CONDENSED MATHEMATICS

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Contents

Remark 0.0.1. This is based on the lectures of Clausen and Scholze.

1. Introduction

Remark 1.0.1. The previous approaches to analytic geometry shares a common problem: there isn't a good theory of descent. It's still unknown how to define a category of quasi-coherent sheaves. Part of the reason comes from the fact that the category of topological modules does not form an abelian category.

Another motivation: in the geometrization of local Langlands, $\mathbb{Q}_p$ is replaced by a family of Fargues–Fontaine curves, which are exotic adic spaces. It's very possible that the global Langlands can be geometrized in the same fashion, by replacing $\mathbb{Z}$ with a new geometric object, which should contain both archimedean and non-archimedean parts. Scholze is currently working on the archimedean case.

The new idea is to replace topological spaces by a new kind of objects. The idea dates back to Grothendieck, that the object should be detected by a collection of good test spaces. The previous choice is profinite spaces.

A light profinite set is a countable limit of finite sets. A light condensed set is a sheaf of sets on the category of light profinite sets with respect to the Grothendieck topology whose covering families are finite collections of jointly surjective continuous maps. More explicitly, a light condensed set is a functor $X : \mathrm{LightProf}^{\mathrm{opp}} \to \mathrm{Set}$ such that

(1) $X(\emptyset) = *$.
(2) $X(S \sqcup T) \to X(S) \times X(T)$ is an isomorphism.
(3) If $T \to S$ is a surjection, then

$$X(S) \to X(T) \rightrightarrows X(T \times_S T)$$

is an equalizer diagram.

Example 1.0.2. Every topological spacae X gives a condensed set by $X(S) = \mathrm{Cont}(S, X)$.

Remark 1.0.3. Two examples of light profinite sets: $S = *$ and $S = \mathbb{N}\cup\{\infty\}$, the one point compactification of $\mathbb{N}$. For a light condensed set, $X(*)$ should be though as the underlying set, and for $S = \mathbb{N} \cup \{\infty\}$, $X(S)$ gives the set of convergent sequences.

Remark 1.0.4. By allowing all surjections as covers, it gives a simplification of the structure of the category of light condensed sets. On the other hand, requiring the Grothendieck topology to be finitary gives good categorical compactness properties for light profinite sets, which embeds into light condensed sets.

From now on the word "light" is dropped. The notion of condensed abelian groups, condensed rings, and condensed modules are immediate.

Remark 1.0.5. Why condensed rings are not right objects. They are not enough to give a good geometry. Reason: the category of condensed rings has pushouts, i.e. relative tensor products, but $(A \otimes_k B)(*) = A(*) \otimes_{k(*)} B(*)$, i.e. $A \otimes_k B$ just gives a condensed structure on the usual (algebraic) tensor product, which is not the correct thing.

To fix this, additional structure on condensed rings are needed. Some class of (condensed) modules are recorded, to be considered as "complete". This gives the notion of analytic rings.

Remark 1.0.6. The general experience in algebraic geometry shows that the generally correct fiber product of schemes is actually the derived fiber product, which corresponds to derived relative tensor product on the affines.

So "rings" should be understood in the derived sense. There are two options: E_∞-algebras and animated commutative rings, i.e. presented by simplicial commutative rings. The latter is used here because it more related to the classical structures. But the first case also works. Think about the possibility of a theory over the sphere spectrum!

A condensed animated ring is a hypersheaf of animated rings on the site of (light) profinite sets. From now on rings always mean animated rings. One basic invariant of a condensed animated ring R is the (full) derived category $D(R)$.

Remark 1.0.7. Hypersheaf instead of sheaf: we can prove Postinov towers.

An analytic ring is a pair $R = (R^\triangleright, D(R))$ where $R^\triangleright$ is a condensed ring and $D(R) \subset D(R^\triangleright)$ is a full category such that

1. $R^\triangleright \in D(R)$.
2. $D(R)$ is closed under all limits and colimits.
3. If $N \in D(R)$ and $M \in D(R^\triangleright)$, then $R\underline{\mathrm{Hom}}_R^\triangleright(M, N) \in D(R)$.
4. If $\widehat{(-)}_R$ denotes the left adjoint to the inclusion $D(R) \subset D(R^\triangleright$, then $\widehat{(-)}_R$ sends $D(R)_{\geq 0})$ to $D(R^\triangleright)_{\geq 0}$.

A map of analytic rings $R \to S$ is a map of condensed rings $R^\triangleright \to S^\triangleright$ such that if $M \in D(S)$ then the restriction of scalars of M to R lies in $D(R)$.

Remark 1.0.8. There exists a t-structure on $D(R)$ such that $D(R)_{\geq 0} = D(R) \cap D(R^\triangleright)_{\geq 0}$ and $D(R)_{\leq 0} = D(R) \cap D(R^\triangleright)_{\leq 0}$. In particular, $D(R)^\heartsuit = D(R) \cap D(R^\triangleright)^\heartsuit$.

For a (light) profinite set S, the free R-module is $R[S] = \widehat{(R^\triangleright[S])}_R$. These generate $D(R)_{\geq 0}$ under colimits.

Remark 1.0.9. $R^\triangleright[S]$ can be viewd as the space of $R^\triangleright$-linear combinations of dirac measures on S. Then $R[S]$ is some completion, i.e. a bigger space of measures.

Remark 1.0.10. $M \in D(R^\triangleright)^\heartsuit$ lies in $D(R)^\heartsuit$ if and only if for all $f : R^\triangleright[S] \to M$, there is a unique extension to $R[S] \to M$. In other words, if $f : S \to M$ and $\mu \in R[S]$, then there is a well-defined $\int_S f d\mu \in M$.

