Directed equality with dinaturality

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Type theory and symmetric equality

- The most interesting aspect of Martin-Löf type theory: equality types.

$$\frac{C : \mathsf{Type} \quad a, b : C}{a =_C b : \mathsf{Prop}} \qquad \frac{x : C}{\mathsf{refl}_x : x =_C x} \qquad \frac{P : \prod_{a,b\,:C}(a =_C b \to \mathsf{Prop}) \quad d : \prod_{z\,:C} P(z, z, \mathsf{refl}_z)}{J(d) : \prod_{a,b\,:C} \prod_{e\,:a =_C b} P(a, b, e)}$$

- Idea behind the ("diagonal") J rule:

To prove a proposition $P(a, b, e)$ with a proof $e : a =_C b$,
it is enough to prove it for the case $a = b = z$ and $e = \mathsf{refl}_z$.

- Equality in MLTT is inherently symmetric: take $P(a, b, e) := (b =_C a)$,

$$\frac{\mathsf{refl} : \prod_{z:C} z =_C z}{\mathsf{sym} := J(\mathsf{refl}) : \prod_{a,b:C} a =_C b \to b =_C a}$$

- Question: *can you prove that every equality $p : a = a$ is the same as* refl*?*

$\mathsf{uip} : \prod_{a:C} \prod_{p\,:a =_C a} p = \mathsf{refl}_a$ No! There are countermodels.

Models of MLTT

What can types be?

Martin-Löf type theory admits a sound interpretation in:

- Types as sets. [Martin-Löf 1971]
- Types as groupoids. [Hoffmann-Streicher 1999]
- Types as ∞-groupoids. [Voevodsky 2013], [van der Berg-Garner 2010]
- Types as **sSet**s, **Top**, cubical sets. [Awodey-Warren 2007], [CCHM 2007]
- Types as reflexive graphs for parametricity. [Atkey-Ghani-Johann 2007]

Intuition: types are groupoids, equalities are always-invertible morphisms.

General frameworks, from closest to syntax:

- Categories with families and natural models, [Dybjer 1996], [Awodey 2014]
- Locally cartesian closed categories, [Seely 1984]
- Homotopy theoretic models in model categories. [Awodey-Warren 2007]

Some familiar structures are missing... *what about categories and posets?!*

Motivation 1: Directed type theory

Martin-Löf type theory with refl$/J$ *is intrinsically about symmetric equality.*
***Directed type theory** is the generalization to "directed equality".*

The interpretation of directed type theory with *(1-)categories*:

Types ⇝ Categories
Terms ⇝ Functors
Relations ⇝ Profunctors
Points of a type ⇝ Objects of a category
Equalities $e : a = b$ ⇝ Morphisms $e : \hom(a, b)$
Equality types $=_A$: $A \times A \to \mathsf{Prop}$ ⇝ Hom types $\hom_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$

→ Now types have a *polarity*, $\mathbb{C}$ and $\mathbb{C}^{\mathsf{op}}$, i.e., the opposite category.
→ Now equalities $e : \hom(a, b)$ have *directionality:* rewrites, trans., processes.

We want to find which syntactic restriction of MLTT
allow for types can be interpreted as categories.

Directed type theory and dinaturality

- Semantically, refl should be $\mathsf{id}_c \in \hom_{\mathbb{C}}(c,c)$ for $c : \mathbb{C}$.
- Transitivity of directed equality ⇝ composition of morphisms in $\mathbb{C}$.
- However, directed type theory is not so straightforward:

$$\frac{a : \mathbb{C}}{\mathsf{refl}_a ...? : \hom_{\mathbb{C}}(a,a)} \quad \rightsquigarrow \quad \frac{a : \mathbb{C}^{\mathsf{core}}}{\mathsf{refl}_a : \hom(\mathsf{i}^{\mathsf{op}}(a), \mathsf{i}(a))}$$ [North 2018]

- *Problem:* rule is not functorial w.r.t. variance of $\hom_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$, since $a : \mathbb{C}$ appears both contravariantly and covariantly.
- Current approach to DTT in **Cat**: [North 2018], [Neumann 2024]
  → Use the maximal subgroupoid $\mathbb{C}^{\mathsf{core}}$ to collapse the two variances, since $(\mathbb{C}^{\mathsf{core}})^{\mathsf{op}} \cong \mathbb{C}^{\mathsf{core}}$, then use $\mathsf{i} : \mathbb{C}^{\mathsf{core}} \to \mathbb{C}$ and $\mathsf{i}^{\mathsf{op}} : \mathbb{C}^{\mathsf{core}} \to \mathbb{C}^{\mathsf{op}}$.
- Then a J-like rule is validated, but *again using groupoidal structure*.

Contribution 1 – Directed type theory

Is there a way to interpret directed type theory in 1-categories without having to collapse categories back to the "undirected" groupoidal case?

Our approach: yes, by validating rules with dinaturality.

- *Intuition:* dinatural transformations allow for the same variable to appear both covariantly and contravariantly → solving the issue with refl.
- We will see how dinaturality validates both
  → a refl rule (directed equality introduction),
  → a J rule (directed equality elimination).
- The directed J rule is extremely similar to the standard MLTT J rule, but with a syntactic restriction which does **not** allow for symmetry.

Contribution 1 – Directed type theory

Comparison between symmetric equality and our directed equality rule:

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \qquad \Gamma(z, z, x) \vdash P(z, z, x)}{[a : \mathbb{A}, b : \mathbb{A}, x : \mathbb{C}]\ a = b,\ \Gamma(a, b, x) \vdash P(a, b, x)} \ \text{(eq)}$$

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \qquad \Gamma(\overline{z}, z, \overline{x}, x) \vdash P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ \hom(a, b),\ \Gamma(\overline{b}, \overline{a}, \overline{x}, x) \vdash P(a, b, \overline{x}, x)} \ (\hom)$$

- *Restriction:* a, b must appear *positively* in P and *negatively* in Γ.
- (Again, dinaturality allows z to appear twice in $P(z, z, x)$.)
- Using directed J, we obtain the same terms provable in MLTT about symmetric equality (e.g., transitivity, transport, congruence).
- Proving properties about them follows precisely the steps in MLTT. (e.g., the computation rules are the same as the classical case.)

Contribution 2 – (Co)end calculus

Another important open question:

What should the quantifiers of directed type theory be?

- Dinaturality again gives us a hint: *ends and coends*.
- We present a series of "logical rules", inspired by the doctrinal approach categorical logic, viewing (co)ends as the *directed quantifiers* of DTT.
- We then use the rules to give *simple logical proofs* of theorems in category theory, using dinaturality and viewing $\hom$ as directed equality.

Why is this useful?

Our goals for directed type theory, in increasing order of importance:

1. A synthetic theory to work with (∞-)categories. [Riehl-Shulman 2017]

   Martin-Löf type theory $\rightsquigarrow$ a synthetic theory of (∞-)groupoids.
   Directed type theory $\rightsquigarrow$ a synthetic theory of (∞-)categories.

2. Directed higher inductive types: ([Altenkirch-Kaposi 2016], but directed)
   - *Idea*: you can *define* new types (categories) with specified equalities.
   - Internalize category theory in type theory: reason about categories via types.
   - Internalize the type of λ-calculus terms, with directed equalities

$$\beta : (\lambda x.M)N \rightsquigarrow M[N/x]$$

   Like quotients, this greatly simplifies metatheory (e.g. congruence for free)
   - In general: internalize semantics of type theory in TT (e.g., as with QIIT)

3. A doctrinal perspective on *directed logic* for profunctors and posets.
4. **Claim**: we will show how *(co)end calculus is the first-order instantiation of a directed type theory with quantifiers, with semantics in 1-categories*.

Motivation 2: Profunctors

Another story, now in CT:

Rel	$\rightsquigarrow$	**Prof**
Sets	$\rightsquigarrow$	Categories
Relations $A \times B \to \{\top, \bot\}$	$\rightsquigarrow$	Profunctors $\mathbb{C}^{\mathsf{op}} \times \mathbb{D} \to$ **Set**
Existential quantifiers $\exists x$	$\rightsquigarrow$	Coends $\int^x$
Composition: $\exists b \in B.\ R(a,b) \wedge Q(b,c)$	$\rightsquigarrow$	$\int^{b:\mathbb{B}} R(a,b) \times Q(b,c)$
Conjunction of truth values	$\rightsquigarrow$	Cartesian products of **Set**s
Identity relation: $=_{\mathbb{C}}$	$\rightsquigarrow$	Identity profunctor: $\hom_{\mathbb{C}}$

Why do unitality and associativity hold?

Because of rules for equality in FOL. $\rightsquigarrow$ Because of the coYoneda lemma.

Because of rules for $\exists$ and $\wedge$ in FOL. $\rightsquigarrow$ Because of properties of coends.

What's missing from the profunctorial story?

