1. Grothendieck topologies

Definition 1.1. A site is given by a (small) category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target $\{U_i \to U\}_{i \in I}$, called coverings of $\mathcal{C}$, satisfying the following conditions.

(1) If $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering.
(2) If $\{U_i \to U\}_{i \in I}$ is a covering and for each i we have a covering $\{V_{ij} \to U_i\}_{j \in J_i}$, then $\{V_{ij} \to U\}_{i \in I, j \in J_i}$ is a covering.
(3) If $\{U_i \to U\}_{i \in I}$ is a covering and $V \to U$ is a morphism of $\mathcal{C}$, then $U_i \times_U V$ exists for all i, and $\{U_i \times_U V \to V\}_{i \in I}$ is a covering.

2. Set theoretical issues

2.1. Ordinals.

Definition 2.1. A set T is transitive if $x \in T$ implies $x \subset T$.

Definition 2.2. A set α is an ordinal if it is transitive and well-ordered by "$\in$".

Definition 2.3. The smallest ordinal is $\emptyset$ which is also denoted by 0.

Let α be an ordinal.

Definition 2.4. The successor of α is $\alpha + 1 = \alpha \cup \{\alpha\}$, which is also an ordinal. The ordinal α is called a successor ordinal if it is the successor of another ordinal.

Definition 2.5. The ordinal α is called a limit ordinal if it is not 0, and not a successor ordinal.

Lemma 2.6. If α is a limit ordinal, then $\alpha = \bigcup_{\gamma \in \alpha} \gamma$.

TODO: The first limit ordinal is ω and it is also the first infinite ordinal. The first uncountable ordinal ω_1 is the set of all countable ordinals. The collection of all ordinals is a proper class. It is well-ordered by "$\in$" in the following sense: any non-empty set (or even class) of ordinals has a least element. Given a set A of ordinals, we define the supremum of A to be $\sup_{\alpha \in A} \alpha = \bigcup_{\alpha \in A} \alpha$. It is the least ordinal bigger or equal to all $\alpha \in A$. Given any well-ordered set $(S, <)$, there is a unique ordinal α such that $(S, <) \simeq (\alpha, \in)$, called the order type of the well-ordered set $(S, <)$.

Definition 2.7. We define by transfinite induction $V_0 = \emptyset$, $V_{\alpha+1} = P(V_\alpha)$, and for a limit ordinal β, $V_\beta = \bigcup_{\gamma < \beta} V_\gamma$, where $P(x)$ denotes the power set of x.

Lemma 2.8. Every set is an element of V_α for some ordinal α.

2.2. The category of schemes.

Definition 2.9. Let S be a scheme. We define the cardinal

$$\mathrm{size}(S) = \max\{\aleph_0, \kappa_1, \kappa_2\}$$

where κ_1 is the cardinality of the set of affine opens of S, and κ_2 is the supremum of all the cardinalities of $\Gamma(U, \mathcal{O}_S)$ for $U \subset S$ affine open.

Lemma 2.10. Let κ be a cardinal. There exists a set A such that every element of A is a scheme and such that for every scheme S with $\mathrm{size}(S) \leq \kappa$, there is an element $X \in A$ such that X and S are isomorphic as schemes.

Definition 2.11. Let α be an ordinal. We denote Sch_α the full sub-category of Sch whose objects are elements of V_α.

Lemma 2.12. Let $B(\kappa) = \max\{\kappa^{\aleph_0}, \kappa^+\}$ for each cardinal κ. Let S_0 be a set of schemes. There exists a limit ordinal α satisfying the following properties.

(1) We have $S_0 \subset V_\alpha$. In particular, $S_0 \subset \mathrm{Ob}(\mathrm{Sch}_\alpha)$.
(2) For any $S \in \mathrm{Ob}(\mathrm{Sch}_\alpha)$ and any scheme T with $\mathrm{size}(T) \leq B(\mathrm{size}(S))$, there exists a scheme $S' \in \mathrm{Ob}(\mathrm{Sch}_\alpha)$ such that $T \simeq S'$.
(3) For any countable category (i.e. both the set of objects and the set of morphisms are countable) $\mathcal{I}$ and any functor $F : \mathcal{I} \to \mathrm{Sch}_\alpha$, the limit $\lim_{\mathcal{I}} F$ exists in Sch_α if and only if it exists in Sch and moreover, in this case, the natural morphism between them is an isomorphism.

(4) For any countable category (i.e. both the set of objects and the set of morphisms are countable) $\mathcal{I}$ and any functor $F : \mathcal{I} \to \mathrm{Sch}_\alpha$, the colimit $\mathrm{colim}_{\mathcal{I}} F$ exists in Sch_α if and only if it exists in Sch and moreover, in this case, the natural morphism between them is an isomorphism.

Lemma 2.13. Let α be the ordinal constructed in the previous lemma. The category Sch_α satisfies the following properties.

