Discrete Mathematics 1 Lectures, Part 1

Tomasz Brengos
Committers: Aliaksei Kudzelka, Mykhailo Moroz, Mihail Orlov

"There's no such thing as a lousy job — only lousy men who don't care to do it."

Ayn Rand

Preface

These lecture notes have been compiled from official course materials and supplemented with contributions from students and educators. They aim to provide a clear and comprehensive introduction to Discrete Mathematics. Although every effort has been made to ensure accuracy, some errors may remain. Your feedback and contributions are highly valued.

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Contents

1 Counting (Combinatorics)

Counting forms the basis of combinatorics. In these lectures we explore several counting rules, examples, and proofs.

1.1 Rule of Sum (Addition Principle)

If a set S is partitioned into disjoint subsets,

$$S = S_1 \cup S_2 \cup \cdots \cup S_k,$$

then the total number of elements in S is the sum of the number of elements in each subset:

$$|S| = |S_1| + |S_2| + \cdots + |S_k|.$$

Example: Suppose we wish to count the number of ways to choose a subset of a set X of size u, but we only consider subsets of a fixed size k. If we let S be the family of all such subsets, then using the rule of sum by dividing the choices according to a distinguished element (say, whether a chosen element is included or not) we can count the subsets by summing over the possibilities. (This idea is used later in proofs for binomial coefficients and the power set.)

Theorem:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof: Consider $S = \binom{X}{k}$, the set of all subsets of X of size k. Take any element $a \in X$. Define: S_1 as the subsets in S that contain a and S_2 as the subsets in S that do not contain a.

Since every subset of S either contains a or does not, we see that S_1 and S_2 are disjoint and their union forms S, i.e.,

$$S_1 \cup S_2 = S.$$

By the rule of sum, we get:

$$|S| = |S_1| + |S_2|.$$

Now, each subset in S_1 must contain a, so we choose the remaining $k-1$ elements from $X \setminus \{a\}$, which has $n-1$ elements. Thus, $|S_1| = \binom{n-1}{k-1}$. Each subset in S_2 does not contain a, so we choose all k elements from $X \setminus \{a\}$. Thus, $|S_2| = \binom{n-1}{k}$.

Therefore,

$$\binom{n}{k} = |S| = |S_1| + |S_2| = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Example: Let $S = \{\triangle, \square, \circ\}$ and $k = 2$, choosing $a = \circ$. Fixing $\circ$ as one of the elements in the subset of size k, we get:

$$S_1 = \{\{\circ, \triangle\}, \{\circ, \square\}\}.$$

Taking all subsets of size k without $\circ$:

$$S_2 = \{\{\triangle, \square\}\}.$$

We have $|S_1| = 2$ and $|S_2| = 1$, so $|S_1| + |S_2| = 3$.
On the other hand,

$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = 3.$$

Thus, $|S| = |S_1| + |S_2| = 3$, verifying the identity.

1.2 Rule of Product (Multiplication Principle)

When an object is constructed by a sequence of choices, where:

- The first choice can be made in a ways,
- The second in b ways,
- ...

the total number of objects is the product:

$$a \times b \times \cdots .$$

Example: A word of length n over the binary alphabet $\{0, 1\}$ is formed by choosing one of 2 possibilities for each position. Hence, there are

$$2^n$$

possible words.

1.3 Rule of Bijection

If there exists a bijection (a one-to-one and onto mapping) between two sets S and T, then they have the same number of elements:

$$|S| = |T|.$$

Example: Consider the power set of a set X, denoted by $\mathcal{P}(X)$. There is a natural bijection between $\mathcal{P}(X)$ and the set of binary sequences of length $|X|$: for each subset $A \subseteq X$, assign the sequence $(a_1, a_2, \ldots, a_n)$ where

$$a_i = \begin{cases} 1, & \text{if } x_i \in A, \\ 0, & \text{if } x_i \notin A. \end{cases}$$

This shows that

$$|\mathcal{P}(X)| = 2^{|X|}.$$

1.4 Counting in Two Ways

Rule of Counting in Two Ways When two formulae enumerate the same quantity, they must be equal.

Example:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Proof: Consider a lattice grid of size $(n + 1) \times (n + 1)$, defined as:

$$X = \{(i, j) \mid i, j \in \{1, 2, \ldots, n + 1\}\}.$$

Clearly, $|X| = (n + 1)^2$.

Now, partition X into three subsets: - X_1, the points strictly below the secondary diagonal. - X_2, the points strictly above the secondary diagonal. - X_3, the points on the secondary diagonal itself.

Since these three sets form a partition, we have:

$$|X| = |X_1| + |X_2| + |X_3|.$$

Observing their sizes:

$$|X_1| = |X_2| = 1 + 2 + \cdots + n, \quad |X_3| = n + 1.$$

Thus,

$$(n+1)^2 = 2(1+2+\cdots+n)+(n+1).$$

Rearranging, we get:

$$1+2+\cdots+n = \frac{(n+1)^2-(n+1)}{2} = \frac{n(n+1)}{2}.$$

Hence, we have proven the formula:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

1.5 Binomial Coefficients and Permutations

Let X be a set with $|X| = n$.

Subsets: The number of ways to choose a k-subset of X is given by the binomial coefficient

$$\binom{n}{k}.$$

Permutations: A k-permutation of a set X of size n is a k-word over the alphabet X whose entries are distinct.

Theorem: There are exactly

$$n(n-1)(n-2)\dots(n-k+1)$$

k-permutations of an n-set.

Question: How are k-permutations of an n-set related to k-subsets of an n-set?

Answer: The difference between a k-permutation and a k-subset is that a permutation is ordered, while a subset is not. To express a k-permutation in terms of a k-subset, we need to account for all possible arrangements of the elements, which is $k!$. Thus,

$$k\text{-permutation} = \binom{n}{k} \cdot k!$$

Expressing $\binom{n}{k}$ as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

we obtain:

$$k\text{-permutation} = \frac{n!}{(n-k)!}.$$

Proof by Counting in Two Ways: Count the number of k-permutations of an n-set in two ways:

(1) Directly, by applying the rule of product:

$$n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}.$$

(2) First choose a k-subset (in $\binom{n}{k}$ ways) and then arrange it (in $k!$ ways), giving

$$\binom{n}{k} \cdot k!.$$

Equate these two counts to obtain the relation.

1.6 Binomial Theorem

For any x, y in a field and nonnegative integer n, the binomial theorem states:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Explanation: This theorem is a direct consequence of counting the number of ways to choose k copies of x (and the remaining $n-k$ copies of y) when expanding the product.

1.7 Multisets

Definition: A multiset of a set X of size n is a function

$$m : X \to \mathbb{N}$$

that assigns a non-negative integer to each element of X, representing its multiplicity in the multiset.

Example: Let $X = \{a, b, c\}$, and consider the multiset $\{a, a, b\}$. Then, the function m is given by:

$$m(a) = 2, \quad m(b) = 1, \quad m(c) = 0.$$

Question: What is the number of k-multisets of a set of size n?

Theorem: The number of all k-multisets of an n-set is

$$\binom{n+k-1}{k}.$$

Proof: Let X be the set of all k-multisets of an n-set. Let Y be the set of all distributions of k identical objects into n buckets.

Claim 1: There is a bijection from X to Y. Thus, by the rule of bijection, we have

$$|X| = |Y|.$$

Claim 2: Let Z be the set of all binary sequences of length $n+k-1$ with exactly $n-1$ ones (or equivalently, k zeros). There is a bijection from Y to Z. Hence,

$$|Y| = |Z| \Rightarrow |X| = |Z|.$$

Since the number of such binary sequences is given by

$$\binom{n+k-1}{k},$$

we conclude that

$$|X| = \binom{n+k-1}{k}.$$

1.8 Lattice Paths

Consider an $m \times n$ grid with lattice points at the intersections.

Problem: How many paths are there from $(0,0)$ to (m,n) if one may only move right or up?

Solution: Every path consists of exactly m right moves and n up moves. Thus, a path can be represented as a sequence of $m+n$ moves, where we choose n positions (out of $m+n$) for the up moves. Hence, the number of paths is:

$$\binom{m+n}{n}.$$

Bijection Explanation: There is a bijection between the set of such lattice paths and the set of binary sequences of length $m+n$ with exactly n ones (representing the up moves).

2 Partitions and Stirling Numbers

2.1 Set Partitions

Definition: A set $\{A_1, A_2, \ldots, A_k\}$ of subsets of N forms a *partition of* the set N if:

$$A_i \neq \emptyset, \quad A_i \cap A_j = \emptyset \text{ for } i \neq j, \quad \text{and} \quad N = A_1 \cup A_2 \cup \cdots \cup A_k.$$

If a partition P of the set N is of size k then we say that P partitions N into k blocks.

Example: For $N = \{1,2,3,4\}$, one partition into 2 blocks could be

$$\{\{1,3\},\{2,4\}\}.$$

All possible partitions of a set N are denoted by $\Pi(N)$.

2.2 Stirling Numbers of the Second Kind

Question: How many k-partitions of an n-set are there?

Answer: Let $S(n,k)$ (or $\{n\ k\}$) denote the answer to our question, called the *Stirling number of the second kind*. We define the base cases as follows:

$$S(0,0) := 1, \quad S(0,k) := 0 \quad \text{for } k > 0.$$

Theorem: The total number of set partitions of N is given by

$$|\Pi(N)| = \sum_{k=0}^{|N|} S(|N|, k).$$

Remark: The quantity

$$B(|N|) := |\Pi(N)| = \sum_{k=0}^{|N|} S(|N|, k)$$

is called the *Bell number*.

Example: Let $N = [5] = \{1, 2, 3, 4, 5\}$. List all possible 2-partitions of N.

Firstly, consider cases where the first subset contains only one element:

$$1|2\,3\,4\,5, \quad 2|1\,3\,4\,5, \quad 3|1\,2\,4\,5, \quad 4|1\,2\,3\,5, \quad 5|1\,2\,3\,4.$$

Now, consider cases where the first subset contains two elements:

$$1\,2|3\,4\,5, \quad 1\,3|2\,4\,5, \quad 1\,4|2\,3\,5, \quad 1\,5|2\,3\,4,$$

$$2\,3|1\,4\,5, \quad 2\,4|1\,3\,5, \quad 2\,5|1\,3\,4, \quad 3\,4|1\,2\,5, \quad 3\,5|1\,2\,4, \quad 4\,5|1\,2\,3.$$

Since we are only interested in the contents of the two subsets (not their arrangement or order), we do not list cases where the first subset has three or four elements, as these would be overcounting. For example, the partitions $1\,2|3\,4\,5$ and $3\,4\,5|1\,2$ are considered the same.