- We use FOL in **Rel**: there are no rules for a *directed logic* in **Prof**.
- A proper treatment of coends as quantifiers *qua quantifiers*.
- A syntactic treatment of directed logic (with directed equality and its rules, and quantifiers) in terms of the doctrinal approach. [Jacobs 1999]

Background: Categorical logic, in one slide

Categorical models of *first order logic*, by [Lawvere 1969]:

- Take **Ctx** the following syntactic category:
    - Objects: contexts $\mathit{\Gamma}, \Delta$, i.e., sequences of types.
    - Morphisms $\mathit{\Gamma} \to \Delta$: a term $\mathit{\Gamma} \vdash t : \Delta$ of type Δ given in context Γ,
    - Identity on $\mathit{\Gamma} := [x_1 : A, x_2 : B, ..., x_n : C]$: n-tuple of vars $\langle x_1, x_2, ..., x_n \rangle$.
    - Composition: substitution of terms.
- For any $\mathit{\Gamma}$: **Ctx**, there is a poset $\mathcal{P}(\mathit{\Gamma})$:
    - Objects: lists of formulas Φ in context $\mathit{\Gamma}$.
    - Morphisms $\Phi \to \Psi$: entailments $\mathit{\Gamma} \mid \Phi \vdash \Psi$ of formulas.
    - Composition and identities: cut rule and axiom rule.
- This association is functorial: there is a functor $\mathcal{P}$: $\mathbf{Ctx}^{\mathsf{op}} \to \mathbf{Pos}$, the action on morphisms is given by substitution of terms into formulas.
- *Idea:* Logical connectives, quantifiers, *equality types*, subset types, quotients, can be characterized solely as categorical properties of $\mathcal{P}$.
- A *doctrine* is a finite product category $\mathbb{C}$ and a functor $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$.
- Morphisms of doctrines → interpretations/models in different categories.

Background: Doctrines with $\wedge, \Rightarrow, \forall, \exists, =$

- Take $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ with $\mathbb{C}$ all binary products, and assume:

1. $[\wedge, \Rightarrow]$: Each $\mathcal{P}(\mathit{\Gamma})$ has products and exponentials for every $\mathit{\Gamma}$,
2. $[\forall, \exists]$: Given the projections $\pi_{A[\mathit{\Gamma}]} : A \times \mathit{\Gamma} \to \mathit{\Gamma}$, each reindexing $\mathcal{P}(\pi_{A[\mathit{\Gamma}]}) : \mathcal{P}(\mathit{\Gamma}) \to \mathcal{P}(A \times \mathit{\Gamma})$ has right and left adjoints

   $$\exists_{A[\mathit{\Gamma}]}, \forall_{A[\mathit{\Gamma}]} : \mathcal{P}(A \times \mathit{\Gamma}) \to \mathcal{P}(\mathit{\Gamma}), \qquad \exists_{A[\mathit{\Gamma}]} \dashv \mathcal{P}(\pi_{A[\mathit{\Gamma}]}) \dashv \forall_{A[\mathit{\Gamma}]}.$$

   such that for $f : \mathit{\Gamma} \to \Delta$ the *Beck-Chevalley condition* holds:

   $$\exists_{A[\Delta]} \mathbin{;} \mathcal{P}(f) \cong \mathcal{P}(\mathsf{id}_X \times f) \mathbin{;} \exists_{A[\mathit{\Gamma}]}, : \mathcal{P}(A \times \Delta) \to \mathcal{P}(\mathit{\Gamma})$$
   $$\forall_{A[\Delta]} \mathbin{;} \mathcal{P}(f) \cong \mathcal{P}(\mathsf{id}_X \times f) \mathbin{;} \forall_{A[\mathit{\Gamma}]}.$$

3. $[=]$: Left adjoints to reindexing along diagonals $\Delta_X : A \to A \times A$:

   $$\mathsf{Eq}_A : \mathcal{P}(A) \to \mathcal{P}(A \times A), \qquad \mathsf{Eq}_A \dashv \mathcal{P}(\Delta_A).$$

   Certain *Frobenius conditions* are automatic thanks to exponentials.

- The above capture the fact that $\mathcal{P}$ is a *doctrine* with $\wedge, \Rightarrow, \forall, \exists, =$.

Motivation 3: Lawvere equality for categories

*Can we interpret logic in **Cat**, viewed as a category of contexts?*

From type theory: why **Cat** is *not* a model for Martin-Löf type theory:

- **Cat** is *not* locally cartesian closed,
- **Cat** is *not* even regular (but it is 2-regular [Street CBS 1982])

From the doctrinal perspective: **Set** plays the role of *generalized truth values* in **Prof**, and presheaves are "generalized logical predicates", but:

- Equality in the presheaf hyperdoctrine $\mathcal{P}\mathsf{sh}(\mathbb{C}) := [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}] : \mathbf{Cat} \to \mathbf{CAT}$ does **not** validate Frobenius and Beck-Chevalley: [Melliès-Zeilberger 2016]

$$\mathsf{Eq}_{\mathbb{C}} := \exists_{\Delta_{\mathbb{C}}}(\top_{\mathbb{C}}) \quad \in \quad \mathcal{P}\mathsf{sh}(\mathbb{C} \times \mathbb{C})$$

$$\mathsf{Eq}_{\mathbb{C}} = (a, b) \mapsto \int^{x \in \mathbb{C}} \hom_{\mathbb{C}}(a, x) \times \hom_{\mathbb{C}}(b, x)$$

i.e., any two objects forming a cospan for some x should be equated.
Note: These conditions also fail in groupoids.

Motivation 3: Lawvere on $\hom$ as equality

Lawvere commenting on the failure of Frobenius/Beck-Chevalley for $\mathcal{P}$sh:

- […] *This should not be taken as indicative of a lack of vitality of $\mathcal{P}$sh as hyperdoctrine, or even of a lack of a satisfactory theory of equality for it.*
- *Rather, it indicates that we have probably been too naive in defining equality in a manner too closely suggested by the classical conception.*
- *Equality should be the "graph" of the identity term. But present categorical conceptions indicate that the graph of a functor $F : \mathbb{C} \to \mathbb{D}$ should be* […] *a binary attribute of* mixed *variance $\mathcal{P}sh(\mathbb{C}^{\mathsf{op}} \times \mathbb{D})$.*
- *Thus in particular "equality" should be the functor $\hom_{\mathbb{C}}$* […].
- *The term which would take the place of $\Delta_{\mathbb{C}}$ in such a more enlightened theory of equality would be the forgetful functor* $\mathsf{Tw}(\mathbb{C}) \to \mathbb{C}^{\mathsf{op}} \times \mathbb{C}$. […]
- *Of course to abstract from this example would require at least the addition of a functor $T \xrightarrow{\mathsf{op}} T$ to the structure of a [doctrine].*

[Lawvere 1970, Equality in Hyperdoctrines]

Background on dinaturality and (co)ends

Background: Dinatural transformations

- Let $-^{\diamond} : \textbf{Cat} \to \textbf{Cat}$ be the comonad defined by $\mathbb{C} \mapsto \mathbb{C}^{\mathsf{op}} \times \mathbb{C}$.
- A **difunctor** from $\mathbb{C}$ to $\mathbb{D}$ is a functor $F, G : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$.
- A **dipresheaf** on $\mathbb{C}$ is an endoprofunctor $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \textbf{Set}$.
- Given difunctors $F, G : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$, a **dinatural transformation** $\alpha : F \overset{\bullet\bullet}{\longrightarrow} G$ is a family of morphisms of $\mathbb{D}$ for each $x : \mathbb{C}$,

$$\alpha_x : F(x, x) \longrightarrow G(x, x)$$

such that for every $a, b : \mathbb{C}$ and $f : a \to b$ this hexagon commutes:

$$\begin{tikzcd}
 & F(b,b) \arrow[r, "\alpha_b"] & G(b,b) \arrow[dr, "G(f{,}\mathsf{id})"] & \\
F(b,a) \arrow[ur, "F(\mathsf{id}{,}f)"] \arrow[dr, "F(f{,}\mathsf{id})"'] & & & G(a,b) \\
 & F(a,a) \arrow[r, "\alpha_a"] & G(a,a) \arrow[ur, "G(\mathsf{id}{,}f)"'] &
\end{tikzcd}$$

Theorem (Naturals are dinaturals [Dubuc and Street 1969])

A dinatural between functors which do not depend on $\mathbb{C}^{\mathsf{op}}$ is just a natural.

Background: Dinaturals don't always compose

- Dinaturals don't always compose: $\alpha : F \xrightarrow{\bullet\bullet} G$, $\beta : G \xrightarrow{\bullet\bullet} H$,

$$\begin{tikzcd}
 & F(b,b) \arrow[rr, "\alpha_b"] & & G(b,b) \arrow[rr, "\beta_b"] \arrow[dr, "G(f,\mathsf{id})"] & & H(b,b) \arrow[dr, "H(f,\mathsf{id})"] & \\
F(b,a) \arrow[ur, "F(\mathsf{id},f)"] \arrow[dr, "F(f,\mathsf{id})"'] & & G(b,a) \arrow[ur, "G(\mathsf{id},f)"] \arrow[dr, "G(f,\mathsf{id})"'] & & G(a,b) & & H(a,b) \\
 & F(a,a) \arrow[rr, "\alpha_a"'] & & G(a,a) \arrow[rr, "\beta_a"'] \arrow[ur, "G(\mathsf{id},f)"'] & & H(a,a) \arrow[ur, "H(\mathsf{id},f)"'] &
\end{tikzcd}$$

- They always compose with naturals: $\alpha : F \xrightarrow{\bullet\bullet} G$, $\beta : G \longrightarrow H$,

$$\begin{tikzcd}
 & F(b,b) \arrow[r, "\alpha_b"] & G(b,b) \arrow[rr, "\beta_{bb}"] \arrow[dr, "G(f,\mathsf{id})"] & & H(b,b) \arrow[dr, "H(f,\mathsf{id})"] & \\
F(b,a) \arrow[ur, "F(\mathsf{id},f)"] \arrow[dr, "F(f,\mathsf{id})"'] & & & G(a,b) \arrow[rr, "\beta_{ab}"] & & H(a,b) \\
 & F(a,a) \arrow[r, "\alpha_a"'] & G(a,a) \arrow[rr, "\beta_{aa}"'] \arrow[ur, "G(\mathsf{id},f)"'] & & H(a,a) \arrow[ur, "H(\mathsf{id},f)"'] &
\end{tikzcd}$$

Background: (Co)wedges and (co)ends

A *cone* is a natural from a constant functor into P.
A *wedge* is a dinatural from a constant difunctor into P.

