CHAPTER 23

23.1

	x	$f(x)$
x_{i-2}	0.261799388	0.965925826
x_{i-1}	0.523598776	0.866025404
x_i	0.785398163	0.707106781
x_{i+1}	1.047197551	0.5
x_{i+2}	1.308996939	0.258819045

true = –sin(π/4) = –0.70710678

The results are summarized as

	first-order	second-order
Forward	–0.79108963	–0.72601275
	–11.877%	–2.674%
Backward	–0.60702442	–0.71974088
	14.154%	–1.787%
Centered	–0.69905703	–0.70699696
	1.138%	0.016%

23.2

	x	$f(x)$
x_{i-2}	21	1.322219295
x_{i-1}	23	1.361727836
x_i	25	1.397940009
x_{i+1}	27	1.431363764
x_{i+2}	29	1.462397998

$$\text{truth} = \frac{\log_{10}(e)}{25} = 0.017371779$$

The results are summarized as

	first-order	second-order
Forward	0.016711878	0.017309258
	3.799%	0.360%
Backward	0.018106086	0.017281994
	–4.227%	0.517%
Centered	0.017408982	0.017371197
	–0.214%	0.003%

23.3

	x	$f(x)$
x_{i-2}	1.8	6.049647464
x_{i-1}	1.9	6.685894442
x_i	2	7.389056099
x_{i+1}	2.1	8.166169913

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x_{i+2}	2.2	9.025013499

Both the first and second derivatives have the same value,

$$\text{truth} = e^2 = 7.389056099$$

The results are summarized as

	first-order	second-order
First derivative	7.401377351	7.389031439
	-0.166750%	0.000334%
Second derivative	7.395215699	7.389047882
	-0.083361%	0.000111%

23.4 The true value is $-\sin(\pi/4) = -0.70710678$.

$$D(\pi/3) = \frac{-0.25882 - 0.965926}{2(1.047198)} = -0.58477$$

$$D(\pi/6) = \frac{0.258819 - 0.965926}{2(0.523599)} = -0.67524$$

$$D = \frac{4}{3}(-0.67524) - \frac{1}{3}(-0.58477) = -0.70539$$

23.5 The true value is $1/x = 1/5 = 0.2$.

$$D(2) = \frac{1.94591 - 1.098612}{2(2)} = 0.211824$$

$$D(1) = \frac{1.791759 - 1.386294}{2(1)} = 0.202733$$

$$D = \frac{4}{3}(0.202733) - \frac{1}{3}(0.211824) = 0.199702$$

23.6 The true value

$$f'(0) = 8(0)^3 - 18(0)^2 - 12 = -12$$

Equation (23.9) can be used to compute the derivative as

$x_0 = -0.5$ $\quad$ $f(x_0) = -1.125$
$x_1 = 1$ $\quad$ $f(x_1) = -24$
$x_2 = 2$ $\quad$ $f(x_2) = -48$

$$f'(0) = -1.125\frac{2(0)-1-2}{(-0.5-1)(-0.5-2)} + (-24)\frac{2(0)-(-0.5)-2}{(1-(-0.5))(1-2)} + (-48)\frac{2(0)-(-0.5)-1}{(2-(-0.5))(2-1)}$$
$$= 0.9 - 24 + 9.6 = -13.5$$

Centered difference:

$$f'(0) = \frac{-24-12}{1-(-1)} = -18$$

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23.7 At $x = x_i$, Eq. (23.9) is

$$f'(x) = f(x_{i-1})\frac{2x_i - x_i - x_{i+1}}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} + f(x_i)\frac{2x_i - x_{i-1} - x_{i+1}}{(x_i - x_{i-1})(x_i - x_{i+1})}$$
$$+ f(x_{i+1})\frac{2x_i - x_{i-1} - x_i}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)}$$

For equispaced points that are h distance apart, this equation becomes

$$f'(x) = f(x_{i-1})\frac{-h}{-h(-2h)} + f(x_i)\frac{2x_i - (x_i - h) - (x_i + h)}{h(-h)} + f(x_{i+1})\frac{h}{2h(h)}$$
$$= \frac{-f(x_{i-1})}{2h} + 0 + \frac{f(x_{i+1})}{2h} = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

23.8 (a)

	x	***f*(*x*)**
x_{i-2}	-0.5	-17.125
x_{i-1}	-0.25	-16.0156
x_i	0	-15
x_{i+1}	0.25	-13.9844
x_{i+2}	0.5	-12.875

$$f'(x) = \frac{-(-12.875) + 8(-13.9844) - 8(-16.0156) - 17.125}{12(0.25)} = 4$$

$$f''(x) = \frac{-(-12.875) + 16(-13.9844) - 30(-15) + 16(-16.0156) - (-17.125)}{12(0.25)^2} = 0$$

(b)

	x	***f*(*x*)**
x_{i-2}	0.2	0.039203
x_{i-1}	0.3	0.08598
x_i	0.4	0.14737
x_{i+1}	0.5	0.219396
x_{i+2}	0.6	0.297121

$$f'(x) = \frac{-(0.297121) + 8(0.219396) - 8(0.08598) + 0.039203}{12(0.1)} = 0.674504$$

$$f''(x) = \frac{-(0.297121) + 16(0.219396) - 30(0.14737) + 16(0.08598) - (0.039203)}{12(0.1)^2} = 1.071654$$

(c)

	x	***f*(*x*)**
x_{i-2}	2	0.786843
x_{i-1}	2.5	1.100778
x_i	3	1.557408
x_{i+1}	3.5	2.338254
x_{i+2}	4	4.131729

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$$f'(x) = \frac{-(4.131729) + 8(2.338254) - 8(1.100778) + 0.786843}{12(0.5)} = 1.092486$$

$$f''(x) = \frac{-(4.131729) + 16(2.338254) - 30(1.557408) + 16(1.100778) - (0.786843)}{12(0.5)^2} = 1.127902$$

(d)

	x	*f*(*x*)
x_{i-2}	0.6	0.62948
x_{i-1}	0.8	0.540569
x_i	1	0.479426
x_{i+1}	1.2	0.433954
x_{i+2}	1.4	0.398355

$$f'(x) = \frac{-(0.398355) + 8(0.433954) - 8(0.540569) + 0.62948}{12(0.2)} = -0.25908$$

$$f''(x) = \frac{-(0.398355) + 16(0.433954) - 30(0.479426) + 16(0.540569) - (0.62948)}{12(0.2)^2} = 0.378652$$