Thus, all possible partitions have been listed, and their total number is 15. Therefore,

$$S(5, 2) = 15.$$

Note: Stirling numbers consider objects that we distribute as distinct, the boxes (subsets) as identical, and the size of subsets as known. Due to this, in the example, we did not consider the cases $1\,2|3\,4\,5$ and $3\,4\,5|1\,2$ as distinct.

Additionally, Bell's number counts all possible partitions, meaning the number of subsets k is not fixed but varies from 0 to $|N|$. This concept may seem similar to multisets; however, Bell's number treats objects being distributed as distinct and the boxes(subsets) as identical, while multisets treat objects as identical and boxes as distinct.

Recurrence Relation: These numbers satisfy the recurrence:

$$S(n, k) = S(n-1, k-1) + k\,S(n-1, k).$$

Proof: Let $N = [n]$ and P be the set of all k-partitions of N. We observe that $|P| = S(n, k)$, where $S(n, k)$ denotes the Stirling number of the second kind.

Consider an element $x \in [n]$. Define the following subsets of P: X_1 consists of partitions in P where x forms a singleton block, i.e., one of the subsets A_i in the partition $\{A_1, A_2, \ldots, A_k\}$ is $\{x\}$. $X_2 = P \setminus X_1$, meaning X_2 consists of partitions where x is not a singleton block but instead belongs to one of the k subsets.

Now, we compute their cardinalities: Since X_1 consists of partitions where x is a singleton, the remaining $n-1$ elements must be partitioned into $k-1$ subsets. Thus,

$$|X_1| = S(n-1, k-1).$$

In X_2, the element x is assigned to one of the k subsets after partitioning the remaining $n-1$ elements into k subsets. Thus,

$$|X_2| = k \cdot S(n-1, k).$$

By the rule of sum,

$$S(n, k) = |X_1| + |X_2| = S(n-1, k-1) + k \cdot S(n-1, k).$$

This completes the proof.

2.3 Counting Maps

Setup: Consider two finite sets N and R of sizes n and r respectively. We want to answer three main questions about the functions from N to R:

Q1: How many functions from N to R are there?

Answer: Each of the n elements of N can be mapped to any of the r elements of R. Hence, there are r^n possible functions in total.

Q2: How many injective (one-to-one) functions from N to R?

Answer: To build an injective function, choose a distinct image in R for each element of N. Thus, the number of injective functions is

$$r \times (r-1) \times (r-2) \times \cdots \times (r-n+1) = \frac{r!}{(r-n)!}.$$

Q3: How many surjective (onto) functions from N to R?

Answer: A function $f : N \to R$ is surjective if every element of R has a nonempty preimage. Equivalently, the sets

$$f^{-1}(y_1), \quad f^{-1}(y_2), \quad \ldots, \quad f^{-1}(y_r)$$

form a partition of N into r nonempty blocks. Since there are $S(n,r)$ ways to partition N into r nonempty subsets (where $S(n,r)$ is the Stirling number of the second kind), and each partition can be labeled in $r!$ ways (assigning each of the r blocks to a different $y_i \in R$), the total number of surjections is

$$\mathrm{Sur}(n,r) = r! \cdot S(n,r)$$

Corollary: Let N and R be finite sets with $|N| = n$ and $|R| = r$. Then the total number of functions from N to R can be expressed as

$$|\mathrm{Map}(N,R)| \ = \ r^n \ = \ \sum_{k=0}^{r} \binom{r}{k} \, k! \, S(n,k).$$

2.4 Number Partitions

Definition. A *number partition* of $n \in \mathbb{N}$ is an expression

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k,$$

where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1.$$

Example. List all possible different number partitions of $n = 5$ into two summands:

$$5 = 4 + 1 \quad \text{and} \quad 5 = 3 + 2.$$

Question. How many k-partitions of n are there?

Answer. Define

$$P(n,k) = \Big\{ (\lambda_1, \ldots, \lambda_k) \,\Big|\, n = \lambda_1 + \lambda_2 + \cdots + \lambda_k, \ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1 \Big\},$$

and let

$$p(n,k) = |P(n,k)|.$$

Moreover, set

$$P(n) = \bigcup_{k=1}^{n} P(n,k) \quad \text{and} \quad p(n) = |P(n)|.$$

An immediate observation is that

$$p(n) = \sum_{k=0}^{n} p(n,k).$$

Example. List all possible different number partitions of $n = 5$:

$$P(5) = \Big\{\{5\},\ \{4,1\},\ \{3,2\},\ \{3,1,1\},\ \{2,2,1\},\ \{2,1,1,1\},\ \{1,1,1,1,1\}\Big\},$$

with, for instance,

$$P(5,1) = \{\{5\}\}, \quad P(5,2) = \{\{4,1\},\{3,2\}\}, \quad P(5,3) = \{\{3,1,1\},\{2,2,1\}\},$$
$$P(5,4) = \{\{2,1,1,1\}\}, \quad P(5,5) = \{\{1,1,1,1,1\}\}.$$

To derive recursive formulas for $p(n)$ and $p(n,k)$, we introduce the notation

$$p(n, \leq k) \overset{\text{def}}{=} |P(n, \leq k)|, \quad \text{where} \quad P(n, \leq k) = \bigcup_{i=1}^{k} P(n,i).$$

- **Observation 1:** $P(n, \leq n) = P(n)$ and hence $p(n, \leq n) = p(n)$.
- **Observation 2:**

$$p(n, \leq k) = \sum_{i=1}^{k} p(n,i) = p(n,1) + p(n,2) + \cdots + p(n,k).$$

Theorem. There is a bijection

$$\Phi\colon P(n,k) \longrightarrow P(n-k, \leq k),$$

defined as follows.

Represent a partition

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in P(n,k)$$

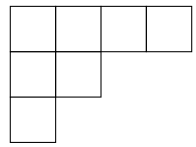
by its Ferrers diagram drawn in the standard way (with the largest row on top). In this diagram, the top row has λ_1 cells, the second row has λ_2 cells, and so on.

Explanation of the Mapping. Our goal is to relate partitions of n into exactly k parts to partitions of $n-k$ with at most k parts. Notice that if we subtract 1 from each part λ_i, then

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k \quad \Longrightarrow \quad n - k = (\lambda_1 - 1) + (\lambda_2 - 1) + \cdots + (\lambda_k - 1).$$

Graphically, subtracting 1 from a part corresponds to removing one cell from its corresponding row. Since the Ferrers diagram is left-justified, every row begins with a cell in the leftmost column. Thus, removing the entire leftmost column is equivalent to subtracting 1 from each λ_i. In this way, the original diagram representing a partition of n with k parts is transformed into a diagram representing a partition of $n-k$ that has at most k parts (some rows may vanish if $\lambda_i = 1$).

Example. Consider the partition $(\lambda_1, \lambda_2, \lambda_3) = (4,2,1)$ of $n = 7$ into 3 parts. Its Ferrers diagram is:



Ferrers Diagram for $(4, 2, 1)$

Removing the leftmost column yields:

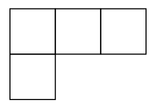


Diagram after removing leftmost column

The new diagram represents the partition $(3, 1)$, which is a partition of $7 - 3 = 4$ (since there were $k = 3$ rows, and we removed one cell per row). Note that $(3, 1)$ is an element of $P(4, \leq 3)$.

Justification of Bijectivity. The mapping

$$\Phi\colon (\lambda_1, \lambda_2, \ldots, \lambda_k) \mapsto (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_k - 1)$$

is invertible. Given any partition μ in $P(n-k, \leq k)$ (which has at most k parts), we can reconstruct a unique partition in $P(n, k)$ by adding 1 to each part and, if necessary, appending enough parts equal to 1 so that the total number of parts becomes exactly k. In other words, the inverse mapping Φ^{-1} is defined by:

$$\Phi^{-1}\colon \mu = (\mu_1, \mu_2, \ldots, \mu_r) \mapsto (\mu_1 + 1, \mu_2 + 1, \ldots, \mu_r + 1, \underbrace{1, 1, \ldots, 1}_{k-r \text{ times}}),$$

with $r \leq k$. It is straightforward to check that Φ and Φ^{-1} are mutual inverses. Thus, the mapping Φ is a bijection, and we have the relation:

$$p(n, k) = |P(n, k)| = |P(n - k, \leq k)| = p(n - k, \leq k).$$

This bijective correspondence is the key step in obtaining a recursive formula for $p(n, k)$.

Corollary 1. *For all integers $n \geq k \geq 1$, we have*

$$p(n, k) = p(n - k, \leq k) = p(n - k, 1) + p(n - k, 2) + \cdots + p(n - k, k - 1) + p(n - k, k).$$

That is, the total number of k-partitions of n can be split into partitions of $n - k$ with at most k parts:

$$p(n, k) = p(n - k, \leq k - 1) + p(n - k, k).$$

Moreover, since

$$p(n - k, \leq k - 1) = p((n - k) + (k - 1), k - 1) = p(n - 1, k - 1),$$

we obtain the recurrence relation

$$p(n, k) = p(n - 1, k - 1) + p(n - k, k).$$

3 Inclusion-Exclusion Principle

Very often, we need to calculate the number of elements in the union of certain sets. Assuming that we know the sizes of these sets, and their mutual intersections, the principle of inclusion and exclusion allows us to do exactly that.

Suppose you have two sets A and B. The size of the union is certainly at most $|A| + |B|$. However, in doing so we count each element of $A \cap B$ twice. To correct for this, we subtract $|A \cap B|$ to obtain

$$|A \cup B| \;=\; |A| + |B| \;-\; |A \cap B|.$$

In general, the formula gets more complicated because we must take into account intersections of multiple sets. The following statement is what we call the *principle of inclusion and exclusion*:

Lemma 1. *For any collection of finite sets $A_1, A_2, \ldots, A_n$, we have*

$$\left|\bigcup_{i=1}^{n} A_i\right| \;=\; \sum_{\substack{I \subseteq [n] \\ I \neq \emptyset}} (-1)^{|I|+1} \left|\bigcap_{i \in I} A_i\right|.$$

Equivalently,

$$|A_1 \cup A_2 \cup \cdots \cup A_n| \;=\; \sum_{i=1}^{n} |A_i| \;-\; \sum_{1 \le i < j \le n} |A_i \cap A_j| \;+\; \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| \;-\; \cdots \;+\; (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

Proof Outline (informal): *Each element that belongs to exactly t of the sets A_i is counted $\binom{t}{1}$ times in the first summation, then subtracted $\binom{t}{2}$ times in the second summation, added $\binom{t}{3}$ times in the third, and so on. In other words, its total contribution is*

$$\binom{t}{1} - \binom{t}{2} + \binom{t}{3} - \cdots + (-1)^{t-1}\binom{t}{t},$$

which equals 1. This alternating sum ensures that each element is ultimately counted exactly once, thereby correcting for any overcounting.

