Reinforcement Learning Applications in Macroeconomics

Topic 1: Basic Concepts

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Purpose of this note

- Understanding how Reinforcement Learning works. - **Topic 1: Basic Concepts.**
- Introduce classical and frontier algorithms in Reinforcement Learning. - **Topic 2: Frontier Algorithms and Application.**
- Discuss multibandit learning. - **Topic 3: Sampling and Adaptive Learning.**
- Discuss multi-agent learning. - **Topic 4: Multi-agent RL and equilibrium.**
- Propose potential directions in which reinforcement learning can contribute to empirical macroeconomic research.
  -**Topic 5: Shocks in the environment and the dynamic back to equilibrium.**
  -**Topic 6: Sub-optimal Choices and Distortion.**
  -**Topic 7: RL with other Machine Learning Methods for Simulation.**

Roadmap

1. Definitions in RL
   - Environment
   - Action
   - Reward
   - States
2. A Simple Q-learning Example

References

Reference: Sutton and Barto (2018)
- *"Reinforcement learning is a computational approach to understanding and automating goal-directed learning and decision making. It is distinguished from other computational approaches by its emphasis on learning by an agent from direct interaction with its environment, without requiring exemplary supervision or complete models of the environment."*

Reference: Dimitri P. Bertsekas (2020)
- *"RL can be viewed as the art and science of sequential decision-making for large and difficult problems, often in the presence of imprecisely known and changing environment conditions."*

Reference: Benjamin Moll (2024)
- *"RL ideas seem to me a promising direction for developing alternative approaches to rational expectations about equilibrium prices in heterogeneous-agent models."*

Basic Concepts

Major concepts in Q-learning:

1. Environment - state variables not for agents, which are usually high-dimensional.
2. Action - choice variables.
3. Reward - utility, payoffs, any objective function economists want to maximize or minimize.
4. States - state variables of the agent.

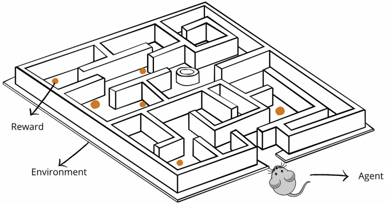
The main difference between Q-learning and value function iteration is how to cope with the transition dynamics. The value function takes the expectation for the future value, but Q-learning updates the quality function with randomization to detect its interaction with the environment.

Basic Concepts

Solving a finite Markov decision problem:
we can use three tabular RL methods. I will introduce the non-tabular methods, which combine ML methods in topic 2.

- **Dynamic Programming**
  - requires full information about the environment, so the transitional probability is explicit.
- **Monte Carlo Methods**
  - requires a definitive end of the trial, but do not need an explicit reward function form.
  Not suitable for macroeconomic questions.
- **Temporal Difference Updating** $\star$ - Minsky (1961),Arthur Samuel (1959)

Exploration: RL evaluates the accumulated reward for actions rather than guiding actions with the expected maximized rewards, so it needs to explore possible actions.



Basic Concepts

k-Armed Bandit problem:

In Sutton and Barto's definition, the k-armed bandit is defined as a trigger for k different expected or mean immediate rewards. - In this case, the agent needs to choose!

$$\downarrow$$

The action brings in uncertainty from its interaction with the environment.

Suppose that the value at time step t is $Q_t(a^k)$, we do not know what the updated Q-value would be, as there is uncertainty in the environment that we may not know anything or only part of it.

⋆ A single-bandit problem is actually about learning the environment.

⋆ A k-bandit problem is actually about learning the joint impact by choices and the environment.

This concept is also developing in experimental economics, see this talk by Dr.Susan Athey: `https://www.youtube.com/watch?v=I6GyDWh8kfw`

