Directed First-Order Logic

Andrea Laretto
Tallinn University of Technology, Estonia

Fosco Loregian
Tallinn University of Technology, Estonia

Niccolò Veltri
Tallinn University of Technology, Estonia

Abstract

We present a first-order logic equipped with an "asymmetric" directed notion of equality, which can be thought of as transitions/rewrites between terms, allowing for types to be interpreted as preorders. We give introduction and elimination rules to such directed equalities in the style of the J-rule of Martin-Löf type theory, but with a syntactic restriction that does not allow for symmetry to be derived. We give a characterization of directed equality as a relative left adjoint, generalizing the idea by Lawvere of equality as left adjoint. The logic is equipped with a precise syntactic system of polarities, inspired by dinaturality, that keeps track of the occurrence of variables (positive/negative/both). The semantics of this logic and its system of variances is then captured categorically using the notion of directed doctrine, which we prove sound and complete with respect to the syntax.

2012 ACM Subject Classification Theory of computation → Type theory

Keywords and phrases directed type theory, doctrines, directed equality, dinaturality

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

Funding *Fosco Loregian*: Loregian was supported by the Estonian Research Council grant PRG1210.
Niccolò Veltri: Veltri was supported by the Estonian Research Council grant PSG749.

1 Introduction

In a series of seminal papers, Lawvere [26–28] unveiled the deep connection between logic and the purely algebraic perspective of category theory [29, 33]. One of the key insights of this connection is to view the *logical connectives* of first-order logic, such as truth $\top$, conjunction $\wedge$, implication $\Rightarrow$, all as instances of the single general notion of *adjunction* [33], an elementary category-theoretical definition. These adjunctions naturally arise in the notion of cartesian closed category [26], which plays a central role in the Curry-Howard-Lambek correspondence [10, 22, 24] and acts as the *formal* framework in which propositions, types, proofs, and programs can be unified under the common language of category theory.

Remarkably, this rephrasing of logical ideas as adjoints extends beyond the propositional connectives: *quantifiers* $\forall$ and $\exists$ can be characterized as right and left adjoints to *weakening* [22, 4.1.8]; the same idea also applies to *equality predicates* $x = y$, which can be captured as the left adjoint to *contraction* [22, 4.1.7], i.e., the act of identifying two variables with a single one by sending a formula $\varphi(x, y)$ to $\varphi(z, z)$ with z fresh.

In standard presentations of logic, equality is often taken to be "just another relation symbol" [39], [9, §48]. Under the modern "post-Lawvere" perspective of logic [47, 4.1.5], equality plays a "special" role similar to that of a *logical connective*, since its behaviour is described using exactly the same kind of rules (via adjoints) that other "more fundamental" connectives are equipped with. Concretely, the fact that equality has a categorical universal property means, for example, that it is characterized up-to-isomorphism [29], and therefore that "having equality" is a *property of a logic*, not structure that can be given arbitrarily [22].

42nd Conference on Very Important Topics (CVIT 2016).
Editors: John Q. Open and Joan R. Access; Article No. 23; pp. 23:1–23:28
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

The core notion behind the categorical perspective on logic is that of *doctrine* [26,34], a concise category-theoretical definition which unifies both syntax and semantics of (first-order) logics. A doctrine is a category $\mathbb{C}$ with finite products, called the “base” category, equipped with a functor $P : \mathbb{C}^{\mathsf{op}} \to \mathsf{Pos}$ to the category Pos of posets and monotone functions. The intuition is that $\mathbb{C}$ acts as a category of *types (i.e. sorts) and terms*, and P sends the type A to the poset $P(A)$ of formulas in context A with logical entailment as relation. A survey of doctrines and their extensions can be found in [22,34].

The presheaf hyperdoctrine. In the same paper introducing equality-as-adjoint [28], Lawvere describes a particularly important example of a doctrine that is *not* equipped with such a notion of equality, namely the *presheaf doctrine* $\mathsf{Psh} : \mathsf{Cat}^{\mathsf{op}} \to \mathsf{CAT}$, which has as base category Cat (the category of small categories) and is defined by sending a category $\mathbb{C}$ to the the large category of presheaves $\mathsf{Psh}(\mathbb{C}) := [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}]$.

This doctrine plays a central role in category theory because it should intuitively capture the “*the logic of categories*” [28]: categories and functors are viewed as *types and terms* of such logic, and, as well-known among category theorists, presheaves can be thought of as *generalized predicates* [6], in the same way that profunctors are generalized relations [28].

We report a particularly suggestive quote by Lawvere [28, p. 11] in Appendix B, where he comments on the failure of the notion of equality as left adjoint in this setting, and what should be done instead: the key idea is that, if types are now categories, the more natural notion of equality should be given by hom-functors $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$, which now plays the role of a *directed, asymmetric equality.*

Directed type theory. This perspective of types-as-categories first envisioned by Lawvere has recently gained new popularity under the topic of *directed type theory* [3,8,17,18,42,44].

The motivation behind directed type theory starts from a seminal paper of Hofmann and Streicher [21], in which the types of Martin-Löf type theory are interpreted as *groupoids* (i.e., categories where every morphism is invertible), laying the conceptual foundation for Homotopy Type Theory [5,23,50]: the key idea is that inhabitants of types are given by objects of groupoids, and proofs of equalities are given by the morphisms between such objects, of which there can be more than a unique one.

The need for morphisms to be invertible comes from the inherently *symmetric* nature of equality: for every proof of equality $x = y$ there has to be a proof of the equality $y = x$. The typical syntactic presentation of equality in type theory [20], which we recall in Appendix A, consists of an introduction rule refl for reflexivity of equality $x = x$, and an elimination rule, often called J-rule [20], that allows one to contract equalities: this last principle allows for symmetry to be *derived*, hence justifying the need for groupoids.

A natural question arises: why not *categories*, rather than groupoids? Such a setting should take the name of *directed* type theory [3,31,41], where types are interpreted as categories, and morphisms capture an *asymmetric* notion of equality (i.e. symmetry is not a theorem, since not all arrows are invertible). Such equalities can then can be used to represent intrinsically directed phenomena, such as processes, transformations, or rewrites [41].

Logic and proof-irrelevance. In both traditional and categorical accounts of logic [51], however, equality is typically *proof-irrelevant*, i.e., there is no distinction between “different” proofs of equality; this is the main distinction between logic and *type theory* in the sense of Jacobs [22, 8.4.11]. In the directed case, one should talk thus about *directed logic*, i.e., a generalization of first-order logic with a proof-irrelevant *asymmetric* notion of equality.

In the same way that groupoids generalize sets by making equality proof relevant (since there might be many morphisms/proofs of equalities), categories generalize *preorders*: in a preorder there is no information on *in what way* two objects a, b are (directionally) connected

Models of logic/TT	Proofs of equality	
	Irrelevant	Relevant
Symmetric Equality	Sets (Set)	Groupoids (Gpd)
Directed Equality	Preorders (Preord)	Categories (Cat)

Figure 1 Elementary models of symmetric and directed logic.

by a morphism $f : a \to b$, but only whether the inequality $a \leq b$ holds or not; this interest in proof-irrelvance is similarly shared in applications of rewriting in computer science, e.g., in rewriting logic [38], where either rewrites happen or not. We summarize the relationship between these elementary models in Figure 1.

Polarity and symmetry problems. Both directed type theory and directed logic have proven to be not so straightforward [1, 41, 44]; we illustrate why the näive approach to the directed setting fails even in the simple case of preorders, which will be the focus of this paper.

With any preorder A there is a naturally associated type A^{op}, called the *opposite type of* A, where the objects are the same but all directed equalities are reversed, as in the notion of opposite preorder. The *type of directed equalities*, often renamed hom-types [41], should then be interpreted via the monotone function $\leq_A : A^{\mathsf{op}} \times A \to \mathbf{I}$, which receives a "contravariant" argument $a : A^{\mathsf{op}}$ and a "covariant" one $b : A$ and returns $\leq(a, b) := (1 \text{ if } a \leq b, 0 \text{ otherwise})$, where $\mathbf{I} = \{0 \to 1\}$ is the preorder with two elements $0, 1 \in \mathbf{I}$ such that $0 \leq 1$. A directed logic/type theory should therefore have some notion of "*polarity*", which allows variables to be distinguished and appear only in the appropriate position, as treated in [31, 40, 41, 43].

However, in the statement for transitivity of directed equality $x \leq y \wedge y \leq z \vdash x \leq z$, the variable y appears both on the right side of $x \leq y$, with type A, and on the right side of $y \leq z$, with seemingly different type A^{op}! A similar problem arises for reflexivity $\overline{x} \leq x$.

Even if we allow variables to appear with both polarities at the same time, the problem then becomes: *how do we make sure that symmetry is not derivable in the syntax*? That is, how should the refl and J rules for equality change in such a way that symmetry cannot be derived, while still being able to recover the expected theorems of directed type theory?

1.1 Contribution

In this paper, we present syntax and semantics for a first-order proof-irrelevant (i.e. logical) version of directed type theory, with a propositional notion of directed equality which allows for types to be interpreted as preorders, but without being able to derive symmetry.

We describe the key features of our logic and its semantics:

- The logic is equipped with a precise system of polarities, inspired by dinaturality [12]. Variables are allowed to appear anywhere in formulas, regardless of their polarity: however, the rule to eliminate directed equalities (reminiscent of the J rule) is equipped with a certain syntactic restriction on the appearance of variables, thus forbidding symmetry.
- We introduce a generalization of the notion of doctrine, called *directed doctrines*, for which the syntax is sound and complete, with the preorder model Preord as our main example: types are interpreted by preorders, with $\leq$ representing the existence of a directed equality.
- Directed doctrines allow us to give a characterization of directed equality, not as a left adjoint to contraction as in Lawvere's approach, but as a left *relative* adjoint [2, 49] to a certain contraction-like operation that collapses two variables of *different* polarity A, A^{op} into a single one of type A, hence generalizing and directifying the notion of symmetric

equality as left adjoint to contraction first introduced by Lawvere. Intuitively, relativeness is used to capture the syntactic restriction on the appearance of variables.

- We devise a way to capture the notion of variance of variables in the doctrinal approach [34]: the idea is to separate contexts between positive, negative, and dinatural variables, and then require that doctrines have specific reindexings that capture such polarity. There are other possible approaches to capture variance; the one explored here allows for the above "contraction-like" operation to be expressed explicitly in the doctrinal semantics.
- The usual logical connectives are generalized to a corresponding "directified" version: *polarized quantifiers* keep track of the variance of the variable being quantified over, and implication in generalized to *polarized implication*, which reverses the polarity of all positions in the curried formula and is characterized by a *relative coadjunction* [4].

Our precise treatment of polarity and directed equality via doctrines allows us to capture a logical flavour to directedness, thus progressing on the line of work first posed by Lawvere [28] on the precise role of variance for the presheaf doctrine. Our work treats the proof-irrelevant case and, instead of considering the *logic of categories*, we give a characterization for the *logic of preorders* by studing the universal property of directed equality. The focus on the proof-irrelevant *logical* case is justified by the use of *posetal* fibrations in typical treatments of categorical logic [7, 35, 36], and that rewriting, e.g., in logic [38], is also typically relational.

1.2 Related work

The failure of equality in the presheaf (hyper)doctrine was recently revived in a paper by Melliès and Zeilberger [37] under the perspective of linear logic and type refinement systems.

North [8, 41] and Altenkirch and Neumann [3] describe a dependent directed type theory with semantics in Cat, but using groupoidal structure to deal with the polarity problems in the rules for directed equality. Here we instead focus on a first-order proof-irrelevant presentation using preorders, and tackle the issue of variance using a proof-irrelevant version of dinaturality rather than groupoids. Another approach to directed equality is the judgemental one [1, 31], which however does not allow for contraction rules to be described and a universal property to be given. New and Licata [40] give a sound and complete presentation for certain double categorical models of which categories (and therefore preorders) are an instance, but at the cost of heavily restricting the syntax (i.e., a symmetricity statement cannot be formulated). Other approaches to directedness based on synthetic intervals and geometric spaces are given in [17, 46, 52]; our paper focuses on syntactic aspects of a logic, generalizing the abstract doctrinal approach and focusing on the elementary model of preorders instead of using geometric spaces. A system of variances was similarly presented in [43], without however providing a formal or semantic account.

We present the syntax of directed first-order logic in Section 2, showing examples of derivations and theories in Section 3. The categorical semantics is given in Section 4, establishing the notion of directed doctrine to capture polarity and directed equality. Syntax and semantics are finally connected in Section 5, concluding with future work in Section 6.

2 Syntax of Directed Logic

We introduce the syntax of directed first-order logic (dFOL) with a natural deduction-style proof system. The main syntactic judgements for types, terms, formulas and entailments are presented in Figures 3–5. As a guiding intuition, the reader can refer to Figure 2 to see how the syntax of directed first-order logic is semantically interpreted in the preorder model.

Concept	Preorder model	Judgement
Type $A, B, P, N, \ldots$	Preorder $[\![A]\!]$	A type
Context Θ, Δ, Γ	Product of preorders	Γ ctx
Term s, t, η, δ, ρ	Monotone function $[\![\Gamma]\!] \to [\![A]\!]$	$\Gamma \vdash t : A$
Equality of terms	Equivalence of monotone functions	$\Gamma \vdash t' = t : A$
Polarized context $\Theta \mid \Delta \mid \Gamma$	$[\![\Theta \mid \Delta \mid \Gamma]\!] := [\![\Theta]\!]^{\mathsf{op}} \times ([\![\Delta]\!]^{\mathsf{op}} \times [\![\Delta]\!]) \times [\![\Gamma]\!]$	
Formulas φ, ψ	Monotone function $[\![\varphi]\!] := [\![\Theta \mid \Delta \mid \Gamma]\!] \to \mathbf{I}$	$[\Theta \mid \Delta \mid \Gamma]\ \varphi\ \mathsf{prop}$
Directed equality $\leq_A$	Monotone function $\leq_A$: $[\![A]\!]^{\mathsf{op}} \times [\![A]\!] \to \mathbf{I}$	
Implication formula $\Rightarrow$	Monotone function $- \Rightarrow - : \mathbf{I}^{\mathsf{op}} \times \mathbf{I} \to \mathbf{I}$	
Conjunction formula $\wedge$	Monotone function $- \wedge - : \mathbf{I} \times \mathbf{I} \to \mathbf{I}$	
Propositional context Φ	Pointwise product $[\![\Phi]\!] := [\![\varphi_1]\!] \wedge [\![\varphi_2]\!] \wedge \ldots$	$[\Theta \mid \Delta \mid \Gamma]\ \Phi\ \mathsf{propctx}$
Entailment	$\forall n, d, p.\ [\![\Phi]\!](n, d, d, p) \leq [\![\varphi]\!](n, d, d, p)$	$[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \varphi$

Figure 2 Intuition for syntax and preorder semantics of directed first-order logic.

The types and terms of directed first-order logic are given in given in Figure 3, and they are a straightforward axiomatization of simply typed λ-calculus (e.g., [45]) with unit, product, and function types. As customary in logic, we omit weakenings that place formulas in the correct context, which we later make explicit in Section 4 for the doctrinal semantics. Judgements are mutually defined with the notion of *signature/theory* [51] to add "generating" symbols for, e.g., base types, terms, propositions, and axioms.