Filtered colimits, or more generally sifted colimits, of analytic rings, has nice properties

1. $(\mathrm{colim}\, R_i)^\triangleright = \mathrm{colim}\, R_i^\triangleright$.
2. $(\mathrm{colim}\, R_i)[S] = \mathrm{colim}\, R_i[S]$.

Pushouts are interesting. Consider the pushout of $A \leftarrow k \to B$. The derived category can be described directly: $D(A \otimes_k B)$ is a full subcategory of $D(A^\triangleright \otimes_{k^\triangleright} B^\triangleright)$ such that the underlying $A^\triangleright$-modules (resp. $B^\triangleright$-modules) lies in $D(A)$ (resp. $D(B)$). But $(A^\triangleright \otimes_{k^\triangleright} B^\triangleright, D(A \otimes_k B))$ is not an analytic ring. The only condition not being satisfied is $A^\triangleright \otimes_{k^\triangleright} B^\triangleright \in D(A \otimes_k B)$. To fix this, apply a completion procedure, i.e. apply left adjoint to $D(A \otimes_k B) \subset D(A^\triangleright \otimes_{k^\triangleright} B^\triangleright)$.

Adic spaces are related to solid analytic rings. Let R be an analytic ring. Consider the space of measures on natural numbers $\mathcal{M}_R(\mathbb{N}) = R[\mathbb{N} \cup \infty]/R[\infty]$. It classifies null sequences in R-modules. Addition on $\mathbb{N}$ induces a ring structure on $\mathcal{M}_R(\mathbb{N})$. There is a sequence $R[T] \to \mathcal{M}_R(\mathbb{N}) \to R[[T]]$. The ring R is called solid if $\mathcal{M}_R(\mathbb{N})/(T-1) = 0$.

Remark 1.0.11. If R is solid, then there is $\mu \in \mathcal{M}_R(\mathbb{N})$ such that $(T-1)\mu = 1 \in \mathcal{M}_R(\mathbb{N})$. μ corresponds to $\sum_n T^n$. So every null sequence is summable.

Theorem 1.0.12. There exists a solid analytic ring $\mathbb{Z}^{\blacksquare} = (\mathbb{Z}, D(\mathbb{Z}^{\blacksquare}))$ such that an analytic ring R is solid if and only if there exists a (necessarily unique) map $\mathbb{Z}^{\blacksquare} \to R$. Moreover, if $S = \lim S_i$ is a profinite set, then

(1) $\mathbb{Z}^{\blacksquare}(S) = \lim \mathbb{Z}[S_i]$. This is also (abstractly) isomorphic to a countable product of $\mathbb{Z}$, i.e. $\prod_I \mathbb{Z}$ where I is countable.
(2) $\prod_I \mathbb{Z}$ is compact projective generator of $D(\mathbb{Z}^{\blacksquare})^{\heartsuit}$, flat with respect to $-\otimes_{\mathbb{Z}^{\blacksquare}} -$. Moreover $\prod_I \mathbb{Z} \otimes_{\mathbb{Z}^{\blacksquare}} \prod_J \mathbb{Z} = \prod_{I\times J} \mathbb{Z}$.
(3) The collection of finitely presented objects in $D(\mathbb{Z}^{\blacksquare})^{\heartsuit}$ is abelian, closed under extensions. Every finitely presented M has a resolution by $\prod_I \mathbb{Z}$ of length ≤ 2.

Remark 1.0.13. $\mathbb{Z}^{\blacksquare}$ behaves like a regular ring of dimension 2!

2. Light condensed sets

Definition 2.0.1. A profinite set is a totally disconnected compact Hausdorff topological space. The category of profinite set, denoted ProFin, is the full subcategory of the category of topolgoical spaces.

Definition 2.0.2. Let FinSet be the category of finite sets.

Definition 2.0.3. A Boolean ring is a commutative ring for which every element is idempotent. The category of Boolean ring is denoted by BRing.

Lemma 2.0.4. The following categories are equivalent:

(1) The category ProFin of profinite sets.
(2) The pro-completion of FinSet.
(3) The opposite $\mathrm{BRing}^{\mathrm{opp}}$ of the category of Boolean algebras.

Definition 2.0.5. The poset $\mathbb{N} = \{0 < 1 < \cdots\}$ is viewed as a category with exactly one arrow from i to j for $i \geq j$.

Lemma 2.0.6. Let S be a profinite set. The following are equivalent:

(1) The set $\mathrm{Cont}(S, \mathbb{F}_2)$ is countable, where $\mathbb{F}_2$ is equipped with the discrete topology. (how about $\mathrm{Cont}(S,\mathbb{Z})$)
(2) S is a metrizable topological space.
(3) S is a second countable topological space.
(4) S is a sequential limit of finite sets equipped with discrete topology.

Remark 2.0.7. Let S be a profinite set. Let I be the set of maps $f : S \to \mathbb{Z}$ such that $\mathrm{im}(f)$ is a finite set and each $f^{-1}(n)$ is open. For $f, g \in I$, define $f \leq g$ if there exists a (necessarily unique) map $\phi_{f,g} : \mathrm{im}(g) \to \mathrm{im}(f)$ such that $f = \phi_{f,g} \circ g$. Then $(\mathrm{im}(f))_{f\in I}$ is a projective system of finite sets, and we have a homeomorphism $S \to \lim_{f\in I} \mathrm{im}(f)$.

Remark 2.0.8. Let S be a profinite set. Let I be the set of partitions $S = \sqcup_{\lambda\in\Lambda} S_\lambda$ of S into finitely many non-empty open subsets. Suppose S is second countable, i.e. S has a countable basis $\mathcal{B}$. Each partition in I is a finite subset of $\mathcal{B}$. So I is countable.

Lemma 2.0.9.

Definition 2.0.10. Let X and Y be light profinite sets. Define $\mathrm{Hom}_{\mathrm{LProFin}}(X, Y) = \lim_j \mathrm{colim}_i \mathrm{Hom}_{\mathrm{Set}}(X_i, Y_j)$.

