Computations of Brumer–Stark Units

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1 Description

We give complete tables for the computations described in §2 of [1]. For each ℓ and p, we consider all fundamental discriminants $D < 1000$ such that $F = \mathbf{Q}(\sqrt{D})$ contains no unit of negative norm, and such that ℓ splits in F and p is inert in F. The group $G = \mathrm{Gal}(H/F) = \mathrm{Cl}^+(F)$ is given as well as the values $\mathrm{ord}_p(\sigma(u_{\mathfrak{p}})) = \zeta_{S,T}(\sigma^{-1}, 0)$ for $\sigma \in G$. The minimal polynomial $P(X)$ of the Brumer–Stark unit $u_{\mathfrak{p}}$ is computed as listed to 100 p-adic digits. All together, the tables contain all fundamental discriminants $D < 1000$ such that $F = \mathbf{Q}(\sqrt{D})$ contains no unit of negative norm. The code used to generate these results is available at [2].

Note that the minimal polynomials of Brumer–Stark units are always palindromic, since $u_{\mathfrak{p}}^{-1}$ is the complex conjugate of $u_{\mathfrak{p}}$. Therefore, to save space, redundant coefficients are often replaced by $\cdots$ in the tables below.

Table 1: $p = 3$, $\ell = 5$

D	G	ord_p	$P(X)$
44	C_2	± 2	$X^2 + \frac{7}{3^2}X + 1$
56	C_2	± 3	$X^2 - \frac{46}{3^3}X + 1$
161	C_2	± 9	$X^2 + \frac{28234}{3^9}X + 1$
209	C_2	± 2	$X^2 + \frac{7}{3^2}X + 1$
221	C_4	$\pm 3, \pm 15$	$X^4 + (\frac{-423812}{3^{13}} + \frac{71680\sqrt{D}}{3^{15}})X^3 + (\frac{76348630}{3^{18}} + \frac{-5218304\sqrt{D}}{3^{16}})X^2 + (\frac{-423812}{3^{13}} + \frac{71680\sqrt{D}}{3^{15}})X + 1$
236	C_2	± 6	$X^2 - \frac{1433}{3^6}X + 1$
284	C_2	± 14	$X^2 - \frac{7162162}{3^{14}}X + 1$
329	C_2	± 15	$X^2 - \frac{26326214}{3^{15}}X + 1$

341	C_2	± 18	$X^2 + \frac{339284078}{3^{18}}X + 1$
344	C_2	± 3	$X^2 + \frac{46}{3^3}X + 1$
476	$C_2 \times C_2$	$\pm 2, \pm 22$	$X^4 + \frac{73720420636}{3^{22}}X^3 + \frac{898195708486}{3^{24}}X^2 + \frac{73720420636}{3^{22}}X + 1$
524	C_2	± 10	$X^2 + \frac{7927}{3^{10}}X + 1$
536	C_2	± 3	$X^2 - \frac{46}{3^3}X + 1$
581	C_2	± 27	$X^2 + \frac{14106563188426}{3^{27}}X + 1$
584	C_4	$\pm 1, \pm 5$	$X^4 + (\frac{-134}{3^4} + \frac{-10\sqrt{D}}{3^5})X^3 + (\frac{1675}{3^6} + \frac{14\sqrt{D}}{3^5})X^2 + \cdots + 1$
689	C_8	$\pm 1, \pm 3, \pm 5, \pm 9$	$X^8 + (\frac{28367}{2\cdot 3^8} + \frac{3587\sqrt{D}}{2\cdot 3^9})X^7 + (\frac{12116975}{2\cdot 3^{13}} + \frac{2563583\sqrt{D}}{2\cdot 3^{14}})X^6 + (\frac{2262488935}{2\cdot 3^{17}} + \frac{9322447\sqrt{D}}{2\cdot 3^{15}})X^5 + (\frac{4758938750}{3^{18}} + \frac{12806215\sqrt{D}}{3^{16}})X^4 + \cdots + 1$
716	C_2	± 10	$X^2 - \frac{65198}{3^{10}}X + 1$
749	C_2	± 27	$X^2 - \frac{2489075116426}{3^{27}}X + 1$
764	C_2	± 26	$X^2 - \frac{2669652903442}{3^{26}}X + 1$
776	C_4	$\pm 3, \pm 15$	$X^4 + (\frac{5253188}{3^{14}} + \frac{1313792\sqrt{D}}{3^{15}})X^3 + (\frac{1380467062}{3^{18}} + \frac{8114176\sqrt{D}}{3^{17}})X^2 + (\frac{5253188}{3^{14}} + \frac{1313792\sqrt{D}}{3^{15}})X + 1$
824	C_2	± 15	$X^2 - \frac{26595314}{3^{15}}X + 1$
869	C_2	± 30	$X^2 + \frac{118462767513202}{3^{30}}X + 1$
956	C_2	± 30	$X^2 - \frac{11092130298802}{3^{30}}X + 1$
989	C_2	± 27	$X^2 - \frac{10308309942374}{3^{27}}X + 1$

Table 2: $p = 7$, $\ell = 5$

D	G	ord_p	$P(X)$
24	C_2	± 1	$X^2 - \frac{11}{7}X + 1$
69	C_2	± 9	$X^2 + \frac{77549186}{7^9}X + 1$
76	C_2	± 2	$X^2 - \frac{73}{7^2}X + 1$
124	C_2	± 6	$X^2 - \frac{50398}{7^6}X + 1$
129	C_2	± 1	$X^2 - \frac{11}{7}X + 1$
136	C_4	$\pm 1, \pm 5$	$X^4 + (\frac{-12776}{7^5} + \frac{192\sqrt{D}}{7^5})X^3 + (\frac{87982}{7^6} + \frac{-1536\sqrt{D}}{7^5})X^2 + (\frac{-12776}{7^5} + \frac{192\sqrt{D}}{7^5})X + 1$
201	C_2	± 1	$X^2 + \frac{11}{7}X + 1$

