

# THE HODGE–TATE PERIOD MAP

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## 1. TECHNICAL TOOLS

### 1.1. Canonical Subgroups.

**Definition 1.1.1.** Let  $A \rightarrow S$  be an Abelian scheme where  $S$  is over  $\mathrm{Spec}(\mathbb{F}_p)$ . We have Frobenius map  $F : A \rightarrow A^{(p)}$  and Verschiebung map  $V : A^{(p)} \rightarrow A$  with the composition  $V \circ F = [p]$ .

The map  $V : A^{(p)} \rightarrow A$  induces a map  $\omega_{A/S} \rightarrow \omega_{A^{(p)}/S} \simeq \omega_{A/S}^{\otimes p}$ . This gives a canonical section of  $\omega_{A/S}^{\otimes(p-1)}$ , called the Hasse invariant of  $A/S$ , denoted  $\mathrm{Ha}(A/S) \in \Gamma(S, \omega_{A/S}^{\otimes(p-1)})$ .

**TODO:** Rewrite using the “ $O(m, \epsilon)$ ” notation from [zhu].

**Definition 1.1.2.** Let  $R$  be a  $p$ -adically complete flat  $\mathbb{Z}_p^{\mathrm{cycl}}$ -algebra. Let  $A \rightarrow \mathrm{Spec}(R)$  be an Abelian scheme. Let  $m \geq 1$  be an integer. Let  $0 \leq \epsilon < 1/2$  such that  $p^\epsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$  makes sense. We say that  $A \rightarrow \mathrm{Spec}(R)$  satisfies the  $O(m, \epsilon)$  condition if  $\mathrm{Ha}(A_1/\mathrm{Spec}(R_1))^{(p^m-1)/(p-1)}$  divides  $p^\epsilon$  as elements in  $R_1 = R/p$ .

**Lemma 1.1.3.** Let  $R$  be a  $p$ -adically complete flat  $\mathbb{Z}_p^{\mathrm{cycl}}$ -algebra. Let  $A \rightarrow \mathrm{Spec}(R)$  be an Abelian scheme satisfying  $O(m, \epsilon)$ . Then there is a unique closed subgroup  $C_m \subset A[p^m]$  such that  $C_m = \ker(F^m) \bmod p^{1-\epsilon}$ .

**Definition 1.1.4** (scholze/III/2.7). Let  $R$  be a  $p$ -adically complete flat  $\mathbb{Z}_p^{\mathrm{cycl}}$ -algebra. We say that an Abelian scheme  $A \rightarrow \mathrm{Spec}(R)$  has a weak canonical subgroup of level  $m$  if  $A \rightarrow \mathrm{Spec}(R)$  satisfies  $O(m, \epsilon)$  for some  $\epsilon < 1/2$ . In that case, we call  $C_m \subset A[p^m]$  in Lemma 1.1.3 the weak canonical subgroup of level  $m$ .

**TODO: strong  $O(m, \epsilon)$**  If moreover  $\mathrm{Ha}(A_1/\mathrm{Spec}(R_1))^{p^m}$  divides  $p^\epsilon$ , then we say that  $C_m$  is a canonical subgroup. Here,  $R_1 = R/p$  and  $A_1 \rightarrow \mathrm{Spec}(R_1)$  is the reduction of  $A \rightarrow \mathrm{Spec}(R)$ .

**Lemma 1.1.5. TODO: Check this!**

Let  $R$  be a  $p$ -adically complete flat  $\mathbb{Z}_p^{\mathrm{cycl}}$ -algebra. Let  $A \rightarrow \mathrm{Spec}(R)$  be an Abelian scheme satisfying strong  $O(m, \epsilon)$ , with strong canonical subgroup  $C \subset A[p]$  of level 1. Then  $(A/C)_{1-\epsilon} \simeq A_{1-\epsilon}^{(p)}$ , and in particular

$$\mathrm{Ha}((A/C)_{1-\epsilon}/R_{1-\epsilon}) = \mathrm{Ha}(A_{1-\epsilon}/R_{1-\epsilon})^p.$$

**Lemma 1.1.6** (scholze/III/2.8). Let  $R$  be a  $p$ -adically complete flat  $\mathbb{Z}_p^{\text{cycl}}$ -algebra. Let  $A$  and  $B$  be Abelian schemes over  $R$ .

- (1) If  $A$  has a canonical subgroup  $C_m \subset A[p^m]$  of level  $m$ , then it has a canonical subgroup  $C_{m'} \subset A[p^{m'}]$  of every level  $m' \leq m$ , and  $C_{m'} \subset C_m$ .
- (2) Let  $f : A \rightarrow B$  be a map of Abelian schemes. Assume that both  $A$  and  $B$  have canonical subgroups  $C_m \subset A[p^m]$  and  $D_m \subset B[p^m]$  of level  $m$ . Then  $C_m$  maps into  $D_m$  under  $f$ .
- (3) Assume that  $A$  has a canonical subgroup  $C_m \subset A[p^m]$  of level  $m$ , and let  $\bar{x}$  be a geometric point of  $\text{Spec}(R[p^{-1}])$ . Then  $C_m(\bar{x}) \simeq (\mathbb{Z}/p^m\mathbb{Z})^g$ , where  $g$  is dimension of the Abelian variety over  $\bar{x}$ .

## 1.2. Hartog's Extension Principle.

**Lemma 1.2.1** ([SGA2, Lemma III.3.1, Proposition III.3.3]). Let  $X$  be a locally Noetherian scheme. Let  $Z \subset X$  be a closed subscheme. Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module. Let  $n \geq 1$  be an integer. Then the following are equivalent:

- (1) For any open subscheme  $V$  of  $X$ , the map

$$H^i(V, \mathcal{F}) \rightarrow H^i(V \setminus Z, \mathcal{F})$$

is bijective for  $i \leq n - 2$  and injective for  $i = n - 1$ .

- (2) For any open subscheme  $V$  of  $X$ , the local cohomology

$$H_{V \cap Z}^i(V, \mathcal{F}) = 0$$

for all  $i \leq n - 1$ .

