Model Documentation

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1. Model Overview

This model introduced a risk calculation system designed to effectively evaluate Value at Risk (VaR) and Expected Shortfall (ES) using three different approaches: Historical Method, Monte Carlo Simulation Method and Parametric Method.

Part 2 demonstrates on how historical data is used in calculating VaR and ES. Part 3 evaluates VaR and ES based on Monte Carlo simulation and introduced a simple at the money option. Part 4 illustrates the parametric method in evaluating VaR and ES by assuming the portfolio and the underlying stocks follows geometric Brownian motion (GBM), respectively. Part 5 describes how to calibrate the parameters of the model to statistics of the historical data. Part 6 discusses the limitations of the model and future possible improvements.

2. Historical VaR/ES

Historical method of computing VaR/ES assumes risk factors follow actual historical distributions. This method simply reorganizes actual historical returns, putting them in order from the worst to the best. From a risk perspective, it then assumes that history will repeat itself.

2.1 Absolute change

The p percentile Value at Risk (VaR) at time T is X means that p percent of the time, losses will be less than or equal to X.

If we denote the portfolio value as V, then

$$\mathrm{VaR}(V,T,p) = G^{-1}(p)$$

where

$$G(X) = P[V_0 - V_T \leq X] = E^P\left[1_{V_0 - V_T \leq X}\right]$$

To derive the formula for VaR, we first denote the absolute change of the portfolio value at time T as

$$V_T - V_0$$

Suppose we have n observations of portfolio values from time 0 to time T, therefore, we denote V_T as

$$V_T = \{V_1, V_2, \dots, V_n\}$$

Then, we denote the relative shares of the portfolio with respect to the initial investment for each observation as

$$Shares = V_0/V_T$$

Then, multiplying by the number of the shares hold on time T, the new value of the portfolio is given by

$$V_0 + \frac{V_0}{V_T}(V_T - V_0)$$

Sorting the result of the scenario from the lowest portfolio value to the highest portfolio value, the p percentile VaR is just the (1 – p) percentile scenario.

Therefore, the p percentile VaR of portfolio from time 0 to time T is given by

$$V a\mathrm{R}(V, T\ , p) = V_0 - \left[V_0 + \frac{V_0}{V_T}(V_T - V_0)\right] Quantile(1-p)$$

where [X] $Quantile(1-p)$ denotes the (1 – p) quantile of X.

The corresponding expected shortfall (ES) is the average loss in the percentile tail:

$$\mathrm{ES}(V, T, p) = V_0 - mean\{\left[V_0 + \frac{V_0}{V_T}(V_T - V_0)\right] Quantile(1-p)\}$$

The idea behind calculating historical VaR and ES for any individual underlying stock is the same, thus omitted.

2.2 Relative change

If we denote the portfolio value as V, then

$$\mathrm{VaR}(V, T, p) = G^{-1}(p)$$

where
$$G(X) = P[V_0 - V_T \leq X] = E^P\left[1_{V_0 - V_T \leq X}\right]$$

To derive the formula for VaR, we first denote the relative change of the portfolio value at time T as

$$log(V_T/V_0)$$

Suppose we have n observations of portfolio values from time 0 to time T, therefore, we denote V_T as

$$V_T = \{V_1, V_2, \dots, V_n\}$$

Then, we denote the relative shares of the portfolio with respect to the initial investment for each observation as

$$Shares = V_0 e^{log(V_T/V_0)}$$

Then, multiplying by the number of the shares hold on time T, the new value of the portfolio is given by

$$V_0 + V_0 e^{log(V_T/V_0)} log(V_T/V_0)$$

Sorting the result of the scenario from the lowest portfolio value to the highest portfolio value, the p percentile VaR is just the (1 – p) percentile scenario.
Therefore, the p percentile VaR of portfolio from time 0 to time T is given by

$$VaR(V, T\ , p) = V_0 - [V_0 + V_0 e^{log(V_T/V_0)} log(V_T/V_0)] Quantile(1-p)$$

The corresponding expected shortfall (ES) is the average loss in the percentile tail:

$$ES(V, T, p) = V_0 - mean\{[V_0 + V_0 e^{log(V_T/V_0)} log(V_T/V_0)] Quantile(1-p)\}$$

The idea behind calculating historical VaR and ES for any individual underlying stock is the same, thus omitted.

