CHAPTER 1

1.1 You are given the following differential equation with the initial condition, $v(t = 0) = v(0)$,

$$\frac{dv}{dt} = g - \frac{c}{m}v$$

The most efficient way to solve this is with Laplace transforms

$$sV(s) - v(0) = \frac{g}{s} - \frac{c}{m}V(s)$$

Solve algebraically for the transformed velocity

$$V(s) = \frac{v(0)}{s + c/m} + \frac{g}{s(s + c/m)} \qquad (1)$$

The second term on the right of the equal sign can be expanded with partial fractions

$$\frac{g}{s(s + c/m)} = \frac{A}{s} + \frac{B}{s + c/m}$$

Combining the right-hand side gives

$$\frac{g}{s(s + c/m)} = \frac{A(s + c/m) + Bs}{s(s + c/m)}$$

By equating like terms in the numerator, the following must hold

$$g = A\frac{c}{m} \qquad\qquad 0 = As + Bs$$

The first equation can be solved for $A = mg/c$. According to the second equation, $B = -A$. Therefore, the partial fraction expansion is

$$\frac{g}{s(s + c/m)} = \frac{mg/c}{s} - \frac{mg/c}{s + c/m}$$

This can be substituted into Eq. 1 to give

$$V(s) = \frac{v(0)}{s + c/m} + \frac{mg/c}{s} - \frac{mg/c}{s + c/m}$$

Taking inverse Laplace transforms yields

$$v(t) = v(0)e^{-(c/m)t} + \frac{mg}{c} + \frac{mg}{c}e^{-(c/m)t}$$

or collecting terms

$$v(t) = v(0)e^{-(c/m)t} + \frac{mg}{c}\left(1 - e^{-(c/m)t}\right)$$

The first part is the general solution and the second part is the particular solution for the constant forcing function due to gravity.

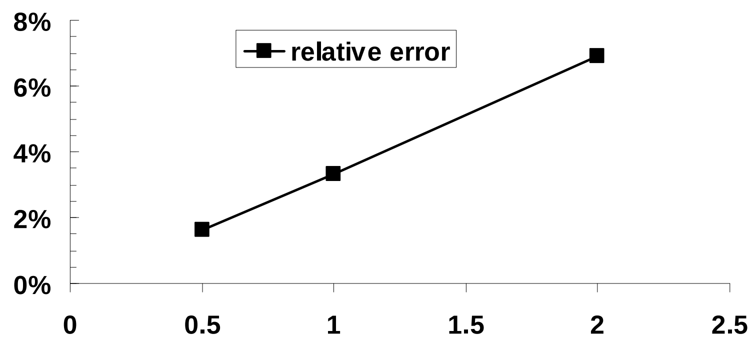
1.2 At t = 10 s, the analytical solution is 44.87 (Example 1.1). The relative error can be calculated with

$$\text{absolute relative error} = \left|\frac{\text{analytical} - \text{numerical}}{\text{analytical}}\right| \times 100\%$$

The numerical results are:

step	$v(10)$	absolute relative error
2	47.9690	6.90%
1	46.3639	3.32%
0.5	45.6044	1.63%

The error versus step size can then be plotted as



Thus, halving the step size approximately halves the error.

1.3 **(*a*)** You are given the following differential equation with the initial condition, $v(t = 0) = 0$,

$$\frac{dv}{dt} = g - \frac{c'}{m}v^2$$

Multiply both sides by m/c'

$$\frac{m}{c'}\frac{dv}{dt} = \frac{m}{c'}g - v^2$$

Define $a = \sqrt{mg/c'}$

$$\frac{m}{c'}\frac{dv}{dt} = a^2 - v^2$$

Integrate by separation of variables,

$$\int \frac{dv}{a^2 - v^2} = \int \frac{c'}{m} dt$$

A table of integrals can be consulted to find that

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a}$$

Therefore, the integration yields

$$\frac{1}{a} \tanh^{-1} \frac{v}{a} = \frac{c'}{m} t + C$$

If $v = 0$ at $t = 0$, then because $\tanh^{-1}(0) = 0$, the constant of integration $C = 0$ and the solution is

