CHAPTER 9

9.1

$$\begin{bmatrix} 2 & 0 & 5 \\ 0 & 1 & 1 \\ 8 & 3 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 40 \\ 10 \\ 20 \end{Bmatrix} \qquad [A]^T = \begin{bmatrix} 2 & 0 & 8 \\ 0 & 1 & 3 \\ 5 & 1 & 0 \end{bmatrix}$$

9.2 (a) Possible multiplications:

$$[A][B] = \begin{bmatrix} 23.5 & -2 \\ 52 & 16 \\ -18 & 8 \end{bmatrix} \qquad [A][C] = \begin{bmatrix} 15 & -13 \\ 0 & -16 \\ -22 & 14 \end{bmatrix} \qquad [B][C] = \begin{bmatrix} -7 & 1 \\ -5 & 1 \end{bmatrix} \qquad [C][B] = \begin{bmatrix} 7 & -4 \\ -11.5 & -2 \end{bmatrix}$$

Note: Some students might recognize that we can also compute $[B][B]$ and $[C][C]$:

$$[B][B] = \begin{bmatrix} 16 & 0 \\ 3 & 4 \end{bmatrix} \qquad [C][C] = \begin{bmatrix} 10 & -6 \\ -9 & 7 \end{bmatrix}$$

(b) $[B][A]$ and $[C][A]$ are impossible because the inner dimensions do not match:

$(2\times2) * (3\times2)$

(c) According to **(a)**, $[B][C] \neq [C][B]$

9.3 (a) $[A] = 3\times2$ $\quad [B] = 3\times3$ $\quad [C] = 3\times1$ $\quad [D] = 2\times4$
$[E] = 3\times3$ $\quad [F] = 2\times3$ $\quad [G] = 1\times3$

(b) Square: $[B]$ and $[E]$ Column: $[C]$ Row: $[G]$

(c) $a_{12} = 5$ $\quad b_{23} = 6$ $\quad d_{32}$ = does not exist
$e_{22} = 2$ $\quad f_{12} = 0$ $\quad g_{12} = 5$

(d)

(1) $[E]+[B] = \begin{bmatrix} 5 & 8 & 16 \\ 8 & 4 & 9 \\ 6 & 0 & 10 \end{bmatrix}$

(2) $[A]+[F] =$ not possible

(3) $[B]-[E] = \begin{bmatrix} 3 & -2 & -2 \\ -6 & 0 & 3 \\ -2 & 0 & -2 \end{bmatrix}$

(4) $7[B] = \begin{bmatrix} 28 & 21 & 49 \\ 7 & 14 & 42 \\ 14 & 0 & 28 \end{bmatrix}$

(5) $[A]\times[B] =$ not possible

(6) $\{C\}^T = \lfloor 3 \quad 5 \quad 1 \rfloor$

(7) $[B]\times[A] = \begin{bmatrix} 54 & 68 \\ 36 & 45 \\ 28 & 34 \end{bmatrix}$

(8) $\{D\}^T = \begin{bmatrix} 9 & 2 \\ 4 & -1 \\ 3 & 6 \\ -6 & 5 \end{bmatrix}$

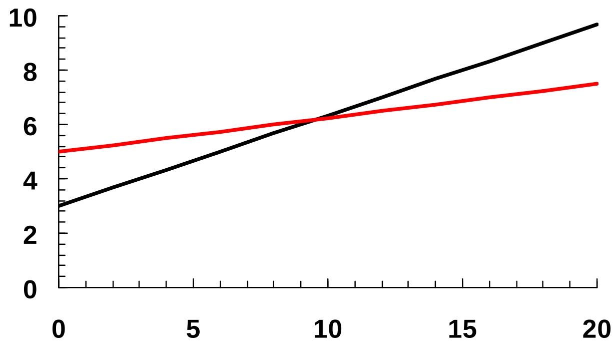
(9) $[A]\times[C] =$ not possible

(10) $[I]\times[B] = [B]$

(11) $[E]^T[E]=\begin{bmatrix}66 & 19 & 54\\ 19 & 29 & 51\\ 54 & 51 & 126\end{bmatrix}$ (12) $[C]^T[C]=35$

9.4 The equations can be rearranged into a format for plotting x_2 versus x_1:

$$x_2 = 3 - 0.3333x_1$$
$$x_2 = 5 - \frac{1}{8}x_1$$

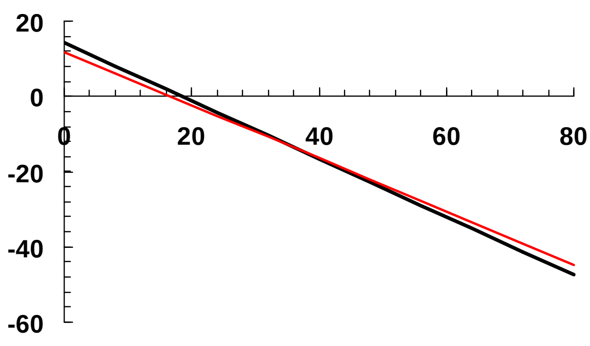


Therefore, the solution is $x_1 = 9.6$, $x_2 = 6.2$. The results can be checked by substituting them back into the original equations:

