# Preparation Tasks 2

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## 1   Problem 1

Prove that the num. of partitions of a positive int $n$ into $k$ *even* parts is eq. to the num. of partitions of $n$ into $k$ odd parts.

- **Idea:** First, express $n$ in two ways by decomposing it into $k$ even parts and into $k$ odd parts. We can write

$$n = 2a_1 + 2a_2 + \cdots + 2a_k,$$
$$n - k = 2a_1 + 2a_2 + \cdots + 2a_k - k,$$
$$n - k = (2a_1 - 1) + (2a_2 - 1) + \cdots + (2a_k - 1).$$

  The first line expresses $n$ as a sum of $k$ even numbers (each $2a_i$). Subtracting $k$ from both sides shows that $n - k$ can be expressed as a sum of $k$ odd numbers (each $2a_i - 1$).

  From this idea, we expect a correspondence between partitions of $n$ into $k$ even parts and partitions of $n - k$ into $k$ odd parts.

- **Formal proof:** Define the sets

$$P_e(n, k) := \{\text{partitions of } n \text{ into } k \text{ even parts}\},$$

  and

$$P_o(n, k) := \{\text{partitions of } n \text{ into } k \text{ odd parts}\}.$$

  For $n$ even, it turns out these two sets have the same size:

$$|P_e(n, k)| = |P_o(n - k, k)|.$$

  To show this, we construct an explicit bijection. Define a function

$$f : P_e(n, k) \to P_o(n - k, k),$$

  by

$$f(b_1, b_2, \ldots, b_k) = (b_1 - 1,\ b_2 - 1,\ \ldots,\ b_k - 1).$$

  Here $(b_1, \ldots, b_k) \in P_e(n, k)$ means $b_1 + \cdots + b_k = n$ with each $b_i$ even, so $b_i \geq 2$ for all $i$. Thus, each $b_i - 1 \geq 1$ and is odd, which ensures $f(b_1, \ldots, b_k) \in P_o(n - k, k)$ is a partition of $n - k$ into $k$ odd parts.

  The function $f$ is well-defined. Moreover, it is invertible by simply adding 1 to each part. In fact, define

$$g : P_o(n - k, k) \to P_e(n, k)$$

  as

$$g(c_1, c_2, \ldots, c_k) = (c_1 + 1,\ c_2 + 1,\ \ldots,\ c_k + 1).$$

  If $(c_1, \ldots, c_k)$ is a partition of $n - k$ into odd parts, then each $c_i$ is odd and $c_i \geq 1$, so $c_i + 1$ is even and $\geq 2$, and $\sum_{i=1}^{k}(c_i + 1) = (n - k) + k = n$. Thus $g(c_1, \ldots, c_k) \in P_e(n, k)$. It is easy to check that $g$ is indeed the inverse of $f$: we have $f(g(c_1, \ldots, c_k)) = (c_1, \ldots, c_k)$ and $g(f(b_1, \ldots, b_k)) = (b_1, \ldots, b_k)$. Therefore, $f$ is bijective.

# 2 Problem 2

Show that the number of partitions of a positive int $n$ with at most $k$ components is eq. to the num. of partitions of $2n$ with at most $k$ even components.

- **Idea:** We consider partitions of $n$ that have at most $k$ parts. Let

$$n = a_1 + a_2 + \cdots + a_L,$$

where $L \leq k$ and $a_1 \geq a_2 \geq \cdots \geq a_L > 0$. (In other words, $a_1, \ldots, a_L$ are the parts of a partition of $n$, listed in non-increasing order, with at most $k$ parts.) For example, if $n = 5$ and $k = 3$, the partitions of 5 with at most 3 parts can be represented (padding with zeros up to 3 parts) as:

$$(2, 2, 1), \qquad (3, 2, 0), \qquad (5, 0, 0),$$

where we use 0 to indicate an empty part (no number in that position).

Notice that doubling each part in these examples produces a partition of $2n = 10$ with only even parts (and still at most 3 components). For instance, $(2, 2, 1)$ doubles to $(4, 4, 2)$, $(3, 2, 0)$ doubles to $(6, 4, 0)$, and $(5, 0, 0)$ doubles to $(10, 0, 0)$. This suggests a direct correspondence between partitions of $n$ (up to $k$ parts) and partitions of $2n$ into even parts (up to $k$ parts).

- **Formal proof:** Let us define the relevant sets in words (as suggested in the notes):

  - $P(n, \leq k)$ — the set of all partitions of $n$ with *at most* $k$ components (parts).
  - $P_e(2n, \leq k)$ — the set of all partitions of $2n$ with at most $k$ *even* components.

  We aim to show that

$$|P(n, \leq k)| = |P_e(2n, \leq k)|,$$

  i.e. the two sets have equal cardinality. To prove this, we construct a bijection $f : P(n, \leq k) \to P_e(2n, \leq k)$. Given any partition $(a_1, a_2, \ldots, a_L)$ of $n$ (with $L \leq k$), map it to

$$f(a_1, a_2, \ldots, a_L) = (2a_1, 2a_2, \ldots, 2a_L).$$

  In other words, $f$ doubles each part of the partition of $n$. If the partition of $n$ has fewer than $k$ parts, we may imagine that it is padded with zeros (as above) which double to zeros, so the resulting partition of $2n$ still has at most $k$ parts. By construction, $f(a_1, \ldots, a_L)$ is a partition of $2n$ in which every part is even, so indeed $f(a_1, \ldots, a_L) \in P_e(2n, \leq k)$. The function $f$ is invertible by halving each even part: for any partition $(b_1, b_2, \ldots, b_M) \in P_e(2n, \leq k)$ (each $b_i$ even), the inverse map $f^{-1}$ gives

$$f^{-1}(b_1, b_2, \ldots, b_M) = \left( \frac{b_1}{2}, \frac{b_2}{2}, \ldots, \frac{b_M}{2} \right),$$

  which is a partition of $2n/2 = n$ with at most $k$ parts. Thus $f$ is a bijection between $P(n, \leq k)$ and $P_e(2n, \leq k)$, and consequently $|P(n, \leq k)| = |P_e(2n, \leq k)|$.