(1) If $X, Y, S \in \mathrm{Ob}(\mathrm{Sch}_\alpha)$, then for any morphisms $f : X \to S$ and $g : Y \to S$, the fibre product $X \times_S Y$ exists in Sch_α, and is a fibre product in the categor of schemes.
(2) Given any at most countable collection $S_1, S_2, \dots$ of elements of $\mathrm{Ob}(\mathrm{Sch}_\alpha)$, the coproduct $\coprod_i S_i$ exists in Sch_α, and is a coproduct in the category of schemes.
(3) For any $S \in \mathrm{Ob}(\mathrm{Sch}_\alpha)$ and any open immersion $U \to S$, there exists a $V \in \mathrm{Ob}(\mathrm{Sch}_\alpha)$ with $V \simeq U$.
(4) For any $S \in \mathrm{Ob}(\mathrm{Sch}_\alpha)$ and any closed immersion $T \to S$, there exists a $T' \in \mathrm{Ob}(\mathrm{Sch}_\alpha)$ with $T' \simeq T$.
(5) And so on.

2.3. **Coverings of sites.** Let $\mathcal{C}$ be a (small) category. Let $\mathrm{Cov}(\mathcal{C})$ be a (proper) class of coverings of $\mathcal{C}$ satisfying the conditions of sites.

Definition 2.14. For an ordinal α, we set $\mathrm{Cov}(\mathcal{C})_\alpha = \mathrm{Cov}(\mathcal{C}) \cap V_\alpha$. Given an ordinal α and a cardinal κ, we set $\mathrm{Cov}(\mathcal{C})_{\alpha,\kappa}$ to be the set of coverings $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})_\alpha$ with $|I| \leq \kappa$.

Lemma 2.15. Let $C_0 \subset \mathrm{Cov}(\mathcal{C})$ be a set. There exists a cardinal κ and a limit cardinal α with the following properties.

(1) We have $C_0 \subset \mathrm{Cov}(\mathcal{C})_{\alpha,\kappa}$.
(2) The set $\mathrm{Cov}(\mathcal{C})_{\alpha,\kappa}$ satisfies the conditions of a site, i.e. $(\mathcal{C}, \mathrm{Cov}(\mathcal{C})_{\alpha,\kappa})$ is a site.
(3) Every covering in $\mathrm{Cov}(\mathcal{C})$ is combinatorially equivalent to a covering in $\mathrm{Cov}(\mathcal{C})_{\alpha,\kappa}$.

3. Valuation rings

Definition 3.1. Let K be a field. Let A, B be subrings of K that are local. We say that B dominates A if $A \subset B$ and $\mathfrak{m}_A = A \cap \mathfrak{m}_B$.

4. Spectral sequences

4.1. **Basics.** Let $\mathcal{A}$ be an abelian category. Let r_0 be an integer.

Definition 4.1. A spectral sequence (starting from r_0) in $\mathcal{A}$ is a family $(E_r, d_r)_{r\geq r_0}$ where each E_r is an object of $\mathcal{A}$, each $d_r : E_r \to E_r$ is a morphism in $\mathcal{A}$ such that $d_r \circ d_r = 0$ and $E_{r+1} \simeq \ker(d_r)/\operatorname{im}(d_r)$ for $r \geq r_0$.

Let $(E_r, d_r)_{r\geq r_0}$ be a spectral sequence in $\mathcal{A}$.

Definition 4.2. We define subobjects

$$0 = B_{r_0} \subset B_{r_0+1} \subset \cdots \subset Z_{r_0+1} \subset Z_{r_0} = E_{r_0}$$

by the following procedure. Set $B_{r_0+1} = \operatorname{im}(d_{r_0})$ and $Z_{r_0+1} = \ker(d_{r_0})$. Then $E_{r_0+1} \simeq Z_{r_0+1}/B_{r_0+1}$. Suppose we have defined Z_r and B_r with $E_r \simeq Z_r/B_r$. Then we set Z_{r+1} and B_{r+1} to be the unique subobject of Z_r containing B_r corresponding to $\ker(d_r)$ and $\operatorname{im}(d_r)$. In particular we have $E_r \simeq Z_r/B_r$ for all $r \geq r_0$.

Definition 4.3. If the subobjects $Z_\infty = \bigcap_r Z_r$ and $B_\infty = \bigcup_r B_r$ of E_{r_0} exist, we define the limit of spectral sequence to be $E_\infty = Z_\infty/B_\infty$.

Definition 4.4. We say that the spectral sequence $(E_r, d_r)_{r\geq r_0}$ degenerates at E_r if $d_{r'} = 0$ for all $r' \geq r$.

4.2. **Exact couples.** Let $\mathcal{A}$ be an abelian category.

Definition 4.5. An exact couple in $\mathcal{A}$ is a datum (A, E, α, f, g) where A, E are objects of $\mathcal{A}$, and α, f, g are morphisms as depicted in the following (non-commutative) diagram

$$\begin{tikzcd} A \arrow[rr, "\alpha"] & & A \arrow[dl, "g"] \\ & E \arrow[ul, "f"] & \end{tikzcd}$$

that is exact at each corner, i.e. the kernel of each morphism is equal to the image of its predecessor, i.e. $\ker(\alpha) = \operatorname{im}(f)$, $\ker(f) = \operatorname{im}(g)$, and $\ker(g) = \operatorname{im}(\alpha)$.

