Preliminaries: two work-horse identities

Throughout the solutions we repeatedly invoke the following elementary—but indispensable—summation formulas. Which have already been discussed on the lecture. They are stated once here for quick reference and are *not* re-derived every time we use them.

Infinite geometric series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \qquad |r| < 1. \tag{1}$$

Finite geometric series

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}, \qquad r \neq 1,\ N \in \mathbb{N}. \tag{2}$$

Tasks:

1. Find generating functions $G(x) = \sum_{n=0}^{\infty} a_n x^n$ for each of the following sequences:

(a) $a_n = \alpha^n, \quad n = 0, 1, 2, \dots,\ \alpha \in \mathbb{R}.$

(b) $a_n = \begin{cases} 1, & n = 0, 1, \dots, N, \\ 0, & n > N. \end{cases}$

(c) $a_n = \begin{cases} n+1, & n = 0, 1, \dots, N, \\ 0, & n > N. \end{cases}$

(d) $a_n = \alpha n, \quad n = 0, 1, 2, \dots,\ \alpha \in \mathbb{R}.$

(e) $a_n = n^2, \quad n = 0, 1, 2, \dots.$

(f) $a_n = n\,\alpha^n, \quad n = 0, 1, 2, \dots,\ \alpha \in \mathbb{R}.$

2. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the ordinary generating function of the sequence $(a_n)_{n \geq 0}$. For each definition of A_n below, determine the generating function $F(x) = \sum_{n=0}^{\infty} A_n x^n$.

(a) $A_n = a_{n+1}.$

(b) $A_n = a_{n+k}$, fixed $k \in \mathbb{N}$.

(c) $A_n = a_{n+1} - a_n.$

(d) $A_n = n\,a_n.$

(e) $A_n = \begin{cases} a_{n-1}, & n \geq 1, \\ 0, & n = 0. \end{cases}$

Solution & Derivations

1. Generating functions of the given sequences

We always write $G(x) = \sum_{n \geq 0} a_n x^n$.

1(a) $\quad a_n = \alpha^n$**.**

Start by writing the series explicitly:

$$f(x) = 1 + \alpha x + (\alpha x)^2 + (\alpha x)^3 + \dots$$

Now compress the same string of terms into $\sum$-notation:

$$f(x) = \sum_{n=0}^{\infty} (\alpha x)^n, \qquad |\alpha x| < 1.$$

Since this is a geometric series with ratio $r = \alpha x$ (and $|r| < 1$ for convergence), we invoke the closed form

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Substituting $r = \alpha x$ gives

$$\boxed{G(x) = \frac{1}{1-\alpha x}}.$$

1(b) Truncated arithmetic sequence $a_n = n + 1$.

$$\begin{aligned} f(x) &= \sum_{n=0}^{N} (n+1)x^n = \\ &= \sum_{n=0}^{N} \frac{d}{dx} x^{n+1} = \\ &= \frac{d}{dx} \sum_{n=0}^{N} x^{n+1} = \\ &= \frac{d}{dx}\left(x \frac{1-x^{N+1}}{1-x} \right) = \\ &= \frac{1-(N+2)x^{N+1} + (N+1)x^{N+2}}{(1-x)^2}. \end{aligned}$$

1(c) $a_n = \alpha n$.

$$f(x) = 0 + \alpha x + 2\alpha x^2 + 3\alpha x^3 + \dots$$

$$f(x) = \sum_{n=0}^{\infty} \alpha n\, x^n, \qquad |x| < 1.$$

$$\begin{aligned} \sum_{n=0}^{\infty} n\, x^n &= x \frac{d}{dx} \sum_{n=0}^{\infty} x^n = x \frac{d}{dx}\Big(\frac{1}{1-x}\Big) = \frac{x}{(1-x)^2}, \\ f(x) &= \alpha \frac{x}{(1-x)^2}. \end{aligned}$$

1(d) $a_n = n^2$.