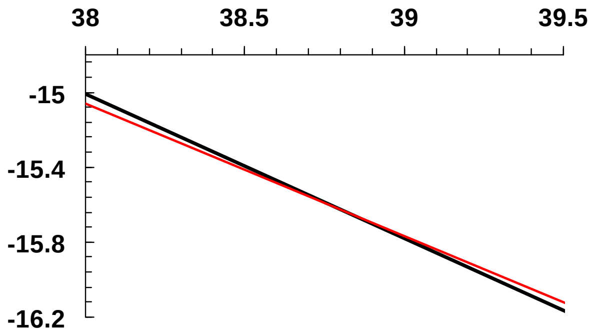
$$2(9.6) - 6(6.2) = -18$$
$$-9.6 + 8(6.2) = 40$$

9.5 (a) The equations can be rearranged into a format for plotting x_2 versus x_1:

$$x_2 = 14.25 - 0.77x_1$$
$$x_2 = 11.76471 - \frac{1.2}{1.7}x_1$$



If you zoom in, it appears that there is a root at about (38.7, –15.6).



The results can be checked by substituting them back into the original equations:

$$0.77(38.7) - 15.6 = 14.2 \cong 14.25$$
$$1.2(38.7) + 1.7(-15.6) = 19.92 \cong 20$$

(b) The plot suggests that the system may be ill-conditioned because the slopes are so similar.

(c) The determinant can be computed as

$$D = 0.77(1.7) - 1(1.2) = 0.11$$

which is relatively small. Note that if the system is normalized first by dividing each equation by the largest coefficient,

$$0.77x_1 + x_2 \quad = 14.25$$
$$0.705882x_1 + x_2 = 11.76471$$

the determinant is even smaller

$$D = 0.77(1) - 1(0.705882) = 0.064$$

(d) Using Eqs. (9.10) and (9.11) yields

$$x_1 = \frac{14.25(1.7) - 1(20)}{0.109} = 38.761$$
$$x_2 = \frac{0.77(20) - 14.25(1.2)}{0.109} = -15.596$$

9.6 (a) The determinant can be computed as:

$$A_1 = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1(0) - 1(1) = -1$$
$$A_2 = \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} = 2(0) - 1(3) = -3$$
$$A_3 = \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 2(1) - 1(3) = -1$$
$$D = 0(-1) - 2(-3) + 5(-1) = 1$$

(b) Cramer's rule

$$x_1 = \frac{\begin{vmatrix} 1 & 2 & 5 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix}}{D} = \frac{-2}{1} = -2$$

$$x_2 = \frac{\begin{vmatrix} 0 & 1 & 5 \\ 2 & 1 & 1 \\ 3 & 2 & 0 \end{vmatrix}}{D} = \frac{8}{1} = 8$$

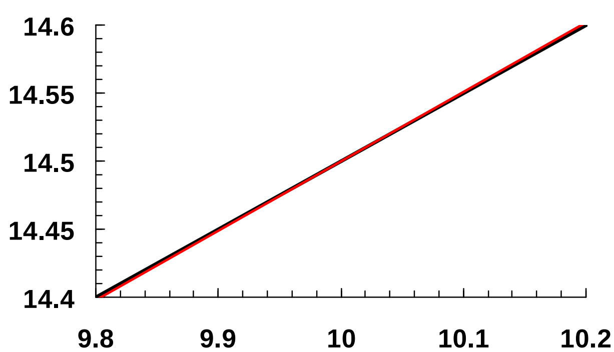
$$x_3 = \frac{\begin{vmatrix} 0 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{vmatrix}}{D} = \frac{-3}{1} = -3$$

(c) The results can be checked by substituting them back into the original equations:

$$2(8) + 5(-3) = 1$$
$$2(-2) + 8 - 3 = 1$$
$$3(-2) + 8 = 2$$

9.7 (a) The equations can be rearranged into a format for plotting x_2 versus x_1:

$$x_2 = 9.5 + 0.5x_1$$
$$x_2 = 9.4 + 0.51x_1$$



The solution is x_1 = 10, x_2 = 14.5. Notice that the lines have very similar slopes.

(b) The determinant can be computed as

$$D = 0.5(-2) - (-1)1.02 = 0.02$$

(c) The plot and the low value of the determinant both suggest that the system is ill-conditioned.