4 Permutations and Derangements

4.1 Permutations

A *permutation* of n elements is an arrangement (ordering) of those elements. For example, there are 6 permutations of the set $\{a, b, c\}$:

$$(a, b, c), \quad (a, c, b), \quad (b, a, c), \quad (b, c, a), \quad (c, a, b), \quad (c, b, a).$$

Since there are 3 choices for the first element, 2 for the second (once the first is chosen), and 1 for the last, by the multiplicative principle there are $3 \cdot 2 \cdot 1 = 3! = 6$ permutations in total.

Factorials and counting. In general, the number of permutations of n (distinct) elements is given by

$$n! \;=\; n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

Partial permutations (k-permutations). Sometimes we only permute k of the n elements, where $1 \le k \le n$. The number of ways to do this is denoted $P(n, k)$ and can be found by thinking:

$$P(n, k) \;=\; n \times (n-1) \times \cdots \times (n-k+1).$$

There are k factors in that product. Using factorial notation, we can write

$$P(n, k) \;=\; \frac{n!}{(n-k)!}.$$

Relationship to combinations. An alternate derivation uses combinations: first *choose* which k elements from the n will appear (that can be done in $\binom{n}{k}$ ways), then *arrange* those k in order (which can be done in $k!$ ways). Hence,

$$P(n,k) \;=\; \binom{n}{k}\,k!.$$

Since $\binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}$, multiplying by $k!$ yields exactly $\dfrac{n!}{(n-k)!}$, consistent with the direct counting approach.

4.2 Derangements

A *derangement* of n elements is a permutation where no element remains in its original position. More precisely, if we think of a permutation as a bijection θ on the set $\{1,2,\ldots,n\}$, then θ is a derangement if and only if

$$\theta(k) \;\neq\; k \quad \text{for all } k \in \{1,2,\ldots,n\}.$$

Equivalently, a derangement has no fixed points.

For example, for $n=3$, the permutations of $\{1,2,3\}$ are:

$$(1,2,3),\quad (1,3,2),\quad (2,1,3),\quad (2,3,1),\quad (3,1,2),\quad (3,2,1).$$

Among these, the derangements are $(2,3,1)$ and $(3,1,2)$; the other permutations fix at least one of the elements.

Counting Derangements via Inclusion-Exclusion

Let $D(n)$ denote the number of derangements of n elements. We will use the principle of inclusion-exclusion. Suppose we label the elements as $1,2,\ldots,n$, and define A_i to be the set of permutations that fix the element i (i.e. $\theta(i)=i$). Then any derangement is a permutation that lies in none of the sets A_i (for $1 \le i \le n$). We have

$$\bigl|A_i\bigr| \;=\; (n-1)!,$$

since if we fix one position i, then we permute the remaining $n-1$ elements freely. In general,

$$\bigl|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\bigr| \;=\; (n-k)!.$$

By inclusion-exclusion, the size of the union $A_1 \cup A_2 \cup \cdots \cup A_n$ is

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)!.$$

Hence the number of permutations that do not lie in this union—i.e. the number of derangements—is

$$D(n) \;=\; n! \;-\; \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n}(n-n)!.$$

$$D(n) \;=\; \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! \;=\; n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Thus, a concise closed-form for the number of derangements is

$$D(n) \;=\; n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Note on the series for e^{-1}:
In Calculus, one learns that the exponential function has a power series expansion

$$e^x \;=\; \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Setting $x = -1$ gives

$$e^{-1} \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}.$$

Hence,

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} \;\xrightarrow[n\to\infty]{}\; e^{-1}.$$

If you have not taken (or do not recall) a full course in Calculus, think of this as a special case of a well-known infinite series expansion for the exponential function.

Since the finite sum $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$ converges to e^{-1} as $n \to \infty$, we conclude that

$$\lim_{n\to\infty} \frac{D(n)}{n!} \;=\; \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \;=\; e^{-1}.$$

Numerically, this means that for large n, about $1/e \approx 36.8\%$ of all permutations of $\{1, \dots, n\}$ are derangements (i.e. have no fixed points).

A Recurrence Relation

We can also show that $D(n)$ satisfies the recurrence

$$D(n) \;=\; (n-1)\big(D(n-1) + D(n-2)\big), \quad \text{with } D(1) = 0,\, D(2) = 1.$$

One way to see this: consider where 1 goes in a derangement of $\{1, 2, \dots, n\}$. It can go to any of $n-1$ positions. If 1 goes to position j, then either (i) the element j goes to position 1 (a swap), which reduces the problem to deranging the remaining $n-2$ elements, or (ii) the element j does *not* go to position 1, effectively reducing the problem to deranging $n-1$ elements. This yields the above recurrence.

5 Preparation Tasks 1

5.1 Task 1

Problem Statement:

There are 20 students and 5 trips. In each of the following scenarios, count the number of ways the students can be assigned to trips.

(a) *Students are different, trips are different.* (I.e., we do care exactly which student goes to which distinct trip.)

(b) *Trips are different, but we only care about how many students go to each trip.* (Not which specific students, only the counts per trip.)

(c) *Trips are not distinguished, and we only care about the number of students in each trip.* (All trips are identical, so only the final “multiset of group sizes” matters.)

(d) *We do not care who goes where, but only with whom they go.* (We only care about the partition of the 20 students into some grouping, ignoring which trip is which.)

(e) *Students are different, trips are different, and each trip has the same number of students.*

Solutions and Explanations

(a) Each of the 20 **distinct** students independently chooses one of 5 **distinct** trips. This gives

$$5^{20}$$

total ways (each student has 5 choices).

(b) Now the students' individual identities no longer matter; we only care that, for example, "Trip 1 has 7 students, Trip 2 has 3 students," etc. Since there are 5 **distinct** trips, we want the number of 5-tuples $(n_1, n_2, n_3, n_4, n_5)$ of nonnegative integers summing to 20:

$$n_1 + n_2 + n_3 + n_4 + n_5 = 20.$$

By the stars-and-bars formula, the count is

$$\binom{20+5-1}{5-1} = \binom{24}{4}.$$

(c) Now the 5 trips themselves are **indistinguishable**, and we only care about how many students end up in each group (not which trip is which). Conceptually, this is the number of ways to partition 20 (distinct) students into up to 5 unlabeled subsets.

Interpreted as integer partitions, we want the number of partitions of the integer 20 into at most 5 parts. Denote by $p(n, i)$ the number of ways to partition n into exactly i (positive) parts. Then the total number of partitions of 20 into at most 5 parts is

$$p_{\leq 5}(20) = \sum_{i=1}^{5} p(20, i).$$

(d) We do not care who goes where, but only with whom they go. That is, we only care about how to group the 20 students, disregarding which trip label is attached to each group.

In terms of set partitions, the total number of ways to partition 20 distinct students into any number of unlabeled subsets is the Bell number B_{20}. The Bell number can be expressed as a sum of Stirling numbers of the second kind:

$$B_{20} = \sum_{k=0}^{20} S(20, k),$$

where $S(n, k)$ (with $S(n, 0) = 0$ for $n > 0$) counts the number of ways to partition n distinct elements into exactly k nonempty unlabeled subsets. Equivalently, one can start the sum at $k = 1$ since there are no partitions of 20 elements into 0 subsets:

$$B_{20} = \sum_{k=1}^{20} S(20, k).$$

(e) Students are different, trips are different, and each trip has the *same* number of students. Since 20 is divisible by 5, that means exactly 4 students must go on each trip. Count the ways to split 20 distinct students into 5 distinct groups of 4 each. One way to see this is:

$$\underbrace{\binom{20}{4}}_{\text{Trip 1}}\underbrace{\binom{16}{4}}_{\text{Trip 2}}\cdots\underbrace{\binom{4}{4}}_{\text{Trip 5}}.$$

Equivalently, the multinomial coefficient

$$\frac{20!}{4!\,4!\,4!\,4!\,4!}.$$

5.2 Task 2

Problem Statement:

We have 6 toy cars and 4 dolls (total 10 toys), and 10 boxes. In each scenario, count how many ways there are to place the 10 toys into the 10 boxes, under various assumptions of distinctness or identicalness:

(a) Everything (cars, dolls, boxes) is different.

(b) Everything is different and every toy goes into a separate box.

(c) All toys are different, but boxes are identical.

(d) Cars are different, but all dolls are identical; boxes are different.

(e) All cars are identical, all dolls are identical; boxes are different.

(f) Cars are different, but all dolls are identical; boxes are identical.

(g) We don't care which toy is which, but only how many toys are in each box; boxes are different.

(h) We don't care which toy is which, but only how many toys are in each box; boxes are identical.

Solutions and Explanations

Let us denote our toys as follows:

$$\text{Cars}: C_1, C_2, C_3, C_4, C_5, C_6; \quad \text{Dolls}: D_1, D_2, D_3, D_4.$$

We have 10 boxes, say $B_1, B_2, \ldots, B_{10}$.

(a) Everything is different. Each of the 10 distinct toys can go into any of 10 distinct boxes. Hence there are

$$10^{10}$$

ways (each toy has 10 choices independently).

(b) Everything is different and every toy goes into a separate box. We must place exactly one of the 10 distinct toys in each of the 10 distinct boxes (no box is left empty, no box has more than one toy). This is simply a permutation of 10 objects into 10 boxes:

$$10!$$

ways.

(c) All toys are different, but boxes are identical. We have 10 *distinct* toys (labeled objects) to be placed into up to 10 *indistinguishable* boxes. Equivalently, this is the number of ways to partition a set of 10 labeled elements into any number of unlabeled subsets (possibly fewer than 10 if some boxes are empty).

The total count is the Bell number B_{10}. It can be expressed as a sum of Stirling numbers of the second kind:

$$B_{10} \;=\; \sum_{k=0}^{10} S(10,k) \;=\; \sum_{k=1}^{10} S(10,k),$$

where $S(n,k)$ is the number of ways to partition n distinct objects into k nonempty subsets. There is no simple closed form for B_{10}, so we typically leave the answer in this summation form or as B_{10} itself.