Let (A, E, α, f, g) be an exact couple in $\mathcal{A}$. Let $d = g \circ f$. Then $d \circ d = 0$.

Definition 4.6. The derived exact couple $(A', E', \alpha', f', g')$ of (A, E, α, f, g) is defined as follows:

(1) $E' = \ker(d)/\operatorname{im}(d)$;
(2) $A' = \operatorname{im}(\alpha)$;
(3) $\alpha' : A' \to A'$ induced by α;
(4) $f' : E' \to A'$ induced by f;
(5) $g' : A' \to E'$ induced by "$g \circ \alpha^{-1}$".

Lemma 4.7. $(A', E', \alpha', f', g')$ is an exact couple.

Remark 4.8. Consider the following commutative diagram

$$\begin{tikzcd} & E \arrow[r, "d"] \arrow[d, "0"] & E \arrow[r] \arrow[d, "f"] & E/\operatorname{im}(d) \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & A/\ker(\alpha) \arrow[r, "\alpha"] & A \arrow[r, "g"] & E & \end{tikzcd}$$

with exact rows, and the snake lemma gives the morphism $f' : E' \to A'$. The map $g' : A' \to E'$ can be obtained by applying the snake lemma to the diagram

$$\begin{tikzcd} & E \arrow[r, "f"] \arrow[d, "d"] & A \arrow[r, "\alpha"] \arrow[d, "g"] & \operatorname{im}(\alpha) \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & \ker(d) \arrow[r] & E \arrow[r, "d"] & E. & \end{tikzcd}$$

Definition 4.9. The spectral sequence associated to the exact couple (A, E, α, f, g) is the spectral sequence $(E_r, d_r)_{r\geq 1}$ defined as follows.

(1) $E_1 = E$ and $d_1 = d$.
(2) $E_{r+1} = E'_r$ and $d_{r+1} = d'_r$ for $r \geq 1$.

Example 4.10. We record an example of exact couples here. Let p be a prime number. All cohomology groups will have coefficients in $\mathbb{Z}/(p)$. Let X be a connective spectrum such that $H^*(X)$ has finite type. Let $E = H\mathbb{Z}/(p)$ be the mod p Eilenberg–Mac Lane spectrum. A mod p Adams resolution $(X_s, g_s)_{s\geq 0}$ for X is a diagram

$$\begin{tikzcd} \cdots \arrow[r] & X_2 \arrow[r, "g_1"] \arrow[d, "f_2"] & X_1 \arrow[r, "g_0"] \arrow[d, "f_1"] & X_0 \arrow[d, "f_0"] \\ & K_2 & K_1 & K_0 \end{tikzcd}$$

where $X_0 = X$, each K_s is a wedge of suspensions of E, $H^*(f_s)$ is surjective, and X_{s+1} is the fibre of f_s. The fibrations $X_{s+1} \to X_s \to K_s$ induces long exact sequences

$$\cdots \to \pi_*(X_{s+1}) \to \pi_*(X_s) \to \pi_*(K_s) \to \pi_{*-1}(X_{s+1}) \to \cdots .$$

If we regard $\pi_*(X_s)$ and $\pi_*(K_s)$ for all s as bigraded abelian groups D_1 and E_1, i.e. $D_1^{s,t} = \pi_{t-s}(X_s)$ and $E_1^{s,t} = \pi_{t-s}(K_s)$, then the long exact sequence induces an exact couple

$$\begin{array}{ccc} D_1 & \xrightarrow{i_1} & D_1 \\ {\scriptstyle k_1}\nwarrow & & \swarrow{\scriptstyle j_1} \\ & E_1 & \end{array}$$

where $i_1^{s,t} = \pi_{t-s}(g_s)$, $j_1^{s,t} = \pi_{t-s}(f_s)$, and k_1 is the connecting morphism.

4.3. **The Serre spectral sequence.**

Lemma 4.11. Let $F \to X \to B$ be a fibration with B a simply-connected CW complex. Let G be an abelian group. There is a spectral sequence $E^r_{p,q}$ with the following properties.

(1) The differentials are $E^r_{p,q} \to E^r_{p-r,q-1+r}$.
(2) $E^2_{p,q} = H_p(B; H_q(F; G))$.
(3) $E^\infty_{p,n-p} \simeq F^p_n / F^{p-1}_n$ where $0 \subset F^0_n \subset \cdots \subset F^n_n = H_n(X; G)$.

Example 4.12. Consider the fibration $S^1 \to S^\infty \to \mathbb{C}P^\infty$. The E^2-page is given by

$$E^2_{p,q} = H_p(\mathbb{C}P^\infty; H_q(S^1; \mathbb{Z})).$$

One immediate observation is that $E^2_{p,q}$ can only be non-trivial if $p \geq 0$ and $q \geq 0$. We know that

$$H_q(S^1; \mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the E^2-page is

$$E^2_{p,q} = \begin{cases} H_p(\mathbb{C}P^\infty; \mathbb{Z}) & q = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Because S^∞ is contractible, we have $H_n(X; \mathbb{Z}) = 0$ for $n > 0$ and $H_0(X; \mathbb{Z}) = \mathbb{Z}$. Then the E^∞-page is

$$E^\infty_{p,q} = \begin{cases} \mathbb{Z} & (p, q) = (0, 0) \\ 0 & \text{otherwise.} \end{cases}$$

The following facts can be checked easily.

