statemech density operators

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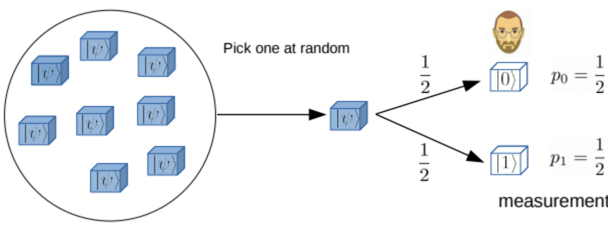
1 problem 1

what are the prbabilities of measureint $|\pm\rangle$ for these three staes

$$|\pm\rangle = |0\rangle \pm |1\rangle \tag{1}$$

case 1:



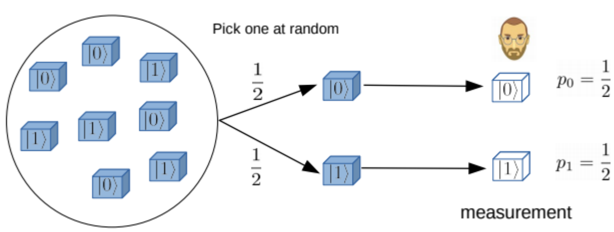
solution:

$$\rho = |\langle\pm|\psi\rangle|^2 = |(\langle 0| \pm \langle 1|) * \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|^2 \tag{2}$$

$$= \frac{1}{2} \pm \frac{1}{2} \tag{3}$$

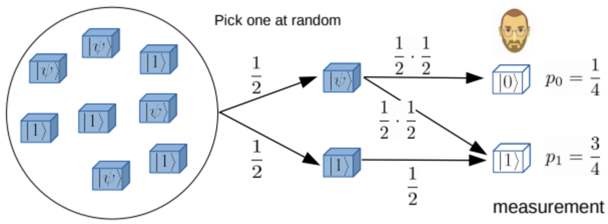
so the probability to measure $|+\rangle$ is 1 while the prbability to mesure $|-\rangle$ is zero

case 2:



solution:
since the state are either $|0\rangle$ or $|1\rangle$ there is 0 probability to be in $|\pm\rangle$ which is a sumation of those two states

case 3:



solution:

$$\rho_{|+\rangle} = |\langle+|\psi\rangle|^2 = 1$$
$$\rho_{|-\rangle} = |\langle-|\psi\rangle|^2 = 0$$

but $|\psi\rangle$ has 1/2 chance of being measured, therfore $|+\rangle$ has 1/2 chance of being measured and $|-\rangle$ has zero chance of being measured.

question: a system is in a pure state $|\Psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$. find teh density matrix in $|0\rangle$ and $|1\rangle$

solution: since it is a pure state the density opertaor isjust the outer product of $|\Psi\rangle$

$$\rho = \frac{1}{2}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right) \tag{4}$$

to get this in matrix formation in the basis $|0\rangle$ and $|1\rangle$ we do $\rho = \sum_i \sum_j |\psi_i\rangle\langle\psi_i| \rho |\psi_j\rangle\langle\psi_j|$. so the density matrix is

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{5}$$

2 problem: show that the purity is less than one, that is tr$\rho^2 < 1$

here $\rho = P_1 |\Psi_1\rangle\langle\Psi_1| + P_2 |\Psi_2\rangle\langle\Psi_2|$
so $\rho^2 = P_1^2 |\Psi_1\rangle\langle\Psi_1| |\Psi_1\rangle\langle\Psi_1| + P_1 P_2 |\Psi_1\rangle\langle\Psi_1| |\Psi_2\rangle\langle\Psi_2| + P_2 P_1 |\Psi_2\rangle\langle\Psi_2| |\Psi_1\rangle\langle\Psi_1| + P_2^2 |\Psi_2\rangle\langle\Psi_2| |\Psi_2\rangle\langle\Psi_2|$

$$\rho^2 = P_1^2 |\Psi_1\rangle\langle\Psi_1| + P_1 P_2 |\Psi_1\rangle\langle\Psi_1| |\Psi_2\rangle\langle\Psi_2| + P_2 P_1 |\Psi_2\rangle\langle\Psi_2| |\Psi_1\rangle\langle\Psi_1| + P_2^2 |\Psi_2\rangle\langle\Psi_2| \tag{6}$$

$$tr(\rho^2) = \langle\Psi_1| \rho^2 |\Psi_1\rangle + \langle\Psi_2| \rho^2 |\Psi_2\rangle \tag{7}$$

$$\text{since } \langle\Psi_1|\Psi_2\rangle = 0$$
$$tr(\rho^2) = P_1^2 + 0 + 0 + 0 + 0 + 0 + 0 + P_2^2$$

since p1 +p2 =1 the trace of ρ^2 is less than one

3 show that if the initial state is pure, the system remains in pure state all time. Similarly, if the initial state is mied, it remaisn in mixed state all time