$$\frac{1}{a} \tanh^{-1} \frac{v}{a} = \frac{c'}{m} t$$

This result can then be rearranged to yield

$$v = \sqrt{\frac{gm}{c'}} \tanh\left(\sqrt{\frac{gc'}{m}} t\right)$$

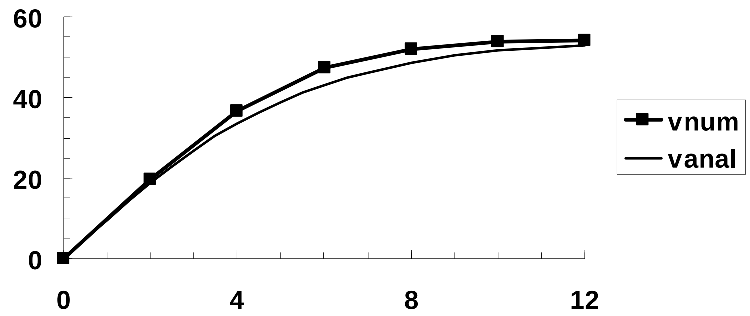
(***b***) Using Euler's method, the first two steps can be computed as

$$v(2) = 0 + \left[9.8 - \frac{0.225}{68.1}(0)^2\right]2 = 19.6$$

$$v(4) = 19.6 + \left[9.8 - \frac{0.225}{68.1}(19.6)^2\right]2 = 36.6615$$

The computation can be continued and the results summarized and plotted as:

t	***v***	***dv/dt***
0	0	9.8
2	19.6	8.53075
4	36.6615	5.35926
6	47.3800	2.38305
8	52.1461	0.81581
10	53.7777	0.24479
12	54.2673	0.07002
∞	54.4622	



Note that the analytical solution is included on the plot for comparison.

1.4 $v(t) = \frac{gm}{c}(1 - e^{-(c/m)\,t})$

jumper #1: $v(t) = \frac{9.8(70)}{12}(1 - e^{-(12/70)\,10}) = 46.8714$

jumper #2: $46.8714 = \frac{9.8(75)}{15}(1 - e^{-(15/75)\,t})$

$46.8714 = 49 - 49e^{-0.2\,t}$

$0.04344 = e^{-0.2\,t}$

$\ln 0.04344 = -0.2t$

$t = \frac{\ln 0.04344}{-0.2} = 15.6818\,\mathrm{s}$

1.5 Before the chute opens ($t < 10$), Euler's method can be implemented as

$$v(t + \Delta t) = v(t) + \left[9.8 - \frac{10}{80}v(t)\right]\Delta t$$

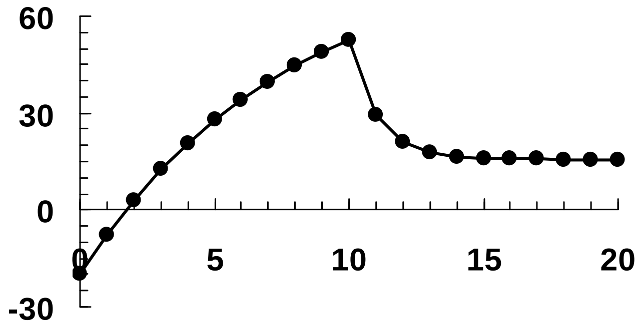
After the chute opens ($t \geq 10$), the drag coefficient is changed and the implementation becomes

$$v(t + \Delta t) = v(t) + \left[9.8 - \frac{50}{80}v(t)\right]\Delta t$$

Here is a summary of the results along with a plot:

Chute closed			Chute opened		
t	***v***	***dv/dt***	***t***	***v***	***dv/dt***
0	-20.0000	12.3000	10	52.5134	-23.0209
1	-7.7000	10.7625	11	29.4925	-8.6328
2	3.0625	9.4172	12	20.8597	-3.2373
3	12.4797	8.2400	13	17.6224	-1.2140
4	20.7197	7.2100	14	16.4084	-0.4552

5	27.9298	6.3088	15	15.9531	-0.1707
6	34.2385	5.5202	16	15.7824	-0.0640
7	39.7587	4.8302	17	15.7184	-0.0240
8	44.5889	4.2264	18	15.6944	-0.0090
9	48.8153	3.6981	19	15.6854	-0.0034
			20	15.6820	-0.0013