(d) Cars are different, but dolls are identical; boxes are different. - We have 6 distinct cars $C_1, \ldots, C_6$. Each can go into any of 10 distinct boxes: 10^6 ways. - We have 4 *identical* dolls. Distributing 4 indistinguishable objects into 10 distinct boxes is given by the "stars-and-bars" formula:

$$\binom{4+10-1}{4} \;=\; \binom{13}{4}.$$

Multiply these independent choices:

$$10^6 \;\times\; \binom{13}{4}.$$

(e) All cars are identical, all dolls are identical; boxes are different. - Distribute 6 identical cars into 10 distinct boxes: $\binom{6+10-1}{6} = \binom{15}{6}$. - Distribute 4 identical dolls into 10 distinct boxes: $\binom{4+10-1}{4} = \binom{13}{4}$. Multiply for the total:

$$\binom{15}{6} \;\times\; \binom{13}{4}.$$

(g) We don't care which toy is which, only about how many toys are in each box; boxes are different. Now all 10 toys are treated as identical. With 10 *distinct* boxes, we only need the distribution of 10 identical items into 10 distinct bins, i.e. the number of solutions to

$$n_1 + n_2 + \cdots + n_{10} = 10$$

where $n_i \geq 0$. By stars-and-bars, that is

$$\binom{10+10-1}{10} \;=\; \binom{19}{10}.$$

(h) We don't care which toy is which, only about how many toys are in each box; boxes are identical. Now both the toys and the boxes are considered identical. We want the number of ways to partition 10 identical objects among up to 10 identical boxes. Equivalently, this is the number of ways to express the integer 10 as a sum of positive integers, ignoring order.

We denote by $p(n)$ the total number of partitions of n. This can also be expressed in terms of $p(n,k)$, which is the number of ways to partition n into exactly k positive parts:

$$p(n) \;=\; \sum_{k=1}^{n} p(n,k).$$

Since we can't have more than 10 parts if each part is positive, for $n = 10$ we write

$$p(10) \ = \ \sum_{k=1}^{10} p(10, k).$$

It is known (by direct enumeration or from tables) that

$$p(10) \ = \ 42.$$

Hence, there are 42 ways to distribute 10 indistinguishable toys among 10 indistinguishable boxes.

Remark. Parts like (f) are more intricate when combining "partially identical" toys with "identical boxes." Typically, such problems require detailed case analysis or generating functions.

5.3 Task 3

There are n rows with n chairs in each row (so n^2 chairs in total), and we have k students (where $k \leq n$) who are all distinct unless otherwise noted. We want to count the number of ways they can sit subject to various conditions. In each part, "at most one person in each chair" is always assumed.

(a) No extra assumptions.

We are simply placing k *distinct* students into n^2 distinct seats. First choose which k seats will be occupied, then permute the k students among those chosen seats. The number of ways is:

$$\binom{n^2}{k} k! \quad = \quad \frac{n^2!}{(n^2 - k)!}.$$

(b) Everyone sits in the first row.

The first row has n chairs and $k \leq n$ students. We must pick which k of the n seats are used, and then assign the k distinct students to those seats. Hence

$$\binom{n}{k} k!.$$

(c) We only care about the *numbers* of students sitting in each row.

Now we ignore which particular seats in each row are used *and* which particular students go to a given row. We only record the integer vector $(x_1, x_2, \ldots, x_n)$ of row-occupancies, where x_i is the number of students in row i, and

$$x_1 + x_2 + \cdots + x_n \ = \ k, \quad 0 \ \leq \ x_i \ \leq \ n.$$

Since $k \leq n$, the constraint $x_i \leq n$ is automatically satisfied. The number of nonnegative solutions to $x_1 + \cdots + x_n = k$ is

$$\binom{k + n - 1}{n - 1}.$$

(Here the students themselves are no longer distinguished, nor are seats in the same row.)

(d) We only care about *which chairs* are taken, not who sits in them.

In this scenario, we ignore the identity of students and only note *the set of occupied seats* (of size k). Hence we need to choose exactly k chairs out of n^2, with no regard to which student goes where. The count is

$$\binom{n^2}{k}.$$

(e) We only care about the numbers of students in each row, with each row having at most as many students as the previous row.

Let x_i be the number of students in row i. Then we record

$$(x_1, x_2, \ldots, x_n) \quad \text{such that} \quad x_1 \geq x_2 \geq \cdots \geq x_n \geq 0 \quad \text{and} \quad x_1 + x_2 + \cdots + x_n = k.$$

Equivalently, we are looking for a partition of k into at most n parts. Let

$$p(k, j)$$

be the number of partitions of k into exactly j positive parts. Then the number of partitions of k into *at most* n parts is

$$p_{\leq n}(k) \;=\; \sum_{j=0}^{n} p(k, j).$$

(Of course, $p(k, 0) = 0$ unless $k = 0$, but we often include $j = 0$ for completeness.)

Hence the number of ways to seat k students under these conditions is

$$\boxed{p_{\leq n}(k) \;=\; \sum_{j=0}^{n} p(k, j).}$$

There is no simpler closed-form expression for these partition numbers, but tables and generating functions can be used to compute them for specific k and n.

(f) We do not care who sits where, only which students sit *together in the same row* (no matter which row).

In this case, the arrangement is determined solely by partitioning the set of k distinct students into nonempty subsets, where each subset corresponds to a group sitting together in one row (and the rows themselves are unlabeled). The number of ways to partition a k-element set into j nonempty subsets is given by the Stirling number of the second kind, denoted $S(k, j)$. Therefore, if we allow any number j of groups (with $1 \leq j \leq k$), the total number of ways is:

$$\sum_{j=1}^{k} S(k, j).$$

This sum is known as the Bell number B_k; that is,

$$B_k = \sum_{j=1}^{k} S(k, j).$$

Thus, the number of ways is:

$$\boxed{B_k.}$$

(g) Everyone sits in the first 5 rows, and each of these 5 rows has the *same* number of students (assume $5 \mid k$).

Let $k = 5m$. We have 5 labeled rows (rows 1 through 5), and each must have exactly m students. We do care about *which* students go to each row, and also which seats they occupy (and the students are distinct). The counting proceeds in two stages:

(a) Distribute the k students into 5 labeled groups of size m each:

$$\binom{k}{m}\binom{k-m}{m} \cdots = \frac{k!}{(m!)^5}.$$

(b) For each group of m destined for row i, choose which m seats (out of n in that row) they occupy, and permute the m distinct students among those m seats:

$$\binom{n}{m} m! \quad \text{ways per row.}$$

Hence, for 5 rows, multiply $\left(\binom{n}{m}m!\right)^5$.

Overall, the number of ways is:

$$\frac{k!}{(m!)^5} \times \left(\binom{n}{m} m!\right)^5 = k! \left[\binom{n}{m}\right]^5, \quad \text{where } m = \frac{k}{5}.$$

(h) Everyone sits in the first 3 rows; we do not care which physical seats are used, but we *do* care about left-right adjacency (ignoring empty chairs).

Here each row is treated as a consecutive *line* of however many students it has. We do care which student is to the left or right of which other student in that row, but we ignore the actual seat numbers and any gaps. Thus, for row i having x_i students, the x_i chosen students can be arranged in $(x_i)!$ ways (linear order). Then we sum over all ways to split k distinct students into 3 labeled groups (for the 3 rows) and order each group:

(a) Choose a triple (x_1, x_2, x_3) with $x_1 + x_2 + x_3 = k$.

(b) Choose which x_1 students go to row 1, x_2 to row 2, etc. That is a multinomial factor $\frac{k!}{x_1!\,x_2!\,x_3!}$.

(c) Permute each group internally in $(x_i)!$ ways.
But multiplying $\frac{k!}{x_1!\,x_2!\,x_3!} \times (x_1)!(x_2)!(x_3)! = k!$.

Finally, we sum over all nonnegative (x_1, x_2, x_3) summing to k. The number of such triples is $\binom{k+3-1}{3-1} = \binom{k+2}{2}$. Thus total seatings:

$$k! \binom{k+2}{2}.$$

(i) If a row is not empty, it contains *at least two* people; we do not distinguish students *and* we do not distinguish chairs in the same row.

So we only care how many students are in each row, with $x_i = 0$ or $x_i \geq 2$. Since $k \leq n$, it is possible that many rows are empty. We want nonnegative $x_1, \ldots, x_n$ with $x_1 + \cdots + x_n = k$ and each nonempty $x_i \geq 2$. Since we do *not* distinguish which particular students go to row i, nor which seats they occupy, we are effectively counting the integer solutions:

$$x_i \in \{0\} \cup \{2, 3, \ldots\}, \quad x_1 + \cdots + x_n = k.$$

Let m be the number of nonempty rows. Then m satisfies $1 \leq m \leq \lfloor k/2 \rfloor$, and we choose which m rows are nonempty in $\binom{n}{m}$ ways. In each of those m chosen rows, let $x_i \geq 2$. Set $y_i = x_i - 2 \geq 0$, so $\sum_{i=1}^{m} y_i = k - 2m$. The number of nonnegative solutions to that is $\binom{(k-2m)+m-1}{m-1} = \binom{k-m-1}{m-1}$, valid as long as $k \geq 2m$.

Summing over all m gives

$$\sum_{m=1}^{\lfloor k/2 \rfloor} \binom{n}{m} \binom{k-m-1}{m-1}, \quad \text{(assuming } k > 0\text{)}.$$

(j) If a row is not empty, it has at least two people; we do not distinguish students, *but* we do care which specific seats are occupied.

(k) If a chair is empty, then all chairs *to its left in the same row* are empty as well.

Equivalently, in each row that has $x_i > 0$ seats occupied, those must be the leftmost x_i chairs of that row, with no gaps. If we do *not* distinguish students and only care about which seats are occupied, then row i is either completely empty ($x_i = 0$) or we occupy seats 1 through x_i. Summing over all rows, we must choose a total of k occupied chairs:

$$x_1 + x_2 + \cdots + x_n = k, \quad 0 \leq x_i \leq n.$$

Since $k \leq n$, the row capacity $x_i \leq n$ is never an issue. The number of nonnegative integer solutions is the standard stars-and-bars count:

$$\binom{k+n-1}{n-1}.$$

Thus there are $\binom{k+n-1}{n-1}$ ways to pick exactly which seats are occupied (each row's occupied block starts at seat 1).

6 Preparation Tasks 2

6.1 Problem 1

Prove that the num. of partitions of a positive int n into k *even* parts is eq. to the num. of partitions of n into k odd parts.

