Generating Functions

Tomasz Brengos
Committers : Mykhailo Moroz, Mihail Orlov

1 Generating Series

Instead of viewing a sequence as a function that returns its nth term, a *generating series* packages all of its terms into a single power series whose coefficients are exactly the sequence entries. Concretely, the sequence

$$2,\ 3,\ 5,\ 8,\ 12,\ \ldots$$

is encoded by the generating series

$$2 + 3x + 5x^2 + 8x^3 + 12x^4 + \cdots .$$

In general, given any sequence $\{c_n\}_{n \geq 0}$, its generating series is the formal power series

$$G(x) \ = \ \sum_{n=0}^{\infty} c_n x^n \ = \ c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots .$$

We say that $G(x)$ "generates" the sequence $\{c_n\}$ because each coefficient of x^n in $G(x)$ is precisely c_n. Generating series turn sequence-based problems into algebraic manipulations of power series, a technique we will exploit heavily in what follows.

Recall of the Basic Series

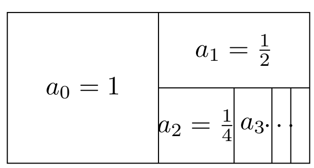

Figure 1: A geometric interpretation of the binary series, showing how $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$.

A Geometric View of the Binary Series For $|x| < 1$, we have the infinite geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n .$$

We now present a quick proof of this result by performing long division of 1 by $1 - x$.

$$
\begin{array}{r|l}
 & 1+x+x^2+x^3+\cdots \\
1-x & 1 \\ \hline
 & \underline{1-x} \\
 & x \\
 & \underline{x-x^2} \\
 & x^2 \\
 & \underline{x^2-x^3} \\
 & x^3 \\
 & \vdots
\end{array}
$$

The process works as follows: The long-division proceeds by repeatedly dividing the current remainder by the leading term of the divisor, producing one new power of x at each step:

1. Divide 1 by $1-x$. The multiplier needed to eliminate the constant term is 1, so

$$1 - 1 \cdot (1-x) = x.$$

   Thus the first summand is 1, leaving a remainder of x.

2. Divide the remainder x by $1-x$. The multiplier is x, so

$$x - x \cdot (1-x) = x^2.$$

   Hence the second summand is x, leaving a remainder of x^2.

3. Divide x^2 by $1-x$. The multiplier is x^2, giving

$$x^2 - x^2 \cdot (1-x) = x^3.$$

   Therefore the third summand is x^2, with remainder x^3.

4. Continuing in this fashion produces the infinite series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots .$$

Continuing indefinitely produces

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n,$$

as claimed.
We will use this fact in further examples throughout the notes.

2 Building Generating Functions

The simplest (or "basic") generating function is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots ,$$

which generates the constant sequence $1, 1, 1, \ldots$.

Replacing x with $-x$:

$$\frac{1}{1-(-x)} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots ,$$

generating $1, -1, 1, -1, \ldots$.

Replacing x with $3x$:

$$\frac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + \cdots ,$$

generating $1, 3, 9, 27, \ldots$.

Scaling a sequence by 3:

$$\frac{3}{1-3x} = 3 + 9x + 27x^2 + 81x^3 + \cdots ,$$

generating $3, 9, 27, 81, \ldots$.

Termwise addition of sequences:
Adding the generating functions for $1, 1, 1, \ldots$ and $1, 3, 9, \ldots$ gives

$$\frac{1}{1-x} + \frac{1}{1-3x} = 2 + 4x + 10x^2 + 28x^3 + \cdots ,$$

which generates $2, 4, 10, 28, \ldots$.

Replacing x with x^2:

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots ,$$

generating $1, 0, 1, 0, 1, 0, \ldots$.

Shifting a sequence:
Multiplying by x shifts all coefficients right by one:

$$\frac{x}{1-3x} = 0 + x + 3x^2 + 9x^3 + \cdots ,$$

generating $0, 1, 3, 9, \ldots$, and

$$\frac{x}{1-x^2} = 0 + x + 0x^2 + x^3 + \cdots ,$$

generating $0, 1, 0, 1, \ldots$.

Combining shifted sequences:
Adding the two "even-odd" generating functions recovers

$$\frac{1}{1-x^2} + \frac{x}{1-x^2} = \frac{1+x}{1-x^2} = \frac{1}{1-x},$$

which generates $1, 1, 1, 1, \ldots$.

Differentiation:
Differentiating the basic Generating Function

$$\frac{d}{dx}\Big(\frac{1}{1-x}\Big) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots ,$$

yields the generating function for $1, 2, 3, 4, \ldots$.

