Directed First-Order Logic

Andrea Laretto, Fosco Loregian, Niccolò Veltri

Tallinn University of Technology

EPN WG6 Meeting

April 17th, 2025

Symmetric equality

- The most interesting aspect of logic/MLTT: *equality*.
- Today we will only talk about **first-order**:

$$\frac{}{[x : A, \Gamma]\ \Phi \vdash x = x}\ (\text{refl}) \qquad \frac{[z : A, \Gamma] \qquad \Phi(z, z) \vdash P(z, z)}{[a : A, b : A, \Gamma]\ a = b,\ \Phi(a, b) \vdash P(a, b)}\ (\text{J})$$

- Transitivity of equality:

$$\frac{\dfrac{}{[z : A, c : A] \qquad z = c \vdash z = c}\ (\text{id})}{[a : A, b : A, c : A]\ a = b,\ b = c \vdash a = c}\ (\text{J})$$

- Equality in first-order logic/MLTT is inherently symmetric:

$$\frac{\dfrac{}{[z : A] \qquad \vdash z = z}\ (\text{refl})}{[a : A, b : A]\ a = b \vdash b = a}\ (\text{J})$$

Motivation: Directed type theory

Martin-Löf type theory with refl$/J$ *is intrinsically about symmetric equality.*
Directed type theory *is the generalization to "directed equality".*

The interpretation of directed type theory with *(1-)categories*:

$$\begin{aligned}
\text{Types} &\rightsquigarrow \text{Categories} \\
\text{Terms} &\rightsquigarrow \text{Functors} \\
\text{Equalities } e : a = b &\rightsquigarrow \text{Morphisms } e : \hom(a, b) \\
\text{Equality types } =_A : A \times A \to \text{Type} &\rightsquigarrow \text{Hom types } \hom_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}
\end{aligned}$$

$\to$ Now types have a *polarity*, $\mathbb{C}$ and $\mathbb{C}^{\mathsf{op}}$, i.e., the opposite category.
$\to$ Now equalities $e : \hom(a, b)$ have *directionality:* rewrites, trans., processes.

We want to find which syntactic restriction of MLTT
allow for types can be interpreted as categories.

Current approaches to directed type theory

- Semantically, refl should be $\mathsf{id}_c \in \hom_{\mathbb{C}}(c, c)$ for $c : \mathbb{C}$.
- Transitivity of directed equality $\rightsquigarrow$ composition of morphisms in $\mathbb{C}$.

$$\dfrac{\dfrac{}{[z : \mathbb{C}^{\mathsf{op}}, c : \mathbb{C}] \quad \hom(z, c) \vdash \hom(z, c)}\ (\mathsf{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}, c : \mathbb{C}] \ \hom(a, b),\ \hom(\bar{b}, c) \vdash \hom(a, c)}\ (\mathsf{J})$$

- However, directed type theory is not so straightforward:

$$\dfrac{a : \mathbb{C}}{\mathsf{refl}_a ...? : \hom_{\mathbb{C}}(a, a)} \quad \rightsquigarrow \quad \dfrac{a : \mathbb{C}^{\mathsf{core}}}{\mathsf{refl}_a : \hom(\mathsf{i}^{\mathsf{op}}(a), \mathsf{i}(a))}$$ [North 2018]

- *Problem:* rule is not functorial w.r.t. variance of $\hom_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$, since $a : \mathbb{C}$ appears both contravariantly and covariantly.
- A possible approach to DTT in **Cat**: use groupoids!
  $\to$ Use the maximal subgroupoid $\mathbb{C}^{\mathsf{core}}$ to collapse the two variances.
- Then a J-like rule is validated, again using groupoidal structure.

Towards dinatural directed type theory

Can we interpret a (first-order) directed type theory in 1-categories without having to use groupoids?

Our approach: yes, by validating rules with dinatural transformations.

