CHAPTER 22

22.1 Analytical solution:

$$I = \int_0^3 xe^x\, dx = \left[xe^x - e^x \right]_0^3 = 41.17107385$$

	1	2	3	4
	ε_t →	**5.8349%**	**0.1020%**	**0.0004%**
n	**ε_a →**	**26.8579%**	**0.3579%**	**0.0015862%**
1	90.38491615	43.57337260	41.21305531	41.17125852
2	55.27625849	41.36057514	41.17191160	
4	44.83949598	41.18370307		
8	42.09765130			

22.2 Analytical solution:

$$I = \int_1^2 \left(2x + \frac{3}{x}\right)^2 dx = \left[\frac{4}{3}x^3 + 12x - \frac{9}{x} \right]_1^2 = 25.8333333$$

The first iteration involves computing 1 and 2 segment trapezoidal rules and combining them as

$$I = \frac{4(26.3125) - 27.625}{3} = 25.875$$

and computing the approximate error as

$$\varepsilon_a = \left| \frac{25.875 - 26.3125}{25.875} \right| \times 100\% = 1.6908\%$$

The computation can be continues as in the following tableau until $\varepsilon_a < 0.5\%$.

	1	2	3
n	**ε_a →**	**1.6908%**	**0.0098%**
1	27.62500000	25.87500000	25.83456463
2	26.31250000	25.83709184	
4	25.95594388		

The true error of the final result can be computed as

$$\varepsilon_t = \left| \frac{25.8333333 - 25.83456463}{25.8333333} \right| \times 100\% = 0.0048\%$$

22.3

	1	2	3
n	**ε_a →**	**7.9715%**	**0.0997%**
1	1.34376994	1.97282684	1.94183605
2	1.81556261	1.94377297	
4	1.91172038		

22.4 Change of variable:

$$x = \frac{2+1}{2} + \frac{2-1}{2} x_d = 1.5 + 0.5x_d$$
$$dx = \frac{2-1}{2} dx_d = 0.5dx_d$$
$$I = \int_{-1}^{1} \left(3 + x_d + \frac{3}{1.5+0.5x_d} \right)^2 0.5dx_d$$

Therefore, the transformed function is

$$f(x_d) = 0.5\left(3 + x_d + \frac{3}{1.5+0.5x_d} \right)^2$$

Two-point formula:

$$I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 12.00146 + 13.80525 = 25.8067$$
$$\varepsilon_t = \left| \frac{25.833333 - 25.8067}{25.833333} \right| \times 100\% = 0.103\%$$

Three-point formula:

$$\begin{aligned} I &= 0.5555556 f(-0.7745967) + 0.8888889 f(0) + 0.5555556 f(0.7745967) \\ &= 0.5555556(12.1108) + 0.8888889(12.5) + 0.5555556(14.38716) \\ &= 6.72822 + 11.11111 + 7.992868 = 25.8322 \end{aligned}$$
$$\varepsilon_t = \left| \frac{25.833333 - 25.8322}{25.833333} \right| \times 100\% = 0.0044\%$$

Four-point formula:

$$\begin{aligned} I &= 0.3478548 f(-0.861136312) + 0.6521452 f(-0.339981044) + 0.6521452 f(0.339981044) \\ &\qquad + 0.3478548 f(0.861136312) \\ &= 0.3478548(12.22202) + 0.6521452(12.08177) + 0.6521452(13.19129) + 0.3478548(14.66156) \\ &= 4.251489 + 7.879067 + 8.602639 + 5.100095 = 25.83329 \end{aligned}$$
$$\varepsilon_t = \left| \frac{25.833333 - 25.83329}{25.833333} \right| \times 100\% = 0.000169\%$$

22.5 Change of variable:

$$x = \frac{3+0}{2} + \frac{3-0}{2} x_d = 1.5 + 1.5x_d$$
$$dx = \frac{3-0}{2} dx_d = 1.5dx_d$$
$$I = \int_{-1}^{1} (1.5 + 1.5x_d) e^{1.5+1.5x_d} \, 1.5dx_d$$

