CHAPTER 14

14.1 The partial derivatives can be evaluated,

$$\frac{\partial f}{\partial x} = 4x = 4(2) = 8$$
$$\frac{\partial f}{\partial y} = 2y = 2(2) = 4$$

The angle in the direction of ***h*** is

$$\theta = \tan^{-1}\left(\frac{2}{3}\right) = 0.588033 \text{ radians } (= \ 33.69007°)$$

The directional derivative can be computed as

$$g'(0) = 8\cos(0.588033) + 4\sin(0.588033) = 8.875203$$

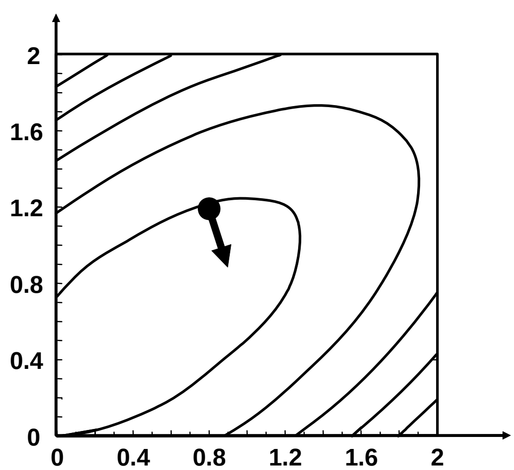
14.2 The elevation can be determined as

$$f(0.8,\ 1.2) = 2(0.8)1.2 + 1.5(1.2) - 1.25(0.8)^2 - 2(1.2)^2 + 5 = 5.04$$

The partial derivatives can be evaluated,

$$\frac{\partial f}{\partial x} = 2y - 2.5x = 2(1.2) - 2.5(0.8) = 0.4$$
$$\frac{\partial f}{\partial y} = 2x + 1.5 - 4y = 2(0.8) + 1.5 - 4(1.2) = -1.7$$

which can be used to determine the gradient as $\nabla f = 0.4\mathbf{i} - 1.7\mathbf{j}$. This corresponds to the direction $\theta = \tan^{-1}(-1.7/0.4) = -1.3397$ radians ($= -76.76°$). This vector can be sketched on a topographical map of the function as shown below:



The slope in this direction can be computed as

$$\sqrt{0.4^2 + (-1.7)^2} = 1.746$$

14.3 The partial derivatives can be evaluated,

$$\frac{\partial f}{\partial x} = -3x + 2.25y$$
$$\frac{\partial f}{\partial y} = 2.25x - 4y + 1.75$$

These can be set to zero to generate the following simultaneous equations

$$3x - 2.25y = 0$$
$$-2.25x + 4y = 1.75$$

which can be solved for x = 0.567568 and y = 0.756757, which is the optimal solution.

14.4 The partial derivatives can be evaluated at the initial guesses, x = 1 and y = 1,

$$\frac{\partial f}{\partial x} = -3x + 2.25y = -3(1) + 2.25(1) = -0.75$$
$$\frac{\partial f}{\partial y} = 2.25x - 4y + 1.75 = 2.25(1) - 4(1) + 1.75 = 0$$

Therefore, the search direction is –0.75**i**.

$$f(1 - 0.75h,\ 1) = 0.5 + 0.5625h - 0.84375h^2$$

This can be differentiated and set equal to zero and solved for h^* = 0.33333. Therefore, the result for the first iteration is x = 1 – 0.75(0.3333) = 0.75 and y = 1 + 0(0.3333) = 1.

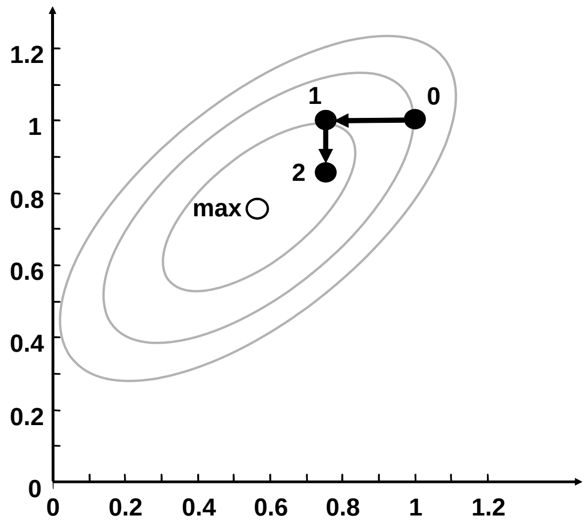
For the second iteration, the partial derivatives can be evaluated as,

$$\frac{\partial f}{\partial x} = -3(0.75) + 2.25(1) = 0$$
$$\frac{\partial f}{\partial y} = 2.25(0.75) - 4(1) + 1.75 = -0.5625$$

Therefore, the search direction is –0.5625**j**.

$$f(0.75,\ 1 - 0.5625h) = 0.59375 + 0.316406h - 0.63281h^2$$

This can be differentiated and set equal to zero and solved for h^* = 0.25. Therefore, the result for the second iteration is x = 0.75 + 0(0.25) = 0.75 and y = 1 + (–0.5625)0.25 = 0.859375.



