NOTES

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Contents

1. Topology

1.1. Compactness.

Remark 1.1.1. We shall try to drop the confusing "quasi-compact" terminology.

Definition 1.1.2. A map $f : X \to Y$ of topological spaces is called quasi-compact if the preimage $f^{-1}(K)$ is compact for every compact open subset K of Y.

Definition 1.1.3. Let X be a topological space. A subspace $Z \subset X$ is called retrocompact if the inclusion $Z \to X$ is quasi-compact.

Definition 1.1.4. Let X be a topological space. A subset of X is called constructible if it is a finite union of subsets of X of the form $U \cap V^c$ where U and V are open and retrocompact in X.

Remark 1.1.5. Our definition of "constructible" is called "globally constructible" in ((todo: cite EGA1 second edition)), cf. [Sta, Tag 04ZC].

Lemma 1.1.6. Let X be a topological space. Let U be an open subset of X. Let E be a constructible subset of X. Then $E \cap U$ is constructible in U.

Proof. ((todo: ...)) □

2. Infinity categories

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2.3. Limits and colimits.

2.4. Yoneda lemma.

2.5. **Adjoint functors.**

2.6. **Presentable ∞-categories.**

3. Commutative algebra

3.1. **Hilbert Nullstellensatz.**

3.2. **todo.**

Lemma 3.2.1. Let $I_1, \dots, I_n$ be ideals of A such that $I_i + I_j = A$ for all $i \neq j$. Then $\cap_i I_i = \prod_i I_i$, and the projections $A \to A/I_i$ induce an isomorphism of rings

$$A/\prod_i I_i \to \prod_i A/I_i.$$

Lemma 3.2.2. Let A be a ring. Let $I_1, \dots, I_n$ be ideals of A. Let $\mathfrak{p}$ be a prime ideal of A containing $\prod_i I_i$. Then $I_i \subset \mathfrak{p}$ for some i. If $\mathfrak{p} = \cap_i I_i$ or $\mathfrak{p} = \prod_i I_i$, then $\mathfrak{p} = I_i$ for some i.

Lemma 3.2.3. Let A be a ring. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be prime ideals of A. Let I be an ideal of A such that $I \subset \cup_i \mathfrak{p}_i$. Then $I \subset \mathfrak{p}_i$ for some i.

Lemma 3.2.4. Let A be a ring. Let M be an A-module of finite type. Let I be an ideal of A. If $IM = M$, then there exists $f \in 1 + I$ such that $fM = 0$.

4. Adic spaces

Definition 4.0.1. A Huber ring A is called Tate if it contains a topologically nilpotent unit.

Lemma 4.0.2. If A is Tate with ring of definition A_0 and pseudo-uniformizer ϖ, then ϖA_0 is an ideal of definition.

Remark 4.0.3. The smallest choice for A^+ is the integral closure of $\mathbb{Z} + A^{\circ\circ}$ in A.

Remark 4.0.4. Let (A, A^+) be a Huber pair. We can define its "completion". Let

$$\widehat{A} = \lim_n A/I^n.$$

(caution: here I is an ideal of A_0, and the quotient A/I^n is only a quotient of Abelian groups.) It is equipped with the unique topology such that $\widehat{A}_0 = \lim_n A_0/I^n$ is open. Hence $\widehat{A}$ is a complete topological Abelian group. There is a unique ring structure on $\widehat{A}$ such that $A \to \widehat{A}$ is continuous. Then $\widehat{A}$ is a Huber ring, with a ring of definition $\widehat{A}_0$ and an ideal of definition $I\widehat{A}_0$. Let $\widehat{A}^+$ be the closure of the imgage of A^+ in $\widehat{A}$. Then $(\widehat{A}, \widehat{A}^+)$ is a Huber pair. We have a natural homeomorphism

$$\mathrm{Spa}(\widehat{A}, \widehat{A}^+) \to \mathrm{Spa}(A, A^+)$$

which also identifies rational subsets on both sides.

Lemma 4.0.5. Let (A, A^+) be a complete Huber pair.

(1) If A is non-zero, then $\mathrm{Spa}(A, A^+)$ is non-empty.
(2) A^+ is equal to the subring of A of elements $f \in A$ such that $|f(x)| \leq 1$ for all $x \in \mathrm{Spa}(A, A^+)$.
(3) $f \in A$ is invertible if and only if $|f(x)| \neq 0$ for all $x \in \mathrm{Spa}(A, A^+)$.

Proof. Proof of (3). For an ideal $\mathfrak{a}$ of A, we can form the "quotient" $(A/\mathfrak{a}, (A/\mathfrak{a})^+)$. Indeed, $(A/\mathfrak{a})_0 = A_0/(A_0 \cap \mathfrak{a})$ with ideal of definition $I/(I \cap \mathfrak{a})$, and take $(A/\mathfrak{a})^+$ to be the integral closure of $A^+/(A^+ \cap \mathfrak{a})$ in $A/\mathfrak{a}$. $\square$

Remark 4.0.6. Let (A, A^+) be a complete Huber pair. Let U be a rational subset of $X = \mathrm{Spa}(A, A^+)$. Then U itself can be "identified" as an adic spectrum.

Lemma 4.0.7. Let (A, A^+) be a complete Huber pair. Let U be a rational subset of $X = \mathrm{Spa}(A, A^+)$. Then there exists a complete Huber pair (B, B^+) together with a map $(A, A^+) \to (B, B^+)$ of Huber pairs such that $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ factors through U and is universal, i.e.

$$\begin{tikzcd} \mathrm{Spa}(B, B^+) \arrow[r] & U \arrow[r] & \mathrm{Spa}(A, A^+) \\ \mathrm{Spa}(C, C^+) \arrow[u, dashed, "\exists!"] \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

Moreover, $\mathrm{Spa}(B, B^+) \to U$ is a homeomorphism.