Remark 2.0.11. Every element in $\mathrm{Hom}_{\mathrm{LProFin}}(X, Y)$ is a compatible sequence $\phi_j \in \mathrm{colim}_i \mathrm{Hom}_{\mathrm{Set}}(X_i, Y_j)$. Each ϕ_j is represented by some $\phi_{j,i_j} \in \mathrm{Hom}_{\mathrm{Set}}(X_{i_j}, Y_j)$. So to specify a map $X \to Y$, we need a map $f : \mathbb{N} \to \mathbb{N}$, and for each $n \in \mathbb{N}$ a map $\phi_n : X_{f(n)} \to Y_n$, satisfying the following compatibility condition. Let $y_n : Y_n \to Y_{n-1}$ be the transition map. Then $y_n \circ \phi_n : X_{f(n)} \to Y_{n-1}$ should represent the same element as $y_{n-1} : X_{f(n-1)} \to Y_{n-1}$ in $\mathrm{colim}_i \mathrm{Hom}_{\mathrm{Set}}(X_i, Y_{n-1})$. In other words, there exists X_m with maps $X_m \to X_{f(n-1)}$ and $X_m \to X_{f(n)}$ such that $X_m \to X_{f(n)} \to Y_n \to Y_{n-1}$ and $X_m \to X_{f(n-1)} \to Y_{n-1}$ are the same.

The representative ϕ_n can be chosen to be canonical as follows. The set of integers $m \in \mathbb{N}$ such that ϕ_n is equivalent to some element $X_m \to Y_n$ is non-empty, and thus has a minimal element, which we denote by $f'(n)$. The compatibility condition then implies that $f'(n) \geq f'(n-1)$.

Definition 2.0.12. A Stone space is a totally disconnected compact Hausdorff topological space.

Definition 2.0.13. A topological space X is totally separated if for distinct points $x, y \in X$ there exists a decomposition $X = X_1 \sqcup X_2$ into clopen subsets such that $x \in X_1$ and $y \in X_2$.

Lemma 2.0.14. A topological space is a Stone space if and only if it is compact and totally separated.

Remark 2.0.15. A totally separated topological space is totally disconnected. The converse is false.

Lemma 2.0.16. Let S be a totally disconnected compact Hausdorff topological space. Then $\mathrm{Cont}(S, \mathbb{F}_2)$ is a Boolean ring.

Lemma 2.0.17. Let A be a Boolean ring. Then the topological space $\mathrm{Spec}(A)$ is a Stone space. There is a natural bijection $\mathrm{Hom}_{\mathrm{CRing}}(A, \mathbb{F}_2) \simeq \mathrm{Spec}(A)$, sending $\phi : A \to \mathbb{F}_2$ to $\ker(\phi)$.

Definition 2.0.18. A profinite space is a Stone space. It is a cofiltered limit of finite discrete spaces.

Definition 2.0.19. Let S be a profinite space. The size of S is the cardinality of S. The weight of S is the cardinality of $\mathrm{Cont}(S, \mathbb{F}_2)$.

Definition 2.0.20. Let $\widehat{\mathbb{N}}$ be the one-point compactification of the discrete set $\mathbb{N}$. It has underlying set $\widehat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. The open subsets are either subsets of $\mathbb{N}$, or subsets of the form $(\mathbb{N}\backslash F) \cup \{\infty\}$ where $F \subset \mathbb{N}$ is a finite subset. Then $\widehat{\mathbb{N}}$ is a profinite space.

Lemma 2.0.21. Let A be a countable Boolean ring. Then $\mathrm{Spec}(A)$ is a metrizable Stone space.

Lemma 2.0.22. A compact Hausdorff topological space is metrizable if and only if it is second countable.

Lemma 2.0.23. Let S be a profinite set. The following are equivalent.

Lemma 2.0.24. The category of light profinite sets admits countable limits.

Remark 2.0.25. The goal is to have a theory of global (i.e. over $\mathrm{Spec}(\mathbb{Z})$) perfectoid spaces.

Remark 2.0.26. In the p-adic setting, all p^n-roots, i.e. $1/p^n$-powers, are adjoined to the base field. So globally, it might be tempting to add all $\mathbb{Q}$-powers, or even $\mathbb{R}$-powers. Such objects can be described in the formalism of condensed sets.

Let ProFin be the category of profinite sets.

Lemma 2.0.27. The following categories are equivalent

(1) Pro(Fin), the pro-completion of the category of finite sets. The objects are "formal cofiltered limits" $\lim_{i\in I} S_i$ where each S_i is a finite set and I is a cofiltered poset. The morphisms are
$$\mathrm{Hom}(\lim_{i\in I} S_i, \lim_{j\in J} T_j) = \lim_{j\in J} \operatorname*{colim}_{i\in I} \mathrm{Hom}(S_i, T_j).$$
(2) The category of totally disconnected compact Hausdorff topological spaces.
(3) The opposite of the category of Boolean algebras. Recall that a Boolean algebra is a commutative ring R such that $x^2 = x$ for all $x \in R$.

The functors are as follows

- Each formal limit $\lim_{i\in I} S_i$ is mapped to the actual limit $\lim_{i\in I} S_i$ in the category of topological spcaes, where each S_i is equipped with the discrete topology.
- Every totally disconnected compact Hausdorff space S is mapped to the Boolean algebra $\mathrm{Cont}(S, \mathbb{F}_2)$ of continuous functions $S \to \mathbb{F}_2$ where $\mathbb{F}_2$ has the discrete topology.
- Every Boolean algebra A is mapped to the topological space $\mathrm{Spec}(A)$, which is also identified with $\mathrm{Hom}(A, \mathbb{F}_2)$.
- The space $\mathrm{Hom}(A, \mathbb{F}_2)$ is mapped to the formal limit $\lim_i \mathrm{Hom}(A_i, \mathbb{F}_2)$, where A_i is a finite subalgebra of A

There are two ways to measure how big a profinite set is. Let $S = \lim_i S_i$ be a profinite set. The size of S is the cardinality of S. The weight of S is the cardinality of $\mathrm{Cont}(S, \mathbb{F}_2)$. A profinite set is light if its weight λ satisfies $\lambda \leq \omega$, where ω is the first infinite ordinal, i.e. the set $\mathrm{Cont}(S, \mathbb{F}_2)$ is countable. If the weight λ is infinite, then λ is also equal to the minimal cardinality of I. So a profinite set is light if and only if it is a countable limit of finite sets.