- *Intuition:* dinaturals allow for the same x to appear co-/contra-variantly.
- Semantically, dinaturality also tells us what a directed J rule should be.
- Directed J rule: very similar to the usual symmetric J rule, but with a syntactic restriction which does **not** allow for symmetry.
- Allows to give a type-theoretical meaning to (co)end calculus.
- *Downside*: dinaturals do not always compose!
  $\rightarrow$ Restricted *cut rules*, only with *naturals.*
- Big but inevitable restriction $\rightarrow$ we don't get usual CwFs/fibrations.

Today: preorders and directed doctrines

- This (and much more!) in our previous paper:

> *"Directed equality with dinaturality"* (`arXiv:2409.10237`)
> Andrea Laretto, Fosco Loregian, Niccolò Veltri (5)

- Today, I'll talk about a spinoff of this story:

That paper	$\rightsquigarrow$	**Today**
Categories	$\rightsquigarrow$	Preorders
Proof-relevance	$\rightsquigarrow$	Proof-irrelevance (rewrites happen or not)
Dinatural trans. in Set	$\rightsquigarrow$	"Diagonal" entailments $P(x,x) \leq Q(x,x)$
Not all cuts	$\rightsquigarrow$	Entailments compose (no hexagon to check)
No abstract model	$\rightsquigarrow$	*Directed doctrines*
Rules for $\hom$	$\rightsquigarrow$	Directed eq. $\leq$ as a *relative left adjoint*
Rules for $\Rightarrow$	$\rightsquigarrow$	Polarized exponentials (which are unique)
Polarities as predicates	$\rightsquigarrow$	Polarities using context separation

- **Focus:** 1. make polarity precise, 2. universal properties of directed equality, for communities interested in FOL/doctrines/rewriting.

Directed first-order logic: syntax and semantics

Just like FOL, but:

- Terms and types are a simple axiomatization of simply-typed λ-calculus.
- A *directed equality formula* $s \leq_A t$, for two terms s, t
  Intuition: "the term s rewrites to the term t (of type A)",
- *Base formulas* $P(s \mid t)$, divided in a positive and a negative side,
- An *implication connective* $\psi \Rightarrow \varphi$ called *polarized exponential*.
- **Semantics:** Our main model for dFOL: the *preorder model*:

Notion	Syntax	Semantics
Types	A type	Preorders
Contexts	Γ	Product of preorders
Terms	$\Gamma \vdash t : A$	Monotone functions
Formulas	[...] φ prop	Monotone functions into $\mathbf{I} := \{0 \to 1\}$
Base formulas	$P(x \mid y)$	$P : [\![A]\!]^{\mathsf{op}} \times [\![A]\!] \to \mathbf{I}$
Directed equality	$x \leq_A y$	$- \leq_A - : [\![A]\!]^{\mathsf{op}} \times [\![A]\!] \to \mathbf{I}$
Polarized exponentials	$\psi \Rightarrow \varphi$	$- \Rightarrow - : \mathbf{I}^{\mathsf{op}} \times \mathbf{I} \to \mathbf{I}$

- Key idea: the semantic "$-^{\mathsf{op}}$" of preorders must be reflected in syntax.

Polarity and variance

- A **position** is any place in which a variable can appear (even in terms). e.g., there are 5 positions in the FOL formula $P(x, y, s(z)) \wedge Q(t(v, w))$.
- A position has a **variance**, either *positive* or *negative*.
  Variance starts as positive and inverts when:
  1. It is used on the left side of $\psi \Rightarrow \varphi$,
  2. It is used on the left side of $s \leq t$,
  3. It is used on the left side of base predicates $P(s \mid t)$.
- *Semantically, variance inverts whenever* $-^{\mathsf{op}}$ *of preorders is involved.*
- Examples of variance:

$$f(x) \leq y \qquad (y \leq s(x)) \Rightarrow \varphi(z) \qquad (\psi(y) \Rightarrow \varphi) \Rightarrow \varphi$$

- A variable has **polarity**, based on *variance of positions where it is* <u>*used:*</u>
  1. A variable x is *positive* if it appears only in positive positions,
  2. A variable x is *negative* if it appears only in negative positions,
  3. A variable x is *dinatural* if it appears in positive *or* negative positions (i.e., always! in a sense, variables are always dinatural.)
- **Note:** there's no *"dinatural variance"*: you <u>use</u> a variable dinaturally.