3. Monte Carlo VaR/ES

The Monte Carlo method to compute VaR/ES is very straightforward. It involves developing a model for future stock price returns running multiple hypothetical trials though the model. Specifically, it is to simulate the risk factors and use the pricers to directly compute the VaR/ES. If positions and payoffs are linear and normally distributed (which can be a big assumption), then it is the same as parametric VaR/ES.

3.1 Long position VaR/ES

Suppose we have a portfolio of value V_t at time t which consists of n underlying stocks with prices $S_{i,t}$ at time t.

The long position VaR/ES refers to the situation where the value of the initial portfolio is greater than zero, which means $V_0 > 0$.

Our portfolio is therefore expressed as a sum of positions in each underlying stock:

$$V_t = \sum_{i=1}^{n} a_i S_{i,t}$$

where a_i is the number of shares of S_i.

Assume each S_i follows a geometric Brownian motion (GBM), we have that

$$S_{i,t} = S_{i,0}e^{\left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i W_{i,t}}$$

Rewriting the above equation in a differential form, we have

$$\mathrm{d}S_{i,t} = \mu_i S_{i,t}\mathrm{dt} + \sigma_i S_i \mathrm{d}W_{i,t}$$

where $W_{i,t}$ is a one-dimensional Wiener process which has correlation ρ_{ij}

$$\mathrm{d}W_{i,t}\mathrm{d}W_{j,t} = \rho_{ij}\mathrm{dt}$$

for $\mathrm{i} \neq j$.

The only randomness exists in W_t.
Using Monte Carlo simulation, we simulate N paths of W_t at time t assuming that it follows a standard normal distribution.
Therefore, we have

$$S_{i,t} = \left(S_{i,1}, S_{i,2}, \dots, S_{i,N}\right)$$

We denote the loss of the portfolio at time t as

$$Loss = V_0 - V_t = \sum_{i=1}^{n} a_i(S_{i,0} - S_{i,t})$$

where V_0 is the initial investment of the portfolio.

Then, sort the realization of the $(S_{i,0} - S_{i,t})$ in ascending order, which is the same as sorting $V_0 - V_t$ in ascending order.

For simulations S = 1, 2, 3, …, N, denote $[V_0 - V_t]^S$ as the Sth smallest loss.
Therefore, the p percentile VaR of long portfolio from time 0 to time t is given by

$$VaR(V, t, p) = [V_0 - V_t]^{Q(p)}$$

where we use $Q(p)$ to denote the index of p quantile.

The p percentile ES of long portfolio from time 0 to time t is the average of loss that are greater than the p percentile VaR.

Hence,

$$ES(V, t, p) = [1/(N - Q(p) + 1)]\left(\sum_{S = Q(p)}^{N} [V_0 - V_t]^{Q(p)}\right)$$

The long position VaR and ES for assuming the overall portfolio follows a GBM is very similar to assuming each underlying stock follows a GBM, except that we are using the portfolio drift and volatility in the formula instead of that of an underlying stock. Thus, the details are omitted.

3.2 Short position VaR/ES

The short position VaR/ES refers to the situation where the value of the initial portfolio is greater than zero, which means $V_0 < 0$.

Similar setting as before, under the short position, the loss of the portfolio at time t is

$$Loss = V_t - V_0 = \sum_{i=1}^{n} a_i(S_{i,t} - S_{i,0})$$

where V_0 is the initial investment of the portfolio.

Then, sort the realization of the $(S_{i,t} - S_{i,0})$ in ascending order, which is the same as sorting $V_t - V_0$ in ascending order.

For simulations S = 1, 2, 3, …, N, denote $[V_t - V_0]^S$ as the Sth smallest loss.
Therefore, the p percentile VaR of short portfolio from time 0 to time t is given by

$$VaR(V, t, p) = [V_t - V_0]^{Q(p)}$$

where we use $Q(p)$ to denote the index of p quantile.

