

# Permutations and Derangements

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## 1 Permutations

A *permutation* of  $n$  elements is an arrangement (ordering) of those elements. For example, there are 6 permutations of the set  $\{a, b, c\}$ :

$$(a, b, c), \quad (a, c, b), \quad (b, a, c), \quad (b, c, a), \quad (c, a, b), \quad (c, b, a).$$

Since there are 3 choices for the first element, 2 for the second (once the first is chosen), and 1 for the last, by the multiplicative principle there are  $3 \cdot 2 \cdot 1 = 3! = 6$  permutations in total.

**Factorials and counting.** In general, the number of permutations of  $n$  (distinct) elements is given by

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

**Partial permutations (k-permutations).** Sometimes we only permute  $k$  of the  $n$  elements, where  $1 \leq k \leq n$ . The number of ways to do this is denoted  $P(n, k)$  and can be found by thinking:

$$P(n, k) = n \times (n-1) \times \cdots \times (n-k+1).$$

There are  $k$  factors in that product. Using factorial notation, we can write

$$P(n, k) = \frac{n!}{(n-k)!}.$$

**Relationship to combinations.** An alternate derivation uses combinations: first *choose* which  $k$  elements from the  $n$  will appear (that can be done in  $\binom{n}{k}$  ways), then *arrange* those  $k$  in order (which can be done in  $k!$  ways). Hence,

$$P(n, k) = \binom{n}{k} k!.$$

Since  $\binom{n}{k} = \frac{n!}{(n-k)! k!}$ , multiplying by  $k!$  yields exactly  $\frac{n!}{(n-k)!}$ , consistent with the direct counting approach.

## 2 Derangements

A *derangement* of  $n$  elements is a permutation where no element remains in its original position. More precisely, if we think of a permutation as a bijection  $\theta$  on the set  $\{1, 2, \dots, n\}$ , then  $\theta$  is a derangement if and only if

$$\theta(k) \neq k \quad \text{for all } k \in \{1, 2, \dots, n\}.$$

Equivalently, a derangement has no fixed points.

For example, for  $n = 3$ , the permutations of  $\{1, 2, 3\}$  are:

$$(1, 2, 3), \quad (1, 3, 2), \quad (2, 1, 3), \quad (2, 3, 1), \quad (3, 1, 2), \quad (3, 2, 1).$$

Among these, the derangements are  $(2, 3, 1)$  and  $(3, 1, 2)$ ; the other permutations fix at least one of the elements.

## Counting Derangements via Inclusion-Exclusion

Let  $D(n)$  denote the number of derangements of  $n$  elements. We will use the principle of inclusion-exclusion. Suppose we label the elements as  $1, 2, \dots, n$ , and define  $A_i$  to be the set of permutations that fix the element  $i$  (i.e.  $\theta(i) = i$ ). Then any derangement is a permutation that lies in none of the sets  $A_i$  (for  $1 \leq i \leq n$ ). We have

$$|A_i| = (n-1)!,$$

since if we fix one position  $i$ , then we permute the remaining  $n-1$  elements freely. In general,

$$|A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| = (n-k)!.$$

By inclusion-exclusion, the size of the union  $A_1 \cup A_2 \cup \dots \cup A_n$  is

$$\sum_{k=1}^n (-1)^{k+1} \binom{n}{k} (n-k)!.$$

Hence the number of permutations that do not lie in this union—i.e. the number of derangements—is

$$\begin{aligned} D(n) &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \dots + (-1)^n \binom{n}{n}(n-n)!. \\ D(n) &= \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)! = n! \sum_{k=0}^n \frac{(-1)^k}{k!}. \end{aligned}$$

Thus, a concise closed-form for the number of derangements is

$$D(n) = n! \sum_{k=0}^n \frac{(-1)^k}{k!}.$$

### Note on the series for $e^{-1}$ :

In Calculus, one learns that the exponential function has a power series expansion

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Setting  $x = -1$  gives

$$e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}.$$

Hence,

$$\sum_{k=0}^n \frac{(-1)^k}{k!} \xrightarrow{n \rightarrow \infty} e^{-1}.$$

If you have not taken (or do not recall) a full course in Calculus, think of this as a special case of a well-known infinite series expansion for the exponential function.

Since the finite sum  $\sum_{k=0}^n \frac{(-1)^k}{k!}$  converges to  $e^{-1}$  as  $n \rightarrow \infty$ , we conclude that

$$\lim_{n \rightarrow \infty} \frac{D(n)}{n!} = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(-1)^k}{k!} = e^{-1}.$$

Numerically, this means that for large  $n$ , about  $1/e \approx 36.8\%$  of all permutations of  $\{1, \dots, n\}$  are derangements (i.e. have no fixed points).

## A Recurrence Relation

We can also show that  $D(n)$  satisfies the recurrence

$$D(n) = (n-1)(D(n-1) + D(n-2)), \quad \text{with } D(1) = 0, D(2) = 1.$$

One way to see this: consider where 1 goes in a derangement of  $\{1, 2, \dots, n\}$ . It can go to any of  $n-1$  positions. If 1 goes to position  $j$ , then either (i) the element  $j$  goes to position 1 (a swap), which reduces the problem to deranging the remaining  $n-2$  elements, or (ii) the element  $j$  does *not* go to position 1, effectively reducing the problem to deranging  $n-1$  elements. This yields the above recurrence.