Polarized contexts

- The polarity of a variable also lifts to whole entailments $\psi \vdash \varphi$.
- *Convenience:* $\overline{x}$ denotes the contravariant use of dinatural variables.
- Examples of polarity:

$$\begin{array}{r|l} x \leq y \wedge \overline{y} \leq z \vdash x \leq z & x \text{ neg}, y \text{ dinat}, z \text{ pos} \\ \overline{x} \leq y \vdash \overline{y} \leq x & x, y \text{ dinat} \\ x \leq y \vdash f(x) \leq f(y) & x \text{ neg}, y \text{ pos} \\ \vdash \overline{x} \leq x & x \text{ dinat} \\ \vdash f(\overline{x}) \leq g(x) & x \text{ dinat} \end{array}$$

- A **context** Γ is just a list of types and variables.
- A **polarized context** $\Theta \mid \Delta \mid \Gamma$: a triple of "physically separated" contexts, one for each polarity:
  - Θ is a list of variables usable *negatively* only,
  - Δ is a list of variables usable *dinaturally*,
  - Γ is a list of variables usable *positively* only.
- Variables from Θ and Γ are said to be *natural*.

Formulas – propositional connectives

- The judgement for formulas is indexed by a polarized context:

$$\boxed{[\Theta \mid \Delta \mid \Gamma]\ \varphi\ \mathsf{prop}}$$

- Propositional connectives of dFOL:

$$\frac{}{[\Theta \mid \Delta \mid \Gamma]\ \top\ \mathsf{prop}}$$

$$\frac{[\Theta \mid \Delta \mid \Gamma]\ \varphi\ \mathsf{prop} \qquad [\Theta \mid \Delta \mid \Gamma]\ \psi\ \mathsf{prop}}{[\Theta \mid \Delta \mid \Gamma]\ \varphi \land \psi\ \mathsf{prop}}$$

$$\frac{[\Gamma \mid \Delta \mid \Theta]\ \varphi\ \mathsf{prop} \qquad [\Theta \mid \Delta \mid \Gamma]\ \psi\ \mathsf{prop}}{[\Theta \mid \Delta \mid \Gamma]\ \varphi \Rightarrow \psi\ \mathsf{prop}}$$

- Note! $x \in \Gamma$ must be used negatively in φ to be positive in $\varphi \Rightarrow \psi$.

Formulas – base cases

What about the base cases? We use polarity here.

$$x \leq y \qquad P(n \mid p)$$

- *What variables can I use in a **positive** position?*
  $\rightarrow$ *Either a positive variable, or a dinatural variable.*
- *What variables can I use in a **negative** position?*
  $\rightarrow$ *Either a negative variable, or a dinatural variable.*

$$\frac{\Theta, \Delta \vdash s : A \qquad \Gamma, \Delta \vdash t : A}{[\Theta \mid \Delta \mid \Gamma]\ s \leq_A t \ \mathsf{prop}}$$

- Negative case: the term $s : A$ can use the *context concatenation* Θ, Δ.

$$\frac{P \in \Sigma_P \qquad \Theta, \Delta \vdash s : \mathsf{neg}(P) \qquad \Gamma, \Delta \vdash t : \mathsf{pos}(P)}{[\Theta \mid \Delta \mid \Gamma]\ P(s \mid t)\ \mathsf{prop}}$$

- *What can I use in place of a variable used **dinaturally**?*
  $\rightarrow$ *Only another dinatural variable: I must be able to use <u>the same variable</u> both negatively and positively.*