▶ **Definition 1** (Judgements and signatures). *We list here the main judgements of our logic which are sets inductively defined by the rules in Figures 3–5.*

- *The set of* type derivations *is* $\{A\ \mathsf{type}\}$ *is inductively defined by the judgement "A type" given in Figure 3, along with the judgements for* contexts $\{\Gamma\ \mathsf{ctx}\}$*, i.e. finite lists of types.*
- *The set of* term derivations $\{\Gamma \vdash t : A\}$*, assuming* A type *and* Γ ctx*.*
- *The set of* term equality judgements $\{\Gamma \vdash t = t' : A\}$*, assuming* $\Gamma \vdash t, t' : A$*.*
- *The set of* formula derivations $\{[\Theta \mid \Delta \mid \Gamma]\ \mathsf{prop}\}$ *given in Figure 4 assumes* Θ, Δ, Γ ctx*.*
- *The set of* propositional contexts $\{[\Theta \mid \Delta \mid \Gamma]\ \Phi\ \mathsf{propctx}\}$ *captures finite lists of formulas.*
- *The set of* entailments $\{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \varphi\}$ *is defined inductively in Figure 5.*

Interleaved with these judgements is the definition for the signatures *of our logic. We do not explicitly indicate the dependence between a judgement and the signatures that come before it.*

- *A* type signature Σ_B *is defined as a set of symbols* Σ_B *representing for base types.*
- *A* term signature Σ_F *is defined as a set of function symbols* Σ_F*, along with two functions* $\mathsf{dom}, \mathsf{cod} : \Sigma_F \to \{A\ \mathsf{type}\}$ *that return domain and codomain types of such symbols.*
- *A* term equality signature Σ_E *is a set* Σ_E *with functions* $\mathsf{eqC} : \Sigma_E \to \{\Gamma\ \mathsf{ctx}\}$*,* $\mathsf{eqT} : \Sigma_E \to \{A\ \mathsf{type}\}$*, and* $\mathsf{eqL}, \mathsf{eqR} : (e : \Sigma_E) \to \{\mathsf{eqC}(e) \vdash t : \mathsf{eqT}(e)\}$ *dependent on* e*.*
- *A* formula signature Σ_P *is a set of predicate symbols* Σ_P *with* $\mathsf{neg}, \mathsf{pos} : \Sigma_P \to \{A\ \mathsf{type}\}$*.*
- Axioms *are given by a set* Σ_A *of symbols and three (dependent) functions* $\mathsf{actx} : \Sigma_A \to \{\Gamma\ \mathsf{ctx}\}$*,* $\mathsf{hyp} : (a : \Sigma_A) \to \{[\mathsf{actx}(a)]\ \Phi\ \mathsf{propctx}\}$*,* $\mathsf{conc} : (a : \Sigma_A) \to \{[\mathsf{actx}(a)]\ \mathsf{prop}\}$*.*

▶ **Definition 2** (Theory). *A* theory Σ *is defined as a tuple* $\Sigma = (\Sigma_B, \Sigma_F, \Sigma_E, \Sigma_P, \Sigma_A)$ *which collects together data from all previous signatures.*

We now formally introduce the concepts of *position, variance, and polarity* with their notation.

$$\boxed{A \text{ type}} \quad \frac{C \in \Sigma_B}{C \text{ type}} \quad \frac{A \text{ type} \quad B \text{ type}}{A \times B \text{ type}} \quad \frac{A \text{ type} \quad B \text{ type}}{A \Rightarrow B \text{ type}} \quad \frac{}{\top \text{ type}} \quad \boxed{\Gamma \text{ ctx}} \quad \frac{}{[\,] \text{ ctx}} \quad \frac{\Gamma \text{ ctx} \quad A \text{ type}}{\Gamma, A \text{ ctx}}$$

$$\boxed{\Gamma \vdash t : A} \quad \frac{}{\Gamma, x : A, \Gamma' \vdash x : A} \quad \frac{f \in \Sigma_F \quad \Gamma \vdash t : \mathsf{dom}(f)}{\Gamma \vdash f(t) : \mathsf{cod}(f)} \quad \frac{}{\Gamma \vdash ! : \top} \quad \frac{\Gamma \vdash s : A \quad \Gamma \vdash t : B}{\Gamma \vdash \langle s, t \rangle : A \times B}$$

$$\frac{\Gamma \vdash p : A \times B}{\Gamma \vdash \pi_1(p) : A} \quad \frac{\Gamma \vdash p : A \times B}{\Gamma \vdash \pi_2(p) : B} \quad \frac{\Gamma \vdash s : A \Rightarrow B \quad \Gamma \vdash t : A}{\Gamma \vdash s \cdot t : B} \quad \frac{\Gamma, x : A \vdash t(x) : B}{\Gamma \vdash \lambda x. t(x) : A \Rightarrow B}$$

$$\boxed{\Gamma \vdash t = t' : A} \quad \frac{\Gamma \vdash t : \top}{\Gamma \vdash t = ! : \top} \quad \frac{\Gamma \vdash s : A \quad \Gamma \vdash t : B}{\Gamma \vdash \pi_1(\langle s, t \rangle) = s : A} \quad \frac{\Gamma \vdash s : A \quad \Gamma \vdash t : B}{\Gamma \vdash \pi_2(\langle s, t \rangle) = t : B} \quad \frac{\Gamma \vdash p : A \times B}{\Gamma \vdash \langle \pi_1(p), \pi_2(p) \rangle = p : A \times B}$$

$$\frac{\Gamma, x : A \vdash f(x) : B \quad \Gamma \vdash t : A}{\Gamma \vdash (\lambda x. f(x)) \cdot t = f[x \mapsto t] : B} \quad \frac{\Gamma, x : A \vdash f(x) : B}{\Gamma, x : A \vdash (\lambda x. f(x)) \cdot x = f(x) : B} \quad \frac{E \in \Sigma_E}{\mathsf{eqC}(E) \vdash \mathsf{eqL}(E) = \mathsf{eqR}(E) : \mathsf{eqT}(E)}$$

Figure 3 Syntax of directed first-order logic – types and terms.

$$\boxed{[\Theta \mid \Delta \mid \Gamma]\ \varphi \text{ prop}} \quad \frac{[\Theta \mid \Delta \mid \Gamma]\ \psi \text{ prop} \quad [\Theta \mid \Delta \mid \Gamma] \text{ prop}}{[\Theta \mid \Delta \mid \Gamma]\ \psi \wedge \varphi \text{ prop}} \quad \frac{[\Gamma \mid \Delta \mid \Theta]\ \psi \text{ prop} \quad [\Theta \mid \Delta \mid \Gamma] \text{ prop}}{[\Theta \mid \Delta \mid \Gamma]\ \psi \Rightarrow \varphi \text{ prop}}$$

$$\frac{}{[\Theta \mid \Delta \mid \Gamma]\ \top \text{ prop}} \quad \frac{\Theta, \Delta \vdash s : A \quad \Gamma, \Delta \vdash t : A}{[\Theta \mid \Delta \mid \Gamma]\ s \leq_A t \text{ prop}} \quad \frac{P \in \Sigma_P \quad \Theta, \Delta \vdash s : \mathsf{neg}(P) \quad \Gamma, \Delta \vdash t : \mathsf{pos}(P)}{[\Theta \mid \Delta \mid \Gamma]\ P(s \mid t) \text{ prop}}$$

$$\frac{p \in \{-, \Delta, +\} \quad [\Theta \mid \Delta \mid \Gamma], [x :^p A]\ \varphi(x) \text{ prop}}{[\Theta \mid \Delta \mid \Gamma]\ \exists^p x. \varphi(x) \text{ prop}} \quad \frac{p \in \{-, \Delta, +\} \quad [\Theta \mid \Delta \mid \Gamma], [x :^p A]\ \varphi(x) \text{ prop}}{[\Theta \mid \Delta \mid \Gamma]\ \forall^p x. \varphi(x) \text{ prop}}$$

Figure 4 Syntax of directed first-order logic – formulas.

$$\boxed{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \varphi} \quad \frac{[\Theta \mid \Delta \mid \Gamma]\ \Psi \vdash \psi \quad [\Theta \mid \Delta \mid \Gamma]\ \Phi, \psi, \Phi' \vdash \varphi}{[\Theta \mid \Delta \mid \Gamma]\ \Phi, \Psi, \Phi' \vdash \varphi}\ (\mathsf{cut}) \quad \frac{A \in \Sigma_A}{[\mathsf{actx}(A)]\ \mathsf{hyp}(A) \vdash \mathsf{conc}(A)}\ (\mathsf{axiom})$$

$$\frac{\sigma : \{1, ..., m\} \to \{1, ..., n\} \quad [\Theta \mid \Delta \mid \Gamma]\ \varphi_{\sigma(1)}, ..., \varphi_{\sigma(m)} \vdash \psi}{[\Theta \mid \Delta \mid \Gamma]\ \varphi_1, ..., \varphi_n \vdash \psi}\ (\mathsf{struct}) \quad \frac{\Theta, \Delta \vdash \eta : N \quad \Delta \vdash \delta : D \quad \Gamma, \Delta \vdash \rho : P \quad [\Theta, n : N \mid \Delta, d : D \mid \Gamma, p : P]\ \Phi(n, \overline{d}, d, p) \vdash \varphi(n, \overline{d}, d, p)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi(\eta, \delta, \delta, \rho) \vdash \varphi(\eta, \delta, \delta, \rho)}\ (\mathsf{reindex})$$

$$\frac{}{[\Theta \mid \Delta \mid \Gamma]\ \Phi, \varphi, \Phi' \vdash \varphi}\ (\mathsf{hyp}) \quad \frac{[\Theta, x : N \mid \Delta \mid \Gamma, y : P]\ \Phi \text{ propctx}, \varphi \text{ prop} \quad [\Theta \mid \Delta, x : N, y : P \mid \Gamma]\ \Phi(x, y) \vdash \varphi(x, y)}{[\Theta, x : N \mid \Delta \mid \Gamma, y : P]\ \Phi(x, y) \vdash \varphi(x, y)}\ (\mathsf{upgrade\text{-}to\text{-}nat})$$

$$\frac{}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \top}\ (\top) \quad \frac{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi \quad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \varphi}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi \wedge \varphi}\ (\wedge)$$

$$\frac{[\Theta \mid \Delta, z : A \mid \Gamma] \qquad \Phi \vdash \varphi(\overline{z}, z)}{[\Theta, a : A \mid \Delta \mid \Gamma, b : A]\ a \leq b, \Phi \vdash \varphi(a, b)}\ (\leq) \quad \frac{[\Gamma \mid \Delta \mid \Theta]\ \psi \text{ prop} \quad [\Theta \mid \Delta \mid \Gamma]\ \Phi \text{ propctx}, \varphi \text{ prop} \quad [\bullet \mid \Delta, \Theta, \Gamma \mid \bullet]\ \psi, \Phi \vdash \varphi}{[\Theta \mid \Delta \mid \Gamma] \qquad \Phi \vdash \psi \Rightarrow \varphi}\ (\Rightarrow)$$

$$\frac{p \in \{-, \Delta, +\} \quad [\Theta \mid \Delta \mid \Gamma]\ \exists^p x. \psi(x), \Phi \vdash \varphi}{[\Theta \mid \Delta \mid \Gamma], [x :^p A]\ \psi(x), \Phi \vdash \varphi}\ (\exists) \quad \frac{p \in \{-, \Delta, +\} \quad [\Theta \mid \Delta \mid \Gamma], [x :^p A]\ \Phi \vdash \varphi(x)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \forall^p x. \varphi(x)}\ (\forall)$$

Figure 5 Syntax of directed first-order logic – entailments.

▶ **Definition 3** (Positions in a formula). *We use the name* position *to indicate any point in which a (term) variable can appear in a formula, e.g., there are four possible positions x, y, z, w for variables to appear in the formula "$x \leq y \wedge P(z, f(w))$".*

▶ **Definition 4** (Polarity of a position). *Positions have a* polarity*, which can either be* positive *or* negative*: intuitively, a position starts out as positive, and flips between being positive and negative precisely in the following cases:*

1. *when it occurs on the left of the formula $x \leq y$: e.g., the variable x indicates a negative position in "$x \leq c$" and "$f(x) \leq y$";*
2. *when it occurs on the left of an implication formula $\psi \Rightarrow \varphi$, e.g., the position indicated by x is negative in "$P(x) \Rightarrow \varphi(y)$" and "$Q(f(x)) \Rightarrow \top$";*
3. *when it occurs on the negative side n of a predicate symbol "$P(n \mid p)$", e.g., x is negative in "$P(x \mid \langle p, q\rangle)$" and "$P(f(g(x), y) \mid f(z))$".*

Polarity can be inverted twice, e.g., x occurs positively *in "$x \leq y \Rightarrow \varphi$" and "$(y \leq x \Rightarrow \varphi) \Rightarrow \varphi$".*

From the semantic perspective of preorders, this flipping of polarity corresponds with the presence of the opposite preorder P^{op} on the left side of functors $\leq_P : P^{\mathsf{op}} \times P \to \mathbf{I}$, and $- \Rightarrow - := \leq_{\mathbf{I}} : \mathbf{I}^{\mathsf{op}} \times \mathbf{I} \to \mathbf{I}$ in particular. The fact that $(P^{\mathsf{op}})^{\mathsf{op}} \equiv P$ justifies the fact that inverting a negative variable makes it positive again.

▶ **Definition 5** (Variance of a variable). *Variables can occur in multiple positions at the same time. The* variance *of a variable x is defined to be either* positive, negative*, or* dinatural*:*

- positive *iff every position in which x occurs is positive;*
- negative *iff every position in which x occurs is negative;*
- dinatural *iff every position in which x occurs is positive or negative (i.e., always).*

We will see in Theorem 9 how indeed any *variable can be lifted to be dinatural. In practice, we use the more intuitive idea that a variable x is called "(strictly) dinatural" iff it is neither positive nor negative, i.e., it occurs at least once with both polarities. Variables that are positive* or *negative are called* natural*. We denote the set of variances as $\{-, \Delta, +\}$.*

▶ Remark 6 (On overlines for variables). We will indicate with $\overline{a} : A$ in formulas $\varphi(\overline{a}, a, \overline{b}, b, \ldots)$ to highlight whenever a dinatural variable is used with a different polarity. These are just conveniences and do not exist syntactically. Positive and negative variables will never have an overline, since they always unambiguously appear with the same variance.

The main technical tool that captures this idea of variance is the notion of *polarized context*, which simply consists of physically dividing the variables in the context based on their polarity. This technique lends itself particularly well in the *doctrinal* semantics, which similarly places specific emphasis on the role of contexts and free variables for formulas [22].

▶ **Definition 7** (Polarized context). *A* polarized context *is a triple of contexts $[\Theta \mid \Delta \mid \Gamma]$ for which Θ, Δ, Γ ctx, where, intuitively, Θ is a list of variables that can be used only negatively, variables in Δ are dinatural (i.e., can be used either positively or negatively in any position), and variables in Γ only positively. The extended polarized context $[\Theta \mid \Delta \mid \Gamma], [x :^p A]$ is obtained by appending a variable $x : A$ to the corresponding context with polarity $p \in \{-, \Delta, +\}$.*

We denote context concatenation as Γ, Δ for Γ ctx, Δ ctx and use $\bullet$ for the empty context. We will use the notation $[\Theta, n : N \mid \Delta, d : D \mid \Gamma, p : P]$ $\varphi(n, \overline{d}, d, p)$ to indicate (some of) the free variables of the formula φ, omitting other variables in Θ, Δ, Γ for brevity.

We describe the key idea behind polarized contexts, used in the formation rule of predicates of Figure 4. Polarized contexts are used in the base formulas $s \leq_A t$ and $P(s \mid t)$, for terms s, t. The idea is that s is typed in context Θ, Δ, and t in context Γ, Δ: intuitively, positive positions in formulas can be filled *either* by a *positive* variable, or by a *dinatural* one, i.e., in the term $\Theta, \Delta \vdash x : A$, either $(x : A) \in \Theta$ or $(x : A) \in \Delta$. However, a dinatural variable can only be replaced by another dinatural variable, since it must appear twice. This is captured by the following definition for substitution of terms inside formulas, later used in Section 4.

▶ **Definition 8** (Substitution of terms in formulas)**.** *Given a formula* $[\Theta, n{:}N \,|\, \Delta, d{:}D \,|\, \Gamma, p{:}P]\ \varphi(n, \overline{d}, d, p)$ prop*, we can substitute any given three terms* $(\Theta, \Delta \vdash \eta : N)$*,* $(\Delta \vdash \delta : D)$*,* $(\Gamma, \Delta \vdash \rho : P)$ *for the variables* n, d, p *in* φ*, denoted as* $\varphi(\eta, \delta, \delta, \rho)$*. Substitution is done by induction on formulas, i.e., there is a function* $\mathsf{subst}_{\eta,\delta,\rho} : \{[\Theta, n{:}N \,|\, \Delta, d{:}D \,|\, \Gamma, p{:}P]\ \mathsf{prop}\} \to \{[\Theta \,|\, \Delta \,|\, \Gamma]\ \mathsf{prop}\}$*. This can be extended in the intuitive way to a multi-variable version using* term substitutions $\Gamma \vdash \sigma : \Gamma'$*, i.e., lists of terms* $\Gamma \vdash t : A_i$ *for* $A_i \in \Gamma' = [A_1, ..., A_n]$*. A multi-variable reindexing function* $\mathsf{subst}_{\theta,\delta,\gamma} : \{[\Theta \,|\, \Delta \,|\, \Gamma]\ \mathsf{prop}\} \to \{[\Theta' \,|\, \Delta' \,|\, \Gamma']\ \mathsf{prop}\}$*, detailed in Definition 37, takes three substitutions* $(\Theta', \Delta' \vdash \theta : \Theta')$*,* $(\Delta' \vdash \delta : \Delta')$ $(\Gamma', \Delta' \vdash \gamma : \Gamma')$*.*

▶ **Theorem 9** (Lifting natural to dinatural)**.** *Any natural variable can be lifted to dinatural by substituting* $\mathsf{subst}_{\langle \pi_1, \eta \rangle, \pi_1, \langle \pi_1, \rho \rangle} : \{[\Theta, n{:}N \,|\, \Delta \,|\, \Gamma, p{:}P]\ \mathsf{prop}\} \to \{[\Theta \,|\, \Delta, n{:}N, p{:}P \,|\, \Gamma]\ \mathsf{prop}\}$ *via* $\eta := (\Theta, \Delta, n{:}N, p{:}P \vdash n{:}N)$*,* $\rho := (\Gamma, \Delta, n{:}N, p{:}P \vdash p{:}P)$*, and* $(\Delta, n{:}N, p{:}P \vdash \pi_1 : \Delta)$*.*

▶ **Theorem 10** (Collapsing two naturals in one dinatural)**.** *A formula with two variables of different variance* $[\Theta, a : A \mid \Delta \mid \Gamma, b : A]\ \varphi(a, b)$ *can be collapsed to one with a single dinatural* x *in context* $[\Theta \mid \Delta, x : A \mid \Gamma]\ \varphi(\overline{x}, x)$*, by substituting with* $\eta := (\Theta, \Delta, a : A \vdash \eta : A)$*,* $\rho := (\Gamma, \Delta, a : A \vdash a : A)$*, and* $\pi_1 := (\Delta, a : A \vdash a : \Delta)$*.*

2.1 Rules

We now describe the main rules of the logic, both in formula construction and entailments.