- (3) For any  $x \in Z$  the depth of  $\mathcal{F}_x$  as an  $\mathcal{O}_{X,x}$ -module is at least  $n$ .

**Lemma 1.2.2** (Serre's criterion). A Noetherian ring  $R$  is normal if and only if  $R_{\mathfrak{p}}$  is regular for every  $\mathfrak{p}$  of height  $\leq 1$  and  $R_{\mathfrak{p}}$  has depth  $\geq 2$  for every  $\mathfrak{p}$  of height  $\geq 2$ .

**Lemma 1.2.3** (gortz-wedhorn-1/6.45). Let  $X$  be a locally Noetherian normal scheme. Let  $U$  be an open subscheme of  $X$  with codimension  $\geq 2$ . Then the map  $H^0(X, \mathcal{O}_X) \rightarrow H^0(U, \mathcal{O}_X)$  is an isomorphism.

*Proof.* We may assume that  $X = \text{Spec}(A)$  where  $A$  is normal integral domain. For every non-empty open  $V$  of  $X$ , the ring  $\Gamma(V, \mathcal{O}_X)$  may be considered as a subring of the function field  $K(X) = \text{Frac}(A)$  such that the restriction maps are given by inclusions of rings. Let  $Z$  be an irreducible closed subset of  $X$  of codimension 1. Then  $U$  intersects  $Z$  non-trivially, so it contains the generic point  $\eta$  of  $Z$ . In other words, the subring  $\Gamma(U, \mathcal{O}_X)$  of the function field  $K(X)$  is contained in the stalk  $\mathcal{O}_{X,\eta}$ . But  $A = \Gamma(X, \mathcal{O}_X)$  is the intersection of all the stalks  $\mathcal{O}_{X,\eta}$ , where  $\eta$  is a prime ideal of height 1; in other words, where  $\eta$  is the generic point of an irreducible closed subset of codimension 1.  $\square$

**Lemma 1.2.4** (scholze/III/2.9; zhu/3.10.4). Let  $R$  be a normal ring, i.e. the localization  $R_{\mathfrak{p}}$  is an integrally closed domain for every prime ideal  $\mathfrak{p}$  of  $R$ . Assume  $R$  is Noetherian. Let  $Z \subset \text{Spec}(R)$  be a closed subscheme of codimension at least 2, i.e. every  $\mathfrak{p} \in Z$  has height at least 2. Then for  $U = \text{Spec}(R) \setminus Z$ ,

$$H^0(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)}) \simeq H^0(U, \mathcal{O}_{\text{Spec}(R)}).$$

*Proof.* Consider  $n = 2$  and  $\mathcal{F} = \mathcal{O}_X$  in Lemma 1.2.1. Serre's criterion, cf. Lemma 1.2.2, guarantees the third condition in Lemma 1.2.1. The first assertion gives the desired result.  $\square$

**Lemma 1.2.5** (scholze/III/2.10; zhu/3.10.5). Let  $R$  be a topologically finitely generated, flat, and  $p$ -adically complete  $\mathbb{Z}_p$ -algebra, such that  $\bar{R} = R/p$  is normal. Fix  $f \in R$  such that its reduction  $\bar{f} \in \bar{R}$  is not a zero-divisor. Let  $0 < \epsilon \leq 1$ . Set  $S = (R \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\text{cycl}}) \langle u \rangle / (u \cdot f - p^\epsilon)$ . Then  $S$  is  $p$ -adically complete and flat over  $\mathbb{Z}_p^{\text{cycl}}$ . Fix a closed subscheme  $Y \subset \text{Spec}(\bar{R})$  of codimension  $\geq 2$ . Let  $Z$  be the inverse image of  $Y$  in  $\text{Spf}(S)$ . Then for  $U = |\text{Spf}(S)| \setminus Z$ ,

$$S = H^0(\text{Spf}(S), \mathcal{O}_{\text{Spf}(S)}) \simeq H^0(U, \mathcal{O}_{\text{Spf}(S)}).$$

*Proof.* We first show that the map

$$S \simeq H^0(\text{Spf}(S), \mathcal{O}_{\text{Spf}(S)}) \rightarrow H^0(U, \mathcal{O}_{\text{Spf}(S)})$$

is injective. Since  $S$  is  $p$ -adically separated and  $H^0(U, \mathcal{O}_{\text{Spf}(S)})$  is flat over  $\mathbb{Z}_p^{\text{cycl}}$ , it suffices to show that

$$S_\epsilon \simeq H^0(\text{Spec}(S_\epsilon), \mathcal{O}_{\text{Spec}(S_\epsilon)}) \rightarrow H^0(U_\epsilon, \mathcal{O}_{\text{Spec}(S_\epsilon)})$$

is injective, where  $S_\epsilon = S/p^\epsilon$ ,  $Z_\epsilon$  is the inverse image of  $Y$  in  $\text{Spec}(S_\epsilon)$ , and  $U_\epsilon = \text{Spec}(S_\epsilon) \setminus Z_\epsilon$ . Note that

$$S_\epsilon = (R \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\text{cycl}}) \langle u \rangle / (uf, p^\epsilon) = R_\epsilon[u] / (uf_\epsilon)$$

where  $R_\epsilon = R \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p^{\text{cycl}}/p^\epsilon)$  and  $f_\epsilon \in R_\epsilon$  is the image of  $f \in R$ .

Let  $W \subset \text{Spec}(S_\epsilon)$  be the preimage of  $V = V(\bar{f}) \subset \text{Spec}(\bar{R})$ . Then  $W = V \times_{\text{Spec}(\mathbb{F}_p)} \mathbb{A}_{\mathbb{Z}_p^{\text{cycl}}/p^\epsilon}^1$  is affine. The map  $S_\epsilon \rightarrow R_\epsilon$  sending  $u$  to zero induces a section  $\text{Spec}(R_\epsilon) \rightarrow \text{Spec}(S_\epsilon)$ . We have a decomposition  $\text{Spec}(S_\epsilon) = N \cup W$ , where  $N = \text{Spec}(R_\epsilon[u]/(u)) \simeq \text{Spec}(R_\epsilon)$  is the image of the section  $\text{Spec}(R_\epsilon) \rightarrow \text{Spec}(S_\epsilon)$ . Take  $V_\epsilon = V \times_{\text{Spec}(\mathbb{F}_p)} \text{Spec}(\mathbb{Z}_p^{\text{cycl}}/p^\epsilon)$ . Then  $W = V_\epsilon \times_{\text{Spec}(\mathbb{Z}_p^{\text{cycl}}/p^\epsilon)} \mathbb{A}_{\mathbb{Z}_p^{\text{cycl}}/p^\epsilon}^1$ , and  $N \cap W = V_\epsilon$ .