The terminal object in Cone(P) is the *limit* of P.
The terminal object in Wedge(P) is the *end* of P.

- Fix $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$.
- A **wedge** for P is a pair (object $A : \mathbb{C}$, dinatural $\alpha : \mathsf{const}_A \to P$).
- A **wedge morphism** $(A, \alpha) \to (B, \beta)$ is an arrow $w : A \to B$, s.t. $\forall x : \mathbb{C}$,

$$\begin{tikzcd}
A \arrow[rr, "w"] \arrow[dr, "\alpha_x"'] & & B \arrow[dl, "\beta_x"] \\
& P(x,x) &
\end{tikzcd}$$

- There are categories Wedge(P) and Cowedge(P).

Background: (Co)ends

The **end** of P is the terminal object in Wedge(P).
The **coend** of P is the initial object in Cowedge(P).

- Given $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$, the terminal (co)wedge object is denoted as

$$\text{End: } \int_{x:\mathbb{C}} P(\overline{x}, x) \qquad \text{Coend: } \int^{x:\mathbb{C}} P(\overline{x}, x)$$

- **Theorem:** (co)ends exist when $\mathbb{D}$ is (co)complete, they are (co)limits.
- We will always take $\mathbb{D} := \mathbf{Set}$, since we consider profunctors as relations.
- (Co)ends act as binders: given $P : (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \times (\mathbb{D}^{\mathsf{op}} \times \mathbb{D}) \to \mathbf{Set}$,

$$\left((\overline{y}, y) \mapsto \int_{x:\mathbb{C}} P(\overline{x}, x, \overline{y}, y) \right) : \mathbb{D}^{\mathsf{op}} \times \mathbb{D} \to \mathbf{Set}.$$

- Taking parametric (co)ends is functorial in P:

$$\int_{\mathbb{C}[\mathbb{D}]}, \int^{\mathbb{C}[\mathbb{D}]} : [(\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \times (\mathbb{D}^{\mathsf{op}} \times \mathbb{D}), \mathbf{Set}] \to [\mathbb{D}^{\mathsf{op}} \times \mathbb{D}, \mathbf{Set}].$$

(Co)end calculus in one slide

- (Co)ends give elegant characterizations of theorems of category theory.
- Natural transformations as ends:

$$\mathsf{Nat}(F, G) \cong \int_{x:\mathbb{C}} \hom_{\mathbb{D}}(F(\overline{x}), G(x))$$

- Quantifier exchange $(Q : (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \times (\mathbb{D}^{\mathsf{op}} \times \mathbb{D}) \to \mathbf{Set})$:

$$\int_{x:\mathbb{C}} \int_{y:\mathbb{D}} Q(\overline{x}, x, \overline{y}, y) \cong \int_{y:\mathbb{D}} \int_{x:\mathbb{C}} Q(\overline{x}, x, \overline{y}, y) \cong \int_{(x,y):\mathbb{C}\times\mathbb{D}} Q(\overline{x}, x, \overline{y}, y)$$

- (Co)Yoneda lemma: $(P : \mathbb{C} \to \mathbf{Set})$

$$P(a) \cong \int_{x:\mathbb{C}} \hom_{\mathbb{C}}(a, \overline{x}) \Rightarrow P(x)$$

$$P(a) \cong \int^{x:\mathbb{C}} \hom_{\mathbb{C}}(\overline{x}, a) \times P(x)$$

- Kan extensions: $(F : \mathbb{C} \to \mathbb{D})$

$$(\mathsf{Ran}_F P)(a) \cong \int_{x:\mathbb{C}} \hom_{\mathbb{D}}(a, F(\overline{x})) \Rightarrow P(x)$$

$$(\mathsf{Lan}_F P)(a) \cong \int^{x:\mathbb{C}} \hom_{\mathbb{D}}(F(\overline{x}), a) \times P(x)$$

Logical interpretation of (co)end calculus

- (Co)Yoneda lemma:

$$\begin{array}{ccccc} P(a) & \cong & \displaystyle\int_{x:\mathbb{C}} & \hom_{\mathbb{C}}(a,\overline{x}) & \Rightarrow & P(x) \\ P(a) & \Leftrightarrow & \forall(x:\mathbb{C}). & a =_{\mathbb{C}} x & \Rightarrow & P(x) \end{array}$$

$$\begin{array}{cccccc} P(a) & \cong & \displaystyle\int^{x:\mathbb{C}} & \hom_{\mathbb{C}}(\overline{x},a) & \times & P(x) \\ P(a) & \Leftrightarrow & \exists(x:\mathbb{C}). & x =_{\mathbb{C}} a & \wedge & P(x) \end{array}$$

- (Pointwise) right/left Kan extensions using ends/coends:

$$\begin{array}{cccccc} (\mathsf{Ran}_F P)(a) & \cong & \displaystyle\int_{x:\mathbb{C}} & \hom_{\mathbb{D}}(a,F(x)) & \Rightarrow & P(x) \\ (\forall_f(P))(y) & \Leftrightarrow & \forall(x:C). & y =_D f(x) & \Rightarrow & P(x) \end{array}$$

$$\begin{array}{cccccc} (\mathsf{Lan}_F P)(a) & \cong & \displaystyle\int^{x:\mathbb{C}} & \hom_{\mathbb{D}}(F(x),a) & \times & P(x) \\ (\exists_f(P))(y) & \Leftrightarrow & \exists(x:C). & f(x) =_D y & \wedge & P(x) \end{array}$$

Motivation 4: Computing adjoints to reindexings

- Consider a doctrine $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ with $\wedge, \Rightarrow, \forall_{A[\Gamma]}, \exists_{A[\Gamma]}, \mathsf{Eq}_A$.
- With just these assumptions, we can compute the adjoints $\forall_f, \exists_f : \mathcal{P}(C) \to \mathcal{P}(D)$ along any $f : C \to D$, not just projections:

$$\forall_f(P) := \forall_{C[D]}(\mathcal{P}(\mathsf{id}_Y \times f)(\mathsf{Eq}_Y(\top_Y)) \Rightarrow \mathcal{P}(\pi_{C[D]})(P))$$
$$\exists_f(P) := \exists_{C[D]}(\mathcal{P}(\mathsf{id}_Y \times f)(\mathsf{Eq}_Y(\top_Y)) \wedge \mathcal{P}(\pi_{C[D]})(P))$$

- In the syntactic model, for $P : \mathcal{P}(C)$, $f : C \to D$:

$$(\forall_f(P))(y) := \forall x.(y =_D f(x) \Rightarrow P(x))$$
$$(\exists_f(P))(y) := \exists x.(f(x) =_D y \wedge P(x))$$

- Compare these to the formula to compute Kan extensions via (co)ends:

$$(\mathsf{Ran}_F P)(y) \cong \int_{x:\mathbb{C}} \hom_{\mathbb{D}}(y, F(\overline{x})) \Rightarrow P(x)$$
$$(\mathsf{Lan}_F P)(y) \cong \int^{x:\mathbb{C}} \hom_{\mathbb{D}}(F(\overline{x}), y) \times P(x)$$

- ▶ there is yet no decomposition of these properties for doctrines, e.g., coends as quantifiers/adjoints, or "having directed equality".

Motivation 4: Computing Kan extensions with (co)ends

- A logical proof that $\forall_f(P)$ is right adjoint to precomposition with f:

$$\begin{array}{ll} [y : D] & \Gamma(y) \vdash (\forall_f P)(y) \\ & \qquad := \forall x.(y = f(x) \Rightarrow P(x)) \\ \hline\hline [x : C, y : D] & \Gamma(y) \vdash y = f(x) \Rightarrow P(x) \\ \hline\hline \multicolumn{2}{l}{[x : C, y : D]\ y = f(x), \Gamma(y) \vdash P(x)} \\ \hline\hline [x : C] & \Gamma(f(x)) \vdash P(x) \end{array}$$

- There is yet no formal system to do this for the directed case;
- We follow exactly this proof for Lan/Ran, using our rules for dinaturality.

Setting

We present the semantics for a *first-order* non-dependent directed type theory using dinaturality, where types are interpreted by categories, directed equality by $\hom$-functors, quantifiers by (co)ends.

