1. p-adic Hodge Theory

1.1. Review of Classical p-adic Hodge Theory.

Remark 1.1.1. What is p-adic hodge theory? The origin is p-adic cohomology theories.

Remark 1.1.2. Let K be a p-adic field, i.e. a complete discrete valued field with characteristic zero, with perfect residue field with characteristic $p > 0$. Let $G_K = \mathrm{Gal}(\overline{K}/K)$. Let X be a proper smooth algebraic variety over K. We have

(1) étale cohomology $H^n_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p)$, equipped with an action of G_K
(2) the algebraic de Rham cohomology $H^n_{\mathrm{dR}}(X/K)$
(3) crystalline cohomology H^n_{crys} and $H^n_{\mathrm{log-crys}}$.
(4) and more

Faltings' observation: one can compare these different cohomology theories, after tensoring with some large enough mysterious "period rings" B_{HT}, B_{dR}, B_{crys}, B_{st}, etc. For example,

$$H^n_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \simeq H\mathrm{dR}^n(X/K) \otimes_K B_{\mathrm{dR}}$$

compatible with G_K-action and filtration. Note that

(1) $H^n_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p), B_{\mathrm{dR}}$ carries G_K action.
(2)
$$B_{\mathrm{dR}}, H^n_{\mathrm{dR}}(X/K)$$
carries filtration.

The G_K action then shows that

$$H\mathrm{dR}^n(X/K) = (H^n_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p})^{G_K}.$$

Remark 1.1.3. Nowadays, p-adic Hodge theory extends to the study of all p-adic Galois representations, not just H^n_{et}.

Definition 1.1.4. The category $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is the category of all continuous representations of G_K on finite dimensional $\mathbb{Q}_p$ vector spaces. There are subcategories

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \supset \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(G_K \supset \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p} \supset \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p} \supset \mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}$$

((st means semi-stable))

Example 1.1.5. $H^n_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p) \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}$

Remark 1.1.6. There is a functor $D_?$ from $\mathrm{Rep}^?$ to the category of certain semi-linear algebraic objects

$$V \mapsto (V \otimes_{\mathbb{Q}_p} B_?)^{G_K}.$$

((example: $D_{\mathrm{dR}}(H^n_{\mathrm{et}}) = H^n_{\mathrm{dR}}$)) This turns information of G_K-representations into (semi)linear algebra. In certain cases this functor is fully faithful.

Definition 1.1.7. Let B be one of the period rings. ((B carries a G_K action, and B^{G_K} is a field)) For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, consider

$$D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{G_K}$$

which is a B^{G_K} vector space. We say V is B-admissible if the natural map

$$\alpha_V : D_B(V) \otimes_{B^{G_K}} B \to V \otimes_{\mathbb{Q}_p} B$$

is an isomorphism. We write $\mathrm{Rep}^B_{\mathbb{Q}_p}(G_K)$ for the full subcategory of B-admissible representations.

Lemma 1.1.8. The map α_V is always injective. Hence

$$\dim_{B^{G_K}} D_B(V) \leq \dim_{\mathbb{Q}_p} V$$

and α_V is an isomorphism if and only if the equality holds.

Lemma 1.1.9. $\mathrm{Rep}^B_{\mathbb{Q}_p}(G_K)$ is stable under

(1) tensor product
(2) duality

(3) sub-quotients.

Lemma 1.1.10. The restriction of D_B to $\mathrm{Rep}^B_{\mathbb{Q}_p}(G_K)$

$$D_B : \mathrm{Rep}^B_{\mathbb{Q}_p}(G_K) \to \mathrm{Vect}_{B^{G_K}}$$

is exact and faithful, preserves tensor products and duality.

Definition 1.1.11. Let $\mathbb{C}_K$ be the completion of $\overline{K}$.

Definition 1.1.12.

$$B_{\mathrm{HT}} = \bigoplus_{j \in \mathbb{Z}} \mathbb{C}_K(j)$$

This is a graded $\mathbb{C}_K$-algebra. (($\mathbb{C}_K(j)$ is Tate twist))

Lemma 1.1.13.

$$B_{\mathrm{HT}}^{G_K} = K$$

by Tate–Sen theory.

Remark 1.1.14. We then have the functor

$$D_{\mathrm{HT}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Gr}_K$$

where Gr_K is the category of finite dimensional K-vector spaces with $\mathbb{Z}$-grading

Note that

$$\mathrm{gr}^j(D_{\mathrm{HT}}(V)) = (V \otimes \mathrm{gr}^j B_{\mathrm{HT}})^{G_K}.$$

Definition 1.1.15. A Hodge–Tate representation is a B_{HT}-admissible representation.