Environment

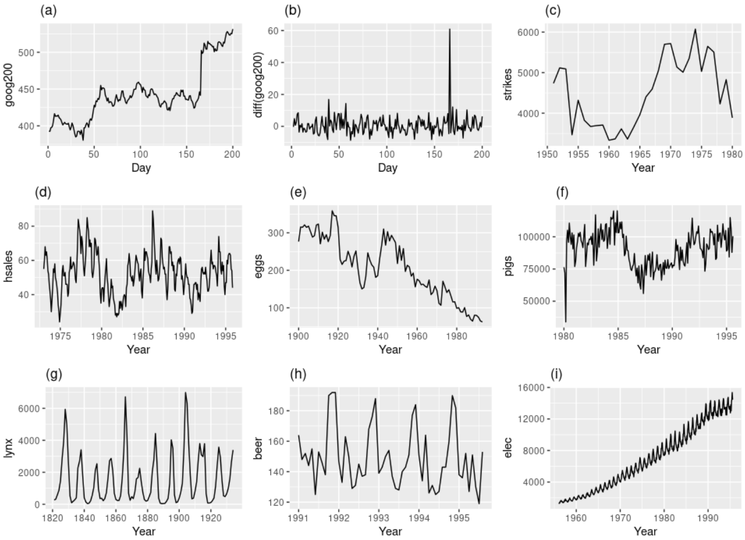
Macroeconomic shocks and environment

In macroeconomics, we usually care about how an aggregate shock affects agents. This means how the change in the environment affects the reward of agents, and thus distorts their actions and consequently affects their state in the language of RL.

Learn the environment

Agents estimate the rewards of actions and make choices (actions) based on the estimated rewards. The rewards contain information about the unobserved environment, each reward is sent back to the agent as a result of the joint effect of the action they take and the environment.

Environment - Stationary & Nonstationary environment



Value of Actions - Stationary Bandit problem

Estimate the value of action a

- By accumulative rewards:

$$Q_t(a) \doteq \frac{\text{sum of rewards when } a \text{ taken prior to } t}{\text{number of times } a \text{ taken prior to } t} = \frac{\sum_{i=1}^{t-1} R_i \cdot \mathbb{1}_{A_i=a}}{\sum_{i=1}^{t-1} \mathbb{1}_{A_i=a}}$$

The value of an action depends on the average historical rewards it induced.

- By incremental updating:
Suppose action a has been selected by n times.

$$\begin{aligned} Q_{n+1}(a) &= \frac{1}{n}\sum_{i=1}^{n} R_i = \frac{1}{n}\left(R_n + \sum_{i=1}^{n-1} R_i\right) = \frac{1}{n}\left(R_n + (n-1)\frac{1}{n-1}\sum_{i=1}^{n-1} R_i\right) \\ &= \frac{1}{n}\left(R_n + (n-1)Q_n\right) = \frac{1}{n}\left(R_n + nQ_n - Q_n\right) = Q_n(a) + \frac{1}{n}\left[R_n - Q_n(a)\right] \end{aligned}$$

The accumulative reward method is unbiased only if the environment is stationary.

Value of Actions - Non-stationary Bandit problem

Estimate the value of action a

- By incremental updating:

$$Q_{n+1}(a) \doteq Q_n(a) + \alpha \left[R_n - Q_n(a) \right]$$

  where $\alpha \in (0, 1]$ is a step-size parameter, while the step size is $\frac{1}{n}$ in the stationary environment. α defines how much the agent predicts relies on the most recent reward.

The general idea is to update the quality function is:

$$\text{NewEstimate} \leftarrow \text{OldEstimate} + \text{StepSize}[\text{Target} - \text{OldEstimate}]$$

Greedy Action

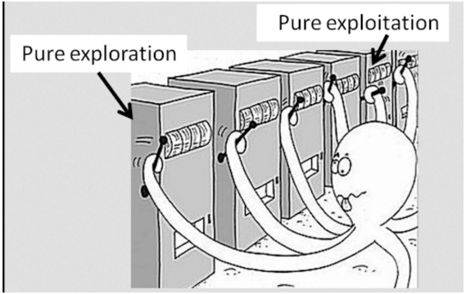
Solving a multi-bandit problem

- Exploitation: Choosing the action that maximized the value function in the past action:

$$A_t \doteq \underset{a}{\operatorname{argmax}} Q_t(a)$$

- Exploration: Choosing a random action to move forward.

ϵ-greedy: define a value $\epsilon \in [0, 1]$, such that at each time step, the agent takes an exploitation action with probability ϵ, and takes an exploration action with probability $1 - \epsilon$. - ϵ could die out (shrink with a defined rate) with more iterations are taken.



Greedy Action

Question: Why do we explore?