- **Idea:** First, express n in two ways by decomposing it into k even parts and into k odd parts. We can write

$$\begin{aligned} n &= 2a_1 + 2a_2 + \cdots + 2a_k, \\ n-k &= 2a_1 + 2a_2 + \cdots + 2a_k - k, \\ n-k &= (2a_1 - 1) + (2a_2 - 1) + \cdots + (2a_k - 1)\,. \end{aligned}$$

  The first line expresses n as a sum of k even numbers (each $2a_i$). Subtracting k from both sides shows that $n-k$ can be expressed as a sum of k odd numbers (each $2a_i - 1$).

  From this idea, we expect a correspondence between partitions of n into k even parts and partitions of $n-k$ into k odd parts.

- **Formal proof:** Define the sets

$$P_e(n,k) := \{\text{partitions of } n \text{ into } k \text{ even parts}\},$$

  and

$$P_o(n,k) := \{\text{partitions of } n \text{ into } k \text{ odd parts}\}.$$

  For n even, it turns out these two sets have the same size:

$$|P_e(n,k)| = |P_o(n-k,k)|\,.$$

  To show this, we construct an explicit bijection. Define a function

$$f : P_e(n,k) \to P_o(n-k,k)\,,$$

  by

$$f(b_1, b_2, \ldots, b_k) = (\,b_1 - 1,\ b_2 - 1,\ \ldots,\ b_k - 1\,)\,.$$

Here $(b_1, \ldots, b_k) \in P_e(n,k)$ means $b_1 + \cdots + b_k = n$ with each b_i even, so $b_i \geq 2$ for all i. Thus, each $b_i - 1 \geq 1$ and is odd, which ensures $f(b_1, \ldots, b_k) \in P_o(n-k,k)$ is a partition of $n-k$ into k odd parts.

The function f is well-defined. Moreover, it is invertible by simply adding 1 to each part. In fact, define

$$g : P_o(n-k,k) \to P_e(n,k)$$

as

$$g(c_1, c_2, \ldots, c_k) = (\, c_1 + 1,\ c_2 + 1,\ \ldots,\ c_k + 1\,)\,.$$

If $(c_1, \ldots, c_k)$ is a partition of $n-k$ into odd parts, then each c_i is odd and $c_i \geq 1$, so $c_i + 1$ is even and ≥ 2, and $\sum_{i=1}^{k}(c_i + 1) = (n-k) + k = n$. Thus $g(c_1, \ldots, c_k) \in P_e(n,k)$. It is easy to check that g is indeed the inverse of f: we have $f(g(c_1, \ldots, c_k)) = (c_1, \ldots, c_k)$ and $g(f(b_1, \ldots, b_k)) = (b_1, \ldots, b_k)$. Therefore, f is bijective.

6.2 Problem 2

Show that the number of partitions of a positive int n with at most k components is eq. to the num. of partitions of $2n$ with at most k even components.

- **Idea:** We consider partitions of n that have at most k parts. Let

$$n = a_1 + a_2 + \cdots + a_L,$$

  where $L \leq k$ and $a_1 \geq a_2 \geq \cdots \geq a_L > 0$. (In other words, $a_1, \ldots, a_L$ are the parts of a partition of n, listed in non-increasing order, with at most k parts.) For example, if $n = 5$ and $k = 3$, the partitions of 5 with at most 3 parts can be represented (padding with zeros up to 3 parts) as:

$$(2,2,1), \qquad (3,2,0), \qquad (5,0,0)\,,$$

  where we use 0 to indicate an empty part (no number in that position).

  Notice that doubling each part in these examples produces a partition of $2n = 10$ with only even parts (and still at most 3 components). For instance, $(2,2,1)$ doubles to $(4,4,2)$, $(3,2,0)$ doubles to $(6,4,0)$, and $(5,0,0)$ doubles to $(10,0,0)$. This suggests a direct correspondence between partitions of n (up to k parts) and partitions of $2n$ into even parts (up to k parts).

- **Formal proof:** Let us define the relevant sets in words (as suggested in the notes):

  - $P(n, \leq k)$ — the set of all partitions of n with *at most* k components (parts).
  - $P_e(2n, \leq k)$ — the set of all partitions of $2n$ with at most k *even* components.

  We aim to show that

$$|P(n, \leq k)| = |P_e(2n, \leq k)|\,,$$

  i.e. the two sets have equal cardinality. To prove this, we construct a bijection $f : P(n, \leq k) \to P_e(2n, \leq k)$. Given any partition $(a_1, a_2, \ldots, a_L)$ of n (with $L \leq k$), map it to

$$f(a_1, a_2, \ldots, a_L) = (\, 2a_1,\ 2a_2,\ \ldots,\ 2a_L\,)\,.$$

  In other words, f doubles each part of the partition of n. If the partition of n has fewer than k parts, we may imagine that it is padded with zeros (as above) which double to zeros, so the resulting partition of $2n$ still has at most k parts. By construction, $f(a_1, \ldots, a_L)$ is a partition of $2n$ in which every part is even, so indeed $f(a_1, \ldots, a_L) \in P_e(2n, \leq k)$. The function f is invertible by halving each even part: for any partition $(b_1, b_2, \ldots, b_M) \in P_e(2n, \leq k)$ (each b_i even), the inverse map f^{-1} gives

$$f^{-1}(b_1, b_2, \ldots, b_M) = \left(\frac{b_1}{2},\ \frac{b_2}{2},\ \ldots,\ \frac{b_M}{2}\right),$$

  which is a partition of $2n/2 = n$ with at most k parts. Thus f is a bijection between $P(n, \leq k)$ and $P_e(2n, \leq k)$, and consequently $|P(n, \leq k)| = |P_e(2n, \leq k)|$.

7 Functions Between Sets

Let N and R be sets with $|N| = n$ and $|R| = r$.

(i) **Total Functions:** The number of functions from N to R is

$$r^n.$$

Explanation: For every element in N, there are $|R| = r$ possible values in R. Thus, for the first element, there are r choices, for the second element, there are r choices, and so on. Applying the rule of product, the total number of functions is r^n.

(ii) **Injective Functions:** When $r \geq n$, an injective function (one-to-one) from N to R can be chosen by assigning distinct images to the n elements.

If a function is injective, then for each value in the range there is only one corresponding argument. This means that function values cannot repeat, ensuring that $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

Since there are $|R| = r$ choices for the first argument, $r - 1$ choices for the second, $r - 2$ for the third, and so on, applying the rule of product, the number of injective functions from N to R is:

$$r \cdot (r-1) \cdots (r-n+1) = \frac{r!}{(r-n)!}.$$

(iii) **Surjective Functions:** A function is surjective (onto) if every element in R has a pre-image in N, meaning every element in R is an image of some element in N. Consider a surjection $f : N \to R = \{y_1, y_2, \ldots, y_r\}$. We observe that the preimages $f^{-1}(y_1), f^{-1}(y_2), \ldots, f^{-1}(y_r)$ form a partition of N into r non-empty subsets, as each element y_i in R corresponds to one or more elements from N. The number of ways to partition N into r parts is given by the Stirling number $S(n, r)$, and since we can permute the r elements in R in $r!$ ways, the total number of surjective functions from N to R is:

$$r!\, S(n, r),$$

where $S(n, r)$ is the Stirling number of the second kind, counting the ways to partition N into r non-empty subsets.

Example: For $N = \{1, 2, 3\}$ and $R = \{y_1, y_2\}$:

Here $|N| = 3$ and $|R| = 2$.

- Total functions: $2^3 = 8$.
- Injective functions: Not possible since $|R| < |N|$.
- Surjective functions: Consider all possible surjective functions:

$f_1 : \{1, 2\} \mapsto y_1, 3 \mapsto y_2$ - Another possible permutation for this partition: $f_2 : \{1, 2\} \mapsto y_2, 3 \mapsto y_1$
$f_3 : \{2, 3\} \mapsto y_1, 1 \mapsto y_2$ - Another possible permutation for this partition: $f_4 : 1 \mapsto y_1, \{2, 3\} \mapsto y_2$
$f_5 : \{1, 3\} \mapsto y_1, 2 \mapsto y_2$ - Another possible permutation for this partition: $f_6 : 2 \mapsto y_1, \{1, 3\} \mapsto y_2$

So, we have 6 surjective functions. Using the formula for surjective functions, we first find the Stirling number $S(3, 2) = 3$, which corresponds to the number of partitions without considering permutations. Then, accounting for the permutations of the $r = 2$ elements in R, we compute:

$$2! \cdot S(3, 2) = 2! \cdot 3 = 6,$$

which matches the number of surjective functions we listed.

8 Generating functions

8.1 Generating Series

Instead of viewing a sequence as a function that returns its nth term, a *generating series* packages all of its terms into a single power series whose coefficients are exactly the sequence entries. Concretely, the sequence

$$2,\ 3,\ 5,\ 8,\ 12,\ \ldots$$

is encoded by the generating series

$$2 + 3x + 5x^2 + 8x^3 + 12x^4 + \cdots .$$

In general, given any sequence $\{c_n\}_{n\geq 0}$, its generating series is the formal power series

$$G(x) \;=\; \sum_{n=0}^{\infty} c_n x^n \;=\; c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots .$$

We say that $G(x)$ "generates" the sequence $\{c_n\}$ because each coefficient of x^n in $G(x)$ is precisely c_n. Generating series turn sequence-based problems into algebraic manipulations of power series, a technique we will exploit heavily in what follows.

Recall of the Basic Series

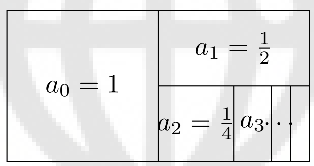


Figure 1: A geometric interpretation of the binary series, showing how $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$.

A Geometric View of the Binary Series For $|x| < 1$, we have the infinite geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n.$$

We now present a quick proof of this result by performing long division of 1 by $1 - x$.

$$\begin{array}{r|l}
 & 1 + x + x^2 + x^3 + \cdots \\
1 - x & 1 \\
\hline
 & \underline{1 - x} \\
 & x \\
 & \underline{x - x^2} \\
 & x^2 \\
 & \underline{x^2 - x^3} \\
 & x^3 \\
 & \vdots
\end{array}$$

The process works as follows: The long-division proceeds by repeatedly dividing the current remainder by the leading term of the divisor, producing one new power of x at each step:

1. Divide 1 by $1-x$. The multiplier needed to eliminate the constant term is 1, so

$$1 - 1 \cdot (1-x) = x.$$

Thus the first summand is 1, leaving a remainder of x.

