Directed First-Order Logic

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Abstract

We present a first-order logic equipped with an "asymmetric" directed notion of equality, which can be thought of as transitions/rewrites between terms, allowing for types to be interpreted as preorders. We then provide a universal property to such "directed equalities" by describing introduction and elimination rules that allows them to be contracted only with certain syntactic restrictions, based on polarity, which do not allow for symmetry to be derived. We give a characterization of such directed equality as a relative left adjoint, generalizing the idea by Lawvere of equality as left adjoint. The logic is equipped with a precise syntactic system of polarities, inspired by dinaturality, that keeps track of the occurrence of variables (positive/negative/both). The semantics of this logic and its system of variances is then captured categorically using the notion of directed doctrine, which we prove sound and complete with respect to the syntax.

1 Introduction

Equality is one of the most interesting aspects of Martin-Löf type theory [19] and in logic in general. For any type $A : \mathsf{Type}$ and $a, b : A$ there is a type of equalities $a =_A b : \mathsf{Type}$, and since this is itself a type, one can talk about the *type of equalities between equalities* $p =_{a=_A b} q$ for any $p, q : a = b$. Are any two proofs of equality p, q themselves equal? This is indeed the case in the so-called *set model* of type theory, where types are interpreted as sets and terms as (dependent) functions, since any two elements either are equal or they are not and any two proofs of equality essentially carry the same information. However, this reasonable fact, called Uniqueness of Identity Proofs (UIP), is not *provable* from the standard rules of MLTT, and must be explicitly taken as axiom: this was shown in Hoffmann and Streicher's seminal paper [20] by soundly interpreting MLTT in a countermodel where UIP is false, the *groupoid model*, in which types are interpreted as groupoids (i.e. categories where every morphism is an isomorphism) and equalities as morphisms between objects, of which there can be more than a unique one and they themselves compose non-trivially. This work, combined with the insight by Voevodsky that equalities can be thought of as paths in a space, and iterated equalities as homotopies [22], laid the foundations for Homotopy Type Theory (HoTT) [45] and later Cubical Type Theory [10], where types are given a geometric interpretation and equalities are precisely interpreted as paths between points in such spaces. This unexpected geometric interpretation of types gave birth to a plethora of works in which types are considered as non-trivial geometric objects, allowing them to be soundly interpreted as ∞-groupoids [7], cubical sets [10], simplicial sets [5], reflexive graphs [4], etc. In the (∞-)groupoidal interpretation, what makes equality inherently *symmetric* is the fact that morphisms are always invertible, similar to the fact that paths between points in a space can always be reversed. A natural question follows: can there be an interpretation of types as *categories*, where morphisms need not be invertible and where equality is not symmetric? Such a system should take the name of *directed type theory* [1, 17, 28, 37], where the directed aspect comes from a non-symmetric interpretation of "equality", which now possesses both a source and a target, as morphisms do in a category.

Directed type theory has been a sought-after goal of recent type theoretical research. In the same way that groupoids generalize sets by making equality *proof relevant* [21, 8.4.11] (since proofs of equalities are not necessarily unique), categories generalize preorders: in a preorder there is no information on *in what way* two objects a, b are (directionally) connected by a morphism $f : a \to b$, but only whether the inequality $a \leq b$ holds or not. We summarize the relationship between these elementary models in the following table, with their corresponding category:

Models of TT	**Proofs of equality**	
	Irrelevant	**Relevant**
Symmetric Equality	Sets (Set)	Groupoids (Gpd)
Directed Equality	Preorders (Preord)	Categories (Cat)

1.1 Variance and dinaturality

A fundamental aspect of directed type theory is the fact that with any type/category $\mathbb{C}$ there is a naturally associated type $\mathbb{C}^{\mathsf{op}}$, called the *opposite type of* $\mathbb{C}$, where the objects are the same but all directed equalities/morphisms are reversed, as in the notion of opposite category. The *type of directed equalities*, often renamed hom-types [37], should then be interpreted via the hom-functor $\hom : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$, which receives a "contravariant" argument $a : \mathbb{C}^{\mathsf{op}}$ and a "covariant" one $b : \mathbb{C}$ and provides a *set* $\hom_{\mathbb{C}}(a, b)$ (i.e., a category with only trivial directed equalities) of directed equalities between the two objects a, b of $\mathbb{C}$. A directed type theory should therefore have some notion of "*polarity*", or occasionally called "*variance*", which allows variables to be distinguished and appear only in the appropriate position, as treated in [28, 36–38]. In the preorder case, hom simply corresponds to the monotone function $\leq_P : P^{\mathsf{op}} \times P \to \mathbf{I}$ associated to any preorder P defined by $\leq(a, b) := (1 \text{ if } a \leq b, 0 \text{ otherwise})$, where $\mathbf{I} = \{0 \to 1\}$ is the preorder with two elements $0, 1 \in \mathbf{I}$ such that $0 \leq 1$. Variance similarly plays a crucial role in programming languages, essentially because function types are contravariant on their arguments: given functors $F, G : \mathbb{C} \to \mathbb{D}$, any family of morphisms $\alpha_x : F(x) \times G(x) \to H(x)$ natural in $x \in \mathbb{C}$ gives rise, by currying, to a family $\mathsf{curry}(\alpha)_x : G(x) \to F(x) \Rightarrow H(x)$ which is *not* natural, but *di*natural in x [12]; for example, in any CCC the counit $\mathsf{eval}_{AB} : A \times (A \Rightarrow B) \to B$ is natural in B, but *di*natural in A [29]. Dinatural transformations are a well-known notion that plays a role both in category theory [39] as well as in the study of parametricity for programming languages [6, 16, 46]; intuitively, these allow for the *same* variable X to appear both co- and contravariantly, as in eval above. We will refer again to dinaturality in Section 2 as the guiding intuition behind a precise system of polarities that can capture the situation in curry.

1.2 Equality, syntactically

We recall the typical syntactic treatment of equality, which we illustrate using a natural deduction style system for non-dependent first-order logic. We denote *contexts* as $\Gamma = [a : A, b : B, ...]$ and consider *formulas-in-context* $[\Gamma]\ \varphi, [\Gamma]\ \psi$, and *lists of formulas-in-context* with $\Phi = \varphi_1, \varphi_2, ..., \varphi_n$. The introduction rule of symmetric equality is typically given by the (refl) rule, stating reflexivity:

$$\frac{}{[\Gamma, x : A]\ \Phi \vdash x = x}\ (\mathsf{refl})$$

In a directed type theory where types are categories, this should be semantically motivated by the fact that there is a directed equality $\mathsf{id}_a \in \hom_{\mathbb{C}}(a, a)$ (i.e., the identity) for any $a : \mathbb{C}$. However, naïvely stating this rule as "$\mathsf{refl}_a : \hom_{\mathbb{C}}(a, a)$" would involve both a contravariant and a covariant occurrence of the same variable $a : \mathbb{C}$, and would not be given functorially with respect to the variance of hom. One solution first considered by North [37] is to use the maximal subgroupoid $\mathbb{C}^{\mathsf{core}}$ to contract the two variances, since $(\mathbb{C}^{\mathsf{core}})^{\mathsf{op}} \cong \mathbb{C}^{\mathsf{core}}$. The refl rule can then be expressed as $\mathsf{refl}_a : \hom(\mathsf{i}^{\mathsf{op}}(a), \mathsf{i}(a))$ via the embeddings $\mathsf{i} : \mathbb{C}^{\mathsf{core}} \to \mathbb{C}$ and $\mathsf{i}^{\mathsf{op}} : \mathbb{C}^{\mathsf{core}} \to \mathbb{C}^{\mathsf{op}}$. The other fundamental rule needed to work with equalities is a way to *eliminate* them; this is typically done with the so-called *J-rule* [19] (illustrated here in FOL),

$$\dfrac{[\Gamma, z : A] \qquad \Phi(z,z) \vdash P(z,z)}{[\Gamma, a : A, b : A]\ a = b,\ \Phi(a,b) \vdash P(a,b)}\ (J)$$

The intuition behind this rule is that to prove a property $P(a,b)$ for any $a, b : A$, if it is assumed in context that $a = b$ then it suffices to prove it for the case $P(z,z)$ where P is instantiated with the same variable $z : A$, and similarly for the context $\Phi(z,z)$. Using these two rules, the usual properties of equality can be derived "for free", e.g., that equality is symmetric, for $\Phi := [\,], P(a,b) := (b = a)$:

$$\dfrac{\dfrac{}{[z : A] \qquad \vdash z = z}\ (\mathsf{refl})}{[a : A, b : A]\ a = b \vdash b = a}\ (J)$$

and transitivity of symmetric equality also follows directly:

$$\dfrac{\dfrac{}{[z : A] \qquad z = c \vdash z = c}\ (\mathsf{hyp})}{[a : A, b : A, c : A]\ a = b,\ b = c \vdash a = c}\ (J)$$

1.3 Doctrines and equality by Lawvere

The two introduction and elimination rules above fully characterize the logical behaviour of equality. A conceptual definition for equality in first-order logic was first noticed in a seminal paper by Lawvere [26], in which models of logic are captured via the notion of *doctrine* [31]. A doctrine is a (pseudo)functor $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Pos}$ from a category $\mathbb{C}$ with finite products into Pos (or Cat), the category of (small) posets(/categories); the syntax of logic is recovered by taking $\mathbb{C} := \mathsf{Ctx}$ to be the category of syntactic contexts and substitutions between them as morphisms, and by having $\mathcal{P}$ send contexts Γ to the poset of formulas $[\Gamma]\ \varphi$ where $\leq$ is given by the existence of an entailment derivation. Through this lens, *equality formulas* are characterized as the *(essentially) unique* operations that provide a left adjoint to certain *"contraction"* functors between posets [21], which send formulas $\varphi(a,b)$ with two free variables into $\varphi(x,x)$ with a single one. Other theorems for equality similarly follow from the universal property of this adjunction. Quoting [43], in a post-Lawvere perspective equality $=$ and quantifiers $\forall, \exists$ play the same "logical" role of connectives $\wedge, \perp, \Rightarrow$, since they all satisfy a precise universal property which characterizes them.

Additional technical conditions have to be satisfied for doctrines to soundly represent logic: the Beck-Chevalley condition and the Frobenius reciprocity condition. The intuition for the former is that quantifiers and equality are well-behaved with respect to substitution [21, 1.9.4], and the latter ensures that rules for colimit-like operations are expressed parametrically with a propositional context in the assumption (this is typically automatic in the presence of implication for the logic [21, 1.9.12(i)]). Lawvere in [26] already identifies a particularly important example of doctrine, the *presheaf doctrine* $\mathsf{Psh} : \mathsf{Cat}^{\mathsf{op}} \to \mathsf{CAT}$ defined by $\mathsf{Psh}(\mathbb{C}) := [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}]$, the (large) category of presheaves on $\mathbb{C}$. At the same time, Lawvere notes that the above conditions are *not* satisfied for the definition of equality given in the case of the presheaf doctrine:

we report a particularly suggestive quote from [26, p. 11] in Appendix A. The definition of equality as a left adjoint used in doctrines allows one to derive that equality is symmetric: it is therefore reasonable for equality to be ill-behaved in "directed" models, such as Cat or Preord, where such conditions similarly fail. This problem in the case of the presheaf doctrine was again revived in a paper by Melliès and Zeilberger [34].

1.4 Contribution

In this paper, we present syntax and semantics for a first-order proof-irrelevant (i.e. logical) version of directed type theory, which is equipped with a propositional notion of directed equality and a system of polarity for positive, negative, and dinatural variables. Directed equality comes with a directed J-like contraction rule, which allows for equalities to be contracted only if certain syntactic restrictions, inspired by dinaturality, are satisfied. Such restrictions allows us to derive the usual properties of equality, except for symmetry. We then describe a class of models called *directed doctrines* for which the syntax is sound and complete, with the preorder model Preord as our main example: types are interpreted by preorders where a (proof-irrelevant) directed equality exists when $a \leq b$. Directed doctrines capture the characterization of directed equality in terms of a left relative adjunction to a certain contraction-like functor, hence generalizing and directifying the notion of symmetric equality as left adjoint to contraction functors by Lawvere. We devise a way to capture and isolate the notion of polarity of variables in the categorical models: the idea is to ask that contexts are separated between positive, negative, and dinatural variables, and then require that the doctrine has specific reindexings that capture such polarity; directed equality is then characterized as left-relative adjoint to these specific reindexings. This is similarly done for a notion of *polarized exponential* and *polarized quantifiers*, which generalize the usual quantifiers with a notion of polarity; such polarity, intuitively, reverses when a formula is curried. Our precise treatment of polarity and directed equality via doctrines allows us to capture a logical notion of directedness, thus progressing on a problem and line of work first posed by Lawvere [26] on the precise role of variance in directed models of logic.

1.5 Related work

North [9, 37] describes a dependent directed type theory with semantics in Cat, but using groupoidal structure to deal with the problem of variance in intro/elim. rules for directed equality. A similar approach was recently taken by Neumann and Altenkirch [2] by using polarities, but rules are restricted to take place in *neutral contexts*, i.e., where the (dependent) type A is typed in a context which is semantically a groupoid. Here we instead focus on a first-order proof-irrelevant presentation using preorders, and tackle the issue of variance using a proof-irrelevant version of dinaturality rather than groupoids. Another approach to directed equality is the judgemental one [1,28], which however does not allow for contraction rules to be described and a universal property to be given. New and Licata [36] give a sound and complete presentation for certain double categorical models of which categories (and therefore preorders) are an instance, but at the cost of heavily restricting the syntax (i.e., a symmetricity statement cannot be formulated). Other approaches to directedness based on synthetic intervals and geometric spaces are given in [17, 41, 47]; our paper focuses on syntactic aspects of a logic, generalizing the abstract doctrinal approach and focusing on the elementary model of preorders instead of using geometric spaces. A system of variances was similarly presented in [38], without however providing a formal or semantic account.

In [23, 24] we present a first-order proof-relevant type theory with semantics in Cat, where entailments are given by dinatural transformations in Set at the cost of having to restrict the syntax to allow only for certain cuts due to the non-compositionality of dinaturals. In this work, we use a proof-irrelevant version of dinatural transformation which always compose by considering preorders only (hence hexagons always commute trivially); instead, we focus on a different approach for the treatment of polarity, using here separate contexts for positive/-dinatural/negative variables rather than classifying variables using predicates, with a careful treatment of polarized exponentials. Moreover, the fact that entailments always compose and form a genuine doctrine with posetal fibers allows us to state that $\leq_A$ is captured precisely as a left-relative adjunction (since here we do not have non-compositionality issues), thereby providing a precise tool for the community interested in doctrines and logic (typically with posetal fibers, as in this paper) to study directedness.

We start by presenting the syntax of directed first-order logic in Section 2, showing examples of derivations and theories in Section 3. The categorical semantics is given in Section 4, establishing the notion of directed doctrine to capture polarity and directed equality. Syntax and semantics are finally connected in Section 5, and conclude with future work in Section 6.