Example 2.0.28. Examples of profinite sets.

(1) Finite sets are profinite sets.
(2) The one-point compactification of natural numbers $S = \mathbb{N} \cup \{\infty\} = \lim_n \{0, 1, \ldots, n, \infty\}$. Its size and weight are both ω.
(3) The Cantor set $S = \{0,1\}^{\mathbb{N}} = \lim_n \{0,1\}^n$. It has size 2^ω and weight ω.
(4) The Stone–Čech compactification $S = \beta\mathbb{N}$. It has size 2^{2^ω} and weight 2^ω since $\mathrm{Cont}(\beta\mathbb{N}, \mathbb{F}_2) \simeq \mathrm{Cont}(\mathbb{N}, \mathbb{F}_2) \simeq \mathcal{P}(\mathbb{N})$, where $\mathcal{P}(\mathbb{N})$ denotes the power set of $\mathbb{N}$.

Lemma 2.0.29. Let κ be the size and let λ be the weight. Then $\lambda \leq 2^\kappa$ and $\kappa \leq 2^\lambda$. Also, if $\kappa = \omega$, then $\lambda = \omega$.

Proof. It's clear that
$$\lambda = |\mathrm{Cont}(S, \mathbb{F}_2)| \leq |\mathrm{Map}(S, \mathbb{F}_2)| = 2^\kappa,$$
and
$$\kappa = |\mathrm{Hom}(A, \mathbb{F}_2)| \leq |\mathrm{Map}(A, \mathbb{F}_2)| = 2^\lambda.$$
Suppose $\kappa = \omega$. Write $S = \{s_0, s_1, \ldots\}$. Choose quotients $S \to S_n$ inductively such that $S \to S_{n-1}$ factors through $S \to S_n$ and $\{s_0, \ldots, s_n\}$ embeds into S_n. Then the induced map $S \to \lim_i S_i$ is bijective. □

Remark 2.0.30. The proof actually shows that if κ is infinite, then $\lambda \leq \kappa$.

Remark 2.0.31. I think the construction of S_i reduces to the following: given two distinct points x, y of a profinite set S, there exists a decomposition $S = \sqcup_i U_i$ into disjoint open subsets, such that x and y belongs to different U_i.

Lemma 2.0.32. The following categories are equivalent

(1) $\mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})$, whose objects are formal limits $\lim_{n\in\mathbb{N}} S_n$, and morphisms are given by
$$\begin{aligned}\mathrm{Hom}(\lim_{n\in\mathbb{N}} S_n, \lim_{m\in\mathbb{N}} T_m) &= \lim_{m\in\mathbb{N}} \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Hom}(S_n, T_m) \\ &= \operatorname*{colim}_{f} \lim_{m\in\mathbb{N}} \mathrm{Hom}(S_{f(m)}, T_m),\end{aligned}$$
where f ranges through all strictly increasing functions $\mathbb{N} \to \mathbb{N}$.
(2) The category of metrizable totally disconnected compact Hausdorff topological spaces.
(3) The opposite of the category of countable Boolean algebras.

Remark 2.0.33. Understand this computation.

Lemma 2.0.34. The category of all light profinite sets has all countable limits, and sequential limits of surjectives are surjective.

Remark 2.0.35. The second assertion is true for the category of all profinite sets.

Lemma 2.0.36. Let S be a light profinite set. Then there exists a surjection $\{0,1\}^{\mathbb{N}} \to S$.

Remark 2.0.37. This means the Cantor set $\{0,1\}^{\mathbb{N}}$ is a "universal model" of light profinite sets.

Lemma 2.0.38. Let S be a light profinite set. Let U be an open subset of S. Then U is a countable disjoint union of light profinite sets.

Proof. Write $S = \lim_n S_n$. Let $\pi_n : S \to S_n$ be the projection. Let $Z = S \backslash U$. Then $Z = \lim_n Z_n$ where Z_n is the image of Z under $\pi_n : S \to S_n$. So $U = \cup_n \pi_n^{-1}(S_n \backslash Z_n)$. Each $\pi_n^{-1}(S_n\backslash Z_n)$ is a clopen subset in S. So $U = \sqcup_n \pi_{n+1}^{-1}(S_{n+1}\backslash Z_{n+1}) \backslash \pi_n^{-1}(S_n \backslash Z_n)$ □

Remark 2.0.39. Closed subsets of profinite sets are profinite.

Remark 2.0.40. In general, for a profinite set S, there exists an open subset U with $H^i(U, \mathbb{Z}) \neq 0$ for $i > 0$.

Lemma 2.0.41. Let S be a non-empty light profinite set. Then S is an injective object in the category of profinite sets.

Proof. Let $Z \to X$ be an injection of profinite sets. The case $S = \{0,1\} = \mathbb{F}_2$ is clear because the map $\mathrm{Cont}(X, \mathbb{F}_2) \to \mathrm{Cont}(Z, \mathbb{F}_2)$ is surjective, i.e. every clopen subset of Z extends to a clopen subset of X. In general, write $S = \lim_n S_n$. Assume all maps $S_{n+1} \to S_n$ are surjective. Use induction on n and note that maps can be extended by fibers. □

A light condensed set is a sheaf on the category of light profinite sets for the Grothendieck topology generated by

(1) finite disjoint unions;
(2) all surjective maps.