(1) $E^2_{0,0} = E^\infty_{0,0}$.
(2) $E^3_{p,q} = E^\infty_{p,q}$ for all p, q.

We have the following complex from the E^2-page

$$0 = E^2_{k+2,-1} \to E^2_{k,0} \to E^2_{k-2,1} \to E^2_{k-4,2} = 0$$

for every k. Note that $E^3_{k,0} = E^\infty_{k,0} = 0$ for $k \neq 0$, and that $E^3_{k-2,1} = E^\infty_{k-2,1} = 0$ for all k. Thus the sequence is exact for $k \geq 1$, i.e. $E^2_{k,0} \simeq E^2_{k-2,1}$ for $k \geq 1$, i.e. $H_k(\mathbb{C}P^\infty; \mathbb{Z}) \simeq H_{k-2}(\mathbb{C}P^\infty)$ for $k \geq 2$ and $H_1(\mathbb{C}P^\infty; \mathbb{Z}) = 0$. This gives all the homologies of $\mathbb{C}P^\infty$.

5. Stable infinity categories

5.1. **Overview.**

Remark 5.1. Let's construct the (symmetric monoidal) derived ∞-category of quasi-coherent sheaves on a scheme X.

First we do the affine case. The most concrete way is to take the 1-category of chain complexes and inverting quasi-isomorphisms (in the ∞-categorical sense). Another (equivalent) definition is to take the stabilization of the animation of the category of finitely generated projective R-modules. Then $D(R)$ has all the properties (functorial, symmetric monoidal, etc.) because the category of finitely generated projective R-modules does.

The next step is to show that $D(-)$, as a functor on affine schemes, satisfies Zariski descent. It boils down to showing the following commutative diagram of stable ∞-categories

$$\begin{tikzcd} D(R) \arrow[r] \arrow[d] & D(R[1/f]) \arrow[d] \\ D(R[1/g]) \arrow[r] & D(R[1/fg]) \end{tikzcd}$$

is cartesian. This agains follows from the case of finitely generated projective R-modules.

Now it's clear how to define the category $D(X)$ where X is a scheme. We simply glue the affines

$$D(X) = \lim_{\mathrm{Spec}(R) \subset X} D(R).$$

So an object of $D(X)$ is simply the collection of an object of $D(R)$ for each affine open $\mathrm{Spec}(R)$ of X, with suitable gluing data. By formal properties the category $D(X)$ has desired properties.

Remark 5.2. An ∞-category is stable if it has a zero object, it admits fibres and cofibres, and every cofibre sequence is a fibre sequence. This is the analogue of abelian categories. The homotopy category of a stable ∞-category is automatically triangulated. Stable ∞-categories have other nice properties, for example, a square is a pullback if and only if it is a pushout, and there exists finite limits and colimits. For a pointed ∞-category $\mathcal{C}$, the spectrum, or stabilization, of $\mathcal{C}$, is a stable ∞-category $\mathrm{Sp}(\mathcal{C})$. One definition is the ∞-category of excisive reduced functors $\mathbb{S}_*^{\mathrm{fin}} \to \mathcal{C}$. Here $\mathbb{S}_*^{\mathrm{fin}}$ is the smallest subcategory of $\mathbb{S}$ that contains the final object, is stable under finite colimits, and consists of pointed objects. Another definition is the homotopy limit of the tower $\cdots \to \mathcal{C} \xrightarrow{\Omega} \mathcal{C}$. Moreover, if $\mathcal{C}$ is presentable, then $\mathrm{Sp}(\mathcal{C})$ is also presentable. We shall assume that $\mathcal{C}$ is presentable. The first approach gives a functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ by evaluating on the zero sphere. In fact, $\mathcal{C}$ is stable if and only if Ω^∞ is an equivalence. The functor Ω^∞ admits a left adjoint Σ^∞.

One should think of animated rings as E_∞-rings with an extra structure.

5.2. **Derived categories.**

Remark 5.3. Let $\mathcal{A}$ be an abelian category with enough injectives. We shall associated to $\mathcal{A}$ an stable ∞-category $D^-(\mathcal{A})$, whose objects can be identified with bounded-above complexes in $\mathcal{A}$. Its homotopy category can be identified (as a triangulated category) with the usual derived category of $\mathcal{A}$. The stable ∞-category $D^-(\mathcal{A})$ is equipped with a t-structure, and there is a canonical equivalence of categories $\mathcal{A} \to D^-(\mathcal{A})^\heartsuit$. It is characterized by the following universal property. If $\mathcal{C}$ is any stable ∞-category equipped with a left-complete t-structure, then any right exact functor $\mathcal{A} \to \mathcal{C}^\heartsuit$ extends (in an essentially unique way) to an exact functor $D^-(\mathcal{A}) \to \mathcal{C}$. This is regarded as the left derived functor.

