Preparation Tasks 2

Tomasz Brengos
Committers : Aliaksei Kudzelka

1 Problem 1

Prove that the num. of partitions of a positive int n into k *even* parts is eq. to the num. of partitions of n into k odd parts.

- **Idea:** First, express n in two ways by decomposing it into k even parts and into k odd parts. We can write

$$\begin{aligned} n &= 2a_1 + 2a_2 + \cdots + 2a_k, \\ n-k &= 2a_1 + 2a_2 + \cdots + 2a_k - k, \\ n-k &= (2a_1 - 1) + (2a_2 - 1) + \cdots + (2a_k - 1)\,. \end{aligned}$$

  The first line expresses n as a sum of k even numbers (each $2a_i$). Subtracting k from both sides shows that $n-k$ can be expressed as a sum of k odd numbers (each $2a_i - 1$).

  From this idea, we expect a correspondence between partitions of n into k even parts and partitions of $n-k$ into k odd parts.

- **Formal proof:** Define the sets

$$P_e(n,k) := \{\text{partitions of } n \text{ into } k \text{ even parts}\},$$

  and

$$P_o(n,k) := \{\text{partitions of } n \text{ into } k \text{ odd parts}\}.$$

  For n even, it turns out these two sets have the same size:

$$|P_e(n,k)| = |P_o(n-k,k)|\,.$$

  To show this, we construct an explicit bijection. Define a function

$$f : P_e(n,k) \to P_o(n-k,k)\,,$$

  by

$$f(b_1, b_2, \ldots, b_k) = (\, b_1 - 1,\ b_2 - 1,\ \ldots,\ b_k - 1\,)\,.$$

  Here $(b_1, \ldots, b_k) \in P_e(n,k)$ means $b_1 + \cdots + b_k = n$ with each b_i even, so $b_i \geq 2$ for all i. Thus, each $b_i - 1 \geq 1$ and is odd, which ensures $f(b_1, \ldots, b_k) \in P_o(n-k,k)$ is a partition of $n-k$ into k odd parts.

  The function f is well-defined. Moreover, it is invertible by simply adding 1 to each part. In fact, define

$$g : P_o(n-k,k) \to P_e(n,k)$$

  as

$$g(c_1, c_2, \ldots, c_k) = (\, c_1 + 1,\ c_2 + 1,\ \ldots,\ c_k + 1\,)\,.$$

  If $(c_1, \ldots, c_k)$ is a partition of $n-k$ into odd parts, then each c_i is odd and $c_i \geq 1$, so $c_i + 1$ is even and ≥ 2, and $\sum_{i=1}^{k}(c_i + 1) = (n-k) + k = n$. Thus $g(c_1, \ldots, c_k) \in P_e(n,k)$. It is easy to check that g is indeed the inverse of f: we have $f(g(c_1, \ldots, c_k)) = (c_1, \ldots, c_k)$ and $g(f(b_1, \ldots, b_k)) = (b_1, \ldots, b_k)$. Therefore, f is bijective. ■

2 Problem 2

Show that the number of partitions of a positive int n with at most k components is eq. to the num. of partitions of $2n$ with at most k even components.

- **Idea:** We consider partitions of n that have at most k parts. Let

$$n = a_1 + a_2 + \cdots + a_L,$$

where $L \le k$ and $a_1 \ge a_2 \ge \cdots \ge a_L > 0$. (In other words, $a_1, \ldots, a_L$ are the parts of a partition of n, listed in non-increasing order, with at most k parts.) For example, if $n = 5$ and $k = 3$, the partitions of 5 with at most 3 parts can be represented (padding with zeros up to 3 parts) as:

$$(2,2,1), \qquad (3,2,0), \qquad (5,0,0)\,,$$

where we use 0 to indicate an empty part (no number in that position).

Notice that doubling each part in these examples produces a partition of $2n = 10$ with only even parts (and still at most 3 components). For instance, $(2,2,1)$ doubles to $(4,4,2)$, $(3,2,0)$ doubles to $(6,4,0)$, and $(5,0,0)$ doubles to $(10,0,0)$. This suggests a direct correspondence between partitions of n (up to k parts) and partitions of $2n$ into even parts (up to k parts).

- **Formal proof:** Let us define the relevant sets in words (as suggested in the notes):

  - $P(n, \le k)$ — the set of all partitions of n with *at most* k components (parts).
  - $P_e(2n, \le k)$ — the set of all partitions of $2n$ with at most k *even* components.

We aim to show that

$$|P(n, \le k)| = |P_e(2n, \le k)|\,,$$

i.e. the two sets have equal cardinality. To prove this, we construct a bijection $f : P(n, \le k) \to P_e(2n, \le k)$. Given any partition $(a_1, a_2, \ldots, a_L)$ of n (with $L \le k$), map it to

$$f(a_1, a_2, \ldots, a_L) = (\,2a_1,\ 2a_2,\ \ldots,\ 2a_L\,)\,.$$

In other words, f doubles each part of the partition of n. If the partition of n has fewer than k parts, we may imagine that it is padded with zeros (as above) which double to zeros, so the resulting partition of $2n$ still has at most k parts. By construction, $f(a_1, \ldots, a_L)$ is a partition of $2n$ in which every part is even, so indeed $f(a_1, \ldots, a_L) \in P_e(2n, \le k)$. The function f is invertible by halving each even part: for any partition $(b_1, b_2, \ldots, b_M) \in P_e(2n, \le k)$ (each b_i even), the inverse map f^{-1} gives

$$f^{-1}(b_1, b_2, \ldots, b_M) = \left(\frac{b_1}{2},\ \frac{b_2}{2},\ \ldots,\ \frac{b_M}{2}\right),$$

which is a partition of $2n/2 = n$ with at most k parts. Thus f is a bijection between $P(n, \le k)$ and $P_e(2n, \le k)$, and consequently $|P(n, \le k)| = |P_e(2n, \le k)|$. ■