Tutorial: Active Inference Controller for Dynamic Systems

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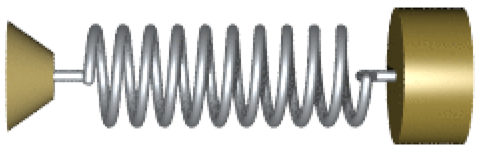
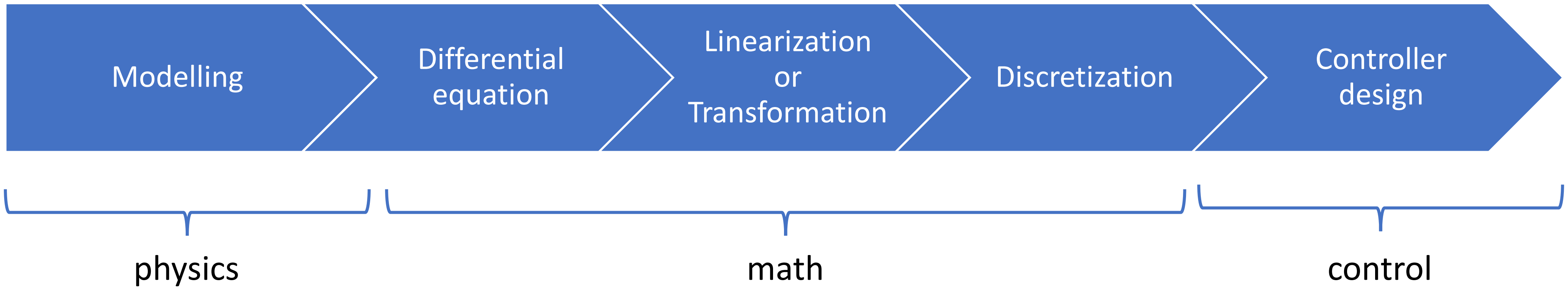
Active inference for robots

Lanillos, P., Meo, C., Pezzato, C., Meera, A. A., Baioumy, M., Ohata, W., Tschantz, A., Millidge, B., Wisse, M., Buckley, C. L. & Tani, J. (2021). **Active Inference in Robotics and Artificial Agents: Survey and Challenges**. *arXiv preprint arXiv:2112.01871*.

Methodology






```
Discretization: sys_d = c2d(ss(model.A,model.B,model.C,[]),dt,'zoh’);
Controller:     dFda = (brain.x - goal_x).'*Pi_g*(-pinv(model.A)*model.B) + Pa*a(:,i-1);
                dFdaa = (-pinv(model.A)*model.B).'*Pi_g*(-pinv(model.A)*model.B) + Pa;
                 a(:,i) = a(:,i-1) + (expm(-k*dFdaa*dt)-eye(ny))*pinv(dFdaa)*dFda;
Generative process: model.x(:,i+1) = sys_d.A*model.x(:,i) + sys_d.B*a(:,i);
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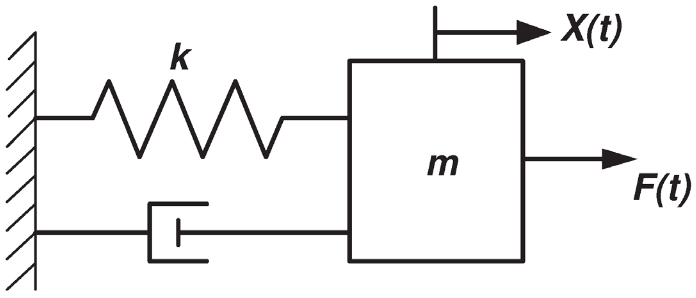
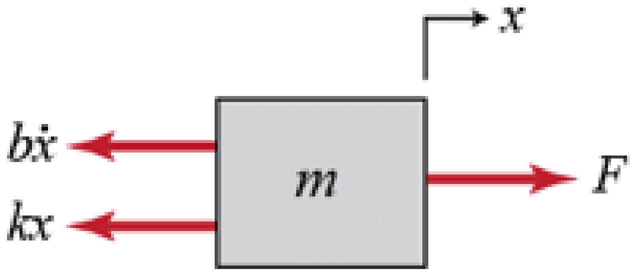
Modelling the dynamic system

Step 1: Derive the equations of motion of the dynamic system





Free body diagram

$$F - b\dot{x} - kx = 0$$

static system

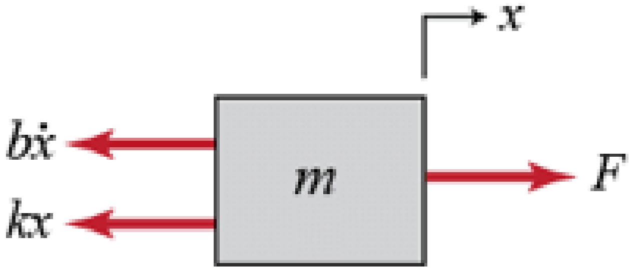
$$F - b\dot{x} - kx = m\ddot{x}$$

dynamic system

Dynamic system → Governing equations of motion

Equations of motion to differential equations

Step 2: convert the dynamic model to a differential equation



$$F - b\dot{x} - kx = m\ddot{x}$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F}{m}$$

Governing equation of motion $\longrightarrow$ Differential equations of motion

Linear time invariant (LTI) state space systems

Step 3a: convert the differential equation to an LTI system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

A, B, C, D are independent of time

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F}{m}$$

$$\dot{X} = AX + Bu$$

$$x_1 = x \qquad x_2 = \dot{x}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \ddot{x} \end{bmatrix}$$

$$\ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x + \frac{F}{m}$$