Equivalently, a light condensed set is a functor $X : \mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})^{\mathrm{opp}} \to \mathrm{Set}$ such that

(1) $X(\emptyset) = *$.
(2) The natural map $X(S_1 \sqcup S_2) \to X(S_1) \times X(S_2)$ is an isomorphism.
(3) For every surjection $T \to S$,
$$X(S) \to X(T) \rightrightarrows X(T \times_S T)$$
is an equalizer diagram.

Let X be a light condensed set. The set $X(*)$ is called the underlying set of X.

Example 2.0.42. Let A be a topological space. The functor $\underline{A} : S \mapsto \mathrm{Cont}(S, A)$ defines a light condensed set. The underlying set of $\underline{A}$ is A. Note that $\underline{A}(\mathbb{N} \cup \{\infty\})$ is the set of convergent sequences (with choice of limit point) in A.

Let $C = \{0, 1\}^{\mathbb{N}}$ be the Cantor set. The set $\underline{A}(C)$ is equipped with an action of the monoid of continuous endomorphisms of C.

Remark 2.0.43. A light condensed set X is completely determined the set $X(C)$ together with the action of the continuous endomorphisms of C, where C is the Cantor set.

Lemma 2.0.44. The functor $\mathrm{Top} \to \mathrm{CondSet}^{\mathrm{light}}$ defined by $A \mapsto \underline{A}$ admits a left adjoint $X \mapsto X(*)_{\mathrm{top}}$, where $X(*)_{\mathrm{top}}$ is the set $X(*)$ equipped with the quotient topology from $\sqcup_{\alpha \in X(S)} S \to X(*)$ where S runs through all light profinite sets.

Remark 2.0.45. The space $X(*)_{\mathrm{top}}$ is a metrizably compactly generated topological space. In the construction of the quotient topology, the light profinite set S can be taken as the Cantor set, instead of running through all light profinite sets. For a metrizably compactly generated topological space A, it holds that $A = \underline{A}(*)_{\mathrm{top}}$.

Hence, metrizable compactly generated topological spaces fully faithfully embeds into the category of light condensed sets.

Remark 2.0.46. Some related literature.

(1) Johnstone's topological topos.
(2) Escande–Xu

Remark 2.0.47. Recall that sheaves of abelian groups on any site form a Grothendieck abelian category. In particular, all limits and colimits exist, and filtered colimits are exact.

The category $\mathrm{CondAb}^{\mathrm{light}}$ is a Grothendieck abelian category.

Example 2.0.48. Let $\mathbb{R}_{\mathrm{disc}}$ be the set of real numbers equipped with the discrete topology. The inclusions $\mathbb{Q} \to \mathbb{R}$ and $\mathbb{R}_{\mathrm{disc}} \to \mathbb{R}$ are continuous. Let's compute the cokernels.

(1) $(\underline{\mathbb{R}}/\underline{\mathbb{Q}})(S) = \mathrm{Cont}(S, \mathbb{R})/\mathrm{Cont}(S, \mathbb{Q})$ for any light profinite set S.
(2) $(\underline{\mathbb{R}}/\underline{\mathbb{R}_{\mathrm{disc}}})(S) = \mathrm{Cont}(S, \mathbb{R})/\mathrm{Cont}(S, \mathbb{R}_{\mathrm{disc}})$ for any light profinite set S.

Lemma 2.0.49. The forgetful functor $\mathrm{CondAb}^{\mathrm{light}} \to \mathrm{CondSet}^{\mathrm{light}}$ has a left adjoint $X \mapsto \mathbb{Z}[X]$.

Lemma 2.0.50. The category $\mathrm{CondAb}^{\mathrm{light}}$ is a Grothendieck abelian category, has a tensor product, and

(1) countable products are exact (and satisfies AB6)
(2) $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]$ is internally projective.

3. Light condensed abelian groups, continued

Remark 3.0.1. It's worth noting that sequential limits of covers are still covers.

There is the Yoneda embedding $\mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin}) \to \mathrm{CondSet}^{\mathrm{light}} = \mathrm{Shv}(\mathrm{Pro}_{\mathbb{N}}(\mathrm{Set}))$.

Remark 3.0.2. Such an embedding exists at least the site is subcanonical.

The category $\mathrm{CondSet}^{\mathrm{light}}$ is generated by the image under colimits.

Remark 3.0.3. some kind of gluing procedure

Let X be a light condensed set. The topology of $X(*)_{\mathrm{top}}$ can be alternatively described as

$$\sqcup_{\beta\in X(\mathbb{N}\cup\{\infty\})}(\mathbb{N}\cup\{\infty\}) \to \sqcup_{\alpha\in X(C)}C \to X(*),$$

where the two maps are both quotient maps, and C is the Cantor set. Informally, this means convergence sequences determine the topology of $X(*)_{\mathrm{top}}$. In particular, being metrizably compactly generated is the same as being sequential. So sequential topological spaces embed into light condensed sets.

Remark 3.0.4. Why do we allow Cantor sets? In any topos, there is an intrinsic notion of being compact and being Hausdorff.

An object X is quasi-compact if any cover $\sqcup_i X_i \to X \twoheadrightarrow X$ admits finite subcover. In the case of light condensed sets (???), it's equivalent to the existence of a surjection from the Cantor set to X or X is empty.

An object X is quasi-separated if for all quasi-compact $Y \to X$ and $Z \to X$ the fiber product $Y \times_X Z$ is quasi-compact. Here, for all maps $f, g : C \to X$ from the Cantor set C, the object $C \times_X C$ is quasi-compact.

If we only allow $\mathbb{N}\cup\{0\}$ in the test category, then quasi-compact objects would all be countable. In particular, the Cantor set would not be quasi-compact. ???