Semantics of polarized contexts and formulas

- In preorders, polarized contexts are interpreted as:

$$[\![[\Theta \mid \Delta \mid \Gamma]]\!] := [\![\Gamma]\!]^{\mathsf{op}} \times [\![\Delta]\!]^{\mathsf{op}} \times [\![\Delta]\!] \times [\![\Theta]\!]$$

- *Crucial:* $[\![\Delta]\!]$ is given with both variances.
- Formulas are interpreted (inductively) as monotone functions into $\mathbf{I}$:

$$[\![[\Theta \mid \Delta \mid \Gamma]\ \varphi\ \mathsf{prop}]\!] : [\![\Gamma]\!]^{\mathsf{op}} \times [\![\Delta]\!]^{\mathsf{op}} \times [\![\Delta]\!] \times [\![\Theta]\!] \to \mathbf{I}$$

- Semantics of directed equality formulas:

$$\begin{aligned} \leq_A &:= (x, y) \mapsto 1 \text{ iff } x \leq y && : [\![A]\!]^{\mathsf{op}} \times [\![A]\!] \to \mathbf{I}, \\ [\![s \leq_A t]\!] &:= ([\![s]\!]^{\mathsf{op}} \times [\![t]\!])\,;\leq_A && : [\![\Gamma]\!]^{\mathsf{op}} \times [\![\Delta]\!]^{\mathsf{op}} \times [\![\Delta]\!] \times [\![\Theta]\!] \to \mathbf{I} \end{aligned}$$

- Semantics of polarized exponentials:

$$\begin{aligned} \Rightarrow &:= \leq_{\mathbf{I}}, \quad : \mathbf{I}^{\mathsf{op}} \times \mathbf{I} \to \mathbf{I} \\ [\![\psi \Rightarrow \varphi]\!] &:= ((\mathsf{reorder}\,; [\![\psi]\!]^{\mathsf{op}}) \times [\![\varphi]\!])\,;\Rightarrow \end{aligned}$$

Polarized quantifiers

- We add also six "*polarized quantifiers*":

$$\begin{array}{ll} \exists^{-}x.\varphi & \forall^{-}x.\varphi \\ \exists^{\Delta}x.\varphi & \forall^{\Delta}x.\varphi \\ \exists^{+}x.\varphi & \forall^{+}x.\varphi \end{array}$$

- In preorders, $\exists^{\Delta}/\forall^{\Delta}$ are decategorifications of (co)ends: lub/glbs in $\mathbf{I}$ that diagonalize $[\![\varphi]\!] : [\![A]\!]^{\mathsf{op}} \times [\![A]\!] \to \mathbf{I}$, e.g.,

$$[\![\exists^{\Delta}x.\varphi(x,x)]\!] := \coprod_{x \in [\![A]\!]} [\![\varphi]\!](x,x).$$

- **Note!** The object x of the preorder $[\![A]\!]$ is used both co/contravariantly.

Entailments

- Judgement for syntactic entailments:

$$\boxed{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \varphi}$$

- In the preorder model, entailments are *"diagonal entailments"*, i.e., a decategorification of dinatural transformations, in $\mathbf{I}$:

$$[\![\Phi]\!], [\![\varphi]\!] : [\![\Gamma]\!]^{\mathsf{op}} \times [\![\Delta]\!]^{\mathsf{op}} \times [\![\Delta]\!] \times [\![\Theta]\!] \to \mathbf{I}$$

$$\begin{aligned} [\![[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \varphi]\!] \text{ holds iff } & \forall n \in [\![\Theta]\!]^{\mathsf{op}}, \\ & \forall d \in [\![\Delta]\!], \\ & \forall p \in [\![\Gamma]\!], \\ & [\![\Phi]\!](n,d,d,p) \leq [\![\varphi]\!](n,d,d,p). \end{aligned}$$

- **Note!** The object d of the preorder $[\![\Delta]\!]$ is used both co/contravariantly.