Define the basic geometric generating function

$$F(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Then view each higher-order sum as a simple transformation of F:

$$F_1(x) = x\,F'(x) = \sum_{n=0}^{\infty} n\,x^n = \frac{x}{(1-x)^2},$$

$$F_2(x) = x\,\frac{d}{dx}F_1(x) = \sum_{n=0}^{\infty} n^2\,x^n = x\,\frac{d}{dx}\Big(\frac{x}{(1-x)^2}\Big) = \frac{x(1+x)}{(1-x)^3}.$$

Thus

$$\boxed{\sum_{n=0}^{\infty} n^2 x^n = F_2(x) = \frac{x(1+x)}{(1-x)^3}, \quad |x|<1.}$$

1(e) $a_n = n\,\alpha^n$.

$$G(x) = 0 + \alpha x + 2\alpha^2 x^2 + 3\alpha^3 x^3 + \dots$$

$$G(x) = \sum_{n=0}^{\infty} n\,\alpha^n x^n = \sum_{n=0}^{\infty} n\,(\alpha x)^n, \qquad |\alpha x| < 1.$$

$$\begin{aligned} G(x) &= x\frac{d}{dx}\sum_{n=0}^{\infty}(\alpha x)^n = x\frac{d}{dx}\Big(\frac{1}{1-\alpha x}\Big) \\ &= x\cdot\frac{\alpha}{(1-\alpha x)^2} = \frac{\alpha x}{(1-\alpha x)^2}. \end{aligned}$$

2. From $f(x)$ to $F(x)$: shifts, differences, multipliers

Let $f(x) = \sum_{n\ge 0} a_n x^n$ be *given*. Each transformation of the indices has a mechanical counterpart on the generating function—think of these as "operator moves" on $f(x)$. No magic, just algebra.

2(a) **Shift forward by one index:** $A_n = a_{n+1}$.

$$F(x) = \sum_{n\ge 0} a_{n+1} x^n$$

Rewriting the sum as:

$$\sum_{n=1}^{\infty} a_n x^{n-1} = x\cdot\left(\sum_{n=1}^{\infty} a_n x^n\right) = x\cdot\left(\sum_{n=0}^{\infty} a_n x^n - a_0\right) = \frac{1}{x}\,(f(x) - a_0)$$

2(b) **Shift forward by** k**:** $A_n = a_{n+k}$.

$$F(x) = \sum_{n=0}^{\infty} a_{n+k} x^n$$

Step 1: Rewriting the sum with a change in the index:

$$F(x) = \sum_{n=k}^{\infty} a_n x^{n-k}$$

Step 2: Factor out x^k:

$$F(x) = x^{-k} \sum_{n=k}^{\infty} a_n x^n$$

Step 3: Final expression using $f(x)$:
Recall that:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Thus, we can express $F(x)$ as:

$$F(x) = x^{-k} \left(f(x) - \sum_{n=0}^{k-1} a_n x^n \right)$$

2(c) **First difference:** $A_n = a_{n+1} - a_n$**.**

$$F(x) = \sum_{n \geq 0} (a_{n+1} - a_n) x^n$$

Step 1: Express $f(x)$ and subtract off a_0:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$f(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n$$

Thus, we can rewrite the sum as:

$$F(x) = [f(x) - a_0] \frac{1}{x} - f(x)$$

Step 2: Factor and simplify:

$$F(x) = \frac{f(x)(1-x) - a_0}{x}$$

2(d) **Index-multiplier:** $A_n = n\, a_n$**.**

$$F(x) = \sum_{n \geq 0} n \cdot a_n x^n = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1} \cdot x = x f'(x)$$

2(e) **Lag operator:** $A_n = a_{n-1}$ **for** $n \geq 1,\ A_0 = 0$**.**

Here we simply *multiply* by x:

$$F(x) = x \sum_{m \geq 0} a_m x^m = x\, f(x).$$

Bottom line. Generating functions live and die by (1)–(2) plus elementary calculus. Once you stop being sentimental about the indices and treat x as an operator knob, everything else collapses to high-school algebra. Don't over-complicate it.

Recurrence-Relation Exercises

Use generating functions to find a closed form for each of the following sequences:

(a) $a_n = 6\,n + a_{n-1}, \quad n \geq 1, \quad a_0 = 0.$

(b) $a_{n+2} = 2\,a_{n+1} + 3\,a_n, \quad n \geq 0, \quad a_0 = 1,\ a_1 = 2.$

(c) $a_n = -\,a_{n-1} + 2\,a_{n-2}, \quad n \geq 2, \quad a_0 = 1,\ a_1 = 2.$

Solution to (a)

Define the ordinary generating function

$$f(x) = \sum_{n=0}^{\infty} a_n\, x^n.$$

Multiply the recurrence $a_n = 6n + a_{n-1}$ by x^n and sum for $n \geq 1$:

$$\sum_{n=1}^{\infty} a_n\, x^n = \sum_{n=1}^{\infty} a_{n-1}\, x^n + 6 \sum_{n=1}^{\infty} n\, x^n \implies f(x) - a_0 = x\, f(x) + 6 \sum_{n=1}^{\infty} n\, x^n.$$