Lemma 3.0.5. The category of quasi-compact quasi-separated light condensed sets is equivalent to the category of metrizable compact Hausdorff spaces.

Remark 3.0.6. This forces us to use finitary covers (otherwise our basic objects will not be compact), and forces us to include the Cantor set.

Lemma 3.0.7. The category of quasi-separated light condensed sets are equivalent to the category $\mathrm{Ind}_{\mathrm{inj}}(\mathcal{D})$, where $\mathcal{D}$ is the category of metrizable compact Hausdorff spaces.

Remark 3.0.8. So quasi-separated objects should be viewed as rising unions of quasi-compact objects. The category of metrizable compact generated weakly Hausdorff spaces embeds.

Remark 3.0.9. The interval $[0, 1]$ is metrizable compact Hausdorff. So it gives a quasi-compact object. Hence there is a surjection from the Cantor set to $[0, 1]$. Such a map exists indeed, and it can be taken as the binary expansion.

(3.0.1). Let M and N be light condensed abelian groups. The tensor product $M \otimes N$ is defined to be the sheafification of the presheaf $S \mapsto M(S) \otimes N(S)$. The unit object is just $\underline{\mathbb{Z}}$, where $\mathbb{Z}$ carries the discrete topology.

(3.0.2). Let X be a light condensed set. The free light condensed abelian group on X, denoted by $\mathbb{Z}[X]$, is the sheafification of $S \mapsto \mathbb{Z}[X(S)]$. The functor $X \mapsto \mathbb{Z}[X]$ is adjoint to the forgetful functor $\mathrm{CondAb}^{\mathrm{light}} \to \mathrm{CondSet}^{\mathrm{light}}$. The idea is that $\mathbb{Z}[X]$ puts some topology on $\mathbb{Z}[X(*)]$.

(3.0.3). Sheafification does not change the value at $*$, because every cover of the point $*$ splits.

(3.0.4). Consider the light condensed abelian group $\mathbb{Z}[\underline{\mathbb{R}}]$. It's underlying set is just the usual $\mathbb{Z}[\mathbb{R}]$, i.e. finite $\mathbb{Z}$-linear combinations of $[x]$ for $x \in \mathbb{R}$. One can view it as finite linear combinations of dirac measures.

The free construction commutes with colimits. So $\mathbb{Z}[\underline{\mathbb{R}}] = \mathrm{colim}_I\, \mathbb{Z}[\underline{I}]$ where I runs through all intervals $[-c, c] \subset \mathbb{R}$. Note that

$$\mathbb{Z}[\underline{I}] = \bigcup_{n\in\mathbb{N}} \mathbb{Z}[\underline{I}]_{\le n},$$

where $\mathbb{Z}[\underline{I}]_{\le n}$ is the subset of elements $\sum_{x\in I} n_x[x]$ with $\sum_{x\in I}|n_x| \le n$. This should be understood both as the underlying sets and as light condensed sets. The spaces $\mathbb{Z}[\underline{I}]_{\le n}$ have canonical topolgy which are compact Hausdorff.

The calculation is related to the surjective maps in the Grothendieck topology.

(3.0.5). Something specific to light condensed sets.

(1) Countable products are exact.
(2) Sequential limits of surjective maps are surjective.
(3) $\mathbb{Z}[\underline{\mathbb{N}\cup\{\infty\}}]$ is internally projective.

Recall that an element P is internally projective if $\underline{\mathrm{Ext}}^i(R,-)=0$ for $i>0$. The third property only holds in the light setting.

The internal Ext can be explicitly written down: $\underline{\mathrm{Ext}}^i(M,N)$ is the sheafification of $S\mapsto \mathrm{Ext}^i(M\otimes \mathbb{Z}[\underline{S}],N)$.

Proof. First reduce (1) to (2) as follows. Let $f_n : M_n \to N_n$ be surjective maps for $n\in\mathbb{N}$. The map $\prod_{n\le m} M_n \times \prod_{n>m} N_n \to \prod_n N_n$ is surjective since finite products are exact. Then $\prod_n M_n \to \prod_n N_n$ is just the limit of $\prod_{n\le m} M_n \times \prod_{n>m} N_n \to \prod_n N_n$, and thus is surjective by (2).

Proof of (2). Let $M_{n+1}\to M_n$ be surjective maps with $M_\infty = \lim_n M_n$. Let $S_0\to M_0$ be a map from a light profinite set. Since $M_1\to M_0$ is surjective, the map $S_0\to M_0$ can be lifted to $S_1\to M_1$ for some light profinite set S_1, with $S_1\to S_0$ surjective. By induction, there is a diagram $\cdots\to S_1\to S_0$. Take $S_\infty=\lim S_n$. Then the map $S_\infty\to S_0$ is surjective, since countable limits of surjective maps of light profinite sets are surjective, and $S_0\to M_0$ lifts to $S_\infty\to M_\infty$.

Proof of (3). Note that $\mathbb{Z}[\infty]$ is a direct factor of $\mathbb{Z}[\mathbb{N}\cup\{\infty\}]$. Let $M=\mathbb{Z}[\mathbb{N}\cup\{\infty\}]/\mathbb{Z}[\infty]$. It classifies null-sequences. It suffices to prove that M is internally projective. First show that M is projective. Let $\widetilde{N}\to N$ be a surjective map of light condensed abelian groups. Let $M\to N$ be a map. The map $\mathbb{N}\cup\{\infty\}\to M\to N$ sends ∞ to 0. The surjectivity of $\widetilde{N}\to N$ implies that there is a lift to $S\to\mathbb{N}\to\mathbb{N}\cup\{\infty\}$ that is surjective. There exists a closed subspace S^1 of S such that $S^1\times_{\mathbb{N}\cup\{\infty\}}\mathbb{N}\to\mathbb{N}$ is an isomorphism (just pick one element from the fiber of each $n\in\mathbb{N}$). So one can assume $S\times_{\mathbb{N}\cup\{\infty\}}\mathbb{N}\to\mathbb{N}$ is an isomorphism. Let $S_\infty = S\times_{\mathbb{N}\cup\{\infty\}}\{\infty\}$. It is a closed subset of $\mathbb{N}$. Since S_∞ is a non-empty light profinite set, it is injective in the category Pro(Fin). Thus there exists a retraction $r:S\to S_\infty$. Let g be the map $S\to\widetilde{N}$. Consider the map $g-i\circ r\circ g : S\to\widetilde{N}$, where $i:S_\infty\to S$ is the inclusion. It maps S_∞ to $0\in\widetilde{N}$. The pullback diagram