Concept	**Preorder model**	**Judgement**
Type $A, B, P, N, ...$	Preorder $[\![A]\!]$	A type
Context Θ, Δ, Γ	Product of preorders	Γ ctx
Term s, t, η, δ, ρ	Monotone function $[\![\Gamma]\!] \to [\![A]\!]$	$\Gamma \vdash t : A$
Equality of terms	Equivalence of monotone functions	$\Gamma \vdash t' = t : A$
Polarized context	$[\![\Theta \mid \Delta \mid \Gamma]\!] := [\![\Theta]\!]^{\mathsf{op}} \times ([\![\Delta]\!]^{\mathsf{op}} \times [\![\Delta]\!]) \times [\![\Gamma]\!]$	
Formulas φ, ψ	Monotone function $[\![\varphi]\!] := [\![\Theta \mid \Delta \mid \Gamma]\!] \to \mathbf{I}$	$[\Theta \mid \Delta \mid \Gamma]\ \varphi$ prop
Directed equality $\leq_A$	Monotone function $\leq_A$: $[\![A]\!]^{\mathsf{op}} \times [\![A]\!] \to \mathbf{I}$	
Implication formula	Monotone function $- \Rightarrow - : \mathbf{I}^{\mathsf{op}} \times \mathbf{I} \to \mathbf{I}$	
Conjunction formula	Monotone function $- \wedge - : \mathbf{I} \times \mathbf{I} \to \mathbf{I}$	
Propositional context	Pointwise product $[\![\Phi]\!] := [\![\varphi_1]\!] \wedge [\![\varphi_2]\!] \wedge ...$	$[\Theta \mid \Delta \mid \Gamma]\ \Phi$ $\mathsf{propctx}$
Entailment	$\forall n, d, p.\ [\![\Phi]\!](n, d, d, p) \leq [\![\varphi]\!](n, d, d, p)$	$[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \varphi$

Figure 1: Intuition for syntax and preorder semantics of directed first-order logic.

2 Syntax of Directed Logic

We introduce the syntax of directed first-order logic with a natural deduction-style proof system. The main syntactic judgements for types, terms, formulas and entailments are presented in Figures 2 to 4. As a guiding intuition, the reader can refer to Figure 1 to see how the syntax of directed first-order logic is semantically interpreted in the preorder model.

The types and terms of directed first-order logic are a straightforward axiomatization of simply typed λ-calculus (e.g., [40]) with unit, product, and function types. We leave the judgements for types and terms in Figure 2 since they follow exactly those of STLC. We shall blur the distinction between a "type" (resp., "term"/"formula") and the derivation tree for the judgement representing it, understanding that a precise definition is given only for the latter. The two notions can be used interchangeably due to a suitable metatheorem on derivation reconstruction and unambiguousness of types (resp., "term"/"formula") not reported here. For simplicity, we will omit the judgements for equality of formulas (and entailments) induced by equality of terms. As customary in logic, we omit for clarity's sake the metatheoretic operations which apply weakening to formulas in order to place them in the correct context; these will become explicit and play a crucial role in the doctrinal semantics of Section 4. We interleave the definition of the judgements with the definition of *theory* [11]: theories allow the logic to be enriched by adding axioms and "generating" symbols from a signature, e.g., base types, terms, propositions, and axioms.

Definition 1 (Signatures and judgements)**.** *The following series of pairs signature/judgement is given, where we do not explicitly indicate the dependence between a definition and the signatures that come before it; all judgements are inductively defined by the rules in Figures 2 to 4:*

- *[Sign.] A* type signature Σ_B *is defined as a set of symbols* Σ_B*, which represents a collection of base types.*
- *[Judg.] The set of* type derivations *generated by a type signature* Σ_B*, denoted as* $\{A \ \mathsf{type}\}$*, is inductively defined by the judgement "*$A \ \mathsf{type}$*" given in Figure 2, along with the judgements for* contexts $\{\Gamma \ \mathsf{ctx}\}$*, i.e. finite lists of types.*
- *[Sign.] A* term signature Σ_F *is a set of function symbols* Σ_F *with functions* $\mathsf{dom}, \mathsf{cod} : \Sigma_F \to \{A \ \mathsf{type}\}$.
- *[Judg.] The set of* term derivations *generated by a term signature* Σ_F *is* $\{\Gamma \vdash t : A\}$*, assuming* $A \ \mathsf{type}$ *and* $\Gamma \ \mathsf{ctx}$.
- *[Sign.] A* term equality signature Σ_E *is a set* Σ_E *with functions* $\mathsf{eqC} : \Sigma_E \to \{\Gamma \ \mathsf{ctx}\}$, $\mathsf{eqT} : \Sigma_E \to \{A \ \mathsf{type}\}$*, and* $\mathsf{eqL}, \mathsf{eqR} : (e : \Sigma_E) \to \{\mathsf{eqC}(e) \vdash t : \mathsf{eqT}(e)\}$ *dependent on* e.
- *[Judg.] The set of* term equality judgements *generated by* Σ_E *is* $\{\Gamma \vdash t = t' : A\}$*, assuming* $\Gamma \vdash t, t' : A$.
- *[Sign.] A* formula signature Σ_P *is a set of predicate symbols* Σ_P *with functions* $\mathsf{neg}, \mathsf{pos} : \Sigma_P \to \{A \ \mathsf{type}\}$.
- *[Judg.] The set of* formula derivations $\{[\Theta \mid \Delta \mid \Gamma] \ \varphi \ \mathsf{prop}\}$ *generated by the formula signature* Σ_P *is described in Figure 3, assuming* $\Theta, \Delta, \Gamma \ \mathsf{ctx}$.
- *[Judg.] The set of* propositional contexts *is denoted as* $\{[\Theta \mid \Delta \mid \Gamma] \ \Phi \ \mathsf{propctx}\}$ *and given in Figure 3. These are simply finite lists of formulas in the same context.*
- *[Sign.]* Axioms *are given by a set* Σ_A *and three (dependent) functions* $\mathsf{actx} : \Sigma_A \to \{\Gamma \ \mathsf{ctx}\}$, $\mathsf{hyp} : (a : \Sigma_A) \to \{[\mathsf{actx}(a)] \ \Phi \ \mathsf{propctx}\}$, $\mathsf{conc} : (a : \Sigma_A) \to \{[\mathsf{actx}(a)] \ \varphi \ \mathsf{prop}\}$.
- *[Judg.] The set of* entailments *generated by axioms* Σ_A *and the previous signatures is defined as in Figure 4.*

We will use the notation $[\Theta, n : N \mid \Delta, d : D \mid \Gamma, p : P] \ \varphi(n, d, d, p)$ to indicate (some of) the free variables of the formula φ, omitting other variables for brevity. Similarly for terms $[\Gamma, a : A, b : B] \ t(a, b) \vdash C$ as in standard presentations.

In the rule (reindex) we use a meta-theoretical function on formula derivations $\varphi(n, \overline{d}, d, p)$ which substitutes terms η, δ, ρ for the variables n, d, p.

$$\boxed{A\ \mathsf{type}} \quad \frac{C \in \Sigma_B}{C\ \mathsf{type}} \quad \frac{A\ \mathsf{type} \qquad B\ \mathsf{type}}{A \times B\ \mathsf{type}} \quad \frac{A\ \mathsf{type} \qquad B\ \mathsf{type}}{A \Rightarrow B\ \mathsf{type}} \quad \frac{}{\top\ \mathsf{type}} \quad \boxed{\Gamma\ \mathsf{ctx}} \quad \frac{}{[]\ \mathsf{ctx}} \quad \frac{\Gamma\ \mathsf{ctx} \qquad A\ \mathsf{type}}{\Gamma, A\ \mathsf{ctx}}$$

$$\boxed{\Gamma \vdash t : A} \quad \frac{}{\Gamma, x : A, \Gamma' \vdash x : A} \quad \frac{f \in \Sigma_F \qquad \Gamma \vdash t : \mathsf{dom}(f)}{\Gamma \vdash f(t) : \mathsf{cod}(f)} \quad \frac{}{\Gamma \vdash ! : \top} \quad \frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \langle s, t \rangle : A \times B}$$

$$\frac{\Gamma \vdash p : A \times B}{\Gamma \vdash \pi_1(p) : A} \quad \frac{\Gamma \vdash p : A \times B}{\Gamma \vdash \pi_2(p) : B} \quad \frac{\Gamma \vdash s : A \Rightarrow B \qquad \Gamma \vdash t : A}{\Gamma \vdash s \cdot t : B} \quad \frac{\Gamma, x : A \vdash t(x) : B}{\Gamma \vdash \lambda x.t(x) : A \Rightarrow B}$$

$$\boxed{\Gamma \vdash t = t' : A} \quad \frac{\Gamma \vdash t : \top}{\Gamma \vdash t = ! : \top} \quad \frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \pi_1(\langle s, t \rangle) = s : A} \quad \frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \pi_2(\langle s, t \rangle) = t : B} \quad \frac{\Gamma \vdash p : A \times B}{\Gamma \vdash \langle \pi_1(p), \pi_2(p) \rangle = p : A \times B}$$

$$\frac{\Gamma, x : A \vdash f(x) : B \qquad \Gamma \vdash t : A}{\Gamma \vdash (\lambda x.f(x)) \cdot t = f[x \mapsto t] : B} \quad \frac{\Gamma, x : A \vdash f(x) : B}{\Gamma, x : A \vdash (\lambda x.f(x)) \cdot x = f(x) : B} \quad \frac{E \in \Sigma_E}{\mathsf{eqC}(E) \vdash \mathsf{eqL}(E) = \mathsf{eqR}(E) : \mathsf{eqT}(E)}$$

Figure 2: Syntax of directed first-order logic – types and terms.

$$\boxed{[\Theta \mid \Delta \mid \Gamma]\ \varphi\ \mathsf{prop}} \quad \frac{[\Theta \mid \Delta \mid \Gamma]\ \psi\ \mathsf{prop} \qquad [\Theta \mid \Delta \mid \Gamma]\ \varphi\ \mathsf{prop}}{[\Theta \mid \Delta \mid \Gamma]\ \psi \wedge \varphi\ \mathsf{prop}} \quad \frac{[\Gamma \mid \Delta \mid \Theta]\ \psi\ \mathsf{prop} \qquad [\Theta \mid \Delta \mid \Gamma]\ \varphi\ \mathsf{prop}}{[\Theta \mid \Delta \mid \Gamma]\ \psi \Rightarrow \varphi\ \mathsf{prop}}$$

$$\frac{}{[\Theta \mid \Delta \mid \Gamma]\ \top\ \mathsf{prop}} \quad \frac{\Theta, \Delta \vdash s : A \qquad \Gamma, \Delta \vdash t : A}{[\Theta \mid \Delta \mid \Gamma]\ s \leq_A t\ \mathsf{prop}} \quad \frac{P \in \Sigma_P \qquad \Theta, \Delta \vdash s : \mathsf{neg}(P) \qquad \Gamma, \Delta \vdash t : \mathsf{pos}(P)}{[\Theta \mid \Delta \mid \Gamma]\ P(s \mid t)\ \mathsf{prop}}$$

$$\frac{p \in \{-, \Delta, +\} \qquad [\Theta \mid \Delta \mid \Gamma], [x :^p A]\ \varphi(x)\ \mathsf{prop}}{[\Theta \mid \Delta \mid \Gamma]\ \exists^p x.\varphi(x)\ \mathsf{prop}} \quad \frac{p \in \{-, \Delta, +\} \qquad [\Theta \mid \Delta \mid \Gamma], [x :^p A]\ \varphi(x)\ \mathsf{prop}}{[\Theta \mid \Delta \mid \Gamma]\ \forall^p x.\varphi(x)\ \mathsf{prop}}$$

Figure 3: Syntax of directed first-order logic – formulas.

$$\boxed{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \varphi} \quad \frac{[\Theta \mid \Delta \mid \Gamma]\ \Psi \vdash \psi \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi, \psi, \Phi' \vdash \varphi}{[\Theta \mid \Delta \mid \Gamma]\ \Phi, \Psi, \Phi' \vdash \varphi}\ (\mathsf{cut}) \quad \frac{A \in \Sigma_A}{[\mathsf{actx}(A)]\ \mathsf{hyp}(A) \vdash \mathsf{conc}(A)}\ (\mathsf{axiom})$$

$$\frac{\sigma : \{1, ..., m\} \to \{1, ..., n\} \\ [\Theta \mid \Delta \mid \Gamma]\ \varphi_{\sigma(1)}, ..., \varphi_{\sigma(m)} \vdash \psi}{[\Theta \mid \Delta \mid \Gamma]\ \varphi_1, ..., \varphi_n \vdash \psi}\ (\mathsf{struct}) \quad \frac{\Theta, \Delta \vdash \eta : N \qquad \Delta \vdash \delta : D \qquad \Gamma, \Delta \vdash \rho : P \\ [\Theta, n : N \mid \Delta, d : D \mid \Gamma, p : P]\ \Phi(n, \overline{d}, d, p) \vdash \varphi(n, \overline{d}, d, p)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi(\eta, \delta, \delta, \rho) \vdash \varphi(\eta, \delta, \delta, \rho)}\ (\mathsf{reindex})$$

$$\frac{}{[\Theta \mid \Delta \mid \Gamma]\ \Phi, \varphi, \Phi' \vdash \varphi}\ (\mathsf{hyp}) \quad \frac{}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \top}\ (\top) \quad \frac{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \varphi}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi \wedge \varphi}\ (\wedge)$$

$$\frac{[\Theta \mid \Delta, z : A \mid \Gamma] \qquad \Phi \vdash \varphi(\overline{z}, z)}{[\Theta, a : A \mid \Delta \mid \Gamma, b : A]\ a \leq b, \Phi \vdash \varphi(a, b)}\ (\leq) \quad \frac{[\quad N, N' \mid \Delta \mid \quad P, P']\ \psi\ \mathsf{prop} \\ [\Theta, N, P' \mid \Delta \mid \Gamma, P, N']\ \Phi\ \mathsf{propctx}, \varphi\ \mathsf{prop} \\ [\Theta, N \mid \Delta, N', P' \mid \Gamma, P]\ \psi, \Phi \vdash \varphi}{[\Theta, P' \mid \Delta, N, P \mid \Gamma, N'] \qquad \Phi \vdash \psi \Rightarrow \varphi}\ (\Rightarrow)$$

$$\frac{p \in \{-, \Delta, +\} \qquad [\Theta \mid \Delta \mid \Gamma]\ \exists^p x.\psi(x), \Phi \vdash \varphi}{[\Theta \mid \Delta \mid \Gamma], [x :^p A]\ \psi(x), \Phi \vdash \varphi}\ (\exists) \quad \frac{p \in \{-, \Delta, +\} \qquad [\Theta \mid \Delta \mid \Gamma], [x :^p A]\ \Phi \vdash \varphi(x)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \forall^p x.\varphi(x)}\ (\forall)$$

Figure 4: Syntax of directed first-order logic – entailments.