(e)

	x	*f*(*x*)
x_{i-2}	1.6	6.553032
x_{i-1}	1.8	7.849647
x_i	2	9.389056
x_{i+1}	2.2	11.22501
x_{i+2}	2.4	13.42318

$$f'(x) = \frac{-(13.42318) + 8(11.22501) - 8(7.849647) + 6.553032}{12(0.2)} = 8.38866$$

$$f''(x) = \frac{-(13.42318) + 16(11.22501) - 30(9.389056) + 16(7.849647) - (6.553032)}{12(0.2)^2} = 7.388924$$

23.9 The first forward difference formula of $O(h^2)$ from Fig. 23.1 can be used to estimate the velocity for the first point at $t = 0$,

$$f'(0) = \frac{-58 + 4(32) - 3(0)}{2(25)} = 1.4\,\frac{\text{km}}{\text{s}}$$

The acceleration can be estimated with the second forward difference formula of $O(h^2)$ from Fig. 23.1

$$f''(0) = \frac{-78 + 4(58) - 5(32) + 2(0)}{(25)^2} = -0.0096\,\frac{\text{km}}{\text{s}^2}$$

For the interior points, centered difference formulas of $O(h^2)$ from Fig. 23.3 can be used to estimate the velocities and accelerations. For example, at the second point at $t = 25$,

$$f'(25) = \frac{58 - 0}{2(25)} = 1.16\,\frac{\text{km}}{\text{s}}$$

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$$f''(25) = \frac{58 - 2(32) + 0}{(25)^2} = -0.0096 \frac{\text{km}}{\text{s}^2}$$

For the final point, backward difference formulas of $O(h^2)$ from Fig. 23.2 can be used to estimate the velocities and accelerations. The results for all values are summarized in the following table.

t	*y*	*v*	*a*
0	0	1.40	-0.0096
25	32	1.16	-0.0096
50	58	0.92	-0.0096
75	78	0.68	-0.0096
100	92	0.44	-0.0096
125	100	0.20	-0.0096

23.10 Here is a VBA program to implement a Romberg algorithm to estimate the derivative of a given function. It is set up to evaluate the derivative from Example 23.1.

```
Option Explicit
Sub RombTest()
Dim maxit As Integer
Dim es As Double, x As Double
x = 0.5
maxit = 3
es = 0.001
MsgBox RomDiff(x, maxit, es)
End Sub
Function RomDiff(x, maxit, es)
Dim n As Integer, j As Integer, k As Integer, iter As Integer
Dim i(10, 10) As Double, ea As Double, del As Double
n = 1
i(1, 1) = DyDx(x, n)
iter = 0
Do
  iter = iter + 1
  n = 2 ^ iter
  i(iter + 1, 1) = DyDx(x, n)
  For k = 2 To iter + 1
    j = 2 + iter - k
    i(j, k) = (4 ^ (k - 1) * i(j + 1, k - 1) - i(j, k - 1)) / (4 ^ (k - 1) - 1)
  Next k
  ea = Abs((i(1, iter + 1) - i(1, iter)) / i(1, iter + 1)) * 100
  If (iter >= maxit Or ea <= es) Then Exit Do
Loop
RomDiff = i(1, iter + 1)
End Function
Function DyDx(x, n)
Dim a As Double, b As Double
a = x - x / n
b = x + x / n
DyDx = (f(b) - f(a)) / (b - a)
End Function

Function f(x)
f = -0.1 * x ^ 4 - 0.15 * x ^ 3 - 0.5 * x ^ 2 - 0.25 * x + 1.2
End Function
```

When the program is run, the result is

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23.11 Here is a VBA program uses Eq. 23.9 to obtain first-derivative estimates for unequally spaced data.

```
Option Explicit
Sub TestDerivUnequal()
Dim n As Integer, i As Integer
Dim x(100) As Double, y(100) As Double, dy(100) As Double
Range("a5").Select
n = ActiveCell.Row
Selection.End(xlDown).Select
n = ActiveCell.Row - n
Range("a5").Select
For i = 0 To n
  x(i) = ActiveCell.Value
  ActiveCell.Offset(0, 1).Select
  y(i) = ActiveCell.Value
  ActiveCell.Offset(1, -1).Select
Next i
For i = 0 To n
  dy(i) = DerivUnequal(x, y, n, x(i))
Next i
Range("c5").Select
For i = 0 To n
  ActiveCell.Value = dy(i)
  ActiveCell.Offset(1, 0).Select
Next i
End Sub
Function DerivUnequal(x, y, n, xx)
Dim ii As Integer
If xx < x(0) Or xx > x(n) Then
  DerivUnequal = "out of range"
Else
  If xx < x(1) Then
    DerivUnequal = DyDx(xx, x(0), x(1), x(2), y(0), y(1), y(2))
  ElseIf xx > x(n - 1) Then
    DerivUnequal = _
        DyDx(xx, x(n - 2), x(n - 1), x(n), y(n - 2), y(n - 1), y(n))
  Else
    For ii = 1 To n - 2
      If xx >= x(ii) And xx <= x(ii + 1) Then
        If xx - x(ii - 1) < x(ii) - xx Then
          'If the unknown is closer to the lower end of the range,
          'x(ii) will be chosen as the middle point
          DerivUnequal = _
          DyDx(xx, x(ii - 1), x(ii), x(ii + 1), y(ii - 1), y(ii), y(ii + 1))
        Else
          'Otherwise, if the unknown is closer to the upper end,
          'x(ii+1) will be chosen as the middle point
          DerivUnequal = _
          DyDx(xx, x(ii), x(ii + 1), x(ii + 2), y(ii), y(ii + 1), y(ii + 2))
        End If
        Exit For
      End If
```