The p percentile ES of short portfolio from time 0 to time t is the average of loss that are greater than the p percentile VaR.

Hence,

$$ES(V, t, p) = [1/(N - Q(p) + 1)](\sum_{S = Q(p)}^{N} [V_t - V_0]^{Q(p)})$$

The short position VaR and ES for assuming the overall portfolio follows a GBM is very similar to assuming each underlying stock follows a GBM, except that we are using the portfolio drift and volatility in the formula instead of that of an underlying stock. Thus, the details are omitted.

3.3 Portfolio with Simple At The Money (ATM) Option

Under this section, we consider a portfolio with a simple at the money (ATM) option which has maturity of one year.

Consider the Black-Scholes formula for call and put options:

$$C = S_0 N(d_1) - e^{-rt} X N(d_2)$$

$$P = e^{-rt} X N(-d_2) - S_0 N(-d_1)$$

where

$$d_1 = \frac{log(S_0/X) + (r + \sigma^2)t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

X is the strike price and r is the risk-free interest rate.

Therefore, our portfolio value at time t is expressed as a sum of positions in each underlying stock and put option or call option depending on different situations.

$$V_t = \sum_{i=1}^{n} a_i S_{i,t} + b_i C_{i,t} + c_i P_{i,t}$$

where a_i, b_i, c_i denotes the shares of underlying stock, call option, and put option, respectively.

Similar setting as before, the loss of the portfolio with ATM option at time t is

$$Loss = V_0 - V_t = \sum_{i=1}^{n} a_i(S_{i,0} - S_{i,t}) + b_i(C_{i,0} - C_{i,t}) + c_i\left(P_{i,0} - P_{i,t}\right)$$

Then, sort the realization of the $loss$ in ascending order, which is the same as sorting $V_0 - V_t$ in ascending order.

For simulations S = 1, 2, 3, …, N, we denote $[V_0 - V_t]^S$ as the Sth smallest loss. Therefore, the p percentile VaR of long portfolio with ATM option from time 0 to time t is given by

$$VaR(V, t, p) = [V_0 - V_t]^{Q(p)}$$

where we use $Q(p)$ to denote the index of p quantile.

The p percentile ES of long portfolio with ATM option from time 0 to time t is the average of loss that are greater than the p percentile VaR.

Hence,

$$ES(V,t,p) \;=\; [1/(N - Q(p) \;+\; 1)]\left(\sum_{S\,=\,Q(p)}^{N} [V_0 - V_t]^{Q(p)}\right)$$

The long position VaR and ES for assuming the overall portfolio follows a GBM is very similar to assuming each underlying stock follows a GBM, except that we are using the portfolio drift and volatility in the formula instead of that of an underlying stock. Thus, the details are omitted.

Also, the formula for short position VaR and ES with ATM option is the same as section 3.2, thus omitted.

4. Parametric VaR/ES

The idea behind the parametric VaR/ES method is to make simplifying assumptions to yield a formula for the VaR/ES based on approximated mean and variance calculations. One of the most used assumptions is that the overall portfolio follows a geometric Brownian motion (GBM). The other one is to assume that each underlying follows a GBM. Altogether, the portfolio follows a normal distribution. We will discuss both assumptions in this model. The fitting of the GBM parameters will be discussed in the next section.

4.1 The overall portfolio follows a GBM

We first assume that our overall portfolio follows a geometric Brownian motion (GBM). The value of the portfolio V_T with drift μ and volatility σ is given by:

$$V_T = \; V_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T}$$

Where W_T is a one-dimensional Wiener process with mean 0 and variance T.