Definition 2 (Substitution of terms in formulas). *Given a derivation tree for a formula* $[\Theta, n : N \mid \Delta, d : D \mid \Gamma, p : P]\ \varphi(n, \overline{d}, d, p)$ prop, *we can substitute any triple of terms* $\Theta, \Delta \vdash \eta : N$, $\Delta \vdash \delta : D$, $\Gamma, \Delta \vdash \rho : P$ *for the variables* n, d, p *in the* φ. *We denote the substituted formula simply as* $\varphi(\eta, \delta, \delta, \rho)$. *This is done by induction on the derivation tree in the intuitive way:*

$$
\begin{aligned}
&\mathsf{subst}_{\eta,\delta,\rho} : \{[\Theta, n : N \mid \Delta, d : D \mid \Gamma, p : P]\ \varphi \ \mathsf{prop}\} \\
&\qquad \to \{[\Theta \mid \Delta \mid \Gamma]\ \varphi \ \mathsf{prop}\} \\
&\mathsf{subst}_{\eta,\delta,\rho}(\top) := \top, \mathsf{subst}(\bot) := \bot \\
&\mathsf{subst}_{\eta,\delta,\rho}(\psi \wedge \varphi) := \mathsf{subst}_{\eta,\delta,\rho}(\psi) \wedge \mathsf{subst}_{\eta,\delta,\rho}(\varphi) \\
&\mathsf{subst}_{\eta,\delta,\rho}(\psi \Rightarrow \varphi) := \mathsf{subst}_{\rho,\delta,\eta}(\psi) \Rightarrow \mathsf{subst}_{\eta,\delta,\rho}(\varphi) \\
&\mathsf{subst}_{\eta,\delta,\rho}(\exists^p x.\varphi(x, n, d, p)) := \exists^p x.\mathsf{subst}_{\eta,\delta,\rho}(\varphi(x, n, d, p)) \\
&\mathsf{subst}_{\eta,\delta,\rho}(\forall^p x.\varphi(x, n, d, p)) := \forall^p x.\mathsf{subst}_{\eta,\delta,\rho}(\varphi(x, n, d, p)) \\
&\mathsf{subst}_{\eta,\delta,\rho}(s(n, d) \le t(d, p)) := s(\eta, \delta) \le t(\delta, \rho) \\
&\mathsf{subst}_{\eta,\delta,\rho}(P(s(n, d) \mid t(d, p))) := P(s(\eta, \delta) \mid t(\delta, \rho))
\end{aligned}
$$

Note the inversion of the terms in the case of implication. In the case of polarized quantifiers we simply substitute under binders in the usual capture-avoidant way. We indicate with $s(\eta, \delta)$ *the substitution of the term* η *in the free variable* $n : N$ *(resp.* δ *in* $d : D$*) in the term* $s(n, d)$, *similarly defined inductively in the intuitive way; we omit this since terms and types are defined exactly as in simply-typed* λ*-calculus. This is similarly done for propositional contexts by applying substitution to each formula judgement appearing in the list.*

Definition 3 (Theory). *A* theory Σ *is defined as a tuple* $\Sigma = (\Sigma_B, \Sigma_F, \Sigma_E, \Sigma_P, \mathsf{EntClo}(\Sigma_A))$ *which collects together data from all previous signatures, using* EntClo *to close the set of axioms* Σ_A *under syntactic entailment [21, 3.2.5].*

We now introduce the concepts of *position, variance, and polarity*, and then describe the rules just presented while giving the fundamental intuition behind them.

Definition 4 (Positions in a formula). *We use the name* position *to indicate any point in which a (term) variable can appear in a formula, e.g., there are four possible positions* x, y, z, w *for variables to appear in the FOL formula "*$x = y \wedge P(z, f(w))$*".*

Definition 5 (Variance of a position). *Positions have a* variance, *which can either be* positive *or* negative*: intuitively, a position starts out as positive, and flips between being positive and negative precisely in the following cases:*

1. *when it occurs on the left of the formula* $x \le y$*: e.g., the variable* x *indicates a negative position in "*$x \le c$*" and "*$f(x) \le y$*";*
2. *when it occurs on the left of an implication formula* $\psi \Rightarrow \varphi$*, e.g., the position indicated by* x *is negative in "*$P(x) \Rightarrow \varphi(y)$*" and "*$Q(f(x)) \Rightarrow \top$*";*
3. *when it occurs on the negative side* n *of a predicate symbol "*$P(n \mid p)$*", e.g.,* x *is negative in "*$P(x, y \mid p, q)$*" and "*$P(f(g(x), y) \mid f(z))$*".*

Variance can be inverted twice: for example, x *occurs* positively *in the formulas "*$x \le y \Rightarrow \varphi$*" and "*$(y \le x \Rightarrow \varphi) \Rightarrow \varphi$*".*

From the semantic perspective of preorders, this flipping of variance corresponds with the presence of the opposite preorder $\mathbb{A}^{\mathsf{op}}$ on the left side of functors $\le_P : P^{\mathsf{op}} \times P \to \mathbf{I}$, and $- \Rightarrow - := \le_{\mathbf{I}} : \mathbf{I}^{\mathsf{op}} \times \mathbf{I} \to \mathbf{I}$ in particular. The fact that $(P^{\mathsf{op}})^{\mathsf{op}} \equiv P$ justifies the fact that inverting a negative variable makes it positive again.

Definition 6 (Polarity of a variable)**.** *Variables can occur in multiple positions at the same time: we say that the* polarity *of variable x is either* positive*,* negative*, or* dinatural*, in the following cases:*

- positive *iff every position in which x occurs is positive;*
- negative *iff every position in which x occurs is negative;*
- dinatural *iff every position in which x occurs is positive or negative (i.e., always: any variable is trivially dinatural).*

We will see in Theorem 1 how indeed any *variable can be "lifted" to be considered dinatural. In practice, we shall use the more intuitive idea that a variable x is called "dinatural" iff it is neither positive nor negative, i.e., it occurs with both variances at the same time (or not at all). A variable x that is either positive or negative is said to be* natural*. We denote the set of polarities as $\{-, \Delta, +\}$.*

In logic and subtyping ""co(ntra)variance" often refers to the polarity of *propositional variables*, intuitively because entailments are contravariant on the left [8], similarly arising for negation $\neg A \Leftrightarrow (A \Rightarrow \bot)$. In directed first-order logic, this polarity aspect is also present at the level of *variables*: in the preorder model, polarity corresponds intuitively to the fact that formulas admit a monotonicity property, e.g., if x is a positive position in a formula $\varphi(x)$, then for any concrete $a, b \in P$ we have that $a \leq b$ and $\varphi(a)$ imply $\varphi(b)$, i.e., φ induces a covariant functor; viceversa, if x is a negative position, then $a \leq b$ and $\varphi(b)$ imply $\varphi(a)$, and φ induces a contravariant functor [30].

Notation 1 (On overlines for variables)**.** *We will indicate with $\overline{a} : A$ in formulas $\varphi(\overline{a}, a, \overline{b}, b, ...)$ to highlight points in which dinatural variables are used with different variances. These are just conveniences and do not exist syntactically. Positive and negative variables will either never or always have an overline, since they always unambiguously appear with the same variance.*

The main formal tool that we use to capture precisely this idea of variance involves dividing contexts in terms of the possible polarity of variables, which we call *polarized contexts*. This technique lends itself particularly well to captured by the *doctrinal* semantics, which similarly puts particular emphasis on the role of contexts and free variables for formulas [21].

Definition 7 (Polarized context)**.** *A* polarized context *is a triple of contexts $[\Theta \mid \Delta \mid \Gamma]$ for which Θ, Δ, Γ ctx, where, intuitively, Θ is a list of variables that can be used only negatively, variables in Δ are dinatural (i.e., can be used either positively or negatively in any position), and variables in Γ only positively. We will occasionaly abbreviate definitions by parameterizing them w.r.t. a polarity $p \in \{-, \Delta, +\}$: for example, $[\Theta \mid \Delta \mid \Gamma], [x :^p A]$ denotes the extended polarized context obtained by appending a variable x of type A to the correct context. We denote with $\bullet$ the empty context.*

We now describe the main rules of the logic, both in formula construction and entailments.

(Predicate symbols for formulas.) Polarized contexts are particularly relevant in the base formulas $s \leq_A t$ and $P(n \mid p)$, where s, t are terms. In particular, s is typed in context Θ, Δ, and t in context Γ, Δ: the core idea behind this definition is that positive positions in formulas can be filled *either* by a *positive* variable, or by a *dinatural* one, i.e., in the case in which a term s is a variable $\Theta, \Delta \vdash x : A$ for some A, either $(x : A) \in \Theta$ or $(x : A) \in \Delta$. This rule allows for the specific kind of reindexing of variables used in the rule (reindex), as well as in the categorical semantics in Section 4.

(Polarized exponentials.) In directed first-order logic, implication of formulas must be given a special treatment due to the contravariance of $\Rightarrow$ in its first argument. This behaviour is similarly mirrored in the case of preorders, due to the presence of $\mathbf{I}^{\mathsf{op}}$ in the semantics for implication $- \Rightarrow - : \mathbf{I}^{\mathsf{op}} \times \mathbf{I} \to \mathbf{I}$. This is implemented in two judgements: when constructing formulas $[\Theta \mid \Delta \mid \Gamma]\ \psi \Rightarrow \varphi\ \mathsf{prop}$, and when currying entailments in the rule $(\Rightarrow)$. We use the name *polarized exponentials* to reflect the inversion of polarity for variables: given $\psi \Rightarrow \varphi$, we swap the negative and positive context of the formula ψ on the left side. A similar idea is applied for currying in entailments, since, as briefly illustrated in Section 1.2, currying a formula ψ *inverts* the polarity of all the variables in ψ. For instance, consider a formula $\psi(x, y, z)$, with x, y negative positions and z positive, and take the entailment $\psi(x, \overline{y}, z) \wedge \Phi(y) \vdash \varphi(z)$ where x is negative, y dinatural (negative in ψ and positive in Φ), z positive; by currying, this is equivalent to $\Phi(y) \vdash \psi(x, y, \overline{z}) \Rightarrow \varphi(z)$, where now x appears *positively* on the right side of the entailment, z now becomes dinatural (due to the positive appearance in φ and the negative appearance in ψ, which is now on the left side of $\Rightarrow$), and y becomes positive.

This behaviour of managing polarity of variables is implemented by the rule $(\Rightarrow)$: the intuition is that we consider both the case in which variables which are natural on the top side (the contexts N, P) become dinatural on the bottom side, *and viceversa* (the contexts N', P'). The fact that positive variables can directly switch to negative is captured in the derived rule $(\Rightarrow^+_-)$. Note the absence of a general context Θ, Γ in ψ: this is because *all* the variables of ψ change polarity, and the only ones that do not are those in Δ since they already appear with both variances.

(Directed equality.) The formula $a \leq_A b$ (where a, b are positions with negative and positive variance, respectively) is the main construct of directed first-order logic, and it is a proposition stating that a "rewrites" in b: in the preorder model, this is indeed represented by the fact that the relation $a \leq b$ holds. We illustrate the intuition behind the rule for directed equality $(\leq)$: in the $\Downarrow$ direction, the rule states that a directed equality $a \leq b$ in context can be contracted *only if* a and b only appear *naturally* in the conclusion; thanks to the presence of polarized exponentials, (as well as in the preorder model) a and b must appear with the opposite variance in the hypothesis context $(\leq^{\mathsf{full}})$. We show in Remark 2 that this syntactic constraint does not allow for symmetry to be derived.

(Polarized quantifiers.) Since polarized contexts keep track of the polarity of variables, quantifiers also need to track this information: we define polarized quantifiers $\forall^+.\phi(x)$, $\forall^- x.\phi(x)$, $\forall^\Delta x.\phi(\overline{x}, x)$, similarly for exists $\exists^p.\phi(x)$ for $p \in \{-, \Delta, +\}$. The rules for polarized quantifiers are captured in $(\forall)$ and $(\exists)$.

Remark 1 (On admissibility of substitution in entailments)**.** *Substitution in formulas of Definition 2 is used in* (reindex)*. In other syntactic accounts of logic (e.g., [43]) this rule is often admissible rather than assumed, but it requires rules for entailments to be parametric on terms. We use this equivalent presentation of rules in adjoint form to make the doctrinal semantics more intuitive.*

3 Examples

In this section we provide derived rules and then use them in examples that exemplify the properties of directed equality and polarized exponentials/quantifiers.

Example 1 (Derived rules)**.** *The following rules are derivable; we report here only some statements, leaving the complete derivations and other examples in Appendix B. The following rule and* ($\leq$-refl) *state reflexivity of directed equality, and they are equivalent to the* $\Uparrow$ *direction of* ($\leq$)*.*

$$\frac{\Delta \vdash t : A}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash t \leq_A t}\ (\leq\text{-refl}_t)$$

Note that t *in* ($\leq$-refl$_t$) *must not depend on any natural variable, since they would have to appear with both variances on the two sides of* $\leq$*; this is mirrored by reindexing. We give a rule for directed equality contraction which takes into account the polarity of the variables contracted (in* ($\leq$) *they always become dinatural).* ($\leq^-$) *is given similarly.*

$$\frac{\begin{array}{c}[\Theta \mid \Delta \mid \Gamma, a : A]\ \Phi\ \mathsf{propctx}, \varphi\ \mathsf{prop} \\ [\Theta \mid \Delta \mid \Gamma, z : A] \qquad \Phi(z) \vdash \varphi(z)\end{array}}{[\Theta \mid \Delta, a : A \mid \Gamma, b : A]\ \overline{a} \leq b, \Phi(a) \vdash \varphi(b)}\ (\leq^+)$$

Using polarized exponentials the following general principle for directed equality elimination can be derived, with an additional propositional context Φ *in which, however,* a, b *must appear* negatively*. This is reminiscent of the Frobenius equality formulation for equality [21, 3.2.4].*

$$\frac{[\Theta \mid \Delta, z : A \mid \Gamma] \qquad \Phi(z, \overline{z}) \vdash \varphi(\overline{z}, z)}{[\Theta \mid \Delta, a : A, b : A \mid \Gamma]\ a \leq b, \Phi(\overline{a}, \overline{b}) \vdash \varphi(a, b)}\ (\leq^{\mathsf{full}})$$

The following more general directed equality elimination with terms can be derived using (reindex)*:*

$$\frac{\begin{array}{c}\Theta, \Delta \vdash \eta : A, \quad \Gamma, \Delta \vdash \rho : A \\ [\Theta, a : A \mid \Delta \mid \Gamma, b : A]\ \Phi(a, b)\ \mathsf{propctx}, \varphi(a, b)\ \mathsf{prop} \\ [\Theta, z : A \mid \Delta \mid \Gamma] \qquad \Phi(\overline{z}, z) \vdash \varphi(\overline{z}, z)\end{array}}{[\Theta \mid \Delta \mid \Gamma]\ \eta \leq_A \rho, \Phi(\eta, \rho) \vdash \varphi(\eta, \rho)}\ (\leq_t)$$

Note that the syntactic restriction here is still captured by requiring for natural *occurrences of* $a : A, b : A$ *in* Φ, φ*.*

The typical rules for quantifiers of FOL can be derived from the adjoint formulation ($\exists$), ($\forall$) *using* (cut) *and* (reindex)*; we report here only the rules used in examples, leaving the full set of rules and their proofs in Appendix B:*

$$\frac{\Theta, \Delta \vdash \eta : N \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi(\eta)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \exists^- x.\psi(x)}\ (\exists^-_t)$$

$$\frac{\Delta \vdash \delta : D \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \forall^\Delta x.\psi(x)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi(\delta, \delta)}\ (\forall^\Delta_t)$$

A rule ($\Rightarrow^+_-$) which directly swaps a variable from negative to positive can be derived ($\Rightarrow$).