We consider the following interpretation:

Types $\rightsquigarrow$ Categories (possibly with $-^{\mathsf{op}}$)
Contexts $\rightsquigarrow$ Lists of categories
Terms $\rightsquigarrow$ Functors $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$
Propositions $\rightsquigarrow$ Endoprofunctors $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$
Entailments $\rightsquigarrow$ Dinatural transformations (not required to compose)
Directed equality $\rightsquigarrow$ $\hom$-functors $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to$ **Set**.

- **Warning:** we do not give a formal presentation of this, e.g., using doctrines, precisely because dinaturals do not compose in general.
- Despite this, we can still validate and apply rules for directed equality, and use them in practice to prove theorems about category theory.

Directed type theory: notation

- We use type theoretical notation to emphasize dinaturals-as-entailments:

$$[x : \mathbb{C}, y : \mathbb{D}]\ F(\overline{x}, \overline{y}, x, y) \vdash \alpha : G(\overline{x}, \overline{y}, x, y)$$

  says that α is a dinatural for $F, G : (\mathbb{C} \times \mathbb{D})^{\mathsf{op}} \times (\mathbb{C} \times \mathbb{D}) \to \mathbb{E}$.

- The term context $[x : \mathbb{C}, y : \mathbb{D}]$ are the indices of the dinatural.
- We give names to assumptions p, q:

$$[x : \mathbb{C}]\ p : P(\overline{x}, x), q : Q(\overline{x}, x) \vdash h[p, q] : R(\overline{x}, x)$$

  prop. context → interpreted as the pointwise product of functors in **Set**.

- The following dinaturals are the same:

$$\begin{aligned} [x : \mathbb{C}]\ F(\overline{x}) &\vdash \alpha\ \ : G(\overline{x}, x) \\ [x : \mathbb{C}^{\mathsf{op}}]\ F(x) &\vdash \alpha' : G(x, \overline{x}) \end{aligned}$$

- We use $F^{\mathsf{op}} : \mathbb{C}^{\mathsf{op}} \to \mathbb{D}^{\mathsf{op}}$ for $F : \mathbb{C} \to \mathbb{D}$, **no** swapping for difunctors.

Directed type theory: propositional rules – Products

- Dinaturals support propositional conjunction, using products in **Set**.

Theorem (Product of dipresheaves) (👋)

There is an isomorphism of sets natural in $\Gamma, P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \textbf{\textit{Set}}$*:*

$$\frac{[x : \mathbb{C}]\ \Gamma(\overline{x}, x) \vdash P(\overline{x}, x) \times Q(\overline{x}, x)}{[x : \mathbb{C}]\ \Gamma(\overline{x}, x) \vdash P(\overline{x}, x), \qquad [x : \mathbb{C}]\ \Gamma(\overline{x}, x) \vdash Q(\overline{x}, x).}$$

Bottom side: product of sets of dinaturals.
Similarly, $\top_{\mathbb{C}} : \mathbb{C}^{\diamond} \to \textbf{\textit{Set}} := (c, c') \mapsto \{*\}$ *has a unique dinat* $\Gamma \xrightarrow{\bullet\bullet} \top_{\mathbb{C}}$*.*

Directed type theory: propositional rules – Exponentials

- Contrary to naturals, exponentials of dipresheaves are *pointwise*:

Theorem (Exponential of dipresheaves) (👋)

*There is an isomorphism of sets natural in $F, G, H : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to$ **Set**:*

$$\begin{array}{c} [x : \mathbb{C}]\ F(\overline{x}, x) \times G(\overline{x}, x) \vdash H(\overline{x}, x) \\ \hline\hline [x : \mathbb{C}] \qquad\qquad G(\overline{x}, x) \vdash F^{\mathsf{op}}(x, \overline{x}) \Rightarrow H(\overline{x}, x) \end{array} \ (\mathrm{exp})$$

Proof. *Obvious by currying the families of morphisms.*

- *Why are exponentials of presheaves with naturals not pointwise?*

$$\begin{array}{c} [x : \mathbb{C}]\ F(x) \times G(x) \vdash H(x) \\ \hline\hline [x : \mathbb{C}] \qquad\qquad G(x) \vdash F^{\mathsf{op}}(\overline{x}) \Rightarrow H(x) \end{array} \ (\mathrm{exp})$$

  but, the bottom family would be *dinatural* in x.

- ▶ We will show how to use our rules to justify the usual exponential.

Directed type theory: term rules – Reindexing

- Entailments/dinaturals can be reindexed by terms/difunctors:

Theorem (Reindexing with difunctors) (🐔)

Take a difunctor $F : \mathbb{C}^\diamond \to \mathbb{D}$ *and a dinat* $\alpha : P \overset{\bullet\bullet}{\longrightarrow} Q$ *for* $P, Q : \mathbb{D}^\diamond \to \mathbb{E}$*.*

$$\frac{[x : \mathbb{D}] \qquad\qquad P(\overline{x}, x) \vdash \alpha : Q(\overline{x}, x)}{[x : \mathbb{C}]\ \ P(F^{\mathsf{op}}(x, \overline{x}), F(\overline{x}, x)) \vdash F^*(\alpha) : Q(F^{\mathsf{op}}(x, \overline{x}), F(\overline{x}, x))}\ \text{(reindex)}$$

defined by $F^*(\alpha)_x := \alpha_{F(x,x)}$.

- As a special case, naturality in two variables can be collapsed into one:

Theorem (Naturality in two → dinaturality) (🐔)

Take $P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$*. For a natural* $\alpha : P \longrightarrow Q$*, there is a* $P \overset{\bullet\bullet}{\longrightarrow} Q$*:*

$$\frac{[x : \mathbb{C}^{\mathsf{op}}, y : \mathbb{C}]\ \ P(x, y) \vdash \qquad \alpha : Q(x, y)}{[z : \mathbb{C}]\ \ P(\overline{z}, z) \vdash \Delta(\alpha) : Q(\overline{z}, z)}\ (\Delta)\ \textit{given by}\ \Delta(\alpha)_x := \alpha_{xx}.$$

Directed type theory with dinaturality – refl

- Dinaturality allows us to solve the variance issue of refl: [North 2016]

Theorem (Directed equality introduction) (♉)

There is a dinatural transformation $\mathsf{refl}_{\mathbb{C}} : \top \xrightarrow{\bullet\bullet} \hom$,

$$\dfrac{}{[x : \mathbb{C}]\ \top \vdash \mathsf{refl}_{\mathbb{C}} : \hom_{\mathbb{C}}(\overline{x}, x)}\ (\mathsf{refl})$$

where $\top$ *denotes the dipresheaf constant in* $\top_{\textbf{\textit{Set}}}$.

Proof. $\alpha_x(*) := \mathsf{id}_x$. *Dinaturality: for any* $f : a \to b$, $f \mathbin{;} \mathsf{id}_b = \mathsf{id}_a \mathbin{;} f$.

- This is reflexivity of directed equality, via identities.

Directed type theory with dinaturality – Intermezzo

- Before introducing the directed J rule, we show its fundamental idea: the connection between naturality and dinaturality.

Theorem (Characterization of dinaturals via naturality) (👋)

*There is an isomorphism, natural in $P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to$ **Set**, between the set of dinaturals $P \xrightarrow{\bullet\bullet} Q$ and certain natural transformations:*

$$\begin{array}{c} [x : \mathbb{C}]\ P(\overline{x}, x) \xrightarrow{\bullet\bullet} Q(\overline{x}, x) \\ \hline\hline [a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}]\, \hom(a, b) \longrightarrow P^{\mathsf{op}}(b, a) \Rightarrow Q(a, b) \end{array}$$

Proof.

($\Downarrow$) *Given $\alpha : P \xrightarrow{\bullet\bullet} Q$ and $f : \hom(a, b)$, the map $P(b, a) \to Q(a, b)$ is exactly the side of the hexagon in the definition of dinaturality. This is obtained via the functorial action of P, Q.*

($\Uparrow$) *Take $a = b$ and precompose with $\mathsf{id}_a \in \hom(a, a)$.*

The isomorphism follows from (di)naturality of both maps.

Directed type theory with dinaturality – Directed J

- Directed equality elimination is just this result, uncurried.

Theorem (Directed equality elimination) (👋)

Take $\Gamma, P : (\mathbb{A}^{\mathsf{op}}) \times (\mathbb{A}) \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to$ ***Set****.*
Given a dinatural $h : \Gamma \xrightarrow{\bullet\bullet} P$*, there is a dinatural* $J(h)$ *as follows:*

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \qquad\qquad k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash h[k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ e : \hom(a, b), k : \Gamma(\overline{b}, \overline{a}, \overline{x}, x) \vdash J(h)[e, k] : P(a, b, \overline{x}, x)}\ (J)$$

The dinatural $J(h)$ *satisfies the following "computation rule",*

$$J(h)_{zzx}[\mathsf{refl}_{\mathbb{A}z}, k] = h_{zx}[k]$$

for any object $z : \mathbb{A}, x : \mathbb{C}$ *and* $k \in \Gamma(z, z, x, x)$*.*
Proof. *Explicitly, the dinatural* $J(h)$ *is given by*

$$J(h)_{abx}[e, k] := (\Gamma(\mathsf{id}_b, e, \mathsf{id}_x, \mathsf{id}_x)\ ;\ h_{bx}\ ;\ P(e, \mathsf{id}_b, \mathsf{id}_x, \mathsf{id}_x))[k].$$

Computation clearly holds for $e = \mathsf{id}_z$*, without dinaturality.*

Directed type theory with dinaturality – Intuition for J

Intution for the directed J rule

- Whenever there are two positions $a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}$ in the conclusion P,
- and I have a directed equality in context $\hom_{\mathbb{A}}(a, b)$,

---

- it is enough to prove that P holds "on the diagonal", where a, b are identified with the same dinatural variable $z : \mathbb{A}$.
- Moreover, a, b can be identified in context *only if they appear negatively*.