4.1.1 Long position VaR/ES

To compute VaR, we first need to know the probability of V_T being lower than a level X.

$$V_T \;=\; V_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T}$$

$$P(V_T < X) = P\big(log(V_T) < log(X)\big)$$

$$= P\left(log(V_0) + \left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T < log(X)\right)$$

$$= P\left(W_T < \frac{log\left(\frac{X}{V_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma}\right)$$

Since $W_T \sim N(0, T)$, so

$$P(W_T < a) = \phi\left(\frac{a}{\sqrt{T}}\right)$$

$$P(V_T < X) = \Phi\left(\frac{log\left(\frac{X}{V_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$P(V_T < X) = 1 - p$$

$$1 - p = \Phi\left(\frac{log\left(\frac{X}{V_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$\Phi^{-1}(1-p) = \frac{log\left(\frac{X}{V_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$V_0 e^{\sigma\sqrt{T}\Phi^{-1}(1-p) + \left(\mu - \frac{\sigma^2}{2}\right)T} = X$$

These yield the p percentile long position VaR of V from time 0 to time T is

$$\mathrm{VaR}(V, T, p) = V_0 - V_0 e^{\sigma\sqrt{T}\Phi^{-1}(1-p) + \left(\mu - \frac{\sigma^2}{2}\right)T}$$

where Φ^{-1} is he inverse of the cumulative distribution function of the stand normal distribution.

To derive the formula for expected shortfall (ES), first note that ES is the average of loss that exceeds VaR.

$$\mathrm{ES}(V, T, p) = E[V_0 - V_T | V_0 - V_T > \mathrm{VaR}(V, T, p)]$$

$$= E[V_0 - V_T | V_T < V_0 - \mathrm{VaR}(V, T, p)]$$

$$= V_0 - E[V_T | V_T < V_0 - \mathrm{VaR}(V, T, p)]$$

$$= \mathrm{V_0} - \mathrm{E}\left[\mathrm{V_T} 1_{V_T < V_0 - \mathrm{VaR}(V, T, p)}\right] / (1 - p)$$

$$= V_0 - \frac{1}{1 - p}\int_0^X V_T P(V_T) dV_T$$

where

$$X = V_0 - \mathrm{VaR}(S,T,p)$$

Therefore, we need to compute $\int_0^X V_T P(V_T)dV_T$ to derive ES.

$$\int_0^X V_T P(V_T)dV_T = V_0 e^{\mu T}\Phi\left[\Phi^{-1}(1-p) - \sigma\sqrt{T}\right]$$

Altogether, the formula for p percentile long position ES of V from time 0 to time T is

$$\mathrm{ES}(V,T,p) = V_0 - \frac{V_0 e^{\mu T}\Phi\left[\Phi^{-1}(1-p) - \sigma\sqrt{T}\right]}{1-p}$$

4.1.2 Short position VaR/ES

Similar to the formula derivation of the long position VaR/ES, the loss for a short position is just the negative of the loss for the corresponding long position. So, by flipping signs, we can easily do the calculation. The VaR for the short at a percentile of p would be the negative of the VaR of the long position at a percentile of 1 – p.

Therefore, the p percentile short position VaR of V from time 0 to time T is

$$VaR(V,T,p) = V_0 e^{\sigma\sqrt{T}\Phi^{-1}(p)+\left(\mu-\frac{\sigma^2}{2}\right)T} - V_0$$

The ES for the short at a percentile of p would be the negative of the ES of the long position at a percentile of 1 – p. The p percentile short position ES of V from time 0 to time T is given by

$$\mathrm{ES(V,T,p)} \ = \frac{V_0 e^{\mu T}\Phi\left[-\Phi^{-1}(p) + \sigma\sqrt{T}\right]}{1-p} - V_0$$

Alternatively, we can use Black-Scholes formula to derive the formula for short position ES.

