THE HODGE–TATE PERIOD MAP

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Contents

1. Technical Tools

1.1. Canonical Subgroups.

Definition 1.1.1. Let $A \to S$ be an Abelian scheme where S is over $\mathrm{Spec}(\mathbb{F}_p)$. We have Frobenius map $F : A \to A^{(p)}$ and Verschiebung map $V : A^{(p)} \to A$ with the composition $V \circ F = [p]$.

The map $V : A^{(p)} \to A$ induces a map $\omega_{A/S} \to \omega_{A^{(p)}/S} \simeq \omega_{A/S}^{\otimes p}$. This gives a canonical section of $\omega_{A/S}^{\otimes(p-1)}$, called the Hasse invariant of A/S, denoted $\mathrm{Ha}(A/S) \in \Gamma(S, \omega_{A/S}^{\otimes(p-1)})$.

TODO: Rewrite using the "$O(m, \epsilon)$" notation from [zhu].

Definition 1.1.2. Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $A \to \mathrm{Spec}(R)$ be an Abelian scheme. Let $m \geq 1$ be an integer. Let $0 \leq \epsilon < 1/2$ such that $p^\epsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$ makes sense. We say that $A \to \mathrm{Spec}(R)$ satisfies the $O(m, \epsilon)$ condition if $\mathrm{Ha}(A_1/\mathrm{Spec}(R_1))^{(p^m-1)/(p-1)}$ divides p^ϵ as elements in $R_1 = R/p$.

Lemma 1.1.3. Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $A \to \mathrm{Spec}(R)$ be an Abelian scheme satisfying $O(m, \epsilon)$. Then there is a unique closed subgroup $C_m \subset A[p^m]$ such that $C_m = \ker(F^m) \mod p^{1-\epsilon}$.

Definition 1.1.4 (scholze/III/2.7)**.** Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. We say that an Abelian scheme $A \to \mathrm{Spec}(R)$ has a weak canonical subgroup of level m if $A \to \mathrm{Spec}(R)$ satisfies $O(m, \epsilon)$ for some $\epsilon < 1/2$. In that case, we call $C_m \subset A[p^m]$ in Lemma 1.1.3 the weak canonical subgroup of level m.

TODO: strong $O(m, \epsilon)$ If moreover $\mathrm{Ha}(A_1/\mathrm{Spec}(R_1))^{p^m}$ divides p^ϵ, then we say that C_m is a canonical subgroup. Here, $R_1 = R/p$ and $A_1 \to \mathrm{Spec}(R_1)$ is the reduction of $A \to \mathrm{Spec}(R)$.

Lemma 1.1.5. TODO: Check this!

Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $A \to \mathrm{Spec}(R)$ be an Abelian scheme satisfying strong $O(m, \epsilon)$, with strong canonical subgroup $C \subset A[p]$ of level 1. Then $(A/C)_{1-\epsilon} \simeq A_{1-\epsilon}^{(p)}$, and in particular

$$\mathrm{Ha}((A/C)_{1-\epsilon}/R_{1-\epsilon}) = \mathrm{Ha}(A_{1-\epsilon}/R_{1-\epsilon})^p.$$

Lemma 1.1.6 (scholze/III/2.8)**.** Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let A and B be Abelian schemes over R.

(1) If A has a canonical subgroup $C_m \subset A[p^m]$ of level m, then it has a canonical subgroup $C_{m'} \subset A[p^{m'}]$ of every level $m' \leq m$, and $C_{m'} \subset C_m$.
(2) Let $f : A \to B$ be a map of Abelian schemes. Assume that both A and B have canonical subgroups $C_m \subset A[p^m]$ and $D_m \subset B[p^m]$ of level m. Then C_m maps into D_m under f.
(3) Assume that A has a canonical subgroup $C_m \subset A[p^m]$ of level m, and let $\overline{x}$ be a geometric point of $\mathrm{Spec}(R[p^{-1}])$. Then $C_m(\overline{x}) \simeq (\mathbb{Z}/p^m\mathbb{Z})^g$, where g is dimension of the Abelian variety over $\overline{x}$.

1.2. **Hartog's Extension Principle.**

Lemma 1.2.1 ([SGA2, Lemma III.3.1, Proposition III.3.3])**.** Let X be a locally Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $n \geq 1$ be an integer. Then the following are equivalent:

(1) For any open subscheme V of X, the map
$$H^i(V, \mathcal{F}) \to H^i(V \backslash Z, \mathcal{F})$$
is bijective for $i \leq n-2$ and injective for $i = n-1$.
(2) For any open subscheme V of X, the local cohomology
$$H^i_{V \cap Z}(V, \mathcal{F}) = 0$$
for all $i \leq n-1$.
(3) For any $x \in Z$ the depth of $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module is at least n.

Lemma 1.2.2 (Serre's criterion)**.** A Noetherian ring R is normal if and only if $R_{\mathfrak{p}}$ is regular for every $\mathfrak{p}$ of height ≤ 1 and $R_{\mathfrak{p}}$ has depth ≥ 2 for every $\mathfrak{p}$ of height ≥ 2.