Definition 1.1.16. Let $V \in \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(G_K)$, i.e. a Hodge–Tate representation. The Hodge–Tate weights of V are those indices $j \in \mathbb{Z}$ such that $\mathrm{gr}^j D_{\mathrm{HT}}(V) \neq 0$, whose dimension is the "multiplicity".

Example 1.1.17. $\mathbb{Q}_p(n)$ is Hodge–Tate with Hodge–Tate weight $-n$.

Example 1.1.18. Let A be an abelian variety over K of dimension d. Let

$$T_p A = \lim_n A[p^n](\overline{K})$$

be the Tate module. Then

$$V_p A = T_p A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is Hodge–Tate with weights 0 and -1, both have multiplicity equal to d.

Definition 1.1.19. Let $\mathcal{O}^\flat_{\mathbb{C}_K}$ be the tilt of $\mathcal{O}_{\mathbb{C}_K}$.

$$\mathcal{O}^\flat_{\mathbb{C}_K} = \lim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/p \simeq \lim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K}$$

where the last isomorphism is only an isomorphism of monoids. The isomorphisms are

$$c \mapsto (c_0, c_1, \dots) \mapsto (c^{(0)}, c^{(1)}, \dots).$$

Consider the θ-map

$$\theta : W(\mathcal{O}^\flat_{\mathbb{C}_K}) \to \mathcal{O}_{\mathbb{C}_K}$$

defined by

$$\sum_{n=0}^{\infty} [c_n] p^n \mapsto \sum_{n=0}^{\infty} c_n^{(0)} p^n$$

Then

$$\theta : W(\mathcal{O}^\flat_{\mathbb{C}_K})[1/p] \to \mathbb{C}_K.$$

Let B^+_{dR} be the completion of $W(\mathcal{O}^\flat_{\mathbb{C}_K})[1/p]$ with respect to $\ker(\theta)$. This is a discrete valuation ring, with maximal ideal $\mathfrak{m}_{\mathrm{dR}}$ and residue field $\mathbb{C}_K$. Let B_{dR} be the fraction field of B^+_{dR}. It is a discretely valued field, with G_K-action.

$$\mathrm{Fil}^i B_{\mathrm{dR}} = \mathfrak{m}^i_{\mathrm{dR}}.$$

Lemma 1.1.20. θ extends to

$$\theta^+_{\mathrm{dR}} : B^+_{\mathrm{dR}} \to \mathbb{C}_K$$

and is surjective, with kernel $\mathfrak{m}_{\mathrm{dR}}$. The ideal $\mathfrak{m}_{\mathrm{dR}}$ is principal, generated by $\zeta = [p^\flat] - p$ where

$$p^\flat = (p, p^{1/p}, p^{1/p^2}, \dots) \in \mathcal{O}^\flat_{\mathbb{C}_K}$$

((Teichimullar lift)) Choose $\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}^\flat_{\mathbb{C}_K}$. Let

$$t = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\epsilon] - 1)^n}{n} \in B^+_{\mathrm{dR}}$$

Then $\mathfrak{m}_{\mathrm{dR}}$ is generated by t.

The upshot is

(1) G_K acts on t via cyclotomic characters.
(2) $\mathrm{gr}^\bullet B_{\mathrm{dR}} = B_{\mathrm{HT}}$.
(3) $B_{\mathrm{dR}}^{G_K} = K$.

Remark 1.1.21. Then

$$D_{\mathrm{dR}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Fil}_K$$

where Fil_K is the category of finite dimensional K vector space with exhaustive and separated filtration.

$$\mathrm{Fil}^j(D_{\mathrm{dR}}(V)) = (V \otimes \mathrm{Fil}^j B_{\mathrm{dR}})^{G_K}.$$

Lemma 1.1.22. For $V \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K)$,

$$\mathrm{gr}^\bullet D_{\mathrm{dR}}(V) = D_{\mathrm{HT}}(V).$$

Hence

$$\mathrm{Rep}^{\mathrm{dR}} \subset \mathrm{Rep}^{\mathrm{HT}}.$$

Lemma 1.1.23. Let K'/K be an extension of p-adic fields inside $\mathbb{C}_K$. For any $V \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K)$, the natural map

$$D_{\mathrm{dR},K}(V) \otimes_K K' \to D_{\mathrm{dR},K'}(V)$$

is an isomorphism. In particular, V is dR as G_K-representation if and only if V is dR as $G_{K'}$-repersentation.