Therefore, the transformed function is

$$f(x_d) = (2.25 + 2.25x_d)e^{1.5+1.5x_d}$$

Two-point formula:

$$I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 1.792647 + 37.81485 = 39.6075$$

$$\varepsilon_t = \left|\frac{41.17107 - 39.6075}{41.17107}\right| \times 100\% = 3.7977\%$$

Three-point formula:

$$\begin{aligned} I &= 0.5555556\,f(-0.7745967) + 0.8888889\,f(0) + 0.5555556\,f(0.7745967) \\ &= 0.5555556(0.711181) + 0.8888889(10.0838) + 0.5555556(57.19111) \\ &= 0.3951 + 8.963378 + 31.77284 = 41.13131 \end{aligned}$$

$$\varepsilon_t = \left|\frac{41.17107 - 41.13131}{41.17107}\right| \times 100\% = 0.09657\%$$

Four-point formula:

$$\begin{aligned} I &= 0.3478548\,f(-0.861136312) + 0.6521452\,f(-0.339981044) + 0.6521452\,f(0.339981044) \\ &\qquad + 0.3478548\,f(0.861136312) \\ &= 0.3478548(0.384798) + 0.6521452(3.996712) + 0.6521452(22.50094) + 0.3478548(68.294) \\ &= 0.133854 + 2.606436 + 14.67388 + 23.7564 = 41.17057 \end{aligned}$$

$$\varepsilon_t = \left|\frac{41.17107 - 41.17057}{41.17107}\right| \times 100\% = 0.001229\%$$

22.6 Change of variable:

$$x = \frac{2+0}{2} + \frac{2-0}{2}x_d = 1 + x_d$$

$$dx = \frac{2-0}{2}dx_d = dx_d$$

$$I = \int_{-1}^{1} \frac{e^{1+x_d}\sin(1+x_d)}{1+(1+x_d)^2}\,dx_d$$

Therefore, the transformed function is

$$f(x_d) = \frac{e^{1+x_d}\sin(1+x_d)}{1+(1+x_d)^2}$$

Five-point formula:

$$I = 0.236927f(-0.90618) + 0.478629f(-0.53847) + 0.568889f(0) + 0.478629f(0.53847) + 0.236927f(0.90618)$$
$$= 0.236927(0.102) + 0.478629(0.582434) + 0.568889(1.143678) + 0.478629(1.382589) + 0.236927(1.370992)$$
$$= 0.024166442 + 0.278769811 + 0.650625504 + 0.661746769 + 0.324824846 = 1.940133372$$
$$\varepsilon_t = \left|\frac{1.940130022 - 1.940133372}{1.940130022}\right| \times 100\% = 0.000173\%$$

22.7 Here is the Romberg tableau for this problem.

	1	2	3
n	ε_a →	**5.5616%**	**0.0188%**
1	224.36568786	288.56033084	289.43080513
2	272.51167009	289.37640049	
4	285.16021789		

Therefore, the estimate is 289.430805.

22.8 Change of variable:

$$x = \frac{3-3}{2} + \frac{3-(-3)}{2} x_d = 3x_d$$
$$dx = \frac{3-(-3)}{2} dx_d = 3dx_d$$
$$I = \int_{-1}^{1} \frac{3}{1+9x_d^2}\, dx_d$$

Therefore, the transformed function is

$$f(x_d) = \frac{3}{1+9x_d^2}$$

Two-point formula:

$$I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 0.75 + 0.75 = 1.5$$
$$\varepsilon_t = \left|\frac{2.49809 - 1.5}{2.49809}\right| \times 100\% = 39.95\%$$

The remaining formulas can be implemented with the results summarized in this table.