Lemma 1.2.3 (gortz-wedhorn-1/6.45)**.** Let X be a locally Noetherian normal scheme. Let U be an open subscheme of X with codimension ≥ 2. Then the map $H^0(X, \mathcal{O}_X) \to H^0(U, \mathcal{O}_X)$ is an isomorphism.

Proof. We may assume that $X = \mathrm{Spec}(A)$ where A is normal integral domain. For every non-empty open V of X, the ring $\Gamma(V, \mathcal{O}_X)$ may be considered as a subring of the function field $K(X) = \mathrm{Frac}(A)$ such that the restriction maps are given by inclusions of rings. Let Z be an irreducible closed subset of X of codimension 1. Then U intersects Z non-trivially, so it contains the generic point η of Z. In other words, the subring $\Gamma(U, \mathcal{O}_X)$ of the function field $K(X)$ is contained in the stalk $\mathcal{O}_{X,\eta}$. But $A = \Gamma(X, \mathcal{O}_X)$ is the intersection of all the stalks $\mathcal{O}_{X,\eta}$, where η is a prime ideal of height 1; in other words, where η is the generic point of an irreducible closed subset of codimension 1. $\square$

Lemma 1.2.4 (scholze/III/2.9; zhu/3.10.4)**.** Let R be a normal ring, i.e. the localization $R_{\mathfrak{p}}$ is an integrally closed domain for every prime ideal $\mathfrak{p}$ of R. Assume R is Noetherian. Let $Z \subset \mathrm{Spec}(R)$ be a closed subscheme of codimension at least 2, i.e. every $\mathfrak{p} \in Z$ has height at least 2. Then for $U = \mathrm{Spec}(R) \backslash Z$,
$$H^0(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}) \simeq H^0(U, \mathcal{O}_{\mathrm{Spec}(R)}).$$

Proof. Consider $n = 2$ and $\mathcal{F} = \mathcal{O}_X$ in Lemma 1.2.1. Serre's criterion, cf. Lemma 1.2.2, guarantees the third condition in Lemma 1.2.1. The first assertion gives the desired result. $\square$

Lemma 1.2.5 (scholze/III/2.10; zhu/3.10.5)**.** Let R be a topologically finitely generated, flat, and p-adically complete $\mathbb{Z}_p$-algebra, such that $\overline{R} = R/p$ is normal. Fix $f \in R$ such that its reduction $\overline{f} \in \overline{R}$ is not a zero-divisor. Let $0 < \epsilon \leq 1$. Set $S = (R \,\widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u \rangle / (u \cdot f - p^{\epsilon})$. Then S is p-adically complete and flat over $\mathbb{Z}_p^{\mathrm{cycl}}$. Fix a closed subscheme $Y \subset \mathrm{Spec}(\overline{R})$ of codimension ≥ 2. Let Z be the inverse image of Y in $\mathrm{Spf}(S)$. Then for $U = |\mathrm{Spf}(S)| \backslash Z$,
$$S = H^0(\mathrm{Spf}(S), \mathcal{O}_{\mathrm{Spf}(S)}) \simeq H^0(U, \mathcal{O}_{\mathrm{Spf}(S)}).$$

Proof. We first show that the map
$$S \simeq H^0(\mathrm{Spf}(S), \mathcal{O}_{\mathrm{Spf}(S)}) \to H^0(U, \mathcal{O}_{\mathrm{Spf}(S)})$$
is injective. Since S is p-adically separated and $H^0(U, \mathcal{O}_{\mathrm{Spf}(S)})$ is flat over $\mathbb{Z}_p^{\mathrm{cycl}}$, it suffices to show that
$$S_\epsilon \simeq H^0(\mathrm{Spec}(S_\epsilon), \mathcal{O}_{\mathrm{Spec}(S_\epsilon)}) \to H^0(U_\epsilon, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$$

is injective, where $S_\epsilon = S/p^\epsilon$, Z_ϵ is the inverse image of Y in $\mathrm{Spec}(S_\epsilon)$, and $U_\epsilon = \mathrm{Spec}(S_\epsilon)\backslash Z_\epsilon$. Note that

$$S_\epsilon = (R \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u\rangle/(uf, p^\epsilon) = R_\epsilon[u]/(uf_\epsilon)$$

where $R_\epsilon = R \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)$ and $f_\epsilon \in R_\epsilon$ is the image of $f \in R$.

Let $W \subset \mathrm{Spec}(S_\epsilon)$ be the preimage of $V = V(\overline{f}) \subset \mathrm{Spec}(\overline{R})$. Then $W = V \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathbb{A}^1_{\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon}$ is affine. The map $S_\epsilon \to R_\epsilon$ sending u to zero induces a section $\mathrm{Spec}(R_\epsilon) \to \mathrm{Spec}(S_\epsilon)$. We have a decomposition $\mathrm{Spec}(S_\epsilon) = N \cup W$, where $N = \mathrm{Spec}(R_\epsilon[u]/(u)) \simeq \mathrm{Spec}(R_\epsilon)$ is the image of the section $\mathrm{Spec}(R_\epsilon) \to \mathrm{Spec}(S_\epsilon)$. Take $V_\epsilon = V \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)$. Then $W = V_\epsilon \times_{\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)} \mathbb{A}^1_{\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon}$, and $N \cap W = V_\epsilon$.