209	C_2	± 2	$X^2 - \frac{73}{7^2}X + 1$
236	C_2	± 6	$X^2 - \frac{211273}{7^6}X + 1$
264	$C_2 \times C_2$	$\pm 1, \pm 3$	$X^4 - \frac{22}{7^3}X^3 + \frac{1131}{7^4}X^2 - \frac{22}{7^3}X + 1$
285	$C_2 \times C_2$	$\pm 6, \pm 18$	$X^4 - \frac{563584621867196}{7^{18}}X^3 - \frac{107153704401579734394}{7^{24}}X^2 + \ldots + 1$
321	C_6	$\pm 1, \pm 3, \pm 7$	$X^6 + (\frac{55935}{2\cdot 7^7} + \frac{-63891\sqrt{D}}{2\cdot 7^7})X^5 + (\frac{1062148509}{2\cdot 7^{10}} + \frac{2960001\sqrt{D}}{2\cdot 7^{10}})X^4 + (\frac{-49244921}{2\cdot 7^{10}} + \frac{-279429993\sqrt{D}}{2\cdot 7^{11}})X^3 + \cdots + 1$
341	C_2	± 18	$X^2 - \frac{1964710753423202}{7^{18}}X + 1$
376	C_2	± 15	$X^2 + \frac{5477442893714}{7^{15}}X + 1$
381	C_2	± 5	$X^2 - \frac{32989}{7^5}X + 1$
444	$C_2 \times C_2$	$\pm 3, \pm 5$	$X^4 + \frac{25322}{7^5}X^3 + \frac{8005131}{7^8}X^2 + \frac{25322}{7^5}X + 1$
489	C_2	± 1	$X^2 + \frac{11}{7}X + 1$
524	C_2	± 10	$X^2 + \frac{487078727}{7^{10}}X + 1$
556	C_2	± 6	$X^2 + \frac{83927}{7^6}X + 1$
584	C_4	$\pm 1, \pm 5$	$X^4 + (\frac{30447}{7^5} + \frac{-1377\sqrt{D}}{7^5})X^3 + (\frac{246364}{7^6} + \frac{-2430\sqrt{D}}{7^5})X^2 + (\frac{30447}{7^5} + \frac{-1377\sqrt{D}}{7^5})X + 1$
636	$C_2 \times C_2$	$\pm 4, \pm 16$	$X^4 - \frac{32680821710204}{7^{16}}X^3 + \frac{21577781512065606}{7^{20}}X^2 - \frac{32680821710204}{7^{16}}X + 1$
649	C_2	± 6	$X^2 - \frac{211273}{7^6}X + 1$
664	C_2	± 9	$X^2 + \frac{21302786}{7^9}X + 1$
689	C_8	$\pm 1, \pm 3, \pm 5, \pm 9$	$X^8 + (\frac{-96585229}{2\cdot 7^9} + \frac{133323\sqrt{D}}{2\cdot 7^7})X^7 + (\frac{523337048995}{2\cdot 7^{13}} + \frac{-216747384087\sqrt{D}}{2\cdot 7^{14}})X^6 + (\frac{-366432628616013}{2\cdot 7^{16}} + \frac{83033328813675\sqrt{D}}{2\cdot 7^{17}})X^5 + (\frac{10610660816305286}{7^{18}} + \frac{-46740169758801\sqrt{D}}{7^{17}})X^4 + \cdots + 1$
696	$C_2 \times C_2$	$\pm 8, \pm 10$	$X^4 + \frac{887497754}{7^{10}}X^3 + \frac{7273880240745819}{7^{18}}X^2 + \frac{887497754}{7^{10}}X + 1$
741	$C_2 \times C_2$	$\pm 3, \pm 21$	$X^4 + \frac{695444597426120828}{7^{21}}X^3 + \frac{404785271316975019206}{7^{24}}X^2 + \frac{695444597426120828}{7^{21}}X + 1$
776	C_4	$\pm 3, \pm 15$	$X^4 + (\frac{1975502872872}{7^{15}} + \frac{-489277800000\sqrt{D}}{7^{15}})X^3 + (\frac{6515911097389294}{7^{18}} + \frac{-5484283200000\sqrt{D}}{7^{17}})X^2 + (\frac{1975502872872}{7^{15}} + \frac{-489277800000\sqrt{D}}{7^{15}})X + 1$
789	C_2	± 39	$X^2 + \frac{116160269647284519468619901052931 4}{7^{39}}X + 1$
796	C_2	± 18	$X^2 + \frac{1851089399171998}{7^{18}}X + 1$
824	C_2	± 15	$X^2 - \frac{205062858286}{7^{15}}X + 1$

885	$C_2 \times C_2$	$\pm 2, \pm 10$	$X^4 - \frac{902769529}{7^{10}}X^3 + \frac{60792434256}{7^{12}}X^2 - \frac{902769529}{7^{10}}X + 1$
969	$C_2 \times C_2$	$0, \pm 4$	$X^4 + \frac{1679}{7^4}X^3 + \frac{1056}{7^4}X^2 + \frac{1679}{7^4}X + 1$

Table 3: $p = 11, \ell = 5$

D	G	ord_p	$P(X)$
21	C_2	± 3	$X^2 + \frac{1962}{11^3}X + 1$
24	C_2	± 1	$X^2 + (\frac{-7}{11} + \frac{-3\sqrt{D}}{11})X + 1$
76	C_2	± 2	$X^2 + \frac{233}{11^2}X + 1$
129	C_2	± 1	$X^2 + (\frac{-21}{2\cdot 11} - \frac{\sqrt{D}}{2\cdot 11})X + 1$
156	$C_2 \times C_2$	$\pm 1, \pm 3$	$X^4 + (\frac{-2808}{11^3} + \frac{-24\sqrt{D}}{11^3})X^3 + (\frac{36137}{11^4} + \frac{6\sqrt{D}}{11^2})X^2 + (\frac{-2808}{11^3} + \frac{-24\sqrt{D}}{11^3})X + 1$
161	C_2	± 9	$X^2 + \frac{2874798918}{11^9}X + 1$
184	C_2	± 9	$X^2 + \frac{354349618}{11^9}X + 1$
204	$C_2 \times C_2$	$\pm 2, \pm 6$	$X^4 + \frac{3034094}{11^6}X^3 + \frac{456208611}{11^8}X^2 + \frac{3034094}{11^6}X + 1$
249	C_2	± 3	$X^2 - \frac{587}{11^3}X + 1$
316	C_6	$\pm 2, \pm 6, \pm 18$	$X^6 + (\frac{1558408275932970933 4}{11^{18}} + \frac{207963230118617088\sqrt{D}}{11^{18}})X^5 + (\frac{3011097530357985107693643 1}{11^{24}} + \frac{156988123811907599268249 6\sqrt{D}}{11^{24}})X^4 + (\frac{2961551759995928199614976628}{11^{26}} + \frac{290862780332883761627086848\sqrt{D}}{11^{26}})X^3 + \cdots + 1$
321	C_6	$\pm 1, \pm 3, \pm 7$	$X^6 + (\frac{-57058071}{2\cdot 11^7} + \frac{1388793\sqrt{D}}{2\cdot 11^7})X^5 + (\frac{43340986329}{11^{10}} + \frac{-3671846499\sqrt{D}}{11^{10}})X^4 + (\frac{-413951644540}{11^{11}} + \frac{4270558635\sqrt{D}}{11^{10}})X^3 + \cdots + 1$
329	C_2	± 15	$X^2 + \frac{7038210629286198}{11^{15}}X + 1$
376	C_2	± 15	$X^2 + \frac{6656187998706302}{11^{15}}X + 1$
381	C_2	± 5	$X^2 + \frac{300198}{11^5}X + 1$
469	C_6	$\pm 3, \pm 9, \pm 21$	$X^6 + (\frac{6062693026756491401330}{11^{21}} + \frac{-42143748202480873472\sqrt{D}}{11^{20}})X^5 + (\frac{27890625333887733737305237254663}{11^{30}} + \frac{-100792761033662143567205031936\sqrt{D}}{11^{29}})X^4 + (\frac{25396498275109170063468719818921148}{11^{33}} + \frac{-23032775709004051831435547527577 6\sqrt{D}}{11^{32}})X^3 + \cdots + 1$
501	C_2	± 33	$X^2 - \frac{44946948494493582973059229141165562}{11^{33}}X + 1$