Remark 4.0.8. Always taking $A^+ = A^\circ$ is not robust under taking rational subsets. So an explicit choice of A^+ is required.

Example 4.0.9. Let K be a non-Archimedean field. Let $K^+ \subset K^\circ = \mathcal{O}_K$ be an open valuation subring of K. If $K^+ = K^\circ = \mathcal{O}_K$, then $\mathrm{Spa}(K, K^+)$ has only one point. In general, $\mathrm{Spa}(K, K^+)$ has more than one point. It looks like

$$\bullet \rightsquigarrow \bullet \rightsquigarrow \bullet \rightsquigarrow \bullet$$

where "$\rightsquigarrow$" means specialization of points.

Remark 4.0.10. If A is Tate, then every point in $\mathrm{Spa}(A, A^+)$ admits a unique rank 1 generalization.

Example 4.0.11. $\mathrm{Spa}(\mathbb{Z}_p[[T]], \mathbb{Z}_p[[T]])$ is not Tate.

Remark 4.0.12. Relation with schemes and formal schemes.

Take A discrete and $A^+ = A$. Then we have maps $\mathrm{Spec}(A) \to \mathrm{Spa}(A, A) \to \mathrm{Spec}(A)$, where the second map is given by taking the kernel of a valuation, and the first map is given by taking the trivial (i.e. rank zero) valuation on the residue field. The composition is identity. This identifies $\mathrm{Spec}(A)$ with the subspace of rank zero valuations of $\mathrm{Spa}(A, A)$.

Remark 4.0.13. If X is a smooth projective connected curve over a field k_0, with function field k, then the Riemann–Zariski space $\mathrm{RZ}(k, k_0)$ is homeomorphic to X.

((todo: check: Buzzard–Verberkmoes, "Stably uniform affinoids are sheafy"))

Definition 4.0.14. Let (A, A^+) be a Huber pair. It is called stably uniform if all the rational subsets are uniform, i.e. B is uniform (i.e. B° is bounded) for all rational subsets $\mathrm{Spa}(B, B^+) \subset \mathrm{Spa}(A, A^+)$.

Remark 4.0.15. The case "stably uniform Tate Huber pairs are sheaf" is extremely important, as it contains adic spaces that are hightly non-Noetherian, e.g. perfectoid spaces.

5. Trace Formula

Let G be a reductive group over $\mathbb{Q}$. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$.

We have a right action of $G(\mathbb{A})$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$.

Let $\mathcal{A}_{\mathrm{cusp}}(G^1)$ be the set of isomorphism classes of irreudiclbe representations of $G(\mathbb{A})^1$ in L^2_{cusp}.

Recall $C_c^\infty(G(\mathbb{A})) = C_c^\infty(G(\mathbb{R})) \otimes_{\mathbb{C}} C_c^\infty(G(\mathbb{A}_f))$. We define $C_c^\infty(G(\mathbb{A})^1)$ to be the set of restrictions of $f \in C_c^\infty(A_G(\mathbb{R})^0\backslash G(\mathbb{R})) \otimes C_c^\infty(G(\mathbb{A}_f))$.

For every $f \in C_c^\infty(G(\mathbb{A})^1)$, we define an operator $R^1_{\mathrm{disc}}(f)$ on L^2_{cusp}. Then $R^1_{\mathrm{disc}}(f)$ is of trace class, and

$$\mathrm{tr} R^1_{\mathrm{disc}}(f) = \sum_{\pi \in \mathcal{A}_{\mathrm{cusp}}(G^1)} m_\pi f_G(\pi).$$

((todo: finish the part, i.e. until page 6 of lecture notes))

Let P_0 be a minimal parabolic subgroup of G. Fix a Levi component M_0 of P_0. For a standard parabolic subgroup P of G, we denote by M_P the standard Levi component of P. The unipotent radical of M is denoted by N_P. Thus $P = M_P N_P$. Fix a maximal compact subgroup K of $G(\mathbb{A})$ such that $G(\mathbb{A}) = P_0(\mathbb{A})K$ (recall: this is called the Iwasawa decomposition).

Definition 5.0.1. A cuspidal datum is an equivalence class of pairs (P, σ) where P is a standard parabolic subgroup of G, and $\sigma \in \mathcal{A}_{\mathrm{cusp}}(M_P^1)$. Two pairs (P, σ) and (P', σ') are equivlant if (M_P, σ) and $(M_{P'}, \sigma')$ are conjugate under $G(\mathbb{Q})$. The set of cuspidal data is denoted $\mathfrak{X}^G$.

Consider the parabolic induction

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(1_{N_P(\mathbb{A})} \boxtimes R^1_{M_P}).$$

Here $R^1_{M_P}$ is the regular representation of $M_P(\mathbb{A})$ on $L^2(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1) \simeq L^2(M_P(\mathbb{Q})A_P(\mathbb{R})\backslash M_P(\mathbb{A}))$. Thus $R^1_{M_P}$ is the induction of the trivial representation from $M_P(\mathbb{Q})A_P(\mathbb{R})^0$ to $M_P(\mathbb{A})$. Therefore

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(1_{N_P(\mathbb{A})} \boxtimes R^1_{M_P}) = \mathrm{Ind}_{N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0}^{G(\mathbb{A})}\mathrm{triv}.$$

This is the regular representation of $G(\mathbb{A})$ on

$$L^2(N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A})).$$

The space is denote by $\mathcal{H}_P$. It carries an inner product defined by

$$\langle \phi, \psi\rangle = \int_K \int_{M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1} \phi(mk)\overline{\psi(mk)}dmdk.$$

The representation, denoted by I_P, is not unitary.

References

[Sta] The Stacks Project Authors. Stacks project.

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