We then have the following interpretations:

(1) Each section in $\Gamma(\mathrm{Spec}(S_\epsilon), \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$ is a pair (f_1, f_2) such that $f_1 \in \Gamma(N, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$ and $f_2 \in \Gamma(W, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$ such that $f_1 = f_2$ on $N \cap W = V_\epsilon$.

(2) Each section in $H^0(U_\epsilon, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$ is a pair (f_1, f_2) such that $f_1 \in H^0(U_\epsilon \cap N, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$, and $f_2 \in H^0(U_\epsilon \cap W, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$, such that $f_1 = f_2$ on $U_\epsilon \cap N \cap W$.

The (classical) Hartog's extension principle, i.e. Lemma 1.2.4 applied to $Y \subset \mathrm{Spec}(\overline{R})$, shows that

$$\Gamma(\mathrm{Spec}(\overline{R})\backslash Y) \simeq \Gamma(\mathrm{Spec}(\overline{R})).$$

Under base-change this gives

$$\Gamma(U_\epsilon \cap N) \simeq \Gamma(\mathrm{Spec}(\overline{Y})\backslash Y) \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon \simeq \Gamma(\mathrm{Spec}(\overline{R})) \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon \simeq \Gamma(N).$$

Thus injectivity reduces to show that

$$\Gamma(V) \otimes_{\mathbb{F}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)[u]\Gamma(W) \to \Gamma(U_\epsilon \cap W) = \Gamma(V\backslash Y) \otimes_{\mathbb{F}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)[u]$$

is injective. It suffices to show that $\Gamma(V) \to \Gamma(V\backslash Y)$ is injective, where both V and $V\backslash Y$ are $\mathbb{F}_p$-schemes. We have $\mathrm{depth}(\mathcal{O}_{V,y}) = \mathrm{depth}(\overline{R}_y) - 1$ for all $y \in V$, cf. [SGA-2/III/2.5]. Thus $\mathrm{depth}(\mathcal{O}_{V,y}) \geq 1$ for every $x \in V \cap Y$ by Serre's criterion, i.e. Lemma 1.2.2. Then the desired injectivity follows from Lemma 1.2.1.

We then show the surjectivity.

TODO □

1.3. **Tate's Normalized Traces.**

Lemma 1.3.1 (scholze/III/2.21; zhu/3.12.2)**.** Let R be a p-adically complete flat $\mathbb{Z}_p$-algebra. Let $Y_1, \ldots, Y_n \in R$. Let $P_1, \ldots, P_n \in R\langle X_1, \ldots, X_n\rangle$ be topologically nilpotent elements, or equivalently, each P_i has topologically nilpotent coefficients in R. Let

$$S = R\langle X_1, \ldots, X_n\rangle/(X_1^p - Y_1 - P_1, \ldots, X_n - Y_n - P_n).$$

Then

(1) The ring S is a finite free R-module of rank p^n, with a basis given by $X_1^{i_1} \cdots X_n^{i_n}$ with $0 \leq i_1, \ldots, i_n \leq p-1$.

(2) Let I be the ideal of R generated by p together with all the coefficients of all P_i. Then the trace map $\mathrm{tr}_{S/R} : S \to R$ sends S to I^n, i.e. $\mathrm{tr}_{S/R}(S) \subset I^n$.

Lemma 1.3.2 (scholze/III/2.22; zhu/3.12.4)**.** Let R be a p-adically complete flat $\mathbb{Z}_p$-algebra topologically of finite type, formally smooth of dimension n over $\mathbb{Z}_p$. Let $f \in R$ such that its reduction $\overline{f} \in \overline{R} = R/p$ is not a zero-divisor. Let $0 \leq \epsilon < 1/2$. Let

$$S_\epsilon = (R \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u_\epsilon\rangle/(u_\epsilon \cdot f - p^\epsilon).$$

Suppose $\varphi : S_\epsilon \to S_{\epsilon/p}$ is a map of $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra such that modulo $p^{1-\epsilon}$ it is given by the relative Frobenius. In other words, $\varphi \mod p^{1-\epsilon}$ is the map

$$R_{1-\epsilon}[u_\epsilon]/(f \cdot u_\epsilon - p^\epsilon) \to R_{1-\epsilon}[u_{\epsilon/p}]/(f \cdot u_{\epsilon/p} - p^{\epsilon/p}),$$

where $R_{1-\epsilon} = \overline{R} \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^{1-\epsilon})$, which sends u_ϵ to $u_{\epsilon/p}^p$, and restricts to $\mathrm{Fr}_{\overline{R}} \otimes \mathrm{id}$ on $R_{1-\epsilon}$. Then

(1) The map
$$\varphi[1/p] : S_\epsilon[1/p] \to S_{\epsilon/p}[1/p]$$
is finite and flat of degree p^n.