$$\dot{X} = \begin{bmatrix} x_2 \\ -\frac{b}{m}\dot{x} - \frac{k}{m}x + \frac{F}{m} \end{bmatrix}$$

Linear time invariant state space systems

$$x_1 = x \qquad x_2 = \dot{x}$$

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$$\dot{X} = \begin{bmatrix} x_2 \\ -\frac{b}{m}\dot{x} - \frac{k}{m}x + \frac{F}{m} \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{F}{m} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F$$

$$\dot{X} = AX + Bu$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}, \qquad u = F, \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Linear time invariant state space systems

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\dot{X} = AX + Bu$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}, \quad u = F, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = CX \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

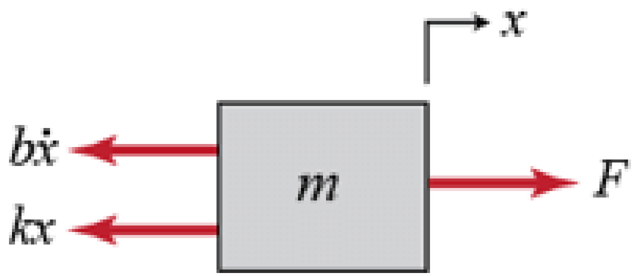
Observation model

Differential equations $\longrightarrow$ LTI system

LTI systems through linearization

Step 3b: linearize the differential equation to form an LTI system

$$\dot{X} = \begin{bmatrix} x_2 \\ -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{F}{m} \end{bmatrix} = \begin{bmatrix} f_1(X,u) \\ f_2(X,u) \end{bmatrix} \qquad y = x_1 = g(X,u)$$



$$\begin{aligned} \dot{X} &= f(X,u) \\ y &= g(X,u) \end{aligned} \qquad \begin{aligned} \ddot{X} &= f(X,u) = AX + Bu \\ y &= g(X,u) = CX + Du \end{aligned}$$

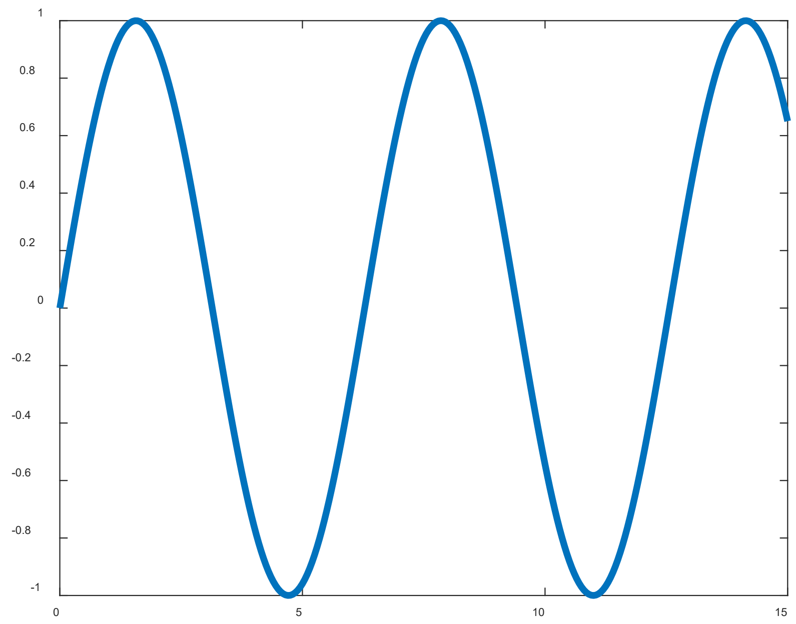
$$A = \frac{\partial f}{\partial X}_{\{X=X^e,u=u^e\}}, \qquad B = \frac{\partial f}{\partial u}_{\{X=X^e,u=u^e\}}, \qquad C = \frac{\partial g}{\partial X}_{\{X=X^e,u=u^e\}}, \qquad D = \frac{\partial g}{\partial u}_{\{X=X^e,u=u^e\}}$$

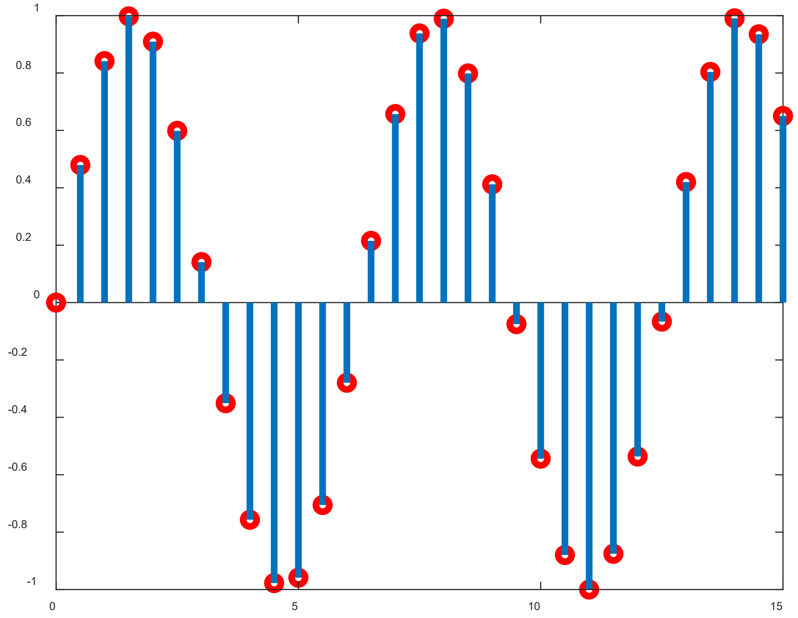
$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}, \qquad C = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix} = [1 \quad 0], \qquad D = \frac{\partial g}{\partial u} = 0$$

Discretization

Step 4: discretize the continuous time model to discrete time model

Why? Because the robot world is discrete!!