3 Recurrence Relations & Generating Functions

We conclude with an example of one of the many reasons studying generating functions is helpful: solving recurrence relations via algebraic manipulation of power series.

Example: Tower of Hanoi The minimum number of moves required to transfer n disks satisfies

$$a_0 = 0, \quad a_1 = 1, \quad a_n = 2\,a_{n-1} + 1 \quad (n \geq 1),$$

giving the sequence

$$0,\ 1,\ 3,\ 7,\ 15,\ 31, \ldots$$

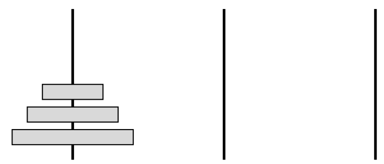

Figure 2: Initial configuration for Tower of Hanoi (3 disks).

Define the generating function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Using the recurrence for $n \geq 1$:

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \big(2a_{n-1} + 1\big) x^n = 2x \sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} x^n,$$

so

$$f(x) - a_0 = 2x\, f(x) + \frac{x}{1-x},$$

and since $a_0 = 0$,

$$f(x) = \frac{x}{(1-x)(1-2x)}.$$

Performing partial fractions:

$$\frac{x}{(1-x)(1-2x)} = \frac{-1}{1-x} + \frac{1}{1-2x},$$

hence

$$f(x) = -\frac{1}{1-x} + \frac{1}{1-2x}.$$

Extracting coefficients yields the closed-form solution

$$a_n = 2^n - 1,$$

confirming the well-known formula for the Tower of Hanoi moves.

4 Introduction to the Fibonacci Sequence

The Fibonacci sequence famously arises from a puzzle involving rabbit populations. Imagine starting with a single pair of rabbits that takes one month to mature. After maturing, each pair produces a new pair of rabbits every month. Mathematically, if F_n represents the number of rabbit pairs in month n, the sequence satisfies the initial conditions

$$F_0 = 0, \quad F_1 = 1,$$

and the recurrence

$$F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 0.$$

Q: is there a non-recursive (closed-form) formula for F_n ?

1) || (2 small rabbits)

2) || + || (1 big pair + 1 small pair)

3) || + || + || (2 big pairs + 1 small pair)

4) || + || + || + || + || (3 big pairs + 2 small pairs)

Figure 3: Illustration of rabbit pairs over successive months. Blue bars represent small rabbits; orange bars represent big (mature) rabbits.

Idea: consider and calculate it

5 Deriving the Closed-Form for the Fibonacci Sequence

Step 1: Define the generating function. Let $\{F_n\}_{n=0}^{\infty}$ be the Fibonacci sequence with

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0).$$

Define the generating function

$$f(x) \;=\; \sum_{n=0}^{\infty} F_n\, x^n.$$

We aim to find a closed-form expression for $f(x)$, and then extract a formula for F_n.

Step 2: Use the Fibonacci recurrence in $f(x)$. Starting from

$$f(x) \;=\; F_0 + F_1 x + \sum_{n=2}^{\infty} F_n\, x^n,$$

and noting $F_0 = 0$, $F_1 = 1$, we have

$$f(x) \;=\; x + \sum_{n=2}^{\infty} \bigl(F_{n-1} + F_{n-2}\bigr)\, x^n$$

because $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Separate the sums:

$$f(x) = x + \sum_{n=2}^{\infty} F_{n-1}\, x^n + \sum_{n=2}^{\infty} F_{n-2}\, x^n.$$

Shift indices to factor out $f(x)$:

$$\sum_{n=2}^{\infty} F_{n-1}\, x^n = x \sum_{n=2}^{\infty} F_{n-1}\, x^{n-1} = x \sum_{m=1}^{\infty} F_m\, x^m = x\bigl(f(x) - F_0\bigr) = x\, f(x),$$

since $F_0 = 0$. Similarly,

$$\sum_{n=2}^{\infty} F_{n-2}\, x^n = x^2 \sum_{n=2}^{\infty} F_{n-2}\, x^{n-2} = x^2 \sum_{k=0}^{\infty} F_k\, x^k = x^2\, f(x).$$

Hence,

$$f(x) = x + x\, f(x) + x^2\, f(x) \quad\Longrightarrow\quad f(x)\bigl(1 - x - x^2\bigr) = x.$$