(d) Using Eqs. (9.10) and (9.11) yields

$$x_1 = \frac{-9.5(-2) - (-1)(-18.8)}{0.02} = 10$$

$$x_2 = \frac{0.5(-18.8) - (-9.5)1.02}{0.02} = 14.5$$

(e) Using Eqs. (9.10) and (9.11) yields

$$x_1 = \frac{-9.5(-2) - (-1)(-18.8)}{-0.02} = -10$$

$$x_2 = \frac{0.52(-18.8) - (-9.5)1.02}{-0.02} = 4.3$$

The ill-conditioned nature of the system is illustrated by the fact that a small change in one of the coefficients results in a huge change in the results.

9.8 (a) The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 10 & 2 & -1 & 27 \\ -3 & -6 & 2 & -61.5 \\ 1 & 1 & 5 & -21.5 \end{bmatrix}$$

Forward elimination:

a_{21} is eliminated by multiplying row 1 by –3/10 and subtracting the result from row 2. a_{31} is eliminated by multiplying row 1 by 1/10 and subtracting the result from row 3.

$$\begin{bmatrix} 10 & 2 & -1 & 27 \\ 0 & -5.4 & 1.7 & -53.4 \\ 0 & 0.8 & 5.1 & -24.2 \end{bmatrix}$$

a_{32} is eliminated by multiplying row 2 by 0.8/(–5.4) and subtracting the result from row 3.

$$\begin{bmatrix} 10 & 2 & -1 & 27 \\ 0 & -5.4 & 1.7 & -53.4 \\ 0 & 0 & 5.351852 & -32.1111 \end{bmatrix}$$

Back substitution:

$$x_3 = \frac{-32.1111}{5.351852} = -6$$

$$x_2 = \frac{-53.4 - 1.7(-6)}{-5.4} = 8$$

$$x_1 = \frac{27 - (-1)(-6) - 2(8)}{10} = 0.5$$

(b) Check:

$$10(0.5) + 2(8) - (-6) = 27$$

$$-3(0.5) - 6(8) + 2(-6) = -61.5$$

$$0.5 + 8 + 5(-6) = -21.5$$

9.9 (a) The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 4 & 1 & -1 & -2 \\ 5 & 1 & 2 & 4 \\ 6 & 1 & 1 & 6 \end{bmatrix}$$

Forward elimination: First, we pivot by switching rows 1 and 3:

$$\begin{bmatrix} 6 & 1 & 1 & 6 \\ 5 & 1 & 2 & 4 \\ 4 & 1 & -1 & -2 \end{bmatrix}$$

Multiply row 1 by 5/6 and subtract from row 2 to eliminate a_{21}. Multiply row 1 by 4/6 and subtract from row 3 to eliminate a_{31}.

$$\begin{bmatrix} 6 & 1 & 1 & 6 \\ 0 & 0.16667 & 1.16667 & -1 \\ 0 & 0.33333 & -1.6667 & -6 \end{bmatrix}$$

Pivot:

$$\begin{bmatrix} 6 & 1 & 1 & 6 \\ 0 & 0.33333 & 1.6667 & -6 \\ 0 & 0.16667 & -1.1667 & -1 \end{bmatrix}$$

Multiply row 2 by 0.16667/0.33333 = 0.5 and subtract from row 3 to eliminate a_{32}.

$$\begin{bmatrix} 6 & 1 & 1 & 6 \\ 0 & 0.33333 & 1.6667 & -6 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

Back substitution:

$$x_3 = \frac{2}{2} = 1$$
$$x_2 = \frac{-6-(-1.6667)1}{0.33333} = -13$$
$$x_1 = \frac{6-1-(-13)}{6} = 3$$

Check:

$$4(3) - 13 - 3 = -2$$
$$5(3) - 13 + 2(1) = 4$$
$$6(3) - 13 + 1 = 6$$

9.10 (a) The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 2 & -6 & -1 & -38 \\ -3 & -1 & 7 & -34 \\ -8 & 1 & -2 & -20 \end{bmatrix}$$

Forward elimination: First, we pivot by switching rows 1 and 3:

$$\begin{bmatrix} -8 & 1 & -2 & -20 \\ -3 & -1 & 7 & -34 \\ 2 & -6 & -1 & -38 \end{bmatrix}$$

Multiply row 1 by –3/(–8) = 0.375 and subtract from row 2 to eliminate a_{21}. Multiply row 1 by 2/(–8) = –0.25 and subtract from row 3 to eliminate a_{31}.

$$\begin{bmatrix} -8 & 1 & -2 & -20 \\ 0 & -1.375 & 7.75 & -26.5 \\ 0 & -5.75 & -1.5 & -43 \end{bmatrix}$$

Pivot:

$$\begin{bmatrix} -8 & 1 & -2 & -20 \\ 0 & -5.75 & -1.5 & -43 \\ 0 & -1.375 & 7.75 & -26.5 \end{bmatrix}$$