14.5 (a)

$$\nabla f = \begin{Bmatrix} 2y^2 + 3ye^{xy} \\ 4xy + 3xe^{xy} \end{Bmatrix} \quad H = \begin{bmatrix} 3y^2e^{xy} & 4y + 3xye^{xy} + 3e^{xy} \\ 4y + 3xye^{xy} + 3e^{xy} & 4x + 3x^2e^{xy} \end{bmatrix}$$

(b)

$$\nabla f = \begin{Bmatrix} 2x \\ 2y \\ 4z \end{Bmatrix} \qquad H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(c)

$$\nabla f = \begin{Bmatrix} \dfrac{2x + 2y}{x^2 + 2xy + 3y^2} \\ \dfrac{2x + 6y}{x^2 + 2xy + 3y^2} \end{Bmatrix} \qquad H = \frac{\begin{bmatrix} -2x^2 - 4xy + 2y^2 & -2x^2 - 12xy - 6y^2 \\ -2x^2 - 12xy - 6y^2 & 2x^2 - 12xy - 18y^2 \end{bmatrix}}{\left(x^2 + 2xy + 3y^2\right)^2}$$

14.6 The partial derivatives can be evaluated at the initial guesses, $x = 1$ and $y = 1$,

$$\frac{\partial f}{\partial x} = 2(x-3) = 2(1-3) = -4$$

$$\frac{\partial f}{\partial y} = 2(y-2) = 2(1-2) = -2$$

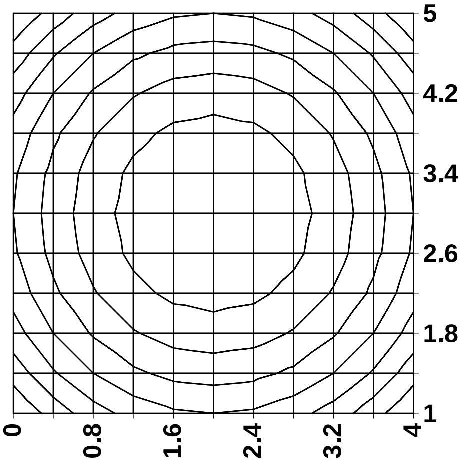
$$f(1-4h,\ 1-2h) = (1-4h-3)^2 + (1-2h-2)^2$$

$$g(h) = (-4h-2)^2 + (-2h-1)^2$$

Setting $g'(h) = 0$ gives $h^* = -0.5$. Therefore,

$x = 1 - 4(-0.5) = 3$
$y = 1 - 2(-0.5) = 2$

Thus, for this special case, the approach converges on the correct answer after a single iteration. This occurs because the function is spherical as shown below. Thus, the gradient for any guess points directly at the solution.



14.7 The partial derivatives can be evaluated at the initial guesses, $x = 0$ and $y = 0$,

$$\frac{\partial f}{\partial x} = 3.5 + 2x - 4x^3 - 2y = 3.5 + 2(0) - 4(0)^3 - 2(0) = 3.5$$

$$\frac{\partial f}{\partial y} = 2 - 2x - 2y = 2 - 2(0) - 2(0) = 2$$

$$f(0 + 3.5h,\ 0 + 2h) = 16.25h - 5.75h^2 - 150.06h^4$$

$$g'(h) = 16.25 - 11.5h - 600.25h^3$$

The root of this equation can be determined by bisection. Using initial guesses of $h = 0$ and 1 yields a root of $h^* = 0.27893$ after 13 iterations with $\varepsilon_a = 0.04\%$. Therefore,

$x = 0 + 3.5(0.27893) = 0.976257$
$y = 0 + 2(0.27893) = 0.557861$

14.8

$$\frac{\partial f}{\partial x} = -8 + 2x - 2y \qquad \frac{\partial f}{\partial y} = 12 + 8y - 2x$$

At $x = y = 0$,

$$\frac{\partial f}{\partial x} = -8 \qquad \frac{\partial f}{\partial y} = 12$$

$$f(0 - 8h, 0 + 12h) = g(h)$$

$$g(h) = 832h^2 + 208h$$

At $g'(h) = 0$, $h^* = -0.125$.

