Functions between sets

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Let N and R be sets with $|N| = n$ and $|R| = r$.

(i) **Total Functions:** The number of functions from N to R is

$$r^n.$$

Explanation: For every element in N, there are $|R| = r$ possible values in R. Thus, for the first element, there are r choices, for the second element, there are r choices, and so on. Applying the rule of product, the total number of functions is r^n.

(ii) **Injective Functions:** When $r \geq n$, an injective function (one-to-one) from N to R can be chosen by assigning distinct images to the n elements.

If a function is injective, then for each value in the range there is only one corresponding argument. This means that function values cannot repeat, ensuring that $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

Since there are $|R| = r$ choices for the first argument, $r - 1$ choices for the second, $r - 2$ for the third, and so on, applying the rule of product, the number of injective functions from N to R is:

$$r \cdot (r-1) \cdots (r-n+1) = \frac{r!}{(r-n)!}.$$

(iii) **Surjective Functions:** A function is surjective (onto) if every element in R has a pre-image in N, meaning every element in R is an image of some element in N. Consider a surjection $f : N \to R = \{y_1, y_2, \ldots, y_r\}$. We observe that the preimages $f^{-1}(y_1), f^{-1}(y_2), \ldots, f^{-1}(y_r)$ form a partition of N into r non-empty subsets, as each element y_i in R corresponds to one or more elements from N. The number of ways to partition N into r parts is given by the Stirling number $S(n, r)$, and since we can permute the r elements in R in $r!$ ways, the total number of surjective functions from N to R is:

$$r!\, S(n, r),$$

where $S(n, r)$ is the Stirling number of the second kind, counting the ways to partition N into r non-empty subsets.

Example: For $N = \{1, 2, 3\}$ and $R = \{y_1, y_2\}$:
Here $|N| = 3$ and $|R| = 2$.

- Total functions: $2^3 = 8$.
- Injective functions: Not possible since $|R| < |N|$.
- Surjective functions: Consider all possible surjective functions:
  $f_1 : \{1, 2\} \mapsto y_1, 3 \mapsto y_2$ - Another possible permutation for this partition: $f_2 : \{1, 2\} \mapsto y_2, 3 \mapsto y_1$
  $f_3 : \{2, 3\} \mapsto y_1, 1 \mapsto y_2$ - Another possible permutation for this partition: $f_4 : 1 \mapsto y_1, \{2, 3\} \mapsto y_2$
  $f_5 : \{1, 3\} \mapsto y_1, 2 \mapsto y_2$ - Another possible permutation for this partition: $f_6 : 2 \mapsto y_1, \{1, 3\} \mapsto y_2$
  So, we have 6 surjective functions. Using the formula for surjective functions, we first find the Stirling number $S(3, 2) = 3$, which corresponds to the number of partitions without considering permutations. Then, accounting for the permutations of the $r = 2$ elements in R, we compute:

$$2! \cdot S(3, 2) = 2! \cdot 3 = 6,$$

which matches the number of surjective functions we listed.