(Structural rules.) Variables can be substituted via (reindex), showing just the one-variable case as in Definition 8. The structural rule (upgrade-to-nat) allows for dinatural variables that are not *used* with both variances (i.e., are *weakened* as dinatural) to be upgraded as natural; note that the entailment in the hypothesis implicitly uses Theorem 9 on both sides.

(Polarized implications.) In directed first-order logic, implication is given a special treatment due to the contravariance of $\Rightarrow$ in its first argument, which we call *polarized implication.* In the preorder semantics, this is motivated by the presence of $\mathbf{I}^{\mathsf{op}}$ in $- \Rightarrow - : \mathbf{I}^{\mathsf{op}} \times \mathbf{I} \to \mathbf{I}$.

This is reflected in the construction of implication formulas $[\Theta \mid \Delta \mid \Gamma]\ \psi \Rightarrow \varphi$ prop, since we swap the negative and positive context in the formula $[\Gamma \mid \Delta \mid \Theta]\ \psi$ prop on the left side.

The intuition behind the rule for entailments $(\Rightarrow)$ is that one can curry a formula ψ between the left and right side of entailments, at the cost of *inverting* the polarity of all the positions in ψ: for example, consider a formula $\psi(x, y, z)$, with x, y negative positions and z positive, and take the entailment $\psi(x, \overline{y}, z) \wedge \Phi(y) \vdash \varphi(z)$ where x has negative variance, y dinatural (negative in ψ and positive in Φ), z positive; by currying ψ, we obtain the entailment $\Phi(y) \vdash \psi(x, y, \overline{z}) \Rightarrow \varphi(z)$, where now x is positive, z becomes dinatural (due to the positive appearance in φ and the *negative* appearance in ψ), and y is now positive.

This idea is concretely implemented by making *all* variables in Φ, ψ, φ *dinatural* in the top side of $(\Rightarrow)$: derived rules that more clearly show the idea of polarity inversion (e.g., $(\Rightarrow^{\pm})$, $(\Rightarrow^{+}_{R})$, $(\Rightarrow_{R})$) follow by (reindex) and (upgrade-to-nat) to make variables natural again.

(Directed equality.) The formula $a \leq_A b$ (where a, b are positions with negative and positive polarity, respectively) is the main construct of dFOL, and intuitively states that a "rewrites" to b: in the preorder model, this is the fact that the relation $a \leq b$ holds. The introduction rule is given by ($\leq$-refl). We illustrate the intuition behind the rule for directed equality contraction ($\leq$): the $\Downarrow$ direction states that a directed equality $a \leq b$ in context can be contracted *only if* a, b only appear *naturally* in the conclusion; using ($\Rightarrow$), a and b can appear in the hypothesis context only if they appear *negatively*, shown in ($\leq^{\Phi}$).

(Polarized quantifiers.) Since polarized contexts keep track of the polarity of variables, quantifiers also need to keep track of this information: we define polarized quantifiers $\forall^{+}.\phi(x)$, $\forall^{-}x.\phi(x)$, $\forall^{\Delta}x.\phi(\overline{x}, x)$, similarly for exists $\exists^{p}.\phi(x)$ for $p \in \{-, \Delta, +\}$, with rules ($\forall$) and ($\exists$). Note that $(\exists^{-}x.x \leq y) \Rightarrow P$ and $\exists^{+}x.(x \leq y \Rightarrow P)$ are both well-formed in the same context.

3 Examples

In this section we provide derived rules and then use them in examples that exemplify the properties of directed equality and polarized implications/quantifiers.

▶ **Example 11** (Derived rules). The rules in Figure 6 are derivable; we report here only (some) statements, leaving complete derivations and other examples in *Appendix D*.

- Rule ($\leq$-refl$_t$) states reflexivity with a general term t; note that t must not depend on any natural variable (as in (reindex)), since variables of t have to appear on both sides of $\leq$.
- Rules ($\leq^{-}$), ($\leq^{+}$) are versions of ($\leq$) which take into account the polarity of the variable being contracted, since ($\leq$) always makes them dinatural.
- Rules ($\Rightarrow^{+}_{R}$), ($\Rightarrow^{\pm}$) intuitively show how polarity inversion follows from ($\Rightarrow$), (upgrade-to-nat).
- Using polarized implications, a general rule ($\leq^{\Phi}$) allows for variables a, b to appear in Φ, however, they must be used only *negatively* because of the swap in polarity. This is reminiscent of a (polarized) Frobenius formulation for equality [22, 3.2.4]. A version with terms is given in ($\leq_t$); the syntactic restriction is the fact that $a, b : A$ are *natural* in Φ, φ.
- The inverse direction of ($\leq$) is logically equivalent to ($\leq$-refl), shown in Appendix F.
- We can use (reindex) to obtain functorial versions ($\Downarrow^{\Delta}_{A,B}$), ($\downdownarrows^{\Delta}_{A}$) of Theorems 9 and 10.
- Typical rules for FOL quantifiers, e.g., ($\exists^{-}_{t}$), ($\forall^{\Delta}_{t}$), follow from ($\exists$), ($\forall$) and (cut), (reindex).

▶ **Example 12** (Directed equality). We illustrate some of the properties specific of directed equality, considering in the signature $A \vdash f : B$ and $P \in \Sigma_F$ s.t. $\mathsf{pos}(P) := A, \mathsf{neg}(P) := \top$.

- Transitivity of directed equality:

$$\dfrac{\dfrac{}{[z : A \mid \bullet \mid c : A] \quad z \leq c \vdash z \leq c}\,(\text{hyp})}{[a : A \mid b : A \mid c : A]\ a \leq b, \overline{b} \leq c \vdash a \leq c}\,(\leq^{-})$$

- Congruence (i.e. monotonicity of terms):

$$\dfrac{\dfrac{}{[\bullet \mid z : A \mid \bullet]\ \vdash f(\overline{z}) \leq_B f(z)}\,(\leq\text{-refl}_t)}{[a : A \mid \bullet \mid b : A]\ a \leq_A b \vdash f(a) \leq_B f(b)}\,(\leq)$$

- Monotonicity for predicates (i.e., transport):

$$\dfrac{\dfrac{}{[\bullet \mid \bullet \mid z : A] \quad P(z) \vdash P(z)}\,(\text{hyp})}{[a : A \mid \bullet \mid b : A]\ a \leq b, P(a) \vdash P(b)}\,(\leq^{+})$$

- Existence of singletons [50, 1.12.1]:

$$\dfrac{\dfrac{\dfrac{}{[\bullet \mid y : A \mid \bullet]\ \vdash \overline{y} \leq y}\,(\leq\text{-refl})}{[\bullet \mid \bullet \mid y : A]\ \vdash \exists^{-}x.x \leq y}\,(\exists^{-}_{t})}{[\bullet \mid \bullet \mid \bullet]\ \vdash \forall^{+}y.\exists^{-}x.x \leq y}\,(\forall)$$

- Pair of rewrites:

$$\dfrac{\dfrac{\dfrac{}{[\bullet \mid x : A, y : B \mid \bullet]\ \vdash (\overline{x}, \overline{y}) \leq_{A \times B} (x, y)}\,(\leq\text{-refl}_t)}{[b : A \mid x : A \mid b' : B]\ b \leq_B b' \vdash (\overline{x}, b) \leq_{A \times B} (x, b')}\,(\leq)}{[a : A, b : A \mid \bullet \mid a' : A, b' : B]\ a \leq_A a',\ b \leq_B b' \vdash (a, b) \leq_{A \times B} (a', b')}\,(\leq)$$

$$\frac{\begin{array}{c}[\Theta, n:N \mid \Delta \mid \Gamma]\ \psi(n)\ \mathsf{prop}\\ \Theta,\Delta \vdash \eta : N \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi(\eta)\end{array}}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \exists^{-} x.\psi(x)}\ (\exists^{-}_{t}) \qquad \frac{\Delta \vdash \delta : D \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \forall^{\Delta} x.\psi(x)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi(\delta,\delta)}\ (\forall^{\Delta}_{t})$$

$$\frac{}{[\Theta \mid \Delta, z:A \mid \Gamma]\ \Phi \vdash \overline{z} \leq z}\ (\leq\text{-refl}) \qquad \frac{[\Theta \mid \Delta, z:A \mid \Gamma] \qquad \Phi(\overline{z},z) \vdash \varphi(\overline{z},z)}{[\Theta \mid \Delta, a:A, b:A \mid \Gamma]\ a \leq b, \Phi(\overline{b},\overline{a}) \vdash \varphi(a,b)}\ (\leq^{\Phi})$$

$$\frac{\begin{array}{c}[\Theta \mid \Delta \mid \Gamma, a:A]\ \Phi(a),\varphi(a)\ \mathsf{prop}\\ [\Theta \mid \Delta \mid \Gamma, z:A] \qquad \Phi(z) \vdash \varphi(z)\end{array}}{[\Theta \mid \Delta, a:A \mid \Gamma, b:A]\ \overline{a} \leq b, \Phi(a) \vdash \varphi(b)}\ (\leq^{+}) \qquad \frac{\begin{array}{c}\Theta,\Delta \vdash \eta : A, \quad \Gamma,\Delta \vdash \rho : A\\ [\Theta, a:A \mid \Delta \mid \Gamma, b:A]\ \Phi(a,b),\ \varphi(a,b)\ \mathsf{prop}\\ [\Theta \mid \Delta, z:A \mid \Gamma] \qquad \Phi(\overline{z},z) \vdash \varphi(\overline{z},z)\end{array}}{[\Theta \mid \Delta \mid \Gamma]\ \eta \leq_{A} \rho,\ \Phi(\eta,\rho) \vdash \varphi(\eta,\rho)}\ (\leq_{t})$$

$$\frac{[\Theta, n:A \mid \Delta \mid \Gamma, p:B]\ \Phi(n,p) \vdash \varphi(n,p)}{[\Theta \mid \Delta, n:A, p:B \mid \Gamma]\ \Phi(n,p) \vdash \varphi(n,p)}\ (\Downarrow^{\Delta}_{A,B}) \qquad \frac{[\Theta, n:A \mid \Delta \mid \Gamma, p:A]\ \Phi(n,p) \vdash \varphi(n,p)}{[\Theta \mid \Delta, a:A \mid \Gamma]\ \Phi(\overline{z},z) \vdash \varphi(\overline{z},z)}\ (\downdownarrows^{\Delta}_{A})$$

$$\frac{\begin{array}{c}[\Theta', x{:}A \mid \Delta \mid \Gamma']\ \psi(x)\ \mathsf{prop}\\ [\Theta \mid \Delta \mid \Gamma, x{:}A]\ \Phi(x)\ \mathsf{propctx}\\ [\Theta,\Theta' \mid \Delta, x{:}A \mid \Gamma,\Gamma']\ \varphi(\overline{x}),\Phi(x) \vdash \varphi(x)\end{array}}{[\Theta,\Gamma' \mid \Delta \mid \Gamma,\Theta', x{:}A]\ \Phi(x) \vdash \psi(x) \Rightarrow \varphi(x)}\ (\Rightarrow^{+}_{R}) \qquad \frac{\begin{array}{c}[x{:}A \mid \Delta \mid y{:}B]\ \varphi(x,y)\ \mathsf{prop}\\ [\Theta, x{:}A \mid \Delta \mid \Gamma, y{:}B]\ \psi(x,y),\Phi \vdash \varphi\end{array}}{[\Theta, y{:}B \mid \Delta \mid \Gamma, x{:}A]\ \Phi \vdash \psi(x,y) \Rightarrow \varphi}\ (\Rightarrow^{\pm})$$

Figure 6 Derivable rules in directed first-order logic.

The converse "directed injectivity of pairs" follows from *congruence* with projections $\pi_{1,2}$.

- Higher-order rewriting:

$$\frac{\dfrac{\dfrac{}{[\bullet \mid h : A \Rightarrow B, x : A \mid \bullet]\ \vdash \overline{h} \cdot \overline{x} \leq_{B} h \cdot x}\ (\leq\text{-refl}_{t})}{[\bullet \mid h : A \Rightarrow B \mid \bullet]\ \vdash \forall^{\Delta} x.\ \overline{h} \cdot \overline{x} \leq_{B} h \cdot x}\ (\forall^{\Delta}_{t})}{[f : A \Rightarrow B \mid \bullet \mid g : A \Rightarrow B]\ f \leq_{A \Rightarrow B} g \vdash \forall^{\Delta} x.\ f \cdot \overline{x} \leq_{B} g \cdot x}\ (\leq)$$

  The other direction is not derivable in general, since it captures a suitable notion of "2-dimensional extensionality" [19], e.g., when A, B are thought of as discrete preorders/sets.

▶ Remark 13 (Failure of symmetry). The rule for directed equality contraction $(\leq)$ cannot be used to derive symmetry, since in the entailment $[\bullet \mid a : A, b : A \mid \bullet]\ \overline{a} \leq b \vdash \overline{b} \leq a$ both a and b appear *dinaturally*. As we will see in Example 25, the Preord model is a countermodel.

▶ Remark 14 (Symmetric equality). We define $a = b := \overline{a} \leq b \wedge \overline{b} \leq a$ in context $[\bullet \mid a, b : A \mid \bullet]$. Equality is symmetric: $[\Theta \mid \Delta, a, b : A \mid \Gamma]\ \overline{a} \leq b \wedge \overline{b} \leq a \vdash \overline{b} \leq a \wedge \overline{a} \leq b$ follows by swap. We prove in Appendix G that $=$ is precisely the left adjoint to contraction in dinatural contexts.

We give examples of signatures to show how dFOL can be used to model directed structure. A suitable extension (e.g., a two-level system [40]) could be used to show non-trivial theorems.

▶ **Example 15** (Theory of λ-terms). We capture a signature of λ-terms in HOAS-style [15]:

- $\Sigma_{A} := \{T\}$ with a type of (untyped) λ-terms,
- $\Sigma_{F} := \{\tilde{\lambda}, \mathsf{app}\}$ for λ-abstraction and application, with $\mathsf{dom}(\tilde{\lambda}) := T \Rightarrow T$, $\mathsf{cod}(\tilde{\lambda}) := T$ and $\mathsf{dom}(\mathsf{app}) := T \times T, \mathsf{cod}(\mathsf{app}) := T$.
- $\Sigma_{E} := \{\eta\}$, such that: $\dfrac{}{[f : T \Rightarrow T]\ \big(\lambda x.\mathsf{app}(\tilde{\lambda}(f), x)\big) = f : T \Rightarrow T}\ (\eta)$
- $\Sigma_{P} := \{\}, \Sigma_{A} := \{\beta\}$, such that: $\dfrac{}{[\bullet \mid s : T \Rightarrow T, t : T \mid \bullet]\ \mathsf{app}(\tilde{\lambda}(\overline{s}), \overline{t}) \leq s \cdot t}\ (\beta)$

We can automatically derive that app and λ are congruences w.r.t. β-reduction:

$$\dfrac{\dfrac{}{[\bullet \mid z : T, t : T \mid \bullet] \vdash \mathsf{app}(\overline{z}, \overline{t}) \leq_T \mathsf{app}(z, t)}\ (\leq\text{-refl}_t)}{[s : T \mid t : T \mid s' : T]\ s \leq_T s' \vdash \mathsf{app}(s, \overline{t}) \leq_T \mathsf{app}(s', t)}\ (\leq)$$

Semantically, the type T can be interpreted by the preorder of λ-terms ordered by β-relation, as an example of the semantics given in Section 4.

▶ **Example 16** (Domain theory). We give a signature of ω-CPOs$_\perp$ from domain theory [16]:

- $\Sigma_A := \{D, \omega\}$, with a domain D and the chain ω, i.e., chains in D are just terms $\omega \Rightarrow D$.
- $\Sigma_F := \{0, \mathsf{succ}\}$ defined in the intuitive way, $\Sigma_E, \Sigma_P := \{\}$, $\Sigma_A := \{\omega_{\mathsf{axiom}}, \perp_{\mathsf{axiom}}, \bigsqcup_{\mathsf{axiom}}\}$,

$$\dfrac{}{\vdash \forall^\Delta(i : \omega).\ \overline{i} \leq \mathsf{succ}(i)}\ (\omega_{\mathsf{axiom}}) \qquad \dfrac{}{\vdash \exists^- b.\ \forall^+ x.\ b \leq_D x}\ (\perp_{\mathsf{axiom}})$$

$$\dfrac{}{\begin{array}{c}\vdash \forall^\Delta(c : \omega \Rightarrow D).\ \exists^\Delta(b : D).\ (\forall^-(i : \omega).\ \overline{c} \cdot i \leq b) \\ \wedge\ (\forall^+(b' : D).(\forall^-(i : \omega).\ \overline{c} \cdot i \leq b) \Rightarrow \overline{b} \leq b')\end{array}}\ (\textstyle\bigsqcup_{\mathsf{axiom}})$$

4 Doctrinal Semantics

In this section we introduce the doctrinal semantics of dFOL, i.e., directed doctrines. The idea is that reindexing in directed doctrines captures precisely the reindexing of Definition 8. We do this simply by "augmenting" the base category of doctrines via the so-called *polarization category*, and then ask for adjunctions hold w.r.t. to these special reindexings.