Multiply row 2 by –1.375/–5.75 = 0.23913 and subtract from row 3 to eliminate a_{32}.

$$\begin{bmatrix} -8 & 1 & -2 & -20 \\ 0 & -5.75 & -1.5 & -43 \\ 0 & 0 & 8.108696 & -16.2174 \end{bmatrix}$$

Back substitution:

$$x_3 = \frac{-16.2174}{8.108696} = -2$$
$$x_2 = \frac{-43-(-1.5)(-2)}{-5.75} = 8$$
$$x_1 = \frac{-20+2(-2)-1(8)}{-8} = 4$$

(b) Check:

$$2(4)-6(8)-(-2) = -38$$
$$-3(4)-(8)+7(-2) = -34$$
$$-8(4)+(8)-2(-2) = -20$$

9.11 (a) The determinant can be computed as:

$$A_1 = \begin{vmatrix} 2 & -1 \\ -2 & 0 \end{vmatrix} = 2(0)-(-1)(-2) = -2$$
$$A_2 = \begin{vmatrix} 1 & -1 \\ 5 & 0 \end{vmatrix} = 1(0)-(-1)(5) = 5$$
$$A_3 = \begin{vmatrix} 1 & 2 \\ 5 & -2 \end{vmatrix} = 1(-2)-2(5) = -12$$

$$D = 0(-2)-(-3)5+7(-12) = -69$$

(b) Cramer's rule

$$x_1 = \frac{\begin{vmatrix} 2 & -3 & 7 \\ 3 & 2 & -1 \\ 2 & -2 & 0 \end{vmatrix}}{D} = \frac{-68}{-69} = 0.985507$$

$$x_2 = \frac{\begin{vmatrix} 0 & 2 & 7 \\ 1 & 3 & -1 \\ 5 & 2 & 0 \end{vmatrix}}{D} = \frac{-101}{-69} = 1.463768$$

$$x_3 = \frac{\begin{vmatrix} 0 & -3 & 2 \\ 1 & 2 & 3 \\ 5 & -2 & 2 \end{vmatrix}}{D} = \frac{-63}{-69} = 0.913043$$

(c) The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 0 & -3 & 7 & 2 \\ 1 & 2 & -1 & 3 \\ 5 & -2 & 0 & 2 \end{bmatrix}$$

Forward elimination: First, we pivot by switching rows 1 and 3:

$$\begin{bmatrix} 5 & -2 & 0 & 2 \\ 1 & 2 & -1 & 3 \\ 0 & -3 & 7 & 2 \end{bmatrix}$$

Multiply row 1 by 1/5 = 0.2 and subtract from row 2 to eliminate a_{21}. Because a_{31} already equals zero, it does not have to be eliminated.

$$\begin{bmatrix} 5 & -2 & 0 & 2 \\ 0 & 2.4 & -1 & 2.6 \\ 0 & -3 & 7 & 2 \end{bmatrix}$$

Pivot:

$$\begin{bmatrix} 5 & -2 & 0 & 2 \\ 0 & -3 & 7 & 2 \\ 0 & 2.4 & -1 & 2.6 \end{bmatrix}$$

Multiply row 2 by 2.4/(–3) = –0.8 and subtract from row 3 to eliminate a_{32}.

$$\begin{bmatrix} 5 & -2 & 0 & 2 \\ 0 & -3 & 7 & 2 \\ 0 & 0 & 4.6 & 4.2 \end{bmatrix}$$

Back substitution:

$$x_3 = \frac{4.2}{4.6} = 0.913043$$

$$x_2 = \frac{2 - 7(0.913043)}{-3} = 1.463768$$

$$x_1 = \frac{2 - 0(0.913043) - (-2)(1.463768)}{5} = 0.985507$$

(d) Check:

$$-3(1.463768) + 7(0.913043) = 2$$

$$(0.985507) + 2(1.463768) - (0.913043) = 3$$

$$5(0.985507) - 2(1.463768) = 2$$

9.12 The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 2 & 1 & -1 & 1 \\ 5 & 2 & 2 & -4 \\ 3 & 1 & 1 & 5 \end{bmatrix}$$

Normalize the first row and then eliminate a_{21} and a_{31},

$$\begin{bmatrix} 1 & 0.5 & -0.5 & 0.5 \\ 0 & -0.5 & 4.5 & -6.5 \\ 0 & -0.5 & 2.5 & 3.5 \end{bmatrix}$$

Normalize the second row and eliminate a_{12} and a_{32},

$$\begin{bmatrix} 1 & 0 & 4 & -6 \\ 0 & 1 & -9 & 13 \\ 0 & 0 & -2 & 10 \end{bmatrix}$$

Normalize the third row and eliminate a_{13} and a_{23},

$$\begin{bmatrix} 1 & 0 & 0 & 14 \\ 0 & 1 & 0 & -32 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

Thus, the answers are $x_1 = 14$, $x_2 = -32$, and $x_3 = -5$. Check:

$$2(14) + (-32) - (-5) = 1$$
$$5(14) + 2(-32) + 2(-5) = -4$$
$$3(14) + (-32) + (-5) = 5$$

9.13 (a) The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 6 & 2 & 2 & 2 \\ -3 & 4 & 1 & 1 \end{bmatrix}$$

Forward elimination: a_{21} is eliminated by multiplying row 1 by 6/1 = 6 and subtracting the result from row 2. a_{31} is eliminated by multiplying row 1 by –3/1 = –3 and subtracting the result from row 3.