- The directed J rule is the key to define maps with directed equality.
- Examples: *transitivity*, *congruence*, *transport* for directed equality.

Failure of symmetry for directed equality

These restrictions do *not* allow us to obtain directed equality is symmetric:

$$[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}]\ e : \hom(a, b) \nvdash \mathsf{sym} : \hom(\overline{b}, \overline{a})$$

$\hom(a, b)$ cannot be contracted: a, b must appear *positively* in conclusion.

Directed type theory with dinaturality – Example

Example (Composition in a category)

Transitivity of directed equality $\rightsquigarrow$ categories $\mathbb{C}$ have *composition maps*. Composition is natural in $a : \mathbb{C}^{\mathsf{op}}, c : \mathbb{C}$ and dinatural in $b : \mathbb{C}$:

$$\dfrac{\dfrac{}{[z : \mathbb{C}, c : \mathbb{C}] \qquad\qquad \hom(\overline{z}, c) \vdash \mathsf{id} : \hom(\overline{z}, c)}\ (\mathsf{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}, c : \mathbb{C}]\ \hom(a, b),\ \hom(\overline{b}, c) \vdash J(\mathsf{id}) : \hom(a, c)}\ (J)$$

We contract $\hom(a, b)$. Rule (J) can be applied: a, b appear only negatively in ctx (a does not) and positively in conclusion ($\overline{b}$ does not).

The computation rule for $f \mathbin{;} g := J(\mathsf{id})[f, g]$ states unitality on the left.

- How do we prove unitality on the right, and associativity?
- Using the same method of MLTT: by *dependent* $\hom$ induction.
- First, we need to show that we can compose certain dinats with refl.

Directed type theory – J as iso

Theorem (J as isomorphism)

As in the classical case, directed J is an isomorphism, natural in Γ, P:

$$\begin{array}{c} [z:\mathbb{A}, x:\mathbb{C}] \qquad k:\Gamma(\overline{z},z,\overline{x},x) \vdash h[k] : P(\overline{z},z,\overline{x},x) \\ \hline\hline [a:\mathbb{A}^{\mathsf{op}}, b:\mathbb{A}, x:\mathbb{C}]\ e:\hom(a,b), k:\Gamma(\overline{b},\overline{a},\overline{x},x) \vdash J(h)[e,k] : P(a,b,\overline{x},x) \end{array}\ (\hom)$$

Proof. *The inverse is $J^{-1}(\alpha)_{zx}[k] := \alpha_{zzx}[\mathsf{refl}_{\mathbb{A}z}, k]$.*
The computation rule for $\hom$-elimination is precisely $J\ ;J^{-1} = \mathsf{id}$.
On the other hand, $J^{-1}\ ; J = \mathsf{id}$ follows using (di)naturality.

- *Crucial*: when equalities can be contracted, $J^{-1}(\alpha)$ is always dinatural!
  - This is needed in the examples to contract and compose with refl.
  - This isomorphism is the core rule for $\hom$ in (co)end calculus.

Theorem ($\hom \Rightarrow$ refl)

Rules (refl) *and* (J) *are logically equivalent to* $(\hom)$.
Proof. (refl) *follows from J^{-1} by picking $\Gamma := \top$ and $J(h) := \mathsf{id}_{\hom(a,b)}$.*

Directed type theory – equality of entailments

We work with *proof relevant entailments*: we need to express equality.

Definition (Judgement for equality of entailments)

The following judgement is interpreted semantically as $\alpha = \beta$:

$$[a : \mathbb{C}, b : \mathbb{C}, ...\Gamma]\ h : P(...), h' : Q(...),\ \cdots$$
$$\vdash \alpha[h, h', ...] = \beta[h, h', ...] : R(...)$$

Theorem (Dependent J rule for judgemental equality) (👏)

*Take $\Gamma, P : (\mathbb{A}^{\mathsf{op}}) \times (\mathbb{A}) \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to$ **Set**. Given dinats α, β where J can be applied, then the above judgement implies the one below:*

$$\frac{[z : \mathbb{A}, x : \mathbb{C}]\ k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash \alpha[\mathsf{refl}_z, k] = \beta[\mathsf{refl}_z, k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ e : \hom(a, b), k : \Gamma(\overline{a}, \overline{b}, \overline{x}, x) \vdash \alpha[e, k] = \beta[e, k] : P(a, b, \overline{x}, x)}\ (J\text{-eq})$$

i.e., $\forall k : P(z, z, x, x).\ \alpha_{zzx}[\mathsf{refl}_z, k] = \beta_{zzx}[\mathsf{refl}_z, k]$ *implies* $\alpha = \beta$.

Proof. *By hypothesis, $J^{-1}(\alpha) = J^{-1}(\beta)$, simply apply J.*

Directed type theory – trans assoc/unit via dependent J

Example (Properties of composition)

Back to composition: we want to prove unitality and associativity.

$$\frac{}{[z : \mathbb{C}, c : \mathbb{C}]\ g : \hom(\overline{z}, c) \vdash \mathsf{refl}_z \mathbin{;} g = g : \hom(\overline{z}, c)}\ (J\text{-comp})$$

Unitality on the right is shown by dependent $\hom$ induction:

$$\dfrac{\dfrac{}{[w : \mathbb{C}]\ \top \vdash \mathsf{refl}_w \mathbin{;} \mathsf{refl}_w = \mathsf{refl}_w : \hom(\overline{w}, w)}\ (J\text{-comp})}{[a : \mathbb{C}^{\mathsf{op}}, z : \mathbb{C}]\ f : \hom(a, z) \vdash f \mathbin{;} \mathsf{refl}_z = f : \hom(a, z)}\ (J\text{-eq})$$

To prove associativity, simply contract $f : \hom(a, b)$:

$$\dfrac{\dfrac{}{[z : \mathbb{C}, c : \mathbb{C}, d : \mathbb{C}]\qquad g : \hom(\overline{z}, c), h : \hom(\overline{c}, d) \vdash \mathsf{refl}_z \mathbin{;} (g \mathbin{;} h) = (\mathsf{refl}_z \mathbin{;} g) \mathbin{;} h : \hom(\overline{z}, d)}\ (J\text{-comp})}{[a : \mathbb{C}, b : \mathbb{C}, c : \mathbb{C}, d : \mathbb{C}]\ f : \hom(\overline{a}, b), g : \hom(\overline{b}, c), h : \hom(\overline{c}, d) \vdash f \mathbin{;} (g \mathbin{;} h) = (f \mathbin{;} g) \mathbin{;} h : \hom(\overline{a}, d)}\ (J\text{-eq})$$

where the top sequent $= g \mathbin{;} h$ by computation rules for comp $:= J(\mathsf{id})$.

- *Note!* These are exactly the steps in MLTT for transitivity of paths.

Directed type theory – congr for directed equality

- Every term respects directed equality, i.e., it is a “congruence”: semantically, this is just the functorial action of terms F on morphisms.

Example (Directed equality is a congruence)

Let $F : \mathbb{C} \to \mathbb{D}$ be a functor.

$$\dfrac{\dfrac{}{[z : \mathbb{C}]\ \top \vdash \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{z}), F(z))}\ \text{(reindex)+(refl)}}{[x : \mathbb{C}, y : \mathbb{C}]\ \hom_{\mathbb{C}}(\overline{x}, y) \vdash \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{x}), F(y))}\ (J)$$

Take $\mathsf{map}_F[f] := J(F^*(\mathsf{refl}_{\mathbb{C}}))$. Computation rule: F *maps* refl *to* refl:

$$\dfrac{}{[z : \mathbb{C}]\ \top \vdash \mathsf{map}_F[\mathsf{refl}_z] = F^*(\mathsf{refl}_z) : \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{x}), F(x))}\ (J\text{-comp})$$

Functoriality holds, since both top sides $=$ $\mathsf{map}_F[g]$ via computation rules:

$$\dfrac{\dfrac{}{[z : \mathbb{C}, c : \mathbb{C}]\quad g : \hom(\overline{z}, c) \vdash \mathsf{map}_F[\mathsf{refl}_z \mathbin{;} g] = \mathsf{refl}_{F(z)} \mathbin{;} \mathsf{map}_F[g] : \hom(\overline{z}, d)}\ (J\text{-comp})}{[a : \mathbb{C}, b : \mathbb{C}, c : \mathbb{C}]\ f : \hom(\overline{a}, b), g : \hom(\overline{b}, c) \vdash \mathsf{map}_F[f \mathbin{;} g] = \mathsf{map}_F[f] \mathbin{;} \mathsf{map}_F[g] : \hom(\overline{a}, d)}\ (J\text{-eq})$$