By an entirely parallel discussion, if $\mathcal{A}$ is an abelian category with enough injective objects, we can associated a bounded-below derived category $D^+(\mathcal{A})$. For a Grothendieck abelian category $\mathcal{A}$, it has enough injective objects, and we can associated an unbounded derived category $D(\mathcal{A})$ which contains $D^+(\mathcal{A})$ as a full sub-category (and $D^-(\mathcal{A})$ if $\mathcal{A}$ has enough projective objects).

Definition 5.4. Let $\mathcal{C}$ and $\mathcal{D}$ be stable ∞-categories equipped with t-structures. A functor $f : \mathcal{C} \to \mathcal{D}$ is called right t-exact if it is exact and maps $\mathcal{C}_{\geq 0}$ to $\mathcal{D}_{\geq 0}$. The functor f is called left t-exact if it is exact and maps $\mathcal{C}_{\leq 0}$ to $\mathcal{D}_{\leq 0}$.

Lemma 5.5 ([Lur17, Theorem 1.3.3.2])**.** Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $\mathcal{C}$ be a stable ∞-category equipped with a left complete t-structure. Let $\mathcal{E} \subset \mathrm{Fun}(D^-(\mathcal{A}), \mathcal{C})$ be full sub-category spanned by those right t-exact functors which carry projective objects of $\mathcal{A}$ into the heart of $\mathcal{C}$. The construction $F \mapsto \tau_{\leq 0}(F|_{D^-(\mathcal{A})^\heartsuit})$ determines an equivalence from $\mathcal{E}$ to the nerve of the ordinary category of right exact functors from $\mathcal{A}$ to $\mathcal{C}^\heartsuit$.

6. Cohomology of schemes

6.1. **Basics.**

Remark 6.1. There are three possible variants of cohomology, gotten by restricting the source category for the derived functors, as demonstrated in the following diagram:

$$\begin{tikzcd} (1) & (2) & (3) \\ \mathrm{QCoh}(X) \arrow[r] & \mathrm{Mod}(\mathcal{O}_X) \arrow[r] & \mathrm{Sh}(X, \mathrm{Ab}) \arrow[d] \\ & & \mathrm{Ab}. \end{tikzcd}$$

Each of (1)-(3) is an abelian category with enough injectives. So we could consider the right derived functors with each (1)-(3) as the source category. There is no guarantee that these right derived functors agree, even if all the horizontal functors are exact. But we still have some compatibilities.

(1) For every scheme X, every $\mathcal{O}_X$-module M admits an injective resolution such that each term in the resolution is also injective as an object in $\mathrm{Sh}(X, \mathrm{Ab})$. This implies that $H^i_{(2)}(X, M) = H^i_{(3)}(X, M)$.

(2) For every locally noetherian scheme X, every $M \in \mathrm{QCoh}(X)$ admits an injective resolution such that each term in the resolution is also injective in $\mathrm{Mod}(\mathcal{O}_X)$. This implies that $H^i_{(1)}(X,M) = H^i_{(2)}(X,M)$.

So for locally noetherian schemes, all three options agree. We shall take $H^i_{(3)}$ as the cohomology unless otherwise specified. The reasons are as follows.

(1) There are non-quasi-coherent sheaves of abelian groups on a scheme X whose cohomology is interesting, for example, $H^1(X, \mathcal{O}_X^\times) \simeq \mathrm{Pic}(X)$.
(2) There are certain maps of $\mathcal{O}_X$-modules in $\mathrm{Sh}(X, \mathrm{Ab})$, i.e. not $\mathcal{O}_X$-linear, that we still want to have an induced map on cohomologies. An example is the de Rham complex whose differential is not $\mathcal{O}_X$-linear.

Let $f : Y \to X$ be a morphism of schemes. We have the left exact functor $f_* : \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_X)$. So we may consider its i-th right derived functor $R^i f_*$ for $i \geq 0$.

Lemma 6.2. If $f : Y \to X$ is affine, then $R^i f_*(\mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_Y$-module $\mathcal{F}$ and every integer $i > 0$.

Lemma 6.3. If X is a scheme of Krull dimension $\leq d$, then $H^i(X, \mathcal{F}) = 0$ for all $i > d$ and all sheaves $\mathcal{F} \in \mathrm{Sh}(X, \mathrm{Ab})$.

Lemma 6.4. Suppose $f : Y \to X$ is proper and X is locally noetherian. If $\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module, then $R^i f_*(\mathcal{F})$ is a coherent $\mathcal{O}_X$-module for all $i \geq 0$.

7. The Zariski topology

Definition 7.1. Let T be a scheme. A Zariski covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is an open immersion and such that $T = \bigcup_{i \in I} f_i(T_i)$.

Lemma 7.2. The (proper) class of Zariski coverings satisfies the conditions of sites.

Definition 7.3. A big Zariski site is a site $\mathrm{Sch}_{\mathrm{Zar}}$ constructed as follows:

(1) Choose a set of schemes S_0, and a set of Zariski coverings C_0 among these schemes.
(2) Take $\mathrm{Sch}_{\mathrm{Zar}}$ to be a category Sch_α constructed in Section 2.2.
(3) Choose a set of coverings starting with the category Sch_α, the class of Zariski coverings, and the set C_0, cf. 2.3.