▶ **Definition 17.** *We define the* polarization category $\mathsf{ndp}(\mathbb{C})$ *for a cartesian category* $\mathbb{C}$. *This lifts to a functor* $\mathsf{ndp}(-) : \mathsf{CC} \to \mathsf{CC}$ *on the category of (small) categories with products, where we denote diagrammatic composition as* $f\,;g$ *and pairing of products as* $\langle f, g\rangle$*:*

- *Objects: triples of objects* $(\Theta \mid \Delta \mid \Gamma) \in \mathbb{C}_0 \times \mathbb{C}_0 \times \mathbb{C}_0$,
- *Morphisms* $(\Theta \mid \Delta \mid \Gamma) \to (\Theta' \mid \Delta' \mid \Gamma')$ *are triples* $(n : \Theta \times \Delta \to \Theta' \mid d : \Delta \to \Delta' \mid p : \Gamma \times \Delta \to \Gamma')$,
- *Identities are given by* $(\pi_1 \mid \mathsf{id}_\Delta \mid \pi_1)$,
- *Composition:* $(n \mid d \mid p)\,; (n' \mid d' \mid p')$ *is defined as* $(\langle n, \pi_2\,; d\rangle\,; n' \mid d\,; d' \mid \langle p, \pi_2\,; d\rangle\,; p')$.

We describe semantic versions of Theorems 9 and 10 to characterize $\leq_A$ in Definition 23:

▶ **Definition 18** (Dinatural lift). *There is a functor* $\mathcal{P}(\Downarrow^\Delta_{N,P}) : \mathcal{P}(\Theta \times N \mid \Delta \mid \Gamma \times P) \to \mathcal{P}(\Theta \mid \Delta \times N \times P \mid \Gamma)$ *by reindexing with* $\Downarrow^\Delta_{N,P} := (\langle \pi_1, \pi_3\rangle : \Theta \times \Delta \times N \times P \to \Theta \times N \mid \pi_1 : \Delta \times N \times P \to \Delta \mid \langle \pi_1, \pi_4\rangle : \Gamma \times \Delta \times N \times P \to \Gamma \times P) : (\Theta \mid \Delta \times N \times P \mid \Gamma) \to (\Theta \times N \mid \Delta \mid \Gamma \times P)$. *This weakening is precisely the one used in the top entailment of* (upgrade-to-nat).

▶ **Definition 19** (Dinatural contract). *There is a functor* $\mathcal{P}(\Downarrow\!\!\!\downarrow^\Delta_A) : \mathcal{P}(\Theta \times A \mid \Delta \mid \Gamma \times A) \to \mathcal{P}(\Theta \mid \Delta \times A \mid \Gamma)$ *given by reindexing with* $\Downarrow\!\!\!\downarrow^\Delta_A := (\langle \pi_1, \pi_3\rangle : \Theta \times \Delta \times A \to \Theta \times A \mid \pi_1 : \Delta \times A \to \Delta \mid \langle \pi_1, \pi_3\rangle : \Gamma \times \Delta \times A \to \Gamma \times A) : (\Theta \mid \Delta \times A \mid \Gamma) \to (\Theta \times A \mid \Delta \mid \Gamma \times A)$.

A technical condition on objects of doctrines is necessary because of the base case for predicates $P(s \mid t)$; this is non-standard, and is used to obtain initiality for directed doctrines.

▶ **Definition 20** (No-dinatural-variance condition). *A functor* $\mathcal{P} : \mathsf{ndp}(\mathbb{C})^{\mathsf{op}} \to \mathsf{Pos}$ *is said to satisfy the* (strict) no-dinatural-variance condition (ndv) *if the "collapse everything to dinatural" functor* $\mathcal{P}(\Downarrow^\pm_\Delta) : \mathcal{P}(\Theta \times \Delta \mid \top \mid \Gamma \times \Delta) \to \mathcal{P}(\Theta \mid \Delta \mid \Gamma)$ *induced by reindexing with* $\Downarrow^\pm_\Delta := (\mathsf{id}_{\Theta \times \Delta} \mid\, !_\Delta \mid \mathsf{id}_{\Gamma \times \Delta})$ *is a* bijection on objects, *where* $\varepsilon : \mathcal{P}(\Theta \mid \Delta \mid \Gamma)_0 \to \mathcal{P}(\Theta \times \Delta \mid \top \mid \Gamma \times \Delta)_0$ *denotes the inverse function of sets such that* $\mathcal{P}(\Downarrow^\pm_\Delta)(\varepsilon(p)) = p$.

This is usually *not* an isomorphism of posets since, in the syntactic model, $\forall^{\Delta} d.\varphi(a, d, \overline{d}, b) \vdash \varphi(a, \overline{d}, d, b)$ might not imply that $\forall^{+} d.\forall^{-} d'.\varphi(a, d', d, b) \vdash \varphi(a, d', d, b)$. The converse holds by $\mathcal{P}(\Downarrow^{\pm}_{\Delta})$. The intuition is that any dinatural variable $x : A$ in a predicate arises as a dinatural collapse, since only P and P^{op} are preorders (with no third option): hence, the case $P(s \mid t)$ (and $s \leq_A t$) only depends on $\mathsf{pos}(P)$ and $\mathsf{neg}(P)$, without asking for a type "$\mathsf{dinat}(P)$".

▶ **Definition 21** (Polarized doctrine). *A (split)* polarized doctrine *is a cartesian (closed) category $\mathbb{C}$ with a functor $\mathcal{P} : \mathsf{ndp}(\mathbb{C})^{\mathsf{op}} \to \mathsf{Pos}$ that satisfies the* no-dinatural-variance *condition and, moreover, interprets the rule* (upgrade-to-nat)*: if $\mathcal{P}(\Downarrow^{\Delta}_{N,P})(\psi) \leq \mathcal{P}(\Downarrow^{\Delta}_{N,P})(\varphi)$, then $\psi \leq \varphi$, i.e., the monotone function $\mathcal{P}(\Downarrow^{\Delta}_{N,P})$ is monotone injective for every N, P.*

Closedness of $\mathbb{C}$ is only used to interpret function types in Examples 15 and 16.

▶ **Definition 22** (Weakenings). *Given a polarity $p \in \{-, \Delta, +\}$ we shall denote with $\mathcal{P}(\Theta \mid \Delta \mid \Gamma \mid\mid^{p} A)$ the fiber obtained by applying the functor $- \times A$ to either Θ, Δ, Γ depending on p in the intuitive way. We denote weakening functors by $\mathsf{wk}^{p}_{A} : \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \to \mathcal{P}(\Theta \mid \Delta \mid \Gamma \mid\mid^{p} A)$.*

▶ **Definition 23** (Logical connectives). *We define conditions on a polarized doctrine $\mathcal{P}$. For each definition we require reindexing functors to preserve all structure up-to-equivalence. Details on relative (co)adjunctions for categories are recalled in Appendix E, [4].*

- $\mathcal{P}$ has conjunctions *iff each fiber has finite products (i.e., glbs in preorders). By a standard argument [22] this induces a functor $- \wedge - : \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \times \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \to \mathcal{P}(\Theta \mid \Delta \mid \Gamma)$.*
- $\mathcal{P}$ has polarized implications ($\Rightarrow$) *iff, given the functor $\mathcal{P}(\Downarrow^{\Delta}_{\Theta,\Gamma}) : \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \to \mathcal{P}(\top \mid \Delta \times \Theta \times \Gamma \mid \top)$ for every $\psi \in \mathcal{P}(\Gamma \mid \Delta \mid \Theta)$ there is a right $\mathcal{P}(\Downarrow^{\Delta}_{\Theta,\Gamma})$-relative coadjoint, denoted as $\psi \Rightarrow -$, to the functor sending $\Phi \mapsto \mathcal{P}(\Downarrow^{\Delta}_{\Gamma,\Theta})(\psi) \times \mathcal{P}(\Downarrow^{\Delta}_{\Theta,\Gamma})(\Phi)$. In the posetal case, this corresponds to the following bi-implication, i.e., for $\Phi, \varphi \in \mathcal{P}(\Theta \mid \Delta \mid \Gamma)$:*

$$\frac{[\top \mid \Delta \times \Theta \times \Gamma \mid \top]\ \mathcal{P}(\Downarrow^{\Delta}_{\Gamma,\Theta})(\psi) \times \mathcal{P}(\Downarrow^{\Delta}_{\Theta,\Gamma})(\Phi) \leq \mathcal{P}(\Downarrow^{\Delta}_{\Theta,\Gamma})(\varphi)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \leq \psi \Rightarrow \varphi}\ (\Rightarrow)$$

  where the relative coadjunction $\psi \Rightarrow - \vdash_{\mathcal{P}(\Downarrow^{\Delta}_{\Theta,\Gamma})} \mathcal{P}(\Downarrow^{\Delta}_{\Gamma,\Theta})(\psi) \times \mathcal{P}(\Downarrow^{\Delta}_{\Theta,\Gamma})(-)$ is the following:

$$\begin{tikzcd}
 & \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \arrow[dr, "\mathcal{P}(\Downarrow^{\Delta}_{\Gamma,\Theta})(\psi) \times \mathcal{P}(\Downarrow^{\Delta}_{\Theta,\Gamma})(-)"] & \\
\mathcal{P}(\Theta \mid \Delta \mid \Gamma) \arrow[ur, "\psi \Rightarrow -"] \arrow[rr, "\mathcal{P}(\Downarrow^{\Delta}_{\Theta,\Gamma})"'] \arrow[r, phantom, shift left=4, "\vdash"] & & \mathcal{P}(\top \mid \Delta \times \Theta \times \Gamma \mid \top)
\end{tikzcd}$$

  By a standard argument, $- \Rightarrow - : \mathcal{P}(\Gamma|\Delta|\Theta)^{\mathsf{op}} \times \mathcal{P}(\Theta|\Delta|\Gamma) \to \mathcal{P}(\Theta|\Delta|\Gamma)$ is a functor.
- $\mathcal{P}$ has polarized quantifiers *iff for every $p \in \{-, \Delta, +\}$ and $A \in \mathbb{C}$ the functor $\mathsf{wk}^{p}_{A} : \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \to \mathcal{P}(\Theta \mid \Delta \mid \Gamma \mid\mid^{p} A)$ has a left $\exists^{p}_{A}$ and a right adjoint $\forall^{p}_{A}$ (i.e., $\exists^{p}_{A} \dashv \mathsf{wk}^{p}_{A} \dashv \forall^{p}_{A}$). Moreover, we ask for suitable Beck-Chevalley conditions, i.e., for $\exists^{p}_{A}$, for any $f : (\Theta \mid \Delta \mid \Gamma) \to (\Theta' \mid \Delta' \mid \Gamma')$, $\varphi \in \mathcal{P}(\Theta' \mid \Delta' \mid \Gamma' \mid\mid^{p} A)$ the following points of $\mathcal{P}(\Theta \mid \Delta \mid \Gamma)$ must be equivalent: $\mathcal{P}(f)(\exists^{p}_{A[\Theta'\Delta'\Gamma']}(\varphi)) \Leftrightarrow \exists^{p}_{A[\Theta\Delta\Gamma]}(\mathcal{P}(f \mid\mid^{p} \mathsf{id}_{A})(\varphi))$. The morphism $(f \mid\mid^{p} \mathsf{id}_{A}) : (\Theta \mid \Delta \mid \Gamma \mid\mid^{p} A) \to (\Theta' \mid \Delta' \mid \Gamma' \mid\mid^{p} A)$ reindexes with f but leaves A untouched. We omit a Frobenius condition that follows the standard case [22, 1.9.12(i)].*
- $\mathcal{P}$ has directed equality *iff, given the functors $\mathcal{P}(\downdownarrows^{\Delta}_{A}) : \mathcal{P}(\Theta \times A \mid \Delta \mid \Gamma \times A) \to \mathcal{P}(\Theta \mid \Delta \times A \mid \Gamma)$ and $\mathcal{P}(\mathsf{wk}^{\Delta}_{A}) : \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \to \mathcal{P}(\Theta \mid \Delta \times A \mid \Gamma)$ there is a $\mathcal{P}(\mathsf{wk}^{\Delta}_{A})$-relative left adjoint $\leq_A \times -$ to the contraction functor $\mathcal{P}(\downdownarrows^{\Delta}_{A})$, i.e., $\leq_A \times - \dashv_{\mathcal{P}(\mathsf{wk}^{\Delta}_{A})} \mathcal{P}(\downdownarrows^{\Delta}_{A})$ holds. In the posetal case this is a bi-implication, for $\Phi \in \mathcal{P}(\Theta \mid \Delta \mid \Gamma), \varphi \in \mathcal{P}(\Theta \times A \mid \Delta \mid \Gamma \times A)$:*

$$\begin{tikzcd} & \mathcal{P}(\Theta \times A \mid \Delta \mid \Gamma \times A) \arrow[dr, "\mathcal{P}(\Downarrow^{\Delta}_{A})"] & \\ \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \arrow[ur, "\leq_A \times -"] \arrow[rr, "\mathcal{P}(\mathsf{wk}^{\Delta}_{A})"'] \arrow[rr, phantom, "\dashv", shift left=20] & & \mathcal{P}(\Theta \mid \Delta \times A \mid \Gamma) \end{tikzcd} \qquad \frac{[\Theta \mid \Delta \times A \mid \Gamma]\ \mathcal{P}(\mathsf{wk}^{\Delta}_{A})(\Phi) \leq \mathcal{P}(\Downarrow^{\Delta}_{A})(\varphi)}{[\Theta \times A \mid \Delta \mid \Gamma \times A]\ \leq_A \times \Phi \leq \varphi}\ (\leq)$$

Moreover, we ask for the following Beck-Chevalley condition [22, 3.4.1]; for any map $f := (n \mid d \mid p) : (\Theta \mid \Delta \mid \Gamma) \to (\Theta' \mid \Delta' \mid \Gamma')$ *and* $\varphi \in \mathcal{P}(\Theta' \mid \Delta' \mid \Gamma')$*, the predicates* $\leq_{A[\Theta\Delta\Gamma]} (\mathcal{P}(f)(\varphi)) \Leftrightarrow \mathcal{P}(n \times \mathsf{id}_A \mid d \mid p \times \mathsf{id}_A)(\leq_{A[\Theta'\Delta'\Gamma']} (\varphi))$ *in* $\mathcal{P}(\Theta \times A \mid \Delta \mid \Gamma \times A)$ *are asked to be equivalent, where* $\mathcal{P}(n \times \mathsf{id}_A \mid d \mid p \times \mathsf{id}_A) : \mathcal{P}(\Theta' \times A \mid \Delta' \mid \Gamma' \times A) \to \mathcal{P}(\Theta \times A \mid \Delta \mid \Gamma \times A)$ *is defined in the intuitive way, applying* f *and leaving the variables* A *unaltered. A "polarized Frobenius" condition, exemplified in* ($\leq^{\Phi}$) *(i.e., only variables are only used negatively in context* Φ*), follows automatically using polarized implications [22, 1.9.12(i)].*

Note the similarity between the above relative adjunction and standard accounts of equality in categorical logic [22, 34, 36]. In our case we cannot relate directed equality to existentials and their characterization as left adjoints [35], since the former is given by a *relative* adjunction and the latter by a standard adjunction, respectively.

A binary attribute of mixed variance (following the idea of Lawvere in Appendix B) can be recovered in each fiber by taking $\top \in \mathcal{P}(\top \mid \top \mid \top)$ and $\leq_A \times \top : \mathcal{P}(A \mid \top \mid A)$, where indeed the two variables are now separated by their polarity.

Thanks to the theory of relative (co)adjunctions [4], having polarized implications and directed equality is property and not structure on a directed doctrine [22].

▶ **Definition 24** (Directed doctrine). *A* (split) directed doctrine *is defined to be a* $(\top, \wedge, \Rightarrow, \exists^p, \forall^p, \leq)$*-polarized doctrine, i.e., it satisfies all above conditions.*

4.1 Examples of directed doctrines

We present the main model of directed first-order logic by interpreting types as preorders:

▶ **Example 25** (Directed doctrine of preorders). The directed doctrine $\mathcal{P}\mathsf{reord} : \mathsf{ndp}(\mathsf{Preord})^{\mathsf{op}} \to \mathsf{Pos}$ of preorders has as base category (i.e. types/terms) the category Preord of preorders and monotone functions, and is defined by $\mathcal{P}\mathsf{reord}(N \mid D \mid P) := [N^{\mathsf{op}} \times (D^{\mathsf{op}} \times D) \times P, \mathbf{I}]_{\Delta}$, where this poset is defined as having objects monotone maps $N^{\mathsf{op}} \times (D^{\mathsf{op}} \times D) \times P \to \mathbf{I}$ and $\psi \leq \varphi := \forall n{:}N, d{:}D, p{:}P. \psi(n, d, d, p) \leq \varphi(n, d, d, p)$. The idea for entailments is reminiscent of dinatural transformations [12], since we use the fact that $A_0^{\mathsf{op}} = A_0$ have the same objects.