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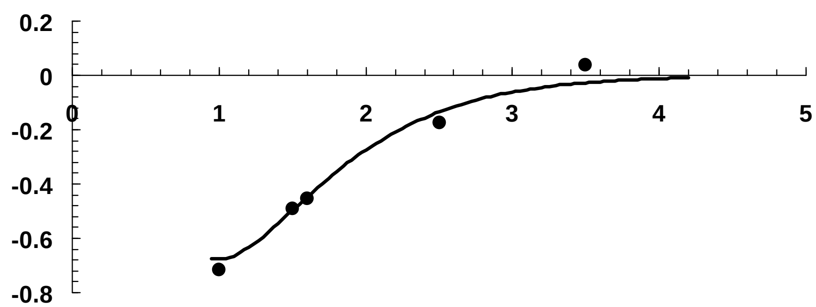
```
      Next ii
    End If
  End If
  End Function
  Function DyDx(x, x0, x1, x2, y0, y1, y2)
  DyDx = y0 * (2 * x - x1 - x2) / (x0 - x1) / (x0 - x2) _
        + y1 * (2 * x - x0 - x2) / (x1 - x0) / (x1 - x2) _
         + y2 * (2 * x - x0 - x1) / (x2 - x0) / (x2 - x1)
  End Function
```

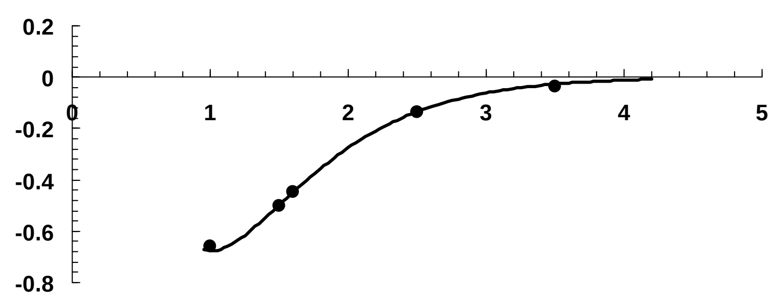
When the program is run, the result is shown below:

	A	B	C
1	Problem 23.11		
2			
3	Data:		
4	x	y	dy/dx
5	1	0.6767	-0.71793
6	1.5	0.3734	-0.49342
7	1.6	0.3261	-0.45258
8	2.5	0.08422	-0.17378
9	3.5	0.01596	0.037264

The results can be compared with the true derivatives which can be calculated with analytical solution, $f'(x) = 5e^{-2x} - 10xe^{-2x}$. The results can be displayed graphically below where the computed values are represented as points and the true values as the curve.



An even more elegant approach is to put cubic splines through the data (recall Sec. 20.2 and the program used for the solution to Prob. 20.10) to evaluate the derivatives. Here is the result of applying that program to this problem.



23.12 (*a*) This problem amounts to solving the following integral:

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$$x = \int_0^t v(t)\ dt$$

We can use Simpson's 1/3 and 3/8 rule to make this evaluation,

$$x = (16-0)\frac{0+4(34.7+82.8)+2(61.8)+99.2}{3(4)} + (28-16)\frac{99.2+3(112+121.9)+129.7}{8}$$

$$= 923.7333 + 1395.9 = 2319.633 \text{ m}$$

(b) The acceleration is equal to

$$a = \frac{135.7-121.9}{32-24} = 1.725\frac{\text{m}}{\text{s}}$$

Because the points around t = 28 are equispaced, a centered finite divided difference provides a good estimate

$$a = \frac{-61.8+4(34.7)-3(0)}{8-0} = 9.625\frac{\text{m}}{\text{s}}$$

(c) For this case, an $O(h^2)$ forward difference can be used,

$$a = \frac{dv}{dt}$$

23.13 (a) Create the M-file function:

```
function y=f(t)
y=9.81*70/12*(1-exp(-12/70*t));
```

Then implement the following MATLAB session:

```
>> Q=quad(@f,0,10)
Q =
  298.5546
```

(b)

$$d(t) = \frac{gm}{c}\int_0^t \left(1 - e^{-(c/m)t}\right)\ dt$$

$$d(t) = \frac{gm}{c}\left[t + \frac{m}{c}e^{-(c/m)t}\right]_0^t$$

$$d(10) = \frac{9.81(70)}{12}\left[10 + \frac{70}{12}e^{-(12/70)10} - 0 - \frac{70}{12}\right] = 298.5546$$

(c) Implement the following MATLAB session:

```
>> x=[9.99 10.01];
>> y=f(x);
>> d=diff(y)./diff(x)

d =
    1.7667
```

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(d)

$$a(t) = \frac{gm}{c}\frac{d}{dt}\left(1 - e^{-(c/m)t}\right)$$

$$a(t) = ge^{-(c/m)t}$$

$$a(10) = 9.81e^{-(12/70)10} = 1.766706$$

23.14 (a) Create the M-file function:

```
function y=fn(x)
y=1/sqrt(2*pi)*exp(-(x.^2)/2);
```

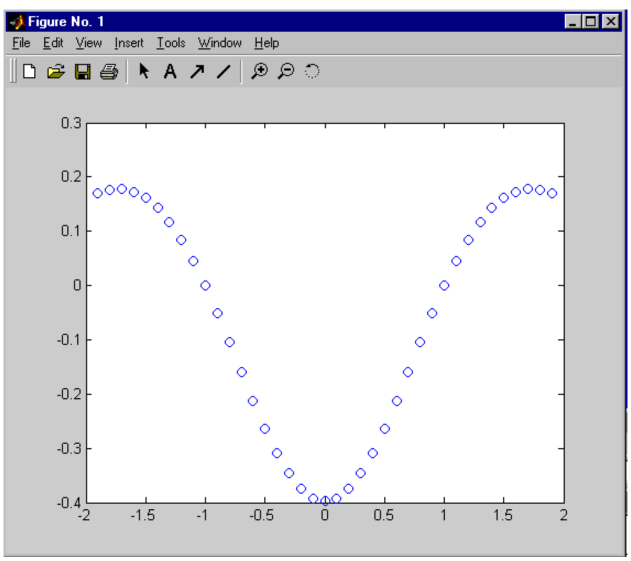
Then implement the following MATLAB session:

```
>> x=-2:.1:2;
>> y=fn(x);
>> Q=quad(@fn,-1,1)
Q =
    0.6827
>> Q=quad(@fn,-2,2)
Q =
    0.9545
```

Thus, about 68.3% of the area under the curve falls between –1 and 1 and about 95.45% falls between –2 and 2.