From the standard computation in Black-Scholes formula, we have

$$E^P\left[V_T 1_{V_t>K}\right] = V_0 e^{\mu T}\Phi(d1)$$

Where $d_1 = \frac{log(V_0/K)+(\mu+\sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$

Now, if K is such that

$$K \ = \ V_0 + \ Short\ VaR(V,T,p) \ = V_0 e^{(\mu-\sigma^2/2)T+\sigma\sqrt{T}\Phi^{-1}(p)}$$

Then, we have $\Phi^{-1}(p) = -d_2 = -\left(d_1 - \sigma\sqrt{T}\right)$

Therefore,

$$E^P\left[V_T 1_{V_T>K}\right] = V_0 e^{\mu T}\Phi\left(-\Phi^{-1}(p)\ + \sigma\sqrt{T}\ \right)$$

Hence,

$$\mathrm{ES(V,T,p)}\ = \frac{V_0 e^{\mu T}\Phi\left[-\Phi^{-1}(p) + \sigma\sqrt{T}\right]}{1-p} - V_0$$

4.2 Each underlying stock follows a GBM

4.2.1 Long portfolio VaR/ES

Under this situation, we assume that each underling stock of the portfolio follows a GBM. Altogether, we assume the overall portfolio follows a normal distribution, which can be a big assumption.

Suppose we have a portfolio of value V_t at time t which consists of n underlying stocks with prices $S_{i,t}$ at time t.

Our portfolio is therefore expressed as a sum of positions:

$$V_t = \sum_{i=1}^{n} a_i S_{i,t}$$

where a_i is the number of stocks of S_i.

By assuming each underlying stock follows a GBM, we have that

$$S_{i,t}\ =\ S_{i,0}e^{\left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i W_{i,t}}$$

Rewritten in a differential form, we have

$$\mathrm{d}S_{i,t} = \mu_i S_{i,t}\mathrm{dt} + \sigma_i S_i \mathrm{d}W_{i,t}$$

where $W_{i,t}$ is a Wiener process which has correlation ρ_{ij}

$$\mathrm{d}W_{i,t}\mathrm{d}W_{j,t} = \rho_{ij}\mathrm{dt}$$

for $\mathrm{i} \neq\ j$. The notation ρ_{ij} is the constant correlation between $W_{i,t}$ and $W_{j,t}$.

At time t, we can calculate the mean, second moment, variance, standard deviation of the portfolio V_t as

$$E[V_t] = \sum_{i=1}^{n} a_i E[S_{i,t}]$$

$$\mathrm{E}[\mathrm{V}_\mathrm{t}^2] = E[\sum_{i=1}^{n} a_i^2\, S_{i,t}^2 \;+ \sum_{1\le i<j\le n} a_i a_j S_{i,t} S_{j,t}]$$

$$\mathrm{var}[V_t] \;=\; \mathrm{E}[\mathrm{V}_\mathrm{t}^2] \;-\; E[V_T^2]$$

$$\mathrm{sd}[V_t] \;=\; \sqrt{\mathrm{var}[V_t]}$$

In order to solve for the mean of the whole portfolio, we can easily derive the formula for the mean of one stock S_i that follows a GBM:

$$E\left[S_{i,t}\right] = S_{i,0} e^{\mu_i t}$$

Therefore, the expectation of the portfolio at time t is

$$E[V_t] = \sum_{i=1}^{n} a_i E[S_{i,t}] \;=\; \sum_{i=1}^{n} a_i S_{i,0} e^{\mu_i t}$$

Next, to derive the formula for $\mathrm{E}[\mathrm{V}_\mathrm{t}^2]$, we also need to compute $E[S_{i,t} S_{j,t}]$.

We know the formula of $S_{i,t} S_{j,t}$ for any $1 \le i \le j \le n$:

$$S_{i,t} S_{j,t} \;=\; S_{i,0} S_{j,0} e^{\left(\mu_i + \mu_j - \left(\sigma_\mathrm{i}^2 + \sigma_\mathrm{j}^2\right)/2\right)t + \sigma_i W_{i,t} \,+\, \sigma_j W_{j,t}}$$

$\sigma_i W_{i,t} \;+\; \sigma_j W_{j,t}$ is normal with mean zero, so variance is given by

$$E\left[\left(\sigma_i W_{i,t} + \sigma_j W_{j,t}\right)^2\right] = E\left[\sigma_i^2 W_{i,t}^2 + \sigma_j^2 W_{j,t}^2\right]$$

$$= \sigma_i^2 t + \sigma_j^2 t + 2\sigma_i \sigma_j E\left[W_{i,t} W_{j,t}\right]$$