(2) The trace map

$$\mathrm{tr} = \mathrm{tr}_{S_{\epsilon/p}[1/p]/S_\epsilon[1/p]} : S_{\epsilon/p}[1/p] \to S_\epsilon[1/p]$$

sends $S_{\epsilon/p}$ into $p^{n-(2n+1)\epsilon} S_\epsilon$. Here $S_{\epsilon/p}[1/p]$ is viewed as an $S_\epsilon[1/p]$-algebra via $\varphi[1/p]$.

1.4. Riemann's Hebbarkeitssatz.

Definition 1.4.1 (scholze/II/3.8)**.** Let p be a prime. Let K be a perfectoid field (of any characteristic). Let t be a non-zero element of K with $|p| \le |t| < 1$. A triple $(\mathcal{X}, \mathcal{Z}, \mathcal{U})$, where $\mathcal{X}$ is an affinoid perfectoid space over K, $\mathcal{Z}$ is a closed subset of $\mathcal{X}$, and $\mathcal{U}$ is a quasi-compact open subset of $\mathcal{X}\backslash\mathcal{Z}$, is said to be good, if

$$H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^+/t)^a \simeq H^0(\mathcal{X}\backslash\mathcal{Z}, \mathcal{O}_{\mathcal{X}}^+/t)^a \hookrightarrow H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}}^+/t)^a.$$

Remark 1.4.2. This notion is independent of the choice of t, and is compatible with tilting.

Situation 1.4.3. Let $K = \mathbb{F}_p((t^{1/p^\infty}))$. Let R_0 be a reduced Tate K-algebra topologically of finite type. Let $\mathcal{X}_0 = \mathrm{Spa}(R_0, R_0^\circ)$ be the associated affinoid adic space of finite type over K. Let R be the completed perfection of R_0, which is a p-finite perfectoid K-algebra. Let $\mathcal{X} = \mathrm{Spa}(R, R^+)$ with $R^+ = R^\circ$, the associated p-finite affinoid perfectoid space over K. Let I_0 be an ideal of R_0. Let $I = I_0 R \subset R$. Let $\mathcal{Z}_0 = V(I_0) \subset \mathcal{X}_0$. Let $\mathcal{Z} = V(I) \subset \mathcal{X}$. Let $\mathcal{U}_0$ be a quasi-compact open subset of $\mathcal{X}_0\backslash\mathcal{Z}_0$ with preimage $\mathcal{U} \subset \mathcal{X}\backslash\mathcal{Z}$.

Lemma 1.4.4 (scholze/II/3.10)**.** Assume Situation 1.4.3. Suppose $(\mathcal{X}, \mathcal{Z}, \mathcal{U})$ is good. Suppose that R_0 is normal, and that $V(I_0) \subset \mathrm{Spec}(R_0)$ is of codimension ≥ 2. Let R_0' be a finite normal R_0-algebra which is étale outside $V(I_0)$, and such that no irreducible component of $\mathrm{Spec}(R_0')$ maps into $V(I_0)$. Let $I_0' = I_0 R_0'$, and $\mathcal{U}_0' \subset \mathcal{X}_0'$ the preimage of $\mathcal{U}_0$. Let R', I', $\mathcal{X}'$, $\mathcal{Z}'$, $\mathcal{U}'$ be the associated perfectoid objects.

(1) There is a perfect trace pairing

$$\mathrm{tr}_{R_0'/R_0} : R_0' \otimes_{R_0} R_0' \to R_0.$$

(2) The trace pairing induces a trace pairing

$$\mathrm{tr}_{R'^\circ/R^\circ} : R'^\circ \otimes_{R^\circ} R'^\circ \to R^\circ.$$

which is almost perfect.

(3) For all open subsets $\mathcal{V} \subset \mathcal{X}$ with preimage $\mathcal{V}' \subset \mathcal{X}'$, the trace pairing induces an isomorphism

$$H^0(\mathcal{V}', \mathcal{O}_{\mathcal{X}'}^+/t)^a \simeq \mathrm{Hom}_{R^\circ/t}(R'^\circ/t, H^0(\mathcal{V}, \mathcal{O}_{\mathcal{X}}^+/t))^a.$$

(4) The triple $(\mathcal{X}', \mathcal{Z}', \mathcal{U}')$ is good.

(5) If $\mathcal{X}' \to \mathcal{X}$ is surjective, then the map

$$H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^+/t) \to H^0(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}^+/t) \cap H^0(\mathcal{U}, \mathcal{O}_{\mathcal{X}}^+/t)$$

is an almost isomorphism.

Lemma 1.4.5 (scholze/II/3.11)**.** Suppose we have a filtered inductive system $(R_0^{(i)})_{i\in I}$ as in the previous lemma, giving rise to $\mathcal{X}^{(i)}$, $\mathcal{Z}^{(i)}$, $\mathcal{U}^{(i)}$. Assume that all transition maps $\mathcal{X}^{(i)} \to \mathcal{X}^{(j)}$ are surjective. Let $\widetilde{\mathcal{X}}$ be the inverse limit of the $\mathcal{X}^{(i)}$ in the category of perfectoid spaces over K, with preimage $\widetilde{\mathcal{Z}} \subset \widetilde{\mathcal{X}}$ of $\mathcal{Z}$, and $\widetilde{\mathcal{U}} \subset \widetilde{\mathcal{X}}\backslash\widehat{\mathcal{U}}$ of $\mathcal{U}$. Then the triple $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{Z}}, \widetilde{\mathcal{U}})$ is good.

