

# Counting (Combinatorics)

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April 11, 2025

## 1 Rule of Sum (Addition Principle)

If a set  $S$  is partitioned into disjoint subsets,

$$S = S_1 \cup S_2 \cup \cdots \cup S_k,$$

then the total number of elements in  $S$  is the sum of the number of elements in each subset:

$$|S| = |S_1| + |S_2| + \cdots + |S_k|.$$

**Example:** Suppose we wish to count the number of ways to choose a subset of a set  $X$  of size  $n$ , but we only consider subsets of a fixed size  $k$ . If we let  $S$  be the family of all such subsets, then using the rule of sum by dividing the choices according to a distinguished element (say, whether a chosen element is included or not) we can count the subsets by summing over the possibilities. (This idea is used later in proofs for binomial coefficients and the power set.)

**Theorem:**

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

**Proof:** Consider  $S = \binom{X}{k}$ , the set of all subsets of  $X$  of size  $k$ . Take any element  $a \in X$ . Define:  $S_1$  as the subsets in  $S$  that contain  $a$  and  $S_2$  as the subsets in  $S$  that do not contain  $a$ .

Since every subset of  $S$  either contains  $a$  or does not, we see that  $S_1$  and  $S_2$  are disjoint and their union forms  $S$ , i.e.,

$$S_1 \cup S_2 = S.$$

By the rule of sum, we get:

$$|S| = |S_1| + |S_2|.$$

Now, each subset in  $S_1$  must contain  $a$ , so we choose the remaining  $k-1$  elements from  $X \setminus \{a\}$ , which has  $n-1$  elements. Thus,  $|S_1| = \binom{n-1}{k-1}$ . Each subset in  $S_2$  does not contain  $a$ , so we choose all  $k$  elements from  $X \setminus \{a\}$ . Thus,  $|S_2| = \binom{n-1}{k}$ .

Therefore,

$$\binom{n}{k} = |S| = |S_1| + |S_2| = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

**Example:** Let  $S = \{\triangle, \square, \circ\}$  and  $k = 2$ , choosing  $a = \circ$ . Fixing  $\circ$  as one of the elements in the subset of size  $k$ , we get:

$$S_1 = \{\{\circ, \triangle\}, \{\circ, \square\}\}.$$

Taking all subsets of size  $k$  without  $\circ$ :

$$S_2 = \{\{\triangle, \square\}\}.$$

We have  $|S_1| = 2$  and  $|S_2| = 1$ , so  $|S_1| + |S_2| = 3$ .

On the other hand,

$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = 3.$$

Thus,  $|S| = |S_1| + |S_2| = 3$ , verifying the identity.

## 2 Rule of Product (Multiplication Principle)

When an object is constructed by a sequence of choices, where:

- The first choice can be made in  $a$  ways,
- The second in  $b$  ways,
- ...

the total number of objects is the product:

$$a \times b \times \cdots$$

**Example:** A word of length  $n$  over the binary alphabet  $\{0, 1\}$  is formed by choosing one of 2 possibilities for each position. Hence, there are

$$2^n$$

possible words.

## 3 Rule of Bijection

If there exists a bijection (a one-to-one and onto mapping) between two sets  $S$  and  $T$ , then they have the same number of elements:

$$|S| = |T|.$$

**Example:** Consider the power set of a set  $X$ , denoted by  $\mathcal{P}(X)$ . There is a natural bijection between  $\mathcal{P}(X)$  and the set of binary sequences of length  $|X|$ : for each subset  $A \subseteq X$ , assign the sequence  $(a_1, a_2, \dots, a_n)$  where

$$a_i = \begin{cases} 1, & \text{if } x_i \in A, \\ 0, & \text{if } x_i \notin A. \end{cases}$$

This shows that

$$|\mathcal{P}(X)| = 2^{|X|}.$$

## 4 Counting in Two Ways

**Rule of Counting in Two Ways** When two formulae enumerate the same quantity, they must be equal.

**Example:**

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

**Proof:** Consider a lattice grid of size  $(n+1) \times (n+1)$ , defined as:

$$X = \{(i, j) \mid i, j \in \{1, 2, \dots, n+1\}\}.$$

Clearly,  $|X| = (n+1)^2$ .

Now, partition  $X$  into three subsets: -  $X_1$ , the points strictly below the secondary diagonal. -  $X_2$ , the points strictly above the secondary diagonal. -  $X_3$ , the points on the secondary diagonal itself.