1.6 (a) The first two steps are

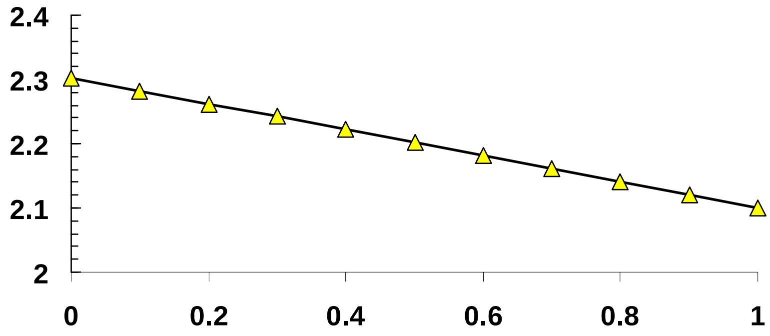
$$c(0.1) = 10 - 0.2(10)0.1 = 9.8 \text{ Bq/L}$$

$$c(0.2) = 9.8 - 0.2(9.8)0.1 = 9.604 \text{ Bq/L}$$

The process can be continued to yield

t	*c*	*dc/dt*
0	10.0000	-2.0000
0.1	9.8000	-1.9600
0.2	9.6040	-1.9208
0.3	9.4119	-1.8824
0.4	9.2237	-1.8447
0.5	9.0392	-1.8078
0.6	8.8584	-1.7717
0.7	8.6813	-1.7363
0.8	8.5076	-1.7015
0.9	8.3375	-1.6675
1	8.1707	-1.6341

(b) The results when plotted on a semi-log plot yields a straight line



The slope of this line can be estimated as

$$\frac{\ln(8.1707) - \ln(10)}{1} = -0.20203$$

Thus, the slope is approximately equal to the negative of the decay rate.

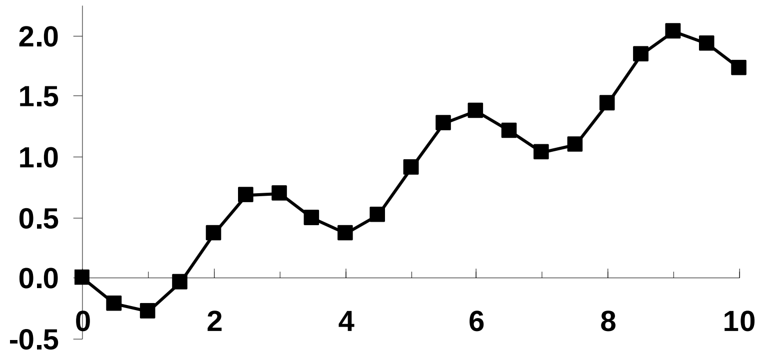
1.7 The first two steps yield

$$y(0.5) = 0 + \left[3\frac{500}{1200}\sin^2(0) - \frac{500}{1200}\right]0.5 = 0 + [0 - 0.41667]0.5 = -0.20833$$

$$y(1) = -0.20833 + \left[\sin^2(0.5) - 0.41667\right]0.5 = -0.27301$$

The process can be continued to give

t	*y*	*dy/dt*	*t*	*y*	*dy/dt*
0	0.00000	-0.41667	5.5	1.27629	0.20557
0.5	-0.20833	-0.12936	6	1.37907	-0.31908
1	-0.27301	0.46843	6.5	1.21953	-0.35882
1.5	-0.03880	0.82708	7	1.04012	0.12287
2	0.37474	0.61686	7.5	1.10156	0.68314
2.5	0.68317	0.03104	8	1.44313	0.80687
3	0.69869	-0.39177	8.5	1.84656	0.38031
3.5	0.50281	-0.26286	9	2.03672	-0.20436
4	0.37138	0.29927	9.5	1.93453	-0.40961
4.5	0.52101	0.77779	10	1.72973	-0.04672
5	0.90991	0.73275			