Entailments

- Structural rules:

$$\frac{}{[\Theta \mid \Delta \mid \Gamma]\ \Phi, \varphi, \Phi' \vdash \varphi}\ \text{(hyp)}$$

$$\frac{[\Theta \mid \Delta \mid \Gamma]\ \Psi \vdash \psi \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi, \psi, \Phi' \vdash \varphi}{[\Theta \mid \Delta \mid \Gamma]\ \Phi, \Psi, \Phi' \vdash \varphi}\ \text{(cut)}$$

- Reindexing uses this idea that "dinaturals are supplied with dinaturals":

$$\frac{\begin{array}{c}\Theta, \Delta \vdash \eta : N \\ \Delta \vdash \delta : D \\ \Gamma, \Delta \vdash \rho : P \\ [\Theta, n : N \mid \Delta, d : D \mid \Gamma, p : P]\ \Phi(n, \overline{d}, d, p) \vdash \varphi(n, \overline{d}, d, p)\end{array}}{[\Theta \mid \Delta \mid \Gamma]\ \Phi(\eta, \delta, \delta, \rho) \vdash \varphi(\eta, \delta, \delta, \rho)}\ \text{(reindex)}$$

Entailments – standard connectives

- Double-lines indicate bi-implications.
- We always show connectives in "adjoint-like" form, as bi-implications.
- Usual adjoint formulation of $\top$, $\wedge$, $\forall$, $\exists$:

$$\frac{}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \top}\ (\top)$$

$$\frac{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi \qquad [\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \varphi}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \psi \wedge \varphi}\ (\wedge)$$

$$\frac{p \in \{-, \Delta, +\} \qquad [\Theta \mid \Delta \mid \Gamma], [x :^p A]\ \Phi \vdash \varphi(x)}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash \forall^p x.\varphi(x)}\ (\forall)$$

$$\frac{p \in \{-, \Delta, +\} \qquad [\Theta \mid \Delta \mid \Gamma]\ \exists^p x.\psi(x), \Phi \vdash \varphi}{[\Theta \mid \Delta \mid \Gamma], [x :^p A]\ \psi(x), \Phi \vdash \varphi}\ (\exists)$$

- The last two rules express that *polarized quantifiers with polarity p are left/right adjoints to weakening on a fresh variable x with polarity p.*

Entailments – polarized exponentials

- Intuition for polarized exponentials: *all positions in ψ switch variance.*

$$\dfrac{[\ldots]\ \psi, \Phi \vdash \varphi}{[\ldots]\quad \Phi \vdash \psi \Rightarrow \varphi}\ (\Rightarrow)$$

$$\begin{array}{ll} [\quad N, N' \mid \Delta \mid \quad P, P']\ \psi & \text{prop} \\ [\Theta, N, P' \mid \Delta \mid \Gamma, P, N']\ \Phi, \varphi & \text{prop} \end{array}$$

$$\dfrac{[\Theta, N \mid \Delta, N', P' \mid \Gamma, P]\ \psi, \Phi \vdash \varphi}{[\Theta, P' \mid \Delta, N, P \mid \Gamma, N']\quad \Phi \vdash \psi \Rightarrow \varphi}\ (\Rightarrow)$$

- We need to consider every possible case nat $\to$ dinat, dinat $\to$ nat:
  - N , P natural above, dinatural below
  - N', P' dinatural above, natural below
  - Φ, ψ, φ share P, N.
- Derived rule: *negative can directly switch to positive* ($N = N'$+etc.)
- No general contexts Θ, Γ in ψ: *all* variables change polarity (except Δ)

Entailments – directed equality

- Most important rule for us: directed equality.
- Intuition: *an equality $x \leq y$ can be contracted only when x and y appear* ***naturally*** *in the conclusion* (same as in previous paper, in **Set**)

$$\frac{[\Theta \mid \Delta, z : A \mid \Gamma] \qquad \Phi \vdash \varphi(\overline{z}, z)}{[\Theta, a : A \mid \Delta \mid \Gamma, b : A] \; a \leq b, \Phi \vdash \varphi(a, b)} \; (\leq)$$

- (Note: a, b do not appear in Φ yet.)
- *Crucial:* this rule allows for most interesting properties of directed equality, except for symmetry! (e.g., $\mathbf{I}$ is a countermodel).