Remark 7.4. The sheaf category $\mathrm{Sh}(\mathrm{Sch}_{\mathrm{Zar}})$ does not depend on the choice of coverings (even the choice of C_0). Thus it only depends on the choice of Sch_α.

Lemma 7.5. Let $\mathrm{Sch}_{\mathrm{Zar}}$ be a big Zariski site. Let $T \in \mathrm{Ob}(\mathrm{Sch}_{\mathrm{Zar}})$. Let $\{T_i \to T\}_{i \in I}$ be a Zariski covering of T. Then there is a covering $\{U_j \to T\}_{j \in J}$ in the site $\mathrm{Sch}_{\mathrm{Zar}}$ that is tautologically equivalent to $\{T_i \to T\}_{i \in I}$.

Definition 7.6. Let S be a scheme. Let $\mathrm{Sch}_{\mathrm{Zar}}$ be a big Zariski site containing S.

(1) The big Zariski site of S, denoted $(\mathrm{Sch}/S)_{\mathrm{Zar}}$, is the localization site $\mathrm{Sch}_{\mathrm{Zar}}/S$ of $\mathrm{Sch}_{\mathrm{Zar}}$ at S.
(2) The small Zariski site of S, denoted S_{Zar}, is the full subcategory of $(\mathrm{Sch}/S)_{\mathrm{Zar}}$ consisting of objects U/S such that $U \to S$ is an open immersion. A covering of S_{Zar} is any covering $\{U_i \to U\}_{i \in I}$ of $(\mathrm{Sch}/S)_{\mathrm{Zar}}$ with $U \in \mathrm{Ob}(S_{\mathrm{Zar}})$.

Lemma 7.7. The category of sheaves on S_{Zar} is equivalent to the category of sheaves on the underlying topological space of S.

8. Simplicial methods

8.1. Basics.

Definition 8.1. For every integer $n \geq 0$, we write $[n]$ for the linearly ordered finite set $\{0 < 1 < \cdots < n-1 < n\}$. We also denote $[-1] = \emptyset$.

Definition 8.2. We define a category Δ^+ as follows.

(1) The objects of Δ^+ are $[n]$ for integers $n \geq -1$.
(2) A morphism from $[m]$ to $[n]$ in the category Δ_+ is a function $\alpha : [m] \to [n]$ that is non-decreasing.

The category Δ^+ is called the augmented simplex category. The simplex category Δ is the full sub-category of Δ_+ consisting of objects $[n]$ with $n \geq 0$.

Remark 8.3. The object $[-1]$ is the initial object of Δ_+, and objects other than $[-1]$ have no morphisms to $[-1]$. The category Δ does not have an initial objects.

Definition 8.4. Let $\mathcal{C}$ be a category. A simplicial object of $\mathcal{C}$ is a functor $\Delta^{\mathrm{opp}} \to \mathcal{C}$, usuall denoted by $X_\bullet : [n] \mapsto X_n$. An augmented simplicial object of $\mathcal{C}$ is a functor $\Delta_+^{\mathrm{opp}} \to \mathcal{C}$, usually denoted by $X_\bullet \to Y$ where Y is the degree -1 part, and $X_\bullet$ is degree ≥ 0 part. An morphism between simplicial objects of $\mathcal{C}$ is a morphism of functors. The category of simplicial objects (resp. augmented simplicial objects) of $\mathcal{C}$ is denoted by $\mathrm{Simp}(\mathcal{C})$ (resp. $\mathrm{Simp}_+(\mathcal{C})$).

Remark 8.5. To give an augmented simplicial object is to give a simplicial object $X_\bullet$ and an additional object X_{-1} of $\mathcal{C}$, equipped with a map $X_0 \to X_{-1}$ such that all possible compositions $X_n \to X_{-1}$ coincide. Then all maps in $X_\bullet$ are over X_{-1}. In other words, an augmented simplicial object of $\mathcal{C}$ with a specified augmentation X_{-1} is simply an simplicial object in the slice category $\mathcal{C}_{/X_{-1}}$.

9. The étale topology

10. The arc-topology

10.1. **Descent.** Let τ be a Grothendieck topology. We can ask when a functor satisfies descent with respect to it, or equivalently, when it is a sheaf. Let's consider Grothendieck topologies on $\mathrm{Sch}_{\mathrm{qcqs}}$ that are finitary, i.e. every cover admits a finite subcover, and such that if $X, Y \in \mathrm{Sch}_{\mathrm{qcqs}}$, then $\{X \to X \sqcup Y \leftarrow Y\}$ forms a covering family, cf. [Lur18, Section A.3.2, Section A.3.3].

Definition 10.1. Let $F : \mathrm{Sch}_{\mathrm{qcqs}}^{\mathrm{opp}} \to \mathcal{C}$ be a presheaf valued in an ∞-category $\mathcal{C}$. We say that F satisfies descent for a morphism $Y \to X$ of qcqs schemes if it satisfies the ∞-categorical sheaf axiom with respect to $Y \to X$, i.e. if the natural map

$$F(X) \to \lim\big[F(Y) \rightrightarrows F(Y \times_X Y) \cdots\big]$$

is an equivalence. If this property holds for all maps $f : Y \to X$ that are covers for the Grothendieck topology τ, and further if F carries finite disjoint unions to finite products, then we say that F satisfies τ-descent, or is a τ-sheaf.