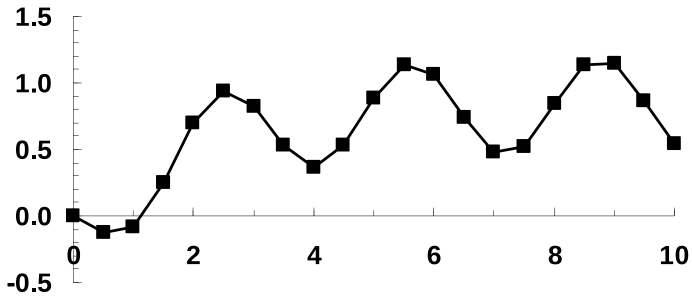
1.8 The first two steps yield

$$y(0.5) = 0 + \left[3\frac{500}{1200}\sin^2(0) - \frac{300(1+0)^{1.5}}{1200} \right] 0.5 = 0 + [0 - 0.25]0.5 = -0.125$$

$$y(1) = -0.125 + \left[\sin^2(0.5) - \frac{300(1-0.125)^{1.5}}{1200} \right] 0.5 = -0.08366$$

The process can be continued to give

t	*y*	*dy/dt*
0	0.00000	-0.25000
0.5	-0.12500	0.08269
1	-0.08366	0.66580
1.5	0.24924	0.89468
2	0.69658	0.48107
2.5	0.93711	-0.22631
3	0.82396	-0.59094
3.5	0.52849	-0.31862
4	0.36918	0.31541
4.5	0.52689	0.72277
5	0.88827	0.50073
5.5	1.13864	-0.15966
6	1.05881	-0.64093
6.5	0.73834	-0.51514
7	0.48077	0.08906
7.5	0.52530	0.62885
8	0.83973	0.59970
8.5	1.13958	0.01457
9	1.14687	-0.57411
9.5	0.85981	-0.62702
10	0.54630	-0.11076



1.9 $Q_{1,\text{in}} = Q_{2,\text{out}} + v_{3,\text{out}} A_3$

$$A_3 = \frac{Q_{1,\text{in}} - Q_{2,\text{out}}}{v_{3,\text{out}}} = \frac{40\text{ m}^3\text{/s} - 20\text{ m}^3\text{/s}}{6\text{ m/s}} = 3.333\text{ m}^2$$

1.10

$$Q_{\text{students}} = 35\text{ ind} \times 80\frac{\text{J}}{\text{ind s}} \times 15\text{ min} \times 60\frac{\text{s}}{\text{min}} \times \frac{\text{kJ}}{1000\text{ J}} = 2520\text{ kJ}$$

$$m = \frac{PV\text{Mwt}}{RT} = \frac{(101.325\text{ kPa})(10\text{m} \times 8\text{m} \times 3\text{m} - 35 \times 0.075\text{ m}^3)(28.97\text{ kg/kmol})}{(8.314\text{ kPa m}^3\text{/(kmol K)}((20 + 273.15)\text{K})} = 285.8908\text{ kg}$$

$$\Delta T = \frac{Q_{\text{students}}}{mC_v} = \frac{2520\text{ kJ}}{(285.8908\text{ kg})(0.718\text{ kJ/(kg K))}} = 12.27654\text{ K}$$

Therefore, the final temperature is 20 + 12.27654 = 32.27654°C**.**

1.11 $\sum \text{M}_{\text{in}} - \sum \text{M}_{\text{out}} = 0$

$$\text{Food} + \text{Drink} + \text{Air In} + \text{Metabolism} = \text{Urine} + \text{Skin} + \text{Feces} + \text{Air Out} + \text{Sweat}$$

$$\text{Drink} = \text{Urine} + \text{Skin} + \text{Feces} + \text{Air Out} + \text{Sweat} - \text{Food} - \text{Air In} - \text{Metabolism}$$

$$\text{Drink} = 1.4 + 0.35 + 0.2 + 0.4 + 0.2 - 1 - 0.05 - 0.3 = 1.2\text{ L}$$

1.12 (a) The force balance can be written as:

$$m\frac{dv}{dt} = -mg(0)\frac{R^2}{(R+x)^2} + cv$$

Dividing by mass gives

$$\frac{dv}{dt} = -g(0)\frac{R^2}{(R+x)^2} + \frac{c}{m}v$$

(b) Recognizing that $dx/dt = v$, the chain rule is

$$\frac{dv}{dt} = v\frac{dv}{dx}$$

Setting drag to zero and substituting this relationship into the force balance gives

$$\frac{dv}{dx} = -\frac{g(0)}{v}\frac{R^2}{(R+x)^2}$$

(c) Using separation of variables

$$v\,dv = -g(0)\frac{R^2}{(R+x)^2}dx$$

Integrating gives

$$\frac{v^2}{2} = g(0)\frac{R^2}{R+x} + C$$

Applying the initial condition yields

$$\frac{v_0^2}{2} = g(0)\frac{R^2}{R+0} + C$$

which can be solved for $C = v_0^2/2 - g(0)R$, which can be substituted back into the solution to give

$$\frac{v^2}{2} = g(0)\frac{R^2}{R+x} + \frac{v_0^2}{2} - g(0)R$$

or

$$v = \pm\sqrt{v_0^2 + 2g(0)\frac{R^2}{R+x} - 2g(0)R}$$

Note that the plus sign holds when the object is moving upwards and the minus sign holds when it is falling.