We then have the following interpretations:

- (1) Each section in  $\Gamma(\text{Spec}(S_\epsilon), \mathcal{O}_{\text{Spec}(S_\epsilon)})$  is a pair  $(f_1, f_2)$  such that  $f_1 \in \Gamma(N, \mathcal{O}_{\text{Spec}(S_\epsilon)})$  and  $f_2 \in \Gamma(W, \mathcal{O}_{\text{Spec}(S_\epsilon)})$  such that  $f_1 = f_2$  on  $N \cap W = V_\epsilon$ .
- (2) Each section in  $H^0(U_\epsilon, \mathcal{O}_{\text{Spec}(S_\epsilon)})$  is a pair  $(f_1, f_2)$  such that  $f_1 \in H^0(U_\epsilon \cap N, \mathcal{O}_{\text{Spec}(S_\epsilon)})$ , and  $f_2 \in H^0(U_\epsilon \cap W, \mathcal{O}_{\text{Spec}(S_\epsilon)})$ , such that  $f_1 = f_2$  on  $U_\epsilon \cap N \cap W$ .

The (classical) Hartog's extension principle, i.e. Lemma 1.2.4 applied to  $Y \subset \text{Spec}(\bar{R})$ , shows that

$$\Gamma(\text{Spec}(\bar{R}) \setminus Y) \simeq \Gamma(\text{Spec}(\bar{R})).$$

Under base-change this gives

$$\Gamma(U_\epsilon \cap N) \simeq \Gamma(\text{Spec}(\bar{Y}) \setminus Y) \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\text{cycl}}/p^\epsilon \simeq \Gamma(\text{Spec}(\bar{R})) \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\text{cycl}}/p^\epsilon \simeq \Gamma(N).$$

Thus injectivity reduces to show that

$$\Gamma(V) \otimes_{\mathbb{F}_p} (\mathbb{Z}_p^{\text{cycl}}/p^\epsilon)[u] \Gamma(W) \rightarrow \Gamma(U_\epsilon \cap W) = \Gamma(V \setminus Y) \otimes_{\mathbb{F}_p} (\mathbb{Z}_p^{\text{cycl}}/p^\epsilon)[u]$$

is injective. It suffices to show that  $\Gamma(V) \rightarrow \Gamma(V \setminus Y)$  is injective, where both  $V$  and  $V \setminus Y$  are  $\mathbb{F}_p$ -schemes. We have  $\text{depth}(\mathcal{O}_{V,y}) = \text{depth}(\bar{R}_y) - 1$  for all  $y \in V$ , cf. [SGA-2/III/2.5]. Thus  $\text{depth}(\mathcal{O}_{V,y}) \geq 1$  for every  $x \in V \cap Y$  by Serre's criterion, i.e. Lemma 1.2.2. Then the desired injectivity follows from Lemma 1.2.1.

We then show the surjectivity.

TODO

□

### 1.3. Tate's Normalized Traces.

**Lemma 1.3.1** (scholze/III/2.21; zhu/3.12.2). Let  $R$  be a  $p$ -adically complete flat  $\mathbb{Z}_p$ -algebra. Let  $Y_1, \dots, Y_n \in R$ . Let  $P_1, \dots, P_n \in R \langle X_1, \dots, X_n \rangle$  be topologically nilpotent elements, or equivalently, each  $P_i$  has topologically nilpotent coefficients in  $R$ . Let

$$S = R \langle X_1, \dots, X_n \rangle / (X_1^p - Y_1 - P_1, \dots, X_n^p - Y_n - P_n).$$

Then

- (1) The ring  $S$  is a finite free  $R$ -module of rank  $p^n$ , with a basis given by  $X_1^{i_1} \cdots X_n^{i_n}$  with  $0 \leq i_1, \dots, i_n \leq p-1$ .
- (2) Let  $I$  be the ideal of  $R$  generated by  $p$  together with all the coefficients of all  $P_i$ . Then the trace map  $\text{tr}_{S/R} : S \rightarrow R$  sends  $S$  to  $I^n$ , i.e.  $\text{tr}_{S/R}(S) \subset I^n$ .

**Lemma 1.3.2** (scholze/III/2.22; zhu/3.12.4). Let  $R$  be a  $p$ -adically complete flat  $\mathbb{Z}_p$ -algebra topologically of finite type, formally smooth of dimension  $n$  over  $\mathbb{Z}_p$ . Let  $f \in R$  such that its reduction  $\bar{f} \in \bar{R} = R/p$  is not a zero-divisor. Let  $0 \leq \epsilon < 1/2$ . Let

$$S_\epsilon = (R \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\text{cycl}}) \langle u_\epsilon \rangle / (u_\epsilon \cdot f - p^\epsilon).$$

Suppose  $\varphi : S_\epsilon \rightarrow S_{\epsilon/p}$  is a map of  $\mathbb{Z}_p^{\text{cycl}}$ -algebra such that modulo  $p^{1-\epsilon}$  it is given by the relative Frobenius. In other words,  $\varphi \bmod p^{1-\epsilon}$  is the map

$$R_{1-\epsilon}[u_\epsilon] / (f \cdot u_\epsilon - p^\epsilon) \rightarrow R_{1-\epsilon}[u_{\epsilon/p}] / (f \cdot u_{\epsilon/p} - p^{\epsilon/p}),$$

where  $R_{1-\epsilon} = \bar{R} \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p^{\text{cycl}}/p^{1-\epsilon})$ , which sends  $u_\epsilon$  to  $u_{\epsilon/p}^p$ , and restricts to  $\text{Fr}_{\bar{R}} \otimes \text{id}$  on  $R_{1-\epsilon}$ . Then

- (1) The map

$$\varphi[1/p] : S_\epsilon[1/p] \rightarrow S_{\epsilon/p}[1/p]$$

is finite and flat of degree  $p^n$ .