- Semantically, symmetric equality in doctrines is a left adjoint.
- *Can we use $(\leq)$ to characterize directed equality also as a left adjoint? Almost!* We give a characterization of $\leq$ as a ***relative*** **left adjoint**.

Derived rules for directed equality

- refl = the upwards direction of $(\leq)$ + (cut):

$$\frac{\Delta \vdash t : A}{[\Theta \mid \Delta \mid \Gamma]\ \Phi \vdash t \leq_A t}\ (\leq\text{-refl}_t)$$

- Directed equality with Frobenius (note! a, b are negative in Φ):

$$\frac{[\Theta \mid \Delta, z : A \mid \Gamma] \qquad \Phi(z, \overline{z}) \vdash \psi(\overline{z}, z)}{[\Theta \mid \Delta, a : A, b : A \mid \Gamma]\ a \leq b, \Phi(\overline{a}, \overline{b}) \vdash \psi(a, b)}\ (\leq^{\text{frob}})$$

follows from $(\Rightarrow)$, pick $\varphi(a, b) := \Phi(a, b) \Rightarrow \psi(a, b)$ to curry Φ to the left.

Examples with directed equality

- Transitivity of directed equality:

$$\dfrac{\dfrac{}{[z : A \mid \bullet \mid c : A] \qquad z \leq c \vdash z \leq c}\,(\text{hyp})}{[a : A \mid b : A \mid c : A]\; a \leq b, \overline{b} \leq c \vdash a \leq c}\,(\leq^-)$$

- Congruence of directed equality (i.e. internal monotonicity for terms):

$$\dfrac{\dfrac{}{[\bullet \mid z : A \mid \bullet] \qquad \vdash f(\overline{z}) \leq_B f(z)}\,(\leq\text{-refl}_t)}{[a : A \mid \bullet \mid b : A]\; a \leq_A b \vdash f(a) \leq_B f(b)}\,(\leq)$$

- Transport of equalities between proofs of predicates:

$$\dfrac{\dfrac{}{[\bullet \mid \bullet \mid z : A] \qquad P(z) \vdash P(z)}\,(\text{hyp})}{[a : A \mid \bullet \mid b : A]\; a \leq b, P(a) \vdash P(b)}\,(\leq^+)$$

Examples with directed equality

- Pair of rewrites:

$$\dfrac{\dfrac{\overline{[\bullet \mid x : A, y : B \mid \bullet] \ \vdash (\overline{x}, \overline{y}) \leq_{A\times B} (x, y)} \ (\leq\text{-refl}_t)}{[b : A \mid x : A \mid b' : B] \ b \leq_B b' \vdash (\overline{x}, b) \leq_{A\times B} (x, b')} \ (\leq)}{[a : A, b : A \mid \bullet \mid a' : A, b' : B] \ a \leq_A a', \ b \leq_B b' \vdash (a, b) \leq_{A\times B} (a', b')} \ (\leq)$$

  For the other direction use congruence with the projection terms.

- Higher-order rewriting:

$$\dfrac{\dfrac{\overline{[\bullet \mid h : A \Rightarrow B, x : A \mid \bullet] \ \vdash \overline{h} \cdot \overline{x} \leq_B h \cdot x} \ (\leq\text{-refl}_t)}{[\bullet \mid h : A \Rightarrow B \mid \bullet] \ \vdash \forall^{\Delta} x. \ \overline{h} \cdot \overline{x} \leq_B h \cdot x} \ (\forall^{\Delta}_t)}{[f : A \Rightarrow B \mid \bullet \mid g : A \Rightarrow B] \ f \leq_{A\Rightarrow B} g \vdash \forall^{\Delta} x. \ f \cdot \overline{x} \leq_B g \cdot x} \ (\leq)$$

  The other direction is not derivable in general, since it is a directed version of extensionality "on 2-cells".