Thus,

$$f(x) = \frac{x}{1 - x - x^2}.$$

Step 3: Partial-Fraction Decomposition (as in the images). First, rewrite

$$\frac{1}{1-x-x^2} \;=\; \frac{1}{-\bigl(x^2+x-1\bigr)} \;=\; -\,\frac{1}{x^2+x-1}.$$

Next, factor x^2+x-1. Observe that the roots of

$$x^2+x-1 \;=\; 0$$

are

$$x \;=\; -\,\frac{1+\sqrt{5}}{2} \quad \text{and} \quad x \;=\; -\,\frac{1-\sqrt{5}}{2}.$$

Hence,

$$x^2+x-1 \;=\; \Bigl(x+\tfrac{1+\sqrt{5}}{2}\Bigr)\,\cdot\,\Bigl(x+\tfrac{1-\sqrt{5}}{2}\Bigr).$$

Therefore,

$$-\,\frac{1}{x^2+x-1} \;=\; -\,\frac{1}{\Bigl(x+\tfrac{1+\sqrt{5}}{2}\Bigr)\Bigl(x+\tfrac{1-\sqrt{5}}{2}\Bigr)}.$$

We look for constants A and B such that

$$-\,\frac{1}{\Bigl(x+\tfrac{1+\sqrt{5}}{2}\Bigr)\Bigl(x+\tfrac{1-\sqrt{5}}{2}\Bigr)} \;=\; \frac{A}{x+\tfrac{1+\sqrt{5}}{2}} \;+\; \frac{B}{x+\tfrac{1-\sqrt{5}}{2}}.$$

Step 4: Solve for A and B. Comparing coefficients of x and the constant term in

$$-1 \;=\; A\Bigl(x+\beta\Bigr) \;+\; B\Bigl(x+\alpha\Bigr),$$

we obtain the system

$$\begin{cases} A+B=0, \\ A\,\beta+B\,\alpha=-1. \end{cases}$$

It follows that

$$B=-A, \quad A\,(\beta-\alpha)=-1 \implies A=\frac{1}{\alpha-\beta} \quad \text{and} \quad B=-\,\frac{1}{\alpha-\beta}.$$

Hence,

$$-\,\frac{1}{(x+\alpha)(x+\beta)} \;=\; \frac{1}{\alpha-\beta}\,\frac{1}{x+\alpha} \;-\; \frac{1}{\alpha-\beta}\,\frac{1}{x+\beta}.$$

Step 5: Combine with the earlier factor -1 and rewrite. Recalling that

$$\frac{1}{1-x-x^2} \;=\; -\,\frac{1}{x^2+x-1} \;=\; -\,\frac{1}{(x+\alpha)\,(x+\beta)},$$

we combine the above result to conclude

$$\frac{1}{1-x-x^2} \;=\; \frac{1}{\alpha-\beta}\left(\frac{1}{x+\alpha} \;-\; \frac{1}{x+\beta}\right).$$

Step 6: Expand each term in a power series. Notice that

$$\frac{1}{x+\alpha} = \frac{1}{\alpha}\frac{1}{1+\frac{x}{\alpha}} = \frac{1}{\alpha}\sum_{n=0}^{\infty}\left(-\tfrac{x}{\alpha}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{\alpha^{n+1}}x^n,$$

valid for $\left|\frac{x}{\alpha}\right| < 1$. Similarly,

$$\frac{1}{x+\beta} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\beta^{n+1}}x^n.$$

Hence,

$$\frac{1}{1-x-x^2} = \frac{1}{\alpha-\beta}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{\alpha^{n+1}}x^n - \sum_{n=0}^{\infty}\frac{(-1)^n}{\beta^{n+1}}x^n\right] = \sum_{n=0}^{\infty}\left[\frac{1}{\alpha-\beta}\left(\frac{(-1)^n}{\alpha^{n+1}} - \frac{(-1)^n}{\beta^{n+1}}\right)\right]x^n.$$

Step 7: Identify Fibonacci numbers. Recall that $\alpha - \beta = \sqrt{5}$, and

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}}.$$

One checks (or uses known identities) to see that the coefficient of x^n in the above power series is exactly F_n. Consequently,

$$\sum_{n=0}^{\infty} F_n x^n = \frac{1}{1-x-x^2},$$

which is the generating function for the Fibonacci sequence.

Conclusion. We have shown that the generating function for the Fibonacci sequence is $\frac{x}{1-x-x^2}$. Through partial fractions and comparing coefficients, we deduced that

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}.$$

This gives a non-recursive (closed-form) expression for F_n, completing the derivation.

