Graphs

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1 Formal definition

A graph is a pair $G = (V, E)$ where V is a finite non-empty set (called the set of *vertices*) and E is a set (called the set of *edges*), where $E \subseteq \binom{V}{2}$ (all two-element subsets of V).

Notation:

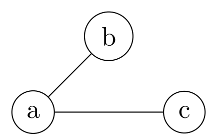
If G=(V, E), then $V =: V(G)$ and $E =: E(G)$.

If $\{u, v\} \in E$, then we write $uv \in E$ ($vu = uv$, as edges are unordered)

If $u, v \in V(G)$ such that $uv \in E(G)$, then we say that u and v are *connected* by an edge in G

Example

Let $G = (V, E)$ where $V = \{a, b, c\}$ and $E = \{\{a, b\}, \{a, c\}\}$. This can be visualized as:



2 Basic Concepts and Terminology

Simple Graphs

A graph is called **simple** if it is non-oriented (undirected), has no loops (edges from a vertex to itself), and is not a multigraph (has at most one edge between any pair of vertices). The definition provided in Section 1 refers to simple graphs.

Isomorphism

We say that two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there exists a bijection $\varphi : V_1 \to V_2$ such that $\{u, v\} \in E_1 \iff \{\varphi(u), \varphi(v)\} \in E_2$

Graph sum

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. We define $G_1 + G_2$, as the graph $G = (V, E)$ where:

- $V = V(G_1) \cup V(G_2)$
- $E = E(G_1) \cup E(G_2)$

Connectedness

A graph G is **connected** if it cannot be represented as the sum $G_1 + G_2$ for some non-empty graphs G_1 and G_2. Otherwise, the graph is called **disconnected**. Equivalently, a graph is connected if there is a path between every pair of distinct vertices.

Connected Components

If a graph G can be expressed as a sum $G = G_1 + G_2 + \cdots + G_n$ for some $n \in \mathbb{N}$, where each G_i is a connected graph, then each G_i is called a **connected component** of G.

Adjacency and Incidence

As previously mentioned, if $u, v \in V(G)$ and $uv \in E(G)$, we say that vertices u and v are **adjacent** (or **neighbours**).

If for a vertex $v \in V(G)$ and an edge $e \in E(G)$, we have $v \in e$ then we say that vertex v is **incident** to edge e.

Vertex Degree

For a vertex $v \in V(G)$, we define its **degree**, as:

$$\deg_G(v) = |\{u \in V(G) : vu \in E(G)\}|$$

Isolated and Leaf Vertices

- If $\deg_G(v) = 0$, then vertex v is called **isolated**.
- If $\deg_G(v) = 1$, then vertex v is called a **leaf**.

Degree Sequence

Let G be a graph with vertices $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then $sort[deg(v_1), \ldots, deg(v_n)]$ is called the **degree sequence** of G

Handshaking Lemma

In any graph $G = (V, E)$, the sum of the degrees of all vertices is equal to twice the number of edges:

$$\sum_{v \in V(G)} \deg_G(v) = 2|E(G)|$$

Proof Sketch. Each edge $e = \{u, v\}$ contributes exactly 1 to the degree of vertex u and exactly 1 to the degree of vertex v. Therefore, when summing all degrees, each edge is counted twice. □

Corollary 1. *In any graph G, the number of vertices with an odd degree is even.*

Subgraphs

A **subgraph** H of a graph G is a graph $H = (V_1, E_1)$, such that

- $V_1 \subseteq V(G)$
- $E_1 \subseteq E(G)$