10.2. **Basics.**

Definition 10.2. (1) An extension of valuation rings is a faithfully flat map of valuation rings, or equivalently, an injective local homomorphism.

(2) A map of qcqs schemes $Y \to X$ is called a v-cover if for any valuation ring V and any map $\mathrm{Spec}(V) \to X$, there is an extension of valuation rings $V \to W$ and a map $\mathrm{Spec}(W) \to Y$ that fits into a commutative square

$$\begin{tikzcd} \mathrm{Spec}(W) \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}(V) \arrow[r] & X. \end{tikzcd}$$

(3) The v-topology on the category of schemes is the Grothendieck topology where the covering families $\{f_i : Y_i \to X\}_{i \in I}$ are those families with the following property: for any affine open $V \subset X$, there exists a map $t : K \to I$ of sets with K finite, and affine opens $U_k \subset f_{t(k)}^{-1}(V)$ for each $k \in K$ such that the induced map $\bigsqcup_k U_k \to V$ is a v-cover in the sense of (2).

Remark 10.3. For finite type maps of noetherian schemes, the v-topology coincides with the h-topology defined by Voevodsky [Voe96]. In general, every v-cover is a limit of h-covers.

Definition 10.4. (1) A map $f : Y \to X$ of qcqs schemes is an arc-cover if for any rank ≤ 1 valuation ring V and a map $\mathrm{Spec}(V) \to X$, there is an extension $V \to W$ of rank ≤ 1 valuation rings and a map $\mathrm{Spec}(W) \to Y$ lifting $\mathrm{Spec}(V) \to X$.

(2) The arc-topology on the category of all schemes is defined similarly to the v-topology.

Remark 10.5. For noetherian targets, there is no distinction between v-covers and arc-covers.

Example 10.6. Let V be a valuation ring of rank 2. Let $\mathfrak{p} \subset V$ be the unique height 1 prime. Then both $V_\mathfrak{p}$ and $V/\mathfrak{p}$ are rank 1 valuation rings, and the map $V \to V_\mathfrak{p} \times V/\mathfrak{p}$ is an arc-cover but not a v-cover.

Remark 10.7. A contravariant functor F on the category Sch of all schemes that is a sheaf for the Zariski topology is automatically determined by its restriction to the subcategory $\mathrm{Sch}_{\mathrm{qcqs}}$ of qcqs schemes. Conversely, any Zariski sheaf on $\mathrm{Sch}_{\mathrm{qcqs}}$ comes from a unique unique Zariski sheaf on Sch.

10.3. **Excision.**

Definition 10.8. An excision datum is a map $f : (A, I) \to (B, J)$ where A and B are commutative rings, $I \subset A$ and $J \subset B$ are ideals, and $f : A \to B$ is a map that carries $I \subset A$ isomorphically onto $J \subset B$. In this situation, we obtain a commutative square of rings

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A/I \arrow[d] \\ B \arrow[r] & B/J \end{tikzcd}$$

that is both cartesian and co-cartesian. Such diagrams are also called Milnor squares, cf. [Mil71]. We say that a $D(\mathbb{Z})$-valued functor on commutative rings is excisive if for any excisive datum as above, the square obtained by apply F is cartesian.

Lemma 10.9. Let V be a valuation ring. Let $\mathfrak{p} \subset \mathfrak{q}$ be primes of V. Then $(V_\mathfrak{q}, \mathfrak{p}V_\mathfrak{q}) \to (V_\mathfrak{p}, \mathfrak{p}V_\mathfrak{p})$ is an excision datum.

Proof. By replacing V with $V_\mathfrak{q}$, we may assume that $\mathfrak{q}$ is the maximal ideal. $\square$

Main result:

Lemma 10.10. Let $\mathcal{C}$ be an ∞-category that has all small limits. Then any arc-sheaf $F : \mathrm{Sch}_{\mathrm{qcqs}}^{\mathrm{opp}} \to \mathcal{C}$ satisfies excision.

11. Six functors

11.1. **Introduction.** Let C be the category of finite-dimensional locally compact Hausdorff topological spaces. We are interested in the cohomology $H^i(X, \mathbb{Z})$ of such a space X. In this generality, where X might be a Cantor set, the good definition of cohomology is not singular cohomology, but instead the sheaf cohomology. We start with the (Grothendieck) abelian category $\mathrm{Ab}(X)$ of abelian sheaves on X, and the global sections functor

$$H^0(X, -) : \mathrm{Ab}(X) \to \mathrm{Ab}.$$

This is a left exact functor. We can define the full derived functor

$$R\Gamma(X, -) : D(\mathrm{Ab}(X)) \to D(\mathrm{Ab}).$$

We shall write $D(X, \mathbb{Z}) = D(\mathrm{Ab}(X))$ for the (unbounded) derived (∞)-category of abelian sheaves on X. One immediately wonders about the functoriality of the construction. For any $f : Y \to X$, there is an exact pullback functor $f^* : \mathrm{Ab}(X) \to \mathrm{Ab}(Y)$ inducing a functor

$$f^* : D(Y, \mathbb{Z}) \to D(X, \mathbb{Z})$$

which admits a right adjoint

$$f_* : D(X, \mathbb{Z}) \to D(Y, \mathbb{Z}).$$

The cohomology of X can be described in terms of these functors

$$R\Gamma(X, \mathbb{Z}) = f_* \underline{\mathbb{Z}} = f_* f^* \mathbb{Z} \in D(*, \mathbb{Z}) = D(\mathrm{Ab}).$$