6 More examples

In earlier sections (see, e.g., *A Geometric View of the Binary Series* on page 14), we explored methods to solve recurrences and introduced generating functions as a tool to transform sequences into functions. In this section, we briefly reiterate these ideas and demonstrate, through several examples, how generating functions serve as a bridge between discrete mathematics and calculus.

Example 1: Constant Sequence. Consider the sequence defined by

$$a_n = 1 \quad \text{for all } n \geq 0,$$

so that the sequence is

$$1, 1, 1, \ldots.$$

By the geometric series formula (proved earlier), its generating function is given by

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

Example 2: Exponential Sequence. Now, let

$$a_n = \frac{1}{n!}.$$

Then the generating function is

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x, \quad x \in \mathbb{R}.$$

A partial justification of this result can be obtained by recalling the Taylor series expansion of the exponential function. Although a complete treatment of Taylor series is a topic in calculus (not yet covered in this course), note that differentiating the power series term-by-term confirms the identity.

Example 3: Binomial Coefficient Sequence. Consider the sequence defined by

$$a_n = \binom{n+k}{k}.$$

Theorem. The generating function for this sequence is

$$f(x) = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}}, \quad |x| < 1.$$

Proof.

- For $k = 1$: Note that

  $$a_n = \binom{n+1}{1} = n + 1,$$

  so that

  $$f(x) = \sum_{n=0}^{\infty} \binom{n+1}{1} x^n = \sum_{n=0}^{\infty} (n+1)x^n.$$

  Recall the geometric series,

  $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

  and observe that by differentiating both sides term-by-term with respect to x, we can derive the generating function for the sequence $(n + 1)$. In detail, differentiate the left-hand side:

  $$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

  On the right-hand side, notice that since

  $$\frac{d}{dx} x^{n+1} = (n+1)x^n,$$

  differentiating the series yields

  $$\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} (n+1)x^n.$$

  Thus, we conclude that

  $$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

  This recovers the generating function for $k = 1$. A less formal derivation was given in the subsection *Building Generating Functions* on page 16.

A complete inductive proof follows similar lines but is omitted here for brevity.

Example 4: Alternating Factorial Sequence. Define the sequence by

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \dfrac{(-1)^{\frac{n-1}{2}}}{n!}, & \text{if } n \text{ is odd.} \end{cases}$$

Then the generating function is

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sin(x), \quad x \in \mathbb{R}.$$

Even though a full treatment of the Taylor series for trigonometric functions is part of calculus (again, a topic not yet covered here), this example illustrates how generating functions capture nontrivial sequence behavior by connecting discrete structures with analytic functions.

7 Generating Function Applications

One key application is the multiplication (or convolution) of generating functions, which naturally arises when we combine two distinct combinatorial constructions into a single, more complex structure.

Question: If a_k counts all objects of type A of size k and b_k counts all objects of type B of size k, how many pairs of objects (A, B) have a total size of n?

Answer: The number of such pairs is given by

$$\sum_{k=0}^{n} a_k\, b_{n-k}.$$

Observation: The generating function for the sequence

$$C_n = \sum_{k=0}^{n} a_k\, b_{n-k}$$

is

$$\sum_{n=0}^{\infty} C_n\, x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k\, b_{n-k} \right) x^n.$$

It shows that multiplying the generating functions corresponding to $\{a_n\}$ and $\{b_n\}$ produces a new generating function whose coefficients are given by the convolution of the two original sequences.

Example 1: Dice Sum Counting. A classic example of this application is counting the number of ways to obtain a given sum when rolling two standard six-sided dice. For a single die, the generating function is:

$$D(x) = x + x^2 + x^3 + x^4 + x^5 + x^6,$$

where the term x^k corresponds to rolling a k. Since the two dice are independent, the generating function for the sum of the two dice is:

$$D(x)^2 = \left(x + x^2 + x^3 + x^4 + x^5 + x^6\right)^2.$$

Expanding this product, the coefficient of x^n in $D(x)^2$ equals the number of ways to achieve a total sum of n. For instance, one can verify that the coefficient of x^7 is 6, which corresponds to the six possible outcomes that sum to 7 (namely, the pairs $(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$).