- The *ndv* condition is satisfied, since $N^{\mathsf{op}} \times (D^{\mathsf{op}} \times D) \times P \cong (N \times D)^{\mathsf{op}} \times \top \times (P \times D)$.
- Rule (upgrade-to-nat) is validated: if φ factors through the projection that forgets D or D^{op}, an entailment can simply be rephrased by having $N \times D$ or $P \times D$ in the domain.
- Conjunction $\psi \wedge \varphi$ is interpreted by the pointwise product of monotone functions in $\mathbf{I}$, and similarly implications $\psi \Rightarrow \varphi$ by postcomposing $\langle \psi^{\mathsf{op}}, \varphi \rangle$ with $\leq_{\mathbf{I}}: \mathbf{I}^{\mathsf{op}} \times \mathbf{I} \to \mathbf{I}$.
- Reindexing on formulas is given by precomposition of monotone functions: given $\eta : N \times D \to N', \delta : D \to D', \rho : P \times D \to P'$ and $\varphi : [N^{\mathsf{op}} \times (D^{\mathsf{op}} \times D) \times P, \mathbf{I}]$, the resulting substitution is given by $\mathcal{P}(\eta, \delta, \rho)(\varphi)(n, d', d, p) := \varphi(\eta^{\mathsf{op}}(n, d'), \delta^{\mathsf{op}}(d), \delta(d), \rho(p, d))$.
- Polarized quantifiers are given using indexed products and coproducts in $\mathbf{I}$ (i.e., glb and lub) in the intuitive way; we illustrate the idea with the following examples: given $\varphi : [N^{\mathsf{op}} \times (D^{\mathsf{op}} \times D) \times (P \times A), \mathbf{I}]$, $\psi : [N^{\mathsf{op}} \times ((D \times A)^{\mathsf{op}} \times (D \times A)) \times P, \mathbf{I}]$ we define $\exists^{+}(\varphi)(n, d', d, p) := \bigvee_{a \in A} \varphi(n, d', d, p, a) \in \mathbf{I}$ and $\forall^{\Delta}(\psi)(n, d', d, p) := \bigwedge_{a \in A} \psi(n, (d', a), (d, a), p) \in \mathbf{I}$. For $\forall^{\Delta}$ we again rely on $A_0^{\mathsf{op}} = A_0$, in a way that resembles a decategorification of ends [32].

- Moreover, this doctrine has directed equality, interpreted by monotone functions $\leq_A$: $[(\Theta \times A)^{\mathsf{op}} \times (\Delta^{\mathsf{op}} \times \Delta) \times (\Gamma \times A), \mathbf{I}]$ defined by $\leq_A := ((n,a),(d,d'),(p,a')) \mapsto a \leq_A a' \in \mathbf{I}$.
- Rule ($\leq$-refl) corresponds to $\forall a.\top \leq_{\mathbf{I}} (a \leq_A a)$, and the relative adjunction for directed equality is interpreted with the following construction: assume $f : \mathcal{P}\mathsf{reord}(\Theta, \Delta, \Gamma), g : \mathcal{P}\mathsf{reord}(\Theta \times A, \Delta, \Gamma \times A)$, such that $\forall a : A, n : N, d : D, p : P. f(n,d,d,p) \leq g((n,a),d,d,(p,a))$. Note that dinatural collapse is applied on g. We show $\forall (n,a) : N \times A, d : D, (p,a') : P \times A.\ (a \leq_A a') \wedge f(n,d,d,p) \leq_{\mathbf{I}} g((n,a),d,d,(p,a'))$: assuming $a \leq a'$, we use monotonicity of g to obtain $f(n,d,d,p) \leq g((n,a),d,d,(p,a)) \leq g((n,a),d,d,(p,a'))$ as desired. This proof motivates the syntactic restriction for a, b to appear naturally in the conclusion, since we rely on monotonicity of either a or a', *but not both*, to use the hypothesis.

▶ **Definition 26** (Syntactic directed doctrine). *Given a theory* Σ*, we define the* syntactic doctrine $\mathsf{Syn}(\Sigma) : \mathsf{ndp}(\mathsf{Ctx})^{\mathsf{op}} \to \mathsf{Pos}$ *as the syntactic directed doctrine inductively generated from* Σ*, as follows:* Ctx *is the category where objects are contexts* Γ, Δ*, morphisms are term substitutions, and* $\mathcal{P}(\Theta \mid \Delta \mid \Gamma) := \{[\Theta \mid \Delta \mid \Gamma]\ \Phi\ \mathsf{propctx}\}$*, where the poset relation is given by multi-entailment judgements* $\Phi \vdash \Psi$*, defined just like substitutions.*

The following lemma establishes the *ndv* condition for Syn; since we want to show initiality, the fact that Syn satisfies this indicates that directed doctrines also require this in general.

▶ **Theorem 27** (Syn has no-dinatural-variance). *The syntactic doctrine* $\mathsf{Syn}(\Sigma)$ *satisfies the* ndv *condition, i.e., there is a bijection* $\{[\Theta \mid \Delta \mid \Gamma]\ \varphi\ \mathsf{prop}\} \cong \{[\Theta, \Delta \mid \bullet \mid \Gamma, \Delta]\ \varphi\ \mathsf{prop}\}$.

Proof. By induction, using that $\{(\Theta, \Delta), \bullet \vdash A\} \cong \{\Theta, \Delta \vdash A\}$ in $\leq_A$ and $P(s \mid t)$. ◀

▶ **Definition 28** (Set doctrines). *The* Set *doctrine of sets and subsets (e.g., [34]) can be lifted to a directed doctrine by precomposing the discrete poset functor* $\mathsf{Set} \to \mathsf{Pos}$.

▶ **Theorem 29.** *A directed doctrine* $\mathcal{P} : \mathsf{ndp}(\mathbb{C})^{\mathsf{op}} \to \mathsf{Pos}$ *induces a doctrine* $\mathcal{P}' : \mathbb{C}^{\mathsf{op}} \to \mathsf{Pos}$ *by precomposing with the functor* $\Downarrow : \mathbb{C} \to \mathsf{ndp}(\mathbb{C})$ *given by* $C \mapsto (\top \mid C \mid \top)$*; intuitively, this captures the non-directed "sub-logic" inside dinatural contexts. Such doctrine will have* $\wedge$, $\Rightarrow$, $\forall$, $\exists$, $=$ *(defined as in Remark 14) if the directed doctrine equipped with such structure (only "dinatural" quantifiers are needed). A doctrine can be lifted to a directed one by precomposing with* $\Uparrow : \mathsf{ndp}(\mathbb{C}) \to \mathbb{C}$ *given by* $(\Theta \mid \Delta \mid \Gamma) \mapsto \Theta \times \Delta \times \Delta \times \Gamma$*, satisfying the* ndv *condition.*

5 Interpretation

We now define models for the syntax and describe the interpretation of a theory in doctrines. These mostly follow the standard approach of functorial semantics à-la-Lawvere [22, 25].

▶ **Definition 30.** *We denote the (2-)category of directed doctrines and their morphisms as* $\mathsf{DDoctrine}$*, as in [35]: a* morphism of directed doctrines $\mathcal{P} \to \mathcal{P}'$ *for* $\mathcal{P} : \mathsf{ndp}(\mathbb{C})^{\mathsf{op}} \to \mathsf{Pos}$, $\mathcal{P}' : \mathsf{ndp}(\mathbb{D})^{\mathsf{op}} \to \mathsf{Pos}$ *is defined as a pair* (F, α)*, where* $F : \mathbb{C} \to \mathbb{D}$ *is a functor that preserves CCC structure, and* $\alpha : P \Rightarrow \mathsf{ndp}(F)^{\mathsf{op}} ; P'$ *is a natural transformation such that each functor* $\alpha_{\Theta,\Delta,\Gamma} : P(\Theta \mid \Delta \mid \Gamma) \to P'(F(\Theta), F(\Delta), F(\Gamma))$ *preserves all the structure (i.e., terminals, products, directed equality, polarized implications and quantifiers) present in each fiber [22]. For instance, preservation of directed equality means that* $\alpha_{\Theta,\Delta,\Gamma}(\leq_A \times \varphi) = \mathcal{P}'(\cong_{\Theta A} \mid \mathsf{id} \mid \cong_{\Gamma A})(\leq'_{F(A)} \times F(\varphi))$ *where* $\cong_{AB} := \langle F(\pi_1), F(\pi_2) \rangle : F(A \times B) \to F(A) \times F(B)$.

A 2-cell $(F, \alpha) \Rightarrow (G, \beta)$ *is defined by a natural transformation* $\theta : F \Rightarrow G$ *such that* $\alpha_{\Theta,\Delta,\Gamma}(p) \leq \mathcal{P}'(\theta_{\Theta,\Delta} \mid \theta_\Delta \mid \theta_{\Gamma,\Delta})(\beta_{\Theta,\Delta,\Gamma}(p))$*, where* $\theta_{\Theta,\Delta}, \theta_{\Gamma,\Delta}$ *are given in the intuitive way.*

▶ **Definition 31.** *We define the* 1-category of theories Theory *where objects are theories* Σ *and a* morphism of theories $M : \Sigma \to \Sigma'$ *is defined as a tuple of set functions* $M := (b : \Sigma_B \to \Sigma'_B, p : \Sigma_P \to \Sigma'_P, f : \Sigma_F \to \Sigma'_F)$ *such that*

- $f \,;\, \mathsf{dom}' \cong \mathsf{dom} \,;\, [\![-]\!]_b$ *and* $f \,;\, \mathsf{cod}' \cong \mathsf{cod} \,;\, [\![-]\!]_b$ *are isomorphisms of sets,*
- $\forall e \in \Sigma_E, [\![\mathsf{actx}(e)]\!]_{\mathsf{bctx}} \vdash [\![\mathsf{eqL}(e)]\!]_f = [\![\mathsf{eqR}(e)]\!]_f : [\![\mathsf{eqt}(e)]\!]_b$ *is derivable using* Σ' *as theory,*
- $p \,;\, \mathsf{pos}' \cong \mathsf{pos} \,;\, [\![-]\!]_b$ *and* $p \,;\, \mathsf{neg}' \cong \mathsf{neg} \,;\, [\![-]\!]_b$*,*
- $\forall a \in \Sigma_A, [[\![\mathsf{actx}(a)]\!]_{\mathsf{bctx}}]\ [\![\mathsf{hyp}(a)]\!]_{\mathsf{pctx}} \vdash [\![\mathsf{conc}(a)]\!]_{\mathsf{pctx}}$ *is derivable using* Σ' *as theory,*

denoting the translations on types as $[\![-]\!]_b$*, contexts* $[\![-]\!]_{\mathsf{bctx}}$*, terms* $[\![-]\!]_f$*, formulas* $[\![-]\!]_p$*.*

▶ **Theorem 32** (Initial theory). *There is a theory* $\emptyset$ *defined by always choosing the empty set in Definition 2. Clearly* $\emptyset$ *is the initial object in the category of theories* Theory*.*

▶ **Definition 33** (Underlying theory). *Given a directed doctrine* $\mathcal{P} : \mathsf{ndp}(\mathbb{C})^{\mathsf{op}} \to \mathsf{Pos}$*, we define the* underlying theory $\mathsf{Lang}(\mathcal{P}) \in \mathsf{Theory}$ *as follows:*

- *base types* $\Sigma_B := \mathbb{C}_0$ *the objects of* $\mathbb{C}$*,*
- *base terms* $\Sigma_F := \mathbb{C}_1$ *is the set of arrows* $\mathbb{C}_0$ *of* $\mathbb{C}$ *with their intuitive* $\mathsf{dom}, \mathsf{cod}$ *functions;*
- *term equalities* $\Sigma_E := \{(A, \Gamma, s, t) \mid s, t \in \{\Gamma \vdash t : A\}, [\![t]\!]_f = [\![s]\!]_f\}$ *are given by the set of terms (formed with* Σ_B/Σ_F*) for which their induced interpretation* $[\![-]\!]_f$ *is equal in* $\mathbb{C}$*.*
- *the set of base formulas* $\Sigma_P := \coprod_{k \in C_0 \times C_0 \times C_0} \mathcal{P}(k)_0$ *is the set of all objects in all fibers* $\mathcal{P}(\Theta \mid \Gamma \mid \Delta)$*, where* $\mathsf{pos}(\Theta, \Delta, \Gamma, p) := \Theta \times \Delta$ *and* $\mathsf{neg}(\Theta, \Delta, \Gamma, p) := \Gamma \times \Delta$*.*
- *finally, the set of axioms* Σ_A *has a symbol whenever* $\psi \leq \varphi$ *holds for* ψ, φ *in any poset.*

▶ **Definition 34** (Model). *A* model *of a theory* Σ *in* $\mathcal{P}$ *is a doctrine morphism* $\mathsf{Syn}(\Sigma) \to \mathsf{Lang}(\mathcal{P})$*. This corresponds exactly with the expected notion of model for* $\mathcal{P} := \mathcal{P}\mathsf{reord}$*, i.e., a preorder for each base type in* Σ_B*, a monotone function for each base term, etc.*

We now establish that directed first-order logic is the internal language for directed doctrines.

▶ **Theorem 35** (Internal language correspondence). *The above constructions are functorial, forming 2-functors* $\mathsf{Syn} : \mathsf{Theory} \rightleftarrows \mathsf{DDoctrine} : \mathsf{Lang}$*. Moreover, they form a bi-adjunction between 2-categories [22, 2.2.5], i.e., there is an equivalence of categories between the category* $\mathsf{Syn}(\Sigma) \longrightarrow \mathcal{P}$ *in* $\mathsf{DDoctrine}$ *and the set* $\Sigma \longrightarrow \mathsf{Lang}(\mathcal{P})$ *in* Theory *for any* Σ *and* $\mathcal{P}$*.*

Proof. Detailed in Appendix H, showing how the *no-dinatural-variance* condition is used. ◀

▶ **Corollary 36** (Soundness and completeness). *An entailment is derivable in directed first-order logic in the empty theory if and only if it holds in every directed doctrine* $\mathcal{P}$*.*

Proof. Detailed in Appendix I, which follows immediately from the theorem above. ◀

6 Conclusion and Future Work

In this paper we introduced a sound and complete syntax for a directed first-order proof-irrelevant type theory with polarities and a "directed" notion of equality $\leq_A$, which we characterized in terms of a left relative adjoint. Our analysis of the universal property of directed equality further enlightens the question posed by Lawvere on the precise role of hom and variance [28] in the categorical treatment of logic, treated here for the *logic of preorders*.

The idea of polarized contexts is a further step towards a satisfactory syntactic treatment of directed type theory, which even in the first-order case with preorder semantics requires a careful treatment of polarity. Type dependency is particularly non-trivial, since it is not obvious what role variance should play when variables appear in the types themselves.

We saw how dFOL is only a slight departure from first-order logic, allowing for it to be expanded in the same way that the latter is the basis of many logical systems. The notion of directed doctrine could be studied more precisely under the 2-categorical perspective, e.g., by giving algebraic characterizations of directed equality [13] or the study of their completions [34, 36], possibly with a directed quantitative extension in the style of [11].

Following the geometric intuition for HoTT, the theory of directed spaces and directed algebraic topology might prove to be a suitable model for our logic, enabling reasoning about concurrent systems in the style of [14, 41] with directed equality as reachability.

Another approach to capturing variances could be based on a multicategorical "graded" approach [30], tagging variables with polarities and using a binary operation that combines the polarity of variables together such that $- \otimes + = \Delta$. A relatively straightfoward extension to this work is the addition of A^{op} types, as in [31]: our work lays the foundation to capture this idea precisely, since we can state how positive occurrences of A^{op} are equivalent to negative ones of A in a polarized context; hence `op`-types becomes a "representable" way (in the sense of [48]) of capturing such swap between positive/negative variables.

References

1 Benedikt Ahrens, Paige Randall North, and Niels van der Weide. Bicategorical type theory: semantics and syntax. *Mathematical Structures in Computer Science*, pages 1–45, October 2023. URL: `https://www.cambridge.org/core/journals/mathematical-structures-in-computer-science/article/bicategorical-type-theory-semantics-and-syntax/725F2E17B25094145F9D037D9465A534`, `doi:10.1017/S0960129523000312`.

2 Thorsten Altenkirch, James Chapman, and Tarmo Uustalu. Monads Need Not Be Endofunctors. In Luke Ong, editor, *Foundations of Software Science and Computational Structures*, Lecture Notes in Computer Science, pages 297–311, Berlin, Heidelberg, 2010. Springer. `doi:10.1007/978-3-642-12032-9\_21`.

3 Thorsten Altenkirch and Jacob Neumann. Synthetic 1-Categories in Directed Type Theory, October 2024. arXiv:2410.19520. `arXiv:2410.19520`, `doi:10.48550/arXiv.2410.19520`.

4 Nathanael Arkor and Dylan McDermott. The formal theory of relative monads. *Journal of Pure and Applied Algebra*, 228(9):107676, September 2024. `doi:10.1016/j.jpaa.2024.107676`.

5 Steve Awodey and Michael A. Warren. Homotopy theoretic models of identity types. *Mathematical Proceedings of the Cambridge Philosophical Society*, 146(1):45–55, 2009. `doi:10.1017/S0305004108001783`.

6 Edwin S. Bainbridge. Feedback and generalized logic. *Information and Control*, 31(1):75–96, May 1976. URL: `https://www.sciencedirect.com/science/article/pii/S0019995876903909`, `doi:10.1016/S0019-9958(76)90390-9`.

7 Filippo Bonchi, Alessio Santamaria, Jens Seeber, and Pawel Sobocinski. On Doctrines and Cartesian Bicategories. In Fabio Gadducci and Alexandra Silva, editors, *9th Conference on Algebra and Coalgebra in Computer Science (CALCO 2021)*, volume 211 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 10:1–10:17, Dagstuhl, Germany, 2021. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. URL: `https://drops.dagstuhl.de/opus/volltexte/2021/15365`, `doi:10.4230/LIPIcs.CALCO.2021.10`.

8 Fernando Chu, Éléonore Mangel, and Paige Randall North. A directed type theory for 1-categories. In *30th International Conference on Types for Proofs and Programs TYPES 2024–Abstracts*, page 205, 2024. URL: `https://types2024.itu.dk/abstracts.pdf#page=215`.