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & -4 & 8 & 20 \\ 0 & 7 & -2 & -8 \end{bmatrix}$$

a_{32} is eliminated by multiplying row 2 by 7/(–4) = –1.75 and subtracting the result from row 3.

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & -4 & 8 & 20 \\ 0 & 0 & 12 & 27 \end{bmatrix}$$

Back substitution:

$$x_3 = \frac{27}{12} = 2.25$$

$$x_2 = \frac{20 - 8(2.25)}{-4} = -0.5$$

$$x_1 = \frac{-3 - (-1)(2.25) - 1(-0.5)}{1} = -0.25$$

(b) The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 6 & 2 & 2 & 2 \\ -3 & 4 & 1 & 1 \end{bmatrix}$$

Forward elimination: First, we pivot by switching rows 1 and 2:

$$\begin{bmatrix} 6 & 2 & 2 & 2 \\ 1 & 1 & -1 & -3 \\ -3 & 4 & 1 & 1 \end{bmatrix}$$

Multiply row 1 by 1/6 = 0.16667 and subtract from row 2 to eliminate a_{21}. Multiply row 1 by –3/6 = –0.5 and subtract from row 3 to eliminate a_{31}.

$$\begin{bmatrix} 6 & 2 & 2 & 2 \\ 0 & 0.66667 & -1.33333 & -3.33333 \\ 0 & 5 & 2 & 2 \end{bmatrix}$$

Pivot:

$$\begin{bmatrix} 6 & 2 & 2 & 2 \\ 0 & 5 & 2 & 2 \\ 0 & 0.66667 & -1.33333 & -3.33333 \end{bmatrix}$$

Multiply row 2 by 0.66667/5 = 0.133333 and subtract from row 3 to eliminate a_{32}.

$$\begin{bmatrix} 6 & 2 & 2 & 2 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & -1.6 & -3.6 \end{bmatrix}$$

Back substitution:

$$x_3 = \frac{-3.6}{1.6} = 2.25$$
$$x_2 = \frac{2 - 2(2.25)}{5} = -0.5$$
$$x_1 = \frac{2 - 2(2.25) - 2(-0.5)}{6} = -0.25$$

(c) The system is first expressed as an augmented matrix:

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 6 & 2 & 2 & 2 \\ -3 & 4 & 1 & 1 \end{bmatrix}$$

Normalize the first row, and then eliminate a_{21} and a_{31},

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & -4 & 8 & 20 \\ 0 & 7 & -2 & -8 \end{bmatrix}$$

Normalize the second row and eliminate a_{12} and a_{32},

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 12 & 27 \end{bmatrix}$$

Normalize the third row and eliminate a_{13} and a_{23},

$$\begin{bmatrix} 1 & 0 & 0 & -0.25 \\ 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 2.25 \end{bmatrix}$$

9.14 In a fashion similar to Example 9.11, vertical force balances can be written to give the following system of equations,

$$\begin{aligned}
m_1 g - T_{12} - c_1 v &= m_1 a \\
m_2 g + T_{12} - c_2 v - T_{23} &= m_2 a \\
m_3 g - c_3 v + T_{23} - T_{34} &= m_3 a \\
m_4 g - c_4 v + T_{34} - T_{45} &= m_4 a \\
m_5 g - c_5 v + T_{45} &= m_5 a
\end{aligned}$$

After substituting the known values, the equations can be expressed in matrix form (g = 9.8),

$$\begin{bmatrix} 55 & 1 & 0 & 0 & 0 \\ 75 & -1 & 1 & 0 & 0 \\ 60 & 0 & -1 & 1 & 0 \\ 75 & 0 & 0 & -1 & 1 \\ 90 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} a \\ T_{12} \\ T_{23} \\ T_{34} \\ T_{45} \end{Bmatrix} = \begin{Bmatrix} 498 \\ 627 \\ 453 \\ 591 \\ 792 \end{Bmatrix}$$

The system can be solved for

$a = 8.225$ $\quad T_{12} = 4.5$ $\quad T_{23} = 14.625$
$T_{34} = -25.875$ $\quad T_{12} = -51.75$

9.15 Using the format of Eq. 9.27,

$$[A] = \begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix} \qquad [B] = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}$$

$$\{U\} = \begin{Bmatrix} 3 \\ 3 \end{Bmatrix} \qquad \{V\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