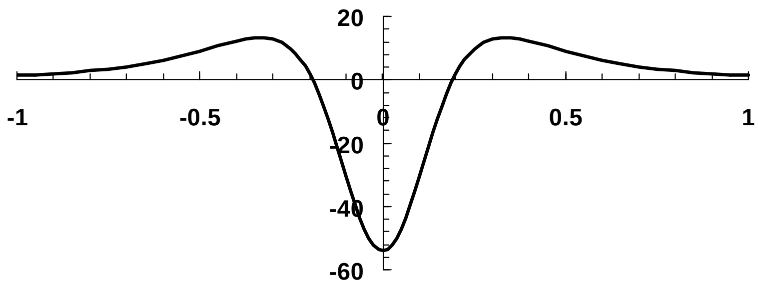
n	Integral	ε_t
2	1.5	39.95%
3	3.1875	27.60%
4	2.189781	12.34%
5	2.671698	6.95%
6	2.411356	3.47%

Thus, the results are converging, but at a very slow rate. Insight into this behavior can be gained by looking at the function and its derivatives.

$$f'(x_d) = -\frac{54x_d}{(1+9x_d^2)^2}$$

$$f''(x_d) = -\frac{54(1+9x_d^2)^2 - 1944x_d^2(1+9x_d^2)}{(1+9x_d^2)^4}$$

We can plot the second derivative as



The second and higher derivatives are large. Thus, the integral evaluation is inaccurate because the error is related to the magnitudes of the derivatives.

22.9 (a)

$$\int_2^\infty \frac{dx}{x(x+2)} = \int_0^{0.5} \frac{1}{t^2}(t)\frac{1}{1/t+2}dt = \int_0^{0.5} \frac{1}{1+2t}dt$$

We can use 8 applications of the extended midpoint rule.

$$\frac{1}{16}(0.941176 + 0.842105 + 0.761905 + 0.695652 + 0.64 + 0.592593 + 0.551724 + 0.516129) = 0.34633$$

This result is close to the analytical solution

$$\int_2^\infty \frac{dx}{x(x+2)} = \left[0.5\ln\left(\frac{x}{x+2}\right)\right]_2^\infty = 0.5\ln\left(\frac{\infty}{\infty+2}\right) - 0.5\ln\left(\frac{2}{2+2}\right) = 0.346574$$

(b)

$$\int_0^\infty e^{-y}\sin^2 y\,dy = \int_0^2 e^{-y}\sin^2 y\,dy + \int_2^\infty e^{-y}\sin^2 y\,dy$$

For the first part, we can use 4 applications of Simpson's 1/3 rule

$$I = (2-0)\frac{0 + 4(0.048 + 0.219 + 0.258 + 0.168) + 2(0.139 + 0.26 + 0.222) + 0.112}{24} = 0.344115$$

For the second part,

$$\int_2^\infty e^{-y}\sin^2 y\,dy = \int_0^{1/2} \frac{1}{t^2}e^{-1/t}\sin^2(1/t)\,dt$$

We can use the extended midpoint rule with h = 1/8.

$$I = \frac{1}{8}(0 + 0.0908 + 0.00142 + 0.303) = 0.0494$$

The total integral is

$$I = 0.344115 + 0.0494 = 0.393523$$

This result is close to the analytical solution of 0.4.

(c)

$$\int_0^\infty \frac{1}{(1+y^2)(1+y^2/2)}\,dy = \int_0^2 \frac{1}{(1+y^2)(1+y^2/2)}\,dy + \int_2^\infty \frac{1}{(1+y^2)(1+y^2/2)}\,dy$$

For the first part, we can use Simpson's 1/3 rule

$$(2-0)\frac{1+4(0.9127+0.4995+0.2191+0.0972)+2(0.7111+0.3333+0.1448)+0.0667}{24} = 0.863262$$

For the second part,

$$\int_2^\infty \frac{1}{(1+y^2)(1+y^2/2)}\,dy = \int_0^{1/2} \frac{1}{t^2(1+(1/t)^2)(1+1/(2t^2))}\,dt$$

We can use the extended midpoint rule with h = 1/8.

$$I = \frac{1}{8}(0.007722 + 0.063462 + 0.148861 + 0.232361) = 0.056551$$

The total integral is

$$I = 0.863262 + 0.056551 = 0.919813$$

This result is close to the analytical solution of 0.920151.