- get rid of the local optimal solution
- learn more about the reward generated by both the dynamic environment and the actions (or choices in our macroeconomic definition).

What is in the back of this algorithm is the Law of Large Numbers.

- This is why we can use the past average reward to estimate the value of actions when the environment is stationary.

Episode

An episode is a finite sequence of interactions between the agent and the environment, starting from an initial state and ending at a terminal state. Formally, an episode consists of:

$$(S_0, A_0, R_1, S_1, A_1, R_2, S_2, \ldots, S_T)$$

- S_t is the state at time t.
- A_t is the action taken at time t.
- R_{t+1} is the reward received after taking the action A_t.
- S_T is the terminal state, where the episode ends.

The terminal states could be a required state that defined by the game (e.g. Exit the maze), or achieve the convergence status (e.g. $|V_{t+1} - V_t| < \epsilon$)

Dynamic Programming

$$
\begin{aligned}
v_\pi(s) &\doteq \mathrm{E}_\pi\left[R_{t+1}+\gamma v_\pi\left(S_{t+1}\right) \mid S_t=s\right] \\
&= \max_a \sum p\left(s', r \mid s, a\right)\left[r+\gamma v_*\left(s'\right)\right] \text{ (value iteration)} \\
&= \sum_a \pi(a|s) \sum_{s', r} p\left(s', r \mid s, a\right)\left[r+\gamma v_\pi\left(s'\right)\right] \text{ (policy iteration)}
\end{aligned}
$$

where $p\left(s', r \mid s, a\right)$ should be explicit

$\pi(a|s)$ is the probability of taking action a in the state s under the policy π.
For this method, we need to have a good knowledge of the transitional probability. This means we need to know the information for the entire state set and how they update. This is impossible most of the time.

Tabular RL - MC

Monte Carlo The Monte Carlo method - sampling with a random walk. This method solves the reinforcement learning problem based on averaging sample returns.

$$v_\pi(s) \doteq \mathbb{E}_\pi \left[G_t \mid S_t = s \right]$$

where G_t is the actual return following time t.

The policy function is determined according to the policy improvement theorem.

There are two types of MC methods:

- The first-visit MC method
- The every-visit MC method

The MC method is more suitable for a static environment as the rewards come out for the multiple visits of a state s should converge to an expected value.

Tabular RL - MC

Reference at Sutton and Barto Chapter 4.2

[Definition] policy improvement theorem:
Let π and π' be any paie of deterministic policies such that for all $s \in S$,

$$q_\pi\left(s, \pi'(s)\right) \geq v_\pi(s)$$

Then the policy π' must be as good as, or better than, π. That is, it must obtain greater or equal expected return from all states $s \in S$:

$$v_{\pi'}(s) \geq v_\pi(s)$$

Note that the strict inequality comes together for any two states.

Tabular RL - TD

Reference link

Temporal-Difference: $V_t \leftarrow V_t + \alpha * E_t = V_t + \alpha * \left(\underbrace{r_{t+1} + \gamma * V_{t+1}}_{\text{true reward} V^*} - V_t \right)$

where γ is the time discount rate, as β in macro.

- **State Action Reward State Action (SARSA):**

$$Q\left(s_t, a_t\right) \leftarrow Q\left(s_t, a_t\right) + \alpha\left[r_{t+1} + \gamma Q\left(s_{t+1}, a_{t+1}\right) - Q\left(s_t, a_t\right)\right]$$

  SARSA is the on-policy rule. - carry what we have for the last term (either exploration or exploitation result forward).

- **Q-learning:**

$$Q\left(S_t, A_t\right) \leftarrow Q\left(S_t, A_t\right) + \alpha\left[R_{t+1} + \gamma \max_a Q\left(S_{t+1}, a\right) - Q\left(S_t, A_t\right)\right]$$

  Q-learning is the off-policy rule - carry the choice that maximizes the quality function no matter what type (explore or exploit).