The set of equations to be solved are

$$\begin{aligned} 3x_1 + 4x_2 - 2y_1 \quad &= 3 \\ x_2 + y_1 \quad &= 3 \\ 2x_1 \quad + 3y_1 + 4y_2 &= 1 \\ -x_1 \quad + y_2 &= 0 \end{aligned}$$

These can be solved for $x_1 = -0.4.6667$, $x_2 = 1.73333$, $y_1 = 1.26667$, and $y_2 = -0.46667$. Therefore, the solution is $z_1 = -0.46667 + 1.73333\, i$ and $z_2 = 1.26667 - 0.46667\, i$.

9.16 Here is a VBA program to implement matrix multiplication and solve Prob. 9.2 for the case of $[A]\times[B]$.

```
Option Explicit

Sub Mult()
Dim i As Integer, j As Integer
Dim l As Integer, m As Integer, n As Integer
Dim a(10, 10) As Double, b(10, 10) As Double
Dim c(10, 10) As Double
l = 2
m = 2
n = 3
a(1, 1) = 6: a(1, 2) = -1
a(2, 1) = 12: a(2, 2) = 8
a(3, 1) = -5: a(3, 2) = 4
b(1, 1) = 4: b(1, 2) = 0
b(2, 1) = 0.5: b(2, 2) = 2
Call Mmult(a, b, c, m, n, l)
For i = 1 To n
  For j = 1 To l
    MsgBox c(i, j)
  Next j
Next i
End Sub

Sub Mmult(a, b, c, m, n, l)
Dim i As Integer, j As Integer, k As Integer
Dim sum As Double
For i = 1 To n
  For j = 1 To l
    sum = 0
    For k = 1 To m
```

```
      sum = sum + a(i, k) * b(k, j)
    Next k
    c(i, j) = sum
  Next j
Next i
End Sub
```

9.17 Here is a VBA program to implement the matrix transpose and solve Prob. 9.2 for the case of $[A]^T$.

```
Option Explicit
Sub TransTest()
Dim i As Integer, j As Integer
Dim m As Integer, n As Integer
Dim a(10, 10) As Double, aT(10, 10) As Double
n = 3
m = 2
a(1, 1) = 6: a(1, 2) = -1
a(2, 1) = 12: a(2, 2) = 8
a(3, 1) = -5: a(3, 2) = 4
Call Transpose(a, aT, n, m)
For i = 1 To m
  For j = 1 To n
    MsgBox aT(i, j)
  Next j
Next i
End Sub

Sub Transpose(a, b, n, m)
Dim i As Integer, j As Integer
For i = 1 To m
  For j = 1 To n
    b(i, j) = a(j, i)
  Next j
Next i
End Sub
```

9.18 Here is a VBA program to implement the Gauss elimination algorithm and solve the test case in the problem statement.

```
Option Explicit
Sub GaussElim()
Dim n As Integer, er As Integer, i As Integer
Dim a(10, 10) As Double, b(10) As Double, x(10) As Double
n = 3
a(1, 1) = 1: a(1, 2) = 2: a(1, 3) = -1
a(2, 1) = 5: a(2, 2) = 2: a(2, 3) = 2
a(3, 1) = -3: a(3, 2) = 5: a(3, 3) = -1
b(1) = 2: b(2) = 9: b(3) = 1
Call Gauss(a, b, n, x, er)
If er = 0 Then
  For i = 1 To n
    MsgBox "x(" & i & ") = " & x(i)
  Next i
Else
  MsgBox "ill-conditioned system"
End If
End Sub

Sub Gauss(a, b, n, x, er)
Dim i As Integer, j As Integer
Dim s(10) As Double
```

```
Const tol As Double = 0.000001
er = 0
For i = 1 To n
  s(i) = Abs(a(i, 1))
  For j = 2 To n
    If Abs(a(i, j)) > s(i) Then s(i) = Abs(a(i, j))
  Next j
Next i
Call Eliminate(a, s, n, b, tol, er)
If er <> -1 Then
  Call Substitute(a, n, b, x)
End If
End Sub

Sub Pivot(a, b, s, n, k)
Dim p As Integer, ii As Integer, jj As Integer
Dim factor As Double, big As Double, dummy As Double
p = k
big = Abs(a(k, k) / s(k))
For ii = k + 1 To n
  dummy = Abs(a(ii, k) / s(ii))
  If dummy > big Then
    big = dummy
    p = ii
  End If
Next ii
If p <> k Then
  For jj = k To n
    dummy = a(p, jj)
    a(p, jj) = a(k, jj)
    a(k, jj) = dummy
  Next jj
  dummy = b(p)
  b(p) = b(k)
  b(k) = dummy
  dummy = s(p)
  s(p) = s(k)
  s(k) = dummy
End If
End Sub

Sub Substitute(a, n, b, x)
Dim i As Integer, j As Integer
Dim sum As Double
x(n) = b(n) / a(n, n)
For i = n - 1 To 1 Step -1
  sum = 0
  For j = i + 1 To n
    sum = sum + a(i, j) * x(j)
  Next j
  x(i) = (b(i) - sum) / a(i, i)
Next i
End Sub

Sub Eliminate(a, s, n, b, tol, er)
Dim i As Integer, j As Integer, k As Integer
Dim factor As Double
For k = 1 To n - 1
  Call Pivot(a, b, s, n, k)
  If Abs(a(k, k) / s(k)) < tol Then
    er = -1
    Exit For
```

```
  End If
  For i = k + 1 To n
    factor = a(i, k) / a(k, k)
    For j = k + 1 To n
      a(i, j) = a(i, j) - factor * a(k, j)
    Next j
    b(i) = b(i) - factor * b(k)
  Next i
Next k
If Abs(a(k, k) / s(k)) < tol Then er = -1
End Sub
```