(d)

$$\int_{-2}^\infty ye^{-y}\,dy = \int_{-2}^2 ye^{-y}\,dy + \int_2^\infty ye^{-y}\,dy$$

For the first part, we can use 4 applications of Simpson's 1/3 rule

$$I = (2-(-2))\frac{-14.78+4(-6.72-0.824+0.303+0.335)+2(-2.72+0+0.368)+0.2707}{24} = -7.807$$

For the second part,

$$\int_2^\infty ye^{-y}\,dy = \int_0^{1/2} \frac{1}{t^3}e^{-1/t}\,dt$$

We can use the extended midpoint rule with h = 1/8.

$$I = \frac{1}{8}(0.000461 + 0.732418 + 1.335696 + 1.214487) = 0.410383$$

The total integral is

$$I = -7.80733 + 0.410383 = -7.39695$$

(e)

$$\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\ dx = \int_0^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\ dx + \int_2^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\ dx$$

For the first part, we can use 4 applications of Simpson's 1/3 rule

$$I = (2-0)\frac{0.399 + 4(0.387 + 0.301 + 0.183 + 0.086) + 2(0.352 + 0.242 + 0.130) + 0.054}{24} = 0.47725$$

For the second part,

$$\int_2^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\ dx = \int_0^{1/2} \frac{1}{\sqrt{2\pi}} \frac{1}{t^2} e^{-1/(2t^2)}\ dt$$

We can use the extended midpoint rule with h = 1/8.

$$I = \frac{1}{8}(0 + 0 + 0.024413063 + 0.152922154) = 0.02217$$

The total integral is

$$I = 0.47725 + 0.02217 = 0.499415$$

This is close to the exact value of 0.5.

22.10 (a) Here is a VBA program to implement the algorithm from Fig. 22.1*a*. It is set up to evaluate the integral in the problem statement,

```
Option Explicit
Sub TrapTest()
Dim a As Double, b As Double
Dim n As Integer
a = 0
b = 1
n = 4
MsgBox TrapEq(n, a, b)
End Sub
Function TrapEq(n, a, b)
Dim h As Double, x As Double, sum As Double
Dim i As Integer
h = (b - a) / n
x = a
sum = f(x)
For i = 1 To n - 1
  x = x + h
```

```
  sum = sum + 2 * f(x)
Next i
sum = sum + f(b)
TrapEq = (b - a) * sum / (2 * n)
End Function
Function f(x)
f = x ^ 0.1 * (1.2 - x) * (1 - Exp(20 * (x - 1)))
End Function
```

When the program is run, the result is

The percent relative error can be computed as

$$\varepsilon_t = \left| \frac{0.602298 - 0.478602}{0.602298} \right| \times 100\% = 20.54\%$$

(b) Here is a VBA program to implement the algorithm from Fig. 22.1*b*. It is set up to evaluate the integral in the problem statement,

```
Option Explicit
Sub SimpTest()
Dim a As Double, b As Double
Dim n As Integer
a = 0: b = 1: n = 4
MsgBox SimpEq(n, a, b)
End Sub
Function SimpEq(n, a, b)
Dim h As Double, x As Double, sum As Double
Dim i As Integer
h = (b - a) / n
x = a
sum = f(x)
For i = 1 To n - 2 Step 2
  x = x + h
  sum = sum + 4 * f(x)
  x = x + h
  sum = sum + 2 * f(x)
Next i
x = x + h
sum = sum + 4 * f(x)
sum = sum + f(b)
SimpEq = (b - a) * sum / (3 * n)
End Function
Function f(x)
f = x ^ 0.1 * (1.2 - x) * (1 - Exp(20 * (x - 1)))
End Function
```

When the program is run, the result is

The percent relative error can be computed as

$$\varepsilon_t = \left| \frac{0.602298 - 0.529287}{0.602298} \right| \times 100\% = 12.12\%$$