We compute the weighted sum by differentiation:

$$\sum_{n=1}^{\infty} n\, x^n = x\, \frac{d}{dx}\Bigl(\sum_{n=0}^{\infty} x^n\Bigr) = x\, \frac{d}{dx}\Bigl(\frac{1}{1-x}\Bigr) = \frac{x}{(1-x)^2}.$$

Since $a_0 = 0$, the functional equation becomes

$$f(x)\,(1-x) = 6\,\frac{x}{(1-x)^2} \quad \implies \quad f(x) = \frac{6x}{(1-x)^3}.$$

$$\boxed{(1-x)^{-\alpha} = \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} x^k \quad (\alpha \in \mathbb{N},\ \text{as covered in lectures and tutorials})}$$

In particular, for $\alpha = 3$ this gives $(1-x)^{-3} = \sum_{k=0}^{\infty} \binom{k+2}{2} x^k$.

Finally, substituting into $f(x)$ yields

$$f(x) = 6x \sum_{k=0}^{\infty} \binom{k+2}{2} x^k = 6 \sum_{n=1}^{\infty} \binom{n+1}{2} x^n \implies a_n = 6 \binom{n+1}{2} = 3\,n(n+1).$$

Solution to (b)

Define the ordinary generating function

$$f(x) = \sum_{n=0}^{\infty} a_n \, x^n.$$

Multiply the recurrence $a_{n+2} = 2a_{n+1} + 3a_n$ by x^n and sum for $n \geq 0$:

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{n+2} x^n &= \sum_{m=2}^{\infty} a_m \, x^{m-2} && \text{(set } m = n+2\text{)} \\
x^{-2} \sum_{m=2}^{\infty} a_m \, x^m &= x^{-2}\big(f(x) - a_0 - a_1 x\big), \\
\sum_{n=0}^{\infty} a_{n+1} x^n &= \sum_{m=1}^{\infty} a_m \, x^{m-1} = x^{-1}\big(f(x) - a_0\big), \\
\sum_{n=0}^{\infty} a_n \, x^n &= f(x).
\end{aligned}$$

Putting these into $\sum a_{n+2} x^n = 2 \sum a_{n+1} x^n + 3 \sum a_n x^n$ gives

$$x^{-2}\big(f(x) - a_0 - a_1 x\big) = 2\, x^{-1}\big(f(x) - a_0\big) + 3\, f(x).$$

Substitute $a_0 = 1$, $a_1 = 2$ and clear denominators:

$$f(x) - 1 - 2x = 2x\big(f(x) - 1\big) + 3x^2 f(x) \implies f(x)\,(1 - 2x - 3x^2) = 1 \implies f(x) = \frac{1}{(1+x)(1-3x)}.$$

$$\boxed{(1-r)^{-1} = \sum_{n=0}^{\infty} r^n \quad \text{(as covered in lectures and tutorials)}}$$

Use partial fractions:

$$\frac{1}{(1+x)(1-3x)} = \frac{\frac{1}{4}}{1+x} + \frac{\frac{3}{4}}{1-3x},$$

and expand each by the boxed identity with $r = -x$ and $r = 3x$:

$$f(x) = \tfrac{1}{4} \sum_{n=0}^{\infty} (-x)^n + \tfrac{3}{4} \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} \Big(\tfrac{1}{4}(-1)^n + \tfrac{3}{4}\, 3^n \Big) x^n.$$

Hence

$$\boxed{a_n = \tfrac{1}{4}(-1)^n + \tfrac{3}{4}\, 3^n.}$$

Solution to (c)

Define the ordinary generating function

$$f(x) = \sum_{n=0}^{\infty} a_n \, x^n.$$

Multiply the recurrence $a_n = -a_{n-1} + 2a_{n-2}$ by x^n and sum for $n \geq 2$:

$$\begin{aligned}
\sum_{n=2}^{\infty} a_n\, x^n &= -\sum_{n=2}^{\infty} a_{n-1}\, x^n + 2\sum_{n=2}^{\infty} a_{n-2}\, x^n \\
&= -x \sum_{m=1}^{\infty} a_m\, x^m + 2x^2 \sum_{m=0}^{\infty} a_m\, x^m \\
&= -x\big(f(x) - a_0\big) + 2x^2 f(x),
\end{aligned}$$

while

$$\sum_{n=2}^{\infty} a_n\, x^n = f(x) - a_0 - a_1 x.$$

Substituting $a_0 = 1,\ a_1 = 2$ yields

$$f(x) - 1 - 2x = -x\big(f(x) - 1\big) + 2x^2 f(x) \implies f(x)\,(1 + x - 2x^2) = 1 + 3x \implies f(x) = \frac{1+3x}{(1-x)(1+2x)}.$$