Remark 1.1.24. For crystalline representations, we'd like to enhace the linear structure on B_{dR}, say, Frobenius.

Definition 1.1.25. Let

$$A^0_{\mathrm{crys}} = W(\mathcal{O}^\flat_{\mathbb{C}_K})[\xi^m/m! \mid m \geq 1]$$

In other words, we join divided powers. Let

$$A_{\mathrm{crys}} = \lim_n A^0_{\mathrm{crys}}/p^n$$

be the p-adic completion.

Lemma 1.1.26. A_{crys} is naturally a subring B^+_{dR}.

Remark 1.1.27. This result is difficult.

Remark 1.1.28. The θ map extends to

$$\theta : A_{\mathrm{crys}} \to \mathcal{O}_{\mathbb{C}_K}.$$

Definition 1.1.29.

$$B^+_{\mathrm{crys}} = A_{\mathrm{crys}}[1/p]$$

$$B_{\mathrm{dR}} \supset B_{\mathrm{crys}} = B^+_{\mathrm{crys}}[1/t] = A_{\mathrm{crys}}[1/t].$$

Lemma 1.1.30. B_{crys} has a G_K action.

$$B^{G_K}_{\mathrm{crys}} = W(k)[1/p] = K_0,$$

((maximal unramified extension??))

Lemma 1.1.31. WE have an injection $B_{\mathrm{crys}} \otimes_{K_0} K \to B_{\mathrm{dR}}$. The Frobenius on B_{crys} is injective, and given by $\varphi(t) = pt$. The Frobenius commutes with the G_K-action.

Remark 1.1.32.
$$D_{\mathrm{crys}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{MF}_K^{\phi}$$
where MF_K^{ϕ} is the category of finite dimensional K_0 vector spaces D, equipped with a filtration of $D \otimes_{K_0} K$, and a $\phi : D \to D$, φ semi-linear bijective map.

Lemma 1.1.33. The functor
$$D_{\mathrm{crys}} : \mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(G_K) \to \mathrm{MF}_K^{\phi}$$
is fully faithful. The inverse functor on the essential image is
$$D \mapsto \mathrm{Fil}^0(D \otimes_{K_0} B_{\mathrm{crys}})^{\phi=1}.$$

Lemma 1.1.34. Let L/K be an unramified extension of p-adic fields inside $\mathbb{C}_K$. For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, the natural map
$$D_{\mathrm{crys},K}(V) \otimes_{K_0} L_0 \to D_{\mathrm{crys},L}(V)$$
is an isomorphism. Hence V is crys as G_K-representation if and only if it is crys as G_L representation.

Definition 1.1.35.
$$B_{\mathrm{st}} = B_{\mathrm{crys}}[\log([p]^\flat/p)] \subset B_{\mathrm{dR}}$$
where
$$\log([p]^\flat/p) = \sum \frac{(-1)^{n+1}}{n}\left(\frac{[p]^\flat}{p} - 1\right) = \sum_{n\geq 1}(-1)^{n+1}\frac{1}{np^n}\xi^n$$
converges in B_{dR}^+

It carries a G_K action, $B_{\mathrm{st}}^{G_K} = K_0$, the Frobenius is
$$\varphi(\log([p]^\flat/p)) = p\log([p]^\flat/p).$$
We have an injection $B_{\mathrm{st}} \otimes_{K_0} K \to B_{\mathrm{dR}}$. The element $\log([p]^\flat/p)$ is transcendental over $\mathrm{Frac}(B_{\mathrm{crys}})$
$$B_{\mathrm{st}} \simeq B_{\mathrm{crys}}[X].$$

We define a "monodromy operator"
$$N : B_{\mathrm{st}} \to B_{\mathrm{st}}$$
by
$$N = -\frac{d}{dX}.$$
Then
(1) $B_{\mathrm{crys}} = B_{\mathrm{st}}^{N=0}$.
(2) $N\varphi = p\varphi N$.
(3) N commutes with G_K action.

$$D_{\mathrm{st}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{MF}_K^{\phi,N}$$
where $\mathrm{MF}_K^{\phi,N}$ is MF_K^{ϕ} with the additional data: $N : D \to D$ is a K_0-linear such that $N\phi = p\phi N$.

