statemech liouville equation divervation

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1 proof that $\nabla \cdot \rho v = \{\rho, H\}$

since this system has a constant number of particles, the number going out must equal in magnitude the number coming in. The rate they are coming in is $\frac{\partial}{\partial t}\int_\omega \rho d\omega$ and the rate out is $\int_\sigma \rho v \cdot \hat{n} d\sigma$. to get the integrals of the same space i used the fact that $\int_V \nabla \cdot A d\tau = \oint A \cdot da$. using this $\int_\sigma \rho v \cdot \hat{n} d\sigma = \int_\omega \nabla \cdot \rho v \, d\omega$

$$\frac{\partial}{\partial t}\int_\omega \rho d\omega = -\int_\omega \nabla \cdot \rho v \, d\omega \tag{1}$$

$$\Rightarrow \nabla \cdot \rho v + \frac{\partial \rho}{\partial t} = 0 \tag{2}$$

now we shall look at $\nabla \cdot \rho v$

$$\nabla \cdot \rho v = \sum_{i=1}^{3N}\left(\frac{\partial}{\partial q_i}\rho \dot{q}_i + \frac{\partial}{\partial p_i}\rho \dot{p}_i\right) \tag{3}$$

I shall now expand it

$$\nabla \cdot \rho v = \sum_{i=1}^{3N}\left(\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i + \frac{\partial \dot{q}_i}{\partial q_i}\rho + \frac{\partial \dot{p}_i}{\partial p_i}\rho\right) \tag{4}$$

these equations are simplified useing the definitions of $\dot{q}$ and the $\dot{p}$

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

$$\frac{\partial \dot{q}}{\partial q} = \frac{\partial^2 H}{\partial q \partial p} \qquad \frac{\partial}{\partial \dot{p}} = -\frac{\partial^2 H}{\partial p \partial q}$$

using these we get that

$$\nabla \cdot \rho v = \sum_{i=1}^{3N}\left(\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i\right) \tag{5}$$

and using the fact the $\{f, g\} = \sum_{i}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$

$$\boldsymbol{\nabla} \cdot \rho v = \{\rho, H\} \tag{6}$$

$$\therefore \boxed{\frac{\partial \rho}{\partial t} = -\{\rho, H\}} \tag{7}$$