(b)

```
>> x=-2:.1:2;
>> y=fn(x);
>> d=diff(y)./diff(x);
>> x=-1.95:.1:1.95;
>> d2=diff(d)./diff(x);
>> x=-1.9:.1:1.9;
>> plot(x,d2,'o')
```



Thus, inflection points ($d^2y/dx^2 = 0$) occur at –1 and 1.

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23.15 (a) Create the M-file function:

```
function y=fn(x)
y=1/sqrt(2*pi)*exp(-(x.^2)/2);
```

Then implement the following MATLAB session:

```
>> x=-1:.5:1;
>> y=fn(x);
>> I=trapz(x,y)
I =
    0.6725
>> x=-2:.5:2;
>> y=fn(x);
>> I=trapz(x,y)
I =
    0.9500
```

Note that there is another MATLAB function called `trap` that can also be used to solve this problem. To learn more about this function type

```
>> help trap
```

The resulting description will be displayed,

```
I = trap(func,a,b,n):
    multiple-application trapezoidal rule.
input:
  func = name of function to be integrated
  a, b = integration limits
  n = number of segments
output:
  I = integral estimate
```

Here are the results of using this function to solve the problem:

```
>> I=trap(@fn,-1,1,4)
I =
    0.6725
>> I=trap(@fn,-2,2,8)
I =
    0.9500
```

Thus, about 67.25% of the area under the curve falls between –1 and 1 and about 95% falls between –2 and 2.

(b)

```
>> d=diff(y)./diff(x);
>> x=-1.75:.5:1.75;
>> d2=diff(d)./diff(x);
>> x=-1.5:.5:1.5;
>> plot(x,d2,'o')
```

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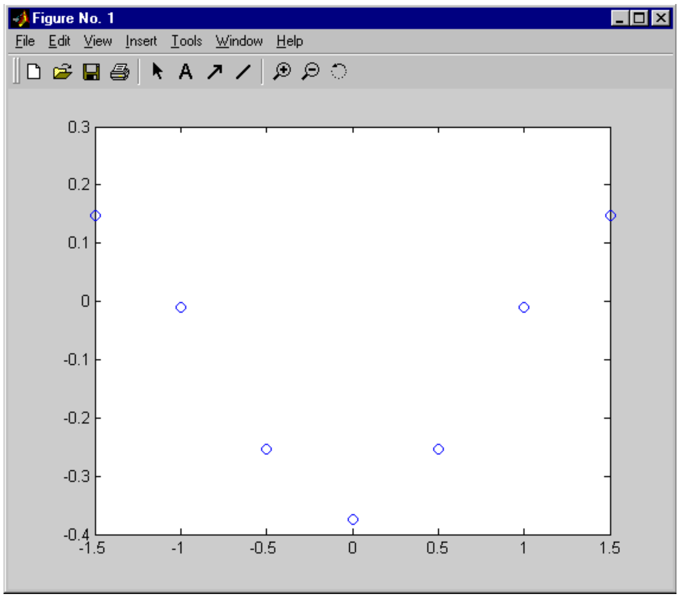

Thus, inflection points ($d^2y/dx^2 = 0$) occur at –1 and 1.

23.16 MATLAB Script:

```
% Prob2316 Integration program
a=0;
b=pi/2;
integral=quad(@ff,a,b)

function y=ff(x)
y=cos(cos(x));

>> prob2316
integral =
    1.2020
```

23.17 MATLAB Script saved as prob2317.m:

```
%Numerical Integration of sin(t)/t = function sint(t)
%Limits: a=0, b=2pi
%Using the "quad" and "quadl" function for numerical integration
%Plot of function
t=0.01:0.01:2*pi;
y=ff2(t);
plot(t,y); grid
%Integration
format long
a=0.01;
b=2*pi;
Iquad=quad('ff2',a,b)
Iquadl=quadl('ff2',a,b)

function y=ff2(t)
y=sin(t)./t;
```

MATLAB execution:

```
>> prob2317
```

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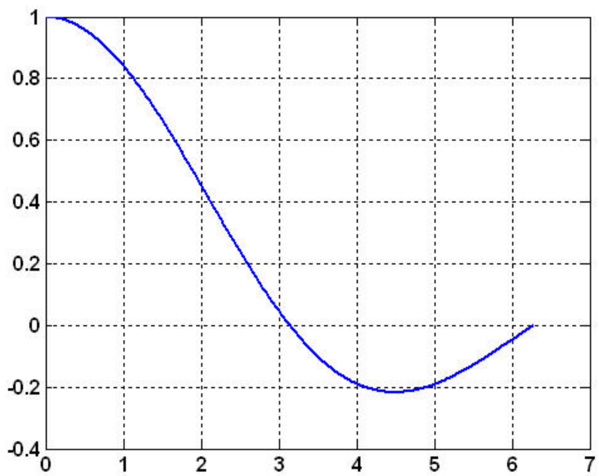
```
Iquad =
   1.40815163720125
Iquadl =
   1.40815163168846
```



23.18

```
%Centered Finite Difference First & Second Derivatives of Order O(dx^2)
%Using diff(y)
dx=1.;
y=[1.4 2.1 3.3 4.8 6.8 6.6 8.6 7.5 8.9 10.9 10];
dyf=diff(y);
% First Derivative Centered FD using diff
n=length(y);
for i=1:n-2
   dydxc(i)=(dyf(i+1)+dyf(i))/(2*dx);
end
%Second Derivative Centered FD using diff
dy2dx2c=diff(dyf)/(dx*dx);
fprintf('first derivative \n');  fprintf('%f\n', dydxc)
fprintf('second derivative \n'); fprintf('%f\n', dy2dx2c)

first derivative
0.950000
1.350000
1.750000
0.900000
0.900000
0.450000
0.150000
1.700000
0.550000
second derivative
0.500000
0.300000
0.500000
-2.200000
2.200000
-3.100000
2.500000
0.600000
-2.900000
```