1.5. The Hodge–Tate Filtration.

Lemma 1.5.1. Let C be an algebraically closed and complete extension of $\mathbb{Q}_p$. Let $A \to \mathrm{Spec}(C)$ be an Abelian variety. Then A has its Hodge–Tate filtration

$$0 \to \mathrm{Lie}(A)(1) \to T_p(A) \otimes_{\mathbb{Z}_p} C \to (\mathrm{Lie}(A^\vee))^* \to 0.$$

TODO

2. Siegel Modular Varieties

Let p be a fixed prime. Let $g \geq 1$ be an integer.

Definition 2.0.1. The symplectic similitude group GSp_{2g} is the reductive group scheme over $\mathbb{Z}$ whose points in a commutative ring R are given by

$$\mathrm{GSp}_{2g}(R) = \{x \in \mathrm{GL}_{2g}(V); \exists \nu(x) \in R^\times, x^t \Omega x = \nu(x)\Omega\}$$

where $\Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ is the standard symplectic matrix of order $2g$.

In the following discussion, we write $G = \mathrm{GSp}_{2g}$. Let $K_p = G(\mathbb{Z}_p)$. Let K^p be a compact open subgroup of $G(\mathbb{A}^{\infty,p})$ that is contained in

$$\Gamma(N)^{(p)} = \{g \in G(\mathbb{A}^{\infty,p}); g \equiv 1 \bmod N\}$$

for some integer $N \geq 3$ not divisible by p.

Definition 2.0.2. Let $m \geq 1$ be an integer.

$$\Gamma_0(p^m) = \left\{ g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \bmod p^m \right\}$$

$$\Gamma_{\mathrm{s}}(p^m) = \left\{ g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \bmod p^m, \nu(g) \equiv 1 \bmod p^m \right\}$$

$$\Gamma_1(p^m) = \left\{ g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \bmod p^m \right\}$$

$$\Gamma(p^m) = \left\{ g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod p^m \right\}$$

Let X be the scheme over $\mathrm{Spec}(\mathbb{Z}_{(p)})$ classifying principally polarized (projective?) Abelian schemes of relative dimension g with level K^p structures. Let X^* be the minimal compactification of X.

TODO: Add references.

For each $U \in \{\Gamma(p^m), \Gamma_{\mathrm{s}}(p^m), \Gamma_0(p^m)\}$, we have a scheme $X_{U,\mathbb{Q}}$ over $\mathbb{Q}$ with certain moduli interpretations.

Let $\mathfrak{X}$ be the formal scheme over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$ defined as the p-completion of $X_{\mathbb{Z}_p^{\mathrm{cycl}}} = X \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} \mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$.

The universal Abelian scheme $A \to X$ gives a line bundle $\omega = \omega_{A/S} = \bigwedge^g \Omega^1_{A/X}$. The sheaf ω extends to the minimal compactification X^*. The Hasse invariant defines a section $\mathrm{Ha} \in H^0(X_{\mathbb{F}_p}, \omega^{\otimes(p-1)})$. The section Ha extends to $\mathrm{Ha} \in H^0(X^*_{\mathbb{F}_p}, \omega^{\otimes(p-1)})$. For $g = 1$, this follows from direct inspection. For $g \geq 2$, this follows from the (classical) Hartog's extension principle. TODO: Clearify this paragraph.

Let $\mathfrak{A} \to \mathfrak{X}$ be the universal formal Abelian scheme.

3. The ϵ-Neighbourhoods

Lemma 3.0.1. Let S be a p-adically complete $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. There is a bijection

$$\mathrm{Hom}_{\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})}(\mathrm{Spf}(S), \mathfrak{X}^*) \simeq \mathrm{Hom}_{\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})}(\mathrm{Spec}(S), X^*_{\mathbb{Z}_p^{\mathrm{cycl}}}).$$

Definition 3.0.2. Let $0 \leq \epsilon < 1$ such that there exists an element $p^\epsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$ with p-adic valuation ϵ. Let $\mathcal{M}_\epsilon$ be the functor sending a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra S to the set of pairs $(f, [u])$, where

- f is a map $\mathrm{Spf}(S) \to \mathfrak{X}^*$; it's equivalent to a map $\mathrm{Spec}(S) \to X^*_{\mathbb{Z}_p^{\mathrm{cycl}}}$ by Lemma 3.0.1.
- Let $\overline{f} : \mathrm{Spec}(S/p) \to X^*_{\mathbb{F}_p}$ be the reduction of $\mathrm{Spec}(S) \to X^*_{\mathbb{Z}_p^{\mathrm{cycl}}}$. Recall that we have the Hasse section $\mathrm{Ha} \in H^0(X^*_{\mathbb{F}_p}, \omega^{\otimes(p-1)})$. It pullbacks to $\overline{f}^*\mathrm{Ha} \in H^0(\mathrm{Spec}(S/p), \overline{f}^*\omega^{\otimes(p-1)})$. Then $[u]$ is an equivalence class of sections $u \in H^0(\mathrm{Spec}(S/p), \overline{f}^*\omega^{\otimes(1-p)})$ satisfying $u \cdot \overline{f}^*\mathrm{Ha} = p^\epsilon \in S_1$ under the equivalence relation that $u \sim u'$ if and only if there exists some $h \in S$ such that $u' = u(1 + p^{1-\epsilon}h)$.