22.11 Here is a VBA program to implement the algorithm from Fig. 22.4. It is set up to evaluate the integral from Prob. 22.10.

```
Option Explicit
Sub RhombTest()
Dim maxit As Integer
Dim a As Double, b As Double, es As Double
a = 0: b = 1
maxit = 10: es = 0.00000001
MsgBox Rhomberg(a, b, maxit, es)
End Sub
Function Rhomberg(a, b, maxit, es)
Dim n As Long, j As Integer, k As Integer, iter As Integer
Dim i(11, 11) As Double, ea As Double
n = 1
i(1, 1) = TrapEq(n, a, b)
iter = 0
Do
  iter = iter + 1
  n = 2 ^ iter
  i(iter + 1, 1) = TrapEq(n, a, b)
  For k = 2 To iter + 1
    j = 2 + iter - k
    i(j, k) = (4 ^ (k - 1) * i(j + 1, k - 1) - i(j, k - 1)) / (4 ^ (k - 1) - 1)
  Next k
  ea = Abs((i(1, iter + 1) - i(2, iter)) / i(1, iter + 1)) * 100
  If (iter >= maxit Or ea <= es) Then Exit Do
Loop
Rhomberg = i(1, iter + 1)
End Function
Function TrapEq(n, a, b)
Dim i As Long
Dim h As Double, x As Double, sum As Double
h = (b - a) / n
x = a
sum = f(x)
For i = 1 To n - 1
  x = x + h
  sum = sum + 2 * f(x)
Next i
sum = sum + f(b)
TrapEq = (b - a) * sum / (2 * n)
End Function
Function f(x)
f = x ^ 0.1 * (1.2 - x) * (1 - Exp(20 * (x - 1)))
End Function
```

When the program is run for the function from Prob. 22.10, the result is

The percent relative error can be computed as

$$\varepsilon_t = \left| \frac{0.602298 - 0.6021617}{0.602298} \right| \times 100\% = 0.0226\%$$

22.12 Here is a VBA program to implement the algorithm from Fig. 22.5. It is set up to evaluate the integral from Prob. 22.10.

```
Option Explicit
Function quadadapt(a, b)
Dim tol As Double, c As Double
Dim fa As Double, fb As Double, fc As Double
Dim Q As Double
tol = 0.000001
' Initialization
c = (a + b) / 2
fa = f(a)
fc = f(c)
fb = f(b)
' Recursive call
Q = quadstep(a, b, tol, fa, fc, fb)
quadadapt = Q
End Function
Function quadstep(a, b, tol, fa, fc, fb)
Dim h As Double, c As Double, fd As Double, fe As Double, Q1 As Double
Dim Q2 As Double, Q As Double, Qa As Double, Qb As Double
' Recursive subfunction used by quadadapt.
h = b - a
c = (a + b) / 2
fd = f((a + c) / 2)
fe = f((c + b) / 2)
Q1 = h / 6 * (fa + 4 * fc + fb)
Q2 = h / 12 * (fa + 4 * fd + 2 * fc + 4 * fe + fb)
If Abs(Q2 - Q1) <= tol Then
   Q = Q2 + (Q2 - Q1) / 15
Else
   Qa = quadstep(a, c, tol, fa, fd, fc)
   Qb = quadstep(c, b, tol, fc, fe, fb)
   Q = Qa + Qb
End If
quadstep = Q
End Function
Function f(x)
f = x ^ 0.1 * (1.2 - x) * (1 - Exp(20 * (x - 1)))
End Function
```

Here is a driver program to invoke the function:

```
Sub QuadAdaptTest()
MsgBox quadadapt(0, 1)
End Sub
```