2. Divide the remainder x by $1-x$. The multiplier is x, so

$$x - x \cdot (1-x) = x^2.$$

Hence the second summand is x, leaving a remainder of x^2.

3. Divide x^2 by $1-x$. The multiplier is x^2, giving

$$x^2 - x^2 \cdot (1-x) = x^3.$$

Therefore the third summand is x^2, with remainder x^3.

4. Continuing in this fashion produces the infinite series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots .$$

Continuing indefinitely produces

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n,$$

as claimed.
We will use this fact in further examples throughout the notes.

8.2 Building Generating Functions

The simplest (or "basic") generating function is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots ,$$

which generates the constant sequence $1, 1, 1, \ldots$.

Replacing x with $-x$:

$$\frac{1}{1-(-x)} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots ,$$

generating $1, -1, 1, -1, \ldots$.

Replacing x with $3x$:

$$\frac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + \cdots ,$$

generating $1, 3, 9, 27, \ldots$.

Scaling a sequence by 3:

$$\frac{3}{1-3x} = 3 + 9x + 27x^2 + 81x^3 + \cdots ,$$

generating $3, 9, 27, 81, \ldots$.

Termwise addition of sequences:
Adding the generating functions for $1, 1, 1, \ldots$ and $1, 3, 9, \ldots$ gives

$$\frac{1}{1-x} + \frac{1}{1-3x} = 2 + 4x + 10x^2 + 28x^3 + \cdots ,$$

which generates $2, 4, 10, 28, \ldots$.

Replacing x with x^2:

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots ,$$

generating $1, 0, 1, 0, 1, 0, \ldots$.

Shifting a sequence:
Multiplying by x shifts all coefficients right by one:

$$\frac{x}{1-3x} = 0 + x + 3x^2 + 9x^3 + \cdots ,$$

generating $0, 1, 3, 9, \ldots$, and

$$\frac{x}{1-x^2} = 0 + x + 0x^2 + x^3 + \cdots ,$$

generating $0, 1, 0, 1, \ldots$.

Combining shifted sequences:
Adding the two "even-odd" generating functions recovers

$$\frac{1}{1-x^2} + \frac{x}{1-x^2} = \frac{1+x}{1-x^2} = \frac{1}{1-x},$$

which generates $1, 1, 1, 1, \ldots$.

Differentiation:
Differentiating the basic Generating Function

$$\frac{d}{dx}\Big(\frac{1}{1-x}\Big) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots ,$$

yields the generating function for $1, 2, 3, 4, \ldots$.

8.3 Recurrence Relations & Generating Functions

We conclude with an example of one of the many reasons studying generating functions is helpful: solving recurrence relations via algebraic manipulation of power series.

Example: Tower of Hanoi The minimum number of moves required to transfer n disks satisfies

$$a_0 = 0, \quad a_1 = 1, \quad a_n = 2\,a_{n-1} + 1 \quad (n \geq 1),$$

giving the sequence

$$0,\ 1,\ 3,\ 7,\ 15,\ 31, \ldots$$

Define the generating function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

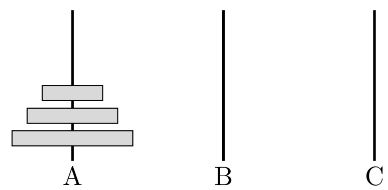


Figure 2: Initial configuration for Tower of Hanoi (3 disks).

Using the recurrence for $n \geq 1$:

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \big(2a_{n-1} + 1\big) x^n = 2x \sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} x^n,$$

so

$$f(x) - a_0 = 2x\, f(x) + \frac{x}{1-x},$$

and since $a_0 = 0$,

$$f(x) = \frac{x}{(1-x)(1-2x)}.$$

Performing partial fractions:

$$\frac{x}{(1-x)(1-2x)} = \frac{-1}{1-x} + \frac{1}{1-2x},$$

hence

$$f(x) = -\frac{1}{1-x} + \frac{1}{1-2x}.$$

Extracting coefficients yields the closed-form solution

$$a_n = 2^n - 1,$$

confirming the well-known formula for the Tower of Hanoi moves.

8.4 Introduction to the Fibonacci Sequence

The Fibonacci sequence famously arises from a puzzle involving rabbit populations. Imagine starting with a single pair of rabbits that takes one month to mature. After maturing, each pair produces a new pair of rabbits every month. Mathematically, if F_n represents the number of rabbit pairs in month n, the sequence satisfies the initial conditions

$$F_0 = 0, \quad F_1 = 1,$$

and the recurrence

$$F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 0.$$

Q: is there a non-recursive (closed-form) formula for F_n ?

Idea: consider and calculate it

1) || (2 small rabbits)

2) || + || (1 big pair + 1 small pair)

3) || + || + || (2 big pairs + 1 small pair)

4) || + || + || + || + || (3 big pairs + 2 small pairs)

Figure 3: Illustration of rabbit pairs over successive months. Blue bars represent small rabbits; orange bars represent big (mature) rabbits.

8.5 Deriving the Closed-Form for the Fibonacci Sequence

Step 1: Define the generating function. Let $\{F_n\}_{n=0}^{\infty}$ be the Fibonacci sequence with

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0).$$

Define the generating function

$$f(x) \;=\; \sum_{n=0}^{\infty} F_n\, x^n.$$

We aim to find a closed-form expression for $f(x)$, and then extract a formula for F_n.

Step 2: Use the Fibonacci recurrence in $f(x)$. Starting from

$$f(x) \;=\; F_0 + F_1 x + \sum_{n=2}^{\infty} F_n\, x^n,$$

and noting $F_0 = 0$, $F_1 = 1$, we have

$$f(x) \;=\; x + \sum_{n=2}^{\infty} \bigl(F_{n-1} + F_{n-2}\bigr)\, x^n$$

because $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Separate the sums:

$$f(x) = x + \sum_{n=2}^{\infty} F_{n-1}\, x^n + \sum_{n=2}^{\infty} F_{n-2}\, x^n.$$

Shift indices to factor out $f(x)$:

$$\sum_{n=2}^{\infty} F_{n-1}\, x^n = x \sum_{n=2}^{\infty} F_{n-1}\, x^{n-1} = x \sum_{m=1}^{\infty} F_m\, x^m = x\bigl(f(x) - F_0\bigr) = x\, f(x),$$

since $F_0 = 0$. Similarly,

$$\sum_{n=2}^{\infty} F_{n-2}\, x^n = x^2 \sum_{n=2}^{\infty} F_{n-2}\, x^{n-2} = x^2 \sum_{k=0}^{\infty} F_k\, x^k = x^2\, f(x).$$

Hence,

$$f(x) = x + x\, f(x) + x^2\, f(x) \;\implies\; f(x)\bigl(1 - x - x^2\bigr) = x.$$

Thus,

$$f(x) = \frac{x}{1 - x - x^2}.$$

Step 3: Partial-Fraction Decomposition (as in the images). First, rewrite

$$\frac{1}{1-x-x^2} \;=\; \frac{1}{-\big(x^2+x-1\big)} \;=\; -\,\frac{1}{x^2+x-1}.$$

Next, factor x^2+x-1. Observe that the roots of

$$x^2+x-1 \;=\; 0$$

are

$$x \;=\; -\,\frac{1+\sqrt{5}}{2} \quad\text{and}\quad x \;=\; -\,\frac{1-\sqrt{5}}{2}.$$

Hence,

$$x^2+x-1 \;=\; \Big(x+\tfrac{1+\sqrt{5}}{2}\Big)\;\cdot\;\Big(x+\tfrac{1-\sqrt{5}}{2}\Big).$$

Therefore,

$$-\,\frac{1}{x^2+x-1} \;=\; -\,\frac{1}{\Big(x+\tfrac{1+\sqrt{5}}{2}\Big)\Big(x+\tfrac{1-\sqrt{5}}{2}\Big)}.$$

We look for constants A and B such that

$$-\,\frac{1}{\Big(x+\tfrac{1+\sqrt{5}}{2}\Big)\Big(x+\tfrac{1-\sqrt{5}}{2}\Big)} \;=\; \frac{A}{x+\tfrac{1+\sqrt{5}}{2}} \;+\; \frac{B}{x+\tfrac{1-\sqrt{5}}{2}}.$$

Step 4: Solve for A and B. Comparing coefficients of x and the constant term in

$$-\,1 \;=\; A\Big(x+\beta\Big) \;+\; B\Big(x+\alpha\Big),$$

we obtain the system

$$\begin{cases} A+B=0, \\ A\,\beta+B\,\alpha=-\,1. \end{cases}$$

It follows that

$$B=-\,A, \quad A\,(\beta-\alpha)=-\,1 \;\Longrightarrow\; A=\frac{1}{\alpha-\beta} \quad\text{and}\quad B=-\,\frac{1}{\alpha-\beta}.$$

Hence,

$$-\,\frac{1}{(x+\alpha)(x+\beta)} \;=\; \frac{1}{\alpha-\beta}\,\frac{1}{x+\alpha} \;-\; \frac{1}{\alpha-\beta}\,\frac{1}{x+\beta}.$$

Step 5: Combine with the earlier factor -1 and rewrite. Recalling that

$$\frac{1}{1-x-x^2} \;=\; -\,\frac{1}{x^2+x-1} \;=\; -\,\frac{1}{(x+\alpha)\,(x+\beta)},$$

we combine the above result to conclude

$$\frac{1}{1-x-x^2} \;=\; \frac{1}{\alpha-\beta}\left(\frac{1}{x+\alpha} \;-\; \frac{1}{x+\beta}\right).$$

Step 6: Expand each term in a power series. Notice that

$$\frac{1}{x+\alpha} \;=\; \frac{1}{\alpha}\,\frac{1}{1+\frac{x}{\alpha}} \;=\; \frac{1}{\alpha}\sum_{n=0}^{\infty}\left(-\tfrac{x}{\alpha}\right)^{n} \;=\; \sum_{n=0}^{\infty}\frac{(-1)^n}{\alpha^{\,n+1}}\,x^n,$$

valid for $\left|\frac{x}{\alpha}\right| < 1$. Similarly,

$$\frac{1}{x+\beta} \;=\; \sum_{n=0}^{\infty}\frac{(-1)^n}{\beta^{\,n+1}}\,x^n.$$

Hence,

$$\frac{1}{1-x-x^2} \;=\; \frac{1}{\alpha-\beta}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{\alpha^{\,n+1}}\,x^n \;-\; \sum_{n=0}^{\infty}\frac{(-1)^n}{\beta^{\,n+1}}\,x^n\right] = \sum_{n=0}^{\infty}\left[\frac{1}{\alpha-\beta}\left(\frac{(-1)^n}{\alpha^{\,n+1}} \;-\; \frac{(-1)^n}{\beta^{\,n+1}}\right)\right]x^n.$$

Step 7: Identify Fibonacci numbers. Recall that $\alpha-\beta=\sqrt{5}$, and

$$F_n \;=\; \frac{\alpha^n-\beta^n}{\alpha-\beta} \;=\; \frac{\alpha^n-\beta^n}{\sqrt{5}}.$$

One checks (or uses known identities) to see that the coefficient of x^n in the above power series is exactly F_n. Consequently,

$$\sum_{n=0}^{\infty}F_n\,x^n \;=\; \frac{1}{1-x-x^2},$$

which is the generating function for the Fibonacci sequence.