Example of signatures

- Signature of λ-terms using HOAS:

$$\begin{array}{lll} \Sigma_{\text{types}} & := \{T\} & \\ \Sigma_{\text{terms}} & := \{\tilde{\lambda}, \mathsf{app}\} & \\ \Sigma_{\text{term-eqs}} & := \{\eta\} & \mathsf{dom}(\tilde{\lambda}) := T \Rightarrow T,\ \mathsf{cod}(\tilde{\lambda}) := T \\ \Sigma_{\text{preds}} & := \{\} & \mathsf{dom}(\mathsf{app}) := T \times T,\ \mathsf{cod}(\mathsf{app}) := T \\ \Sigma_{\text{axioms}} & := \{\beta\} & \end{array}$$

$$\frac{}{[f : T \Rightarrow T]\ \Big(\lambda x.\mathsf{app}(\tilde{\lambda}(f), x)\Big) = f : T \Rightarrow T}\ (\eta)$$

$$\frac{}{[\bullet \mid s : T \Rightarrow T, t : T \mid \bullet]\ \mathsf{app}(\tilde{\lambda}(\overline{s}), \overline{t}) \leq s \cdot t}\ (\beta)$$

We can *prove* that rewriting is trans./refl., a congruence on app, $\tilde{\lambda}$ for free:

$$\frac{\dfrac{}{[\bullet \mid z : T, t : T \mid \bullet]\ \vdash \mathsf{app}(\overline{z}, \overline{t}) \leq_T \mathsf{app}(z, t)}\ (\leq\text{-refl}_t)}{[s : T \mid t : T \mid s' : T]\ s \leq_T s' \vdash \mathsf{app}(s, \overline{t}) \leq_T \mathsf{app}(s', t)}\ (\leq)$$

Doctrines with restrictions

- Doctrines $\approx$ models of first-order logic [Lawvere 1970]
- Doctrine = category $\mathbb{C}$ with products + (pseudo)functor $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$.
- *Intuition:* objects Γ of $\mathbb{C}$ are contexts, $\mathcal{P}(\Gamma)$ is the poset of formulas with implication as relation.
- *How to model polarity using doctrinal semantics?*
- Idea: change the base category with a specific construction on Ctx,
- $\to$ ask for (relative) adjunctions only for *specific reindexings*.

Polarized doctrines

Definition (*Polarization category* of $\mathbb{C}$)

Given a category $\mathbb{C}$ with products, the category ndp($\mathbb{C}$) is defined as:

- Objects: triples of objects $(\Theta \mid \Delta \mid \Gamma) \in \mathbb{C}_0 \times \mathbb{C}_0 \times \mathbb{C}_0$,
- Morphisms $(\Theta \mid \Delta \mid \Gamma) \to (\Theta' \mid \Delta' \mid \Gamma')$ are triples $(n \mid d \mid p)$ with

$$\begin{aligned} n &: \Theta \times \Delta \to \Theta' \\ d &: \qquad\quad \Delta \to \Delta' \\ p &: \ \Gamma \times \Delta \to \Gamma' \end{aligned}$$

Definition (Polarized doctrines)

A *(split) polarized doctrine* is a category $\mathbb{C}$ with finite products and a functor $\mathcal{P} : \mathrm{ndp}(\mathbb{C})^{\mathsf{op}} \to \mathbf{Pos}$ which satisfies a certain technical condition called the **no-dinatural-variance** condition.

No-dinatural-variance condition

- *Intuition*: the **ndv** condition is necessary because in the base case $P(s \mid t)$ we do not ask for any "dinatural" term (which *would* be there for standard doctrines on $\mathrm{ndp}(\mathbb{C})$).