524	C_2	± 10	$X^2 - \frac{50395911602}{11^{10}}X + 1$
536	C_2	± 3	$X^2 + \frac{2338}{11^3}X + 1$
556	C_2	± 6	$X^2 - \frac{2882153}{11^6}X + 1$
589	C_2	± 18	$X^2 + \frac{8650843028015765362}{11^{18}}X + 1$
604	C_2	± 14	$X^2 - \frac{758322870048082}{11^{14}}X + 1$
681	C_2	± 5	$X^2 + (\frac{-315541}{2\cdot 11^5} + \frac{-4293\sqrt{D}}{2\cdot 11^5})X + 1$
689	C_8	$\pm 1, \pm 3, \pm 5, \pm 9$	$X^8 + (\frac{-2933818079}{2\cdot 11^9} + \frac{456087681\sqrt{D}}{2\cdot 11^9})X^7 + (\frac{1569924958656353}{2\cdot 11^{14}} + \frac{-5557235487489\sqrt{D}}{2\cdot 11^{13}})X^6 + (\frac{-2521174786518864489}{2\cdot 11^{17}} + \frac{22499250144556689\sqrt{D}}{2\cdot 11^{17}})X^5 + (\frac{747989229794352702}{11^{18}} + \frac{-429775960839285879\sqrt{D}}{11^{18}})X^4 + \cdots + 1$
721	C_2	± 15	$X^2 + \frac{7038210629286198}{11^{15}}X + 1$
744	$C_2 \times C_2$	$\pm 2, \pm 10$	$X^4 + (\frac{39145946134}{11^{10}} + \frac{1912932126\sqrt{D}}{11^{10}})X^3 + (\frac{10578582159267}{11^{12}} + \frac{13157802\sqrt{D}}{11^8})X^2 + (\frac{39145946134}{11^{10}} + \frac{1912932126\sqrt{D}}{11^{10}})X + 1$
776	C_4	$\pm 3, \pm 15$	$X^4 + (\frac{-1509314973211004}{11^{15}} + \frac{74678370681600\sqrt{D}}{11^{15}})X^3 + (\frac{-359099615028221130}{11^{18}} + \frac{-264066306525858816\sqrt{D}}{11^{18}})X^2 + \cdots + 1$
789	C_2	± 39	$X^2 - \frac{133564381727497426314296924385305418228 2}{11^{39}}X + 1$
805	$C_2 \times C_2$	$\pm 18, \pm 36$	$X^4 + \frac{737250285200458922118843526183827376 4}{11^{36}}X^3 + \frac{569413470241751339855253186293881637235427126776403663286}{11^{54}}X^2 + \ldots + 1$
824	C_2	± 15	$X^2 - \frac{6656187998706302}{11^{15}}X + 1$
844	C_2	± 6	$X^2 - \frac{3284647}{11^6}X + 1$
849	C_2	± 3	$X^2 + (\frac{2379}{2\cdot 11^3} + \frac{-71\sqrt{D}}{2\cdot 11^3})X + 1$
876	$C_4 \times C_2$	$\pm 4, \pm 4, \pm 16, \pm 24$	$X^8 + (\frac{-4707695763569256584804748}{11^{24}} + \frac{219105226393728925972 80\sqrt{D}}{11^{23}})X^7 + (\frac{1121271730754524644101438059176949941947498}{11^{40}} + \frac{-166940628128261101695597300421333860580 80\sqrt{D}}{11^{40}})X^6 + (\frac{-164236860740648941737638166737166913064468 19504}{11^{44}} + \frac{217782993846013704768977588394705852276681792\sqrt{D}}{11^{44}})X^5 + (\frac{263007783544403877579203452712902759574862607043283}{11^{48}} + \frac{-799892424858707914411116009016059862280892209 2512\sqrt{D}}{11^{48}})X^4 + \cdots + 1$
921	C_2	± 3	$X^2 + (\frac{717}{11^3} + \frac{-64\sqrt{D}}{11^3})X + 1$
956	C_2	± 30	$X^2 + \frac{1870243971525391468602921821998}{11^{30}}X + 1$

989	C_2	± 27	$X^2 - \frac{819765810753101677987730044 2}{11^{27}}X + 1$

Table 4: $p = 3, \ell = 7$

D	G	ord_p	$P(X)$
44	C_2	± 4	$X^2 - \frac{113}{3^4}X + 1$
77	C_2	± 12	$X^2 + \frac{781282}{3^{12}}X + 1$
92	C_2	± 12	$X^2 + \frac{1062686}{3^{12}}X + 1$
221	C_4	$\pm 3, \pm 21$	$X^4 + (\frac{-927839396}{3^{19}} + \frac{1984663552\sqrt{D}}{3^{21}})X^3 + (\frac{1166767652582}{3^{24}} + \frac{-782188544\sqrt{D}}{3^{21}})X^2 + (\frac{-927839396}{3^{19}} + \frac{1984663552\sqrt{D}}{3^{21}})X + 1$
284	C_2	± 28	$X^2 - \frac{5542979604322}{3^{28}}X + 1$
305	C_4	$\pm 1, \pm 7$	$X^4 + (\frac{173\sqrt{D}}{2\cdot 3^6} + \frac{8105}{2\cdot 3^7})X^3 + (\frac{47699}{2\cdot 3^8} + \frac{185\sqrt{D}}{2\cdot 3^6})X^2 + (\frac{173\sqrt{D}}{2\cdot 3^6} + \frac{8105}{2\cdot 3^7})X + 1$
344	C_2	± 4	$X^2 + \frac{34}{3^4}X + 1$
380	$C_2 \times C_2$	$\pm 5, \pm 11$	$X^4 - \frac{24992\sqrt{D}}{3^{11}}X^3 + \frac{150822542}{3^{16}}X^2 - \frac{24992\sqrt{D}}{3^{11}}X + 1$
413	C_2	± 27	$X^2 + \frac{11300832017270}{3^{27}}X + 1$
428	C_2	± 12	$X^2 + \frac{762319}{3^{12}}X + 1$
473	C_6	$\pm 2, \pm 4, \pm 10$	$X^6 + (\frac{10496}{3^9} + \frac{692\sqrt{D}}{3^{10}})X^5 + (\frac{1387768}{3^{13}} + \frac{-36836\sqrt{D}}{3^{14}})X^4 + (\frac{20670335}{3^{16}} + \frac{769472\sqrt{D}}{3^{15}})X^3 + (\frac{1387768}{3^{13}} + \frac{-36836\sqrt{D}}{3^{14}})X^2 + (\frac{10496}{3^9} + \frac{692\sqrt{D}}{3^{10}})X + 1$
536	C_2	± 4	$X^2 - \frac{34}{3^4}X + 1$
620	$C_2 \times C_2$	$\pm 5, \pm 13$	$X^4 + \frac{71786\sqrt{D}}{3^{13}}X^3 + \frac{893917513}{3^{18}}X^2 + \frac{71786\sqrt{D}}{3^{13}}X + 1$
632	C_2	± 20	$X^2 - \frac{6074010274}{3^{20}}X + 1$
716	C_2	± 20	$X^2 - \frac{2722789598}{3^{20}}X + 1$
737	C_2	± 4	$X^2 + \frac{113}{3^4}X + 1$
764	C_2	± 52	$X^2 + \frac{5795117153597046022417118}{3^{52}}X + 1$
869	C_2	± 60	$X^2 - \frac{70748889263545459865344573598}{3^{60}}X + 1$
893	C_2	± 45	$X^2 + \frac{1984895108328084801686}{3^{45}}X + 1$
905	C_8	$\pm 1, \pm 5, \pm 7, \pm 13$	$X^8 + (\frac{4681537}{2\cdot 3^{13}} + \frac{54235\sqrt{D}}{2\cdot 3^{11}})X^7 + (\frac{26680788119}{2\cdot 3^{19}} + \frac{1186723595\sqrt{D}}{2\cdot 3^{20}})X^6 + (\frac{5200676047009}{2\cdot 3^{24}} + \frac{1043139376375\sqrt{D}}{2\cdot 3^{25}})X^5 + (\frac{54515270640146}{3^{26}} + \frac{102857840135\sqrt{D}}{3^{24}})X^4 + \cdots + 1$