When the program is run for the function from Prob. 22.10, the result is

Here is a MATLAB M-file to solve the same problem:

```
function [Q,fcount] = quadadapt(F,a,b,tol,varargin)
% quadadapt  Evaluate definite integral numerically.
%   Q = quadadapt(F,A,B) approximates the integral of F(x) from A to B
%   to within a tolerance of 1.e-6.

% Default tolerance
if nargin < 4 | isempty(tol)
   tol = 1.e-6;
end
% Initialization
c = (a + b)/2;
fa = feval(F,a,varargin{:});
fc = feval(F,c,varargin{:});
fb = feval(F,b,varargin{:});
% Recursive call
[Q,k] = quadstep(F, a, b, tol, fa, fc, fb, varargin{:});
fcount = k + 3;

function [Q,fcount] = quadstep(F,a,b,tol,fa,fc,fb,varargin)
% Recursive subfunction used by quadadapt.
h = b - a;
c = (a + b)/2;
fd = feval(F,(a+c)/2,varargin{:});
fe = feval(F,(c+b)/2,varargin{:});
Q1 = h/6 * (fa + 4*fc + fb);
Q2 = h/12 * (fa + 4*fd + 2*fc + 4*fe + fb);
if abs(Q2 - Q1) <= tol
   Q  = Q2 + (Q2 - Q1)/15;
   fcount = 2;
else
   [Qa,ka] = quadstep(F, a, c, tol, fa, fd, fc, varargin{:});
   [Qb,kb] = quadstep(F, c, b, tol, fc, fe, fb, varargin{:});
   Q  = Qa + Qb;
   fcount = ka + kb + 2;
end
```

Here is a MATLAB session that uses the foregoing M-file to solve Prob. 22.10

```
>> format long
>> f=@(x) x .^ 0.1 .* (1.2 - x) .* (1 - exp(20 * (x - 1)));
>> quadadapt(f,0,1)
ans =
   0.602297369958960
```

22.13 Here is a VBA program to implement an algorithm for Gauss quadrature. It is set up to evaluate the integral from Prob. 22.10.

```
Option Explicit
Sub GaussQuadTest()
Dim i As Integer, j As Integer, k As Integer
Dim a As Double, b As Double, a0 As Double, a1 As Double, sum As Double
Dim c(11) As Double, x(11) As Double, j0(5) As Double, j1(5) As Double
'set constants
c(1) = 1#: c(2) = 0.888888889: c(3) = 0.555555556: c(4) = 0.652145155
c(5) = 0.347854845: c(6) = 0.568888889: c(7) = 0.478628671: c(8) =
0.236926885
c(9) = 0.467913935: c(10) = 0.360761573: c(11) = 0.171324492
x(1) = 0.577350269: x(2) = 0: x(3) = 0.774596669: x(4) = 0.339981044
x(5) = 0.861136312: x(6) = 0: x(7) = 0.53846931: x(8) = 0.906179846
x(9) = 0.238619186: x(10) = 0.661209386: x(11) = 0.932469514
j0(1) = 1: j0(2) = 3: j0(3) = 4: j0(4) = 7: j0(5) = 9
j1(1) = 1: j1(2) = 3: j1(3) = 5: j1(4) = 8: j1(5) = 11
a = 0
b = 1
Sheets("Sheet1").Select
Range("a1").Select
For i = 1 To 5
  ActiveCell.Value = GaussQuad(i, a, b, c, x, j0, j1)
  ActiveCell.Offset(1, 0).Select
Next i
End Sub
Function GaussQuad(n, a, b, c, x, j0, j1)
Dim k As Integer, j As Integer
Dim a0 As Double, a1 As Double
Dim sum As Double
a0 = (b + a) / 2
a1 = (b - a) / 2
sum = 0
If Int(n / 2) - n / 2# = 0 Then
  k = (n - 1) * 2
  sum = sum + c(k) * a1 * f(fc(x(k), a0, a1))
End If
For j = j0(n) To j1(n)
  sum = sum + c(j) * a1 * f(fc(-x(j), a0, a1))
  sum = sum + c(j) * a1 * f(fc(x(j), a0, a1))
Next j
GaussQuad = sum
End Function
Function fc(xd, a0, a1)
fc = a0 + a1 * xd
End Function
Function f(x)
f = x ^ 0.1 * (1.2 - x) * (1 - Exp(20 * (x - 1)))
End Function
```

When the program is run, the results along with the error estimate are

n	integral	ε_t
2	0.621078	3.12%
3	0.609878	1.26%
4	0.605242	0.49%
5	0.603822	0.25%
6	0.603317	0.17%