Conclusion. We have shown that the generating function for the Fibonacci sequence is $\frac{x}{1-x-x^2}$. Through partial fractions and comparing coefficients, we deduced that

$$F_n = \frac{\alpha^n-\beta^n}{\sqrt{5}}.$$

This gives a non-recursive (closed-form) expression for F_n, completing the derivation.

8.6 More examples

In earlier sections (see, e.g., *A Geometric View of the Binary Series* on page 14), we explored methods to solve recurrences and introduced generating functions as a tool to transform sequences into functions. In this section, we briefly reiterate these ideas and demonstrate, through several examples, how generating functions serve as a bridge between discrete mathematics and calculus.

Example 1: Constant Sequence. Consider the sequence defined by

$$a_n = 1 \quad \text{for all } n \geq 0,$$

so that the sequence is

$$1,\,1,\,1,\ldots.$$

By the geometric series formula (proved earlier), its generating function is given by

$$f(x) = \sum_{n=0}^{\infty}x^n = \frac{1}{1-x}, \quad |x|<1.$$

Example 2: Exponential Sequence. Now, let

$$a_n = \frac{1}{n!}.$$

Then the generating function is

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x, \quad x \in \mathbb{R}.$$

A partial justification of this result can be obtained by recalling the Taylor series expansion of the exponential function. Although a complete treatment of Taylor series is a topic in calculus (not yet covered in this course), note that differentiating the power series term-by-term confirms the identity.

Example 3: Binomial Coefficient Sequence. Consider the sequence defined by

$$a_n = \binom{n+k}{k}.$$

Theorem. The generating function for this sequence is

$$f(x) = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}}, \quad |x| < 1.$$

Proof.

- For $k = 1$: Note that
$$a_n = \binom{n+1}{1} = n + 1,$$
so that
$$f(x) = \sum_{n=0}^{\infty} \binom{n+1}{1} x^n = \sum_{n=0}^{\infty} (n+1)x^n.$$
Recall the geometric series,
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$
and observe that by differentiating both sides term-by-term with respect to x, we can derive the generating function for the sequence $(n+1)$. In detail, differentiate the left-hand side:
$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$
On the right-hand side, notice that since
$$\frac{d}{dx} x^{n+1} = (n+1)x^n,$$
differentiating the series yields
$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} (n+1)x^n.$$
Thus, we conclude that
$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n.$$
This recovers the generating function for $k = 1$. A less formal derivation was given in the subsection *Building Generating Functions* on page 16.

A complete inductive proof follows similar lines but is omitted here for brevity.

Example 4: Alternating Factorial Sequence. Define the sequence by

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \dfrac{(-1)^{\frac{n-1}{2}}}{n!}, & \text{if } n \text{ is odd.} \end{cases}$$

Then the generating function is

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sin(x), \quad x \in \mathbb{R}.$$

Even though a full treatment of the Taylor series for trigonometric functions is part of calculus (again, a topic not yet covered here), this example illustrates how generating functions capture nontrivial sequence behavior by connecting discrete structures with analytic functions.

8.7 Generating Function Applications

One key application is the multiplication (or convolution) of generating functions, which naturally arises when we combine two distinct combinatorial constructions into a single, more complex structure.

Question: If a_k counts all objects of type A of size k and b_k counts all objects of type B of size k, how many pairs of objects (A, B) have a total size of n?

Answer: The number of such pairs is given by

$$\sum_{k=0}^{n} a_k \, b_{n-k}.$$

Observation: The generating function for the sequence

$$C_n = \sum_{k=0}^{n} a_k \, b_{n-k}$$

is

$$\sum_{n=0}^{\infty} C_n \, x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k \, b_{n-k} \right) x^n.$$

It shows that multiplying the generating functions corresponding to $\{a_n\}$ and $\{b_n\}$ produces a new generating function whose coefficients are given by the convolution of the two original sequences.

Example 1: Dice Sum Counting. A classic example of this application is counting the number of ways to obtain a given sum when rolling two standard six-sided dice. For a single die, the generating function is:

$$D(x) = x + x^2 + x^3 + x^4 + x^5 + x^6,$$

where the term x^k corresponds to rolling a k. Since the two dice are independent, the generating function for the sum of the two dice is:

$$D(x)^2 = \left(x + x^2 + x^3 + x^4 + x^5 + x^6\right)^2.$$

Expanding this product, the coefficient of x^n in $D(x)^2$ equals the number of ways to achieve a total sum of n. For instance, one can verify that the coefficient of x^7 is 6, which corresponds to the six possible outcomes that sum to 7 (namely, the pairs $(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$).

Example 2: Candy Selection Problem. Selecting 30 candies from 20 large types (each type can be picked at most once) and 40 small types (each type can be picked in any quantity) can be modeled with generating functions. For a single large candy type (available at most once), the generating function is

$$1 + x,$$

and for 20 independent large types, the combined generating function is

$$(1 + x)^{20}.$$

For a small candy type (with unlimited supply), the generating function is

$$1 + x + x^2 + \cdots = \frac{1}{1 - x},$$

so for 40 small types it is

$$\left(\frac{1}{1 - x}\right)^{40} = (1 - x)^{-40}.$$

Thus, the overall generating function becomes

$$G(x) = (1+x)^{20}\,(1-x)^{-40}.$$

To determine the number of ways to select 30 candies, we need the coefficient of x^{30} in $G(x)$. Expanding,

$$(1+x)^{20} = \sum_{i=0}^{20} \binom{20}{i} x^i, \quad (1-x)^{-40} = \sum_{j\geq 0} \binom{39+j}{39} x^j,$$

the convolution gives:

$$[x^{30}]\,G(x) = \sum_{i=0}^{20} \binom{20}{i} \binom{39+30-i}{39}.$$

Using a Vandermonde's convolution argument, one can show that

$$\sum_{i=0}^{20} \binom{20}{i} \binom{69-i}{39} = \binom{59}{30}.$$

Example 3: Selection with Limited Green Items. Consider selecting 20 objects from three categories:

1. An infinite pile of red objects,
2. An infinite pile of blue objects,
3. A pile of green objects with only 5 available.

For the red and blue objects (with unlimited supply), the generating function is:

$$\frac{1}{1-x},$$

so for both together we have:

$$\frac{1}{(1-x)^2}.$$

For the green objects (at most 5), the generating function is:

$$1 + x + x^2 + x^3 + x^4 + x^5 = \frac{1-x^6}{1-x}.$$

Thus, the overall generating function becomes:

$$G(x) = \frac{1}{(1-x)^2} \cdot \frac{1-x^6}{1-x} = \frac{1-x^6}{(1-x)^3}.$$

Using the expansion

$$(1-x)^{-3} = \sum_{n\geq 0} \binom{n+2}{2} x^n,$$

the coefficient of x^{20} in $G(x)$ is computed by writing:

$$G(x) = (1-x)^{-3} - x^6(1-x)^{-3}.$$

The coefficient from the first term is $\binom{22}{2} = 231$ (since $[x^{20}](1-x)^{-3} = \binom{20+2}{2}$) and from the second term, it is $\binom{16}{2} = 120$ (as the x^6 shifts the index, so $[x^{20}](x^6(1-x)^{-3}) = [x^{14}](1-x)^{-3}$). Hence,

$$[x^{20}]\,G(x) = 231 - 120 = 111.$$

Example 4: Coin Change with Limited Denominations. Determine the number of ways to make 10 (units) using coins of value 1, 2, and 5, where 1- and 2-unit coins are available in unlimited supply but 5-unit coins are limited to at most 2. The generating functions are:

$$G_1(x) = \frac{1}{1-x} \quad \text{(for 1-unit coins)},$$

$$G_2(x) = \frac{1}{1-x^2} \quad \text{(for 2-unit coins)},$$

$$G_5(x) = 1 + x^5 + x^{10} \quad \text{(for 5-unit coins, at most 2)}.$$

Thus, the overall generating function is:

$$G(x) = \frac{1}{(1-x)(1-x^2)}\,(1 + x^5 + x^{10}).$$

To find the coefficient of x^{10}, write:

$$G(x) = \frac{1}{(1-x)(1-x^2)} + \frac{x^5}{(1-x)(1-x^2)} + \frac{x^{10}}{(1-x)(1-x^2)}.$$

The first term contributes the number of ways to form 10 with 1- and 2-unit coins, which is 6; the second term contributes the number of ways to form 5 (which is 3); and the third term contributes 1 (making 0 with 1- and 2-unit coins). Therefore, the total number of ways is:

$$6 + 3 + 1 = 10.$$

Example 5: Composition with Constrained Part Sizes. Count the number of compositions of 7 into exactly 3 positive parts, where each part is at most 4. The generating function for a single part that can take values 1 through 4 is:

$$F(x) = x + x^2 + x^3 + x^4.$$

For a composition with exactly 3 parts, the generating function is:

$$G(x) = [F(x)]^3 = (x + x^2 + x^3 + x^4)^3.$$

A term x^7 in the expansion corresponds to a composition of 7 into 3 parts. Without the upper bound, the number of compositions of 7 into 3 parts (with each part at least 1) is given by

$$\binom{7-1}{3-1} = \binom{6}{2} = 15.$$

However, we must exclude compositions where any part exceeds 4. In this specific case, the only forbidden compositions are those where one part is 5 and the other two are 1 (i.e., $5 + 1 + 1$ and its permutations), which count to 3. Hence, the number of valid compositions is:

$$15 - 3 = 12.$$

9 Catalan Numbers

9.1 Introduction and Motivating Problem

How many ways can we correctly place n pairs of parentheses? This is a classic combinatorial question about **valid parentheses combinations**. For example, with $n = 1$ pair, there is only one valid arrangement: ‘()‘. With $n = 2$ pairs, there are two valid arrangements: ‘()()‘ and ‘(())‘. With $n = 3$ pairs, there are five valid arrangements (e.g. ‘()()()‘, ‘()(())‘, ‘(())()‘, ‘(()())‘, ‘((()))‘). In general, the number of distinct well-formed parentheses sequences grows quickly with n.