(d) Euler's method can be developed as

$$v(x_{i+1}) = v(x_i) + \left[-\frac{g(0)}{v(x_i)}\frac{R^2}{(R+x_i)^2}\right](x_{i+1} - x_i)$$

The first step can be computed as

$$v(10{,}000) = 1{,}400 + \left[-\frac{9.8}{1{,}400}\frac{(6.37\times10^6)^2}{(6.37\times10^6 + 0)^2}\right](10{,}000 - 0) = 1{,}400 + (-0.007)10{,}000 = 1{,}330$$

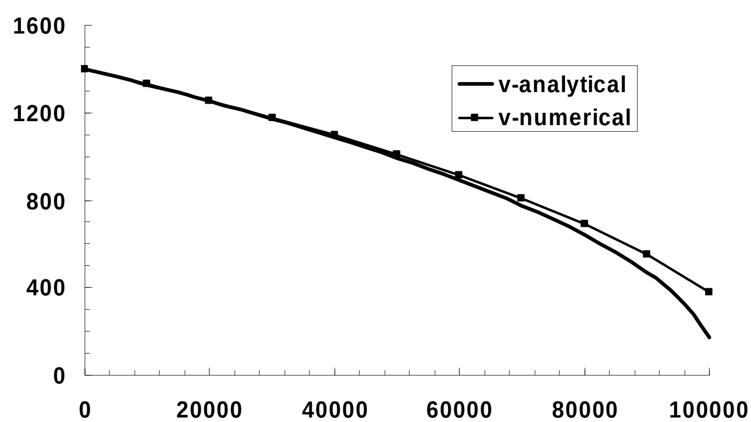
The remainder of the calculations can be implemented in a similar fashion as in the following table

x	*v*	*dv/dx*	*v*-analytical
0	1400.000	-0.00700	1400.000
10000	1330.000	-0.00735	1328.272
20000	1256.547	-0.00775	1252.688
30000	1179.042	-0.00823	1172.500
40000	1096.701	-0.00882	1086.688
50000	1008.454	-0.00957	993.796
60000	912.783	-0.01054	891.612
70000	807.413	-0.01188	776.473
80000	688.661	-0.01388	641.439
90000	549.864	-0.01733	469.650
100000	376.568	-0.02523	174.033

For the analytical solution, the value at 10,000 m can be computed as

$$v=\sqrt{1{,}400^2+2(9.8)\frac{(6.37\times10^6)^2}{(6.37\times10^6+10{,}000)^2}-2(9.8)(6.37\times10^6)}=1{,}328.272$$

The remainder of the analytical values can be implemented in a similar fashion as in the last column of the above table. The numerical and analytical solutions can be displayed graphically.



1.13 The volume of the droplet is related to the radius as

$$V=\frac{4\pi r^3}{3} \qquad (1)$$

This equation can be solved for radius as

$$r=\sqrt[3]{\frac{3V}{4\pi}} \qquad (2)$$

The surface area is

$$A = 4\pi r^2 \tag{3}$$

Equation (2) can be substituted into Eq. (3) to express area as a function of volume

$$A = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3}$$

This result can then be substituted into the original differential equation,

$$\frac{dV}{dt} = -k4\pi\left(\frac{3V}{4\pi}\right)^{2/3} \tag{4}$$

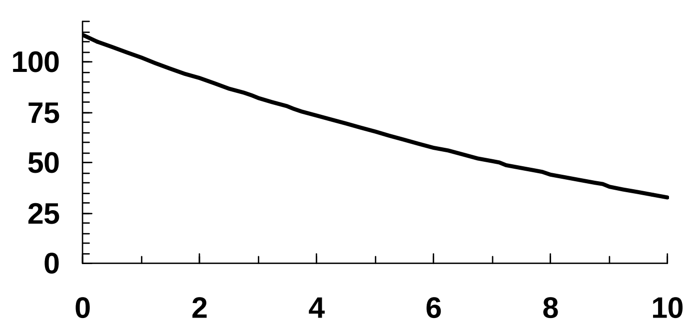
The initial volume can be computed with Eq. (1),