Example 2: Candy Selection Problem. Selecting 30 candies from 20 large types (each type can be picked at most once) and 40 small types (each type can be picked in any quantity) can be modeled with generating functions. For a single large candy type (available at most once), the generating function is

$$1 + x,$$

and for 20 independent large types, the combined generating function is

$$(1 + x)^{20}.$$

For a small candy type (with unlimited supply), the generating function is

$$1 + x + x^2 + \cdots = \frac{1}{1 - x},$$

so for 40 small types it is

$$\left(\frac{1}{1 - x}\right)^{40} = (1 - x)^{-40}.$$

Thus, the overall generating function becomes

$$G(x) = (1+x)^{20}\,(1-x)^{-40}.$$

To determine the number of ways to select 30 candies, we need the coefficient of x^{30} in $G(x)$. Expanding,

$$(1+x)^{20} = \sum_{i=0}^{20} \binom{20}{i} x^i, \quad (1-x)^{-40} = \sum_{j\geq 0} \binom{39+j}{39} x^j,$$

the convolution gives:

$$[x^{30}]\,G(x) = \sum_{i=0}^{20} \binom{20}{i}\binom{39+30-i}{39}.$$

Using a Vandermonde's convolution argument, one can show that

$$\sum_{i=0}^{20} \binom{20}{i}\binom{69-i}{39} = \binom{59}{30}.$$

Example 3: Selection with Limited Green Items. Consider selecting 20 objects from three categories:

1. An infinite pile of red objects,
2. An infinite pile of blue objects,
3. A pile of green objects with only 5 available.

For the red and blue objects (with unlimited supply), the generating function is:

$$\frac{1}{1-x},$$

so for both together we have:

$$\frac{1}{(1-x)^2}.$$

For the green objects (at most 5), the generating function is:

$$1 + x + x^2 + x^3 + x^4 + x^5 = \frac{1-x^6}{1-x}.$$

Thus, the overall generating function becomes:

$$G(x) = \frac{1}{(1-x)^2}\cdot\frac{1-x^6}{1-x} = \frac{1-x^6}{(1-x)^3}.$$

Using the expansion

$$(1-x)^{-3} = \sum_{n\geq 0}\binom{n+2}{2} x^n,$$

the coefficient of x^{20} in $G(x)$ is computed by writing:

$$G(x) = (1-x)^{-3} - x^6(1-x)^{-3}.$$

The coefficient from the first term is $\binom{22}{2} = 231$ (since $[x^{20}](1-x)^{-3} = \binom{20+2}{2}$) and from the second term, it is $\binom{16}{2} = 120$ (as the x^6 shifts the index, so $[x^{20}](x^6(1-x)^{-3}) = [x^{14}](1-x)^{-3}$). Hence,

$$[x^{20}]\,G(x) = 231 - 120 = 111.$$

Example 4: Coin Change with Limited Denominations. Determine the number of ways to make 10 (units) using coins of value 1, 2, and 5, where 1- and 2-unit coins are available in unlimited supply but 5-unit coins are limited to at most 2. The generating functions are:

$$G_1(x) = \frac{1}{1-x} \quad \text{(for 1-unit coins)},$$

$$G_2(x) = \frac{1}{1-x^2} \quad \text{(for 2-unit coins)},$$

$$G_5(x) = 1 + x^5 + x^{10} \quad \text{(for 5-unit coins, at most 2)}.$$

Thus, the overall generating function is:

$$G(x) = \frac{1}{(1-x)(1-x^2)}\,(1 + x^5 + x^{10}).$$

To find the coefficient of x^{10}, write:

$$G(x) = \frac{1}{(1-x)(1-x^2)} + \frac{x^5}{(1-x)(1-x^2)} + \frac{x^{10}}{(1-x)(1-x^2)}.$$

The first term contributes the number of ways to form 10 with 1- and 2-unit coins, which is 6; the second term contributes the number of ways to form 5 (which is 3); and the third term contributes 1 (making 0 with 1- and 2-unit coins). Therefore, the total number of ways is:

$$6 + 3 + 1 = 10.$$

Example 5: Composition with Constrained Part Sizes. Count the number of compositions of 7 into exactly 3 positive parts, where each part is at most 4. The generating function for a single part that can take values 1 through 4 is:

$$F(x) = x + x^2 + x^3 + x^4.$$

For a composition with exactly 3 parts, the generating function is:

$$G(x) = [F(x)]^3 = (x + x^2 + x^3 + x^4)^3.$$

A term x^7 in the expansion corresponds to a composition of 7 into 3 parts. Without the upper bound, the number of compositions of 7 into 3 parts (with each part at least 1) is given by

$$\binom{7-1}{3-1} = \binom{6}{2} = 15.$$

However, we must exclude compositions where any part exceeds 4. In this specific case, the only forbidden compositions are those where one part is 5 and the other two are 1 (i.e., $5+1+1$ and its permutations), which count to 3. Hence, the number of valid compositions is:

$$15 - 3 = 12.$$