The functor f_* is the relative version of cohomology. We should think of it as interpolating the cohomology of all the fibres.

Lemma 11.1 (Proper base change)**.** Let $f : X \to Y$ be a proper map in C. Let $g : Y' \to Y$ be a map in C, with base-change $f' : X' = X \times_Y Y' \to Y'$, as depicted in the following pullback diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

Then the natural map $g^* f_* \to f'_*(g')^*$ of functors $D(X, \mathbb{Z}) \to D(Y', \mathbb{Z})$ is an isomorphism.

Lemma 11.2 (Künneth formula)**.** For proper X and Y, there is a natural isomorphism

$$R\Gamma(X, \mathbb{Z}) \otimes R\Gamma(Y, \mathbb{Z}) \simeq R\Gamma(X \times Y, \mathbb{Z}).$$

Here $\otimes : D(\mathrm{Ab}) \times D(\mathrm{Ab}) \to D(\mathrm{Ab})$ is the tensor product on $D(\mathrm{Ab})$, i.e. the derived tensor product.

To actually formulate this result, we already need the functor

$$- \otimes - : D(X, \mathbb{Z}) \times D(X, \mathbb{Z}) \to D(X, \mathbb{Z})$$

which has a (partial) right adjoint

$$\underline{\mathrm{Hom}}(-, -) : D(X, \mathbb{Z})^{\mathrm{opp}} \times D(X, \mathbb{Z}) \to D(X, \mathbb{Z})$$

characterized by the adjunction

$$\mathrm{Hom}(A, \underline{\mathrm{Hom}}(B, C)) \simeq \mathrm{Hom}(A \otimes B, C).$$

One naturally wonders how these new functors interact with the previous ones. The pullback functor f^* is symmetric monoidal, i.e.

$$f^*(A \otimes B) \simeq f^* A \otimes f^* B.$$

Note that these isomorphisms are actually new data. There is also a compatibility between tensor product and pushforward, again in the proper case.

Lemma 11.3 (Projection formula)**.** Let $f : X \to Y$ be proper. Let $A \in D(X, \mathbb{Z})$ and $B \in D(Y, \mathbb{Z})$. The natural map

$$f_* A \otimes B \to f_*(A \otimes f^* B)$$

is an isomorphism.

This map is adjoint to

$$f^*(f_* A \otimes B) \simeq f^* f_* A \otimes f^* B \to A \otimes f^* B.$$

For any map $f : X \to Y$, there is a "proper pushforward" functor

$$f_! : D(X, \mathbb{Z}) \to D(Y, \mathbb{Z})$$

which admits a right adjoint

$$f^! : D(Y, \mathbb{Z}) \to D(X, \mathbb{Z}).$$

Lemma 11.4 (Verdier duality)**.** Let $f : X \to Y$ be a "manifold bundle" of relative dimension d. Then $f^!$ is isomorphic to $f^* \otimes \omega_{X/Y}$ where $\omega_{X/Y} = f^! \mathbb{Z}$ is locally isomorphic to $\mathbb{Z}[d]$.

A curious phenomenon is that in most cases the association $X \mapsto D(X)$ can be factored as a composite

$$C \to \{\text{analytic stacks}\} \to \mathrm{Cat}.$$

12. Surveys

12.1. Crystalline local systems.

Definition 12.1. A p-adic field is a field of characteristic zero that is complete with respect to a fixed (non-archimedean) discrete valuation such that the residue field is perfect of characteristic $p > 0$.

Let K be a p-adic field with ring of integers $\mathcal{O}_K$ and residue field k.

Let X be a smooth p-adic formal scheme over $\mathcal{O}_K$. Motivated by the de Rham and Hodge theorem in complex geometry, Grothendieck asked if there exists a "mysterious functor" relating the $\mathbb{Q}_p$-étale cohomology of the generic fibre X_η over K, and the crystalline cohomology of the special fibre X_s over k. This is first formulated by Fontaine [Fon82] using the p-adic period ring B_{crys}. Fontaine made the following prediction, now known as the C_{crys}-conjecture.

Conjecture 12.2 (Fontaine)**.** Let X be a smooth proper scheme over $\mathcal{O}_K$. There is a natural isomorphism of B_{crys}-modules

$$H^i($$

Remark 12.3. Approaches:

(1) Associations with filtered F-isocrystals.
(2) Associations in the sense of Faltings.

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