statemech: expectation value of nj

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march 2020

1 boson

goal is to derive $\langle n_j \rangle = \frac{1}{\exp(\mu - \epsilon) - 1}$

$$\begin{aligned}
\langle n_j \rangle &= \sum_{\{n_k\}} n_j P\{n_k\} \\
&= \sum_{\{n_k\}} n_j \exp\left\{ \sum_{\{n_k\}} \beta\left(\mu - \epsilon_k\right) n_k \right\} \frac{1}{\sum_{\{n_k\}} \exp\left\{ \sum_{\{n_k\}} \beta\left(\mu - \epsilon_k\right) n_k \right\}} \\
&= -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_k} \ln(\mathcal{Z})
\end{aligned}$$

$$\langle n_j \rangle = -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_k} \ln(\mathcal{Z}) \tag{1}$$

i got this be making use of the fact that $P\{n_k\} = \exp\left\{ \sum_{\{n_k\}} \beta\left(\mu - \epsilon_k\right) n_k \right\} \frac{1}{\sum_{\{n_k\}} \exp\left\{ \sum_{\{n_k\}} \beta(\mu - \epsilon_k) n_k \right\}}$ and the fact the $\frac{\frac{\partial}{\partial x} y}{y} = \frac{\partial}{\partial x} \ln(y)$. Equation 1 wil be used for both the boson and fermion case.

$$\mathcal{Z} = \prod_{k=0}^{\infty} \frac{1}{1 - \exp\{\beta\left(\mu - \epsilon_k\right)\}}$$

plug this into equation 1

$$\begin{aligned}
\langle n_j \rangle &= -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_k} \ln\left(\prod_{k=0}^{\infty} \frac{1}{1 - \exp\{\beta\left(\mu - \epsilon_k\right)\}} \right) \\
&= \frac{1}{\beta} \frac{\partial}{\partial \epsilon_k} \ln\left(\prod_{k=0}^{\infty} 1 - \exp\{\beta\left(\mu - \epsilon_k\right)\} \right) \\
&= \frac{1}{\beta} \frac{\prod_{k=0}^{\infty} \exp\{\beta\left(\mu - \epsilon_k\right)\}}{\prod_{k=0}^{\infty} \left(1 - \exp\{\beta\left(\mu - \epsilon_k\right)\}\right)}
\end{aligned}$$

which simplifies into

$$\boxed{= \frac{1}{\exp(-\beta\left(\mu - \epsilon\right)) - 1}}$$

2 fermion

very similar to boson except $\mathcal{Z} = \prod_{k=0}^{\infty} (1 + \exp\{\beta(\mu - \epsilon_k)\})$

plug this into equation 1

$$\langle n_j \rangle = -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_k} \ln\left(\prod_{k=0}^{\infty} (1 + \exp\{\beta(\mu - \epsilon_k)\}) \right)$$

$$= \frac{1}{\beta} \frac{\prod_{k=0}^{\infty} \exp\{\beta(\mu - \epsilon_k)\}}{\prod_{k=0}^{\infty} (1 + \exp\{\beta(\mu - \epsilon_k)\})}$$

which simplifies into

$$\boxed{= \frac{1}{\exp(-\beta(\mu - \epsilon)) + 1}}$$