(2) The trace map

$$\mathrm{tr} = \mathrm{tr}_{S_{\epsilon/p}[1/p]/S_{\epsilon}[1/p]} : S_{\epsilon/p}[1/p] \rightarrow S_{\epsilon}[1/p]$$

sends  $S_{\epsilon/p}$  into  $p^{n-(2n+1)\epsilon}S_{\epsilon}$ . Here  $S_{\epsilon/p}[1/p]$  is viewed as an  $S_{\epsilon}[1/p]$ -algebra via  $\varphi[1/p]$ .

#### 1.4. Riemann's Hebbbarkeitssatz.

**Definition 1.4.1** (scholze/II/3.8). Let  $p$  be a prime. Let  $K$  be a perfectoid field (of any characteristic). Let  $t$  be a non-zero element of  $K$  with  $|p| \leq |t| < 1$ . A triple  $(\mathcal{X}, \mathcal{Z}, \mathcal{U})$ , where  $\mathcal{X}$  is an affinoid perfectoid space over  $K$ ,  $\mathcal{Z}$  is a closed subset of  $\mathcal{X}$ , and  $\mathcal{U}$  is a quasi-compact open subset of  $\mathcal{X} \setminus \mathcal{Z}$ , is said to be good, if

$$H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^+/t)^a \simeq H^0(\mathcal{X} \setminus \mathcal{Z}, \mathcal{O}_{\mathcal{X}}^+/t)^a \hookrightarrow H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}}^+/t)^a.$$

**Remark 1.4.2.** This notion is independent of the choice of  $t$ , and is compatible with tilting.

**Situation 1.4.3.** Let  $K = \mathbb{F}_p((t^{1/p^\infty}))$ . Let  $R_0$  be a reduced Tate  $K$ -algebra topologically of finite type. Let  $\mathcal{X}_0 = \mathrm{Spa}(R_0, R_0^\circ)$  be the associated affinoid adic space of finite type over  $K$ . Let  $R$  be the completed perfection of  $R_0$ , which is a  $p$ -finite perfectoid  $K$ -algebra. Let  $\mathcal{X} = \mathrm{Spa}(R, R^+)$  with  $R^+ = R^\circ$ , the associated  $p$ -finite affinoid perfectoid space over  $K$ . Let  $I_0$  be an ideal of  $R_0$ . Let  $I = I_0 R \subset R$ . Let  $\mathcal{Z}_0 = V(I_0) \subset \mathcal{X}_0$ . Let  $\mathcal{Z} = V(I) \subset \mathcal{X}$ . Let  $\mathcal{U}_0$  be a quasi-compact open subset of  $\mathcal{X}_0 \setminus \mathcal{Z}_0$  with preimage  $\mathcal{U} \subset \mathcal{X} \setminus \mathcal{Z}$ .

**Lemma 1.4.4** (scholze/II/3.10). Assume Situation 1.4.3. Suppose  $(\mathcal{X}, \mathcal{Z}, \mathcal{U})$  is good. Suppose that  $R_0$  is normal, and that  $V(I_0) \subset \mathrm{Spec}(R_0)$  is of codimension  $\geq 2$ . Let  $R'_0$  be a finite normal  $R_0$ -algebra which is étale outside  $V(I_0)$ , and such that no irreducible component of  $\mathrm{Spec}(R'_0)$  maps into  $V(I_0)$ . Let  $I'_0 = I_0 R'_0$ , and  $\mathcal{U}'_0 \subset \mathcal{X}'_0$  the preimage of  $\mathcal{U}_0$ . Let  $R', I', \mathcal{X}', \mathcal{Z}', \mathcal{U}'$  be the associated perfectoid objects.

(1) There is a perfect trace pairing

$$\mathrm{tr}_{R'_0/R_0} : R'_0 \otimes_{R_0} R'_0 \rightarrow R_0.$$

(2) The trace pairing induces a trace pairing

$$\mathrm{tr}_{R'^\circ/R^\circ} : R'^\circ \otimes_{R^\circ} R'^\circ \rightarrow R^\circ.$$

which is almost perfect.

(3) For all open subsets  $\mathcal{V} \subset \mathcal{X}$  with preimage  $\mathcal{V}' \subset \mathcal{X}'$ , the trace pairing induces an isomorphism

$$H^0(\mathcal{V}', \mathcal{O}_{\mathcal{X}'}^+/t)^a \simeq \mathrm{Hom}_{R^\circ/t}(R'^\circ/t, H^0(\mathcal{V}, \mathcal{O}_{\mathcal{X}}^+/t))^a.$$

(4) The triple  $(\mathcal{X}', \mathcal{Z}', \mathcal{U}')$  is good.

(5) If  $\mathcal{X}' \rightarrow \mathcal{X}$  is surjective, then the map

$$H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^+/t) \rightarrow H^0(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}^+/t) \cap H^0(\mathcal{U}, \mathcal{O}_{\mathcal{X}}^+/t)$$

is an almost isomorphism.

**Lemma 1.4.5** (scholze/II/3.11). Suppose we have a filtered inductive system  $(R_0^{(i)})_{i \in I}$  as in the previous lemma, giving rise to  $\mathcal{X}^{(i)}, \mathcal{Z}^{(i)}, \mathcal{U}^{(i)}$ . Assume that all transition maps  $\mathcal{X}^{(i)} \rightarrow \mathcal{X}^{(j)}$  are surjective. Let  $\tilde{\mathcal{X}}$  be the inverse limit of the  $\mathcal{X}^{(i)}$  in the category of perfectoid spaces over  $K$ , with preimage  $\tilde{\mathcal{Z}} \subset \tilde{\mathcal{X}}$  of  $\mathcal{Z}$ , and  $\tilde{\mathcal{U}} \subset \tilde{\mathcal{X}} \setminus \tilde{\mathcal{Z}}$  of  $\mathcal{U}$ . Then the triple  $(\tilde{\mathcal{X}}, \tilde{\mathcal{Z}}, \tilde{\mathcal{U}})$  is good.