$$V = \frac{4\pi r^3}{3} = \frac{4\pi(3)^3}{3} = 113.0973\ \text{mm}^3$$

Euler's method can be used to integrate Eq. (4). Here are the beginning and last steps

t	***V***	***dV/dt***
0	113.0973	-11.3097
0.25	110.2699	-11.1204
0.5	107.4898	-10.9327
0.75	104.7566	-10.7466
1	102.07	-10.5621
•		
•		
•		
9	38.29357	-5.49416
9.25	36.92003	-5.36198
9.5	35.57954	-5.2314
9.75	34.27169	-5.1024
10	32.99609	-4.97499

A plot of the results is shown below:



Eq. (2) can be used to compute the final radius as

$$r = \sqrt[3]{\frac{3(32.99609)}{4\pi}} = 1.9897$$

Therefore, the average evaporation rate can be computed as

$$k = \frac{(3 - 1.9897)\text{ mm}}{10\text{ min}} \frac{60\text{ min}}{\text{hr}} = 0.10103 \frac{\text{mm}}{\text{min}}$$

which is approximately equal to the given evaporation rate.

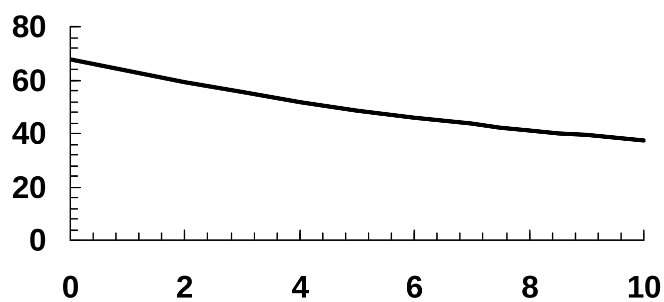
1.14 The first two steps can be computed as

$$T(1) = 68 + [-0.1(68 - 21)]1 = 68 + (-4.7)1 = 63.3$$

$$T(2) = 63.3 + [-0.1(63.1 - 21)]1 = 63.3 + (-4.23)1 = 59.07$$

The remaining results are displayed below along with a plot

t	***T***	***dT/dt***
0	68	-4.7
1	63.3	-4.23
2	59.07	-3.807
3	55.263	-3.4263
4	51.8367	-3.08367
5	48.75303	-2.7753
6	45.97773	-2.49777
7	43.47995	-2.248
8	41.23196	-2.0232
9	39.20876	-1.82088
10	37.38789	-1.63879



1.15 For body weight:

$$4.5 + 4.5 + 12 + 4.5 + 33 + TW = 60$$

TW = 1.5%

For total body water:

$$7.5 + 7.5 + 20 + 7.5 + 2.5 + IW = 100$$

IW = 55%

1.16 (*a*) The solution of the differential equation is

$$N = N_0 e^{\mu t}$$

The doubling time can be computed as the time when $N = 2N_0$,

$$2N_0 = N_0 e^{\mu(20)}$$

$$\mu = \frac{\ln 2}{20\ \text{hrs}} = \frac{0.693}{20\ \text{hrs}} = 0.034657/\text{hr}$$

(*b*) The volume of an individual spherical cell is

$$\text{cell volume} = \frac{\pi d^3}{6} \tag{1}$$

The total volume is

$$\text{volume} = \frac{\pi d^3}{6} N \tag{2}$$

The rate of change of N is defined as

$$\frac{dN}{dt} = \mu N \tag{3}$$

If $N = N_0$ at $t = 0$, Eq. 3 can be integrated to give

$$N = N_0 e^{\mu t} \tag{4}$$

Therefore, substituting (4) into (2) gives an equation for volume

$$\text{volume} = \frac{\pi d^3}{6} N_0 e^{\mu t} \tag{5}$$

(*c*) This equation can be solved for time

$$t = \frac{\ln \dfrac{6 \times \text{volume}}{\pi d^3 N_0}}{\mu} \tag{6}$$

The volume of a 500 μm diameter tumor can be computed with Eq. 2 as 65,449,847. Substituting this value along with d = 20 μm, N_0 = 1 and μ = 0.034657/hr gives

$$t = \frac{\ln\left(\dfrac{6 \times 65{,}449{,}847}{\pi 20^3 (1)}\right)}{0.034657} = 278.63\,\text{hr} = 11.6\,\text{d} \qquad (6)$$