917	C_2	± 45	$X^2 + \frac{5232379916079099988570}{3^{45}}X + 1$
956	C_2	± 60	$X^2 + \frac{84659281195866805682786231198}{3^{60}}X + 1$
989	C_2	± 36	$X^2 - \frac{299691187379140958}{3^{36}}X + 1$

Table 5: $p = 5, \ell = 7$

D	G	ord_p	$P(X)$
57	C_2	± 1	$X^2 + (\frac{-\sqrt{D}}{2\cdot 5} + \frac{9}{2\cdot 5})X + 1$
88	C_2	± 4	$X^2 - \frac{1151}{5^4}X + 1$
92	C_2	± 12	$X^2 + \frac{64250786}{5^{12}}X + 1$
93	C_2	± 9	$X^2 - \frac{3355686}{5^9}X + 1$
172	C_2	± 4	$X^2 + \frac{1054}{5^4}X + 1$
177	C_2	± 3	$X^2 + \frac{191}{5^3}X + 1$
253	C_2	± 36	$X^2 + \frac{26021007276285290110207774}{5^{36}}X + 1$
268	C_2	± 4	$X^2 + \frac{1054}{5^4}X + 1$
312	$C_2 \times C_2$	$\pm 5, \pm 11$	$X^4 + (\frac{-84067994}{5^{11}} + \frac{-4627596\sqrt{D}}{5^{11}})X^3 + (\frac{515388352131}{5^{16}} + \frac{1308\sqrt{D}}{5^6})X^2 + \cdots + 1$
393	C_2	± 5	$X^2 + (\frac{-2077}{5^5} + \frac{-204\sqrt{D}}{5^5})X + 1$
408	$C_2 \times C_2$	$\pm 2, \pm 6$	$X^4 + (\frac{11241}{5^6} + \frac{-2498\sqrt{D}}{5^6})X^3 + (\frac{1849376}{5^8} + \frac{-36\sqrt{D}}{5^4})X^2 + \cdots + 1$
417	C_2	± 3	$X^2 + (\frac{111}{2\cdot 5^3} + \frac{19\sqrt{D}}{2\cdot 5^3})X + 1$
428	C_2	± 12	$X^2 - \frac{64250786}{5^{12}}X + 1$
473	C_6	$\pm 2, \pm 4, \pm 10$	$X^6 + (\frac{4471208}{5^{10}} + \frac{-113292\sqrt{D}}{5^9})X^5 + (\frac{2655379352}{5^{14}} + \frac{10457208\sqrt{D}}{5^{12}})X^4 + (\frac{-347645111839}{5^{16}} + \frac{251940672\sqrt{D}}{5^{15}})X^3 + \cdots + 1$
492	$C_2 \times C_2$	$\pm 4, \pm 12$	$X^4 + (\frac{384801698}{5^{12}} + \frac{11433792\sqrt{D}}{5^{12}})X^3 + (\frac{319962417987}{5^{16}} + \frac{7920927552\sqrt{D}}{5^{16}})X^2 + (\frac{384801698}{5^{12}} + \frac{11433792\sqrt{D}}{5^{12}})X + 1$
508	C_2	± 20	$X^2 + \frac{182008936336226}{5^{20}}X + 1$
568	C_6	$\pm 4, \pm 6, \pm 18$	$X^6 + (\frac{6052077803198}{5^{18}} + \frac{-106991215488\sqrt{D}}{5^{17}})X^5 + (\frac{317211039072290591}{5^{24}} + \frac{-2778454174036224\sqrt{D}}{5^{23}})X^4 + (\frac{298668778721858988516}{5^{28}} + \frac{-4703922018048\sqrt{D}}{5^{19}})X^3 + \cdots + 1$
597	C_2	± 27	$X^2 - \frac{12205041021933540534}{5^{27}}X + 1$
632	C_2	± 20	$X^2 - \frac{162337197810334}{5^{20}}X + 1$

652	C_2	± 4	$X^2 - \frac{1054}{5^4}X + 1$
732	$C_2 \times C_2$	$\pm 5, \pm 11$	$X^4 + (\frac{988998}{5^{11}} + \frac{2637328\sqrt{D}}{5^{11}})X^3 + (\frac{261091780931}{5^{16}} + \frac{48\sqrt{D}}{5^6})X^2 + (\frac{988998}{5^{11}} + \frac{2637328\sqrt{D}}{5^{11}})X + 1$
737	C_2	± 4	$X^2 + \frac{1151}{5^4}X + 1$
753	C_2	± 7	$X^2 + (\frac{84961}{2\cdot 5^7} + \frac{-8277\sqrt{D}}{2\cdot 5^7})X + 1$
793	C_8	$\pm 1, \pm 5, \pm 7, \pm 13$	$X^8 + (\frac{1906875217}{2\cdot 5^{13}} + \frac{56608499\sqrt{D}}{2\cdot 5^{13}})X^7 + (\frac{204522591029961}{2\cdot 5^{20}} + \frac{6707208376983\sqrt{D}}{2\cdot 5^{20}})X^6 + (\frac{391626287626167399}{2\cdot 5^{25}} + \frac{275763922205239\sqrt{D}}{2\cdot 5^{24}})X^5 + (\frac{952634142621099598}{5^{26}} + \frac{20753234633821799\sqrt{D}}{5^{26}})X^4 + \cdots + 1$
813	C_2	± 11	$X^2 + (\frac{-22197723}{5^{11}} + \frac{-2641876\sqrt{D}}{5^{11}})X + 1$
893	C_2	± 45	$X^2 + \frac{70387540570654878982640244775 74}{5^{45}}X + 1$
897	$C_4 \times C_2$	$\pm 7, \pm 9, \pm 11, \pm 21$	$X^8 + (\frac{2549757626558363}{2\cdot 5^{21}} + \frac{1416002374557\sqrt{D}}{2\cdot 5^{21}})X^7 + (\frac{51143699935554731498041}{5^{32}} + \frac{5670903011142486 4533\sqrt{D}}{5^{31}})X^6 + (-\frac{11738117897361345671334368371}{2\cdot 5^{41}} + \frac{49351162786458138729675149 31\sqrt{D}}{2\cdot 5^{41}})X^5 + (\frac{-44895867640480714989621403 28642159}{5^{48}} + \frac{49988908282076855221482\sqrt{D}}{5^{34}})X^4 + \cdots + 1$
933	C_2	± 19	$X^2 + \frac{38039425368074}{5^{19}}X + 1$
988	$C_2 \times C_2$	$\pm 6, \pm 18$	$X^4 + \frac{1820481770524}{5^{18}}X^3 + \frac{80918765552399046}{5^{24}}X^2 + \frac{1820481770524}{5^{18}}X + 1$