9 Alonzo Church. *Introduction to Mathematical Logic*, volume 1 of *Introduction to Mathematical Logic*. Princeton University Press, 1956.

10 Roy L. Crole. Deriving Category Theory from Type Theory. In Geoffrey Burn, Simon Gay, and Mark Ryan, editors, *Theory and Formal Methods 1993*, Workshops in Computing, pages 15–26, London, 1993. Springer. doi:10.1007/978-1-4471-3503-6_2.

11 Francesco Dagnino and Fabio Pasquali. Quantitative equality in substructural logic via lipschitz doctrines. *Logical Methods in Computer Science*, Volume 21, Issue 1, Jan 2025. URL: https://lmcs.episciences.org/10396, doi:10.46298/lmcs-21(1:7)2025.

12 Eduardo Dubuc and Ross Street. Dinatural transformations. In S. MacLane, H. Applegate, M. Barr, B. Day, E. Dubuc, Phreilambud, A. Pultr, R. Street, M. Tierney, and S. Swierczkowski, editors, *Reports of the Midwest Category Seminar IV*, Lecture Notes in Mathematics, pages 126–137, Berlin, Heidelberg, 1970. Springer. doi:10.1007/BFb0060443.

13 Jacopo Emmenegger, Fabio Pasquali, and Giuseppe Rosolini. Elementary doctrines as coalgebras. *Journal of Pure and Applied Algebra*, 224(12):106445, 2020. URL: https://www.sciencedirect.com/science/article/pii/S0022404920301444, doi:10.1016/j.jpaa.2020.106445.

14 Lisbeth Fajstrup, Eric Goubault, Emmanuel Haucourt, Samuel Mimram, and Martin Raussen. *Directed Algebraic Topology and Concurrency*. Springer International Publishing, 2016. doi:10.1007/978-3-319-15398-8.

15 M. Fiore, G. Plotkin, and D. Turi. Abstract syntax and variable binding. In *Proceedings. 14th Symposium on Logic in Computer Science (Cat. No. PR00158)*, pages 193–202, July 1999. ISSN: 1043-6871. URL: https://ieeexplore.ieee.org/document/782615, doi:10.1109/LICS.1999.782615.

16 Marcelo Fiore and Giuseppe Rosolini. The category of cpos from a synthetic viewpoint. *Electronic Notes in Theoretical Computer Science*, 6:133–150, 1997. MFPS XIII, Mathematical Foundations of Progamming Semantics, Thirteenth Annual Conference. URL: https://www.sciencedirect.com/science/article/pii/S1571066105801653, doi:10.1016/S1571-0661(05)80165-3.

17 Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz. Directed univalence in simplicial homotopy type theory, July 2024. doi:10.48550/arXiv.2407.09146.

18 Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz. The Yoneda embedding in simplicial type theory, January 2025. doi:10.48550/arXiv.2501.13229.

19 Martin Hofmann. *Extensional Constructs in Intensional Type Theory*. Springer-Verlag, Berlin, Heidelberg, 1997.

20 Martin Hofmann. Syntax and Semantics of Dependent Types. In Andrew M. Pitts and Peter Dybjer, editors, *Semantics and Logics of Computation*, Publications of the Newton Institute, pages 79–130. Cambridge University Press, Cambridge, 1997. URL: https://www.cambridge.org/core/books/semantics-and-logics-of-computation/syntax-and-semantics-of-dependent-types/119C8085C6A1A0CD7F24928EF866748F, doi:10.1017/CBO9780511526619.004.

21 Martin Hofmann and Thomas Streicher. The groupoid interpretation of type theory. In Giovanni Sambin and Jan M Smith, editors, *Twenty-five years of constructive type theory (Venice, 1995)*, volume 36 of *Oxford Logic Guides*, pages 83–111. Oxford Univ. Press, New York, October 1998. doi:10.1093/oso/9780198501275.003.0008.

22 Bart P. F. Jacobs. *Categorical Logic and Type Theory*, volume 141 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, 1999.

23 Chris Kapulkin and Peter LeFanu Lumsdaine. The Simplicial Model of Univalent Foundations (after Voevodsky). Technical report, 2012. arXiv:1211.2851. arXiv:1211.2851, doi:10.48550/arXiv.1211.2851.

24 Joachim Lambek and Philip J. Scott. *Introduction to Higher-Order Categorical Logic*, volume 7 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1986. URL: 5https://www.cambridge.org/ee/academic/subjects/mathematics/logic-categories-and-sets/introduction-higher-order-categorical-logic,https://www.cambridge.org/ee/academic/subjects/mathematics/logic-categories-and-sets.

25 F. William Lawvere. *Functorial Semantics of Algebraic Theories*. PhD thesis, Columbia University, 1963.

26 F. William Lawvere. Adjointness in Foundations. *Dialectica*, 23(3/4):281–296, 1969. URL: https://www.jstor.org/stable/42969800.

27 F. William Lawvere. Diagonal arguments and cartesian closed categories. In *Category Theory, Homology Theory and their Applications II: Proceedings of the Conference held at the Seattle Research Center of the Battelle Memorial Institute, June 24 – July 19, 1968 Volume Two*, pages 134–145. Springer Berlin Heidelberg, Berlin, Heidelberg, 1969. Reprinted in *Reprints in Theory and Applications of Categories*, No. 15 (2006), pp 1-13, http://tac.mta.ca/tac/reprints/articles/15/tr15abs.html.

28 F. William Lawvere. Equality in hyperdoctrines and comprehension schema as an adjoint functor. In A. Heller, editor, *Applications of Categorical Algebra*, pages 1–14. American Mathematical Society, Providence, R.I., 1970.

29 Tom Leinster. *Basic Category Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2014. URL: https://www.cambridge.org/core/books/basic-category-theory/A72533879BBC7BD956CC415777B7DA99, doi:10.1017/CBO9781107360068.

30 Paul-Blain Levy. Locally graded categories, 2019. URL: https://www.cs.bham.ac.uk/âĹpbl/papers/locgrade.pdf.

31 Daniel R. Licata and Robert Harper. 2-Dimensional Directed Type Theory. *Electronic Notes in Theoretical Computer Science*, 276:263–289, September 2011. URL: https://www.sciencedirect.com/science/article/pii/S1571066111001174, doi:10.1016/j.entcs.2011.09.026.

32 Fosco Loregian. *(Co)end Calculus*. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2021. URL: https://www.cambridge.org/core/books/coend-calculus/C662E90767358B336F17B606D19D8C43, doi:10.1017/9781108778657.

33 Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag New York, 2nd edition, 1998. doi:10.1007/978-1-4757-4721-8.

34 Maria Emilia Maietti and Giuseppe Rosolini. Quotient Completion for the Foundation of Constructive Mathematics. *Logica Universalis*, 7(3):371–402, September 2013. doi:10.1007/s11787-013-0080-2.

35 Maria Emilia Maietti and Giuseppe Rosolini. Unifying Exact Completions. *Applied Categorical Structures*, 23(1):43–52, February 2015. doi:10.1007/s10485-013-9360-5.

36 Maria Emilia Maietti and Davide Trotta. A characterization of generalized existential completions. *Annals of Pure and Applied Logic*, 174(4):103234, 2023. URL: https://www.sciencedirect.com/science/article/pii/S016800722200149X, doi:10.1016/j.apal.2022.103234.

37 Paul-André Melliès and Noam Zeilberger. A bifibrational reconstruction of Lawvere's presheaf hyperdoctrine. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '16, pages 555–564, New York, NY, USA, July 2016. Association for Computing Machinery. doi:10.1145/2933575.2934525.

38 José Meseguer. Twenty years of rewriting logic. *The Journal of Logic and Algebraic Programming*, 81(7):721–781, 2012. Rewriting Logic and its Applications. URL: https://www.sciencedirect.com/science/article/pii/S1567832612000707, doi:10.1016/j.jlap.2012.06.003.

39 Samuel Mimram. *Program = Proof*. Independently Published, 2020. URL: https://books.google.ee/books?id=nzZzzgEACAAJ.

40 Max S. New and Daniel R. Licata. A Formal Logic for Formal Category Theory. In Orna Kupferman and Pawel Sobocinski, editors, *Foundations of Software Science and Computation Structures*, Lecture Notes in Computer Science, pages 113–134. Springer Nature Switzerland, 2023. doi:10.1007/978-3-031-30829-1_6.

41 Paige Randall North. Towards a Directed Homotopy Type Theory. *Electronic Notes in Theoretical Computer Science*, 347:223–239, November 2019. URL: `https://www.sciencedirect.com/science/article/pii/S1571066119301288`, `doi:10.1016/j.entcs.2019.09.012`.

42 Paige Randall North and Fernando Chu. Dependent two-sided fibrations for directed type theory. 2025.

43 Andreas Nuyts. Towards a directed homotopy type theory based on 4 kinds of variance. Master's thesis, KU Leuven, 2015.

44 Andreas Nuyts. Higher pro-arrows: Towards a model for naturality pretype theory. 2023.

45 Andrew M. Pitts. Categorical logic. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science: Volume 5: Logic and Algebraic Methods*, pages 39–123. Oxford University Press, May 1995. `doi:10.1093/oso/9780198537816.003.0002`.

46 Emily Riehl and Michael Shulman. A type theory for synthetic ∞-categories. *Higher structures*, 1(1), 2017. arXiv:1705.07442.

47 Michael Shulman. *Categorical Logic from a Categorical Point of View*. 2016. URL: `https://github.com/mikeshulman/catlog`.

48 Michael Shulman. Contravariance through enrichment. *Theory and Applications of Categories*, 33(5):95–130, 2018. `arXiv:1606.05058`, `doi:10.48550/arxiv.1606.05058`.

49 Friedrich Ulmer. Properties of dense and relative adjoint functors. *Journal of Algebra*, 8(1):77–95, January 1968. `doi:10.1016/0021-8693(68)90036-7`.

50 The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

51 Dirk van Dalen. *Logic and Structure*. Springer, London, 2013. `doi:10.1007/978-1-4471-4558-5\_7`.

52 Matthew Z. Weaver and Daniel R. Licata. A Constructive Model of Directed Univalence in Bicubical Sets. In *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '20, pages 915–928, New York, NY, USA, July 2020. Association for Computing Machinery. `doi:10.1145/3373718.3394794`.

A Standard rules for symmetric equality

We illustrate the rules of symmetric equality in first-order logic for the reader to compare them our directed rules. We use the typical type theoretical formulation using (refl) and the J-rule, which characterize equality using the following introduction and elimination rules:

$$\frac{}{[\Gamma, x : A]\ \Phi \vdash x = x}\ (\mathsf{refl}) \qquad \frac{[\Gamma, z : A] \qquad \Phi(z,z) \vdash P(z,z)}{[\Gamma, a : A, b : A]\ a = b,\ \Phi(a,b) \vdash P(a,b)}\ (J)$$

The intuition behind the J-rule is the following: to prove a property $P(a,b)$ for any $a,b : A$ assuming that $a = b$, then it is enough to prove P for the case $P(z,z)$ where P is instantiated with the same variable $z : A$, and similarly for the context $\Phi(z,z)$.

Using these two rules one can derive the usual properties of equality, e.g., that equality is symmetric, for $\Phi := [\,], P(a,b) := (b = a)$. Transitivity of symmetric equality similarly follows directly:

$$\frac{\dfrac{}{[z : A] \qquad \vdash z = z}\ (\mathsf{refl})}{[a : A, b : A]\ a = b \vdash b = a}\ (J) \qquad \frac{\dfrac{}{[z : A] \qquad z = c \vdash z = c}\ (\mathsf{hyp})}{[a : A, b : A, c : A]\ a = b,\ b = c \vdash a = c}\ (J)$$

B Lawvere on the presheaf hyperdoctrine

> [...] This should not be taken as indicative of a lack of vitality of [Psh] as a hyperdoctrine, or even of a lack of a satisfactory theory of equality for it. Rather, it indicates that we have probably been too naive in defining equality in a manner too closely suggested by the classical conception. [...] But present categorical conceptions indicate that [...] the graph of a functor $f : \mathbb{B} \to \mathbb{C}$ should be [...] a binary attribute of mixed variance in $\mathsf{Psh}(\mathbb{B}^{\mathsf{op}} \times \mathbb{C})$. Thus in particular "equality" should be the functor $\hom_{\mathbb{B}}$ [...]. The term which would take the place of δ in such a more enlightened theory of equality would then be the forgetful functor $\mathsf{Tw}(\mathbb{B}) \to \mathbb{B}^{\mathsf{op}} \times \mathbb{B}$ from the "twisted morphism category" [...]. Of course to abstract from this example would require at least the addition of a functor $T \xrightarrow{\mathsf{op}} T$ to the structure of a [doctrine]. [28, p. 11]

C Syntax of directed first-order logic

▶ **Definition 37** (Substitution of terms in formulas of Definition 8). *Since it is not standard, we explicitly describe the metatheoretical operation of substitution in formulas, which is used implicitly in* (reindex). *We treat in particular the multi-variable case, substituting a triple of substitutions* $(\Theta', \Delta' \vdash \eta : \Theta)$, $(\Delta' \vdash \delta : \Delta)$, $(\Gamma', \Delta' \vdash \rho : \Gamma)$*:*

$$\begin{aligned}
&\mathsf{subst}_{\eta,\delta,\rho} : \{[\Theta \mid \Delta \mid \Gamma]\ \mathsf{prop}\} \to \{[\Theta' \mid \Delta' \mid \Gamma']\ \mathsf{prop}\}\\
&\mathsf{subst}_{\eta,\delta,\rho}(\top) := \top, \mathsf{subst}(\bot) := \bot\\
&\mathsf{subst}_{\eta,\delta,\rho}(\psi \wedge \varphi) := \mathsf{subst}_{\eta,\delta,\rho}(\psi) \wedge \mathsf{subst}_{\eta,\delta,\rho}(\varphi)\\
&\mathsf{subst}_{\eta,\delta,\rho}(\psi \Rightarrow \varphi) := \mathsf{subst}_{\rho,\delta,\eta}(\psi) \Rightarrow \mathsf{subst}_{\eta,\delta,\rho}(\varphi)\\
&\mathsf{subst}_{\eta,\delta,\rho}(\exists^p x.\varphi(x,n,d,p)) := \exists^p x.\mathsf{subst}_{\eta,\delta,\rho}(\varphi(x,n,d,p))\\
&\mathsf{subst}_{\eta,\delta,\rho}(\forall^p x.\varphi(x,n,d,p)) := \forall^p x.\mathsf{subst}_{\eta,\delta,\rho}(\varphi(x,n,d,p))\\
&\mathsf{subst}_{\eta,\delta,\rho}(s(n,d) \leq t(d,p)) := s(\eta,\delta) \leq t(\delta,\rho)\\
&\mathsf{subst}_{\eta,\delta,\rho}(P(s(n,d) \mid t(d,p))) := P(s(\eta,\delta) \mid t(\delta,\rho))
\end{aligned}$$

In the case of terms we indicate with $s(\eta,\delta)$ *and* $t(\delta,\rho)$ *the usual substitutions of terms in the intuitive way, e.g., if* $\Theta', \Delta' \vdash \eta : \Theta$ *and* $\Delta' \vdash \delta : \Delta$*, and a term* $n : \Theta, d : \Delta \vdash s(n,d) : N$

then $\Theta', \Delta' \vdash s(\eta, \delta) : N$. Note the inversion of the terms in the case of implication. In the case of polarized quantifiers we substitute under binders in the usual capture-avoidant way.

D Derived rules

We show derivations for the rules given in Figure 6 as well as additional ones.