Since these three sets form a partition, we have:

$$|X| = |X_1| + |X_2| + |X_3|.$$

Observing their sizes:

$$|X_1| = |X_2| = 1 + 2 + \dots + n, \quad |X_3| = n + 1.$$

Thus,

$$(n+1)^2 = 2(1 + 2 + \dots + n) + (n+1).$$

Rearranging, we get:

$$1 + 2 + \dots + n = \frac{(n+1)^2 - (n+1)}{2} = \frac{n(n+1)}{2}.$$

Hence, we have proven the formula:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}.$$

## 5 Binomial Coefficients and Permutations

Let  $X$  be a set with  $|X| = n$ .

**Subsets:** The number of ways to choose a  $k$ -subset of  $X$  is given by the binomial coefficient

$$\binom{n}{k}.$$

**Permutations:** A  $k$ -permutation of a set  $X$  of size  $n$  is a  $k$ -word over the alphabet  $X$  whose entries are distinct.

**Theorem:** There are exactly

$$n(n-1)(n-2)\dots(n-k+1)$$

$k$ -permutations of an  $n$ -set.

**Question:** How are  $k$ -permutations of an  $n$ -set related to  $k$ -subsets of an  $n$ -set?

**Answer:** The difference between a  $k$ -permutation and a  $k$ -subset is that a permutation is ordered, while a subset is not. To express a  $k$ -permutation in terms of a  $k$ -subset, we need to account for all possible arrangements of the elements, which is  $k!$ . Thus,

$$k\text{-permutation} = \binom{n}{k} \cdot k!$$

Expressing  $\binom{n}{k}$  as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

we obtain:

$$k\text{-permutation} = \frac{n!}{(n-k)!}.$$

**Proof by Counting in Two Ways:** Count the number of  $k$ -permutations of an  $n$ -set in two ways:

(1) Directly, by applying the rule of product:

$$n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}.$$

(2) First choose a  $k$ -subset (in  $\binom{n}{k}$  ways) and then arrange it (in  $k!$  ways), giving

$$\binom{n}{k} \cdot k!.$$

Equate these two counts to obtain the relation.

## 6 Binomial Theorem

For any  $x, y$  in a field and nonnegative integer  $n$ , the binomial theorem states:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

**Explanation:** This theorem is a direct consequence of counting the number of ways to choose  $k$  copies of  $x$  (and the remaining  $n-k$  copies of  $y$ ) when expanding the product.

## 7 Multisets

**Definition:** A multiset of a set  $X$  of size  $n$  is a function

$$m : X \rightarrow \mathbb{N}$$

that assigns a non-negative integer to each element of  $X$ , representing its multiplicity in the multiset.

**Example:** Let  $X = \{a, b, c\}$ , and consider the multiset  $\{a, a, b\}$ . Then, the function  $m$  is given by:

$$m(a) = 2, \quad m(b) = 1, \quad m(c) = 0.$$

**Question:** What is the number of  $k$ -multisets of a set of size  $n$ ?

**Theorem:** The number of all  $k$ -multisets of an  $n$ -set is

$$\binom{n+k-1}{k}.$$

**Proof:** Let  $X$  be the set of all  $k$ -multisets of an  $n$ -set. Let  $Y$  be the set of all distributions of  $k$  identical objects into  $n$  buckets.

**Claim 1:** There is a bijection from  $X$  to  $Y$ . Thus, by the rule of bijection, we have

$$|X| = |Y|.$$

**Claim 2:** Let  $Z$  be the set of all binary sequences of length  $n+k-1$  with exactly  $n-1$  ones (or equivalently,  $k$  zeros). There is a bijection from  $Y$  to  $Z$ . Hence,

$$|Y| = |Z| \Rightarrow |X| = |Z|.$$

Since the number of such binary sequences is given by

$$\binom{n+k-1}{k},$$

we conclude that

$$|X| = \binom{n+k-1}{k}.$$

## 8 Lattice Paths

Consider an  $m \times n$  grid with lattice points at the intersections.

**Problem:** How many paths are there from  $(0,0)$  to  $(m,n)$  if one may only move right or up?

**Solution:** Every path consists of exactly  $m$  right moves and  $n$  up moves. Thus, a path can be represented as a sequence of  $m+n$  moves, where we choose  $n$  positions (out of  $m+n$ ) for the up moves. Hence, the number of paths is:

$$\binom{m+n}{n}.$$

**Bijection Explanation:** There is a bijection between the set of such lattice paths and the set of binary sequences of length  $m+n$  with exactly  $n$  ones (representing the up moves).