Discretization

$$\dot{X} = A_c X + B_c u$$

$$x[t+1] = A_d x[t] + B_d u[t]$$

$$y[t] = C_d x[t] + D_d u[t]$$

$$A_d = e^{A_c \Delta t}$$

$$B_d = {A_c}^{-1}(A_d - I)B_c$$

$$C_d = C_c, \qquad D_d = D_c$$

$e^{A_c \Delta t}$ is a matrix exponential expm(.) and not exp(.)

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

Stability of an LTI system

Step 5: check the stability of the system

The system is stable if the output is finite for all possible finite inputs

$$\dot{X} = A_c X + B_c u$$

The system is asymptotically stable if and only if all the eigen values of A_c are in the left-hand plane

$$\dot{X} = A_d X + B_d u$$

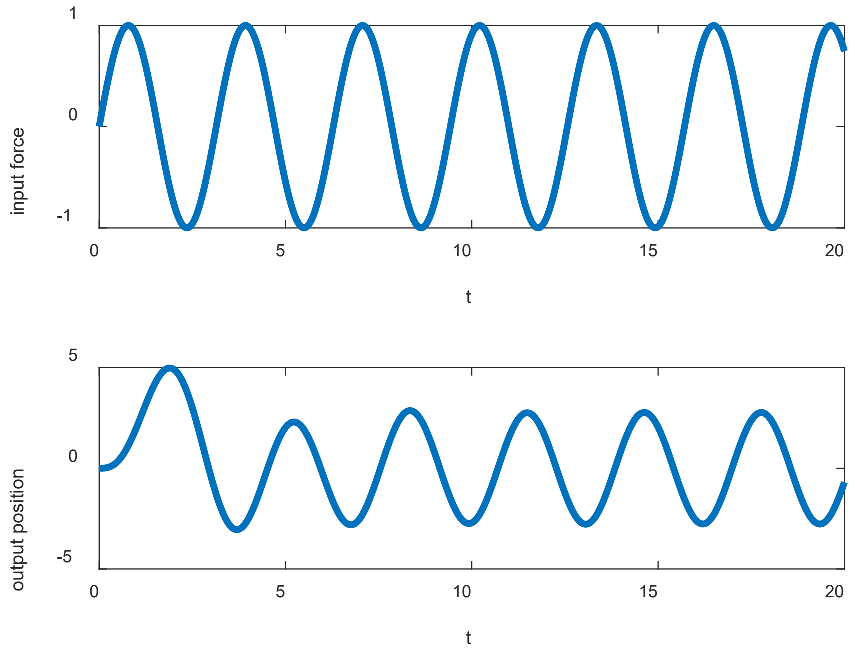
The system is asymptotically stable if and only if all the eigen values of A_d are inside the unit circle

Stability of an SMD system

$$A_c = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}$$

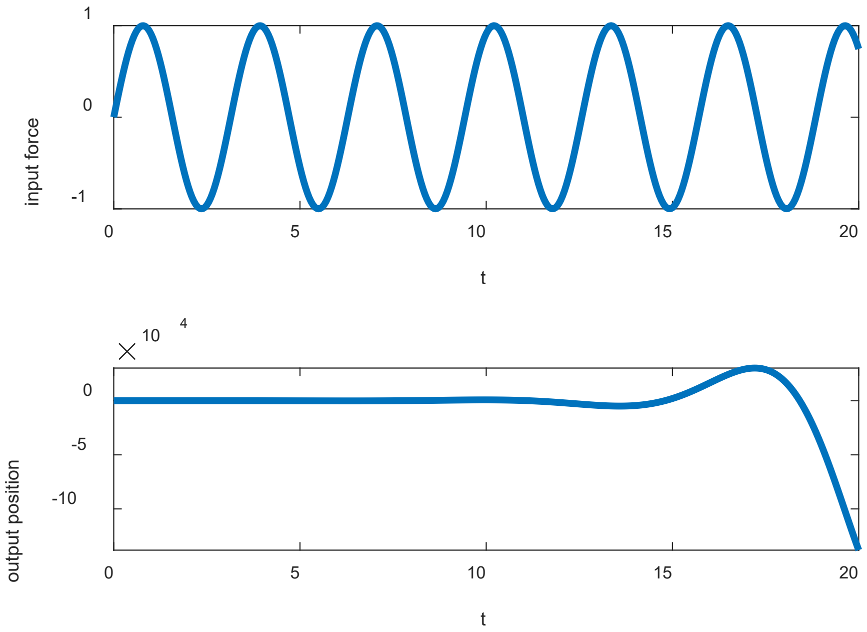
$$m = 0.1kg, \qquad k = 0.1\frac{N}{m}, \qquad b = 0.1\frac{Ns}{m},$$

$$eig(A_c) = \{-0.5 + 0.866i, -0.5 - 0.866i\}$$



$$m = 0.1kg, \qquad k = 0.1\frac{N}{m}, \qquad b = -0.1\frac{Ns}{m},$$

$$eig(A_c) = \{0.5 + 0.866i, 0.5 - 0.866i\}$$



Active Inference controller

Step 6: derive the control law

- Take control actions that follow the gradient of free energy

$$\frac{du}{dt} = -\gamma \frac{\partial F}{\partial u}$$

F – free energy
γ – learning rate
u – control action

$$F = \frac{1}{2}(\tau - \tau^g)^T P^{\tau^g}(\tau - \tau^g) \quad + \frac{1}{2}(u - \eta^u)^T P^u (u - \eta^u)$$

τ – task variable
τ^g - goal for task variable
P^{τ^g} - goal precision (inverse covariance matrix) of τ^g
η^u - prior on control action
P^u - prior control precision

Active Inference controller

$$\frac{du}{dt} = -\gamma \frac{\partial F}{\partial u}$$

$$F = \frac{1}{2}(\tau - \tau^g)^T P^{\tau g}(\tau - \tau^g) \quad + \frac{1}{2}(u - \eta^u)^T P^u (u - \eta^u)$$

$\eta^u = 0$ with high P^u forces the controller to minimize the control action
$\eta^u = 0$ with low P^u allows the controller to explore

$$\frac{du}{dt} = -\gamma \frac{\partial F}{\partial \tau}\frac{\partial \tau}{\partial u} + \frac{\partial F}{\partial u}$$

$$\frac{\partial F}{\partial \tau} = (\tau - \tau^g)^T P^{\tau g}$$

$$\frac{\partial F}{\partial u} = (u - \eta^u)^T P^u$$

System dynamics

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Active inference control for position control

$$\frac{du}{dt} = -\gamma \frac{\partial F}{\partial \tau}\frac{\partial \tau}{\partial u} \qquad \tau = x \qquad \frac{du}{dt} = -\gamma \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial u}$$

$$F = \frac{1}{2}(x - x^g)^T P^{x^g}(x - x^g) \quad + \frac{1}{2}(u - \eta^u)^T P^u (u - \eta^u)$$

$$\dot{x} = Ax + Bu$$

$$\frac{\partial \dot{x}}{\partial u} = A\frac{\partial x}{\partial u} + B, \qquad \frac{\partial \dot{x}}{\partial u} = 0 \; \Rightarrow \frac{\partial x}{\partial u} = -A^{-1}B$$

$$\frac{du}{dt} = -\gamma\left(\frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial u}\right) = -\gamma\left((x - x^g)^T P^{x^g} A^{-1}B + u^T P^u\right)$$

Ajith Anil Meera and Pablo Lanillos, "*Towards metacognitive robot decision making for tool selection*" IWAI 2023.