This problem was our motivation to study a famous sequence of numbers. (It has connection to a similar LeetCode programming problem about generating parentheses.) The numbers counting valid parentheses structures for $n = 1, 2, 3, \ldots$ are:

$$1,\ 2,\ 5,\ 14,\ 42,\ 132,\ \ldots$$

These are known as the **Catalan numbers**. In these notes, we will explore Catalan numbers, their recursive definition, how they relate to binary tree structures, and a derivation of their formula using generating functions.

9.2 Definition of Catalan Numbers

The Catalan numbers C_n can be defined recursively. For $n = 0$ (zero pairs of parentheses), we define $C_0 = 1$ by convention (there is exactly one valid arrangement of zero pairs: an empty sequence). For $n \geq 1$, the Catalan number C_n satisfies the recursion:

$$C_n \;=\; \sum_{k=0}^{n-1} C_k \;\cdot\; C_{n-1-k}\,.$$

In other words, each C_n is obtained by summing over all products $C_k \cdot C_{n-1-k}$ for $0 \leq k \leq n - 1$. This recurrence is the heart of what makes Catalan numbers arise in so many combinatorial structures. We will soon see why this recurrence formula makes sense in terms of counting valid parentheses or binary tree configurations.

(Additional note: There is also a direct formula for C_n which we will derive later. The first few values $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$ confirm the sequence given above.)

Combinatorial reasoning (parenthesis perspective)

Why does the above recurrence hold for valid parentheses? Consider a valid parentheses sequence of n pairs. Focus on the very first “(‘” character. It must have a matching “)‘”. Say this matching “)‘” occurs after forming k pairs inside (between this “(‘” and its matching “)‘”) – those k pairs inside must themselves form a valid sequence. The remaining parentheses (after the matching “)‘”) will form another valid sequence with $n - 1 - k$ pairs. See the schematic below for a sequence split by the first pair:

$$\underbrace{(\quad S_{\text{inside}}\quad)}_{1+k \text{ pairs}}\ S_{\text{outside}}\,,$$

where S_{inside} is a valid sequence of k pairs, and S_{outside} is a valid sequence of $n - 1 - k$ pairs. Any valid sequence can be uniquely decomposed in this way. There are C_k possibilities for S_{inside} and C_{n-1-k} possibilities for S_{outside}. Multiplying and summing over all k from 0 to $n - 1$ gives the recurrence $C_n = \sum_{k=0}^{n-1} C_k\, C_{n-1-k}$, as stated above.

(Additional note: The same recurrence will be explained again using binary trees below, which is an equivalent interpretation. The key idea is splitting a structure (parentheses or tree) at a certain point, resulting in two smaller independent structures.)

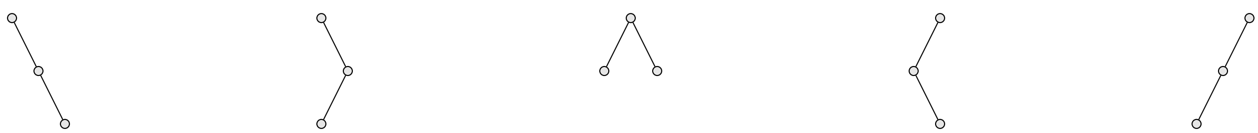
9.3 Catalan Numbers and Binary Trees

Catalan numbers also count the number of distinct **binary tree** structures with a given number of nodes. To appreciate this connection, we first review what a binary tree is:

A **binary tree** is a hierarchical structure consisting of nodes, where each node may have up to two children: a left child and a right child. A node with no children is called a *leaf*. We often draw binary trees in a planar way with the root at the top, left children branching to the left, and right children to the right. In counting binary trees for Catalan numbers, we consider different shapes of the tree (different arrangements of nodes and child connections) as distinct, but we do *not* label the nodes with any specific values.

For example, with only 1 node, there is exactly one possible binary tree (just the root by itself). With 2 nodes, there are exactly two distinct binary tree shapes:

In the first tree above, the root has a left child but no right child. In the second tree, the root has a right child but no left child. These are the only two possible configurations for 2 nodes. Now, for 3 nodes, it turns out there are 5 distinct binary tree shapes. We can enumerate all five (to visualize them, each diagram below shows the shape, with nodes represented by circles):



(All 5 distinct binary trees with 3 nodes)

For 3 nodes, the five tree shapes can be described as: 1. A right-skewed chain (root $\to$ right child $\to$ right grandchild). 2. A tree where the root has only a right child, and that child in turn has a left child. 3. A balanced tree (root with one left child and one right child). 4. A tree where the root has only a left child, and that child has a right child. 5. A left-skewed chain (root $\to$ left child $\to$ left grandchild).

If we proceed to 4 nodes, the number of distinct binary trees grows to 14. Detailing all 14 shapes is cumbersome, but we can categorize them by how the root splits the nodes between left and right subtrees: - 5 of those trees have all 3 of the other nodes in the left subtree (and right subtree empty), and another 5 have all 3 in the right subtree (left empty). These 10 are essentially a root with one side empty and the other side being one of the 5 shapes from the 3-node case. - 2 of the trees have the root with 1 node in the left subtree and 2 in the right subtree. - The remaining 2 have 2 nodes in the left subtree and 1 in the right subtree.

Adding these cases: $5 + 5 + 2 + 2 = 14$ total shapes for 4 nodes. (Indeed, 14 is the next Catalan number after 5.)

This pattern is no coincidence. In fact, the recurrence relation for C_n can be understood by considering how a binary tree of n nodes can be formed. Suppose a binary tree has n nodes in total. Pick a number k between 0 and $n-1$ (inclusive) to be the number of nodes in the left subtree of the root. Then the right subtree will have $n-1-k$ nodes (since one node is the root itself). There are C_k possible shapes for the left subtree (by definition of Catalan numbers for k nodes) and C_{n-1-k} possible shapes for the right subtree. These choices are independent, so there are $C_k \cdot C_{n-1-k}$ possible trees with that particular split of k and $n-1-k$. Summing over all $k = 0, 1, 2, \ldots, n-1$ gives exactly $C_n = \sum_{k=0}^{n-1} C_k \, C_{n-1-k}$. This is the same recursive formula we encountered earlier, now interpreted in terms of binary trees. Thus, the number of valid parentheses arrangements with n pairs is equal to the number of binary tree structures with n nodes, and both are given by the nth Catalan number.

9.4 Deriving the Formula using Generating Functions

While the recursive definition of Catalan numbers is useful, we can go further and derive a closed-form formula for C_n. A powerful method to solve such recurrences is to use a **generating function**. Define the generating function $C(x)$ for the Catalan sequence as:

$$C(x) \;=\; C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots \;=\; \sum_{n \ge 0} C_n \, x^n \,.$$

Using the recurrence $C_n = \sum_{k=0}^{n-1} C_k \, C_{n-1-k}$, we can derive an equation for $C(x)$. First, note that:

$$C(x) - C_0 \;=\; \sum_{n \ge 1} C_n x^n \;=\; \sum_{n \ge 1} \Big(\sum_{k=0}^{n-1} C_k \, C_{n-1-k} \Big) x^n \,.$$

Now, change the summation index by letting $m = n-1$. Then $n \ge 1$ corresponds to $m \ge 0$, and $n = m+1$. The above becomes:

$$C(x) - 1 \;=\; \sum_{m \ge 0} \Big(\sum_{k=0}^{m} C_k \, C_{m-k} \Big) x^{\,m+1} \;=\; x \sum_{m \ge 0} \sum_{k=0}^{m} C_k \, C_{m-k} \, x^m \,.$$

But the double sum $\sum_{m \ge 0} \sum_{k=0}^{m} C_k C_{m-k} x^m$ is recognized as the product of two power series. In fact, by the Cauchy convolution formula,

$$\sum_{m \ge 0} \sum_{k=0}^{m} C_k \, C_{m-k} \, x^m \;=\; \Big(\sum_{k \ge 0} C_k x^k \Big) \Big(\sum_{j \ge 0} C_j x^j \Big) \;=\; C(x)\, C(x) \;=\; [C(x)]^2 \,.$$

Therefore, we have the following functional equation for $C(x)$:

$$C(x) - 1 \;=\; x\,[C(x)]^2 \,,$$

or equivalently,

$$C(x) \;=\; 1 \;+\; x\,[C(x)]^2 \,.$$

This equation is derived directly from the Catalan recurrence. Now we solve for $C(x)$ as an explicit function of x. The equation $C(x) = 1 + x[C(x)]^2$ can be rearranged into a quadratic equation in $C(x)$:

$$x[C(x)]^2 - C(x) + 1 = 0 \,.$$

Solving this quadratic for $C(x)$, we use the quadratic formula (treating $C(x)$ as the unknown and x as a constant):

$$C(x) \;=\; \frac{1 \pm \sqrt{1-4x}}{2x} \,.$$

There are two solutions, but we must choose the one that gives a valid power series expansion. Since $C(0)$ should equal $C_0 = 1$, we take the **negative** branch of the $\pm$ (this ensures $C(x)$ is finite at $x = 0$):

$$C(x) \;=\; \frac{1 - \sqrt{1-4x}}{2x} \,.$$

This is the generating function for the Catalan numbers. We can expand this to obtain a formula for C_n. The series expansion of the square root can be derived using the binomial series:

$$\sqrt{1-4x} \;=\; 1 \;-\; 2x \;-\; 2x^2 \;-\; 4x^3 \;-\; 8x^4 \;-\; \cdots \,,$$

but a more straightforward way is to recognize the known power series for Catalan numbers. In fact, the coefficient extraction can be done by comparing with the binomial theorem. The final result (which one can derive by expanding or by known combinatorial identities) is:

$$C_n \;=\; \frac{1}{n+1}\binom{2n}{n},$$

for $n \geq 0$. This elegant formula gives the nth Catalan number directly. For example, for $n = 4$ it gives $C_4 = \frac{1}{5}\binom{8}{4} = \frac{1}{5} \cdot 70 = 14$, consistent with our earlier count of binary trees or parentheses combinations.

(Additional note: The closed-form formula above was not explicitly given in the lecture, but it is a well-known result for Catalan numbers. It can be proven by induction or other methods as well. Catalan numbers appear in numerous other combinatorial problems, such as counting paths in a grid, polygon triangulations, full binary trees with $n + 1$ leaves, and many more.)