Lemma 3.0.3. Let $0 \le \epsilon < 1$ such that $p^\epsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$ makes sense. Then the functor $\mathcal{M}_\epsilon$ is representable by a formal scheme flat over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$. Moreover, locally it is ...

Let $\mathfrak{X}(\epsilon) \to \mathfrak{X}$ be the pullback of $\mathfrak{X}^*(\epsilon) \to \mathfrak{X}^*$ along $\mathfrak{X} \to \mathfrak{X}^*$. Let $\mathfrak{A}(\epsilon) \to \mathfrak{X}(\epsilon)$ be the pullback of $\mathfrak{A} \to \mathfrak{X}$ along $\mathfrak{X}(\epsilon) \to \mathfrak{X}$.

Remark 3.0.4. TODO: moduli interpretation of $\mathfrak{X}(\epsilon)$. Should be almost identical to $\mathcal{M}_\epsilon$.

Definition 3.0.5. For a formal scheme $\mathfrak{Y}$ over $\mathbb{Z}_p^{\mathrm{cycl}}$ and $a \in \mathbb{Z}_p^{\mathrm{cycl}}$, we write $\mathfrak{Y}/a$ for $\mathfrak{Y} \times_{\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})} \mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/a)$.

Definition 3.0.6. For a formal scheme $\mathfrak{Y}$ over $\mathbb{Z}_p^{\mathrm{cycl}}/p$, we write $\mathfrak{Y}^{(p)}$ for the pullback of $\mathfrak{Y}$ along the (absolute) Frobenius $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p) \to \mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$.

Lemma 3.0.7. Let $0 \le \epsilon < 1$ such that $p^\epsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$ makes sense. We have a natural isomorphism

$$(\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \simeq \mathfrak{X}^*(\epsilon)/p$$

of formal schemes over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$. Furthermore, by pullback we get the following commutative diagram

$$\begin{array}{ccccc} (\mathfrak{A}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{A}(\epsilon)/p & \longrightarrow & \mathfrak{X}(\epsilon)/p & \longrightarrow & \mathfrak{X}^*(\epsilon)/p \end{array}$$

where each vertical map is an isomorphism.

Proof. TODO: This should be checked using moduli interpretations. $\square$

Lemma 3.0.8. Let $0 \le \epsilon < 1$ such that $p^\epsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$ makes sense. The Frobenius map $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p) \to \mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$ induces the following commutative diagram

$$\begin{array}{ccccc} \mathfrak{A}(p^{-1}\epsilon)/p & \longrightarrow & \mathfrak{X}(p^{-1}\epsilon)/p & \longrightarrow & \mathfrak{X}^*(p^{-1}\epsilon)/p \\ \downarrow & & \downarrow & & \downarrow \\ (\mathfrak{A}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \end{array}$$

Lemma 3.0.9. Let $0 \le \epsilon < 1/2$ such that $p^\epsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$ makes sense. There is a unique commutative diagram

$$\begin{array}{ccccc} \mathfrak{A}(p^{-1}\epsilon) & \longrightarrow & \mathfrak{X}(p^{-1}\epsilon) & \longrightarrow & \mathfrak{X}^*(p^{-1}\epsilon) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{A}(\epsilon) & \longrightarrow & \mathfrak{X}(\epsilon) & \longrightarrow & \mathfrak{X}^*(\epsilon) \end{array}$$

that gets identified with the following commutative diagram, cf. Lemma ? and Lemma ? after modulo $p^{1-\epsilon}$

$$\begin{array}{ccccc} \mathfrak{A}(p^{-1}\epsilon)/p & \longrightarrow & \mathfrak{X}(p^{-1}\epsilon)/p & \longrightarrow & \mathfrak{X}^*(p^{-1}\epsilon)/p \\ \downarrow & & \downarrow & & \downarrow \\ (\mathfrak{A}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{A}(\epsilon)/p & \longrightarrow & \mathfrak{X}(\epsilon)/p & \longrightarrow & \mathfrak{X}^*(\epsilon)/p \end{array}$$

Proof. TODO: The map $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon)$ comes from the moduli interpretation, the weak canonical subgroup, and the Hasse invariant. Then $\mathfrak{A}(p^{-1}\epsilon) \to \mathfrak{A}(\epsilon)$ is obtained by base-change. The extension to $\mathfrak{X}^*(p^{-1}\epsilon) \to \mathfrak{X}^*(\epsilon)$ is done using Hartog's extension principle.