Example 2 (Derivations)**.** *We illustrate some derivations with directed first-order logic, taking as reference the examples in Section 1.2. We consider a simple signature with two base types* $A, B \in \Sigma_A$*, a function symbol* $f \in \Sigma_F$ *for which* $\mathsf{dom}(f) := A, \mathsf{cod}(f) := B$*, and a predicate symbol* $P \in \Sigma_P$ *for which* $\mathsf{pos}(P) := A, \mathsf{neg}(P) := \top$*.*

- *Congruence of directed equality (i.e. internal monotonicity for terms):*

$$\dfrac{\dfrac{}{[\bullet \mid z : A \mid \bullet] \qquad \vdash f(\overline{z}) \leq_B f(z)}\ (\leq\text{-refl}_t)}{[a : A \mid \bullet \mid b : A]\ a \leq_A b \vdash f(a) \leq_B f(b)}\ (\leq)$$

- *Transport of equalities between proofs of predicates: (i.e. internal monotonicity for predicates):*

$$\dfrac{\dfrac{}{[\bullet \mid \bullet \mid z : A] \qquad P(z) \vdash P(z)}\ (\mathsf{hyp})}{[a : A \mid \bullet \mid b : A]\ a \leq b,\ P(a) \vdash P(b)}\ (\leq^{+})$$

- *Transitivity of directed equality:*

$$\dfrac{\dfrac{}{[z : A \mid \bullet \mid c : A] \qquad z \leq c \vdash z \leq c}\ (\mathsf{hyp})}{[a : A \mid b : A \mid c : A]\ a \leq b,\ \overline{b} \leq c \vdash a \leq c}\ (\leq^{-})$$

- *Existence of singletons:*

$$\dfrac{\dfrac{\dfrac{}{[\bullet \mid y : A \mid \quad \bullet \quad]\ \vdash \overline{y} \leq y}\ (\leq\text{-refl})}{[\bullet \mid \quad \bullet \quad \mid y : A]\ \vdash \exists^{-} x. x \leq y}\ (\exists^{-}_{t})}{[\bullet \mid \quad \bullet \quad \mid \quad \bullet \quad]\ \vdash \forall^{+} y. \exists^{-} x. x \leq y}\ (\forall)$$

- *Pair of rewrites:*

$$\dfrac{\dfrac{\dfrac{}{[\bullet \mid x : A, y : B \mid \bullet]\ \vdash (\overline{x}, \overline{y}) \leq_{A \times B} (x, y)}\ (\leq\text{-refl}_t)}{[b : A \mid x : A \mid b' : B]\ b \leq_B b' \vdash (\overline{x}, b) \leq_{A \times B} (x, b')}\ (\leq)}{[a : A, b : A \mid \bullet \mid a' : A, b' : B]\ a \leq_A a',\ b \leq_B b' \vdash (a, b) \leq_{A \times B} (a', b')}\ (\leq)$$

The other direction ("directed injectivity of pairs") follows from congruence of directed equality *using projections* π_1, π_2 *and the term equalities.*

- *Higher-order rewriting:*

$$\dfrac{\dfrac{\dfrac{}{[\bullet \mid h : A \Rightarrow B, x : A \mid \bullet]\ \vdash \overline{h} \cdot \overline{x} \leq_B h \cdot x}\ (\leq\text{-refl}_t)}{[\bullet \mid h : A \Rightarrow B \mid \bullet]\ \vdash \forall^{\Delta} x.\ \overline{h} \cdot \overline{x} \leq_B h \cdot x}\ (\forall^{\Delta}_{t})}{[f : A \Rightarrow B \mid \bullet \mid g : A \Rightarrow B]\ f \leq_{A \Rightarrow B} g \vdash \forall^{\Delta} x.\ f \cdot \overline{x} \leq_B g \cdot x}\ (\leq)$$

The other direction is not derivable in general, since it captures a suitable notion of "2-dimensional extensionality" [18] and, in particular, it also would capture an $\Leftrightarrow$*-extensionality.*

Remark 2 (Failure of symmetry)**.** *We illustrate how symmetry of directed equality is not derivable in this logic, and indeed the* Preord *model is a countermodel: the entailment*

$$\dfrac{}{[\bullet \mid a : A, b : A \mid \bullet]\ a \leq b \vdash \overline{b} \leq \overline{a}}$$

can only be given type in the term context where both a *and* b *appear dinaturally, since* $[a : A \mid \bullet \mid b : A]\ a \leq b\ \ \mathsf{prop}$ *and* $[b : A \mid \bullet \mid a : A]\ b \leq a\ \ \mathsf{prop}$ *must be weakened in the same context, namely* $[\bullet \mid a, b : A \mid \bullet]$*, for the judgement to be well-formed.*

The following special cases of $\mathsf{subst}_{\eta,\delta,\rho}$ will be instructive for the categorical semantics in Section 4.

Theorem 1 (Lifting natural to dinatural)**.** *Any variable appearing naturally can be lifted to be dinatural: there is a function on sets of formulas defined by* $\mathsf{lift}_{\Theta,\Delta,\Gamma,N,P} := \mathsf{subst}_{\eta_-,\pi_1,\rho_+} : \{[\Theta, n : N \mid \Delta \mid \Gamma, p : P]\ \varphi\ \mathsf{prop}\} \to \{[\Theta \mid \Delta, n : N, p : P \mid \Gamma]\ \varphi\ \mathsf{prop}\}$*, where* $\eta_- := (\Theta, \Delta, n : N, p : P \vdash n : N)$*,* $\Delta, n : N, p : P \vdash \pi_1 : \Delta$*,* $\rho_+ := (\Gamma, \Delta, n : N, p : P \vdash p : P)$*. We use* $\Delta \cong [\Delta, a : \top]$ *to add a dummy variable for* subst *whenever we do not need to substitute a variable in context.*

Theorem 2 (Collapsing two naturals in one dinatural)**.** *A formula with two natural variables* $\varphi(a, b)$ *can be collapsed to one* $\varphi(\overline{x}, x)$ *in the context with a single dinatural* x*: there is a function on formulas* $\mathsf{contract}_{\Theta\Delta\Gamma A} := \mathsf{subst}_{\eta_\Delta,\pi_1,\rho_\Delta} : \{[\Theta, n{:}A \mid \Delta \mid \Gamma, p{:}A]\ \varphi\ \mathsf{prop}\} \to \{[\Theta \mid \Delta, a{:}A \mid \Gamma]\ \varphi\ \mathsf{prop}\}$*, where* $\eta_\Delta := (\Theta, \Delta, a : A \vdash a : A)$*,* $\pi_1 := (\Delta, a : A \vdash a : \Delta)$*, and* $\rho_\Delta := (\Gamma, \Delta, a : A \vdash a : A)$*. Moreover, this contraction on formulas is functorial on entailments:*

$$\frac{[\Theta, n : A \mid \Delta \mid \Gamma, p : A]\ \Phi(n, p) \vdash \varphi(n, p)}{[\Theta \mid \Delta, a : A \mid \Gamma]\ \Phi(\overline{z}, z) \vdash \varphi(\overline{z}, z)}\ (\mathsf{reindex})$$

This contraction operation is crucial for the characterization of directed equality as adjoint in Definition 13.

We remark how directed equivalence of elements still does *not* in general recover symmetric equality:

Remark 3 (Symmetric equality)**.** *In the entailment*

$$[\Theta \mid \Delta, a : A, b : A \mid \Gamma]\ a \leq_A b \wedge \overline{b} \leq_A \overline{a}, \Phi \vdash \varphi(\overline{a}, a, \overline{b}, b)$$

one cannot contract the directed equalities in context unless a, b *also appear naturally in* φ*; in such a case, one of the two equalities can be moved on the right side using polarized exponentials, and then the other one can be contracted.*

We give some examples of signatures to exemplify how directed first-order logic can be used to model directed structure. A suitable extension of directed first-order logic (e.g., a two-level logical system in the style of [36]) could be used to show non-trivial theorems.

Example 3 (Theory of λ-terms)**.** *We capture a signature of untyped* λ*-terms in the style of HOAS [14], as follows:*

- $\Sigma_A := \{T\}$ *with a type of* λ*-terms,*
- $\Sigma_F := \{\tilde{\lambda}, \mathsf{app}\}$ *for* λ*-abstraction and application with:*

$$\begin{aligned} \mathsf{dom}(\tilde{\lambda}) &:= T \Rightarrow T, & \mathsf{cod}(\tilde{\lambda}) &:= T \\ \mathsf{dom}(\mathsf{app}) &:= T \times T, & \mathsf{cod}(\mathsf{app}) &:= T \end{aligned}$$

- $\Sigma_E := \{\eta\}$*, such that:*

$$\frac{}{[f : T \Rightarrow T]\ \big(\lambda x.\mathsf{app}(\tilde{\lambda}(f), x)\big) = f : T \Rightarrow T}\ (\eta)$$

- $\Sigma_P := \{\}, \Sigma_A := \{\beta\}$*, such that:*

$$\frac{}{[\bullet \mid s : T \Rightarrow T, t : T \mid \bullet]\ \mathsf{app}(\tilde{\lambda}(\overline{s}), \overline{t}) \leq s \cdot t}\ (\beta)$$

Note that from this simple signature we automatically have that app *and* λ *are congruences w.r.t. the reduction relation represented by directed equality:*

$$\dfrac{\dfrac{}{[\bullet \mid z : T, t : T \mid \bullet]\ \vdash \mathsf{app}(\overline{z}, \overline{t}) \leq_T \mathsf{app}(z, t)}\ (\leq\text{-refl}_t)}{[s : T \mid t : T \mid s' : T]\ s \leq_T s' \vdash \mathsf{app}(s, \overline{t}) \leq_T \mathsf{app}(s', t)}\ (\leq)$$

Semantically, the type T *can be interpreted by the preorder of* λ*-terms ordered by* β*-relation, as an example of the semantics given in Section 4.*

Example 4 (Theory of ω-CPOs$_\perp$)**.** *We capture a signature of* ω*-CPOs*$_\perp$ *from domain theory [15], as follows:*

- $\Sigma_A := \{D, \mathbb{N}\}$, *representing, in the preorder model, a domain* D *and the discrete preorder of naturals* $\mathbb{N}$.
- $\Sigma_F := \{0, \mathsf{succ}\}$ *defined in the intuitive way,* $\Sigma_E := \{\}, \Sigma_P := \{\}$, $\Sigma_A := \{\perp_{\mathsf{axiom}}, \bigsqcup_{\mathsf{axiom}}\}$, *for which, (omitting the empty term context):*

$$\frac{}{\vdash \exists^- b.\ \forall^+ x.\ b \leq_D x}\ (\perp_{\mathsf{axiom}})$$

$$\frac{}{\begin{array}{l}\vdash \forall^\Delta(c : \mathbb{N} \Rightarrow D).(\forall^\Delta(i : \mathbb{N}).\ \overline{c} \cdot \overline{i} \leq c \cdot \mathsf{succ}(i)) \\ \quad \Rightarrow (\exists^\Delta(b : D).\ (\forall^-(i : \mathbb{N}).\ \overline{c} \cdot i \leq b) \\ \quad \wedge\ (\forall^+(b' : D).(\forall^-(i : \mathbb{N}).\ \overline{c} \cdot i \leq b) \Rightarrow b \leq b'))\end{array}}\ (\textstyle\bigsqcup_{\mathsf{axiom}})$$

4 Doctrinal Semantics

In this section we overview the doctrinal semantics of directed first-order logic. The crucial aspect is the fact that reindexing in directed first-order logic is given precisely as in Figure 4, which captures exactly the way in which variables of a formula φ can be supplied. The way to capture this is by requiring for the doctrine to have a specific reindexing structure, which we simply implement by changing the base category. Later, we ask for certain adjunctions or rules to hold with respect to these specific reindexings. The intuition for the polarization category is that it captures precisely the reindexing action in Definition 2, interpreting concatenation of contexts via products.

Definition 8. *Given a category* $\mathbb{C}$ *with finite products, we define the* polarization category of $\mathbb{C}$*, denoted as* $\mathsf{ndp}(\mathbb{C})$*:*

- *Objects: triples of objects*
$$(\Theta \mid \Delta \mid \Gamma) \in \mathbb{C}_0 \times \mathbb{C}_0 \times \mathbb{C}_0$$
- *Morphisms* $(\Theta \mid \Delta \mid \Gamma) \to (\Theta' \mid \Delta' \mid \Gamma')$ *are triples*
$$(n : \Theta \times \Delta \to \Theta' \mid d : \Delta \to \Delta' \mid p : \Gamma \times \Delta \to \Gamma')$$
- *Identities are given by* $(\pi_1 \mid \mathsf{id} \mid \pi_1)$.
- *Composition:* $(n \mid d \mid p) ; (n' \mid d' \mid p')$ *is defined as*
$$(\langle n, \pi_2 ; d\rangle ; n' \quad | \quad d ; d' \quad | \quad \langle p, \pi_2 ; d\rangle ; p').$$

where we will always use composition $f\,;g$ in diagrammatic order, and $\langle f, g\rangle$ is the universal map of cartesian products.

Definition 9 ($\mathsf{ndp}(\mathbb{C})$ as functor)**.** *The above definition lifts to a functor $\mathsf{ndp}(-) : \mathsf{CC} \to \mathsf{CC}$ on the category of small categories with finite products CC, i.e., any product-preserving functor $F : \mathbb{C} \to \mathbb{D}$ induces a functor $\mathsf{ndp}(F) : \mathsf{ndp}(\mathbb{C}) \to \mathsf{ndp}(\mathbb{D})$ defined in the intuitive way $(\Theta \mid \Delta \mid \Gamma) \mapsto (F(\Theta) \mid F(\Delta) \mid F(\Gamma))$ and similarly on morphisms.*

A technical condition for doctrines is necessary because of the base case for predicates $P(s \mid t)$; this is the only non-standard aspect in the construction of syntactic doctrines [21], hence the condition involving objects in fibers.

Definition 10 (No-dinatural-variance condition)**.** *A functor $\mathcal{P} : \mathsf{ndp}(\mathbb{C})^{\mathsf{op}} \to \mathsf{Pos}$ is said to satisfy the* (strict) no-dinatural-variance condition *if the "full dinatural collapse" functor $\mathcal{P}(\Uparrow^{\pm!}_{\Delta})$: $\mathcal{P}(\Theta \times \Delta \mid \top \mid \Gamma \times \Delta) \to \mathcal{P}(\Theta \mid \Delta \mid \Gamma)$ induced by reindexing with $\Uparrow^{\pm!}_{\Delta} := (\mathsf{id}_{\Theta\times\Delta} \mid !_{\Delta} \mid \mathsf{id}_{\Gamma\times\Delta})$ is a* bijection on objects*, denoting with $\varepsilon : \mathcal{P}(\Theta \mid \Delta \mid \Gamma)_0 \to \mathcal{P}(\Theta \times \Delta \mid \top \mid \Gamma \times \Delta)_0$ the inverse function of sets with $\mathcal{P}(\Uparrow^{\pm!}_{\Delta})(\varepsilon(p)) = p$.*

This is usually *not* an isomorphism of posets since, in the syntactic model,

$$\forall^{\Delta} d.\varphi(\overline{a}, d, \overline{d}, \overline{b}) \Rightarrow \varphi(a, \overline{d}, d, b)$$

might not imply that

$$\forall^{+} d.\forall^{-} d'.\varphi(\overline{a}, \overline{d}', \overline{d}, \overline{b}) \Rightarrow \varphi(a, d', d, b).$$

The converse implication holds by $\mathcal{P}(\Uparrow^{\pm!}_{\Delta})$. The intuition is that, at the level of formulas, dinatural variables only play a structural role, and there is "no third kind" of variance since predicates only depend on two kinds of variables, and any dinatural $x : A$ in a formula arises as a dinatural collapse. This requirement arises only at the level of predicates, since in the base case $P(s \mid t)$ we do not ask for a type "$\mathsf{dinat}(\sigma)$" (and term $\Delta \vdash \mathsf{dinat}(\sigma)$), but only $\mathsf{pos}(\sigma), \mathsf{neg}(\sigma)$ and s, t.