$$\begin{array}{ccc} S_\infty & \longrightarrow & S \\ \downarrow & & \downarrow \\ \{\infty\} & \longrightarrow & \mathbb{N}\cup\{\infty\} \end{array}$$

is also a pushout diagram in the category of light profinite sets. Thus $g-g\circ i\circ r$ factors through $S\to\mathbb{N}\cup\{\infty\}$ and induces $\widetilde{f}:\mathbb{N}\cup\{\infty\}\to\widetilde{N}$. Moreover, $\widetilde{f}$ is a null-sequence, i.e. it is a map $M=\mathbb{Z}[\mathbb{N}\cup\{\infty\}]/\mathbb{Z}[\infty]\to\widetilde{N}$. Hence M is projective. For internal projectivity, it's essentially the same argument. ☐

(3.0.6). As a corollary, countable limits of covers are covers. This forces one to use totally disconnected spaces as building blocks.

(3.0.7). A piece of warning: $\mathbb{N}\cup\{\infty\}$ is not projective in $\mathrm{CondSet}^{\mathrm{light}}$, and $\mathbb{Z}[\mathbb{N}\cup\{\infty\}]$ is not projective in the category of all condensed abelian groups. The identity map $\mathbb{N}\cup\{\infty\}\to\mathbb{N}\cup\{\infty\}$ cannot be lifted along the surjection $(2\mathbb{N}\cup\{\infty\})\sqcup((2\mathbb{N}+1)\cup\{\infty\})\to\mathbb{N}\cup\{\infty\}$.

(3.0.8). In the category of all condensed sets, all products are exact, and it admits projective generators $\mathbb{Z}[S]$ for $S=\beta I$, the Stone–Čech compactification of a discrete set. None of those objects are internally projective.

(3.0.9). This is related to Banach spaces. The only (known??) injective Banach spaces are $\mathrm{Cont}(S,\mathbb{R})$ for S extremely disconnected. Here the relation between Banach spaces and condensed sets has some duality process.

(3.0.10). The Banach space $c_0(\mathbb{N})$ of null-sequences, corresponding to the object $\mathbb{Z}[\mathbb{N}\cup\{\infty\}]$, is not an injective Banach space. But it is separably injective, which means one only test this injectivity against separable Banach spaces. This is related to that $\mathbb{Z}[\mathbb{N}\cup\{\infty\}]$ is not projective in CondAb, but it is in $\mathrm{CondAb}^{\mathrm{light}}$.

(3.0.11). Let X be a light condensed set and let M be an abelian group. Define

$$H^i(X,M)=\mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{light}}}(\mathbb{Z}[X],\underline{M}).$$

(3.0.12) Lemma. If X is a CW complex, then $H^i(\underline{X},M)\simeq H^i_{\mathrm{sing}}(X,M)$.

(3.0.13). The Dold–Thom theorem says $\mathbb{Z}[\underline{X}]$ is a model for homology of X.

4. Computations of Ext

Lemma 4.0.1. Let X be a CW complex. Let M be an Abelian group. Then $\mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{light}}}(\mathbb{Z}[\underline{X}], M) \simeq H^i_{\mathrm{sing}}(X, M)$.

Proof. Write $X = \cup_i X_i$ with X_i (metrizable) compact Hausdorff. Both sides take filtered colimits to cofiltered limits. So it reduces the case X is compact Hausdorff.

A more general statement: X metrizable compact Hausdorff and M abelian group. Then

$$\mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{light}}}(\mathbb{Z}[\underline{X}], M) \simeq H^i_{\mathrm{sheaf}}(X, M).$$

It's known that if X is a CW complex then $H^i_{\mathrm{sheaf}}(X, M) \simeq H^i_{\mathrm{sing}}(X, M)$, but not in general. If X is totally disconnected compact Hausdorff, then $\Gamma(X, -) : \mathrm{Sh}(X, \mathrm{Ab}) \to \mathrm{Ab}$ is exact. Hence $H^i_{\mathrm{sheaf}}(X, M) = 0$ for $i > 0$ and $H^0_{\mathrm{sheaf}}(X, M)$ consists of locally constant maps $X \to M$. But $H^i_{\mathrm{sing}}(X, M) = 0$ for $i > 0$ and $H^0_{\mathrm{sing}}(X, M)$ consists of all maps $X \to M$.

There is a further upgrade. Fix X metrizable compact Huasdorff. There are two sites: $\mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})_{/X}$ and $\mathrm{Op}(X)$, the site of open subsets of X. There is a map of topos $\lambda : \mathrm{Sh}(\mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})_{/X}) \to \mathrm{Sh}(\mathrm{Op}(X)) = \mathrm{Sh}(X)$. The left hand side $\mathrm{Sh}(\mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})_{/X})$ can be identified with $\mathrm{CondSet}^{\mathrm{light}}{}_{/X}$. For an open subset $U \subset X$, the pullback is $\lambda^* U = \underline{U}$. It further induces $\lambda^* : D(\mathrm{Sh}(X, \mathrm{Ab})) \to D(\mathrm{Ab}(\mathrm{CondSet}^{\mathrm{light}}{}_{/X}))$. The claim is that on D^+, λ^* is fully faithful. Thus for $\mathcal{F} \in \mathrm{Sh}(X, \mathrm{Ab})$, we have $H^i(\mathrm{CondSet}^{\mathrm{light}}{}_{/X}, \lambda^*\mathcal{F}) \simeq H^i_{\mathrm{sheaf}}(X, \mathcal{F})$. Apply to the constant sheaf on M, we get the desired result.