Directed type theory – transport along directed equalities

- Transporting points of predicates (i.e., presheaves) along directed equalities is the functorial action of $P : \mathbb{C} \to \mathbf{Set}$:

Example (Transporting along directed equality)

For any $P : \mathbb{C} \to \mathbf{Set}$:

$$\dfrac{\dfrac{}{[z : \mathbb{C}]\ P(z) \vdash P(z)}\ (\mathsf{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}]\ \ \hom(a,b), P(\overline{a}) \vdash P(b)}\ (J)$$

Computation rule for $\mathsf{subst}[f,k] := J(\mathsf{id})$:
"transporting a point of $P(a)$ along the path refl_z *is the identity"*:

$$\dfrac{}{[z : \mathbb{C}]\ k : P(z) \vdash \mathsf{subst}[\mathsf{refl}_z, k] = k : P(z)}\ (J\text{-comp})$$

Directed type theory – Groupoidal case

- When $\mathbb{A} \cong \mathbb{A}^{\mathsf{op}}$ is a groupoid, $\hom$ is the characterization of symmetric equality as left adjoint to reindexing on diagonals. (with Frobenius)

$$\frac{\dfrac{[z:\mathbb{A}, x:\mathbb{C}] \qquad \Gamma(\overline{z},z,\overline{x},x) \vdash P(\overline{z},z,\overline{x},x)}{[a:\mathbb{A}^{\mathsf{op}}, b:\mathbb{A}, x:\mathbb{C}]\ \hom(a,b),\ \Gamma(\overline{b},\overline{a},\overline{x},x) \vdash P(a,b,\overline{x},x)}\ (\hom)}{\dfrac{[z:\mathbb{A}, x:\mathbb{C}] \qquad \Gamma(z,z,x) \vdash P(z,z,x)}{[a:\mathbb{A}, b:\mathbb{A}, x:\mathbb{C}]\ a \cong b,\ \Gamma(b,a,x) \vdash P(a,b,x)}\ (\text{eq})}$$

- The *proof-relevant directness* of **Cat** seems to be the fundamental obstacle to a fully compositional theory of dinaturals:

Theorem (Dinaturals in groupoids compose) (🐓)

Given a groupoid $\mathbb{C} \cong \mathbb{C}^{\mathsf{op}}$ *and any* $\mathbb{D}$ *for* $F, G, H : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$, *all dinaturals* $\alpha : F \xrightarrow{\bullet\bullet} G$, $\beta : G \xrightarrow{\bullet\bullet} H$ *compose.*

- *Can the directed rule also be characterized as an adjunction? Yes!...*

Directed equality as relative left adjoint

Symmetric equality is a left adjoint to identifying two variables together.

Claim: *Directed equality* is a *relative* left adjoint to identifying two *natural* variables together with a *dinatural* one.

- The *relative* part imposes the syntactic restrictions of directed J: the relative functor is just reindexing along projections.
- Unfortunately, for **Cat** we can only state this as a relative *para*-adjunction, because of compositionality of dinaturals.
- *Para*- indicates that composition is partial: paracategories, parafunctors, para-adjunctions. [Hermida 2003]

Background: Relative adjunctions

Definition (Left relative adjunction [Arkor 2024])

Consider this situation of functors and categories:

$$\begin{tikzcd}
 & \mathbb{D} \arrow[dr, "R"] & \\
\mathbb{C} \arrow[ur, "L"] \arrow[rr, "J"'] & \dashv & \mathbb{X}
\end{tikzcd}$$

We say that L *is the* J*-relative left adjoint to* R, written $L \dashv_J R$, if

$$\mathbb{D}(L(x), y) \cong \mathbb{X}(J(x), R(y))$$

is a bijection natural in both arguments $x : \mathbb{C}, y : \mathbb{D}$.

Directed equality as adjoint (1)

Theorem (Directed equality as relative left adjoint) (👋≈)

- *Let* $[\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \mathbf{Set}]$ *the paracategory where morphisms are dinats natural in* $\mathbb{A}^{\mathsf{op}}, \mathbb{A}$ *and dinatural in* $\mathbb{C}$*, and* $[\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \mathbf{Set}]$ *as dinaturals.*
- *Take the parafunctor* $\pi_{\mathbb{A}}^{*} : [\mathbb{C}^{\diamond}, \mathbf{Set}] \to [\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \mathbf{Set}]$ *defined in the intuitive way by precomposing with projections.*
- *There is a dipresheaf* $\hom_{\mathbb{A}} \in [\mathbb{A}^{\mathsf{op}} \times \mathbb{A}, \mathbf{Set}]$ *such that the functor*

$$\hom_{\mathbb{A}} \times - : [\mathbb{C}^{\diamond}, \mathbf{Set}] \to [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \mathbf{Set}]$$

$$\hom_{\mathbb{A}} \times \mathit{\Gamma} := (\overline{a}, a, \overline{x}, x) \mapsto \hom(\overline{a}, a) \times \mathit{\Gamma}(\overline{x}, x),$$

$$(\hom_{\mathbb{A}} \times \alpha_x)_{abc} := \lambda(e \in \hom(a, b), k \in \mathit{\Gamma}(c, c)).(e, \alpha_c(k))$$

determines a $\pi_{\mathbb{A}}^{*}$*-relative left adjoint to the functor*

$$\Delta_{\mathbb{A}} \times - : [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \mathbf{Set}] \to [\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \mathbf{Set}]$$

$$\Delta_{\mathbb{A}} \times P := P$$

$$(\Delta_{\mathbb{A}} \times \alpha_{abc})_{zx} := \alpha_{zzx}.$$

Directed equality as adjoint (2)

Theorem (Directed equality as relative left adjoint, cont.) (🐔≈)

- *Thus the left relative adjointness situation*

$$\hom_{\mathbb{A}} \times - \dashv_{\pi^*_{\mathbb{A}}} \Delta_{\mathbb{A}} \times -$$

is as follows:

$$\begin{tikzcd}
 & {[\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \boldsymbol{\mathit{Set}}]} \arrow[dr, "\Delta_{\mathbb{A}} \times -"] & \\
{[\mathbb{C}^{\diamond}, \boldsymbol{\mathit{Set}}]} \arrow[ur, "\hom_{\mathbb{A}} \times -"] \arrow[rr, "\pi^*_{\mathbb{A}}"'] & \dashv & {[\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \boldsymbol{\mathit{Set}}]}
\end{tikzcd}$$

Proof. *The required isomorphism is the following:*

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \quad (\pi^*_{\mathbb{A}}(\Gamma)) = \Gamma(\overline{x}, x) \vdash P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}] \;\; \hom_{\mathbb{A}}(a, b) \times \Gamma(\overline{x}, x) \vdash P(a, b, \overline{x}, x)} \;(\hom\text{-rel-adj})$$

which is an instance of directed J*, where* Γ *is mute in* $\overline{a} : \mathbb{A}, \overline{b} : \mathbb{A}^{\mathsf{op}}$*.*
$(\hom\text{-rel-adj}) \Leftrightarrow (\hom)$*: pick* $P := \Gamma^{\mathsf{op}}(b, a, x, \overline{x}) \Rightarrow P(a, b, \overline{x}, x)$*, use* (exp)*.*

(Co)end calculus via dinaturality

- *What are the quantifiers of directed type theory?*
- Dinaturality allows us to view (co)ends as "adjoints" to weakening:

Theorem (Ends and coends as quantifiers) (🐓)

Take $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \textbf{\textit{Set}}$, *and the functor precomposing with projections*

$$\pi_{\mathbb{A}}^{*}(P) : \mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond} \to \textbf{\textit{Set}}$$
$$\pi_{\mathbb{A}}^{*}(P) := (\overline{a}, a, \overline{x}, x) \mapsto P(\overline{x}, x),$$

There are isos of sets of dinats, natural in $P : \mathbb{C}^{\diamond} \to \textbf{\textit{Set}}$, $Q : \mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond} \to \textbf{\textit{Set}}$*:*

$$\begin{array}{c} [a : \mathbb{A}, x : \mathbb{C}] \;\; P(\overline{x}, x) \vdash Q(\overline{a}, a, \overline{x}, x) \\ \hline\hline [x : \mathbb{C}] \;\; P(\overline{x}, x) \vdash \displaystyle\int_{a:\mathbb{A}} Q(\overline{a}, a, \overline{x}, x) \end{array} \;\text{(end)}$$

$$\begin{array}{c} [x : \mathbb{C}] \;\; \displaystyle\int^{a:\mathbb{A}} Q(\overline{a}, a, \overline{x}, x) \vdash P(\overline{x}, x) \\ \hline\hline [a : \mathbb{A}, x : \mathbb{C}] \qquad Q(\overline{a}, a, \overline{x}, x) \vdash P(\overline{x}, x) \end{array} \;\text{(coend)}$$

(Co)ends as quantifiers

Theorem (Beck-Chevalley and Frobenius condition for (co)ends)

