Functions between sets

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Let N and R be sets with |N| = n and |R| = r.

(i) **Total Functions:** The number of functions from N to R is

$$r^n$$
.

Explanation: For every element in N, there are |R| = r possible values in R. Thus, for the first element, there are r choices, for the second element, there are r choices, and so on. Applying the rule of product, the total number of functions is r^n .

(ii) **Injective Functions:** When $r \ge n$, an injective function (one-to-one) from N to R can be chosen by assigning distinct images to the n elements.

If a function is injective, then for each value in the range there is only one corresponding argument. This means that function values cannot repeat, ensuring that $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

Since there are |R| = r choices for the first argument, r - 1 choices for the second, r - 2 for the third, and so on, applying the rule of product, the number of injective functions from N to R is:

$$r \cdot (r-1) \cdots (r-n+1) = \frac{r!}{(r-n)!}.$$

(iii) Surjective Functions: A function is surjective (onto) if every element in R has a pre-image in N, meaning every element in R is an image of some element in N. Consider a surjection $f: N \to R = \{y_1, y_2, \ldots, y_r\}$. We observe that the preimages $f^{-1}(y_1), f^{-1}(y_2), \ldots, f^{-1}(y_r)$ form a partition of N into r non-empty subsets, as each element y_i in R corresponds to one or more elements from N. The number of ways to partition N into r parts is given by the Stirling number S(n, r), and since we can permute the r elements in R in r! ways, the total number of surjective functions from N to R is:

where S(n,r) is the Stirling number of the second kind, counting the ways to partition N into r non-empty subsets.

Example: For $N = \{1, 2, 3\}$ and $R = \{y_1, y_2\}$:

Here |N| = 3 and |R| = 2.

- Total functions: $2^3 = 8$.
- Injective functions: Not possible since |R| < |N|.
- Surjective functions: Consider all possible surjective functions:

 $f_1:\{1,2\}\mapsto y_1,3\mapsto y_2$ - Another possible permutation for this partition: $f_2:\{1,2\}\mapsto y_2,3\mapsto y_1$ $f_3:\{2,3\}\mapsto y_1,1\mapsto y_2$ - Another possible permutation for this partition: $f_4:1\mapsto y_1,\{2,3\}\mapsto y_2$ $f_5:\{1,3\}\mapsto y_1,2\mapsto y_2$ - Another possible permutation for this partition: $f_6:2\mapsto y_1,\{1,3\}\mapsto y_2$ So, we have 6 surjective functions. Using the formula for surjective functions, we first find the Stirling number S(3,2)=3, which corresponds to the number of partitions without considering permutations. Then, accounting for the permutations of the r=2 elements in R, we compute:

$$2! \cdot S(3,2) = 2! \cdot 3 = 6,$$

which matches the number of surjective functions we listed.