Active inference control for velocity control

$$\frac{du}{dt} = -\gamma \frac{\partial F}{\partial \tau}\frac{\partial \tau}{\partial u} \qquad \tau = \dot{x} \qquad \frac{du}{dt} = -\gamma \frac{\partial F}{\partial x}\frac{\partial \dot{x}}{\partial u} + \frac{\partial F}{\partial u}$$

$$F = \frac{1}{2}(\dot{x} - \dot{x}^g)^T P^{\dot{x}^g}(\dot{x} - \dot{x}^g) \quad + \frac{1}{2}(u - \eta^u)^T P^u (u - \eta^u)$$

$$\dot{x} = Ax + Bu$$

$$\frac{\partial \dot{x}}{\partial u} = A\frac{\partial x}{\partial u} + B, \qquad \frac{\partial x}{\partial u} = \begin{bmatrix}1\\1\end{bmatrix} \Rightarrow \frac{\partial \dot{x}}{\partial u} = A\begin{bmatrix}1\\1\end{bmatrix} + B$$

$$\frac{du}{dt} = -\gamma\left(\frac{\partial F}{\partial x}\frac{\partial \dot{x}}{\partial u} + \frac{\partial F}{\partial u}\right) = -\gamma\left((\dot{x} - \dot{x}^g)^T P^{\dot{x}^g}\left(A\begin{bmatrix}1\\1\end{bmatrix} + B\right) + u^T P^u\right)$$

Ajith Anil Meera and Pablo Lanillos, “*Towards metacognitive robot decision making for tool selection*” IWAI 2023.

Discretization of the controller

$$u(t+\Delta t) = u(t) - \gamma \frac{\partial F}{\partial u} \Delta t$$

Euler method

Step 7: discretize the control law for discrete time update rule

$$u(t+\Delta t) = u(t) + \left(e^{\left(-k\frac{\partial^2 F}{\partial u^2}\Delta t\right)} - I\right)\left(\frac{\partial^2 F}{\partial u^2}\right)^{-1} \frac{\partial F}{\partial u}$$

Unknowns: $\frac{\partial F}{\partial u}, \frac{\partial^2 F}{\partial u^2}$

<u>Position control:</u>

$$F = \frac{1}{2}(x - x^g)^T P^{x^g}(x - x^g) \quad + \frac{1}{2}(u - \eta^u)^T P^u (u - \eta^u)$$

$$\frac{\partial F}{\partial u} = (x - x^g)^T P^{x^g} A^{-1} B + u^T P^u$$

$$\frac{\partial^2 F}{\partial u^2} = (A^{-1}B)^T P^{x^g} A^{-1} B + P^u$$

<u>Velocity control:</u>

$$F = \frac{1}{2}(\dot{x} - \dot{x}^g)^T P^{\dot{x}^g}(\dot{x} - \dot{x}^g) \quad + \frac{1}{2}(u - \eta^u)^T P^u (u - \eta^u)$$