Lemma 1.1.36. (1)
$$D_{\mathrm{st}}(V)^{N=0} = D_{\mathrm{crys}}(V)$$
Then $\mathrm{Rep}^{\mathrm{crys}} \subset \mathrm{Rep}^{\mathrm{st}}$.
(2)
$$D_{\mathrm{st}}(V) \otimes_{K_0} K \to D_{\mathrm{dR}}(V)$$
is injective. Then $\mathrm{Rep}^{\mathrm{st}} \subset \mathrm{Rep}^{\mathrm{dR}}$.
(3)
$$D_{\mathrm{st}} : \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K) \to \mathrm{MF}_K^{\phi,N}$$
is fully faithful
(4) V is st as G_K rep if and only if it is st as G_L rep, for L/K unramified

Lemma 1.1.37 (Colmez–Fontaine)**.** Let $\mathrm{MF}_K^{\phi,\mathrm{ad}}$ be the essential image of

$$D_{\mathrm{crys}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(G_K) \to \mathrm{MF}_K^{\phi}.$$

Let $\mathrm{MF}_K^{\phi,\mathrm{wa}}$ be another subcategory of MF_K^{ϕ}, called "weakly admissible" objects. For $(D, \phi, \mathrm{Fil}^\bullet) \in \mathrm{MF}_K^{\phi}$, the filtration gives Hodge polygon $P_H(D)$, the Frobenius gives the Newton polygon $P_N(D)$. Roughly speaking, V is weakly admissible if the Newton polygon lies above the Hodge polygon. Then

$$\mathrm{MF}_K^{\phi,\mathrm{ad}} = \mathrm{MF}_K^{\phi,\mathrm{wa}}.$$

Remark 1.1.38. There is a similar result for D_{st}.

Lemma 1.1.39 (p-adic monodromy theorem, Beger, Andre–Kedlaya–Mekbhout, Fargues–Fontaine, Colmez)**.** $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is called potentially semi-stable if it is semi-stable as $G_{K'}$ rep for some finite extension K'/K. Then V is dR if and only if V is potentially semi-stable.

Remark 1.1.40. Other topics in p-adic Hodge theory.

(1) (φ, Γ)-modules
(2) the Fargues–Fontaine curve

1.2. **Relative p-adic Hodge Theory.**

Remark 1.2.1. The idea is to study geometric families of p-adic Galois representations.

Definition 1.2.2. Let X be a locally Noetherian adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. A $\mathbb{Z}_p$-local system $\mathbb{L}$ on X_{et} is an inverse system of sheafs of $\mathbb{Z}/p^n\mathbb{Z}$-modules $\mathbb{L}_n$ on X_{et} such that

(1) Each $\mathbb{L}_n$ is a locally constant sheaf on X_{et} associated with finitely generated $\mathbb{Z}/p^n\mathbb{Z}$-modules.
(2) The inverse system is isomorphic to an inverse system for which

$$\mathbb{L}_{n+1}/p^n \simeq \mathbb{L}_n.$$

The category of $\mathbb{Z}_p$-local systems is denoted by $\mathrm{Loc}_{\mathbb{Z}_p}(X_{\mathrm{et}})$. After inverting p, we obtain $\mathbb{Q}_p$-local systems $\mathrm{Loc}_{\mathbb{Q}_p}(X_{\mathrm{et}})$.

Remark 1.2.3. On the pro-étale topolgy, a $\mathbb{Z}_p$ (or $\mathbb{Q}_p$) local system is a actual locally constant sheaf.

Remark 1.2.4. Let K be a finite extension of $\mathbb{Q}_p$. Let X be a smooth rigid analytic variety over K. Let $\mathbb{L} \in \mathrm{Loc}_{\mathbb{Q}_p}(X_{\mathrm{et}})$. For every classical point $x \in X$, let $\kappa(x)$ denote the residue field at x, which is a finite extension of $\mathbb{Q}_p$. Let $\overline{x}$ be a geometric point over x. Then the stalk $\mathbb{L}_{\overline{x}}$ is naturally a p-adic Galois representation of $G_{\kappa(x)}$.

Remark 1.2.5. How to generalize the classical p-adic Hodge theory to this relative setting? Can we define HT, dR, st, crys?