Definition 11 (Polarized doctrine)**.** *A (split)* polarized doctrine *is defined as a cartesian closed category $\mathbb{C}$ equipped with a functor $\mathcal{P} : \mathsf{ndp}(\mathbb{C})^{\mathsf{op}} \to \mathsf{Pos}$ satisfying the* no-dinatural-variance *condition.*

Definition 12 (Weakenings)**.** *Given a polarity $p \in \{-, \Delta, +\}$ we shall denote with $\mathcal{P}(\Theta \mid \Delta \mid \Gamma \mid\mid^{p} A)$ the fiber obtained by applying the functor $- \times A$ to either Θ, Δ, Γ depending on p in the intuitive way. We denote weakening functors by $\mathsf{wk}^{p}_{A} : \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \to \mathcal{P}(\Theta \mid \Delta \mid \Gamma \mid\mid^{p} A)$.*

We need semantic equivalents for Theorems 1 and 2 in order to later characterize logical connectives:

Theorem 3 (Dinatural lift)**.** *There is a functor*

$$\mathcal{P}(\Uparrow^{\Delta}_{N,P}) : \mathcal{P}(\Theta \times N \mid \Delta \mid \Gamma \times P) \to \mathcal{P}(\Theta \mid \Delta \times N \times P \mid \Gamma)$$

given by reindexing with the term

$$\begin{aligned} \Uparrow^{\Delta}_{N,P} := (\ & \langle \pi_1, \pi_3\rangle : \Theta \times \Delta \times N \times P \to \Theta \times N, \\ \mid\ & \pi_1 : \Delta \times N \times P \to \Delta, \\ \mid\ & \langle \pi_1, \pi_4\rangle : \Gamma \times \Delta \times N \times P \to \Gamma \times P) \\ :\ & (\Theta \mid \Delta \times N \times P \mid \Gamma) \to (\Theta \times N \mid \Delta \mid \Gamma \times P). \end{aligned}$$

One-variable versions $\Uparrow^{\Delta}_{N}, \Uparrow^{\Delta}_{P}$ can be given similarly. Moreover, $\Uparrow^{\Delta}_{N,P}\,; (\pi_1 \mid !_{\Delta} \mid \pi_1) = \Uparrow^{\pm!}_{\Delta}$.

Theorem 4 (Dinatural contract)**.** *There is a functor*

$$\mathcal{P}(\downdownarrows^{\Delta}_{A}) : \mathcal{P}(\Theta \times A \mid \Delta \mid \Gamma \times A) \to \mathcal{P}(\Theta \mid \Delta \times A \mid \Gamma)$$

given by reindexing with the term

$$\begin{aligned}\downdownarrows^{\Delta}_{A} := (\ & \langle \pi_1, \pi_3 \rangle : \Theta \times \Delta \times A \to \Theta \times A, \\ \mid\ & \pi_1 : \Delta \times A \to \Delta, \\ \mid\ & \langle \pi_1, \pi_3 \rangle : \Gamma \times \Delta \times A \to \Gamma \times A) \\ :\ & (\Theta \mid \Delta \times A \mid \Gamma) \to (\Theta \times A \mid \Delta \mid \Gamma \times A).\end{aligned}$$

Definition 13 (Logical connectives for polarized doctrines)**.** *We define conditions on a polarized doctrine $\mathcal{P} : \mathsf{ndp}(\mathbb{C})^{\mathsf{op}} \to \mathsf{Pos}$. For each definition we also ask that each reindexing functor preserves the relevant structure.*

- $\mathcal{P}$ has conjunctions *iff each fiber has finite products (i.e., greatest lower bounds as a preorder). By a standard argument [21] this induces a conjunction functor $- \wedge - : \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \times \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \to \mathcal{P}(\Theta \mid \Delta \mid \Gamma)$.*

- $\mathcal{P}$ has polarized exponentials $(\Rightarrow)$ *iff there is a functor*

$$\begin{aligned}- \Rightarrow - : & \mathcal{P}(N \times N' \mid \Delta \mid P \times P')^{\mathsf{op}} \\ & \times \mathcal{P}(\Theta \times N \times P' \mid \Delta \mid \Gamma \times P \times N') \\ & \to \mathcal{P}(\Theta \times P' \mid \Delta \times N \times P \mid \Gamma \times N')\end{aligned}$$

  such that, for every $\Theta, \Delta, \Gamma, N, N', P, P' \in \mathbb{C}$ and $\Phi, \varphi \in \mathcal{P}(\Theta \times N \times P' \mid \Delta \mid \Gamma \times P \times N')$, $\psi \in \mathcal{P}(N \times N' \mid \Delta \mid P \times P')$, the top relation holds iff the bottom one holds:

$$\frac{\mathcal{P}(\pi_2, \mathsf{id}, \pi_2)(\mathcal{P}(\Uparrow^{\Delta}_{N',P'})(\psi)) \times \mathcal{P}(\Uparrow^{\Delta}_{N',P'})(\Phi) \leq \mathcal{P}(\Uparrow^{\Delta}_{N',P'})(\varphi)}{\mathcal{P}(\Uparrow^{\Delta}_{N,P})(\Phi) \leq \psi \Rightarrow \varphi}$$

  where the top entailment takes place in $\mathcal{P}(\Theta \times N \mid \Delta \times N' \times P' \mid \Gamma \times P)$ and the bottom in $\mathcal{P}(\Theta \times P' \mid \Delta \times N \times P \mid \Gamma \times N')$. We remark in Theorem 5 that having polarized exponentials is property and not structure.

- $\mathcal{P}$ has polarized quantifiers *iff for every $p \in \{-, \Delta, +\}$ and $A \in \mathbb{C}$ the functor $\mathsf{wk}^p_A : \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \to \mathcal{P}(\Theta \mid \Delta \mid \Gamma \mid\mid^p A)$ has a left adjoint $\exists^p_A$ and a right adjoint $\forall^p_A$, i.e., the adjunction $\exists^p_A \dashv \mathsf{wk}^p_A \dashv \forall^p_A$ holds. Moreover, suitable Beck-Chevalley conditions are satisfied, i.e., the following points of $\mathcal{P}(\Theta \mid \Delta \mid \Gamma)$ are equal for any $f : (\Theta \mid \Delta \mid \Gamma) \to (\Theta' \mid \Delta' \mid \Gamma')$ and $\varphi \in \mathcal{P}(\Theta' \mid \Delta' \mid \Gamma' \mid\mid^p A)$:*

$$\begin{aligned}\mathcal{P}(f)(\exists^p_{A[\Theta'\Delta'\Gamma']}(\varphi)) &= \exists^p_{A[\Theta\Delta\Gamma]}(\mathcal{P}(f \mid\mid^p \mathsf{id}_A)(\varphi)) \\ \mathcal{P}(f)(\forall^p_{A[\Theta'\Delta'\Gamma']}(\varphi)) &= \forall^p_{A[\Theta\Delta\Gamma]}(\mathcal{P}(f \mid\mid^p \mathsf{id}_A)(\varphi))\end{aligned}$$

  where the morphism $(f \mid\mid^p \mathsf{id}_A) : (\Theta \mid \Delta \mid \Gamma \mid\mid^p A) \to (\Theta' \mid \Delta' \mid \Gamma' \mid\mid^p A)$ reindexes the relevant variables with F and leaves A untouched. We omit an intuitive Frobenius condition which in this case holds automatically thanks to the presence of exponentials, following a similar proof in [21, 1.9.12(i)].

- $\mathcal{P}$ has directed equality *iff, given the following functors,*

$$\begin{aligned} \mathcal{P}(\downdownarrows^{\Delta}_{A}) &: \mathcal{P}(\Theta \times A \mid \Delta \mid \Gamma \times A) \to \mathcal{P}(\Theta \mid \Delta \times A \mid \Gamma) \\ \mathcal{P}(\mathsf{wk}^{\Delta}_{A}) &: \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \to \mathcal{P}(\Theta \mid \Delta \times A \mid \Gamma) \end{aligned}$$

  there is a $\mathcal{P}(\mathsf{wk}^{\Delta}_{A})$*-relative left adjoint* $\leq_A \times -$ *to the contraction functor* $\mathcal{P}(\downdownarrows^{\Delta}_{A})$*, i.e.,*

$$\begin{tikzcd} & \mathcal{P}(\Theta \times A \mid \Delta \mid \Gamma \times A) \arrow[dr, "\mathcal{P}(\downdownarrows^{\Delta}_{A})"] \arrow[d, phantom, "\dashv"] & \\ \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \arrow[ur, "\leq_A \times -"] \arrow[rr, "\mathcal{P}(\mathsf{wk}^{\Delta}_{A})"'] & & \mathcal{P}(\Theta \mid \Delta \times A \mid \Gamma) \end{tikzcd}$$

  Details on relative adjunctions for categories are given in Appendix C, [3]. In our case of interest, posets, such condition reduces to the following bi-implication of conditions for $\Phi \in \mathcal{P}(\Theta \mid \Delta \mid \Gamma), \varphi \in \mathcal{P}(\Theta \times A \mid \Delta \mid \Gamma \times A)$*:*

$$\frac{[\Theta \mid \Delta \times A \mid \Gamma]\ \mathcal{P}(\mathsf{wk}^{\Delta}_{A})(\Phi) \leq \mathcal{P}(\downdownarrows^{\Delta}_{A})(\varphi)}{[\Theta \times A \mid \Delta \mid \Gamma \times A] \leq_A \times \Phi \leq \varphi}\ (\leq)$$

  Moreover, we ask for the following Beck-Chevalley condition [21, 3.4.1]; for any map $f := (n \mid d \mid p) : (\Theta \mid \Delta \mid \Gamma) \to (\Theta' \mid \Delta' \mid \Gamma')$ *and* $\varphi \in \mathcal{P}(\Theta' \mid \Delta' \mid \Gamma')$*, the following points in* $\mathcal{P}(\Theta \times A \mid \Delta \mid \Gamma \times A)$ *are asked to be equivalent:*

$$\leq_{A[\Theta\Delta\Gamma]}(\mathcal{P}(f)(\varphi)) \Leftrightarrow \mathcal{P}(n \times \mathsf{id}_A \mid d \mid p \times \mathsf{id}_A)(\leq_{A[\Theta'\Delta'\Gamma']}(\varphi)),$$

  where $\mathcal{P}(n \times \mathsf{id}_A \mid d \mid p \times \mathsf{id}_A) : \mathcal{P}(\Theta' \times A \mid \Delta' \mid \Gamma' \times A) \to \mathcal{P}(\Theta \times A \mid \Delta \mid \Gamma \times A)$ *is defined in the intuitive way by applying* f *and leaving the variables* A *unaltered. A similar "polarized Frobenius" condition, exemplified in* $(\leq^{\mathsf{full}})$*, holds using polarized exponentials [21, 1.9.12(i)].*

Note the similarity between the above left-relative adjunction and standard accounts of equality in categorical logic [21,31,33]. In our case we cannot relate directed equality to existentials and their characterization as left adjoints [32], since the former is given by a *relative* adjunction and the latter by a standard adjunction, respectively.

A binary attribute of mixed variance (following the idea of Lawvere reported in Appendix A) can be recovered in each fiber by taking $\top \in \mathcal{P}(\top \mid \top \mid \top)$ and $\leq_A \times \top : \mathcal{P}(A \mid \top \mid A)$, where indeed the two variables are now separated by their polarity.

Definition 14 (Directed doctrine)**.** *A* (split) directed doctrine *is defined to be a* $(\top, \wedge, \Rightarrow, \exists^p, \forall^p, \leq)$*-polarized doctrine, i.e., it satisfies all above conditions.*

Theorem 5 (Uniqueness of polarized exponentials)**.** *Although the condition for polarized exponentials does not fit the scheme for either standard nor relative adjunction, having polarized exponentials is a property, not a structure; we show that any other functor* $(-\Rightarrow'-) \cong (-\Rightarrow-)$ *by a standard "uniqueness-of-adjoints"-like argument which we show in Appendix D, crucially relying on the no-dinatural-variance condition and the fact that the specific reindexing* $\mathcal{P}(\Uparrow^{\pm!}_{\Delta})$ *is a bijection-on-objects, and not* any *functor.*

Remark 4. *As a special case of* $-\Rightarrow-$ *one can pick* $N' := \Theta, P' := \Gamma, N = P := \top$ *to obtain a functor* $-\Rightarrow_{\varphi}- : \mathcal{P}(\Gamma \mid \Delta \mid \Theta)^{\mathsf{op}} \times \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \to \mathcal{P}(\Theta \mid \Delta \mid \Gamma)$ *which captures a sort of "global closed structure", in the spirit of monoidal fibrations [42]. This functor captures exactly the structure of formulas used in the syntax in Figure 3.*

4.1 Examples of directed doctrines

We present the main model of directed first-order logic, given by interpreting types as preorders:

Example 5 (Directed doctrine of preorders)**.** *The following is a directed doctrine, with* Preord *the category of preorders and monotone functions as base (i.e. types/terms):*

$$\begin{aligned}&\mathcal{P}\mathsf{reord} : \mathsf{ndp}(\mathsf{Preord})^{\mathsf{op}} \to \mathsf{Pos}\\&\mathcal{P}\mathsf{reord}(N \mid D \mid P) := [N^{\mathsf{op}} \times (D^{\mathsf{op}} \times D) \times P, \mathbf{I}]_{\Delta}, \textit{ where}\\&\quad \mathsf{Obj} := \textit{monotone maps } N^{\mathsf{op}} \times (D^{\mathsf{op}} \times D) \times P \to \mathbf{I},\\&\quad \psi \le \varphi := \forall n{:}N, d{:}D, p{:}P. \psi(n,d,d,p) \le \varphi(n,d,d,p).\end{aligned}$$

The idea for entailments is that of dinatural transformations [12], since we rely on the fact that $A_0 = A_0^{\mathsf{op}}$ have the same objects. The no-dinatural-variance condition is satisfied, since $N^{\mathsf{op}} \times (D^{\mathsf{op}} \times D) \times P \cong (N \times D)^{\mathsf{op}} \times \top \times (P \times D)$.
Conjunction $\psi \wedge \varphi$ is interpreted by the pointwise product of monotone functions in $\mathbf{I}$, and similarly exponentials $\psi \Rightarrow \varphi$ by postcomposing $\langle \psi^{\mathsf{op}}, \varphi \rangle$ with $\le_{\mathbf{I}} \colon \mathbf{I}^{\mathsf{op}} \times \mathbf{I} \to \mathbf{I}$. Reindexing on formulas is given by precomposition of monotone functions: given $\eta : N \times D \to N', \delta : D \to D', \rho : P \times D \to P'$ and $\varphi : [N^{\mathsf{op}} \times (D^{\mathsf{op}} \times D) \times P, \mathbf{I}]$,

$$\mathcal{P}(\eta,\delta,\rho)(\varphi)(n,d',d,p) := \varphi(\eta^{\mathsf{op}}(n,d'), \delta^{\mathsf{op}}(d), \delta(d), \rho(p,d)).$$