$$\frac{\partial F}{\partial u} = (\dot{x} - \dot{x}^g)^T P^{\dot{x}^g} \left(A \begin{bmatrix}1\\1\end{bmatrix} + B\right) + u^T P^u$$

$$\frac{\partial^2 F}{\partial u^2} = \left(A \begin{bmatrix}1\\1\end{bmatrix} + B\right)^T P^{\dot{x}^g} \left(A \begin{bmatrix}1\\1\end{bmatrix} + B\right) + P^u$$

Pitfalls of discretization

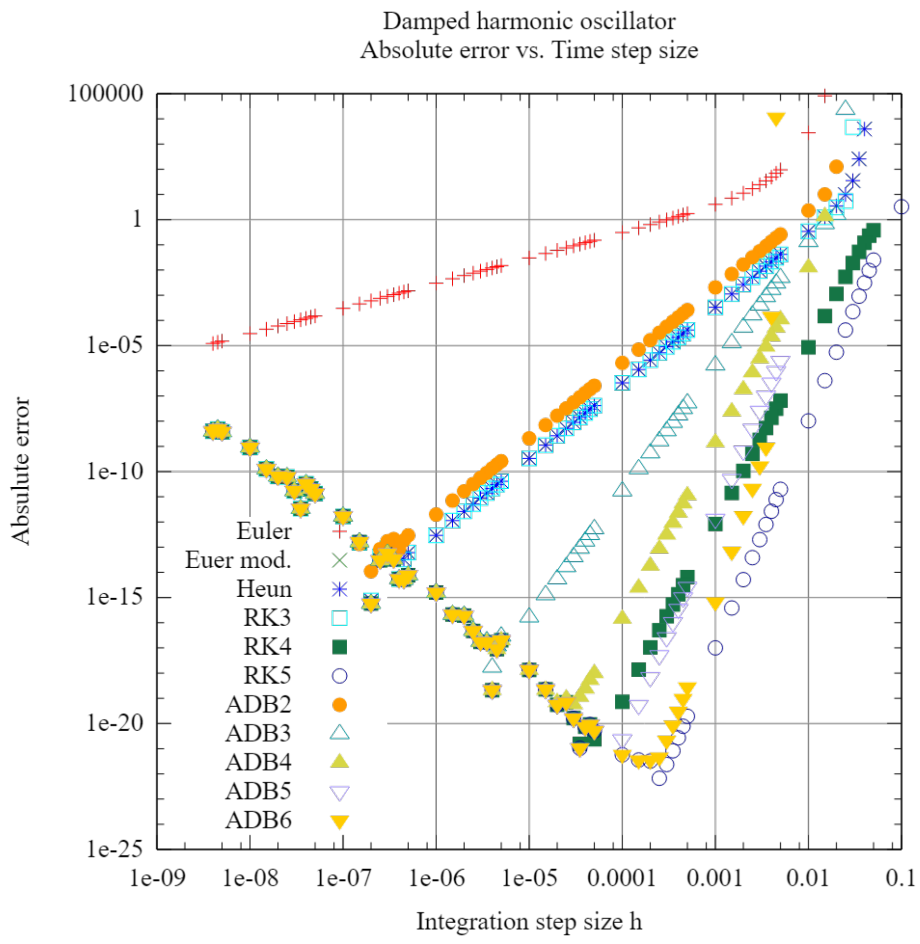


Image source: https://beltoforion.de/en/runge-kutta_vs_euler/

Active inference controller for dynamic systems

- Step 1: model the motion of the system
- Step 2: convert the dynamic model to a differential equation
- Step 3a: convert the differential equation to an LTI system
- Step 3b: linearize the differential equation to form an LTI system
- Step 4: discretize the continuous time model to discrete time model
- Step 5: check the stability of the system
- Step 6: derive the control law
- Step 7: discretize the control law for discrete time update rule

MATLAB implementation

```
% Define the model
model.A = [0 1; -k/m -b/m];                model.B = [0; 1/m];        model.C = [1 0];

% define all the required variables
Pa = .00001*eye(nu);     Pi_g = diag([.01 .01]);    goal_x = [.5; 0];    k_h = 1; dt = .01; nt = 2000;

% discretize the system
sys_d = c2d(ss(model.A,model.B,model.C,[]),dt,'zoh');

for i = 1:nt
        dFda = (brain.x - goal_x).'*Pi_g*(-pinv(model.A)*model.B) + Pa*a(:,i-1);
        dFdaa = (-pinv(model.A)*model.B).'*Pi_g*(-pinv(model.A)*model.B) + Pa;
        a(:,i) = a(:,i-1) + (expm(-k*dFdaa*dt)-eye(ny))*pinv(dFdaa)*dFda;

        % Generative process - take action in the world
        model.x(:,i+1) = sys_d.A*model.x(:,i) + sys_d.B*a(:,i);
end
```

Running code: https://github.com/ajitham123/mTAIC_IWAI2023

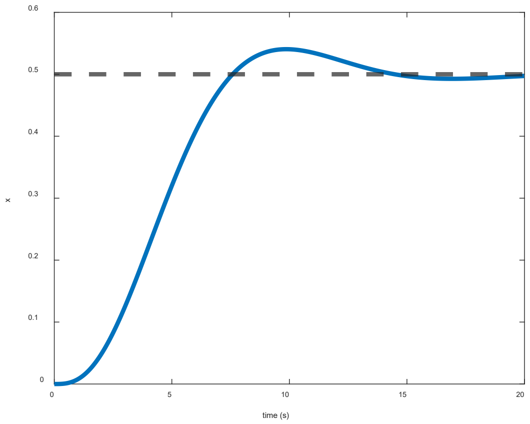
Ajith Anil Meera and Pablo Lanillos, "*Towards metacognitive robot decision making for tool selection*" IWAI 2023.

