1. Algebraic Number Theory

Lemma 1.0.1. Let K be a field. Let $|-|$ be an absolute value on K. Let $x \in K$. Then $x^n \to 0$ as $n \to +\infty$ if and only if $|x| < 1$.

Lemma 1.0.2. Let K be a field. Let $|-|_1$ and $|-|_2$ be non-trivial absolute values on K. The following are equivalent:

(1) The absolute values $|-|_1$ and $|-|_2$ induce the same topology on K.
(2) For every $x \in K$, if $|x|_1 < 1$, then $|x|_2 < 1$.
(3) There exists $\lambda > 0$ such that $|-|_2 = |-|_1^\lambda$.

Proof. Proof of (1) $\Rightarrow$ (2). Let $x \in K$. We have $|x|_1 < 1$ if and only if $x^n \to 0$, which is again equivalent to $|x|_2 < 1$.

Proof of (2) $\Rightarrow$ (3). Since $|-|_1$ is non-trivial, there exists $y \in K$ such that $|y|_1 > 1$. Set

$$\lambda = \frac{\log|y|_2}{\log|y|_1} > 0.$$

Then $|y|_2 = |y|_1^\lambda$. Let $x \in K$. There exists $\mu \in \mathbb{R}$ such that $|x|_1 = |y|_1^\mu$. Let m/n be a rational number with $m/n > \mu$ and $n > 0$. Then

$$|x|_1 = |y|_1^\mu \leq |y|_1^{m/n}.$$

So $|x^n/y^m|_1 < 1$. Hence $|x_n/y^m|_2 < 1$, and thus $|x|_2 < |y|_2^{m/n}$. So $|x|_2 \leq |y|_2^\mu$. Similarly, we have $|x|_2 \geq |y|_2^\mu$. Then $|x|_2 = |y|_2^\mu$. Therefore

$$|x|_2 = |y|_2^\mu = |y|_1^{\lambda\mu} = |x|_1^\lambda.$$

Proof of (3) $\Rightarrow$ (1). This is clear. □

Definition 1.0.3. Let K be a field. Two non-trivial absolute values $|-|_1$ and $|-|_2$ on K are said to be equivalent if they satisfy the equivalent conditions in Lemma 1.0.2.

Lemma 1.0.4. Let K be a field. Let $|-|_1$ and $|-|_2$ be inequivalent non-trivial absolute values on K. There exists $\theta \in K$ such that $|\theta|_1 < 1$ and $|\theta|_2 \geq 1$.

Proof. Clear by Lemma 1.0.2. □

Lemma 1.0.5. Let K be a field. Let $n \geq 2$. Let $|-|_1, \ldots, |-|_n$ be pairwise inequivalent non-trivial absolute values on K. There exists $\theta \in K$ such that $|\theta|_1 > 1$ and $|\theta|_i < 1$ for every $2 \leq i \leq n$.

Proof. Use induction on $n \geq 2$. Suppose $n = 2$. Since $|-|_1$ and $|-|_2$ are inequivalent, there exist $\alpha, \beta \in K$ such that

(1) $|\alpha|_1 < 1$ and $|\alpha|_2 \geq 1$.
(2) $|\beta|_1 \geq 1$ and $|\beta|_2 < 1$.

Then $\theta = \beta\alpha^{-1}$ satisfies $|\theta|_1 > 1$ and $|\beta|_2 < 1$.

Suppose $n \geq 3$. By the induction hypothesis, there exists $\phi \in K$ such that $|\phi|_1 < 1$ and $|\phi|_i > 1$ for every $2 \leq i \leq n-1$. There also exists $\psi \in K$ such that $|\psi|_1 < 1$ and $|\psi|_n > 1$. Then take

$$\theta = \begin{cases} \phi & |\phi|_n < 1 \\ \phi^k\psi & |\phi|_n = 1 \\ \frac{\phi^k}{1+\phi^k} & |\phi|_n > 1 \end{cases}$$

for $k \geq 1$ sufficiently large. □

Lemma 1.0.6. Let K be a field. Let $|-|_1, \ldots, |-|_n$ be pairwise inequivalent non-trivial absolute values on K. Let K_i be the topological space with underlying set K and with topology induced by $|-|_i$. Then the diagonal $\Delta \subset \prod_{i=1}^n K_i$ is dense.

Proof. By the previous lemma, we can find elements $\theta_1, \ldots, \theta_n \in K$ such that $|\theta_i|_i < 1$ and $|\theta_i|_j > 1$ for all $i \neq j$. Let $(\alpha_i)_i \in \prod_{i=1}^n K_i$. Take

$$\zeta_k = \sum_{i=1}^n \frac{\theta_i^k}{1+\theta_i^k}\alpha_i.$$

Then $|\zeta_k - \alpha_i|_i \to 0$ as $k \to +\infty$ for every $1 \leq i \leq n$. □

Lemma 1.0.7. Let L/K be a finite extension of fields with $[L : K] = n$. Let $|-|$ be a complete absolute value on K. Then the absolute value $|-|$ admits a unique extension to an absolute value on L, given by the formula

$$|\alpha| = |\mathrm{Nm}_{L/K}(\alpha)|^{1/n}.$$