Polarized quantifiers are given using indexed products and coproducts in $\mathbf{I}$ (i.e., glb and lub), e.g.: given $\varphi : [N^{\mathsf{op}} \times (D^{\mathsf{op}} \times D) \times (P \times A), \mathbf{I}]$, $\psi : [N^{\mathsf{op}} \times ((D \times A)^{\mathsf{op}} \times (D \times A)) \times P, \mathbf{I}]$ we illustrate the idea with the following examples:

$$\begin{aligned}&\exists^{+}(\varphi)(n,d',d,p) := \textstyle\coprod_{a \in A} \varphi(n,d',d,p,a) \in \mathbf{I}\\&\forall^{\Delta}(\psi)(n,d',d,p) := \textstyle\prod_{a \in A} \psi(n,(d',a),(d,a),p) \in \mathbf{I}\end{aligned}$$

For $\forall^{\Delta}$ we again rely on $A_0^{\mathsf{op}} = A_0$, in a way that resembles a decategorification of ends [29]. Moreover, this doctrine has directed equality, interpreted by formulas

$$\begin{aligned}\le_A \colon & [(\Theta \times A)^{\mathsf{op}} \times (\Delta^{\mathsf{op}} \times \Delta) \times (\Gamma \times A), \mathbf{I}]\\&:= ((n,a),(d,d'),(p,a')) \mapsto a \le_A a' \in \mathbf{I}.\end{aligned}$$

and the left-relative adjunction for directed equality is interpreted with the following construction: assume $f : \mathcal{P}\mathsf{reord}(\Theta, \Delta, \Gamma), g : \mathcal{P}\mathsf{reord}(\Theta \times A, \Delta, \Gamma \times A)$, such that

$$\forall n{:}N, d{:}D, p{:}P. f(n,d,d,p) \le g((n,a),d,d,(p,a)).$$

(Note that dinatural collapse is applied on g.) We show

$$\begin{aligned}&\forall (n,a){:}N \times A, d{:}D, (p,a'){:}P \times A.\\&\quad (a \le_A a') \wedge f(n,d,d,p) \le_{\mathbf{I}} g((n,a),d,d,(p,a')).\end{aligned}$$

Assuming $a \le a'$, we use monotonicity of g to obtain

$$f(n,d,d,p) \le g((n,a),d,d,(p,a)) \le g((n,a),d,d,(p,a'))$$

as desired. Rule ($\le$-refl) *corresponds to $\forall a. \top \le_{\mathbf{I}} (a \le_A a)$. This proof motivates the syntactic restriction for a, b to appear naturally in the conclusion, since we rely on monotonicity of either a or a' to be able to use the hypothesis; however, by following the proof, one can notice that a weaker restriction could suffice, i.e., by allowing $a \le b$ to be contracted if* at least one *of the variables appears naturally. We choose the current approach in order to simplify the treatment and to mirror the case of proof-relevant dinaturals, in which functoriality on both variables is needed for the hexagon to be stated.*

Definition 15 (Syntactic directed doctrine). *Given a theory Σ, we define the* syntactic doctrine $\mathsf{Syn}(\Sigma) : \mathsf{ndp}(\mathsf{Ctx})^{\mathsf{op}} \to \mathsf{Pos}$ *as the syntactic directed doctrine inductively generated from Σ, as follows:* Ctx *is the category where objects are contexts Γ, Δ and morphisms are term substitutions $\Gamma \vdash [\Delta_1, ..., \Delta_n]$, i.e., lists of terms $\Gamma \vdash t : \Delta_i$ for $\Delta_i \in [\Delta_1, ..., \Delta_n]$ (and composition is substitution, as in [40]), and $\mathcal{P}(\Theta \mid \Delta \mid \Gamma) := \{[\Theta \mid \Delta \mid \Gamma]\ \Phi\ \mathsf{propctx}\}$, where the poset relation is given by the existence of multi-entailment judgements $\Phi \vdash \Psi$, defined in a similar way as substitutions. The rest of the structure follows directly from the rules of entailments, paying particular attention to the points in which weakenings and dinatural lifts are implicitly used in both the syntax and the semantics.*

The following lemma establishes the no-dinatural-variance condition for Syn; since we want to show soundness, the fact that Syn satisfies this property indicates that directed doctrines also need this requirement.

Theorem 6 (Syn has no-dinatural-variance). *The syntactic doctrine* $\mathsf{Syn}(\Sigma)$ *satisfies the no-dinatural-variance condition, i.e., there is a bijection between sets of judgements* $\{[\Theta \mid \Delta \mid \Gamma]\ \varphi\ \mathsf{prop}\} \cong \{[\Theta, \Delta \mid \bullet \mid \Gamma, \Delta]\ \varphi\ \mathsf{prop}\}$.

Proof. Straightforward by induction on formulas, using $\{(\Theta, \Delta), \top \vdash A\} \cong \{\Theta, \Delta \vdash A\}$ for $\leq_A$ and $P(s \mid t)$. $\square$

Definition 16 (Set doctrines). *The* Set *doctrine of sets and subsets (e.g., [31]) can be lifted to a directed doctrine by precomposing the discrete poset functor* $\mathsf{Set} \to \mathsf{Pos}$.

Theorem 7. *A directed doctrine $\mathcal{P} : \mathsf{ndp}(\mathbb{C})^{\mathsf{op}} \to \mathsf{Pos}$ induces a doctrine $\Downarrow^{\mathsf{op}} ; \mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Pos}$ by precomposing with the functor $\Downarrow : \mathbb{C} \to \mathsf{ndp}(\mathbb{C})$ given by $C \mapsto (\top \mid C \mid \top)$; intuitively, this captures the non-directed "sub-logic" consisting only of dinatural contexts. Such doctrine will have conjunction, exponentials, existential/universal quantifiers if the original directed doctrine $\mathcal{P}$ is equipped with such structure (only quantifiers with dinatural polarity are needed). A doctrine can be lifted to a directed one by precomposing with $\Uparrow : \mathsf{ndp}(\mathbb{C}) \to \mathbb{C}$ given by $(\Theta \mid \Delta \mid \Gamma) \mapsto \Theta \times \Delta \times \Delta \times \Gamma$, satisfying the no-dinatural-variance condition.*

5 Interpretation

In the following we define models for the syntax and sketch the construction of the interpretation of a theory in a doctrine. These steps follow mostly the standard approach of functorial semantics à-la-Lawvere [21, 25].

Definition 17. *We denote the (2-)category of directed doctrines and their morphisms as* $\mathsf{DDoctrine}$*, as in [32]. A* morphism of directed doctrines $\mathcal{P} \to \mathcal{P}'$ *for* $\mathcal{P} : \mathsf{ndp}(\mathbb{C})^{\mathsf{op}} \to \mathsf{Pos}$, $\mathcal{P}' : \mathsf{ndp}(\mathbb{D})^{\mathsf{op}} \to \mathsf{Pos}$ *is defined as a pair (F, α), where $F : \mathbb{C} \to \mathbb{D}$ is a functor that preserves products, exponentials, terminal objects, and $\alpha : P \Rightarrow \mathsf{ndp}(F)^{\mathsf{op}} ; P'$ is a natural transformation such that each functor $\alpha_{\Theta,\Delta,\Gamma} : P(\Theta \mid \Delta \mid \Gamma) \to P'(F(\Theta), F(\Delta), F(\Gamma))$ for each Θ, Δ, Γ preserves all the structure (i.e., terminal objects, products, polarized exponentials, polarized quantifiers, directed equality) present in each fiber [21]. For instance, preservation of directed equality means that*

$$\alpha_{\Theta,\Delta,\Gamma}(\leq_A \times \varphi) = \mathcal{P}'(\cong_{\Theta A} \mid \mathsf{id} \mid \cong_{\Gamma A})(\leq'_{F(A)} \times F(\varphi))$$

where $\cong_{AB} := \langle F(\pi_1), F(\pi_2) \rangle : F(A \times B) \to F(A) \times F(B)$. A 2-cell $(F, \alpha) \Rightarrow (G, \beta)$ *is defined by a natural transformation $\theta : F \Rightarrow G$ such that*

$$\alpha_{\Theta,\Delta,\Gamma}(p) \leq \mathcal{P}'(\theta_{\Theta,\Delta} \mid \theta_{\Delta} \mid \theta_{\Gamma,\Delta})(\beta_{\Theta,\Delta,\Gamma}(p))$$

holds, where $\theta_{\Theta,\Delta}, \theta_{\Gamma,\Delta}$ are defined in the only possible way.

Definition 18. *We define the* 1-category of theories Theory *where objects are theories Σ and a* morphism of theories $M : \Sigma \to \Sigma'$ *is defined as a tuple*

$$M := (b : \Sigma_B \to \Sigma'_B, p : \Sigma_P \to \Sigma'_P, f : \Sigma_F \to \Sigma'_F)$$

of set functions, such that

- $f \,;\, \mathsf{dom}' \cong \mathsf{dom} \,;\, [\![-]\!]_b$ *and* $f \,;\, \mathsf{cod}' \cong \mathsf{cod} \,;\, [\![-]\!]_b$ *are isomorphisms of sets,*
- $\forall e \in \Sigma_E, [\![\mathsf{actx}(e)]\!]_{\mathsf{bctx}} \vdash [\![\mathsf{eqL}(e)]\!]_f = [\![\mathsf{eqR}(e)]\!]_f : [\![\mathsf{eqt}(\mathsf{e})]\!]_b$ *is syntactically derivable from Figure 4 using Σ' as theory,*
- $p \,;\, \mathsf{pos}' \cong \mathsf{pos} \,;\, [\![-]\!]_b$ *and* $p \,;\, \mathsf{neg}' \cong \mathsf{neg} \,;\, [\![-]\!]_b$*,*
- $\forall a \in \Sigma_A, [[\![\mathsf{actx}(a)]\!]_{\mathsf{bctx}}]\ [\![\mathsf{hyp}(a)]\!]_{\mathsf{pctx}} \vdash [\![\mathsf{conc}(a)]\!]_{\mathsf{pctx}}$ *is syntactically derivable from Figure 4 using Σ' as theory,*

where we denote the induced translations on full types $[\![-]\!]_b$ and contexts $[\![-]\!]_{\mathsf{bctx}}$, terms $[\![-]\!]_f$, formulas $[\![-]\!]_p$ and $[\![-]\!]_{\mathsf{pctx}}$.

Theorem 8 (Initial theory)**.** *There is a theory $\emptyset$ defined by always choosing the empty set in Definition 3. Clearly $\emptyset$ is the initial object in the category of theories* Theory.

Definition 19 (Underlying theory)**.** *Given a directed doctrine $\mathcal{P} : \mathsf{ndp}(\mathbb{C})^{\mathsf{op}} \to \mathsf{Pos}$, we define the* underlying theory $\mathsf{Lang}(\mathcal{P}) \in \mathsf{Theory}$ *as follows:*

- *base types $\Sigma_B := \mathbb{C}_0$ the objects of $\mathbb{C}$,*
- *base terms $\Sigma_F := \mathbb{C}_1$ is the set of arrows $\mathbb{C}_0$ of $\mathbb{C}$ with their intuitive* $\mathsf{dom}, \mathsf{cod}$ *functions;*
- *term equalities $\Sigma_E := \{(A, \Gamma, s, t) \mid s, t \in \{\Gamma \vdash t : A\}, [\![t]\!]_f = [\![s]\!]_f\}$ are given by the set of terms formed with Σ_B/Σ_F as base types/terms for which their interpretation $[\![-]\!]_f$ induced by the previous points is equal in $\mathbb{C}$.*
- *the set of base formulas $\Sigma_P := \coprod_{k \in C_0 \times C_0 \times C_0} \mathcal{P}(k)_0$ is the set of all objects in all fibers $\mathcal{P}(\Theta \mid \Gamma \mid \Delta)$, where* $\mathsf{pos}(p) := \Theta \times \Delta$ *and* $\mathsf{neg}(p) := \Gamma \times \Delta$ *(more precisely,* $\mathsf{pos}(k, p) := k_1 \times k_2$, neg *similarly); this is another crucial point in which we intuitively rely on no-dinatural-variance, because by the isomorphism $\mathcal{P}(\Theta \mid \Delta \mid \Gamma) \cong \mathcal{P}(\Theta \times \Delta \mid \top \mid \Gamma \times \Delta)$ the choice of* $\mathsf{pos}, \mathsf{neg}$ *uniquely represents all predicates, without leaving any predicate uncovered.*
- *finally, the set of base axioms Σ_A contains a symbol whenever $\psi \leq \varphi$ holds for some ψ, φ in any poset.*

Definition 20 (Model)**.** *A* model *of a theory Σ in a directed doctrine $\mathcal{P}$ is a doctrine morphism* $\mathsf{Syn}(\mathcal{P}) \to \mathsf{Lang}(\Sigma)$*. Such a doctrine morphism corresponds exactly with the classical notion of model one expects: e.g., in the posetal case, a model prescribes a poset for each base type $\in \Sigma_B$, a monotone function for each term, suitable monotone functions to interpret product types and conjunction formulas, etc.*

The main theorem below establishes the correspondence between models and syntactic choices (i.e., morphisms of theories): by giving an interpretation of a theory in a doctrine, one can use directed first-order logic to reason about any directed doctrine, where doctrine morphisms now represent a notion of model. From the main theorem, we establish that directed first-order logic is the sound and complete internal language for directed doctrines.

Theorem 9 (Internal language correspondence)**.** *The above constructions are functorial, forming (2-)functors:*

$$\mathsf{Syn} : \mathsf{Theory} \rightleftarrows \mathsf{DDoctrine} : \mathsf{Lang}$$

Moreover, they form a (bi-)adjunction between (2-)categories, i.e., there are equivalences of categories between the above category and the set below [21, 2.2.5] for any theory Σ *and directed doctrine* $\mathcal{P} : \mathsf{ndp}(\mathbb{C})^{\mathsf{op}} \to \mathsf{Pos}$*:*

$$\frac{\mathsf{Syn}(\Sigma) \longrightarrow \mathcal{P} \ \textit{in}\ \mathsf{DDoctrine}}{\overline{\Sigma \longrightarrow \mathsf{Lang}(\mathcal{P}) \ \textit{in}\ \mathsf{Theory}}}$$

Proof. We start by describing the two constructions, focusing in particular on the case of base predicates, since it is where we need Definition 10.
(Construction $\Downarrow$*).* One evaluates the doctrine morphism $(F, \alpha) : \mathsf{Syn}(\Sigma) \longrightarrow \mathcal{P}$ at the judgements involving Σ to obtain the actions of the following signature morphism:

- $b(\sigma) := F_0(\sigma\ \mathsf{type} \in \mathsf{Obj}(\mathsf{Syn}(\Sigma))) \in (\mathsf{Obj}(\mathbb{C}) \equiv \mathsf{Lang}(\mathcal{P})_B)$.

- $f(\sigma) := F_1(\llbracket \mathsf{cod}(\sigma) \rrbracket_b \vdash \sigma(\mathsf{id}) : \llbracket \mathsf{cod}(\sigma) \rrbracket_b) \in \mathsf{Lang}(\mathcal{P})_F$. For the induced interpretation one has that $\llbracket A \rrbracket_b = F(A)$, $\llbracket t \rrbracket_f = F(t)$ hold by induction on A, t, since F_0, F_1 preserve products, exponentials, etc., in each inductive step.

- Term equality condition: given $e \in \Sigma_E$, its associated base equality holds in $\mathsf{Syn}(\Sigma)$ therefore the terms/arrows are equal. Functors F preserve equalities of arrows into $\mathcal{P}$, hence their $\llbracket - \rrbracket_f$ is equal in $\mathsf{Lang}(\mathcal{P})_E$ by definition.