Active inference to control an SMD

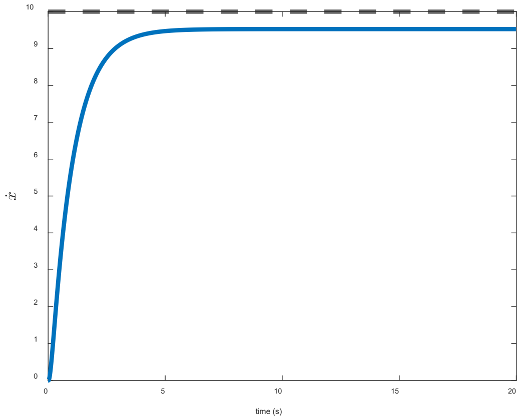
Position control

velocity control

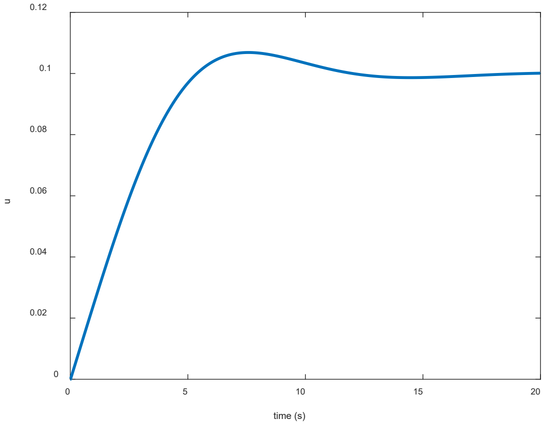
Position



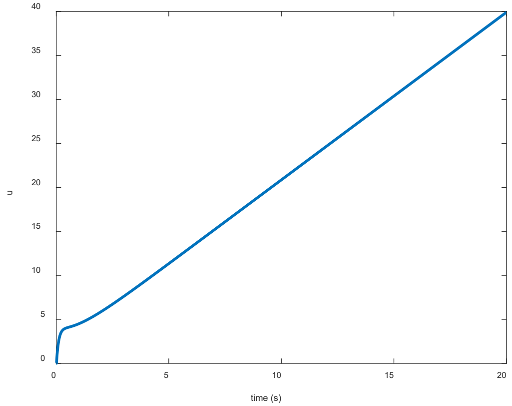
velocity



Control
force



Control
force



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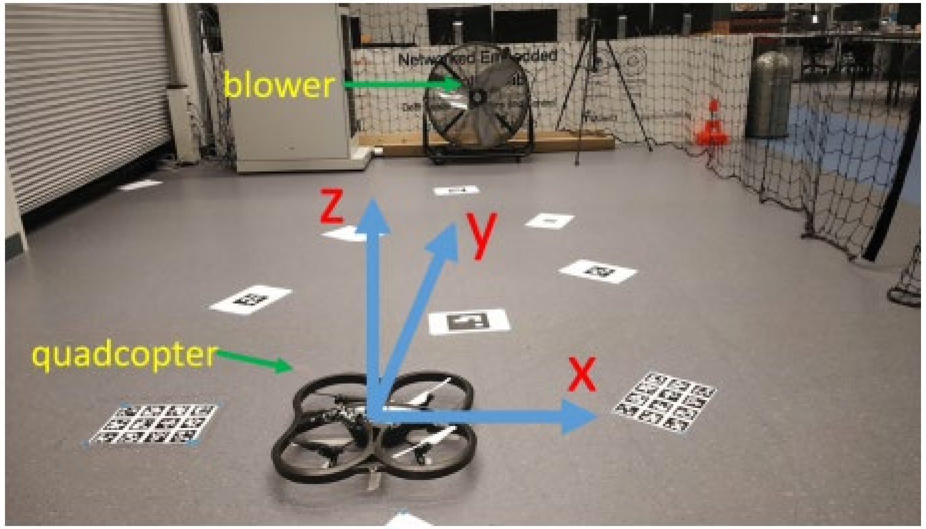
Active inference for robots

Modelling: Quadrotors as LTI system



$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{v} + \mathbf{w}$$
$$\mathbf{y} = C\mathbf{x} + \mathbf{z}.$$

$$\begin{bmatrix} \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{c_{B\phi}}{I_{xx}} & -\frac{c_{B\phi}}{I_{xx}} & -\frac{c_{B\phi}}{I_{xx}} & \frac{c_{B\phi}}{I_{xx}} \end{bmatrix} \begin{bmatrix} pwm_1 \\ pwm_2 \\ pwm_3 \\ pwm_4 \end{bmatrix}$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \dot{\phi} \end{bmatrix}$$

ϕ and $\dot{\phi}$ are roll angle and roll velocity
Pwm are the input signals to the rotors

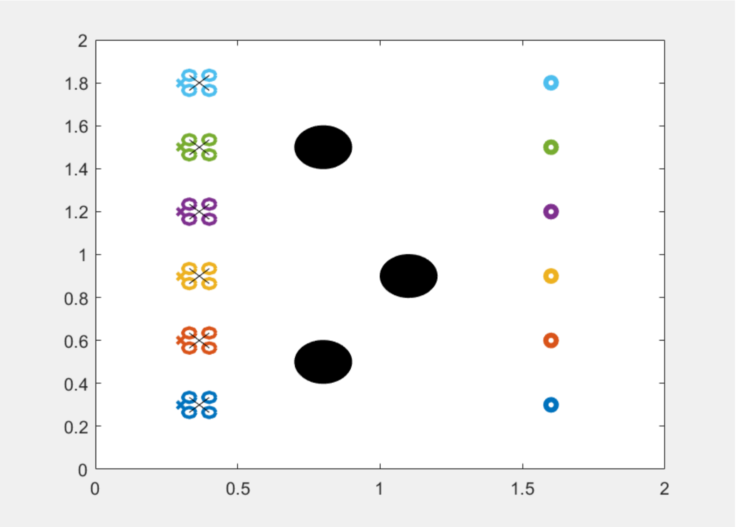
Bos, Fred, et al. "Free energy principle for state and input estimation of a quadcopter flying in wind." *2022 International Conference on Robotics and Automation (ICRA)*. IEEE, 2022.