(Co)ends satisfy a Beck-Chevalley condition: for $F : \mathbb{C}^\diamond \to \mathbb{D}$ *there is a strict isomorphism in the (large) functor category* $[[\mathbb{A}^\diamond \times \mathbb{D}^\diamond, \textbf{\textit{Set}}],[\mathbb{D}^\diamond, \textbf{\textit{Set}}]]$

$$\int_{\mathbb{A}[\mathbb{D}]} ; F^* \cong (\mathrm{id}_{\mathbb{A}^\diamond} \times F)^* ; \int_{\mathbb{A}[\mathbb{C}]}$$

where $\int_{\mathbb{A}[\mathbb{C}]}, \int^{\mathbb{A}[\mathbb{C}]} : [\mathbb{A}^\diamond \times \mathbb{C}^\diamond, \textbf{\textit{Set}}] \to [\mathbb{C}^\diamond, \textbf{\textit{Set}}]$ *are parametric (co)ends, and* $F^* : [\mathbb{D}^\diamond, \textbf{\textit{Set}}] \to [\mathbb{C}^\diamond, \textbf{\textit{Set}}]$ *is precomposition with* $F^\diamond$*.*
Moreover, a Frobenius condition for coends holds: there is an isomorphism natural in $\mathit{\Gamma} : \mathbb{A}^\diamond \times \mathbb{C}^\diamond \to \textbf{\textit{Set}}, P : \mathbb{C}^\diamond \to \textbf{\textit{Set}}$*,*

$$\int^{\mathbb{A}[\mathbb{C}]} (\pi^*_{\mathbb{A}[\mathbb{C}]}(P) \wedge \mathit{\Gamma}) \cong \pi^*_{\mathbb{A}[\mathbb{C}]}(P) \wedge \int^{\mathbb{A}[\mathbb{C}]} (\mathit{\Gamma}),$$

where $- \wedge - : [\mathbb{C}, \textbf{\textit{Set}}] \times [\mathbb{C}, \textbf{\textit{Set}}] \to [\mathbb{C}, \textbf{\textit{Set}}]$ *is the pointwise product.*
Proof. *Beck-Chevalley is easy. For Frobenius, we can use logical rules to mirror the argument in [Jacobs 1999, 1.9.12(i)] with exponentials.*

Our rules, so far

Rules for (co)ends as quantifiers + directed equality for logical proofs of:

- (Co)Yoneda,
- Adjointess of Kan extensions via (co)ends,
- Presheaves are closed under exponentials,
- Associativity of composition of profunctors,
- Right lifts in profunctors,
- (Co)ends preserve limits,
- Adjointness of (co)ends in natural transformations,
- Characterization of dinaturals as certain ends,
- Frobenius property of (co)ends using exponentials.

On our rules

- Our rules mirror *everyday (co)end calculus*:

  use Yoneda via a chain of isomorphisms in **Set**, either:
    - to prove (natural) isomorphisms of objects/functors.
    - to show adjunctions.
- Advantages of using our rules:
    - Structural properties are automatic because of contexts (e.g., Fubini)
    - $\hom$ is seen as directed equality, using dinaturality.
    - The rules are not ad-hoc lemmas, but follow the logical presentation.
- This technique sidesteps compositionality of dinaturals,
  no equational theory needed.
- *Note:* isomorphisms must end in sets of *naturals*, since composition is needed for Yoneda (no notion of *"dinatural isomorphism"*)

On naturality of the rules

- Crucially, we need to ensure naturality of the rules:

Theorem (Naturality for rules) (👋)

All our isomorphisms of sets of dinats are natural in each functor involved.

Proof. *Easy to verify. This relies on the existence of a functor*

$$\mathit{Dinats} : [\mathbb{C}^{\mathsf{op}} \times \mathbb{C}, \mathbb{D}]^{\mathsf{op}} \times [\mathbb{C}^{\mathsf{op}} \times \mathbb{C}, \mathbb{D}] \to \textbf{\textit{Set}}$$

defined on functor categories, where morphisms are <u>*naturals*</u>
(i.e., they always compose with dinaturals.)
For instance, the (exp) *rule:*

$$\mathit{Dinats}(-_1 \times -_2, -_3) \cong \mathit{Dinats}(-_2, -_1^{\mathsf{op}} \Rightarrow -_3)$$

as functors $[\mathbb{C}^{\mathsf{op}} \times \mathbb{C}, \mathbb{D}]^{\mathsf{op}} \times [\mathbb{C}^{\mathsf{op}} \times \mathbb{C}, \mathbb{D}] \times [\mathbb{C}^{\mathsf{op}} \times \mathbb{C}, \mathbb{D}] \to \textbf{\textit{Set}}$*,*
where $-_1 \times -_2$ *is the pointwise product,* $-_1 \Rightarrow -_3$ *is the pointwise hom.*

(Co)end calculus with dinaturality (1)

Yoneda lemma: $(P, \Gamma : \mathbb{C} \to \mathbf{Set})$

$$\begin{array}{c}
[a : \mathbb{C}]\ \Gamma(a) \vdash \int_{x:\mathbb{C}} \hom_{\mathbb{C}}^{\mathsf{op}}(a, \overline{x}) \Rightarrow P(x) \\ \hline\hline
[a : \mathbb{C}, x : \mathbb{C}]\ \Gamma(a) \vdash \hom_{\mathbb{C}}^{\mathsf{op}}(a, \overline{x}) \Rightarrow P(x) \\ \hline\hline
[a : \mathbb{C}]\ \ \hom_{\mathbb{C}}(\overline{a}, x) \times \Gamma(a) \vdash P(x) \\ \hline\hline
[z : \mathbb{C}]\ \Gamma(z) \vdash P(z)
\end{array}
\quad
\begin{array}{l}
\text{(end)} \\ \text{(exp)} \\ \text{(hom)}
\end{array}$$

CoYoneda lemma:

$$\begin{array}{c}
[a : \mathbb{C}]\ \ \int^{x:\mathbb{C}} \hom_{\mathbb{C}}(\overline{x}, a) \times P(x) \vdash \Gamma(a) \\ \hline\hline
[a : \mathbb{C}, x : \mathbb{C}]\ \ \hom_{\mathbb{C}}(\overline{a}, x) \times P(a) \vdash \Gamma(x) \\ \hline\hline
[z : \mathbb{C}]\ P(z) \vdash \Gamma(z)
\end{array}
\quad
\begin{array}{l}
\text{(coend)} \\ \text{(hom)}
\end{array}$$

(Co)end calculus with dinaturality (2)

Presheaves are cartesian closed: $(\mathit{\Gamma}, A, B : \mathbb{C} \to \textbf{Set})$

$$\begin{array}{c}
\begin{aligned}
[x : \mathbb{C}]\ \mathit{\Gamma}(x) \vdash (A \Rightarrow B)(x) & \\
:= \ & \mathsf{Nat}(\hom_{\mathbb{C}}(x, -) \times A, B) \\
\cong \ & \int_{y:\mathbb{C}} \hom_{\mathbb{C}}^{\mathsf{op}}(x, \overline{y}) \times A^{\mathsf{op}}(\overline{y}) \Rightarrow B(y)
\end{aligned} \\
\hline\hline \text{(end)} \\
[x : \mathbb{C}, y : \mathbb{C}]\ \mathit{\Gamma}(x) \vdash \hom_{\mathbb{C}}^{\mathsf{op}}(x, \overline{y}) \times A^{\mathsf{op}}(\overline{y}) \Rightarrow B(y) \\
\hline\hline \text{(exp)} \\
[x : \mathbb{C}, y : \mathbb{C}]\ A(y) \times \hom_{\mathbb{C}}(\overline{x}, y) \times \mathit{\Gamma}(x) \vdash B(y) \\
\hline\hline \text{(coend+frob.)} \\
[y : \mathbb{C}]\ A(y) \times \left(\int^{x:\mathbb{C}} \hom_{\mathbb{C}}(\overline{x}, y) \times \mathit{\Gamma}(x) \right) \vdash B(y) \\
\hline\hline \text{(coYoneda)} \\
[y : \mathbb{C}]\ A(y) \times \mathit{\Gamma}(y) \vdash B(y)
\end{array}$$

Note: $(\hom)$ cannot be applied, y appears positive in context.
→ use (coYoneda) “extensionally”, i.e., $\mathit{\Gamma} \cong$ a certain coend, independently of the point in which it is evaluated.
Note: not obvious how to capture this via **Prof** as proarrow equipment.