D.1 Derived rules for quantifiers

- The rule $(\exists^-_t)$ rule follows by (cut) the hypothesis with a generic propositional context on the left, assuming to have a term $\Theta, \Delta \vdash \eta : N$:

$$\dfrac{[\Theta, x : N \mid \Delta \mid \Gamma]\ \Phi \vdash \psi(\eta) \qquad \dfrac{\dfrac{\dfrac{}{[\Theta \mid \Delta \mid \Gamma]\ \exists^- x.\psi(x), \Phi \vdash \exists^- x.\psi(x)}\,(\mathsf{hyp})}{[\Theta, x : N \mid \Delta \mid \Gamma]\ \psi(x), \Phi \vdash \exists^- x.\psi(x)}\,(\exists)}{[\Theta \mid \Delta \mid \Gamma]\ \psi(\eta), \Phi \vdash \exists^- x.\psi(x)}\,(\mathsf{reindex})}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \exists^- x.\psi(x)}\,(\mathsf{cut})$$

- The rule $(\forall^\Delta_t)$ follows by reindexing with an arbitrary term $\Delta \vdash \delta : A$:

$$\dfrac{\Delta \vdash \delta : A \qquad \dfrac{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \forall^\Delta x.\psi(\overline{x}, x)}{[\Theta \mid \Delta, x : A \mid \Gamma]\ \Phi \vdash \psi(\overline{x}, x)}\,(\forall)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi(\delta, \delta)}\,(\mathsf{reindex})$$

- The following rules for quantifiers follow precisely the same idea:

$$\dfrac{\Delta, \Gamma \vdash \rho : P \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \varphi(\rho)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \exists^+ x.\psi(x)}\,(\exists^+_t) \qquad \dfrac{\Delta, \Gamma \vdash \rho : P \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \forall^+ x.\psi(x)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi(\rho)}\,(\forall^+_t)$$

$$\dfrac{\Theta, \Delta \vdash \eta : N \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \varphi(\eta)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \exists^- x.\psi(x)}\,(\exists^-_t) \qquad \dfrac{\Theta, \Delta \vdash \eta : N \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \forall^- x.\psi(x)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi(\eta)}\,(\forall^-_t)$$

$$\dfrac{\Delta \vdash \delta : D \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \varphi(\delta, \delta)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \exists^\Delta x.\psi(\overline{x}, x)}\,(\exists^\Delta_t) \qquad \dfrac{\Delta \vdash \delta : D \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \forall^\delta x.\psi(x)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi(\delta, \delta)}\,(\forall^\Delta_t)$$

D.2 Derived rules for polarized implication

- We explicitly show the derivation for $(\Rightarrow^+_R)$, where x appears dinaturally on the top and $[\Theta \mid \Delta \mid \Gamma, x{:}A]\ \Phi(x)$ propctx, $[\Theta', x{:}A \mid \Delta \mid \Gamma']\ \psi(x)$ prop:

$$\begin{array}{c} [\Theta, \Theta' \mid \Delta, x{:}A \mid \Gamma, \Gamma']\ \varphi(\overline{x}), \Phi(x) \vdash \varphi(x) \\ \hline\hline [\Theta, \Gamma' \mid \Delta \mid \Gamma, \Theta', x{:}A]\ \Phi(x) \vdash \psi(x) \Rightarrow \varphi(x) \end{array}\,(\Rightarrow^+_R)$$

We show both directions separately.
Top-to-bottom:

$$\dfrac{\dfrac{[\Theta, \Theta' \mid \Delta, x{:}A \mid \Gamma, \Gamma']\ \varphi(\overline{x}), \Phi(x) \vdash \varphi(x)}{[\bullet \mid \Delta, \Theta, \Gamma, x{:}A \mid \bullet]\ \varphi(\overline{x}), \Phi(x) \vdash \varphi(x)}\,(\mathsf{reindex})}{[\Theta, \Gamma' \mid \Delta \mid \Gamma, \Theta', x{:}A]\ \Phi(x) \vdash \varphi(x) \Rightarrow \varphi(x)}\,(\Rightarrow)$$

Bottom-to-top:

$$\dfrac{\dfrac{[\Theta,\Gamma' \mid \Delta \mid \Gamma,\Theta', x\!:\!A]\ \Phi(x) \vdash \varphi(x) \Rightarrow \varphi(x)}{[\bullet \mid \Delta,\Theta,\Gamma,x\!:\!A \mid \bullet]\ \varphi(\overline{x}),\Phi(x) \vdash \varphi(x)}\ (\Rightarrow)}{[\Theta,\Theta' \mid \Delta, x\!:\!A \mid \Gamma,\Gamma']\ \varphi(\overline{x}),\Phi(x) \vdash \varphi(x)}\ (\text{upgrade-to-nat})$$

The rule $(\Rightarrow)$ can be applied on the left side because both ψ and φ are reindexed with the same weakening/contraction operation that is used implicitly in the top side of $(\Rightarrow)$. Rule (upgrade-to-nat) can be used because the variables of Θ,Γ in Φ,φ are implicitly lifted to be dinatural, which is precisely the operation used in the top side of (upgrade-to-nat).

- Other variants $(\Rightarrow^-_L),(\Rightarrow^-_R)$ with a negative variable in the context are defined identically.

- We show the rule $(\Rightarrow^\pm)$ that inverts singular appearances of variables in ψ, where x,y do not appear in Φ and φ, i.e., $[\Theta \mid \Delta \mid \Gamma]\ \Phi$ propctx and $[x\!:\!A \mid \Delta \mid y\!:\!B]\ \psi(x,y)$ prop. Note that ψ does not depend on Θ,Γ, since we want occurrences to switch polarity from negative to positive directly, hence if any variable appeared in Φ,φ they would appear as dinatural in either top or bottom side of the rule.

$$\dfrac{[\Theta, x\!:\!A \mid \Delta \mid \Gamma, y\!:\!B]\ \psi(x,y),\Phi \vdash \varphi}{[\Theta, y\!:\!B \mid \Delta \mid \Gamma, x\!:\!A]\ \Phi \vdash \psi(x,y) \Rightarrow \varphi}\ (\Rightarrow^\pm)$$

we show both directions separately.
Top-to-bottom:

$$\dfrac{\dfrac{[\Theta, x\!:\!A \mid \Delta \mid \Gamma, y\!:\!B]\ \psi(x,y),\Phi \vdash \varphi}{[\bullet \mid \Delta,\Theta,\Gamma,x\!:\!A,y\!:\!B \mid \bullet]\ \psi(x,y),\Phi \vdash \varphi}\ (\text{reindex})}{[\Theta, y\!:\!B \mid \Delta \mid \Gamma, x\!:\!A]\qquad \Phi \vdash \psi(x,y) \Rightarrow \varphi}\ (\Rightarrow)$$

Bottom-to-top:

$$\dfrac{\dfrac{[\Theta, y\!:\!B \mid \Delta \mid \Gamma, x\!:\!A]\qquad \Phi \vdash \psi(x,y) \Rightarrow \varphi}{[\bullet \mid \Delta,\Theta,\Gamma,x\!:\!A,y\!:\!B \mid \bullet]\ \psi(x,y),\Phi \vdash \varphi}\ (\Rightarrow)}{[\Theta, x\!:\!A \mid \Delta \mid \Gamma, y\!:\!B]\ \psi(x,y),\Phi \vdash \varphi}\ (\text{upgrade-to-nat})$$

- We prove more general rules $(\Rightarrow_L),(\Rightarrow_R)$ for $\Rightarrow$, of which $(\Rightarrow^+_R),(\Rightarrow^+_L)$ and $(\Rightarrow^-_R),(\Rightarrow^-_L)$ are special cases in which Φ,φ do not use N or P, respectively:

$$\begin{array}{c}[N \mid \Delta \mid P]\ \psi\ \mathsf{prop} \\ [\Theta, N \mid \Delta \mid \Gamma, P]\ \Phi\ \mathsf{propctx},\ \varphi\ \mathsf{prop} \\ \dfrac{[\Theta, N \mid \Delta \mid \Gamma, P]\ \psi,\Phi \vdash \varphi}{[\Theta \mid \Delta, N, P \mid \Gamma]\qquad \Phi \vdash \psi \Rightarrow \varphi}\ (\Rightarrow_L)\end{array} \qquad \begin{array}{c}[P \mid \Delta \mid N]\ \psi\ \mathsf{prop} \\ [\Theta, N \mid \Delta \mid \Gamma, P]\ \Phi\ \mathsf{propctx},\ \varphi\ \mathsf{prop} \\ \dfrac{[\Theta \mid \Delta, N, P \mid \Gamma]\ \psi,\Phi \vdash \varphi}{[\Theta, N \mid \Delta \mid \Gamma, P]\qquad \Phi \vdash \psi \Rightarrow \varphi}\ (\Rightarrow_R)\end{array}$$

We show $(\Rightarrow_L)$, showing both directions for $[N \mid \Delta \mid P]\ \psi$, and $[\Theta, N \mid \Delta \mid \Gamma, P]\ \Phi$, φ:

$$\dfrac{\dfrac{[\Theta, N \mid \Delta \mid \Gamma, P]\ \psi,\Phi \vdash \varphi}{[\bullet \mid \Delta, N, P, \Theta, \Gamma \mid \bullet]\qquad \Phi \vdash \psi \Rightarrow \varphi}\ (\Rightarrow)}{[\Theta \mid \Delta, N, P \mid \Gamma]\qquad \Phi \vdash \psi \Rightarrow \varphi}\ (\text{upgrade-to-nat}) \qquad \dfrac{\dfrac{[\Theta \mid \Delta, N, P \mid \Gamma]\qquad \Phi \vdash \psi \Rightarrow \varphi}{[\bullet \mid \Delta, N, P, \Theta, \Gamma \mid \bullet]\ \psi,\Phi \vdash \varphi}\ (\Rightarrow)}{[\Theta, N \mid \Delta \mid \Gamma, P]\ \psi,\Phi \vdash \varphi}\ (\text{upgrade-to-nat})$$

We show $(\Rightarrow_R)$, showing both directions for $[P \mid \Delta \mid N]\ \psi$, and $[\Theta, N \mid \Delta \mid \Gamma, P]\ \Phi$, φ:

$$\dfrac{\dfrac{[\Theta \mid \Delta, N, P \mid \Gamma]\ \psi,\Phi \vdash \varphi}{[\bullet \mid \Delta, N, P, \Theta, \Gamma \mid \bullet]\ \psi,\Phi \vdash \varphi}\ (\text{reindex})}{[\Theta, N \mid \Delta \mid \Gamma, P]\qquad \Phi \vdash \psi \Rightarrow \varphi}\ (\Rightarrow) \qquad \dfrac{\dfrac{[\Theta, N \mid \Delta \mid \Gamma, P]\qquad \Phi \vdash \psi \Rightarrow \varphi}{[\bullet \mid \Delta, N, P, \Theta, \Gamma \mid \bullet]\ \psi,\Phi \vdash \varphi}\ (\Rightarrow)}{[\Theta \mid \Delta, N, P \mid \Gamma]\ \psi,\Phi \vdash \varphi}\ (\text{upgrade-to-nat})$$

In both derivations above, the rule (upgrade-to-nat) can be used because ψ does not use Θ, Γ, hence both Θ and Γ are implicitly weakened/contracted to dinatural. On the other hand, N, P already appear naturally in Φ, φ.
Similarly, the rule $(\Rightarrow)$ can only be applied because both Φ and φ are reindexed with the same weakening/contraction operation that is used implicitly in the top side of $(\Rightarrow)$.

D.3 Derived rules for directed equality

- The rule ($\leq$-refl$_t$) is obtained by (reindex)-ing the only dinatural variable in ($\leq$-refl) with an arbitrary term $\Delta \vdash \delta : A$:

$$\dfrac{\dfrac{}{[\Theta \mid \Delta, z : A \mid \Gamma]\ \Phi \vdash \overline{z} \leq_A z}\ (\leq\text{-refl})}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \delta \leq_A \delta}\ (\mathsf{reindex})$$

- The rule $(\leq^{\Phi})$ is a version $(\leq)$ that allows for variables to appear *negatively* in the context hypothesis Φ, using polarized exponentials for $[\Theta, a : A \mid \Delta \mid \Gamma, b : A]\ \Phi(a, b)$ propctx, $\varphi(a, b)$ prop in the same context:

$$\dfrac{\dfrac{\dfrac{[\Theta \mid \Delta, z : A \mid \Gamma]\ \Phi(\overline{z}, z) \vdash \varphi(\overline{z}, z)}{[\Theta \mid \Delta, z : A \mid \Gamma]\ \vdash \Phi(z, \overline{z}) \Rightarrow \varphi(\overline{z}, z)}\ (\Rightarrow)}{[\Theta, a : A \mid \Delta \mid \Gamma, b : A]\ a \leq b \vdash \Phi(b, a) \Rightarrow \varphi(a, b)}\ (\leq)}{[\Theta \mid \Delta, a : A, b : A \mid \Gamma]\ a \leq b, \Phi(\overline{b}, \overline{a}) \vdash \varphi(a, b)}\ (\Rightarrow_R)$$

- The rule $(\leq^{+})$ follows precisely the same idea as $(\leq)$ and $(\leq^{\Phi})$, but allows for positive variables to be directly kept positively (rather than dinatural, as it would be the case in $(\leq)$) whenever they appear as such.

$$\begin{array}{c}[\Theta \mid \Delta \mid \Gamma, a : A]\ \Phi \text{ propctx} \\ [\Theta \mid \Delta \mid \Gamma, b : A]\ \varphi \text{ propctx}\end{array}$$

$$\dfrac{\dfrac{\dfrac{[\Theta \mid \Delta \mid \Gamma, z : A]\ \ \Phi(z) \vdash \varphi(z)}{[\bullet \mid \Delta, \Theta, \Gamma, z : A \mid \bullet]\ \ \vdash \Phi(\overline{z}) \Rightarrow \varphi(z)}\ (\Rightarrow)}{[a : A \mid \Delta, \Theta, \Gamma \mid b : A]\ a \leq b \vdash \Phi(a) \Rightarrow \varphi(b)}\ (\leq)}{[\Theta \mid \Delta, a : A \mid \Gamma, b : A]\ \overline{a} \leq b, \Phi(a) \vdash \varphi(b)}\ (\Rightarrow_R)$$

- The following rule $(\leq^{-})$ is identical but using directed equality on "contravariant" predicates which have negative positions, instead.

$$\begin{array}{c}[\Theta, b : A \mid \Delta \mid \Gamma]\ \Phi \text{ propctx} \\ [\Theta, a : A \mid \Delta \mid \Gamma]\ \varphi \text{ propctx}\end{array}$$

$$\dfrac{\dfrac{\dfrac{[\Theta, z : A \mid \Delta \mid \Gamma]\ \ \Phi(z) \vdash \varphi(z)}{[\bullet \mid \Delta, \Theta, \Gamma, z : A \mid \bullet]\ \ \vdash \Phi(z) \Rightarrow \varphi(\overline{z})}\ (\Rightarrow)}{[a : A \mid \Delta, \Theta, \Gamma \mid b : A]\ a \leq b \vdash \Phi(b) \Rightarrow \varphi(a)}\ (\leq)}{[\Theta, a : A \mid \Delta, b : A \mid \Gamma]\ a \leq b, \Phi(\overline{b}) \vdash \varphi(a)}\ (\Rightarrow_R)$$

- Rule $(\leq_t)$ follows from $(\leq^{\Phi})$ simply by reindexing with terms $\Theta, \Delta \vdash \eta : A$ and $\Gamma, \Delta \vdash \eta : A$.

E Appendix on relative adjunctions

We recall here for the reader's convenience an elementary description of (co)relative adjunction [4, 49].

▶ **Definition 38** (Relative adjunction [4, 5.1])**.** *Consider an arrangement of categories and functors as follows:*

$$\begin{tikzcd} & \mathbb{D} \arrow[dr, "R"] \arrow[d, phantom, "\dashv"] & \\ \mathbb{C} \arrow[ur, "L"] \arrow[rr, "J"'] & {} & \mathbb{X} \end{tikzcd}$$

In this situation, we say that L is the left J-relative left adjoint to R*, written $L \dashv_J R$ and indicated in the above diagram with a central '$\dashv$', if there is a bijection*

$$\mathbb{D}(L(x), y) \cong \mathbb{X}(J(x), R(y))$$

natural in both arguments $x : \mathbb{C}, y : \mathbb{D}$.

▶ **Definition 39** (Relative coadjunction [4, 7.3])**.** *Consider an arrangement of categories and functors as follows:*

$$\begin{tikzcd} & \mathbb{D} \arrow[dr, "L"] \arrow[d, phantom, "\vdash"] & \\ \mathbb{C} \arrow[ur, "R"] \arrow[rr, "J"'] & {} & \mathbb{X} \end{tikzcd}$$

In this situation, we say that L is the left J-relative coadjoint to R*, written $L_J \dashv R$ and indicated in the above diagram with a central '$\vdash$', if there is a bijection*

$$\mathbb{X}(L(x), J(y)) \cong \mathbb{D}(x, R(y))$$

natural in both arguments $x : \mathbb{D}, y : \mathbb{C}$.

One obtains the standard definition of adjunction when $\mathbb{X} = \mathbb{C}$, $J = \mathsf{id}_{\mathbb{C}}$. In the case of posets, the above adjointness condition simply becomes a bi-implication of conditions.

F Directed equality as isomorphism

▶ **Theorem 40.** *We show that the rules for directed equality* (≤-refl) *and* ($\leq$) *can be equivalently captured by the top-to-bottom direction of* ($\leq$) *and its bottom-to-top* ($\leq^{-1}$)*, i.e.,* ($\leq$) *being an isomorphism fully characterizes directed equality.*

Proof. In one direction, we assume the following rule:

$$\frac{[\Theta, a : A \mid \Delta \mid \Gamma, b : A]\ a \leq b, \Phi \vdash \varphi(a, b)}{[\Theta \mid \Delta, z : A \mid \Gamma] \qquad \Phi \vdash \varphi(\overline{z}, z)}\ (\leq^{-1})$$

we derive (≤-refl) simply by picking $\varphi(a, b) := a \leq b$ in context $[\Theta, a : A \mid \Delta \mid \Gamma, b : A]$:

$$\dfrac{\dfrac{}{[\Theta, a : A \mid \Delta \mid \Gamma, b : A]\ a \leq b, \Phi \vdash a \leq b}\ (\mathsf{hyp})}{[\Theta \mid \Delta, z : A \mid \Gamma] \qquad \Phi \vdash \overline{z} \leq z}\ (\leq^{-1})$$

In the other direction, we assume the rule ($\leq$-refl) and derive ($\leq^{-1}$):

$$\dfrac{\dfrac{}{[\Theta \mid \Delta, z : A \mid \Gamma]\ \Phi \vdash \overline{z} \leq z}(\leq\text{-refl}) \qquad \dfrac{[\Theta, a : A \mid \Delta \mid \Gamma, b : A]\ a \leq b, \Phi \vdash \varphi(a,b)}{[\Theta \mid \Delta, z : A \mid \Gamma]\ \overline{z} \leq z, \Phi \vdash \varphi(\overline{z}, z)}(\text{reindex})}{[\Theta \mid \Delta, z : A \mid \Gamma]\ \Phi \vdash \varphi(\overline{z}, z)}(\text{cut})$$

◀

G Symmetric equality via directed equality

▶ **Theorem 41** (Symmetric equality via directed equality). *We show the how the notion of symmetric equality using directed equality $a = b := \overline{a} \leq b \wedge \overline{b} \leq a$ in context $[\bullet \mid a, b : A \mid \bullet]$, defined in Remark 14, satisfies the characterization of symmetric equality as left adjoint to reindexing, i.e., we prove the following statement,*

$$\dfrac{\overline{\quad [z : A]\ \Phi \vdash P(z,z) \quad}}{[a : A, b : A]\ a = b, \Phi \vdash P(a,b)}$$

which in our directed case looks like this, with no restrictions on the appearance of variables,

$$\dfrac{\overline{\quad [\Theta \mid \Delta, z : A \mid \Gamma]\ \Phi \vdash P(\overline{z}, z, z, \overline{z}) \quad}}{[\Theta \mid \Delta, a : A, b : A \mid \Gamma]\ \overline{a} \leq b \wedge \overline{b} \leq a, \Phi \vdash P(\overline{a}, a, \overline{b}, b)}$$

assuming $[\Theta, b : A, c : A \mid \Delta \mid \Gamma, a : A, d : A]\ P(a,b,c,d)$ prop. *Note that this corresponds precisely with the characterization of symmetric equality as left adjoint to contraction* but in the dinatural context *[22]*.