#### 1.5. The Hodge–Tate Filtration.

**Lemma 1.5.1.** Let  $C$  be an algebraically closed and complete extension of  $\mathbb{Q}_p$ . Let  $A \rightarrow \mathrm{Spec}(C)$  be an Abelian variety. Then  $A$  has its Hodge–Tate filtration

$$0 \rightarrow \mathrm{Lie}(A)(1) \rightarrow T_p(A) \otimes_{\mathbb{Z}_p} C \rightarrow (\mathrm{Lie}(A^\vee))^* \rightarrow 0.$$

TODO

## 2. SIEGEL MODULAR VARIETIES

Let  $p$  be a fixed prime. Let  $g \geq 1$  be an integer.

**Definition 2.0.1.** The symplectic similitude group  $\mathrm{GSp}_{2g}$  is the reductive group scheme over  $\mathbb{Z}$  whose points in a commutative ring  $R$  are given by

$$\mathrm{GSp}_{2g}(R) = \{x \in \mathrm{GL}_{2g}(V); \exists \nu(x) \in R^\times, x^t \Omega x = \nu(x) \Omega\}$$

where  $\Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$  is the standard symplectic matrix of order  $2g$ .

In the following discussion, we write  $G = \mathrm{GSp}_{2g}$ . Let  $K_p = G(\mathbb{Z}_p)$ . Let  $K^p$  be a compact open subgroup of  $G(\mathbb{A}^{\infty,p})$  that is contained in

$$\Gamma(N)^{(p)} = \{g \in G(\mathbb{A}^{\infty,p}); g \equiv 1 \pmod{N}\}$$

for some integer  $N \geq 3$  not divisible by  $p$ .

**Definition 2.0.2.** Let  $m \geq 1$  be an integer.

$$\begin{aligned} \Gamma_0(p^m) &= \left\{ g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \pmod{p^m} \right\} \\ \Gamma_s(p^m) &= \left\{ g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \pmod{p^m}, \nu(g) \equiv 1 \pmod{p^m} \right\} \\ \Gamma_1(p^m) &= \left\{ g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \pmod{p^m} \right\} \\ \Gamma(p^m) &= \left\{ g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{p^m} \right\} \end{aligned}$$

Let  $X$  be the scheme over  $\mathrm{Spec}(\mathbb{Z}_{(p)})$  classifying principally polarized (projective?) Abelian schemes of relative dimension  $g$  with level  $K^p$  structures. Let  $X^*$  be the minimal compactification of  $X$ .

**TODO: Add references.**

For each  $U \in \{\Gamma(p^m), \Gamma_s(p^m), \Gamma_0(p^m)\}$ , we have a scheme  $X_{U,\mathbb{Q}}$  over  $\mathbb{Q}$  with certain moduli interpretations.

Let  $\mathfrak{X}$  be the formal scheme over  $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$  defined as the  $p$ -completion of  $X_{\mathbb{Z}_p^{\mathrm{cycl}}} = X \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} \mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$ .

The universal Abelian scheme  $A \rightarrow X$  gives a line bundle  $\omega = \omega_{A/S} = \bigwedge^g \Omega_{A/X}^1$ . The sheaf  $\omega$  extends to the minimal compactification  $X^*$ . The Hasse invariant defines a section  $\mathrm{Ha} \in H^0(X_{\mathbb{F}_p}, \omega^{\otimes(p-1)})$ . The section  $\mathrm{Ha}$  extends to  $\mathrm{Ha} \in H^0(X_{\mathbb{F}_p}^*, \omega^{\otimes(p-1)})$ . For  $g = 1$ , this follows from direct inspection. For  $g \geq 2$ , this follows from the (classical) Hartog's extension principle. **TODO: Clarify this paragraph.**

Let  $\mathfrak{A} \rightarrow \mathfrak{X}$  be the universal formal Abelian scheme.

3. THE  $\epsilon$ -NEIGHBOURHOODS

**Lemma 3.0.1.** Let  $S$  be a  $p$ -adically complete  $\mathbb{Z}_p^{\mathrm{cycl}}$ -algebra. There is a bijection

$$\mathrm{Hom}_{\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})}(\mathrm{Spf}(S), \mathfrak{X}^*) \simeq \mathrm{Hom}_{\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})}(\mathrm{Spec}(S), X_{\mathbb{Z}_p^{\mathrm{cycl}}}^*).$$

**Definition 3.0.2.** Let  $0 \leq \epsilon < 1$  such that there exists an element  $p^\epsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$  with  $p$ -adic valuation  $\epsilon$ . Let  $\mathcal{M}_\epsilon$  be the functor sending a  $p$ -adically complete flat  $\mathbb{Z}_p^{\mathrm{cycl}}$ -algebra  $S$  to the set of pairs  $(f, [u])$ , where

- $f$  is a map  $\mathrm{Spf}(S) \rightarrow \mathfrak{X}^*$ ; it's equivalent to a map  $\mathrm{Spec}(S) \rightarrow X_{\mathbb{Z}_p^{\mathrm{cycl}}}^*$  by Lemma 3.0.1.
- Let  $\bar{f} : \mathrm{Spec}(S/p) \rightarrow X_{\mathbb{F}_p}^*$  be the reduction of  $\mathrm{Spec}(S) \rightarrow X_{\mathbb{Z}_p^{\mathrm{cycl}}}^*$ . Recall that we have the Hasse section  $\mathrm{Ha} \in H^0(X_{\mathbb{F}_p}^*, \omega^{\otimes(p-1)})$ . It pullbacks to  $\bar{f}^* \mathrm{Ha} \in H^0(\mathrm{Spec}(S/p), \bar{f}^* \omega^{\otimes(p-1)})$ . Then  $[u]$  is an equivalence class of sections  $u \in H^0(\mathrm{Spec}(S/p), \bar{f}^* \omega^{\otimes(1-p)})$  satisfying  $u \cdot \bar{f}^* \mathrm{Ha} = p^\epsilon \in S_1$  under the equivalence relation that  $u \sim u'$  if and only if there exists some  $h \in S$  such that  $u' = u(1 + p^{1-\epsilon}h)$ .