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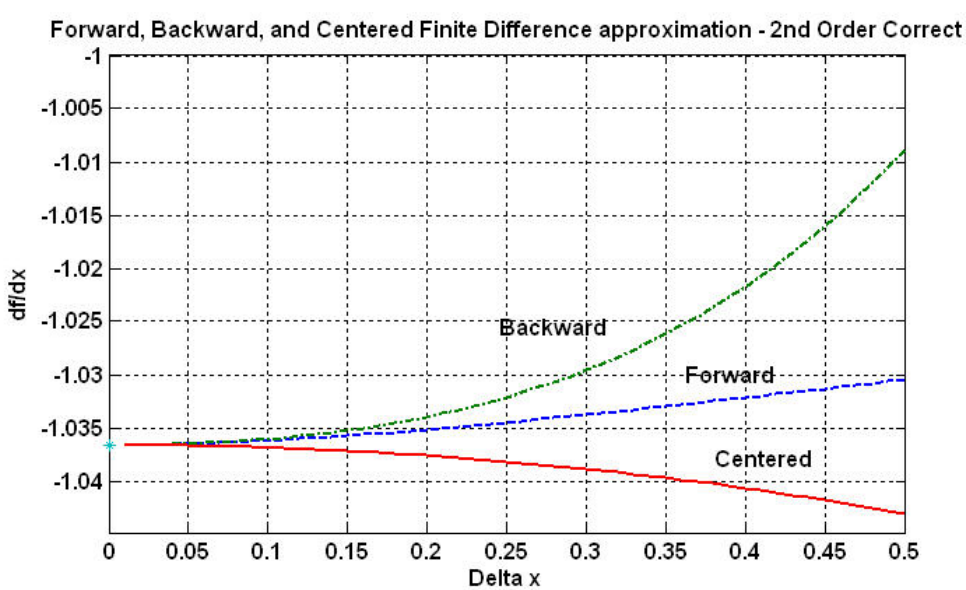
23.19

```
% Finite Difference Approximation of slope
% For f(x)=exp(-2*x)-x
%     f'(x)=-2*exp(-2*x)-1
% Centered diff. df/dx=(f(i+1)-f(i-1))/2dx         + O(dx^2)
% Fwd. diff.     df/dx=(-f(i+2)+4f(i+1)-3f(i))/2dx + O(dx^2)
% Bkwd. diff.    df/dx=(3f(i)-4f(i-1)+f(i-2))/2dx  + O(dx^2)
x=2;
fx=exp(-2*x)-x;
dfdx2=-2*exp(-2*x)-1;
%approximation
dx=0.5:-0.01:.01;
for i=1:length(dx)
  %x-values at i+-dx and +-2dx
  xp(i)=x+dx(i);
  x2p(i)=x+2*dx(i);
  xn(i)=x-dx(i);
  x2n(i)=x-2*dx(i);
  %f(x)-values at i+-dx and +-2dx
  fp(i)=exp(-2*xp(i))-xp(i);
  f2p(i)=exp(-2*x2p(i))-x2p(i);
  fn(i)=exp(-2*xn(i))-xn(i);
  f2n(i)=exp(-2*x2n(i))-x2n(i);
  %Finite Diff. Approximations
  Cdfdx(i)=(fp(i)-fn(i))/(2*dx(i));
  Fdfdx(i)=(-f2p(i)+4*fp(i)-3*fx)/(2*dx(i));
  Bdfdx(i)=(3*fx-4*fn(i)+f2n(i))/(2*dx(i));
end
dx0=0;
plot(dx,Fdfdx,'--',dx,Bdfdx,'-.',dx,Cdfdx,'-',dx0,dfdx2,'*')
grid
title('Forward, Backward, and Centered Finite Difference approximation - 2nd
Order Correct')
xlabel('Delta x')
ylabel('df/dx')
gtext('Centered'); gtext('Forward');  gtext('Backward')
```



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23.20 First, we will use forward expansions. The Taylor series expansion about $a = x_i$ and $x = x_{i+2}$ ($2\Delta x$ steps forward) can be written as:

$$f(x_{i+2}) = f(x_i) + f'(x_i)2\Delta x + \frac{1}{2}f''(x_i)(2\Delta x)^2 + \frac{1}{6}f'''(x_i)(2\Delta x)^3 + \frac{1}{24}f^{(4)}(x_i)(2\Delta x)^4 + \frac{1}{120}f^{(5)}(x_i)(2\Delta x)^5 + \cdots$$

$$f(x_{i+2}) = f(x_i) + 2f'(x_i)\Delta x + 2f''(x_i)\Delta x^2 + \frac{8}{6}f'''(x_i)\Delta x^3 + \frac{16}{24}f^{(4)}(x_i)\Delta x^4 + \frac{32}{120}f^{(5)}(x_i)\Delta x^5 + \cdots \qquad \textbf{(1)}$$

Taylor series expansion about $a = x_i$ and $x = x_{i+1}$ (Δx steps forward):

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{1}{2}f''(x_i)\Delta x^2 + \frac{1}{6}f'''(x_i)\Delta x^3 + \frac{1}{24}f^{(4)}(x_i)\Delta x^4 + \frac{1}{120}f^{(5)}(x_i)\Delta x^5 + \cdots \qquad \textbf{(2)}$$

Multiply Eq. 2 by 2 and subtract the result from Eq. 1 to yield

$$f(x_{i+2}) - 2f(x_{i+1}) = -f(x_i) + f''(x_i)\Delta x^2 + \frac{6}{6}f'''(x_i)\Delta x^3 + \frac{14}{24}f^{(4)}(x_i)\Delta x^4 + \frac{30}{120}f^{(5)}(x_i)\Delta x^5 + \cdots \qquad \textbf{(3)}$$

Next, we will use backward expansions. The Taylor series expansion about $a = x_i$ and $x = x_{i-2}$ ($2\Delta x$ steps backward) can be written as:

$$f(x_{i-2}) = f(x_i) + f'(x_i)(-2\Delta x) + \frac{1}{2}f''(x_i)(-2\Delta x)^2 + \frac{1}{6}f'''(x_i)(-2\Delta x)^3 + \frac{1}{24}f^{(4)}(x_i)(-2\Delta x)^4 + \frac{1}{120}f^{(5)}(x_i)(-2\Delta x)^5 + \cdots$$

$$f(x_{i-2}) = f(x_i) - 2f'(x_i)\Delta x + 2f''(x_i)\Delta x^2 - \frac{8}{6}f'''(x_i)\Delta x^3 + \frac{16}{24}f^{(4)}(x_i)\Delta x^4 - \frac{32}{120}f^{(5)}(x_i)\Delta x^5 + \cdots \qquad \textbf{(4)}$$