We compute $E\left[W_{i,t} W_{j,t}\right]$ using Ito's lemma:

$$E\left[W_{i,t} W_{j,t}\right] \;= W_{i,t} dt \;+\; W_{j,t}\; dt \;+\; dW_{i,t} W_{j,t}$$

$$=\; W_{i,t} dt \;+\; W_{j,t}\; dt \;+\; \rho_{ij} dt$$

So

$$W_{i,t} W_{j,t} \;=\; \int_0^t W_{i,t} dt \;\;+\; \int_0^t W_{j,t} dt \;\;+ \int_0^t \rho_{ij} ds$$

$$=\; martingale \;+\; martinggale \;+\; \rho_{ij} t$$

Therefore,

$$E\left[W_{i,t}W_{j,t}\right] = \rho_{ij}t$$

$$E\left[\left(\sigma_i W_{i,t} + \sigma_j W_{j,t}\right)^2\right] = \sigma_i^2 t + \sigma_j^2 t + 2\sigma_i\sigma_j\rho_{ij}t$$

Altogether, we have

$$E[S_{i,t}S_{j,t}] = S_{i,0}S_{j,0}e^{(\mu_i+\mu_j+\rho_{ij}\sigma_i\sigma_j)t}$$

Now, we can solve for the second moment of the portfolio at time t:

$$\begin{aligned}\mathrm{E}[\mathrm{V}_t^2] &= E[\sum_{i=1}^{n} a_i^2\, S_{i,t}^2 + 2\sum_{1\le i<j\le n} a_i a_j S_{i,t}S_{j,t}] \\ &= \sum_{i=1}^{n} a_i^2 E[S_{i,t}^2] + 2\sum_{1\le i<j\le n} a_i a_j E[S_{i,t}S_{j,t}] \\ &= \sum_{i=1}^{n} a_i^2\, S_{i,0}^2 e^{(2\mu_i+\sigma_i^2)t} + 2\sum_{1\le i<j\le n} a_i a_j S_{i,0}S_{j,0}e^{(\mu_i+\mu_j+\rho_{ij}\sigma_i\sigma_j)t}\end{aligned}$$

With all the information above, since the portfolio follows normal distribution, the p percentile VaR of long portfolio from time 0 to time t is

$$\mathrm{VaR}[V_t, p] = \mathrm{V}_0 - (\mathrm{E}[\mathrm{V}_t] + \Phi^{-1}(1 - p)\mathrm{sd}[\mathrm{V}_t])$$

Similarly, the p percentile ES of long portfolio from time 0 to time t is

$$\mathrm{ES}[V_t, p] = \mathrm{V}_0 - \left(\mathrm{E}[\mathrm{V}_t] - \frac{\phi(\Phi^{-1}(p))}{1-\mathrm{p}}\mathrm{sd}[\mathrm{V}_t]\right)$$

where ϕ is the probability density function of the standard normal distribution.

4.2.2 Short portfolio VaR/ES

Since we assume the portfolio follows a normal distribution, we know that the distribution of the portfolio is symmetric. Thus, the short portfolio VaR/ES is the same as the long portfolio VaR/ES.

Therefore, the p percentile VaR of short portfolio from time 0 to time t is

$$\mathrm{VaR}[V_t, p] = \mathrm{V}_0 - (\mathrm{E}[\mathrm{V}_t] + \Phi^{-1}(1 - p)\mathrm{sd}[\mathrm{V}_t])$$

Similarly, the p percentile ES of short portfolio from time 0 to time t is

$$\mathrm{ES}[V_t, p] = \mathrm{V}_0 - \left(\mathrm{E}[\mathrm{V_t}] - \frac{\phi(\Phi^{-1}(p))}{1-\mathrm{p}}\mathrm{sd}[\mathrm{V_t}]\right)$$

5. Fitting GBM Parameters

In order to fit GBM portfolio to history, we need to relate the parameters of the model to statistics of the historical data.