Its application yields a solution of (1, 1, 1).

9.19 The position of the three masses can be modeled by the following steady-state force balances

$$
\begin{aligned}
0 &= k(x_2 - x_1) + m_1 g - kx_1 \\
0 &= k(x_3 - x_2) + m_2 g - k(x_2 - x_1) \\
0 &= m_3 g - k(x_3 - x_2)
\end{aligned}
$$

Terms can be combined to yield

$$
\begin{aligned}
2kx_1 - kx_2 \qquad\quad &= m_1 g \\
-kx_1 + 2kx_2 - kx_3 &= m_2 g \\
- kx_2 + kx_3 &= m_3 g
\end{aligned}
$$

Substituting the parameter values

$$
\begin{bmatrix} 20 & -10 & 0 \\ -10 & 20 & -10 \\ 0 & -10 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 19.62 \\ 29.43 \\ 24.525 \end{Bmatrix}
$$

The system can be solved for 7.3575, 12.7530 and 15.2055.

9.20 Here is MATLAB M-file function to implement the Newton-Raphson method to solve simultaneous nonlinear systems:

```
function [x,f,ea,iter]=newtmult(func,x0,es,maxit,varargin)
% newtmult: Newton-Raphson root zeroes nonlinear systems
% [x,f,ea,iter]=newtmult(func,x0,es,maxit,p1,p2,...):
% uses the Newton-Raphson method to find the roots of
% a system of nonlinear equations
% input:
% func = name of function that returns f and J
% x0 = initial guess
% es = desired percent relative error (default = 0.0001%)
% maxit = maximum allowable iterations (default = 50)
% p1,p2,... = additional parameters used by function
% output:
% x = vector of roots
% f = vector of functions evaluated at roots
% ea = approximate percent relative error (%)
% iter = number of iterations
if nargin<2,error('at least 2 input arguments required'),end
if nargin<3|isempty(es),es=0.0001;end
if nargin<4|isempty(maxit),maxit=50;end
```

```
iter = 0;x=x0;
while (1)
  [J,f]=func(x,varargin{:});
  dx=J\f;
  x=x-dx;
  iter = iter + 1;
  ea=100*max(abs(dx./x));
  if iter>=maxit|ea<=es, break, end
end
```