Taylor series expansion about $a = x_i$ and $x = x_{i-1}$ (Δx steps backward):

$$f(x_{i-1}) = f(x_i) - f'(x_i)\Delta x + \frac{1}{2}f''(x_i)\Delta x^2 - \frac{1}{6}f'''(x_i)\Delta x^3 + \frac{1}{24}f^{(4)}(x_i)\Delta x^4 - \frac{1}{120}f^{(5)}(x_i)\Delta x^5 + \cdots \qquad \textbf{(5)}$$

Multiply Eq. 5 by 2 and subtract the result from Eq. 4 to yield

$$2f(x_{i-1}) - f(x_{i-2}) = f(x_i) - f''(x_i)\Delta x^2 + \frac{6}{6}f'''(x_i)\Delta x^3 - \frac{14}{24}f^{(4)}(x_i)\Delta x^4 + \frac{30}{120}f^{(5)}(x_i)\Delta x^5 + \cdots \qquad \textbf{(6)}$$

Add Eqs (3) and (6)

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$$f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2}) = 2f'''(x_i)\Delta x^3 + \frac{60}{120} f^{(5)}(x_i)\Delta x^5 + \cdots \qquad \textbf{(7)}$$

Equation 7 can be solved for

$$f'''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{2\Delta x^3} - \frac{\frac{60}{120} f^{(5)}(x_i)\Delta x^5}{2\Delta x^3} + \cdots$$

$$f'''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{2\Delta x^3} - \frac{1}{4} f^{(5)}(x_i)\Delta x^2 + \cdots$$

23.21 (a)

$$v = \frac{dx}{dt} = x'(t_i) = \frac{x(t_{i+1}) - x(t_{i-1})}{2h} = \frac{7.3 - 5.1}{4} = 0.55 \frac{\text{m}}{\text{s}}$$

$$a = \frac{d^2x}{dt^2} = x''(t_i) = \frac{x(t_{i+1}) - 2x(t_i) + x(t_{i-1})}{h^2} = \frac{7.3 - 2(6.3) + 5.1}{2^2} = -0.05 \frac{\text{m}}{\text{s}^2}$$

(b)

$$v = \frac{-x(t_{i+2}) + 4x(t_{i+1}) - 3x(t_i)}{2h} = \frac{-8 + 4(7.3) - 3(6.3)}{4} = 0.575 \frac{\text{m}}{\text{s}}$$

$$a = \frac{-x(t_{i+3}) + 4x(t_{i+2}) - 5x(t_{i+1}) + 2x(t_i)}{h^2} = \frac{-8.4 + 4(8) - 5(7.3) + 2(6.3)}{2^2} = -0.075 \frac{\text{m}}{\text{s}^2}$$

(c)

$$v = \frac{3x(t_i) - 4x(t_{i-1}) + x(t_{i-2})}{2h} = \frac{3(6.3) - 4(5.1) + 3.4}{4} = 0.475 \frac{\text{m}}{\text{s}}$$

$$a = \frac{2x(t_i) - 5x(t_{i-1}) + 4x(t_{i-2}) - x(t_{i-3})}{h^2} = \frac{2(6.3) - 5(5.1) + 4(3.4) - 1.8}{2^2} = -0.275 \frac{\text{m}}{\text{s}^2}$$

23.22

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{\theta(t_{i+1}) - \theta(t_{i-1})}{2h} = \frac{0.67 - 0.70}{4} = -0.0075 \text{ rad/s}$$

$$\dot{r} = \frac{dr}{dt} = \frac{r(t_{i+1}) - r(t_{i-1})}{2h} = \frac{6030 - 5560}{4} = 117.5 \text{ m/s}$$

$$\ddot{\theta} = \frac{d^2\theta}{dt^2} = \frac{\theta(t_{i+1}) - 2\theta(t_i) + \theta(t_{i-1})}{h^2} = \frac{0.67 - 2(0.68) + 0.70}{(2)^2} = 0.0025 \text{ rad/s}^2$$

$$\ddot{r} = \frac{d^2r}{dt^2} = \frac{r(t_{i+1}) - 2r(t_i) + r(t_{i-1})}{h^2} = \frac{6030 - 2(5800) + 5560}{(2)^2} = -2.5 \text{ m/s}^2$$

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$\vec{v} = 117.5\,\vec{e}_r - 43.5\,\vec{e}_\theta$

$\vec{a} = -2.82625\,\vec{e}_r + 12.7375\,\vec{e}_\theta$

23.23 Use the same program as was developed in the solution of Prob. 23.11

```
Option Explicit

Sub TestDerivUnequal()
Dim n As Integer, i As Integer
Dim x(100) As Double, y(100) As Double, dy(100) As Double
Range("a5").Select
n = ActiveCell.Row
Selection.End(xlDown).Select
n = ActiveCell.Row - n
Range("a5").Select
For i = 0 To n
  x(i) = ActiveCell.Value
  ActiveCell.Offset(0, 1).Select
  y(i) = ActiveCell.Value
  ActiveCell.Offset(1, -1).Select
Next i
For i = 0 To n
  dy(i) = DerivUnequal(x, y, n, x(i))
Next i
Range("c5").Select
For i = 0 To n
  ActiveCell.Value = dy(i)
  ActiveCell.Offset(1, 0).Select
Next i
End Sub
Function DerivUnequal(x, y, n, xx)
Dim ii As Integer
If xx < x(0) Or xx > x(n) Then
  DerivUnequal = "out of range"
Else
  If xx < x(1) Then
    DerivUnequal = DyDx(xx, x(0), x(1), x(2), y(0), y(1), y(2))
  ElseIf xx > x(n - 1) Then
    DerivUnequal = _
        DyDx(xx, x(n - 2), x(n - 1), x(n), y(n - 2), y(n - 1), y(n))
  Else
    For ii = 1 To n - 2
      If xx >= x(ii) And xx <= x(ii + 1) Then
        If xx - x(ii - 1) < x(ii) - xx Then
          'If the unknown is closer to the lower end of the range,
          'x(ii) will be chosen as the middle point
          DerivUnequal = _
          DyDx(xx, x(ii - 1), x(ii), x(ii + 1), y(ii - 1), y(ii), y(ii + 1))
        Else
          'Otherwise, if the unknown is closer to the upper end,
          'x(ii+1) will be chosen as the middle point
          DerivUnequal = _
          DyDx(xx, x(ii), x(ii + 1), x(ii + 2), y(ii), y(ii + 1), y(ii + 2))
        End If
        Exit For
      End If
    Next ii
  End If
End If
End Function
```