Tabular RL - TD

- Code for SARSA iteration:

```
    Q[s, a] = Q[s, a] + alpha * (reward - Q[s, a]
            + gamma * np.max([Q[s_next, a_next]
            for a_next in range(action_space)]))
```

- Code for Q-learning iteration:

```
    Q[s, a] = Q[s, a] + alpha * (reward - Q[s, a]
            + gamma * np.max([Q[s_next, a_next]
            for a_next in range(action_space)]))
```

Reference link

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Compare three tabula RL - Monte Carlo, TD, and DP

Feature	**Monte Carlo (MC)**	**Temporal Difference (TD)**	**Dynamic Programming (DP)**
Update Type	Sample-based	Sample-based	Expected update according to $P(S'\|a)$
When to Update	End of episode	After each step	Every iteration
	(only update the path the agent is on)		(could update for all states)
Uses Model?	Not necessary	Not necessary	Yes
			$P(S'\|a)$ is estimated
Bootstrapping?	No	Yes	Yes
Exploration Requirement	Requires full episodes	Learns online	Not needed (model-based)
			(model-based)
Efficiency	Slow convergence	Faster	Computation-heavy
	(high variance)	(lower variance)	
Suitability	Episodic tasks	Both episodic	continuous tasks

Table: Comparison of Monte Carlo, Temporal Difference, and Dynamic Programming

Why not using DP?

In our macroeconomic textbooks, we usually use DP with an explicit transitional probability $P(S'|a,S)$. Some times, it is hard to get especially when there are a high-dimensional confounding variables.

The MC and TD methods could make it flexible, they can be either online - learning the unknown environment without full information, or with some flexible format of model that predicts the environment.
Example: we could predict

$$P\left(S' \mid A, \hat{L}'\right) \quad \text{or} \quad P\left(S' \mid A, L, S\right)$$

where $\hat{L}'$ is the predicted environment state. This prediction could adopt machine learning methods that are more precise and less interpretable.

Reward function

In economics, we have the natural reward function - the objective function.

But it could be more flexible...

Reward is a function of state

In the RL, the reward is not necessarily a function of the state. In economics, as we have theoretical-based assumptions and models, we can write the reward of a function of state variables. This is indirect is you have a high-quality data on the state variables in the learning step.

Examples:

- Households' problem:

$$R(c) = u(c) = u((1+r)a - a')$$

Given the budget constraint:

$$a' = a(1+r) - c$$

- Firms' problem:

$$R(k,l) = \pi(k,l) = f(k,l) - (wl + rk)$$

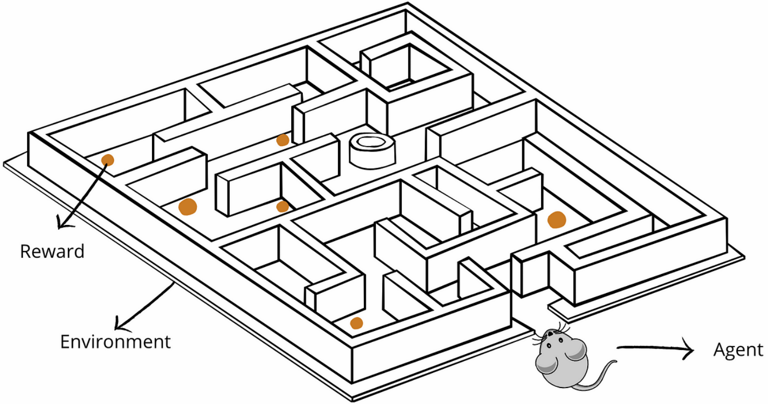
This function is on both state variables and choice variables.

Reward is a function of choice (action)

One virtue of RL is we can get rid of some economic assumptions to simulate. If we relax the assumption of the form of the budget constraints, we can feed the reward function directly as the choice variable (consumption $u(c)$). If we further relax the assumption of the utility function form, the reward could be the choice variable c directly.

States of agents

The states are important for transitional dynamics and are also important in the economic sense. The agent-specific state variable defines the transitional probability. - It may not depict your reward, but it depicts the position of the mouse in the maze, which attach to next step that may or may not have good reward.



States of agents

The transition dynamics are crucial in the VFI; while in the QFI, the convergence could be achieved with the random draw (i.e., by ϵ-greedy).