**Lemma 3.0.3.** Let  $0 \leq \epsilon < 1$  such that  $p^\epsilon \in \mathbb{Z}_p^{\text{cycl}}$  makes sense. Then the functor  $\mathcal{M}_\epsilon$  is representable by a formal scheme flat over  $\text{Spf}(\mathbb{Z}_p^{\text{cycl}})$ . *Moreover, locally it is ...*

Let  $\mathfrak{X}(\epsilon) \rightarrow \mathfrak{X}$  be the pullback of  $\mathfrak{X}^*(\epsilon) \rightarrow \mathfrak{X}^*$  along  $\mathfrak{X} \rightarrow \mathfrak{X}^*$ . Let  $\mathfrak{A}(\epsilon) \rightarrow \mathfrak{X}(\epsilon)$  be the pullback of  $\mathfrak{A} \rightarrow \mathfrak{X}$  along  $\mathfrak{X}(\epsilon) \rightarrow \mathfrak{X}$ .

**Remark 3.0.4.** *TODO: moduli interpretation of  $\mathfrak{X}(\epsilon)$ . Should be almost identical to  $\mathcal{M}_\epsilon$ .*

**Definition 3.0.5.** For a formal scheme  $\mathfrak{Y}$  over  $\mathbb{Z}_p^{\text{cycl}}$  and  $a \in \mathbb{Z}_p^{\text{cycl}}$ , we write  $\mathfrak{Y}/a$  for  $\mathfrak{Y} \times_{\text{Spf}(\mathbb{Z}_p^{\text{cycl}})} \text{Spf}(\mathbb{Z}_p^{\text{cycl}}/a)$ .

**Definition 3.0.6.** For a formal scheme  $\mathfrak{Y}$  over  $\mathbb{Z}_p^{\text{cycl}}/p$ , we write  $\mathfrak{Y}^{(p)}$  for the pullback of  $\mathfrak{Y}$  along the (absolute) Frobenius  $\text{Spf}(\mathbb{Z}_p^{\text{cycl}}/p) \rightarrow \text{Spf}(\mathbb{Z}_p^{\text{cycl}}/p)$ .

**Lemma 3.0.7.** Let  $0 \leq \epsilon < 1$  such that  $p^\epsilon \in \mathbb{Z}_p^{\text{cycl}}$  makes sense. We have a natural isomorphism

$$(\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \simeq \mathfrak{X}^*(\epsilon)/p$$

of formal schemes over  $\text{Spf}(\mathbb{Z}_p^{\text{cycl}}/p)$ . Furthermore, by pullback we get the following commutative diagram

$$\begin{array}{ccccc} (\mathfrak{A}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{A}(\epsilon)/p & \longrightarrow & \mathfrak{X}(\epsilon)/p & \longrightarrow & \mathfrak{X}^*(\epsilon)/p \end{array}$$

where each vertical map is an isomorphism.

*Proof.* *TODO: This should be checked using moduli interpretations.* □

**Lemma 3.0.8.** Let  $0 \leq \epsilon < 1$  such that  $p^\epsilon \in \mathbb{Z}_p^{\text{cycl}}$  makes sense. The Frobenius map  $\text{Spf}(\mathbb{Z}_p^{\text{cycl}}/p) \rightarrow \text{Spf}(\mathbb{Z}_p^{\text{cycl}}/p)$  induces the following commutative diagram

$$\begin{array}{ccccc} \mathfrak{A}(p^{-1}\epsilon)/p & \longrightarrow & \mathfrak{X}(p^{-1}\epsilon)/p & \longrightarrow & \mathfrak{X}^*(p^{-1}\epsilon)/p \\ \downarrow & & \downarrow & & \downarrow \\ (\mathfrak{A}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \end{array}$$

**Lemma 3.0.9.** Let  $0 \leq \epsilon < 1/2$  such that  $p^\epsilon \in \mathbb{Z}_p^{\text{cycl}}$  makes sense. There is a unique commutative diagram

$$\begin{array}{ccccc} \mathfrak{A}(p^{-1}\epsilon) & \longrightarrow & \mathfrak{X}(p^{-1}\epsilon) & \longrightarrow & \mathfrak{X}^*(p^{-1}\epsilon) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{A}(\epsilon) & \longrightarrow & \mathfrak{X}(\epsilon) & \longrightarrow & \mathfrak{X}^*(\epsilon) \end{array}$$

that gets identified with the following commutative diagram, cf. Lemma ? and Lemma ? after modulo  $p^{1-\epsilon}$

$$\begin{array}{ccccc} \mathfrak{A}(p^{-1}\epsilon)/p & \longrightarrow & \mathfrak{X}(p^{-1}\epsilon)/p & \longrightarrow & \mathfrak{X}^*(p^{-1}\epsilon)/p \\ \downarrow & & \downarrow & & \downarrow \\ (\mathfrak{A}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{A}(\epsilon)/p & \longrightarrow & \mathfrak{X}(\epsilon)/p & \longrightarrow & \mathfrak{X}^*(\epsilon)/p \end{array}$$

*Proof.* *TODO: The map  $\mathfrak{X}(p^{-1}\epsilon) \rightarrow \mathfrak{X}(\epsilon)$  comes from the moduli interpretation, the weak canonical subgroup, and the Hasse invariant. Then  $\mathfrak{A}(p^{-1}\epsilon) \rightarrow \mathfrak{A}(\epsilon)$  is obtained by base-change. The extension to  $\mathfrak{X}^*(p^{-1}\epsilon) \rightarrow \mathfrak{X}^*(\epsilon)$  is done using Hartog's extension principle.*

We first construct the map  $\mathfrak{X}(p^{-1}\epsilon) \rightarrow \mathfrak{X}(\epsilon)$ . Let  $S$  be a  $p$ -adically complete flat  $\mathbb{Z}_p^{\text{cycl}}$ -algebra. Let  $(f, [u])$  be a pair where

- $f : \text{Spf}(S) \rightarrow \mathfrak{X}$  is a map of formal schemes over  $\text{Spf}(\mathbb{Z}_p^{\text{cycl}})$ ; its equivalent to a map  $f : \text{Spec}(S) \rightarrow X_{\mathbb{Z}_p^{\text{cycl}}}$ .