Sketch of the proof of the claim. Let $A \in D^+(\mathrm{Sh}(X, \mathrm{Ab}))$. We need to show that $A \to R\lambda_*\lambda^* A$ is an isomorphism. This can be checked on stalks. The key point is the base change property: taking stalks at x commutes with $R\lambda_*$. Here, use general "cohomology commutes with filtered colimits" results for general "coherent topoi" (i.e. quasi-compact and quasi-separated topoi). The key "geometric" input is that $\underline{X}$ is qcqs.

A concrete proof. Let X be metrizable compact Hausdorff. We'd like to find a projective, or at least acyclic, resolution of $\mathbb{Z}[\underline{X}]$.

Step 1. If X is totally disconnected, then

$$\mathrm{Ext}^i(\mathbb{Z}[S], \mathbb{Z}) = \begin{cases} \mathrm{Cont}(S, \mathbb{Z}) & i = 0 \\ 0 & i > 0. \end{cases}$$

This comes down to the following: for all hypercovers $S_\bullet \to S$ (the asscoiated complex $\cdots \to \mathbb{Z}[S_1] \to \mathbb{Z}[S_0] \to \mathbb{Z}[S] \to 0$ is automatically exact), the complex

$$0 \to \mathrm{Cont}(S, \mathbb{Z}) \to \mathrm{Cont}(S_0, \mathbb{Z}) \to \mathrm{Cont}(S_1, \mathbb{Z}) \to \cdots$$

is exact. One way to prove this is that we treat every $\mathrm{Cont}(S_i, \mathbb{Z})$ as a sheaf on S, and then check the exactness on stalks. Another way is to show that every hypercover of profinite sets by profinite sets can be written as a cofiltered limit of hypercovers of finite sets by finite sets.

Step 2. Treat general metrizable compact Hausdorff X. We want to resolve $\mathbb{Z}[\underline{X}]$ by $\mathbb{Z}[S]$ where S is a light profinite set. Consider the simplicial object $S_\bullet$ such that S_n is the n-fold fiber product of S_0 over X, where $S_0 \to \underline{X}$ is a surjection (we can choose S_0 to be the Cantor set), cf. the Čech nerve. We obtain a resolution $\cdots \to \mathbb{Z}[S_1] \to \mathbb{Z}[S_0] \to \mathbb{Z}[\underline{X}] \to 0$. Now $\mathrm{Ext}^i(\mathbb{Z}[\underline{X}], \mathbb{Z})$ is computed by $0 \to \mathrm{Cont}(S_0, \mathbb{Z}) \to \mathrm{Cont}(S_1, \mathbb{Z}) \to \cdots$. The rest argument is similar. ☐

5. Locally Compact Abelian Groups

Definition 5.0.1. Let $\mathrm{LCA}_{\mathrm{m}}$ be the category of metrizable locally compact Abelian groups. It contains groups like discrete Abelian groups, $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{A}$. Every object admits a filtration with three pieces: a discrete Abelian group, a finte-dimensional real vector space, and a compact metrizable Abelian group.

Lemma 5.0.2. Let $A, B \in \mathrm{LCA}_{\mathrm{m}}$. Then

$$\mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{light}}}(\underline{A}, \underline{B}) = \begin{cases} \mathrm{Hom}_{\mathrm{LCA}}(A, B) & i = 0 \\ \mathrm{Ext}^i_{\mathrm{LCA}}(A, B) & i = 1 \\ 0 & i \geq 2. \end{cases}$$

Proof. We need to find something close to a projective resolution of $\underline{A}$. The key result is Breen–Deligne resolution: there is a resolution of the form

$$\cdots \to \mathbb{Z}[M^{\oplus 2}] \to \mathbb{Z}[M] \to M \to 0$$

functorial in the Abelian group M, where each term is $\mathbb{Z}[M^{n_i}]$. Note that there is no known explicit formulas for the resolution. By functoriality this also works for Abelian scheaves on any site.

Another fact is that for any metrizable compact Hausdorff X, we have

$$\mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{light}}}(\mathbb{Z}[\underline{X}], \mathbb{R}) = \begin{cases} \mathrm{Cont}(X, \mathbb{R}) & i = 0 \\ 0 & i > 0. \end{cases}$$

This also wors with $\underline{\mathbb{R}}$ replaced by $\underline{V}$, where V is a Banach V, but this uses partitions of unity and thus requies the local convexity of V. □

Remark 5.0.3. This seems to hold even for internal objects.

Example 5.0.4. Let $A \in \mathrm{LCA}_{\mathrm{m}}$. Then

$$\mathrm{Ext}^i(\underline{A}, \underline{\mathbb{R}/\mathbb{Z}}) = \begin{cases} A^\vee & i = 0 \\ 0 & i > 0. \end{cases}$$

Also, $\mathrm{Ext}^i(\underline{\mathbb{R}}, \underline{\mathbb{Z}}) = 0$ for all $i \geq 0$.

Remark 5.0.5. This is another resolution $Q(M)$ that is explicit and also can be used, namely the MacLane's Q-construction, also defined by Commelin:

$$\cdots \to \mathbb{Z}[M^8] \to \mathbb{Z}[M^4] \to \mathbb{Z}[M^2] \to \mathbb{Z}[M].$$

A fact is that $Q(M) \simeq Q(\mathbb{Z}) \otimes^L_{\mathbb{Z}} M$.

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