(Co)end calculus with dinaturality (3)

Right Kan extensions via ends are right adjoints to precomposition with $F : \mathbb{C} \to \mathbb{D}$ ($P : \mathbb{C} \to \textbf{Set}, \mathit{\Gamma} : \mathbb{D} \to \textbf{Set}$):

$$
\begin{array}{cl}
[y : \mathbb{D}]\ \mathit{\Gamma}(y) \vdash (\mathsf{Ran}_F P)(y) & \\
\qquad := \displaystyle\int_{x:\mathbb{C}} \hom_{\mathbb{D}}^{\mathsf{op}}(y, F^{\mathsf{op}}(\overline{x})) \Rightarrow P(x) & \\
\hline\hline
[x : \mathbb{C}, y : \mathbb{D}]\ \mathit{\Gamma}(y) \vdash \hom_{\mathbb{D}}^{\mathsf{op}}(y, F^{\mathsf{op}}(\overline{x})) \Rightarrow P(x) & (\text{end}) \\
\hline\hline
[x : \mathbb{C}, y : \mathbb{D}]\ \hom_{\mathbb{D}}(\overline{y}, F(x)) \times \mathit{\Gamma}(y) \vdash P(x) & (\text{exp}) \\
\hline\hline
[x : \mathbb{C}]\ \displaystyle\int^{y:\mathbb{D}} \hom_{\mathbb{D}}(\overline{y}, F(x)) \times \mathit{\Gamma}(y) \vdash P(x) & (\text{coend}) \\
\hline\hline
[x : \mathbb{C}]\ \mathit{\Gamma}(F(x)) \vdash P(x) & (\text{coYoneda})
\end{array}
$$

(Co)end calculus with dinaturality (4)

Composition (on both sides) in **Prof** has a right adjoint (right lifts). ($P : \mathbb{C}^{\mathsf{op}} \times \mathbb{A} \to \mathbf{Set}, Q : \mathbb{A}^{\mathsf{op}} \times \mathbb{D} \to \mathbf{Set}$, $\mathit{\Gamma} : \mathbb{C}^{\mathsf{op}} \times \mathbb{D} \to \mathbf{Set}$):

$$
\begin{array}{cl}
[x : \mathbb{C}^{\mathsf{op}}, z : \mathbb{D}] \qquad (P \mathbin{\boldsymbol{;}} -)(Q)(x,z) := & \\
\displaystyle\int^{y:\mathbb{A}} P(x,y) \times Q(\overline{y},z) \vdash \mathit{\Gamma}(x,z) & \\
\hline\hline
[x : \mathbb{C}^{\mathsf{op}}, y : \mathbb{A}, z : \mathbb{D}]\ P(x,y) \times Q(\overline{y},z) \vdash \mathit{\Gamma}(x,z) & \text{(coend)} \\
\hline\hline
[x : \mathbb{C}^{\mathsf{op}}, y : \mathbb{A}, z : \mathbb{D}]\ Q(\overline{y},z) \vdash P^{\mathsf{op}}(\overline{x},\overline{y}) \Rightarrow \mathit{\Gamma}(x,z) & \text{(exp)} \\
\hline\hline
[y : \mathbb{A}, z : \mathbb{D}]\ Q(\overline{y},z) \vdash \displaystyle\int_{x:\mathbb{C}} P^{\mathsf{op}}(\overline{x},\overline{y}) \Rightarrow \mathit{\Gamma}(x,z) & \text{(end)} \\
\hline\hline
[y : \mathbb{A}^{\mathsf{op}}, z : \mathbb{D}]\ Q(y,z) \vdash \displaystyle\int_{x:\mathbb{C}} P^{\mathsf{op}}(\overline{x},y) \Rightarrow \mathit{\Gamma}(x,z) & (\cong) \\
:= \mathrm{Rift}_P(\mathit{\Gamma})(y,z) &
\end{array}
$$

the last (end) can be applied since $x : \mathbb{C}$ does not appear on the left.

(Co)end calculus with dinaturality (5)

Fubini for ends ($\mathit{\Gamma} : \textbf{Set}$, $P : (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \times (\mathbb{D}^{\mathsf{op}} \times \mathbb{D}) \to \textbf{Set}$)

$$
\begin{array}{cl}
[\,]\ \mathit{\Gamma} \vdash \displaystyle\int_{x:\mathbb{C}} \int_{y:\mathbb{D}} P(\overline{x}, x, \overline{y}, y) & \\
\hline\hline
[x : \mathbb{C}]\ \mathit{\Gamma} \vdash \displaystyle\int_{y:\mathbb{D}} P(\overline{x}, x, \overline{y}, y) & \text{(end)} \\
\hline\hline
[x : \mathbb{C}, y : \mathbb{D}]\ \mathit{\Gamma} \vdash P(\overline{x}, x, \overline{y}, y) & \text{(end)} \\
\hline\hline
[y : \mathbb{D}, x : \mathbb{C}]\ \mathit{\Gamma} \vdash P(\overline{x}, x, \overline{y}, y) & \text{(structural property)} \\
\hline\hline
[y : \mathbb{D}]\ \mathit{\Gamma} \vdash \displaystyle\int_{x:\mathbb{C}} P(\overline{x}, x, \overline{y}, y) & \text{(end)} \\
\hline\hline
[\,]\ \mathit{\Gamma} \vdash \displaystyle\int_{y:\mathbb{D}} \int_{x:\mathbb{C}} P(\overline{x}, x, \overline{y}, y) & \text{(end)}
\end{array}
$$

(Co)end calculus with dinaturality (6)

Composition of profunctors is associative: ($\mathit{\Gamma} : \mathbb{A}^{\mathsf{op}} \times \mathbb{D} \to \textbf{Set}$, $P : \mathbb{A}^{\mathsf{op}} \times \mathbb{B} \to \textbf{Set}, Q : \mathbb{B}^{\mathsf{op}} \times \mathbb{C} \to \textbf{Set}, R : \mathbb{C}^{\mathsf{op}} \times \mathbb{D} \to \textbf{Set}$)

$$\begin{array}{cl}
[a : \mathbb{A}, d : \mathbb{D}]\ \displaystyle\int^{b:\mathbb{B}} P(\overline{a}, b) \times \left(\int^{c:\mathbb{C}} Q(\bar{b}, c) \times R(\overline{c}, d) \right) \vdash \mathit{\Gamma}(\overline{a}, d) & \\
\hline\hline
 & \text{(coend)} \\
[a : \mathbb{A}, b : \mathbb{B}, d : \mathbb{D}]\ P(\overline{a}, b) \times \left(\displaystyle\int^{c:\mathbb{C}} Q(\bar{b}, c) \times R(\overline{c}, d) \right) \vdash \mathit{\Gamma}(\overline{a}, d) & \\
\hline\hline
 & \text{(coend)} \\
[a : \mathbb{A}, b : \mathbb{B}, c : \mathbb{C}, d : \mathbb{D}]\ P(\overline{a}, b) \times (Q(\bar{b}, c) \times R(\overline{c}, d)) \vdash \mathit{\Gamma}(\overline{a}, d) & \\
\hline\hline
 & \text{(struct. prop.)} \\
[a : \mathbb{A}, b : \mathbb{B}, c : \mathbb{C}, d : \mathbb{D}]\ (P(\overline{a}, b) \times Q(\bar{b}, c)) \times R(\overline{c}, d) \vdash \mathit{\Gamma}(\overline{a}, d) & \\
\hline\hline
 & \text{(coend)} \\
[a : \mathbb{A}, c : \mathbb{C}, d : \mathbb{D}]\ \left(\displaystyle\int^{b:\mathbb{B}} P(\overline{a}, b) \times Q(\bar{b}, c) \right) \times R(\overline{c}, d) \vdash \mathit{\Gamma}(\overline{a}, d) & \\
\hline\hline
 & \text{(coend)} \\
[a : \mathbb{A}, d : \mathbb{D}]\ \displaystyle\int^{c:\mathbb{C}} \left(\int^{b:\mathbb{B}} P(\overline{a}, b) \times Q(\bar{b}, c) \right) \times R(\overline{c}, d) \vdash \mathit{\Gamma}(\overline{a}, d) &
\end{array}$$

Conclusion and future work

We have seen how dinaturality allows us to give a semantic interpretation towards a directed type theory with quantifiers, where directed equality is given by $\hom$*-functors and quantifiers by (co)ends.*

Future work:

1. Big piece missing from the story: compositionality of dinaturals.
   - *Claim:* non-compositionality is intrinsic to **Cat**, like failure of UIP.
   - Find suitable structures that axiomatize sufficient conditions for composition (e.g., in the style of operads/multicategories but with explicit management of variances and variables).
2. Everything shown so far can be done in **Pos**, where all dinats compose.
   - This can be used to characterize *the directed logic of posets*.
   - Axiomatize this logic of directed equality via a doctrinal approach.
3. *Claim:* dinaturality arises whenever a notion of variance is involved.
   - instead of **Cat**, pick $\mathcal{V}$-**Cat** for suitable $\mathcal{V}$: e.g., a theory of metric spaces.

The $\int$.

Formalization: `github.com/iwilare/dinaturality`
Paper: *“Directed equality with dinaturality”* (`arXiv:2409.10237`)

Thank you for the attention!

Related works

- **[Caccamo-Winskel 2001]**: axiomatic system to manipulate (co)ends; quantifier exchange is postulated, no type theory presented.
- **[Licata 2011]**: a model in **Cat** with directed equality defined judgementally and not propositionally.
- **[Nuyts 2015]**: a preliminary system of contexts variances is given, with no formal syntax nor models.
- **[North 2016]**: dependent type theory, but uses groupoid structure to type the refl rule. We use dinaturality precisely to avoid this problem.
- **[Riehl-Shulman 2017]**: a synthetic theory of $(\infty, 1)$-categories with a directed interval type, no model in **Cat**.
- **[New-Licata 2022]**: a DTT with models in virtual equipments, with directed equality and quantifiers, but very different syntax w.r.t. MLTT
- **[Ahrens 2023]**: judgemental structure for directed equality reminiscent of a bicategorical model.