22.14 Change of variable:

$$x = \frac{1.5+0}{2} + \frac{1.5-0}{2}x_d = 0.75 + 0.75x_d$$
$$dx = \frac{1.5-0}{2}dx_d = 0.75dx_d$$
$$I = \int_{-1}^{1} \frac{1.5}{\sqrt{\pi}} e^{-(0.75+0.75x_d)^2}\ dx_d$$

Therefore, the transformed function is

$$f(x_d) = \frac{1.5}{\sqrt{\pi}} e^{-(0.75+0.75x_d)^2}$$

Two-point formula:

$$I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 0.765382 + 0.208792 = 0.974173$$
$$\varepsilon_t = \left|\frac{0.966105 - 0.974173}{0.966105}\right| \times 100\% = 0.835\%$$

22.15 The function to be integrated is

$$M = \int_2^8 \left(9 + 4\cos^2(0.4t)\right)\left(5e^{-0.5t} + 2e^{0.15t}\right)\ dt$$

The first iteration involves computing 1 and 2 segment trapezoidal rules and combining them as

$$I = \frac{4(340.68170896) - 411.26095167}{3} = 317.15529472$$

and computing the approximate error as

$$\varepsilon_a = \left|\frac{317.15529472 - 340.68170896}{317.15529472}\right| \times 100\% = 7.4179\%$$

The computation can be continues as in the following tableau until $\varepsilon_a < 0.1\%$.

	1	2	3	4
n	**ε_a →**	**7.4179%**	**0.1054%**	**0.0012119%**
1	411.26095167	317.15529472	322.59571622	322.34570788
2	340.68170896	322.25568988	322.34961426	
4	326.86219465	322.34374398		
8	323.47335665			

22.16 The first iteration involves computing 1 and 2 segment trapezoidal rules and combining them as

$$I = (16-0)\frac{0+0}{2} = 0$$

$$I = (16-0)\frac{0+2(2.4)+0}{4} = 19.2$$

$$I = \frac{4(19.2)-0}{3} = 25.6$$

and computing the approximate error as

$$\varepsilon_a = \left|\frac{25.6-19.2}{25.6}\right| \times 100\% = 25\%$$

The computation can be continues as in the following tableau until $\varepsilon_a < 1\%$.

	1	**2**	**3**
n	$\varepsilon_a \rightarrow$	**25.0000%**	**0.7888%**
1	0.00000000	25.60000000	29.29777778
2	19.20000000	29.06666667	
4	26.60000000		

22.17

	1	2	3
	$\varepsilon_a \rightarrow$	3.5090%	0.0036%
1	198.43528583	230.83562235	230.96922983
2	222.73553822	230.96087937	
4	228.90454408		

22.18 $S = \frac{h}{6}\left(f(a)+4f(c)+f(b)\right)$ $\qquad$ $S_2 = \frac{h}{12}\left(f(a)+4f(d)+2f(c)+4f(e)+f(b)\right)$

$$Q = S_2 + \frac{S_2 - S}{15}$$

$$= \frac{h}{12}\left(f(a)+4f(d)+2f(c)+4f(e)+f(b)\right)$$

$$+ \frac{\frac{h}{12}\left(f(a)+4f(d)+2f(c)+4f(e)+f(b)\right) - \frac{h}{6}\left(f(a)+4f(c)+f(b)\right)}{15}$$

Collecting terms

$$Q = \frac{4h}{45}\left(f(a)+4f(d)+2f(c)+4f(e)+f(b)\right) - \frac{h}{90}\left(f(a)+4f(c)+f(b)\right)$$

$$Q = \frac{7}{90}hf(a) + \frac{16}{45}hf(d) + \frac{2}{15}hf(c) + \frac{16}{45}hf(e) + \frac{7}{90}hf(b)$$

$$Q = \frac{h}{90}\left[7f(a)+32f(d)+12f(c)+32f(e)+7f(b)\right]$$

which is Boole’s Rule.