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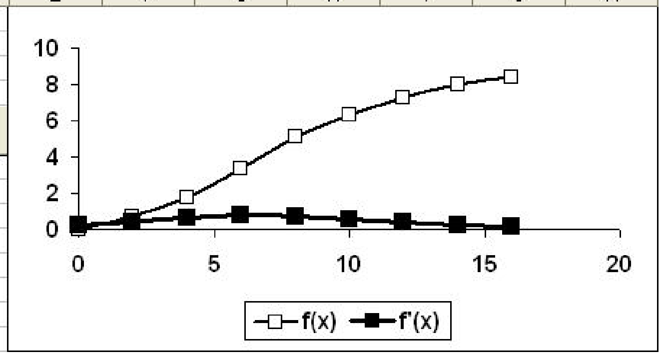
```
Function DyDx(x, x0, x1, x2, y0, y1, y2)
DyDx = y0 * (2 * x - x1 - x2) / (x0 - x1) / (x0 - x2) _
      + y1 * (2 * x - x0 - x2) / (x1 - x0) / (x1 - x2) _
       + y2 * (2 * x - x0 - x1) / (x2 - x0) / (x2 - x1)
End Function
```

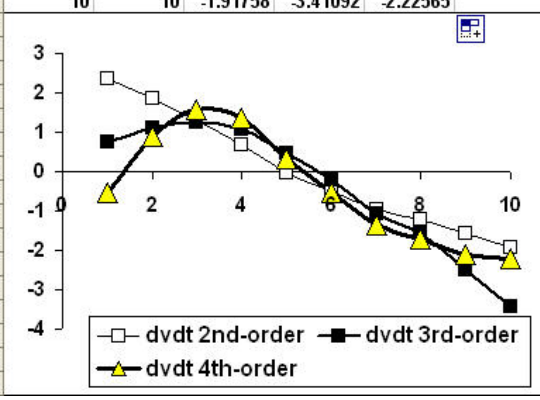
The result of running this program is shown below:

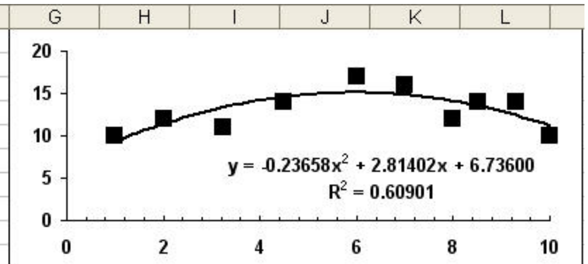
	A	B	C	D
1	Prob 23.23			
2				
3				
4	x	f(x)	f'(x)	
5	0	0	0.250	Run
6	2	0.7	0.425	
7	4	1.8	0.675	
8	6	3.4	0.825	
9	8	5.1	0.725	
10	10	6.3	0.550	
11	12	7.3	0.425	
12	14	8	0.275	
13	16	8.4	0.125	
14				

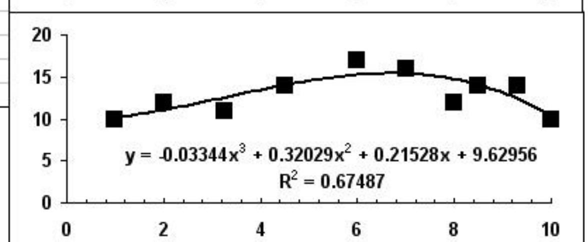


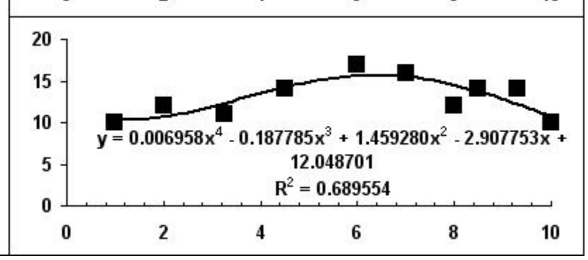
23.24

	A	B	C	D	E
1	Prob. 23.24				
2					
3			dvdt	dvdt	dvdt
4	t	v	2nd-order	3rd-order	4th-order
5	1	10	2.34086	0.75554	-0.52472
6	2	12	1.8677	1.09516	0.898603
7	3.25	11	1.27625	1.237535	1.58255
8	4.5	14	0.6848	1.06641	1.354019
9	6	17	-0.02494	0.44724	0.334539
10	7	16	-0.4981	-0.21634	-0.53585
11	8	12	-0.97126	-1.08056	-1.36401
12	8.5	14	-1.20784	-1.58791	-1.71006
13	9.3	14	-1.58637	-2.504	-2.10285
14	10	10	-1.91758	-3.41092	-2.22565









23.25 The flow rate is equal to the derivative of volume with respect to time. Equation (23.9) can be used to compute the derivative as

$x_0 = 1 \quad f(x_0) = 1$
$x_1 = 5 \quad f(x_1) = 8$
$x_2 = 8 \quad f(x_2) = 16.4$

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$$f'(7) = 1\frac{2(7)-5-8}{(1-5)(1-8)} + 8\frac{2(7)-1-8}{(5-1)(5-8)} + 16.4\frac{2(7)-1-5}{(8-1)(8-5)} = 0.035714 - 3.33333 + 6.247619 = 2.95$$

Therefore, the flow is equal to 2.95 cm^3/s.