- $u \in H^0(\mathrm{Spec}(S/p), \bar{f}^* \omega^{\otimes(1-p)})$  is a section such that  $u \cdot \bar{f}^* \mathrm{Ha} = p^{p^{-1}\epsilon} \in S/p$ .

The map  $f : \mathrm{Spec}(S) \rightarrow X_{\mathbb{Z}_p^{\mathrm{cycl}}}$  gives an Abelian scheme  $A \rightarrow \mathrm{Spec}(S)$  with principal polarization and level  $K^p$  structure. We claim that  $A \rightarrow \mathrm{Spec}(S)$  satisfies strong  $O(1, \epsilon)$ , i.e.  $\mathrm{Ha}(A_1/\mathrm{Spec}(S_1))^p$  divides  $p^\epsilon$ . This follows from

$$p^{p^{-1}\epsilon} = u \cdot \bar{f}^* \mathrm{Ha} = u \cdot \mathrm{Ha}(A_1/\mathrm{Spec}(S_1)).$$

Let  $C \subset A[p]$  be the strong canonical subgroup of level 1. We get an Abelian scheme  $A/C \rightarrow \mathrm{Spec}(S)$  equipped with induced polarization and level structure **TODO: Clarify; use totally isotropic**, which corresponds to a map  $g : \mathrm{Spec}(S) \rightarrow X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ . Then we declare that the pair  $(f, [u])$  gets mapped to the pair  $(g, [u^p])$ .  $\square$

**Lemma 3.0.10.** Let  $0 \leq \epsilon < 1$  such that  $p^\epsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$  makes sense. Let  $m \geq 0$ . The formal Abelian scheme  $\mathfrak{A}(p^{-m}\epsilon) \rightarrow \mathfrak{X}(p^{-m}\epsilon)$  admits a canonical subgroup  $C_m \subset \mathfrak{A}(p^{-m}\epsilon)[p^m]$  of level  $m$ , in the sense that **TODO: reduce to non-formal situation and explain**. This induces a map  $\mathcal{X}(p^{-m}\epsilon) \rightarrow \mathcal{X}_{\Gamma_s(p^m)}$  of the adic generic fiber given by the pair  $(\mathcal{A}(p^{-m}\epsilon)/C_m, \mathcal{A}(p^{-m}\epsilon)[p^m]/C_m)$ . This map extends uniquely to a map  $\mathcal{X}^*(p^{-m}\epsilon) \rightarrow \mathcal{X}_{\Gamma_s(p^m)}^*$ . The two maps are both open immersions of adic spaces. Moreover, for  $m \geq 1$ , the diagram

$$\begin{array}{ccc} \mathcal{X}^*(p^{-m-1}\epsilon) & \longrightarrow & \mathcal{X}_{\Gamma_s(p^{m+1})}^* \\ \downarrow & & \downarrow \\ \mathcal{X}^*(p^{-m}\epsilon) & \longrightarrow & \mathcal{X}_{\Gamma_s(p^m)}^* \end{array}$$

is a pullback diagram, where the vertical map on the left is induced from the map  $\mathfrak{X}(p^{-m-1}\epsilon) \rightarrow \mathfrak{X}(p^{-m}\epsilon)$ , cf. Lemma ??.

#### 4. THE ANTI-CANONICAL TOWERS

#### 5. THE TOPOLOGICAL HODGE–TATE MAP

#### 6. THE PERFECTOID HODGE–TATE MAP

#### APPENDIX A. REVIEW OF ABELIAN SCHEMES

**Definition A.0.1.** Let  $S$  be a scheme. An Abelian scheme over  $S$  is a proper smooth group scheme over  $S$  that is geometrically connected.

#### APPENDIX B. REVIEW OF DEFORMATION THEORY

**Definition B.0.1** ([Ill71, pp. II.1.2.1, II.1.2.3]). Let  $A \rightarrow B$  be a map of rings. The simplicial  $A$ -algebra  $P_A(B)$  is defined by  $P_A(B)_0 = A[B]$  and  $P_A(B)_n = A[P_A(B)_{n-1}]$  for  $n \geq 1$ . The standard resolution of  $B$  over  $A$  is the argumentation  $P_A(B) \rightarrow B$  where  $B$  is viewed as a constant simplicial  $A$ -algebra. The cotangent complex of  $B$  over  $A$  is the simplicial  $B$ -module  $L_{B/A} = \Omega_{P_A(B)/A}^1 \otimes_{P_A(B)} B$ .

**Remark B.0.2.** This definition works in a general topos.

**Definition B.0.3** ([Ill72, p. VII.1.1.1]). Let  $S$  be a scheme. Let  $S_{\mathrm{zar}}$  be the small Zariski site over  $S$ . Let  $S_{\mathrm{fpqc}}$  be the big fpqc site over  $S$ . The natural inclusion  $S_{\mathrm{zar}} \rightarrow S_{\mathrm{fpqc}}$  induces a geometric map  $(\epsilon^*, \epsilon_*) : \mathrm{Sh}(S_{\mathrm{zar}}) \rightleftarrows \mathrm{Sh}(S_{\mathrm{fpqc}})$ .

**Definition B.0.4.** Let  $f : X \rightarrow Y$  be a map of schemes. The cotangent complex is  $L_{X/Y}$ .

**Definition B.0.5.** Let  $S$  be a scheme. Let  $G$  be a group scheme over  $S$  that is flat and locally of finite presentation. Let  $e : S \rightarrow G$  be the unit. The co-Lie complex is  $\ell_G = Le^* L_{G/S}$ , and the Lie complex is  $\ell_G^\vee = R\mathrm{Hom}(\ell_G, \mathcal{O}_S)$ . Define  $\underline{\ell}_G = L\epsilon^* \ell_G$ .