1.17 Continuity at the nodes can be used to determine the flows as follows:

$$Q_1 = Q_2 + Q_3 = 0.7 + 0.5 = 1.2\frac{\text{m}^3}{\text{s}}$$

$$Q_{10} = Q_1 = 1.2\frac{\text{m}^3}{\text{s}}$$

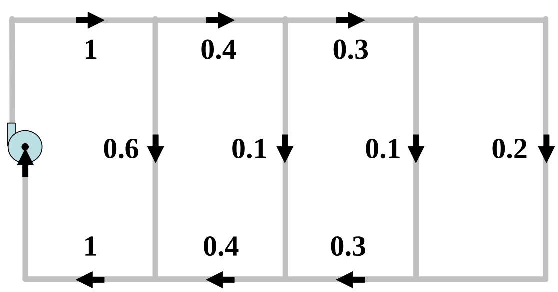
$$Q_9 = Q_{10} - Q_2 = 1.2 - 0.7 = 0.5\frac{\text{m}^3}{\text{s}}$$

$$Q_4 = Q_9 - Q_8 = 0.5 - 0.3 = 0.2\frac{\text{m}^3}{\text{s}}$$

$$Q_5 = Q_3 - Q_4 = 0.5 - 0.2 = 0.3\frac{\text{m}^3}{\text{s}}$$

$$Q_6 = Q_5 - Q_7 = 0.3 - 0.1 = 0.2\frac{\text{m}^3}{\text{s}}$$

Therefore, the final results are



1.18 (*a*) This is a transient computation. For the period ending June 1:

Balance = Previous Balance + Deposits – Withdrawals + Interest
Balance = 1512.33 + 220.13 – 327.26 + 0.01(1512.33) = 1420.32

The balances for the remainder of the periods can be computed in a similar fashion as tabulated below:

Date	Deposit	Withdrawal	Interest	Balance
1-May				$1,512.33
	$220.13	$327.26	$15.12	
1-Jun				$1,420.32

	$216.80	$378.61	$14.20	
1-Jul				$1,272.72
	$450.25	$106.80	$12.73	
1-Aug				$1,628.89
	$127.31	$350.61	$16.29	
1-Sep				**$1,421.88**

(*b*) $\dfrac{dB}{dt} = D(t) - W(t) - iB$

(*c*) t = 0 to 0.5:

$$\frac{dB}{dt} = 220.13 - 327.26 + 0.01(1512.33) = -92.01$$
$$B(0.5) = 1512.33 - 92.01(0.5) = 1466.33$$

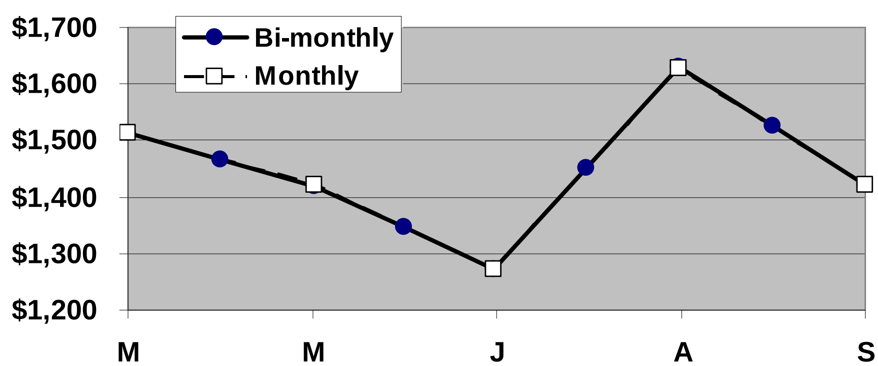
t = 0.5 to 1:

$$\frac{dB}{dt} = 220.13 - 327.260 + 0.01(1466.33) = -92.47$$
$$B(0.5) = 1466.33 - 92.47(0.5) = 1420.09$$