- For predicates, $p(\sigma) := (F(\mathsf{pos}(\sigma)), F(\top), F(\mathsf{neg}(\sigma)), \tilde{p}) \in \mathsf{Lang}(\mathcal{P})_F$, where the point $\tilde{p} := \alpha(\sigma(\pi_1, \pi_1)\ \mathsf{prop}) \in \mathcal{P}(F(\mathsf{pos}(\sigma)) \mid F(\top) \mid F(\mathsf{neg}(\sigma)))_0$ is obtained by applying α to the base case $P(s \mid t)$ prop, but picking $\Delta := \top$ and the projections $\mathsf{pos}(\sigma), \top \vdash \pi_1 : \mathsf{pos}(\sigma)$ for s, t. This choice essentially corresponds to $\Uparrow^{\pm !}_{\top}$. Moreover,

$$\mathsf{pos}'(\tilde{p}) := F(\mathsf{pos}(\sigma)) \times F(\top) \cong F(\mathsf{pos}(\sigma)) = \llbracket \mathsf{pos}(\sigma) \rrbracket_b.$$

  This $\cong$ could be resolved more precisely by working with a weaker 2-Cat of theories.

- For axioms, we apply monotonicity of α on the relation $\mathsf{hyp}(a) \vdash \mathsf{conc}(a)$ in Syn, which holds by the axiom case for $\sigma \in \Sigma_F$. From this we obtain that the desired $\leq$ relation in $\mathcal{P}$, which is exactly how $\mathsf{Lang}(\Sigma)_A$ was defined.
  (Construction $\Uparrow$*).* Each component of the doctrine morphism $(\llbracket - \rrbracket, \llbracket - \rrbracket_\varphi)$ is given by induction on derivations, using the theory morphism $M : \Sigma \to \mathsf{Lang}(\mathcal{P})$ for the base cases and the structure of $\mathcal{P}$ for the inductive steps. The functor $\llbracket - \rrbracket : \{A\ \mathsf{type}\}_\Sigma \to \mathbb{C}_0$ is defined as:

- $\llbracket - \rrbracket_0$ acts by induction on types, using the product functor $- \times - : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ to interpret product types, etc., using for base types $A \in \Sigma_B$ the action $M_b(A) \in \mathbb{C}_0$.

- $\llbracket - \rrbracket_1$ acts by induction on terms, similarly as above. Functoriality is ensured by a substitution lemma.

- The components of the natural transformation

$$\llbracket \Theta \mid \Delta \mid \Gamma \rrbracket_\varphi : \mathsf{Syn}(\Theta \mid \Delta \mid \Gamma) \to \mathcal{P}(\llbracket \Theta \rrbracket \mid \llbracket \Delta \rrbracket \mid \llbracket \Gamma \rrbracket)$$

are the functors which, on objects, act by induction on formula derivations in Syn and use the structure of $\mathcal{P}$, e.g., using $- \wedge -$, implication $- \Rightarrow -$, (Remark 4), and directed equality in $\mathcal{P}$ to combine objects in the inductive step. For the base judgement $\gamma := \sigma(s \mid t)$ prop with $\Theta, \Delta \vdash s : [\![\mathsf{neg}(\sigma)]\!]$, take $\tilde{P} := p(\sigma \in \Sigma_P) \in \mathcal{P}(\Theta', \Delta', \Gamma')$. By the no-dinatural-variance condition, $\varepsilon(\tilde{P}) \in \mathcal{P}(\Theta' \times \Delta' =: \mathsf{neg}'(b(\sigma)) \cong [\![\mathsf{neg}(\sigma)]\!] \mid \top \mid \cdots)$. Finally,

$$[\![\gamma]\!]_\varphi := \mathcal{P}([\![s]\!] \mid ! \mid [\![t]\!])(\varepsilon(\tilde{P})) \in \mathcal{P}([\![\Theta]\!] \mid [\![\Delta]\!] \mid [\![\Gamma]\!]).$$

Naturality is a semantic substitution lemma, by induction.

- The action on morphisms (entailments) is given by interpreting each rule with the properties of $\mathcal{P}$.

We show the equivalence with details in Appendix E for base predicates; both directions use the fact that ε is a bijection.

- *($\Downarrow ; \Uparrow = \mathsf{id}$).* The no-dinatural-variance condition is used here with naturality of α to cover all the predicates in each fiber given by the original interpretation.
- *($\Uparrow ; \Downarrow = \mathsf{id}$).* Similarly straightforward since the interpretation $[\![-]\!]$ is essentially determined (by induction) via the action on the base predicates of M. □

The following corollaries establish soundness (i.e., an initiality theorem [40]) and completeness for the syntax.

Corollary 1 (Soundness)**.** *If an entailment is derivable in directed first-order logic on the empty theory, it holds in every directed doctrine.*

Proof. By Theorem 8, $\emptyset$ is initial in Theory, therefore the bottom, and top, sets in Theorem 9 are singletons, hence $\mathsf{Syn}(\emptyset)$ is also initial. A doctrine morphism $\mathsf{Syn}(\emptyset) \longrightarrow \mathcal{P}$ is exactly a model of the empty theory in $\mathcal{P}$, hence we can send the entailment to the one internal to $\mathcal{P}$. □

Corollary 2 (Completeness)**.** *If an entailment holds in all directed doctrines, it is derivable in directed logic.*

Proof. Since $\mathsf{Syn}(\emptyset)$ is a directed doctrine, the statement is true also in $\mathsf{Syn}(\emptyset)$. By definition of $\mathsf{Syn}(\emptyset)$, a statement is true internally precisely when it is derivable. □

6 Conclusion and Future Work

In this paper we introduced a sound and complete syntax for a directed first-order proof-irrelevant type theory with polarities and a "directed" notion of equality $\leq_A$, which we characterized in terms of a left relative adjoint. This treatment of directed-relations-as-equality further enlightens a question first posed by Lawvere on the precise role of hom and variance [26] in the categorical treatment of logic, treated here for the case of preorders.

The concept of polarized contexts described in this paper is a further step towards a satisfactory syntactic treatment of directed type theory, which even in the first-order case with preorder semantics is particularly non-trivial due to the treatment of polarity. A possible generalization would be to incorporate type dependency and use a notion of indexed posets as models: however, it is not clear what role variance should play when variables appear in the types themselves. Following the geometric intuition for HoTT, the theory of directed spaces and directed algebraic topology [13] might prove to be a suitable model for (an extension) of our logic that can reason about concurrent systems, since directed equality lends itself particularly well to be interpreted as a sort of rewrite as in rewriting logic [35], for which we investigated here just the universal property. Syntactically, we saw how directed first-order logic is a only a slight departure from first-order logic, allowing for it to be expanded in the same way that the latter is the basis of many logical systems. Another approach to capturing variances could be based on a multicategorical "graded" approach [27], tagging variables with polarities and using a binary operation that combines the polarity of variables together such that $- \otimes + = \Delta$. A relatively straightfoward but important extension to this work is given by having a type A^{op} for any type A as in [28]: our work lays the foundation for such a treatment, since we can state precisely how a positive occurrence of A^{op} is equivalent to a negative occurrence of A (hence op-types simply swap positive and negative contexts), and such $-^{\mathsf{op}}$ operation becomes a "representable" way (in a sense similar to [44]) to express such swap of polarities between positive and negative variables.

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A Lawvere on the presheaf hyperdoctrine

> [...] This should not be taken as indicative of a lack of vitality of [Psh] as a hyperdoctrine, or even of a lack of a satisfactory theory of equality for it. Rather, it indicates that we have probably been too naive in defining equality in a manner too closely suggested by the classical conception. [...] But present categorical conceptions indicate that [...] the graph of a functor $f : \mathbb{B} \to \mathbb{C}$ should be [...] a binary attribute of mixed variance in $\mathsf{Psh}(\mathbb{B}^{\mathsf{op}} \times \mathbb{C})$. Thus in particular "equality" should be the functor $\hom_{\mathbb{B}}$ [...]. The term which would take the place of δ in such a more enlightened theory of equality would then be the forgetful functor $\mathsf{Tw}(\mathbb{B}) \to \mathbb{B}^{\mathsf{op}} \times \mathbb{B}$ from the "twisted morphism category" [...]. Of course to abstract from this example would require at least the addition of a functor $T \xrightarrow{\mathsf{op}} T$ to the structure of a [doctrine]. [26, p. 11]

B Derived rules

We show an example derivation for the rules of polarized exponentials with polarities, in this case $\forall^+$ and $\exists^\Delta$:

$$\dfrac{\dfrac{\Delta, \Gamma \vdash \rho : P \qquad [\Theta \mid \Delta \mid \Gamma] \; \Phi \vdash \forall^+ x.\psi(x)}{[\Theta \mid \Delta \mid \Gamma, x : P] \; \Phi \vdash \psi(x)}\,(\forall)}{[\Theta \mid \Delta \mid \Gamma] \; \Phi \vdash \psi(\rho)}\,(\mathsf{reindex})$$

To establish the $\exists^\Delta$ rule, we first start with the following derivation:

$$\dfrac{\dfrac{\dfrac{\Delta \vdash \delta : D}{[\Theta \mid \Delta \mid \Gamma] \; \exists^\Delta x.\psi(\overline{x}, x), \Phi \vdash \exists^\Delta x.\psi(\overline{x}, x)}\,(\mathsf{hyp})}{[\Theta \mid \Delta, x : D \mid \Gamma] \; \psi(\overline{x}, x), \Phi \vdash \exists^\Delta x.\psi(\overline{x}, x)}\,(\exists)}{[\Theta \mid \Delta \mid \Gamma] \; \psi(\delta, \delta), \Phi \vdash \exists^\Delta x.\psi(\overline{x}, x)}\,(\mathsf{reindex})$$

and then one can derive the $\exists^\Delta$ rule by applying (cut) with a generic propositional context on the left.

We show that the $\Uparrow$ direction of $(\leq)$ implies reflexivity of directed equality: take the rule

$$\dfrac{[\Theta, a : A \mid \Delta \mid \Gamma, b : A] \; a \leq b, \Phi \vdash \varphi(a, b)}{[\Theta \mid \Delta, z : A \mid \Gamma] \qquad \Phi \vdash \varphi(\overline{z}, z)}\,(\leq)$$

By instantiating it with $\varphi(a, b) := a \leq b$,

$$\dfrac{\dfrac{}{[\Theta, a : A \mid \Delta \mid \Gamma, b : A] \; a \leq b, \Phi \vdash a \leq b}\,(\mathsf{hyp})}{[\Theta \mid \Delta, z : A \mid \Gamma] \qquad \Phi \vdash \overline{z} \leq z}$$

as desired. The rule ($\leq$-refl_t) is obtained by (reindex) the only dinatural variable in the last derivation with an arbitrary term $\Delta \vdash \delta : A$.

We show additional derived rules in Figures 5 to 7.

$$
\dfrac{\begin{array}{c}[N \mid \Delta \mid P]\ \psi\ \mathsf{prop} \\ [\Theta, N \mid \Delta \mid \Gamma, P]\ \Phi\ \mathsf{propctx}, \varphi\ \mathsf{prop} \\ [\Theta, N \mid \Delta \mid \Gamma, P]\ \psi, \Phi \vdash \varphi\end{array}}{[\Theta \mid \Delta, N, P \mid \Gamma] \quad \Phi \vdash \psi \Rightarrow \varphi}\ (\Rightarrow_L)
\qquad
\dfrac{\begin{array}{c}[P \mid \Delta \mid N]\ \psi\ \mathsf{prop} \\ [\Theta, N \mid \Delta \mid \Gamma, P]\ \Phi\ \mathsf{propctx}, \varphi\ \mathsf{prop} \\ [\Theta \mid \Delta, N, P \mid \Gamma]\ \psi, \Phi \vdash \varphi\end{array}}{[\Theta, N \mid \Delta \mid \Gamma, P] \quad \Phi \vdash \psi \Rightarrow \varphi}\ (\Rightarrow_R)
$$

$$
(\Rightarrow^+_-) := \dfrac{\dfrac{[\Gamma \mid \Delta \mid \Gamma, a : A]\ \ \psi(a), \Phi \vdash \varphi}{[\Gamma \mid \Delta, a : A \mid \Gamma]\ \ \psi(a), \Phi \vdash \varphi}\ (\mathsf{reindex})}{[\Gamma, a : A \mid \Delta \mid \Gamma]\ \ \Phi \vdash \psi(\overline{a}) \Rightarrow \varphi}\ (\Rightarrow_L)
\quad
\dfrac{\dfrac{[\Gamma, a : A \mid \Delta \mid \Gamma]\ \ \Phi \vdash \psi(\overline{a}) \Rightarrow \varphi}{[\Gamma \mid \Delta, a : A \mid \Gamma]\ \ \Phi \vdash \psi(\overline{a}) \Rightarrow \varphi}\ (\mathsf{reindex})}{[\Gamma \mid \Delta \mid \Gamma, a : A]\ \ \psi(a), \Phi \vdash \varphi}\ (\Rightarrow_L)
$$

Figure 5: Derived rules for polarized exponentials.

$$
\dfrac{}{[\Theta \mid \Delta, z : A \mid \Gamma]\ \Phi \vdash \overline{z} \leq z}\ (\leq\text{-}\mathsf{refl})
\qquad
\dfrac{\dfrac{\dfrac{[\Theta \mid \Delta, z : A \mid \Gamma]\ \Phi(z, \overline{z}) \vdash \varphi(\overline{z}, z)}{[\Theta \mid \Delta, z : A \mid \Gamma]\ \vdash, \Phi(z, \overline{z}) \Rightarrow \varphi(\overline{z}, z)}\ (\leq)}{[\Theta, a : A \mid \Delta \mid \Gamma, b : A]\ a \leq b \vdash, \Phi(a, b) \Rightarrow \varphi(a, b)}\ (\Rightarrow)}{[\Theta \mid \Delta, a : A, b : A \mid \Gamma]\ a \leq b, \Phi(\overline{a}, \overline{b}) \vdash \varphi(a, b)}\ (\leq^{\mathsf{full}})
$$

$$
(\leq^+) := \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}[\Theta \mid \Delta \mid \Gamma, a : A]\ \Phi\ \mathsf{propctx} \\ [\Theta \mid \Delta \mid \Gamma, b : A]\ \varphi\ \mathsf{propctx} \\ [\Theta \mid \Delta \mid \Gamma, z : A]\ \ \Phi(z) \vdash \varphi(z)\end{array}}{[\bullet \mid \Delta, \Theta, \Gamma, z : A \mid \bullet] \qquad \vdash \Phi(\overline{z}) \Rightarrow \varphi(z)}\ (\Rightarrow_L)}{[a : A \mid \Delta, \Theta, \Gamma \mid b : A]\ a \leq b \vdash \Phi(a) \Rightarrow \varphi(b)}\ (\leq)}{[\Theta \mid \Delta, a : A \mid \Gamma, b : A]\ a \leq b, \Phi(\overline{a}) \vdash \varphi(b)}\ (\Rightarrow_R)}{[\Theta \mid \Delta, a : A \mid \Gamma, b : A]\ \overline{a} \leq b, \Phi(a) \vdash \varphi(b)}
$$

$$
(\leq^-) := \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}[\Theta, b : A \mid \Delta \mid \Gamma]\ \Phi\ \mathsf{propctx} \\ [\Theta, a : A \mid \Delta \mid \Gamma]\ \varphi\ \mathsf{propctx} \\ [\Theta, z : A \mid \Delta \mid \Gamma]\ \ \Phi(z) \vdash \varphi(z)\end{array}}{[\bullet \mid \Delta, \Theta, z : A, \Gamma \mid \bullet] \qquad \vdash \Phi(\overline{z}) \Rightarrow \varphi(z)}\ (\Rightarrow_L)}{[\bullet \mid \Delta, \Theta, \Gamma, z : A \mid \bullet] \qquad \vdash \Phi(z) \Rightarrow \varphi(\overline{z})}}{[a : A \mid \Delta, \Theta, \Gamma \mid b : A]\ a \leq b \vdash \Phi(b) \Rightarrow \varphi(a)}\ (\leq)}{[\Theta, a : A \mid \Delta, b : A \mid \Gamma]\ a \leq b, \Phi(\overline{b}) \vdash \varphi(a)}\ (\Rightarrow_R)
$$

Figure 6: Derived rules for directed equality.

$$
\dfrac{\Delta, \Gamma \vdash \rho : P \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \varphi(\rho)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \exists^+ x. \psi(x)}\ (\exists^+_t)
\qquad
\dfrac{\Delta, \Gamma \vdash \rho : P \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \forall^+ x. \psi(x)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi(\rho)}\ (\forall^+_t)
$$

$$
\dfrac{\Theta, \Delta \vdash \eta : N \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \varphi(\eta)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \exists^- x. \psi(x)}\ (\exists^-_t)
\qquad
\dfrac{\Theta, \Delta \vdash \eta : N \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \forall^- x. \psi(x)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi(\eta)}\ (\forall^-_t)
$$

$$
\dfrac{\Delta \vdash \delta : D \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \varphi(\delta, \delta)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \exists^\Delta x. \psi(\overline{x}, x)}\ (\exists^\Delta_t)
\qquad
\dfrac{\Delta \vdash \delta : D \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \forall^\delta x. \psi(x)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi(\delta, \delta)}\ (\forall^\Delta_t)
$$

Figure 7: Derived rules for quantifiers.