Decompose into partial fractions:

$$\frac{1+3x}{(1-x)(1+2x)} = \frac{A}{1-x} + \frac{B}{1+2x} \quad\implies\quad A = \tfrac{4}{3},\ B = -\tfrac{1}{3},$$

so

$$f(x) = \frac{4/3}{1-x} - \frac{1/3}{1+2x}.$$

Use the geometric-series identity

$$\boxed{\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \quad \text{(as in lectures and tutorials)}}$$

with $r = x$ and $r = -2x$ to get

$$f(x) = \frac{4}{3}\sum_{n=0}^{\infty} x^n - \frac{1}{3}\sum_{n=0}^{\infty} (-2x)^n = \sum_{n=0}^{\infty} \Big(\tfrac{4}{3} - \tfrac{(-2)^n}{3}\Big) x^n.$$

Therefore

$$\boxed{a_n = \frac{4 - (-2)^n}{3}.}$$

Tasks: Coefficient Extraction

Use generating-function techniques to find the coefficient of x^{12} in each of the following:

(a) $\big(1 + x^3 + x^6 + x^9 + \cdots\big)^7$.

(b) $\big(x + x^2 + x^3 + x^4\big)^5$.

(c) $x^2\,(1-x)^{12}$.

Solutions

(a) $\left(1+x^3+x^6+x^9+\cdots\right)^7$

We use the identity

$$1+x^3+x^6+\cdots=\frac{1}{1-x^3},$$

so

$$\left(1+x^3+x^6+\cdots\right)^7=(1-x^3)^{-7}=\sum_{m=0}^{\infty}\binom{7+m-1}{m}(x^3)^m=\sum_{m=0}^{\infty}\binom{6+m}{m}x^{3m}.$$

To pick out the coefficient of x^{12}, set $3m=12$, so $m=4$. Hence

$$\boxed{[x^{12}]\,(1+x^3+x^6+\cdots)^7=\binom{6+4}{4}=\binom{10}{4}=210.}$$

(b) $\left(x+x^2+x^3+x^4\right)^5$

Factor out x:

$$(x+x^2+x^3+x^4)^5=x^5\,(1+x+x^2+x^3)^5=x^5\left(\tfrac{1-x^4}{1-x}\right)^5=x^5\,(1-x^4)^5\,(1-x)^{-5}.$$

Expand $(1-x^4)^5=\sum_{j=0}^{5}(-1)^j\binom{5}{j}x^{4j}$ and $(1-x)^{-5}=\sum_{k=0}^{\infty}\binom{4+k}{4}x^k$. The coefficient of x^{12} in the product is the coefficient of x^7 in $(1-x^4)^5(1-x)^{-5}$:

$$[x^{12}]=[x^7]\,\sum_{j=0}^{5}(-1)^j\binom{5}{j}x^{4j}\,\sum_{k=0}^{\infty}\binom{4+k}{4}x^k=\sum_{j=0}^{\lfloor 7/4\rfloor}(-1)^j\binom{5}{j}\binom{4+(7-4j)}{4}.$$

Only $j=0,1$ contribute:

$$\begin{aligned} j=0: &\quad \binom{5}{0}\binom{11}{4}=330,\\ j=1: &\quad -\binom{5}{1}\binom{7}{4}=-175. \end{aligned}$$

Therefore

$$\boxed{[x^{12}]\,(x+x^2+x^3+x^4)^5=330-175=155.}$$

(c) $x^2\,(1-x)^{12}$

Write out the binomial expansion:

$$x^2\,(1-x)^{12}=x^2\sum_{n=0}^{12}\binom{12}{n}(-1)^n\,x^n=\sum_{n=0}^{12}\binom{12}{n}(-1)^n\,x^{n+2}.$$

The coefficient of x^{12} comes from $n+2=12$, i.e. $n=10$:

$$[x^{12}]=\binom{12}{10}(-1)^{10}=\binom{12}{2}=66.$$

Hence

$$\boxed{[x^{12}]\,x^2\,(1-x)^{12}=66.}$$