Therefore,

$x = 0 - 8(-0.125) = 1$
$y = 0 + 12(-0.125) = -1.5$

14.9 The following code implements the random search algorithm in VBA. It is set up to solve Prob. 14.7.

```
Option Explicit

Sub RandSearch()
Dim n As Long
Dim xmin As Double, xmax As Double, ymin As Double, ymax As Double
Dim maxf As Double, maxx As Double, maxy As Double
xmin = -2: xmax = 2: ymin = -2: ymax = 2
n = InputBox("n=")
Call RndSrch(n, xmin, xmax, ymin, ymax, maxy, maxx, maxf)
MsgBox maxf
MsgBox maxx
MsgBox maxy
End Sub

Sub RndSrch(n, xmin, xmax, ymin, ymax, maxy, maxx, maxf)
Dim j As Long
Dim x As Double, y As Double, fn As Double
maxf = -1000000000#
For j = 1 To n
  x = xmin + (xmax - xmin) * Rnd
  y = ymin + (ymax - ymin) * Rnd
  fn = f(x, y)
  If fn > maxf Then
    maxf = fn
    maxx = x
    maxy = y
   End If
Next j
End Sub

Function f(x, y)
f = 4 * x + 2 * y + x ^ 2 - 2 * x ^ 4 + 2 * x * y - 3 * y ^ 2
End Function
```

The result of running this program for different number of iterations yields the results in the following table. We have also included to exact result.

n	*f*(*x*, *y*)	*x*	*y*
1000	4.31429	1.011782	0.613747
10000	4.34185	0.961359	0.678104
100000	4.34338	0.964238	0.641633
1000000	4.34397	0.965868	0.656765
10000000	4.34401	0.967520	0.655715
truth	4.34401	0.967580	0.655860

14.10 The following code implements the grid search algorithm in VBA:

```
Option Explicit

Sub GridSearch()
Dim nx As Long, ny As Long
Dim xmin As Double, xmax As Double, ymin As Double, ymax As Double
Dim maxf As Double, maxx As Double, maxy As Double
xmin = -2: xmax = 2: ymin = 1: ymax = 3
nx = 1000
ny = 1000
Call GridSrch(nx, ny, xmin, xmax, ymin, ymax, maxy, maxx, maxf)
MsgBox maxf
```

```
MsgBox maxx
MsgBox maxy
End Sub
Sub GridSrch(nx, ny, xmin, xmax, ymin, ymax, maxy, maxx, maxf)
Dim i As Long, j As Long
Dim x As Double, y As Double, fn As Double
Dim xinc As Double, yinc As Double
xinc = (xmax - xmin) / nx
yinc = (ymax - ymin) / ny
maxf = -1000000000#
x = xmin
For i = 0 To nx
  y = ymin
  For j = 0 To ny
    fn = f(x, y)
    If fn > maxf Then
      maxf = fn
      maxx = x
      maxy = y
    End If
    y = y + yinc
  Next j
  x = x + xinc
Next i
End Sub
Function f(x, y)
f = y - x - 2 * x ^ 2 - 2 * x * y - y ^ 2
End Function
```

14.11

$$f(x,y) = 6x^2y - 9y^2 - 8x^2$$

$$\frac{\partial f}{\partial x} = 12xy - 16x \Rightarrow 12(2)(4) - 16(4) = 32$$

$$\frac{\partial f}{\partial y} = 6x^2 - 18y \Rightarrow 6(4)^2 - 18(2) = 60$$

$$\nabla f = 32\hat{i} + 60\hat{j}$$

$$f\left(x_o + \frac{\partial f}{\partial x}h, y_o + \frac{\partial f}{\partial y}h\right) = f(4+32h, 2+60h)$$

$$= 6(4+32h)^2(2+60h) - 9(2+60h)^2 - 8(4+32h)^2$$

$$g(x) = 368,640h^3 + 63,856h^2 + 4,624h + 28$$

14.12

$$f(x,y) = 2x^3y^2 - 7yx + x^2 + 3y$$

$$\frac{\partial f}{\partial x} = 6x^2y^2 - 7y + 2x \Rightarrow 6(1)(1) - 7(1) + 2(1) = 1$$

$$\frac{\partial f}{\partial x} = 4x^3y - 7x + 3 \Rightarrow 4(1)(1) - 7(1) + 3 = 0$$

$$\nabla f = 1\hat{i} + 0\hat{j}$$

$$f(x_o + \frac{\partial f}{\partial x}h, y_o + \frac{\partial f}{\partial y}h) = f(1+h,\ 1+0h)$$

$$= 2(1+h)^3(1)^2 - 7(1+h)(1) + (1+h)^2 + 3(1)$$

$g(x) = 2h^3 + 7h^2 + h - 1$