Active inference for robot navigation

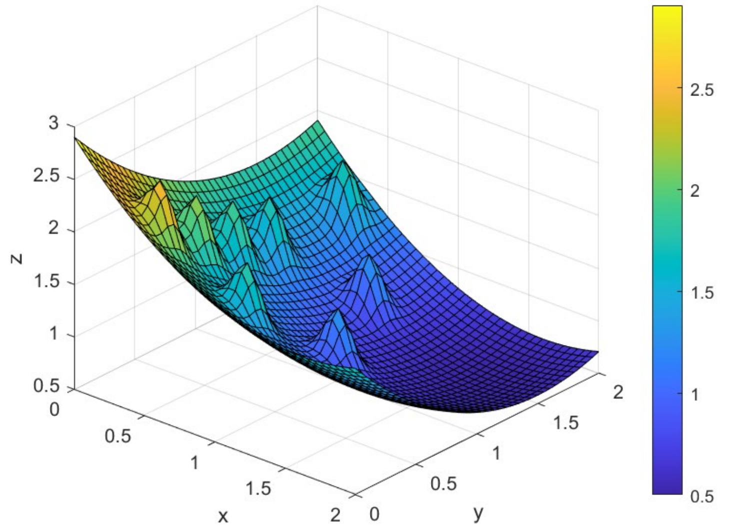
$$F^g = \frac{1}{2}(x - x^g)^T P^{x^g}(x - x^g) \quad + \frac{1}{2}(y - y^g)^T P^{y^g}(y - y^g)$$

$$\frac{\partial u^x}{\partial t} = -\gamma \frac{\partial F}{\partial x}\frac{\partial x}{\partial u^x} = -\gamma \frac{\partial F}{\partial x}$$

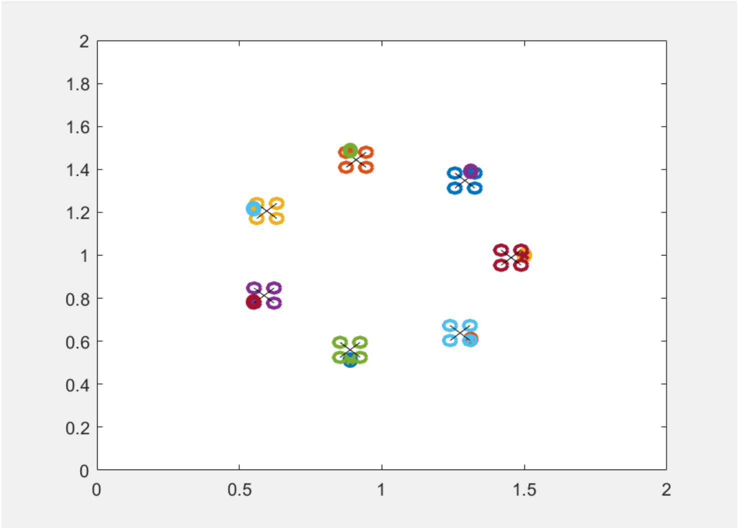
$$F = F^g + F^{os} + F^{od}$$



Static obstacles



Free energy

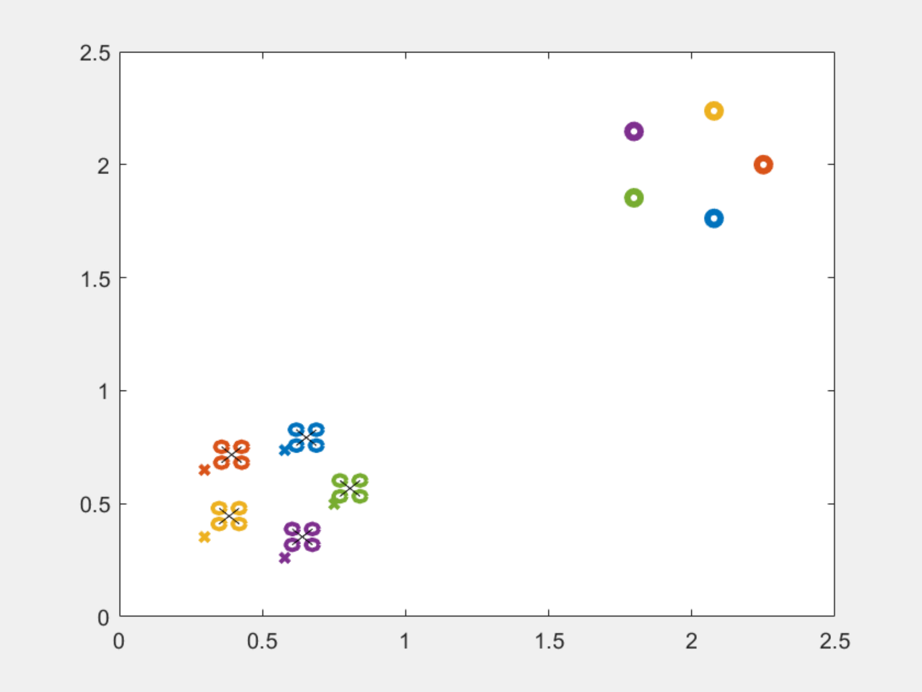


Dynamic obstacles

"Free Energy Principle Based Precision Modulation for Robot Attention: towards brain inspired robot intelligence", PhD thesis, Department of Cognitive Robotics, TU Delft, 2023

Multi robot navigation in formation

- Expects to reach the goal
- Expects to avoid obstacles
- Expects to keep the formation



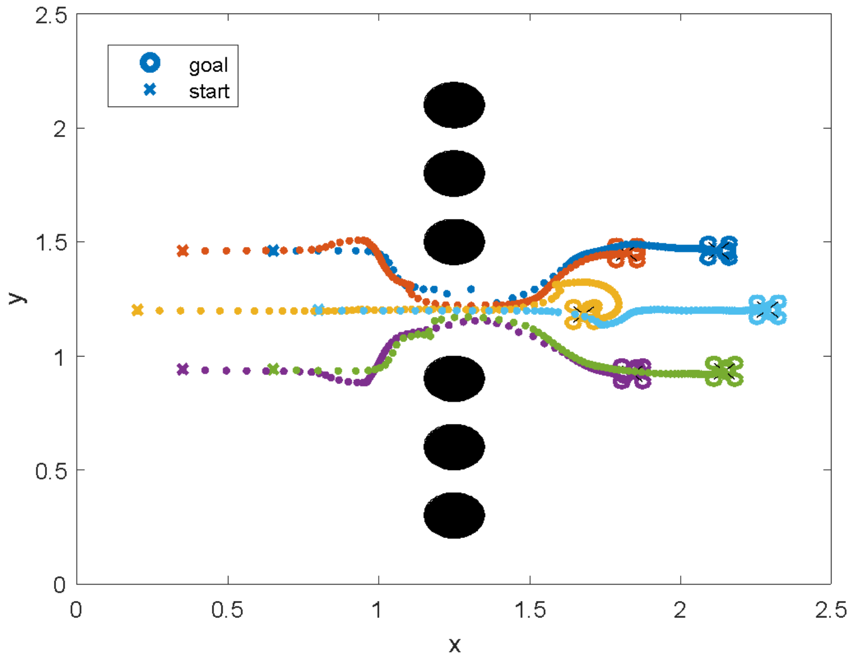