Same as before, by assuming each underlying stock follows a GBM, we have that

$$S_{i,t} = S_{i,0}e^{\left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i W_{i,t}}$$

$$\mathrm{d}S_{i,t} = \mu_i S_{i,t}\mathrm{dt} + \sigma_i S_i \mathrm{d}W_{i,t}$$

where $W_{i,t}$ is a Wiener process which has correlation ρ_{ij}

$$\mathrm{d}W_{i,t}\mathrm{d}W_{j,t} = \rho_{ij}\mathrm{dt}$$

Then, for time interval $t_2 - t_1$ where $0 \leq \mathrm{t}_1 < \mathrm{t}_2$, the log return is

$$log\big(S_i(t_2)/S_i(t_1)\big) = (\mu_i - \sigma_i^2/2)(t_2 - t_1) + \sigma_i\big(W_i(t_2) - W_i(t_1)\big)$$

Let R_i denote the return of stock i, which is

$$R_i = log\big(S_i(t_2)/S_i(t_1)\big)$$

Therefore, we have

$$E[R_i] = (\mu_i - \sigma_i^2/2)(t_2 - t_1)$$

With sample mean and variance being $\bar{\mu}_\iota$ and $\bar{\sigma}_\iota$,

$$\bar{\mu}_\iota \approx E[R_i] = (\mu_i - \sigma_i^2/2)(t_2 - t_1)$$

$$\overline{\sigma_\iota^2} \approx var[R_i] = \sigma_i^2(t_2 - t_1)$$

Rearranging the equation, we can get the formula for parametric drift and volatility:

$$\mu_i = \bar{\mu}_\iota/(t_2 - t_1) + \sigma_i^2/2$$

$$\sigma_i = \bar{\sigma}_\iota\sqrt{t_2 - t_1}$$

Next, we need to derive the formula for the correlation between any two underlying stocks S_i *and* S_j.

First, calculate the covariance between the two underlying stocks.

$$E\big[(R_i - E[R_i])(R_j - E[R_j])\big] = E[\sigma_i(W_i(t_2) - W_i(t_1))\sigma_j(W_j(t_2) - W_j(t_1))]$$
$$= \sigma_i\sigma_j\rho_{ij}(t_2 - t_1)$$

Therefore, by definition of correlation, ρ_{ij} is given by

$$\rho_{ij} = \frac{cov(R_i, R_j)}{\sigma_i\sigma_j(t_2 - t_1)}$$

$$= \frac{E(R_iR_j) - \bar{\mu}_i\bar{\mu}_j}{\sigma_i\sigma_j(t_2 - t_1)}$$

6. Limitations

6.1 Historical Method

Due to the difficulty of implementation, the model only evaluates a simple at-the-money option under the Monte Carlo Method. Under other methods, the portfolio only consists of underlying stocks. And the term 'historical method' in this model only refers to estimation based on relative change of individual stocks.

6.2 Monte Carlo Simulations

When conducting Monte Carlo simulations with the portfolio, assuming underlying stocks follow correlated GBM, this model only supports portfolio with exactly 2 stocks, as the covariance would be a number, which could be easily stored in dataframe.

6.3 Options

The method used in this model to bring down VaR with put option only support long position in stocks, accompanied with put options. And the goal of this method is to decrease VaR by 20%. Both interest rates and implied volatilities are assumed to be constant throughout the time, with interest rate set at 0.5%.
Because of the fact that more concentration on option within the portfolio would change the structure, in other words, gamma, of the portfolio, which is tricky to deal with when computing the percentage to be liquidated numerically, the lower and upper bound to solve for the root was set as 0.01 and 0.1.

6.4 Others

Only the latest 25 years of data is reserved to improve model efficiency. When comparing historical methods, simulated results and parametric method, in the built-in demo functions, rolling window method was used to represent parametric method and exponential weighting method was omitted.

Equivalent lambda's were given as fixed, corresponding to default estimation window of 2, 5, and 10 years respectively. If estimation windows were to be changed, equivalent lambda's need to be re-calculated and modified respectively.