23.26 The velocity at the surface can be computed with Eq. (23.9) as

$x_0 = 0$ $f(x_0) = 0$
$x_1 = 0.002$ $f(x_1) = 0.287$
$x_2 = 0.006$ $f(x_2) = 0.899$

$$f'(0) = 0\frac{2(0)-0.002-0.006}{(0-0.002)(0-0.006)} + 0.287\frac{2(0)-0-0.006}{(0.002-0)(0.002-0.006)} + 0.899\frac{2(0)-0-0.002}{(0.006-0)(0.006-0.002)}$$
$$= 0 + 215.25 - 74.9167 = 140.3333$$

Therefore, the shear stress can be computed as

$$\tau = 1.8\times10^{-5}\frac{\text{N}\cdot\text{s}}{\text{m}^2}140.3333\frac{1}{\text{s}} = 0.00253\frac{\text{N}}{\text{m}^2}$$

23.27 The first forward difference formula of $O(h^2)$ from Fig. 23.1 can be used to estimate the velocity for the first point at $t = 10$,

$$\frac{dc}{dt}(10) = \frac{-1.75 + 4(2.48) - 3(3.52)}{2(10)} = -0.1195$$

For the interior points, centered difference formulas of $O(h^2)$ from Fig. 23.3 can be used to estimate the derivatives. For example, at the second point at $t = 20$,

$$\frac{dc}{dt}(20) = \frac{1.75 - 3.52}{2(10)} = -0.0885$$

For the final point, backward difference formulas of $O(h^2)$ from Fig. 23.2 can be used to estimate the derivative. The results for all values are summarized in the following table.

t	***c***	**–*dc*/*dt***	**log *c***	**log(–*dc*/*dt*)**
10	3.52	0.1195	0.546543	-0.92263
20	2.48	0.0885	0.394452	-1.05306
30	1.75	0.0625	0.243038	-1.20412
40	1.23	0.044	0.089905	-1.35655
50	0.87	0.031	-0.06048	-1.50864
60	0.61	0.021	-0.21467	-1.67778

A log-log plot can be developed

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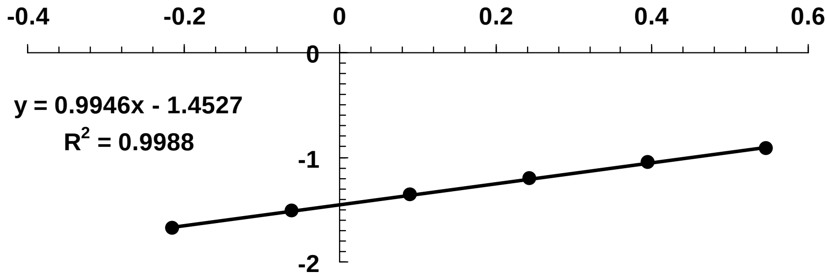


The resulting best-fit equation can be used to compute $k = 10^{-1.45269} =$ 0.035262 and $n = 0.994579$.

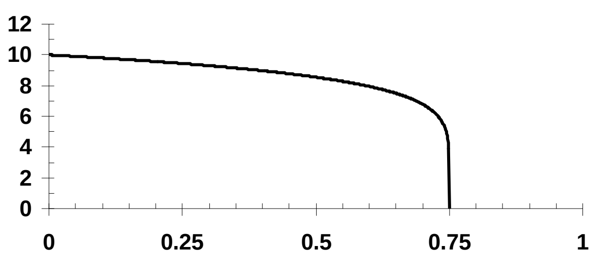
23.28 The solution amounts to implementing the following integral

$$Q = \int_0^{0.75} 2\pi r v \, dr = \int_0^{0.75} 20\pi r \left(1 - \frac{r}{0.75}\right)^{1/7} dr$$

(a)

1	2	3	4	5	6	7	8	9
$\varepsilon_t \rightarrow$	26.0633%	10.7475%	4.7547%	2.1389%	0.9666%	0.4374%	0.1980%	0.0896%
$\varepsilon_a \rightarrow$	25.0000%	1.0725%	0.0983%	0.0104%	0.0012%	0.0001%	0.0000%	0.0000%
0.000000	10.670306	12.880638	13.745507	14.123004	14.292194	14.368567	14.403113	14.418751
8.002729	12.742492	13.731994	14.121529	14.292029	14.368548	14.403111	14.418751	
11.557551	13.670150	14.115443	14.291363	14.368473	14.403103	14.418750		
13.142000	14.087612	14.288614	14.368172	14.403069	14.418746			
13.851209	14.276052	14.366929	14.402933	14.418731				
14.169841	14.361249	14.402370	14.418669					
14.313397	14.399800	14.418414						
14.378199	14.417251							
14.407488								

Notice that the approximate error is higher than the true error. This is a not a desirable result and is an artifact of the shape of the function which, as shown below, is very steep at the end of the integration interval.



If the computation is continued, a more acceptable result of 14.430490 (ε_t = 0.0083%) is attained after 12 iterations.

(b) Change of variable:

$$r = \frac{0.75+0}{2} + \frac{0.75-0}{2} r_d = 0.375 + 0.375 r_d$$

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$$dr = \frac{0.75-0}{2} dr_d = 0.375 dr_d$$

$$Q = \int_{-1}^{1} 2.8125\pi(1+r_d)\left(0.5-0.5r_d\right)^{1/7} \, dr_d$$

Therefore, the transformed function is

$$f(r_d) = 2.8125(1+r_d)\pi\left(0.5-0.5r_d\right)^{1/7}$$

Two-point formula:

$$I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 3.609892 + 11.16182 = 14.77171 \quad (\varepsilon_t = 2.36\%)$$

(c)

```
>> format long
>> fr=@(r) 20 * pi * r .* (1 - r / 0.75) .^ (1 / 7);
>> Flow=quad(fr,0,0.75)
Flow =
  14.431685608362534
```

23.29 (a)

	$\varepsilon_a \rightarrow$	7.4179%	0.1054%	0.0012119%
1	411.26095167	317.15529472	322.59571622	322.34570788
2	340.68170896	322.25568988	322.34961426	
4	326.86219465	322.34374398		
8	323.47335665			

(b)

```
>> format long
>> f = @(t) (9+4*cos(0.4 * t) .^ 2) .* (5*exp(-0.5 * t)+2*exp(0.15 * t));
>> Flow=quad(f,2,8)
Flow =
    3.223483672690013e+002
```

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