Definition 1.2.6. There is a relative version of D_{dR}. Let $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ be the "de Rham period sheaf" on the pro-étale site X_{proet}. There is a functor

$$D_{\mathrm{dR}} : \mathrm{Loc}_{\mathbb{Q}_p}(X_{\mathrm{et}}) \to \mathrm{MIC}(X)$$

where $\mathrm{MIC}(X)$ is the category of vector bundles on X_{an} equipped with an integrable connection, given by

$$\mathbb{L} \mapsto v_*(\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}})$$

where $v : X_{\mathrm{proet}} \to X_{\mathrm{et}}$. We say $\mathbb{L}$ is de Rham if $\mathrm{rk} D_{\mathrm{dR}}(\mathbb{L}) = \mathrm{rk}_{\mathbb{Q}_p}\mathbb{L}$.

Remark 1.2.7. WE want to define crys and st. there are several equivalent definitions

(1) in terms of associatioin: Faltings, Du–Liu–Moon–Shimizu, Guo–Reinocke, Guo–Yang
(2) in terms of period sheaves $\mathcal{O}\mathbb{B}_{\mathrm{crys}}, \mathcal{O}\mathbb{B}_{\mathrm{log}}$: Andreatta–Iovita, Tan–Tong, Guo–Yang
(3) prismatic interpretation: Bhatt–Scholze, Guo–Reinocke, Du–Liu–Moon–Shimizu

1.3. **Main Results and Open Questions.**

Lemma 1.3.1 (Rigidity, Liu–Zhu)**.** Let X be a smooth rigid variety over K. Let $\mathbb{L} \in \mathrm{Loc}_{\mathbb{Q}_p}(X_{\mathrm{et}})$. If $\mathbb{L}_{\overline{x}}$ is de Rham for some $x \in X$, then $\mathbb{L}$ is de Rham.

Remark 1.3.2. Rigidity is false for crys or st. Consider elliptic curves.

Lemma 1.3.3 (Purity, Du–Liu–Moon–Shimizu)**.** Suppose X has good reduction. The generic point of the special fiber corresponds to a type II point $\xi \in X$. If $\mathbb{L}|_{\xi}$ is crys, then $\mathbb{L}$ is crys.

Remark 1.3.4. Similar result for st.

Lemma 1.3.5 (Guo–Yang)**.** Suppose X has good reduction. Then $\mathbb{L}$ is crys if and only if $\mathbb{L}_{\overline{x}}$ is crys for every classical point $x \in X$.

Remark 1.3.6. Similar result for st.

Remark 1.3.7. Other questions.

(1) Relative version of wa = ad?
(2) Relative version of dR = potentially st?
(3) Relative (φ, Γ)-modules?
(4) Relative Fargues–Fontaine curve?

2. Algebraic Number Theory

Remark 2.0.1. Let $A \subset B$ be rings. Suppose that B is free of rank m as an A-module. Let $\beta \in B$. Then multiplication by β gives a map $B \to B$ of A-modules. The trace of this map is denoted by $\mathrm{tr}_{B/A}(\beta)$.

Definition 2.0.2. Let $A \subset B$ be rings. Suppose that B is free of rank m as an A-module. Let $\beta_1, \ldots, \beta_m$ be elements of B. The discriminant of $\beta_1, \ldots, \beta_m$ is

$$D(\beta_1, \ldots, \beta_m) = \det(\mathrm{tr}_{B/A}(\beta_i \beta_j)).$$

Lemma 2.0.3. Let K be a field equipped with an Archimedean absolute value $|-|$. Then K is isomorphic to a subfield of $\mathbb{C}$, and the absolute value is equivalent to the absolute value induced from usual absolute value on $\mathbb{C}$.

Lemma 2.0.4. Let K be a field. Let $|-|_1$ and $|-|_2$ be inequivalent non-trivial absolute values on K. There exists $\theta \in K$ such that

Lemma 2.0.5. Let K be a field. Let $|-|_1, \ldots, |-|_n$ be pairwise inequivalent non-trivial absolute values on K. There exists an element $\theta \in K$ such that $|\theta|_1 > 1$ and $|\theta|_i < 1$ for every $2 \leq i \leq n$.

Proof. Use induction on $n \geq 2$. Suppose $n = 2$. □

Lemma 2.0.6. Let K be a field. Let $|-|_1, \ldots, |-|_n$ be pairwise inequivalent non-trivial absolute values on K. Let K_i be the topological space with underlying set K and with topology induced by the absolute value $|-|_i$. Then the diagonal $\Delta \subset \prod_{i=1}^n K_i$ is a dense subset.