Proof. We prove this by going through a version of transport (as in Section 3) with two variables (one positive and one negative), and then reindex these variables by collapsing indices together: x, w are contracted into a, and y, z are contracted into b.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[\Theta \mid \Delta, p : A, r : A \mid \Gamma]\ \Phi \vdash P(a, \overline{a}, p, \overline{r}) \Rightarrow P(\overline{a}, a, \overline{p}, r)}(\text{hyp})}{[\Theta, z : A \mid \Delta, r : A \mid \Gamma, w : A]\ z \leq w, \Phi \vdash P(a, \overline{a}, w, \overline{r}) \Rightarrow P(\overline{a}, a, z, r)}(\leq)}{[\Theta, x, z : A \mid \Delta \mid \Gamma, y, w : A]\ x \leq y \wedge z \leq w, \Phi \vdash P(a, \overline{a}, w, x) \Rightarrow P(\overline{a}, a, z, y)}(\leq)}{[\Theta, z : A \mid \Delta, x, w : A \mid \Gamma, y : A]\ \overline{x} \leq y \wedge z \leq \overline{w}, P(\overline{a}, a, w, x), \Phi \vdash P(\overline{a}, a, z, y)}(\Rightarrow)}{[\Theta \mid \Delta, a : A, b : A \mid \Gamma]\ \overline{a} \leq b \wedge \overline{b} \leq a, P(\overline{a}, a, a, \overline{a}), \Phi \vdash P(\overline{a}, a, \overline{b}, b)}(\text{reindex})}{[\Theta \mid \Delta, a : A, b : A \mid \Gamma]\ \overline{a} \leq b \wedge \overline{b} \leq a, \Phi \vdash P(\overline{a}, a, \overline{b}, b)}(\text{cut})$$

where in the last step we cut with the hypothesis $[\Theta \mid \Delta, z : A \mid \Gamma]\ \Phi \vdash P(\overline{z}, z, z, \overline{z})$.

We use ($\Rightarrow$) and ($\leq$) just for clarity purposes, to highlight the natural occurrences of variables (instead of using ($\leq^{\Phi}$)). Note that, following the polarity of P, $x \leq y$ is used to transport covariantly (as in ($\leq^{+}$)) and $z \leq w$ is used contravariantly (as in ($\leq^{-}$)). ◀

H Details for the equivalence in Theorem 35

▶ **Theorem 42** (Internal language correspondence, proof). *The constructions* Syn *and* Lang *in Theorem 35 form a (bi-)adjunction between the categories of theories and directed doctrines,* *i.e., there are equivalences of categories between the above category and the set below for any theory* Σ *and directed doctrine* $\mathcal{P} : \mathsf{ndp}(\mathbb{C})^{\mathsf{op}} \to \mathsf{Pos}$*:*

$$\frac{\mathsf{Syn}(\Sigma) \longrightarrow \mathcal{P} \text{ in } \mathsf{DDoctrine}}{\Sigma \longrightarrow \mathsf{Lang}(\mathcal{P}) \text{ in } \mathsf{Theory}}$$

Proof. We start by describing the two constructions, focusing in particular on the case of base predicates, since it is where we need Definition 20.
(Construction $\Downarrow$*).* One evaluates the doctrine morphism $(F, \alpha) : \mathsf{Syn}(\Sigma) \longrightarrow \mathcal{P}$ at the judgements involving Σ to obtain the actions of the following signature morphism:

- $b(\sigma) := F_0(\sigma \text{ type} \in \mathsf{Obj}(\mathsf{Syn}(\Sigma))) \in (\mathsf{Obj}(\mathbb{C}) \equiv \mathsf{Lang}(\mathcal{P})_B)$.
- $f(\sigma) := F_1([\![\mathsf{cod}(\sigma)]\!]_b \vdash \sigma(\mathsf{id}) : [\![\mathsf{cod}(\sigma)]\!]_b) \in \mathsf{Lang}(\mathcal{P})_F$. For the induced interpretation one has that $[\![A]\!]_b = F(A)$, $[\![t]\!]_f = F(t)$ hold by induction on A, t, since F_0, F_1 preserve products, implications, etc., in each inductive step.
- Term equality condition: given $e \in \Sigma_E$, its associated base equality holds in $\mathsf{Syn}(\Sigma)$ therefore the terms/arrows are equal. Functors F preserve equalities of arrows into $\mathcal{P}$, hence their $[\![-]\!]_f$ is equal in $\mathsf{Lang}(\mathcal{P})_E$ by definition.
- For predicates, $p(\sigma) := (F(\mathsf{pos}(\sigma)), F(\top), F(\mathsf{neg}(\sigma)), \tilde{p}) \in \mathsf{Lang}(\mathcal{P})_F$, where the point $\tilde{p} := \alpha(\sigma(\pi_1, \pi_1) \text{ prop}) \in \mathcal{P}(F(\mathsf{pos}(\sigma)) \mid F(\top) \mid F(\mathsf{neg}(\sigma)))_0$ is obtained by applying α to the base case $P(s \mid t)$ prop, but picking $\Delta := \top$ and the projections $\mathsf{pos}(\sigma), \top \vdash \pi_1 : \mathsf{pos}(\sigma)$ for s, t. This choice essentially corresponds to $\Downarrow^{\pm}_{\top}$. Moreover,

  $$\mathsf{pos}'(\tilde{p}) := F(\mathsf{pos}(\sigma)) \times F(\top) \cong F(\mathsf{pos}(\sigma)) = [\![\mathsf{pos}(\sigma)]\!]_b.$$

  This $\cong$ could be resolved more precisely by working with a weaker 2-Cat of theories.
- For axioms, we apply monotonicity of α on the relation $\mathsf{hyp}(a) \vdash \mathsf{conc}(a)$ in Syn, which holds by the axiom case for $\sigma \in \Sigma_F$. From this we obtain that the desired $\leq$ relation in $\mathcal{P}$, which is exactly how $\mathsf{Lang}(\Sigma)_A$ was defined.
  (Construction $\Uparrow$*).* Each component of the doctrine morphism $([\![-]\!], [\![-]\!]_\varphi)$ is given by induction on derivations, using the theory morphism $M : \Sigma \to \mathsf{Lang}(\mathcal{P})$ for the base cases and the structure of $\mathcal{P}$ for the inductive steps. The functor $[\![-]\!] : \{A \text{ type}\}_\Sigma \to \mathbb{C}_0$ is defined as:
- $[\![-]\!]_0$ acts by induction on types, using the product functor $- \times - : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ to interpret product types, etc., using for base types $A \in \Sigma_B$ the action $M_b(A) \in \mathbb{C}_0$.
- $[\![-]\!]_1$ acts by induction on terms, similarly as above. Functoriality is ensured by a substitution lemma.
- The components of the natural transformation

  $$[\![\Theta \mid \Delta \mid \Gamma]\!]_\varphi : \mathsf{Syn}(\Theta \mid \Delta \mid \Gamma) \to \mathcal{P}([\![\Theta]\!] \mid [\![\Delta]\!] \mid [\![\Gamma]\!])$$

  are the functors which, on objects, act by induction on formula derivations in Syn and use the structure of $\mathcal{P}$, e.g., using $- \wedge -$, implication $- \Rightarrow -$, (**??**), and directed equality in $\mathcal{P}$ to combine objects in the inductive step. For the base judgement $\gamma := \sigma(s \mid t)$ prop with $\Theta, \Delta \vdash s : [\![\mathsf{neg}(\sigma)]\!]$, take $\tilde{P} := p(\sigma \in \Sigma_P) \in \mathcal{P}(\Theta', \Delta', \Gamma')$. By the no-dinatural-variance condition, $\varepsilon(\tilde{P}) \in \mathcal{P}(\Theta' \times \Delta' =: \mathsf{neg}'(b(\sigma)) \cong [\![\mathsf{neg}(\sigma)]\!] \mid \top \mid \cdots)$. Finally,

  $$[\![\gamma]\!]_\varphi := \mathcal{P}([\![s]\!] \mid ! \mid [\![t]\!])(\varepsilon(\tilde{P})) \in \mathcal{P}([\![\Theta]\!] \mid [\![\Delta]\!] \mid [\![\Gamma]\!]).$$

  Naturality is a semantic substitution lemma, which follows by induction.
- The action on morphisms (entailments) is given by interpreting each rule with the properties of $\mathcal{P}$.

We show the equivalence with details in Theorem 43 for base predicates; both directions use the fact that ε is a bijection. The general idea is as follows:

- *($\Downarrow$; $\Uparrow = \mathsf{id}$).* The no-dinatural-variance condition is used here with naturality of α to cover all the predicates in each fiber given by the original interpretation.
- *($\Uparrow$; $\Downarrow = \mathsf{id}$).* Similarly straightforward since the interpretation $[\![-]\!]$ is essentially determined (by induction) via the action on the base predicates of M.

◀

▶ **Theorem 43** (Details for the bijection in Theorem 35). *We describe more in detail the equivalence in Theorem 35. We again concentrate on the case of base predicates to highlight the need for the no-dinatural-variance condition, since the rest of the structure is defined in the only possible way.*
($\Downarrow$; $\Uparrow = \mathsf{id}$). *Suppose we are given a morphism of doctrines (F, α); recall that we construct the component on predicates for the corresponding morphism of theories as*

$$\begin{aligned} b(\sigma) &:= \alpha(\sigma\ \mathsf{type}) \\ p(\sigma) &:= (F(\mathsf{pos}(\sigma)), \top, F(\mathsf{neg}(\sigma)), \alpha(\sigma(\pi_1, \pi_1)\ \mathsf{prop})) \end{aligned}$$

by applying F on the base predicate formula. In the other direction, we recall that in general the constructed doctrine morphism $([\![-]\!], [\![-]\!]_\varphi)$ sends judgements $\sigma(s \mid t)$ prop to

$$\mathcal{P}([\![s]\!] \mid ! \mid [\![t]\!])(\varepsilon(p(\sigma \in \Sigma_P))),$$

where $p(\sigma) := p(\sigma)_4 \in \mathcal{P}(p(\sigma)_1 \mid p(\sigma)_2 \mid p(\sigma)_3)$ for $p(\sigma)_1 := [\Theta' \mid \Delta' \mid \Gamma']$. To show that we end up with the original doctrine we need to show that α has the same action on predicates. In particular, it suffices to prove it for the base cases (since the rest of the structure is fixed and pertains to the structure of the doctrine). Hence, we need to show

$$\alpha(\sigma(s \mid t)\ \mathsf{prop}) = \mathcal{P}([\![s]\!] \mid ! \mid [\![t]\!])(\varepsilon(\alpha(\sigma(\pi_1, \pi_1)\ \mathsf{prop}))).$$

We use the definition of Definition 20 to "cancel out" ε and the full dinatural contraction $\mathcal{P}(\Downarrow^{\pm}_{\Delta}) := \mathcal{P}(\mathsf{id}_{\Theta\times\Delta} \mid !_\Delta \mid \mathsf{id}_{\Gamma\times\Delta})$ to obtain

$$\begin{aligned} & \mathcal{P}([\![s]\!] \mid ! \mid [\![t]\!])(\varepsilon(\alpha(\dots))) \\ = \quad & \mathcal{P}([\![s]\!]\ ;\ \mathsf{id}_{[\![\mathsf{neg}(\sigma)]\!]} \mid !\ ;\ ! \mid [\![t]\!]\ ;\ \mathsf{id}_{[\![\mathsf{pos}(\sigma)]\!]})(\varepsilon(\alpha(\dots))) \\ = \quad & \mathcal{P}([\![s]\!]\ ;\ \mathsf{id}_{\Theta'\times\Delta'} \mid !\ ;\ ! \mid [\![t]\!]\ ;\ \mathsf{id}_{\Gamma'\times\Delta'})(\varepsilon(\alpha(\dots))) \\ = \quad & \mathcal{P}([\![s]\!] \mid ! \mid [\![t]\!])(\mathcal{P}(\mathsf{id}_{\Theta'\times\Delta'} \mid ! \mid \mathsf{id}_{\Gamma'\times\Delta'})(\varepsilon(\alpha(\dots)))) \\ = \quad & \mathcal{P}([\![s]\!] \mid ! \mid [\![t]\!])(\alpha(\dots)). \end{aligned}$$

We implicitly used the isomorphisms between $\Theta' \times \Delta' =: \mathsf{neg}(p(\sigma)) := [\![\mathsf{neg}(\sigma)]\!]$ which corresponds exactly with the codomain of s. The final result

$$\alpha(\sigma(s \mid t)\ \mathsf{prop}) = \mathcal{P}([\![s]\!] \mid ! \mid [\![t]\!])(\alpha(\sigma(\pi_1, \pi_1)\ \mathsf{prop}))$$

follows by the naturality of α which expresses well-behavedness with respect to context reindexing.
($\Uparrow$; $\Downarrow = \mathsf{id}$). *Suppose we are given a theory morphism M; we construct $[\![-]\!]$ and the syntactic doctrine by induction, defining $[\![-]\!]_\varphi$ on the base cases $\sigma(s \mid t)$ prop to*

$$\mathcal{P}([\![s]\!] \mid ! \mid [\![t]\!])(\varepsilon(p(\sigma \in \Sigma_P)))$$

where $p(\sigma) \in \mathcal{P}(p(\sigma)_1 \mid p(\sigma)_2 \mid p(\sigma)_3)$ *and* $\varepsilon(p(\sigma)) \in \mathcal{P}([\![\mathsf{pos}(\sigma)]\!], \top, [\![\mathsf{neg}(\sigma)]\!]) \cong \mathcal{P}(p(\sigma)_1 \times p(\sigma)_2, \top, p(\sigma_3) \times p(\sigma)_2)$*. We now give a theory morphism from the doctrine just constructed, which in general recall that it is given by*

$$p'(\sigma) := (F(\mathsf{pos}(\sigma)), \top, F(\mathsf{neg}(\sigma)), \alpha(\sigma(\pi_1, \pi_1)\ \mathsf{prop}))$$

In our case α *is* $[\![-]\!]_\varphi$*, hence*

$$\begin{aligned} p'(\sigma) := (&[\![\mathsf{pos}(\sigma)]\!], \top, [\![\mathsf{neg}(\sigma)]\!],\\ &\mathcal{P}([\![\pi_1]\!] \mid ! \mid [\![\pi_1]\!])(\varepsilon(p(\sigma \in \Sigma_P))) \end{aligned}$$

where $[\![\pi_1]\!] : [\![\mathsf{neg}(\sigma)]\!] \times \top \to [\![\mathsf{neg}(\sigma)]\!]$ *is essentially the identity. As in the previous proof, this reindexing has* $\top$ *as middle morphism, hence decomposes via* $\mathcal{P}(\Downarrow^{\pm}_{\Delta})$ *and cancels out with* ε*, thus obtaining that* $p'(\sigma) = p(\sigma)$*.*

I Details for soundness and completeness

▶ **Corollary 44** (Soundness). *If entailment is derivable in directed first-order logic in the empty theory then it holds in every directed doctrine* $\mathcal{P}$*.*

Proof. By Theorem 32, $\emptyset$ is initial in Theory, hence $\mathsf{Syn}(\emptyset)$ is also initial. From Theorem 35 we have a doctrine morphism $\mathsf{Syn}(\emptyset) \longrightarrow \mathcal{P}$ which corresponds precisely with a model of the empty theory in $\mathcal{P}$. Hence, we can apply this doctrine morphism to the derivation tree of the entailment given to obtain the a corresponding internal statement in $\mathcal{P}$. ◀

▶ **Corollary 45** (Completeness). *If entailment holds in every directed doctrine* $\mathcal{P}$ *then it is derivable in directed first-order logic in the empty theory.*

Proof. Since $\mathsf{Syn}(\emptyset)$ is a directed doctrine, the statement is also true in $\mathsf{Syn}(\emptyset)$. By definition of $\mathsf{Syn}(\emptyset)$, a statement is true internally of such doctrine precisely when it is derivable in the empty theory $\emptyset$. ◀