Table 6: $p = 11, \ell = 7$

D	G	ord_p	$P(X)$
57	C_2	± 1	$X^2 + (\frac{3}{2\cdot 11} + \frac{-5\sqrt{D}}{2\cdot 11})X + 1$
120	$C_2 \times C_2$	$\pm 3, \pm 5$	$X^4 + (\frac{-244494}{11^5} + \frac{-27166\sqrt{D}}{11^5})X^3 + (\frac{665357555}{11^8} + \frac{18\sqrt{D}}{11^2})X^2 + (\frac{-244494}{11^5} + \frac{-27166\sqrt{D}}{11^5})X + 1$
156	$C_2 \times C_2$	$\pm 3, \pm 5$	$X^4 + (\frac{533520}{11^5} + \frac{4560\sqrt{D}}{11^5})X^3 + (\frac{1011136793}{11^8} + \frac{6\sqrt{D}}{11^2})X^2 + \cdots + 1$
172	C_2	± 4	$X^2 + \frac{10319}{11^4}X + 1$
184	C_2	± 12	$X^2 - \frac{2578458526558}{11^{12}}X + 1$
204	$C_2 \times C_2$	$0, \pm 8$	$X^4 + (-\frac{124488737}{11^8} - \frac{27778920\sqrt{D}}{11^8})X^3 + (\frac{433699968}{11^8} + \frac{21811920\sqrt{D}}{11^8})X^2 + \cdots + 1$

205	C_4	± 3, ± 21	$X^4 + (\frac{-1505584466164962468204}{11^{20}} + \frac{16233247244926883840\sqrt{D}}{11^{21}})X^3 + (\frac{28219622414347276747833446}{11^{24}} + \frac{-473808391120495161344 00\sqrt{D}}{11^{23}})X^2 + \cdots + 1$
249	C_2	± 3	$X^2 + (\frac{587}{2\cdot 11^3} + \frac{285\sqrt{D}}{2\cdot 11^3})X + 1$
305	C_4	$\pm 1, \pm 7$	$X^4 + (\frac{11415043}{2\cdot 11^7} + \frac{-8104275\sqrt{D}}{2\cdot 11^7})X^3 + (\frac{2259394747}{2\cdot 11^8} + \frac{-1293525\sqrt{D}}{2\cdot 11^7})X^2 + (\frac{11415043}{2\cdot 11^7} + \frac{-8104275\sqrt{D}}{2\cdot 11^7})X + 1$
316	C_6	$0, \pm 4$, ± 24	$X^6 + (\frac{-29577474141849264729900 10}{11^{24}} + \frac{-765856195179857629200384\sqrt{D}}{11^{24}})X^5 + (\frac{39757085820713117025451167679}{11^{28}} + \frac{56769634854629248749200670 72\sqrt{D}}{11^{28}})X^4 + (\frac{-11081892536966641848569549662 0}{11^{28}} + \frac{40571816109864133660731310 08\sqrt{D}}{11^{28}})X^3 + (\frac{39757085820713117025451167679}{11^{28}} + \cdots + 1$
380	$C_2 \times C_2$	$\pm 5, \pm 11$	$X^4 - \frac{231913275084}{11^{11}}X^3 - \frac{14958851326111354}{11^{16}}X^2 + \cdots + 1$
393	C_2	± 5	$X^2 - \frac{319798}{11^5}X + 1$
417	C_2	± 3	$X^2 + (\frac{-813}{2\cdot 11^3} + \frac{215\sqrt{D}}{2\cdot 11^3})X + 1$
428	C_2	± 12	$X^2 + \frac{3447805043279}{11^{12}}X + 1$
492	$C_2 \times C_2$	$\pm 4, \pm 12$	$X^4 + (\frac{3205064080399}{11^{12}} + \frac{37751088600\sqrt{D}}{11^{12}})X^3 + (\frac{5106707319514560}{11^{16}} + \frac{1058364259390800\sqrt{D}}{11^{16}})X^2 + \cdots + 1$
501	C_2	± 33	$X^2 - \frac{44946948494493582973059229141165562}{11^{33}}X + 1$
505	C_8	$\pm 1, \pm 7$, $\pm 9, \pm 15$	$X^8 + (\frac{-1625141472569785}{11^{15}} + \frac{-403370661924110\sqrt{D}}{11^{15}})X^7 + (\frac{37324030324256318933106581}{11^{24}} + \frac{15586593908923025056050\sqrt{D}}{11^{23}})X^6 + (\frac{-15269248395590120406323822429 8440}{11^{31}} + \frac{-42448879943056999806516861383440\sqrt{D}}{11^{31}})X^5 + (\frac{11418699930460911608809034099164311}{11^{32}} + \frac{76338473101440899600877017010 00\sqrt{D}}{11^{31}})X^4 + (\frac{-15269248395590120406323822429 8440}{11^{31}} + \cdots + 1$
508	C_2	± 20	$X^2 - \frac{117102781407955941602}{11^{20}}X + 1$
536	C_2	± 4	$X^2 + \frac{27166}{11^4}X + 1$
568	C_6	$\pm 4, \pm 6$, ± 18	$X^6 + (\frac{5960088522346210594}{11^{18}} + \frac{607038060236443008\sqrt{D}}{11^{18}})X^5 + (\frac{46923359255440205349587231}{11^{24}} + \frac{94951852274066589403392\sqrt{D}}{11^{23}})X^4 + (\frac{42028572177272858023772404 3740}{11^{28}} + \frac{15343687713947095296\sqrt{D}}{11^{19}})X^3 + \cdots + 1$
589	C_2	± 27	$X^2 - \frac{13743563243985537951181272858}{11^{27}} + 1$
604	C_2	± 28	$X^2 - \frac{28663370302496179239763975 0562}{11^{28}}X + 1$
645	$C_2 \times C_2$	$\pm 9, \pm 33$	$X^4 + \frac{40583064292578727911909826073398796}{11^{33}}X^3 + \frac{11465633896257459153114781161904932751152 6486}{11^{42}}X^2 + \cdots + 1$
681	C_2	± 5	$X^2 + (\frac{-315541}{2\cdot 11^5} + \frac{-4293\sqrt{D}}{2\cdot 11^5})X + 1$