- **Value function iteration**

$$\begin{aligned} v_\pi(s) &\doteq \mathbb{E}_\pi \left[\sum \gamma^k R_{t+k+1} \mid S_t = s \right] \\ &= \sum_a \pi(a \mid s) \sum_{s',r} p\left(s', r \mid s, a\right) \left[r + \gamma v_\pi\left(s'\right) \right], \quad \text{for all } s \in \delta^\infty \end{aligned}$$

- **Quality function iteration**

$$\begin{aligned} q_*(s, a) &= \mathbb{E}\left[R_{t+1} + \gamma \max_{a'} q_*\left(S_{t+1}, a'\right) \mid S_t = s, A_t = a \right] \\ &= \sum_{a',r} p\left(s', r \mid s, a\right) \left[r + \gamma \max_{a'} q_*\left(s', a'\right) \right] \end{aligned}$$

Value function and Quality function

$$q_*(s, a) = \mathbb{E}\left[R_{t+1} + v_*\left(S_{t+1}\right) \mid S_t = s, A_t = a \right]$$

States of agents - transitional dynamics

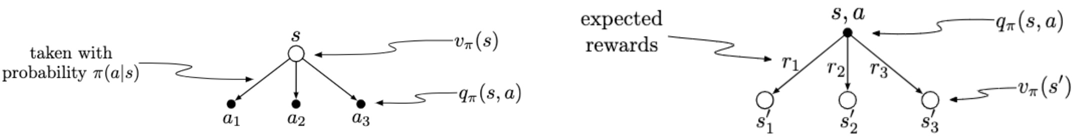


Figure: $v^*(s)$ and $q^*(s,a)$

with the definition of a discrete probability distribution:

$$p\left(s', r \mid s, a\right) \doteq \operatorname{Pr}\left\{S_t = s', R_t = r \mid S_{t-1} = s, A_{t-1} = a\right\}$$

and policy function:

$$\pi(a \mid s)$$

States of agents - transitional dynamics

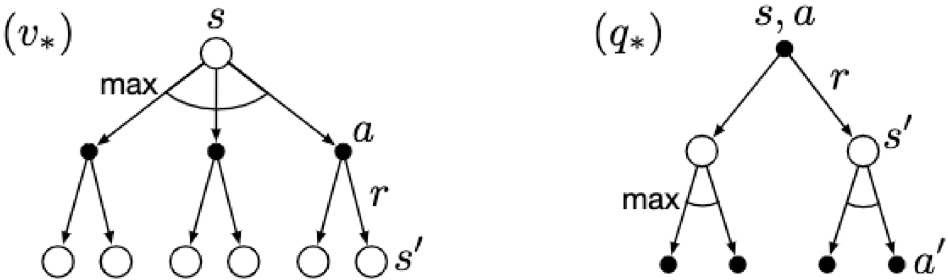


Figure 3.4: Backup diagrams for v_* and q_*

TD-learning: SARSA iteration

Sarsa (on-policy TD control) for estimating $Q \approx q_*$

Algorithm parameters: step size $\alpha \in (0, 1]$, small $\epsilon > 0$
Initialize $Q(s, a)$, for all $s \in S^+$, $a \in A(s)$, arbitrarily except that $Q(\text{terminal}, \cdot) = 0$
Loop for each episode:

- Initialize S
- Choose A from S using policy derived from Q (e.g., ϵ-greedy)
- **Loop for each step of episode:**
    - Take action A, observe R, S'
    - Choose A' from S' using policy derived from Q (e.g., ϵ-greedy)
    - $Q(S, A) \leftarrow Q(S, A) + \alpha\big[R + \gamma Q(S', A') - Q(S, A)\big]$
    - $S \leftarrow S'$; $A \leftarrow A'$

Until S is terminal

TD-learning: Q-learning iteration

Q-learning (off-policy TD control) for estimating $\pi \approx \pi_*$

Algorithm parameters: step size $\alpha \in (0, 1]$, small $\epsilon > 0$
Initialize $Q(s, a)$, for all $s \in S^+$, $a \in A(s)$, arbitrarily except that $Q(\text{terminal}, \cdot) = 0$
Loop for each episode:

- Initialize S
- **Loop for each step of episode:**
  - Choose A from S using policy derived from Q (e.g., ϵ-greedy)
  - Take action A, observe R, S'
  - $Q(S, A) \leftarrow Q(S, A) + \alpha\big[R + \gamma \max_a Q(S', a) - Q(S, A)\big]$
  - $S \leftarrow S'$

Until S is terminal

The End