) is called a PEL Shimura datum.

Definition C.3.2 ([Roz20, Section 1.2])**.** In order the define the integral model, we need to add the following new data. Let $\mathcal{O}_B$ be a $\mathbb{Z}_{(p)}$-order in B that is stable under the involution $*$ and becomes maximal after tensoring with $\mathbb{Z}_{(p)}$. We impose two more conditions which is omitted for now.

Let E be the reflex field.

Let $\mathcal{F}_{K^p}$ be the following category fibered in groupoids over the category of schemes over $\mathcal{O}_E \otimes_\mathbb{Z} \mathbb{Z}_{(p)}$:

- The objects over a scheme S are the tuples $(A, \lambda, \iota, \eta)$, where

Appendix D. Artin's Criterion

Theorem D.0.1. Let S be a scheme of finite type over a field or an excellent Dedekind domain. Let X be a category fibered in groupoids over $\mathrm{Sch}_{/S}$. Then X is an algebraic stack locally of finite type over S if and only if the following conditions hold:

(1) X is a stack for the étale topology.
(2) X is locally of finite presentation.
(3) ...

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