We first construct the map $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon)$. Let S be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $(f, [u])$ be a pair where

- $f : \mathrm{Spf}(S) \to \mathfrak{X}$ is a map of formal schemes over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$; its equivalent to a map $f : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$.

- $u \in H^0(\mathrm{Spec}(S/p), \overline{f}^* \omega^{\otimes(1-p)})$ is a section such that $u \cdot \overline{f}^* \mathrm{Ha} = p^{p^{-1}\epsilon} \in S/p$.

The map $f : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ gives an Abelian scheme $A \to \mathrm{Spec}(S)$ with principal polarization and level K^p structure. We claim that $A \to \mathrm{Spec}(S)$ satisfies strong $O(1, \epsilon)$, i.e. $\mathrm{Ha}(A_1/\mathrm{Spec}(S_1))^p$ divides p^ϵ. This follows from

$$p^{p^{-1}\epsilon} = u \cdot \overline{f}^* \mathrm{Ha} = u \cdot \mathrm{Ha}(A_1/\mathrm{Spec}(S_1)).$$

Let $C \subset A[p]$ be the strong canonical subgroup of level 1. We get an Abelian scheme $A/C \to \mathrm{Spec}(S)$ equipped with induced polarization and level structure TODO: Clarify; use totally isotropic, which corresponds to a map $g : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$. Then we declare that the pair $(f, [u])$ gets mapped to the pair $(g, [u^p])$. □

Lemma 3.0.10. Let $0 \leq \epsilon < 1$ such that $p^\epsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$ makes sense. Let $m \geq 0$. The formal Abelian scheme $\mathfrak{A}(p^{-m}\epsilon) \to \mathfrak{X}(p^{-m}\epsilon)$ admits a canonical subgroup $C_m \subset \mathfrak{A}(p^{-m}\epsilon)[p^m]$ of level m, in the sense that TODO: reduce to non-formal situation and explain. This induces a map $\mathcal{X}(p^{-m}\epsilon) \to \mathcal{X}_{\Gamma_s(p^m)}$ of the adic generic fiber given by the pair $(\mathcal{A}(p^{-m}\epsilon)/C_m, \mathcal{A}(p^{-m}\epsilon)[p^m]/C_m)$. This map extends uniquely to a map $\mathcal{X}^*(p^{-m}\epsilon) \to \mathcal{X}^*_{\Gamma_s(p^m)}$. The two maps are both open immersions of adic spaces. Moreover, for $m \geq 1$, the diagram

$$\begin{tikzcd} \mathcal{X}^*(p^{-m-1}\epsilon) \arrow[r] \arrow[d] & \mathcal{X}^*_{\Gamma_s(p^{m+1})} \arrow[d] \\ \mathcal{X}^*(p^{-m}\epsilon) \arrow[r] & \mathcal{X}^*_{\Gamma_s(p^m)} \end{tikzcd}$$

is a pullback diagram, where the vertical map on the left is induced from the map $\mathfrak{X}(p^{-m-1}\epsilon) \to \mathfrak{X}(p^{-m}\epsilon)$, cf. Lemma ??.

4. The Anti-Canonical Towers

5. The Topological Hodge–Tate Map

6. The Perfectoid Hodge–Tate Map

Appendix A. Review of Abelian Schemes

Definition A.0.1. Let S be a scheme. An Abelian scheme over S is a proper smooth group scheme over S that is geometrically connected.

Appendix B. Review of Deformation Theory

Definition B.0.1 ([Ill71, pp. II.1.2.1, II.1.2.3])**.** Let $A \to B$ be a map of rings. The simplicial A-algebra $P_A(B)$ is defined by $P_A(B)_0 = A[B]$ and $P_A(B)_n = A[P_A(B)_{n-1}]$ for $n \geq 1$. The standard resolution of B over A is the argumentation $P_A(B) \to B$ where B is viewed as a constant simplicial A-algebra. The cotangent complex of B over A is the simplicial B-module $L_{B/A} = \Omega^1_{P_A(B)/A} \otimes_{P_A(B)} B$.

Remark B.0.2. This definition works in a general topos.

Definition B.0.3 ([Ill72, p. VII.1.1.1])**.** Let S be a scheme. Let S_{zar} be the small Zariski site over S. Let S_{fpqc} be the big fqpc site over S. The natural inclusion $S_{\mathrm{zar}} \to S_{\mathrm{fpqc}}$ induces a geometric map $(\epsilon^*, \epsilon_*) : \mathrm{Sh}(S_{\mathrm{zar}}) \rightleftarrows \mathrm{Sh}(S_{\mathrm{fpqc}})$.

Definition B.0.4. Let $f : X \to Y$ be a map of schemes. The cotangent complex is $L_{X/Y}$.

Definition B.0.5. Let S be a scheme. Let G be a group scheme over S that is flat and locally of finite presentation. Let $e : S \to G$ be the unit. The co-Lie complex is $\ell_G = Le^* L_{G/S}$, and the Lie complex is $\ell_G^\vee = R\underline{\mathrm{Hom}}(\ell_G, \mathcal{O}_S)$. Define $\underline{\ell}_G = L\epsilon^* \ell_G$.