C Appendix on relative adjunctions

We recall here for the reader's convenience the elementary notion of (right/left)-relative adjunction.

Definition 21 (Left-relative adjunction [3, 5.1]). *Consider an arrangement of categories and functors as follows:*

$$\begin{tikzcd} & \mathbb{D} \arrow[dr, "R"] \arrow[d, phantom, "\dashv"] & \\ \mathbb{C} \arrow[ur, "L"] \arrow[rr, "J"'] & {} & \mathbb{X} \end{tikzcd}$$

In this situation, we say that L is the J-relative left adjoint to R, *written $L \dashv_J R$ and indicated in the above diagram with a central '$\dashv$', if there is a bijection*

$$\mathbb{D}(L(x), y) \cong \mathbb{X}(J(x), R(y))$$

natural in both arguments $x : \mathbb{C}, y : \mathbb{D}$.

One obtains the standard definition of adjunction when $\mathbb{X} = \mathbb{C}$, $J = \mathsf{id}_{\mathbb{C}}$. In the case of posets, the above natural adjoint simply becomes a bi-implication of conditions.

D Uniqueness for polarized exponentials

Theorem 10 (Uniqueness for polarized exponentials). *We show the uniqueness of the polarized exponentials using the no-dinatural-variance property. Suppose we had another polarized exponential operation $-\Rightarrow'-$ with the same universal property; we show that $(\psi \Rightarrow' \varphi) \Leftrightarrow (\psi \Rightarrow \varphi)$ for the most general case of ψ, φ. The following derivation shows one direction, the other is identical by simply swapping the two operations.*

$$[\quad N, N' \mid \Delta \mid \quad P, P'] \ \psi \ \mathsf{prop}$$
$$[\Theta, N, P' \mid \Delta \mid \Gamma, P, N'] \ \varphi \ \mathsf{prop}$$
$$\overline{\hspace{6cm}} \ (\mathsf{hyp})$$
$$[\Theta, P' \mid \Delta, N, P \mid \Gamma, N'] \ \psi \Rightarrow \varphi \leq \psi \Rightarrow \varphi$$
$$=\!=\!=\!=\!=\!=\!= \ ((\psi \Rightarrow \varphi) = \mathcal{P}(\Uparrow^{\pm!}_{\Delta,N,P})(\varepsilon(\psi \Rightarrow \varphi)))$$
$$[\Theta, P' \mid \Delta, N, P \mid \Gamma, N'] \ \mathcal{P}(\Uparrow^{\pm!}_{\Delta,N,P})(\varepsilon(\psi \Rightarrow \varphi)) \leq \psi \Rightarrow \varphi$$
$$=\!=\!=\!=\!=\!=\!= \ (\psi = \mathcal{P}(\Uparrow^{\pm!}_{\Delta})(\varepsilon(\psi)))$$
$$[\Theta, P' \mid \Delta, N, P \mid \Gamma, N'] \ \mathcal{P}(\Uparrow^{\Delta}_{\Delta,N,P} \mathbin{;} (\pi_1 \mid !_{\Delta,N,P} \mid \pi_1))(\varepsilon(\psi \Rightarrow \varphi)) \leq \mathcal{P}(\Uparrow^{\pm!}_{\Delta})(\varepsilon(\psi)) \Rightarrow \mathcal{P}(\Uparrow^{\pm!}_{\Delta})(\varepsilon(\varphi))$$
$$=\!=\!=\!=\!=\!=\!=$$
$$[\Theta, P' \mid \Delta, N, P \mid \Gamma, N'] \ \mathcal{P}(\Uparrow^{\Delta}_{N,P} \mathbin{;} \Uparrow^{\Delta}_{\Delta} \mathbin{;} (\pi_1 \mid !_{\Delta,N,P} \mid \pi_1))(\varepsilon(\psi \Rightarrow \varphi)) \leq \mathcal{P}(\Uparrow^{\pm!}_{\Delta})(\varepsilon(\psi)) \Rightarrow \mathcal{P}(\Uparrow^{\pm!}_{\Delta})(\varepsilon(\varphi))$$
$$=\!=\!=\!=\!=\!=\!=$$
$$[\Theta, P' \mid \Delta, N, P \mid \Gamma, N'] \ \mathcal{P}(\Uparrow^{\Delta}_{N,P})(\mathcal{P}(\Uparrow^{\Delta}_{\Delta} \mathbin{;} (\pi_1 \mid !_{\Delta,N,P} \mid \pi_1))(\varepsilon(\psi \Rightarrow \varphi))) \leq \mathcal{P}(\Uparrow^{\pm!}_{\Delta})(\varepsilon(\psi)) \Rightarrow \mathcal{P}(\Uparrow^{\pm!}_{\Delta})(\varepsilon(\varphi))$$
$$=\!=\!=\!=\!=\!=\!= \ (\Rightarrow)$$
$$[\Theta, P' \mid \Delta, N, P \mid \Gamma, N'] \quad \cdots \quad \leq \mathcal{P}(\Uparrow^{\Delta}_{N',P'})(\mathcal{P}(\Uparrow^{\pm!}_{\Delta})(\varepsilon(\varphi)))$$
$$=\!=\!=\!=\!=\!=\!= \ (\Rightarrow')$$
$$[\Theta, P' \mid \Delta, N, P \mid \Gamma, N'] \ \mathcal{P}(\Uparrow^{\Delta}_{N,P})(\mathcal{P}(\Uparrow^{\Delta}_{\Delta} \mathbin{;} (\pi_1 \mid !_{\Delta,N,P} \mid \pi_1))(\varepsilon(\psi \Rightarrow \varphi))) \leq \mathcal{P}(\Uparrow^{\pm!}_{\Delta})(\varepsilon(\psi)) \Rightarrow' \mathcal{P}(\Uparrow^{\pm!}_{\Delta})(\varepsilon(\varphi))$$
$$=\!=\!=\!=\!=\!=\!=$$
$$[\Theta, P' \mid \Delta, N, P \mid \Gamma, N'] \ \mathcal{P}(\Uparrow^{\pm!}_{\Delta,N,P})(\varepsilon(\psi \Rightarrow \varphi)) \leq \mathcal{P}(\Uparrow^{\pm!}_{\Delta})(\varepsilon(\psi)) \Rightarrow' \mathcal{P}(\Uparrow^{\pm!}_{\Delta})(\varepsilon(\varphi))$$
$$=\!=\!=\!=\!=\!=\!=$$
$$[\Theta, P' \mid \Delta, N, P \mid \Gamma, N'] \ \psi \Rightarrow \varphi \leq \psi \Rightarrow' \varphi$$

Note that we applied $(\Rightarrow)$ *precisely when there was a dinatural lift operation (as in* Theorem 1*)* *on the left and an exponential on the right, fulfilling the conditions prescribed by the categorical semantics for polarized exponentials. We omit the left hand side of the derivation in the case of implication since we immediately reverse it using the* other *implication operator, which has the same formula on the left.*

E Details for the equivalence in Theorem 9

Theorem 11 (Details for the equivalence in Theorem 9)**.** *We describe more in detail the equivalence in Theorem 9. We again concentrate on the case of base predicates to highlight the need for the no-dinatural-variance condition, since the rest of the structure is defined in the only possible way.*
$(\Downarrow ; \Uparrow = \mathsf{id})$. *Suppose we are given a morphism of doctrines (F, α); recall that we construct the component on predicates for the corresponding morphism of theories as*

$$\begin{aligned} b(\sigma) &:= \alpha(\sigma\ \mathsf{type}) \\ p(\sigma) &:= (F(\mathsf{pos}(\sigma)), \top, F(\mathsf{neg}(\sigma)), \alpha(\sigma(\pi_1, \pi_1)\ \mathsf{prop})) \end{aligned}$$

by applying it on the base predicate formula. In the other direction, we recall that in general the constructed doctrine morphism $(\llbracket - \rrbracket, \llbracket - \rrbracket_\varphi)$ sends judgements $\sigma(s \mid t)$ prop to

$$\mathcal{P}(\llbracket s \rrbracket \mid ! \mid \llbracket t \rrbracket)(\varepsilon(p(\sigma \in \Sigma_P))),$$

where $p(\sigma) := p(\sigma)_4 \in \mathcal{P}(p(\sigma)_1 \mid p(\sigma)_2 \mid p(\sigma)_3)$ for $p(\sigma)_1 := [\Theta' \mid \Delta' \mid \Gamma']$. To show that we end up with the original doctrine we need to show that α has the same action on predicates. In particular, it suffices to prove it for the base cases (since the rest of the structure is fixed and pertains to the structure of the doctrine). Hence, we need to show

$$\alpha(\sigma(s \mid t)\ \mathsf{prop}) = \mathcal{P}(\llbracket s \rrbracket \mid ! \mid \llbracket t \rrbracket)(\varepsilon(\alpha(\sigma(\pi_1, \pi_1)\ \mathsf{prop})))$$

However, we use the definition of Definition 10 to cancel out ε and the full dinatural contraction $\mathcal{P}(\Uparrow^{\pm !}_{\Delta}) := \mathcal{P}(\mathsf{id}_{\Theta \times \Delta} \mid !_{\Delta} \mid \mathsf{id}_{\Gamma \times \Delta})$ to obtain

$$\begin{aligned} & \mathcal{P}(\llbracket s \rrbracket \mid ! \mid \llbracket t \rrbracket)(\varepsilon(\alpha(\dots))) \\ = \ & \mathcal{P}(\llbracket s \rrbracket \mathbin{;} \mathsf{id}_{\llbracket \mathsf{neg}(\sigma) \rrbracket} \mid ! \mathbin{;} ! \mid \llbracket t \rrbracket \mathbin{;} \mathsf{id}_{\llbracket \mathsf{pos}(\sigma) \rrbracket})(\varepsilon(\alpha(\dots))) \\ = \ & \mathcal{P}(\llbracket s \rrbracket \mathbin{;} \mathsf{id}_{\Theta' \times \Delta'} \mid ! \mathbin{;} ! \mid \llbracket t \rrbracket \mathbin{;} \mathsf{id}_{\Gamma' \times \Delta'})(\varepsilon(\alpha(\dots))) \\ = \ & \mathcal{P}(\llbracket s \rrbracket \mid ! \mid \llbracket t \rrbracket)(\mathcal{P}(\mathsf{id}_{\Theta' \times \Delta'} \mid ! \mid \mathsf{id}_{\Gamma' \times \Delta'})(\varepsilon(\alpha(\dots)))) \\ = \ & \mathcal{P}(\llbracket s \rrbracket \mid ! \mid \llbracket t \rrbracket)(\alpha(\dots)) \end{aligned}$$

We implicitly used the isomorphisms between $\Theta' \times \Delta' =: \mathsf{neg}(p(\sigma)) := \llbracket \mathsf{neg}(\sigma) \rrbracket$ which corresponds exactly with the codomain of s. The final result

$$\alpha(\sigma(s \mid t)\ \mathsf{prop}) = \mathcal{P}(\llbracket s \rrbracket \mid ! \mid \llbracket t \rrbracket)(\alpha(\sigma(\pi_1, \pi_1)\ \mathsf{prop}))$$

follows by the naturality of α which expresses well-behavedness with respect to context reindexing.
$(\Uparrow ; \Downarrow = \mathsf{id})$. *Suppose we are given a theory morphism M; we construct $\llbracket - \rrbracket$ and the syntactic doctrine by induction, defining $\llbracket - \rrbracket_\varphi$ on the base cases $\sigma(s \mid t)$ prop to*

$$\mathcal{P}(\llbracket s \rrbracket \mid ! \mid \llbracket t \rrbracket)(\varepsilon(p(\sigma \in \Sigma_P)))$$

where $p(\sigma) \in \mathcal{P}(p(\sigma)_1 \mid p(\sigma)_2 \mid p(\sigma)_3)$ and $\varepsilon(p(\sigma)) \in \mathcal{P}(\llbracket \mathsf{pos}(\sigma) \rrbracket, \top, \llbracket \mathsf{neg}(\sigma) \rrbracket) \cong \mathcal{P}(p(\sigma)_1 \times p(\sigma)_2, \top, p(\sigma_3) \times p(\sigma)_2)$. We now give a theory morphism from the doctrine just constructed, which in general recall that it is given by

$$p'(\sigma) := (F(\mathsf{pos}(\sigma)), \top, F(\mathsf{neg}(\sigma)), \alpha(\sigma(\pi_1, \pi_1)\ \mathsf{prop}))$$

In our case α *is* $[\![-]\!]_\varphi$, *hence*

$$\begin{aligned}p'(\sigma) := ([\![\mathsf{pos}(\sigma)]\!], \top, [\![\mathsf{neg}(\sigma)]\!], \\ \mathcal{P}([\![\pi_1]\!] \mid ! \mid [\![\pi_1]\!])(\varepsilon(p(\sigma \in \Sigma_P)))\end{aligned}$$

where $[\![\pi_1]\!] : [\![\mathsf{neg}(\sigma)]\!] \times \top \to [\![\mathsf{neg}(\sigma)]\!]$ *is essentially the identity. As in the previous proof, this reindexing has* $\top$ *as middle morphism, hence decomposes via* $\mathcal{P}(\Uparrow^{\pm !}_{\Delta})$ *and cancels out with* ε, *thus obtaining that* $p'(\sigma) = p(\sigma)$.