732	$C_2 \times C_2$	$\pm 5, \pm 11$	$X^4 + (\frac{-220455763632}{11^{11}} + \frac{-14999401040\sqrt{D}}{11^{11}})X^3 + (\frac{64386835000389689}{11^{16}} + \frac{64770\sqrt{D}}{11^6})X^2 + \cdots + 1$
744	$C_2 \times C_2$	$\pm 5, \pm 13$	$X^4 + \frac{97786155456260}{11^{13}}X^3 + \frac{20604315955553085174}{11^{18}}X^2 + \cdots + 1$
813	C_2	± 11	$X^2 + (\frac{255394155339}{11^{11}} + \frac{-7726070020\sqrt{D}}{11^{11}})X + 1$
844	C_2	± 12	$X^2 - \frac{4512049161167}{11^{12}}X + 1$
849	C_2	± 3	$X^2 + (\frac{2379}{2 \cdot 11^3} + \frac{-71\sqrt{D}}{2 \cdot 11^3})X + 1$
876	$C_4 \times C_2$	$\pm 2, \pm 6, \pm 22, \pm 34$	$X^8 + (\frac{-3335016613464780443495691152324360 76}{11^{34}} + \frac{-790812215916973121588768378973984\sqrt{D}}{11^{33}})X^7 + (\frac{5191579916897227569013843555050200501612306633222475801713 0}{11^{56}} + \frac{22599582935318965754811789665229802918155799168145696803 2\sqrt{D}}{11^{56}})X^6 + (\frac{-8209306899677791937622239368259148455270555161001976438675614904 0}{11^{62}} + \frac{-21608296378806983856666310537676610823101515480991115205048745 60\sqrt{D}}{11^{62}})X^5 + (\frac{1746276808549746411361322727889718359375305627921454904777879766197 1}{11^{64}} + \frac{15470217538651689196893338257336028035520402883952525718114488224\sqrt{D}}{11^{63}})X^4 + \cdots + 1$
893	C_2	± 45	$X^2 + \frac{1855744600011063825688347943145422878152471890 2}{11^{45}}X + 1$
897	$C_4 \times C_2$	$\pm 7, \pm 9 \pm 11, \pm 21$	$X^8 + (\frac{-28048924735800574227289}{2 \cdot 11^{21}} + \frac{-270821080206654213\sqrt{D}}{2 \cdot 11^{20}})X^7 + (\frac{5523541886478606305486136264317668}{11^{32}} + \frac{15825428394893871666843986516 88\sqrt{D}}{11^{31}})X^6 + (\frac{-2357949394135219212872918516209119 2547790319}{2 \cdot 11^{41}} + \frac{-42588424899656588380103298704116424701721 7\sqrt{D}}{2 \cdot 11^{41}})X^5 + (\frac{4307307686451966507193894135650538042609314929396 69}{2 \cdot 11^{48}} + \frac{2425377175406668459773597524580716 7\sqrt{D}}{2 \cdot 11^{34}})X^4 + \cdots + 1$
956	C_2	± 60	$X^2 + \frac{6054654665698091298544836538641246249432997875838267892471831 98}{11^{60}}X + 1$
989	C_2	± 36	$X^2 + \frac{6149132552037776967568952791794168478}{11^{36}}X + 1$

The tables below are truncated and include only fundamental discriminants not covered by previous tables.

Table 7: $p = 3, \ell = 2$

D	G	ord_p	$P(X)$
140	$C_2 \times C_2$	$\pm 2, \pm 10$	$X^4 - \frac{123556}{3^{10}}X^3 + \frac{1570726}{3^{12}}X^2 + \ldots + 1$
152	C_2	± 6	$X^2 - \frac{658}{3^6}X + 1$
188	C_2	± 10	$X^2 + \frac{29294}{3^{10}}X + 1$

248	C_2	± 6	$X^2 - \frac{658}{3^6}X + 1$
332	C_2	± 18	$X^2 - \frac{748812178}{3^{18}}X + 1$
377	C_4	$\pm 4, \pm 8$	$X^4 + (\frac{1477}{2\cdot 3^8} + \frac{-595\sqrt{D}}{2\cdot 3^7})X^3 + (\frac{3663335}{2\cdot 3^{12}} + \frac{-4165\sqrt{D}}{2\cdot 3^{11}})X^2 + \ldots + 1$
440	$C_2 \times C_2$	$\pm 2, \pm 10$	$X^4 + \frac{46172}{3^{10}}X^3 + \frac{874918}{3^{12}}X^2 + \ldots + 1$
497	C_2	± 14	$X^2 - \frac{7162162}{3^{14}}X + 1$
545	C_4	$\pm 4, \pm 8$	$X^4 + (\frac{-23051}{2\cdot 3^8} + \frac{-133\sqrt{D}}{2\cdot 3^6})X^3 + (\frac{3651671}{2\cdot 3^{12}} + \frac{11837\sqrt{D}}{2\cdot 3^{10}})X^2 + \ldots + 1$
572	$C_2 \times C_2$	$\pm 2, \pm 22$	$X^4 + \frac{86489479964}{3^{22}}X^3 + \frac{1038192047686}{3^{24}}X^2 + \ldots + 1$
665	$C_2 \times C_2$	$\pm 2, \pm 14$	$X^4 + \frac{5506844}{3^{14}}X^3 + \frac{13029766}{3^{16}}X^2 + \ldots + 1$
668	C_2	± 22	$X^2 + \frac{62761269134}{3^{22}}X + 1$
713	C_2	± 18	$X^2 - \frac{22317778}{3^{18}}X + 1$
728	$C_2 \times C_2$	$0, \pm 12$	$X^4 + \frac{-685636}{3^{12}}X^3 + \frac{455846}{3^{12}}X^2 + \ldots + 1$
812	$C_2 \times C_2$	$0, \pm 12$	$X^4 + \frac{1935164}{3^{12}}X^3 + \frac{2817446}{3^{12}}X^2 + \ldots + 1$
860	$C_2 \times C_2$	$\pm 2, \pm 26$	$X^4 + \frac{5731461505244}{3^{26}}X^3 + \frac{57823095669286}{3^{28}}X^2 + \ldots + 1$
908	C_2	± 30	$X^2 + \frac{411625911957902}{3^{30}}X + 1$
920	$C_2 \times C_2$	$0, \pm 12$	$X^4 + \frac{9212}{3^{12}}X^3 + \frac{-629722}{3^{12}}X^2 + \ldots + 1$

Table 8: $p = 5, \ell = 2$

D	G	ord_p	$P(X)$
12	C_2	± 2	$X^2 + \frac{14}{5^2}X + 1$
28	C_2	± 2	$X^2 + \frac{14}{5^2}X + 1$
33	C_2	± 2	$X^2 + (\frac{-49}{2\cdot 5^2} + \frac{-3\sqrt{D}}{2\cdot 5^2})X + 1$
168	$C_2 \times C_2$	$\pm 2, \pm 6$	$X^4 - \frac{35236}{5^6}X^3 + \frac{1128006}{5^8}X^2 + \ldots + 1$
217	C_2	± 6	$X^2 - \frac{29266}{5^6}X + 1$
273	$C_2 \times C_2$	$\pm 2, \pm 6$	$X^4 + (\frac{25543}{5^6} + \frac{1869\sqrt{D}}{5^6})X^3 + (\frac{2835657}{2\cdot 5^8} + \frac{123\sqrt{D}}{2\cdot 5^4})X^2 + \ldots + 1$
348	$C_2 \times C_2$	$\pm 6, \pm 18$	$X^4 + \frac{6503995819804}{5^{18}}X^3 + \frac{92246424230059206}{5^{24}}X^2 + \ldots + 1$
412	C_2	± 10	$X^2 + \frac{19306574}{5^{10}}X + 1$
472	C_2	± 18	$X^2 + \frac{3688161086734}{5^{18}}X + 1$
537	C_2	± 10	$X^2 + (\frac{17325791}{2\cdot 5^{10}} + \frac{673881\sqrt{D}}{2\cdot 5^{10}})X + 1$
552	$C_2 \times C_2$	$\pm 2, \pm 10$	$X^4 + \frac{5698204}{5^{10}}X^3 - \frac{200259354}{5^{12}}X^2 + \ldots + 1$