Lemma B.0.6 ([Ill72, Theorem VII.4.2.5])**.** Let $f : S \to T$ be a map of schemes. Let $i : S \to S'$ be a T-extension by a quasi-coherent module I. Let A be a "schéma en anneaux" over T that is, as a scheme over T, tor-independent (c.f. [SGA6, Definition III.1.5]) with both S and S'. Let F (resp. G') be "schéma en A-modules" that are flat and locally of finite presentation over S (resp. S'). Let G be a "schéma en A-module" over S induced by G'. Let $u : F \to G$ be a morphism of "schémas en A-modules". Let K be the complex fitting into the distinguished triangle $K \to \ell_F^\vee \to \ell_G^\vee \to K[1]$. It is an object in $D(A \otimes_{\mathbb{Z}}^L \mathcal{O})$. Then there is an obstruction $\omega(u, G') \in \operatorname{Ext}_A^2(F, K \otimes_{\mathcal{O}}^L \epsilon^* I)$ which is zero if and only if there exists a pair (F', u') where F' is a deformation of F as "un schéma en A-modules" flat over S' and a map $u' : F' \to G'$ extending u.

Lemma B.0.7. Let S be a scheme. Let $i : S \to S'$ be an extension by a quasi-coherent module I. Suppose S and S' are both tor-indepeendent with $\operatorname{Spec}(\mathbb{Z})$. Let F (resp. G') be commutative group schemes over S (resp. S') that are flat and locally of finite presentation. Let G be a commutative group scheme over S induced by G'. Let $u : F \to G$ be a morphism of group schemes over S. Let K be the cone of the map $\ell_F^\vee \to \ell_G^\vee$. There is an obstruction $\omega(u, G') \in \operatorname{Ext}^1(F, K \otimes^L I)$ which vanishes if and only if there exists a pair (F', u') where F' is a deformation of F as a commutative group scheme that is flat over S', and $u' : F' \to G'$ is a map extending u.

Lemma B.0.8 ([Sch15, Theorem III.2.1])**.** Let A be a ring. Let G and H be commutative group schemes over A that are flat and of finite presentation, with a group map $u : H \to G$. Let $B \to A$ be a square-zero thickening with the argumentation ideal J. Let $\widetilde{G}$ be a lift of G to B. Let K be a cone of the map $\ell_H^\vee \to \ell_G^\vee$ of Lie complexes. Then there is an obstruction class $\omega \in \operatorname{Ext}^1(H, K \otimes^L J)$ which vanishes if and only if there exists a pair $(\widetilde{H}, \widetilde{u})$ where $\widetilde{H}$ is a flat commutative group scheme over B, and $\widetilde{u} : \widetilde{H} \to \widetilde{G}$ is a map lifting $u : H \to G$. Moreover, the obstruction class is functorial in J, in the following sense. If $B' \to A$ is another square-zero thickening with the argumentation ideal J', with a map $B \to B'$ over A, then $\omega' \in \operatorname{Ext}^1(H, K \otimes^L J')$ is the image of $\omega \in \operatorname{Ext}^1(H, K \otimes^L J)$ under the map $J \to J'$.

Appendix C. Moduli Interpretation of PEL Type Shimura Varieties

For PEL type Shimura variety, see Milne.

This is [Kot92]; also, "PEL-type $\mathcal{O}$-lattice", cf. [Lan13, Definition 1.2.1.3]

Let p be a prime. Let B be a finite-dimensional simple $\mathbb{Q}$-algebra with center F. Let $\mathcal{O}_B$ be a $\mathbb{Z}_{(p)}$-order in B. Let $*$ be a positive involution on B that preserves $\mathcal{O}_B$. Let V be a non-degenerate skew-Hermitian B-module. Let G be the group of automorphisms of the skew-Hermitian B-module V. Let K^p be a compact open subgroup of $G(\mathbb{A}_f^p)$. Let $h : \mathbb{C} \to \operatorname{End}_B(V_{\mathbb{R}})$ be an $\mathbb{R}$-algebra homomorphism such that $(h(\overline{z}) = h(z)^*$, and that the symmetric bilinear form $(v, h(i)w)$ on $V_{\mathbb{R}}$ is positive definite. The map h determines a decomposition $V_{\mathbb{C}} = V_1 \oplus V_2$. Here V_1 is the subspace of $V_{\mathbb{C}}$ on which $h(z)$ acts by z. The field of definition of the isomorphism class of the complex representation V_1 of B is a number field E, with ring of integers $\mathcal{O}_E$.

Consider the following moduli problem over $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

Also, see [Lan13, Definition 1.4.1.4, Theorem 1.4.1.11, Remark 1.4.1.13].

Definition C.0.1 ([Lan13, Definition 1.2.1.3])**.** Let B be a finite-dimensional semi-simple algebra over $\mathbb{Q}$ with positive involution $*$ and center F, where "positive" means $\operatorname{tr}_{B/\mathbb{Q}}(xx^*) > 0$ for all $x \neq 0$ in B. Let $\mathcal{O}$ be an order in B mapped to itself under $*$.

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