$$F = F^g + F^{os} + F^{od} + F^f$$

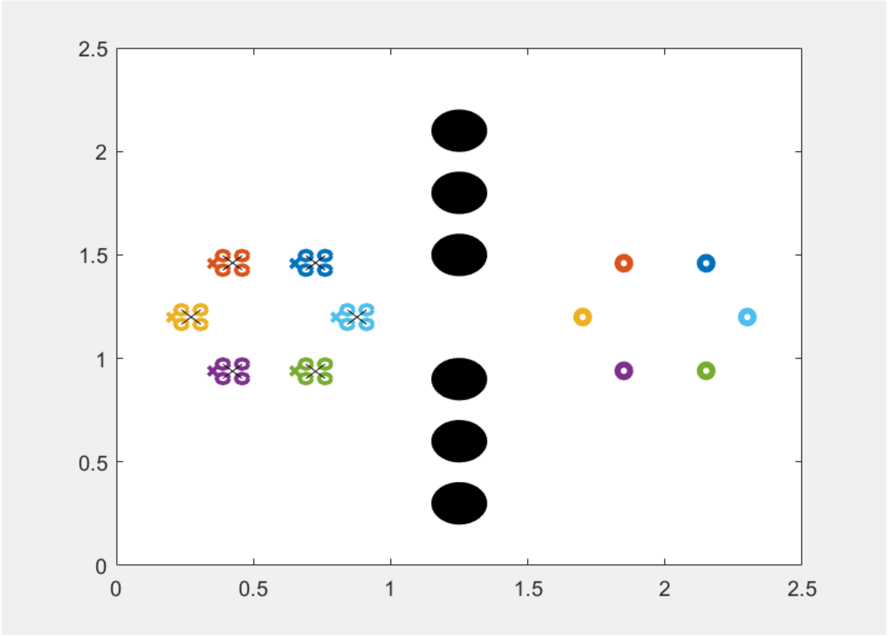
$$F^f = \frac{1}{2}\Pi^f \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left(\|p^j - p^i\| - \|pf^j - pf^i\| \right)^2$$

p^i $and p^j$ are position vector of drone i and j
p^{fi} $and p^{fj}$ are the initial position vector of drone i and j

"Free Energy Principle Based Precision Modulation for Robot Attention: towards brain inspired robot intelligence", PhD thesis, Department of Cognitive Robotics, TU Delft, 2023

Escape manoeuvre for tight spaces





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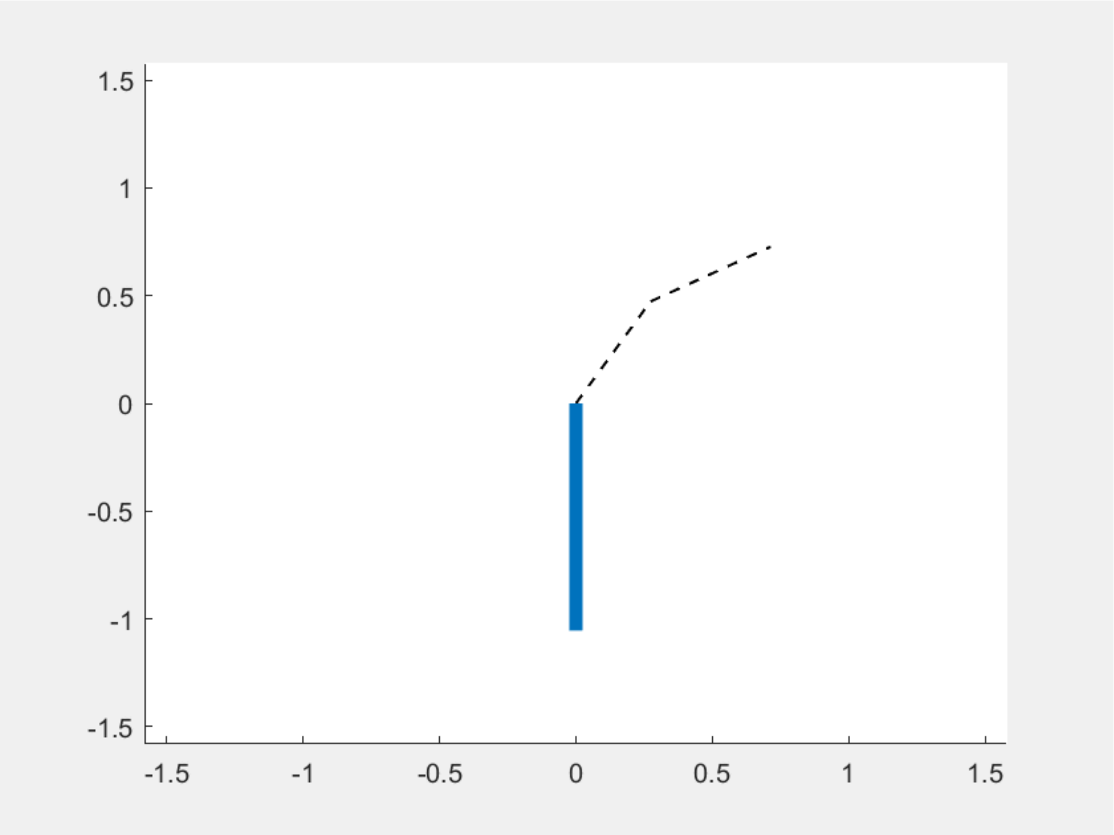
Active Inference on a 2DOF robot arm



Ajith Anil Meera and Pablo Lanillos, "*Towards metacognitive robot decision making for tool selection*" IWAI 2023.

Thank you

Martijn Wisse, TU Delft

Pablo Lanillos, Radboud University