553	C_2	± 10	$X^2 - \frac{5328946}{5^{10}}X + 1$
633	C_2	± 6	$X^2 + (\frac{-29729}{2\cdot 5^6} + \frac{-663\sqrt{D}}{2\cdot 5^6})X + 1$
712	C_4	$0, \pm 12$	$X^4 - \frac{318425156}{5^{12}}X^3 + \frac{545061862}{5^{12}}X^2 + \ldots + 1$
748	$C_2 \times C_2$	$\pm 18, \pm 30$	$X^4 + \frac{2073864202390737978844}{5^{30}}X^3 + \frac{87189780754611656469128573880019 26}{5^{48}}X^2 + \ldots + 1$
777	$C_4 \times C_2$	$\pm 2, \pm 2, \pm 6, \pm 14$	$X^8 + (\frac{32373499271}{2\cdot 5^{14}} + \frac{-803900451\sqrt{D}}{2\cdot 5^{14}})X^7 + (\frac{451241249413462}{5^{20}} + \frac{-17567029323522\sqrt{D}}{5^{20}})X^6 + (\frac{32064555133173737}{2\cdot 5^{22}} + \frac{-260313640519401\sqrt{D}}{2\cdot 5^{21}})X^5 + (\frac{916359669758100389}{2\cdot 5^{24}} + \frac{-35216524712299341\sqrt{D}}{2\cdot 5^{24}})X^4 + \ldots + 1$
817	C_{10}	$\pm 2, \pm 2, \pm 6, \pm 10, \pm 14$	$X^{10} + (\frac{-31172501257}{2\cdot 5^{14}} + \frac{56382809\sqrt{D}}{2\cdot 5^{13}})X^9 + (\frac{184775991679157711}{5^{24}} + \frac{-107439544944354\sqrt{D}}{5^{22}})X^8 + (\frac{-3055388213159228888567}{2\cdot 5^{30}} + \frac{16124426915227153971\sqrt{D}}{2\cdot 5^{29}})X^7 + (\frac{22495680668232664456339}{2\cdot 5^{32}} + \frac{-9033979473758246009 1\sqrt{D}}{2\cdot 5^{31}})X^6 + (\frac{24942854300292502858053}{2\cdot 5^{34}} + \frac{-78805833248560277741\sqrt{D}}{2\cdot 5^{32}})X^5 + \ldots + 1$
888	$C_2 \times C_2$	$\pm 2, \pm 18$	$X^4 - \frac{5801402184676}{5^{18}}X^3 + \frac{105912131880486}{5^{20}}X^2 + \ldots + 1$
892	C_6	$\pm 6, \pm 6, \pm 26$	$X^6 - \frac{6512538557172802966}{5^{26}}X^5 + \frac{216005849710927156938127}{5^{32}}X^4 - \frac{42608848962657975061818 14836}{5^{38}}X^3 + \ldots + 1$
913	C_2	± 18	$X^2 - \frac{1755797021426}{5^{18}}X + 1$
952	$C_2 \times C_2$	$\pm 6, \pm 14$	$X^4 + \frac{7359212764}{5^{14}}X^3 + \frac{81769663095846}{5^{20}}X^2 + \ldots + 1$
993	C_6	$\pm 2, \pm 6, \pm 10$	$X^6 + (\frac{13329309}{5^{10}} + \frac{-1479\sqrt{D}}{5^9})X^5 + (\frac{318359314959}{2\cdot 5^{16}} + \frac{17101049343\sqrt{D}}{2\cdot 5^{16}})X^4 + (\frac{72049234621}{2\cdot 5^{18}} + \frac{558417325707\sqrt{D}}{2\cdot 5^{18}})X^3 + \ldots + 1$

Table 9: $p = 7, \ell = 2$

D	G	ord_p	$P(X)$
220	$C_2 \times C_2$	$\pm 4, \pm 8$	$X^4 + \frac{3063684}{7^8}X^3 + \frac{8933236166}{7^{12}}X^2 + \ldots + 1$
460	$C_2 \times C_2$	$\pm 6, \pm 18$	$X^4 + \frac{3072421354263484}{7^{18}}X^3 + \frac{4913472611831619 51366}{7^{24}}X^2 + \ldots + 1$
696	$C_2 \times C_2$	$\pm 4, \pm 8$	$X^4 - \frac{11379076}{7^8}X^3 + \frac{37909278150}{7^{12}}X^2 + \ldots + 1$
705	$C_2 \times C_2$	$\pm 8, \pm 12$	$X^4 + \frac{42681977084}{7^{12}}X^3 + \frac{340235382048949830}{7^{20}}X^2 + \ldots + 1$
745	C_4	$\pm 4, \pm 8$	$X^4 + (\frac{14767749}{2\cdot 7^8} + \frac{-48983\sqrt{D}}{2\cdot 7^7})X^3 + (\frac{56988487831}{2\cdot 7^{12}} + \frac{-62257393\sqrt{D}}{2\cdot 7^{11}})X^2 + \ldots + 1$

780	$(C_2)^3$	$\pm 2, \pm 2,$ $\pm 6, \pm 26$	$X^8 + \frac{27992629546888641417592}{7^{26}} X^7 + \frac{26473752991767777257939465756}{7^{32}} X^6 - \frac{1263652910044993116491925400 88}{7^{34}} X^5 - \frac{144689532372410668342461449957 06}{7^{36}} X^4 + \ldots + 1$

Table 10: $p = 13, \ell = 2$

D	G	ord_p	$P(X)$
60	$C_2 \times C_2$	$\pm 4, \pm 8$	$X^4 + \frac{-3063684}{13^8} X^3 + \frac{8933236166}{13^{12}} X^2 + \ldots + 1$
280	$C_2 \times C_2$	$\pm 4, \pm 8$	$X^4 - \frac{591479684}{13^8} X^3 + \frac{31637621198886}{13^{12}} X^2 + \ldots + 1$
345	$C_2 \times C_2$	$\pm 4, \pm 8$	$X^4 + \frac{1238502524}{13^8} X^3 + \frac{42751436419110}{13^{12}} X^2 + \ldots + 1$
385	$C_2 \times C_2$	$\pm 2, \pm 10$	$X^4 - \frac{60475859876}{13^{10}} X^3 - \frac{15528772781274}{13^{12}} X^2 + \ldots + 1$
561	$C_2 \times C_2$	$\pm 4, \pm 8$	$X^4 + \frac{138863228}{13^8} X^3 - \frac{28303428746202}{13^{12}} X^2 + \ldots + 1$
609	$C_2 \times C_2$	$\pm 2, \pm 10$	$X^4 - \frac{22973630884}{13^{10}} X^3 - \frac{9567310846170}{13^{12}} X^2 + \ldots + 1$
616	$C_2 \times C_2$	$\pm 10, \pm 14$	$X^4 - \frac{506351797168868}{13^{14}} X^3 + \frac{22624660815186234349430 6118}{13^{24}} X^2 + \ldots + 1$
760	$C_2 \times C_2$	$\pm 2, \pm 14$	$X^4 + \frac{1896790671611164}{13^{14}} X^3 + \frac{42731543174244486}{13^{16}} X^2 + \ldots + 1$
840	$(C_2)^3$	$\pm 2, \pm 2,$ $\pm 6, \pm 14,$	$X^8 + \frac{17662345129411384}{13^{14}} X^7 + \frac{20717066902623474880220}{13^{20}} X^6 + \frac{56559203940176692042971016}{13^{22}} X^5 + \frac{11140418382648156751583091334}{13^{24}} X^4 + \ldots + 1$
856	C_2	± 18	$X^2 - \frac{35712428192685597458}{13^{18}} X + 1$
889	C_2	± 10	$X^2 - \frac{270022758866}{13^{10}} X + 1$