**Lemma B.0.6** ([Ill72, Theorem VII.4.2.5]). Let  $f : S \rightarrow T$  be a map of schemes. Let  $i : S \rightarrow S'$  be a  $T$ -extension by a quasi-coherent module  $I$ . Let  $A$  be a “schéma en anneaux” over  $T$  that is, as a scheme over  $T$ , tor-independent (c.f. [SGA6, Definition III.1.5]) with both  $S$  and  $S'$ . Let  $F$  (resp.  $G'$ ) be “schéma en  $A$ -modules” that are flat and locally of finite presentation over  $S$  (resp.  $S'$ ). Let  $G$  be a “schéma en  $A$ -module” over  $S$  induced by  $G'$ . Let  $u : F \rightarrow G$  be a morphism of “schémas en  $A$ -modules”. Let  $K$  be the



complex fitting into the distinguished triangle  $K \rightarrow \ell_F^\vee \rightarrow \ell_G^\vee \rightarrow K[1]$ . It is an object in  $D(A \otimes_{\mathbb{Z}}^L \mathcal{O})$ . Then there is an obstruction  $\omega(u, G') \in \text{Ext}_A^2(F, K \otimes_{\mathcal{O}}^L \epsilon^* I)$  which is zero if and only if there exists a pair  $(F', u')$  where  $F'$  is a deformation of  $F$  as “un schéma en  $A$ -modules” flat over  $S'$  and a map  $u' : F' \rightarrow G'$  extending  $u$ .

**Lemma B.0.7.** Let  $S$  be a scheme. Let  $i : S \rightarrow S'$  be an extension by a quasi-coherent module  $I$ . Suppose  $S$  and  $S'$  are both tor-independent with  $\text{Spec}(\mathbb{Z})$ . Let  $F$  (resp.  $G'$ ) be commutative group schemes over  $S$  (resp.  $S'$ ) that are flat and locally of finite presentation. Let  $G$  be a commutative group scheme over  $S$  induced by  $G'$ . Let  $u : F \rightarrow G$  be a morphism of group schemes over  $S$ . Let  $K$  be the cone of the map  $\ell_F^\vee \rightarrow \ell_G^\vee$ . There is an obstruction  $\omega(u, G') \in \text{Ext}^1(F, K \otimes^L I)$  which vanishes if and only if there exists a pair  $(F', u')$  where  $F'$  is a deformation of  $F$  as a commutative group scheme that is flat over  $S'$ , and  $u' : F' \rightarrow G'$  is a map extending  $u$ .

**Lemma B.0.8** ([Sch15, Theorem III.2.1]). Let  $A$  be a ring. Let  $G$  and  $H$  be commutative group schemes over  $A$  that are flat and of finite presentation, with a group map  $u : H \rightarrow G$ . Let  $B \rightarrow A$  be a square-zero thickening with the argumentation ideal  $J$ . Let  $\tilde{G}$  be a lift of  $G$  to  $B$ . Let  $K$  be a cone of the map  $\ell_H^\vee \rightarrow \ell_G^\vee$  of Lie complexes. Then there is an obstruction class  $\omega \in \text{Ext}^1(H, K \otimes^L J)$  which vanishes if and only if there exists a pair  $(\tilde{H}, \tilde{u})$  where  $\tilde{H}$  is a flat commutative group scheme over  $B$ , and  $\tilde{u} : \tilde{H} \rightarrow \tilde{G}$  is a map lifting  $u : H \rightarrow G$ . Moreover, the obstruction class is functorial in  $J$ , in the following sense. If  $B' \rightarrow A$  is another square-zero thickening with the argumentation ideal  $J'$ , with a map  $B \rightarrow B'$  over  $A$ , then  $\omega' \in \text{Ext}^1(H, K \otimes^L J')$  is the image of  $\omega \in \text{Ext}^1(H, K \otimes^L J)$  under the map  $J \rightarrow J'$ .

## APPENDIX C. MODULI INTERPRETATION OF PEL TYPE SHIMURA VARIETIES

For PEL type Shimura variety, see Milne.

This is [Kot92]; also, “PEL-type  $\mathcal{O}$ -lattice”, cf. [Lan13, Definition 1.2.1.3]

Let  $p$  be a prime. Let  $B$  be a finite-dimensional simple  $\mathbb{Q}$ -algebra with center  $F$ . Let  $\mathcal{O}_B$  be a  $\mathbb{Z}_{(p)}$ -order in  $B$ . Let  $*$  be a positive involution on  $B$  that preserves  $\mathcal{O}_B$ . Let  $V$  be a non-degenerate skew-Hermitian  $B$ -module. Let  $G$  be the group of automorphisms of the skew-Hermitian  $B$ -module  $V$ . Let  $K^p$  be a compact open subgroup of  $G(\mathbb{A}_f^p)$ . Let  $h : \mathbb{C} \rightarrow \text{End}_B(V_{\mathbb{R}})$  be an  $\mathbb{R}$ -algebra homomorphism such that  $(h(\bar{z}) = h(z))^*$ , and that the symmetric bilinear form  $(v, h(i)w)$  on  $V_{\mathbb{R}}$  is positive definite. The map  $h$  determines a decomposition  $V_{\mathbb{C}} = V_1 \oplus V_2$ . Here  $V_1$  is the subspace of  $V_{\mathbb{C}}$  on which  $h(z)$  acts by  $z$ . The field of definition of the isomorphism class of the complex representation  $V_1$  of  $B$  is a number field  $E$ , with ring of integers  $\mathcal{O}_E$ .

Consider the following moduli problem over  $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ .

Also, see [Lan13, Definition 1.4.1.4, Theorem 1.4.1.11, Remark 1.4.1.13].

**Definition C.0.1** ([Lan13, Definition 1.2.1.3]). Let  $B$  be a finite-dimensional semi-simple algebra over  $\mathbb{Q}$  with positive involution  $*$  and center  $F$ , where “positive” means  $\text{tr}_{B/\mathbb{Q}}(xx^*) > 0$  for all  $x \neq 0$  in  $B$ . Let  $\mathcal{O}$  be an order in  $B$  mapped to itself under  $*$ .

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