9.21 MATLAB M-files can be used to compute both the function values and the Jacobian as

```
function [J,f]=jfrain(x,varargin)
del=0.00001;
df1dx1=(f1(x(1)+del*x(1),x(2),x(3),x(4),x(5),varargin{:})-...
    f1(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(1));
df1dx2=(f1(x(1),x(2)+del*x(2),x(3),x(4),x(5),varargin{:})-...
    f1(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(2));
df1dx3=(f1(x(1),x(2),x(3)+del*x(3),x(4),x(5),varargin{:})-...
    f1(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(3));
df1dx4=(f1(x(1),x(2),x(3),x(4)+del*x(4),x(5),varargin{:})-...
    f1(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(4));
df1dx5=(f1(x(1),x(2),x(3),x(4),x(5)+del*x(5),varargin{:})-...
    f1(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(5));
df2dx1=(f2(x(1)+del*x(1),x(2),x(3),x(4),x(5),varargin{:})-...
    f2(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(1));
df2dx2=(f2(x(1),x(2)+del*x(2),x(3),x(4),x(5),varargin{:})-...
    f2(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(2));
df2dx3=(f2(x(1),x(2),x(3)+del*x(3),x(4),x(5),varargin{:})-...
    f2(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(3));
df2dx4=(f2(x(1),x(2),x(3),x(4)+del*x(4),x(5),varargin{:})-...
    f2(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(4));
df2dx5=(f2(x(1),x(2),x(3),x(4),x(5)+del*x(5),varargin{:})-...
    f2(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(5));
df3dx1=(f3(x(1)+del*x(1),x(2),x(3),x(4),x(5),varargin{:})-...
    f3(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(1));
df3dx2=(f3(x(1),x(2)+del*x(2),x(3),x(4),x(5),varargin{:})-...
    f3(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(2));
df3dx3=(f3(x(1),x(2),x(3)+del*x(3),x(4),x(5),varargin{:})-...
    f3(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(3));
df3dx4=(f3(x(1),x(2),x(3),x(4)+del*x(4),x(5),varargin{:})-...
    f3(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(4));
df3dx5=(f3(x(1),x(2),x(3),x(4),x(5)+del*x(5),varargin{:})-...
    f3(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(5));
df4dx1=(f4(x(1)+del*x(1),x(2),x(3),x(4),x(5),varargin{:})-...
    f4(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(1));
df4dx2=(f4(x(1),x(2)+del*x(2),x(3),x(4),x(5),varargin{:})-...
    f4(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(2));
df4dx3=(f4(x(1),x(2),x(3)+del*x(3),x(4),x(5),varargin{:})-...
    f4(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(3));
df4dx4=(f4(x(1),x(2),x(3),x(4)+del*x(4),x(5),varargin{:})-...
    f4(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(4));
df4dx5=(f4(x(1),x(2),x(3),x(4),x(5)+del*x(5),varargin{:})-...
    f4(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(5));
df5dx1=(f5(x(1)+del*x(1),x(2),x(3),x(4),x(5),varargin{:})-...
    f5(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(1));
df5dx2=(f5(x(1),x(2)+del*x(2),x(3),x(4),x(5),varargin{:})-...
    f5(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(2));
df5dx3=(f5(x(1),x(2),x(3)+del*x(3),x(4),x(5),varargin{:})-...
    f5(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(3));
df5dx4=(f5(x(1),x(2),x(3),x(4)+del*x(4),x(5),varargin{:})-...
```

```
        f5(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(4));
    df5dx5=(f5(x(1),x(2),x(3),x(4),x(5)+del*x(5),varargin{:})-...
        f5(x(1),x(2) ,x(3),x(4),x(5),varargin{:}))/(del*x(5));
    J=[df1dx1 df1dx2 df1dx3 df1dx4 df1dx5
    df2dx1 df2dx2 df2dx3 df2dx4 df2dx5
    df3dx1 df3dx2 df3dx3 df3dx4 df3dx5
    df4dx1 df4dx2 df4dx3 df4dx4 df4dx5
    df5dx1 df5dx2 df5dx3 df5dx4 df5dx5];
    f11=f1(x(1),x(2),x(3),x(4),x(5),varargin{:});
    f22=f2(x(1),x(2),x(3),x(4),x(5),varargin{:});
    f33=f3(x(1),x(2),x(3),x(4),x(5),varargin{:});
    f44=f4(x(1),x(2),x(3),x(4),x(5),varargin{:});
    f55=f5(x(1),x(2),x(3),x(4),x(5),varargin{:});
    f=[f11;f22;f33;f44;f55];
    end

    function f=f1(H,OH,cT,HCO3,CO3,KH,K1,K2,Kw,pco2)
    f = 1e6*H*HCO3/KH/pco2-K1;
    end
    function f=f2(H,OH,cT,HCO3,CO3,KH,K1,K2,Kw,pco2)
    f = H*CO3/HCO3-K2;
    end
    function f=f3(H,OH,cT,HCO3,CO3,KH,K1,K2,Kw,pco2)
    f = H*OH-Kw;
    end
    function f=f4(H,OH,cT,HCO3,CO3,KH,K1,K2,Kw,pco2)
    f = KH*pco2/1e6+HCO3+CO3-cT;
    end
    function f=f5(H,OH,cT,HCO3,CO3,KH,K1,K2,Kw,pco2)
    f = HCO3+2*CO3+OH-H;
    end
```

A script can then be written to invoke the function developed in Prob. 9.20,

```
    format short g
    x0=[10^-7;10^-7;1e-5;2e-6;5e-11];
    KH=10^-1.46;K1=10^-6.3;K2=10^-10.3;Kw=10^-14;pco2=315;
    [x,f,ea,iter]=newtmult(@jfrain,x0,[],[],KH,K1,K2,Kw,pco2);
    roots=x'
    fprintf('Maximum relative error = %8.4g percent\n',ea)
    fprintf('Number of iterations = %5d',iter)
    pH=-log10(x(1))
```

This script, which was saved as `Prob0921NM6.m` can then be run to determine the roots and display the pH

```
    >> Prob0921NM6

    roots =
      2.3419e-006  4.2701e-009   1.326e-005  2.3375e-006  5.0025e-011
    Maximum relative error = 1.265e-012 percent
    Number of iterations =     6
    pH =
           5.6304
```