Table 11: $p = 17, \ell = 2$

D	G	ord_p	$P(X)$
364	$C_2 \times C_2$	$\pm 2, \pm 10$	$X^4 - \frac{4848971065988}{17^{10}} X^3 + \frac{1949513874602118}{17^{12}} X^2 + \ldots + 1$
456	$C_2 \times C_2$	$0, \pm 12$	$X^4 - \frac{1631585980592132}{17^{12}} X^3 + \frac{126954620655654}{17^{11}} X^2 + \ldots + 1$
465	$C_2 \times C_2$	$\pm 2, \pm 10$	$X^4 + (\frac{2641496138183}{17^{10}} + \frac{163985532045\sqrt{D}}{17^{10}}) X^3 + (\frac{2901827204337561}{2 \cdot 17^{12}} + \frac{1084430235\sqrt{D}}{2 \cdot 17^8}) X^2 + \ldots + 1$
924	$(C_2)^3$	$\pm 2, \pm 6,$ $\pm 10,$ ± 30	$X^8 + \frac{12655204070358125288445178967203131128}{17^{30}} X^7 + \frac{42825047009391976186283995167057791743166432875036}{17^{40}} X^6 + \frac{67866890877792614646811008062758226725226805304023190 7528}{17^{46}} X^5 + \frac{25684048298852310363239903881344224384623297816129370990 4454}{17^{48}} X^4 + \ldots + 1$

940	$C_6 \times C_2$	$\pm 2, \pm 2,$ $\pm 6, \pm 10,$ ± 10 ± 30	$X^{12} + \frac{10008885281772595156559693918447252340}{17^{30}} X^{11} +$ $\frac{6165587658501679892286160818200142066156876506349 0}{17^{40}} X^{10} +$ $\frac{11908748284361335397527287156413143144059201601884796017810883 6}{17^{50}} X^9 +$ $\frac{48241865325845778936589455314477429548368295551523092732762575071115 35}{17^{56}} X^8 +$ $\frac{12134392367868579254000216714213693359183095936693595232735732326217507 60}{17^{58}} X^7 +$ $\frac{44778031284162829120955263104058075476099577343116562525114925505831245327 6}{17^{60}} X^6 + \ldots + 1$

Table 12: $p = 19, \ell = 2$

D	G	ord_p	$P(X)$
984	$C_2 \times C_2$	$\pm 2, \pm 10$	$X^4 - \frac{22732562940388}{19^{10}} X^3 + \frac{12015402255844518}{19^{12}} X^2 + \ldots + 1$

Table 13: $p = 5, \ell = 3$

D	G	ord_p	$P(X)$
133	C_2	± 18	$X^2 + \frac{1994534557454}{5^{18}} X + 1$
213	C_2	± 21	$X^2 - \frac{891875426652794}{5^{21}} X + 1$
237	C_2	± 30	$X^2 - \frac{1373294054294170159214}{5^{30}} X + 1$
357	$C_2 \times C_2$	$\pm 3, \pm 33$	$X^4 - \frac{130199308302851104250 12}{5^{33}} X^3 - \frac{25818387357991987132526874}{5^{36}} X^2 + \ldots + 1$
453	C_2	± 42	$X^2 - \frac{45393954624965595184240961 7614}{5^{42}} X + 1$
517	C_2	± 45	$X^2 - \frac{55654850583892508010923658394874}{5^{45}} X + 1$
573	C_2	± 13	$X^2 + \frac{487589446}{5^{13}} X + 1$
717	C_2	± 45	$X^2 - \frac{32054707606024927097813270076986}{5^{45}} X + 1$
957	$C_2 \times C_2$	$\pm 3, \pm 57$	$X^4 - \frac{52618015142634231866728074445949949811 96}{5^{57}} X^3 -$ $\frac{41717752602266302275094480601656620461423 4}{5^{60}} X^2 + \ldots + 1$
973	C_2	± 18	$X^2 - \frac{7081619692786}{5^{18}} X + 1$

Table 14: $p = 13, \ell = 3$

D	G	ord_p	$P(X)$
141	C_2	± 5	$X^2 + \frac{141961}{13^5} X + 1$
301	C_2	± 18	$X^2 - \frac{217267564943832904942}{13^{18}} X + 1$
669	C_2	± 42	$X^2 + \frac{54902185250642228584380463415156011072699683 38}{13^{42}} X + 1$

Table 15: $p = 17, \ell = 3$

D	G	ord_p	$P(X)$
105	$C_2 \times C_2$	$\pm 5, \pm 7$	$X^4 - \frac{1159190842}{17^7}X^3 + \frac{2127975014339691}{17^{12}}X^2 + \ldots + 1$
165	$C_2 \times C_2$	$\pm 3, \pm 21$	$X^4 + \frac{16904152775988245568375652}{17^{21}}X^3 - \frac{131904599303155106682692683194}{17^{24}}X^2 + \ldots + 1$
309	C_2	± 30	$X^2 - \frac{14444108071492462834146898407805570498}{17^{30}}X + 1$
861	$C_2 \times C_2$	$\pm 3, \pm 39$	$X^4 - \frac{252023691729111024673545357410997984486939636 2812}{17^{39}}X^3 + \frac{1526325248023816553737631567335879306864068840555 0534}{17^{42}}X^2 + \ldots + 1$

Table 16: $p = 19, \ell = 3$

D	G	ord_p	$P(X)$
781	C_2	± 21	$X^2 - \frac{722681759644271384279219062}{19^{21}}X + 1$

Table 17: $p = 23, \ell = 3$

D	G	ord_p	$P(X)$
429	$C_2 \times C_2$	$\pm 3, \pm 33$	$X^4 - \frac{209328915737906714522535974919090863406536 2268}{23^{33}}X^3 + \frac{359130246830070154424560542609376080194900379081 66}{23^{36}}X^2 + \ldots + 1$

Table 18: $p = 3, \ell = 19$

D	G	ord_p	$P(X)$
437	C_2	± 90	$X^2 + \frac{250558229260119840879093404276937338380 2898}{3^{90}}X + 1$

References

[1] Samit Dasgupta and Mahesh Kakde, *The Integral Gross–Stark Conjecture, an Exact Formula for Brumer-Stark Units, and Hilbert's 12th Problem*, preprint.

[2] https://github.com/liuyj8526/Computation-of-Elliptic-Units.