The balances for the remainder of the periods can be computed in a similar fashion as tabulated below:

Date	Deposit	Withdrawal	Interest	dB/dt	Balance
1-May	$220.13	$327.26	$15.12	-$92.01	$1,512.33
16-May	$220.13	$327.26	$14.66	-$92.47	$1,466.33
1-Jun	$216.80	$378.61	$14.20	-$147.61	$1,420.09
16-Jun	$216.80	$378.61	$13.46	-$148.35	$1,346.29
1-Jul	$450.25	$106.80	$12.72	$356.17	$1,272.12
16-Jul	$450.25	$106.80	$14.50	$357.95	$1,450.20
1-Aug	$127.31	$350.61	$16.29	-$207.01	$1,629.18
16-Aug	$127.31	$350.61	$15.26	-$208.04	$1,525.67
1-Sep					**$1,421.65**

(*d*) As in the plot below, the results of the two approaches are very close.



1.19 (a) Substituting Eq. (1.10) into Eq. (P1.19) gives

$$\frac{dx}{dt} = \frac{gm}{c}(1 - e^{-(c/m)t})$$

Separation of variables gives

$$\int_0^x dx = \frac{gm}{c}\int_0^t 1 - e^{-(c/m)t}\,dt$$

Integration yields

$$x = \frac{gm}{c}t - \frac{gm^2}{c^2}(1 - e^{-(c/m)t})$$

(b) Euler's method can be applied for the first step as

$$\frac{dv}{dt}(0) = g - \frac{c}{m}v = 9.8 - \frac{12.5}{68.1}0$$
$$\frac{dx}{dt}(0) = v = 0$$
$$v(2) = v(0) + \frac{dv}{dt}(0)\Delta t = 0 + 9.8(2) = 19.6$$
$$x(2) = x(0) + \frac{dx}{dt}(0)\Delta t = 0 + 0(2) = 0$$

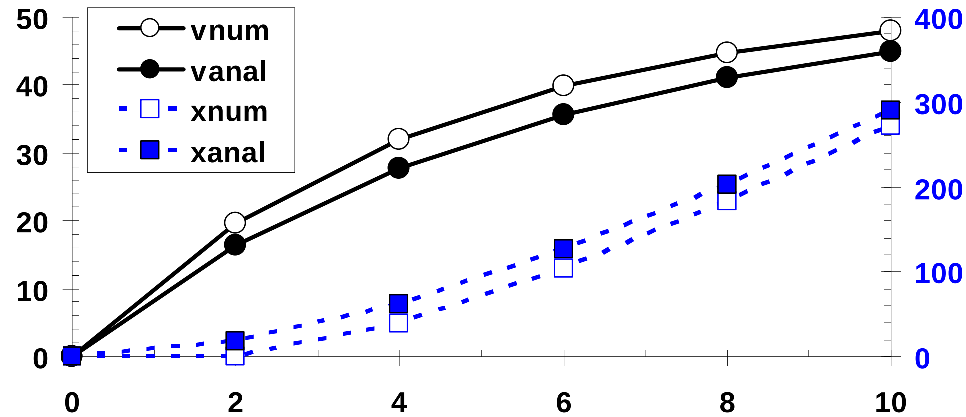
For the second step:

$$\frac{dv}{dt}(2) = 9.8 - \frac{12.5}{68.1}19.6 = 6.2023$$
$$\frac{dx}{dt}(0) = 19.6$$
$$v(4) = 19.6 + 6.2023(2) = 32.0047$$
$$x(4) = 0 + 19.6(2) = 39.2$$

The remaining steps can be computed in a similar fashion as tabulated below along with the analytical solution:

t	vnum	xnum	dv/dt	dx/dt	vanal	xanal
0	0.0000	0.0000	9.8000	0.0000	0.0000	0.0000
2	19.6000	0.0000	6.2023	19.6000	16.4050	17.4065
4	32.0047	39.2000	3.9254	32.0047	27.7693	62.2745
6	39.8555	103.2094	2.4844	39.8555	35.6418	126.1661
8	44.8243	182.9205	1.5723	44.8243	41.0953	203.2361
10	47.9690	272.5691	0.9951	47.9690	44.8731	289.4351

(*c*)



50
40
30
20
10
0
vnum
vanal
xnum
xanal
400
300
200
100
0
0
2
4
6
8
10