Definition (*ndv* condition)

A functor $\mathcal{P} : \mathrm{ndp}(\mathbb{C})^{\mathrm{op}} \to \mathbf{Pos}$ is said to satisfy the *no-dinatural-variance condition* if the functor

$$\mathcal{P}(\mathrm{diag}_\Delta) : \mathcal{P}(\Theta \times \Delta \mid \top \mid \Gamma \times \Delta) \to \mathcal{P}(\Theta \mid \Delta \mid \Gamma)$$

that reindexes with $\mathrm{diag}_\Delta := (\mathrm{id}_{\Theta\times\Delta} \mid !_\Delta \mid \mathrm{id}_{\Gamma\times\Delta})$ is a *bijection of sets*.

$$\frac{\mathrm{diag}_\Delta \quad : (\Theta \mid \Delta \mid \Gamma) \to (\Theta \times \Delta \mid \top \mid \Gamma \times \Delta)}{\begin{array}{ll} n := \mathrm{id}_{\Theta\times\Delta} & : (\Theta) \times (\Delta) \to (\Theta \times \Delta) \\ d := !_\Delta & : \top \to \Delta \\ p := \mathrm{id}_{\Gamma\times\Delta} & : (\Gamma) \times (\Delta) \to (\Gamma \times \Delta) \end{array}}$$

- It's almost never an isomorphism of posets.

No-dinatural-variance condition

Theorem (*ndv* for the syntactic doctrine)

There is a bijection of formulas as follows:

$$[\Theta \times \Delta \mid \top \mid \Gamma \times \Delta]\ \varphi\ \text{prop} \cong [\Theta \mid \Delta \mid \Gamma]\ \text{prop}.$$

Proof. *By induction. Base case: given a derivation tree for* $s \leq t$,

$$\frac{\Theta, \Delta \vdash s : A \qquad \Gamma, \Delta \vdash t : A}{[\Theta \mid \Delta \mid \Gamma]\ s \leq_A t\ \text{prop}} \rightsquigarrow \frac{(\Theta \times \Delta), \top \vdash \tilde{s} : A \qquad \top, (\Gamma \times \Delta) \vdash \tilde{t} : A}{[\Theta \times \Delta \mid \top \mid \Gamma \times \Delta]\ \tilde{s} \leq_A \tilde{t}\ \text{prop}}$$

I construct a formula in context $[\Theta \times \Delta \mid \top \mid \Gamma \times \Delta]\ \tilde{s} \leq_A \tilde{t}\ \text{prop}$*.*
This function is inverse to the reindexing functor shown in ndv.

Directed equality as left adjoint

- Using $\mathrm{ndp}(\mathbb{C})$ seems useless... I'm just changing the base!
- But now I can *express* the *reindexing that collapses natural variables into a single dinatural one*.
- **Collapse** of two naturals with opposite variance into one dinatural:

$$\mathcal{P}(\mathsf{contr}_A) : \mathcal{P}(\Theta \times A \mid \Delta \mid \Gamma \times A) \to \mathcal{P}(\Theta \mid \Delta \times A \mid \Gamma)$$

$$\begin{array}{ll}
\mathsf{contr}_\Delta & : (\Theta \mid \Delta \times A \mid \Gamma) \to (\Theta \times A \mid \Delta \mid \Gamma \times A) \\
\hline
n := \langle \pi_1, \pi_3 \rangle & : (\Theta) \times (\Delta \times A) \to (\Theta \times A) \\
d := \pi_1 & : \Delta \times A \to \Delta \\
p := \langle \pi_1, \pi_3 \rangle & : (\Gamma) \times (\Delta \times A) \to (\Gamma \times A)
\end{array}$$

- **Weakening** with an extra dinatural variable of type A:

$$\mathcal{P}(\mathsf{wk}_A) : \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \to \mathcal{P}(\Theta \mid \Delta \times A \mid \Gamma)$$

- **Directed equality**:
  we ask that there is a relative left adjoint to **collapse**,
  relative to the **weakening** functor.

Relative adjunctions

Given a situation like this in **Cat**:

$$\begin{tikzcd}
& \mathbb{D} \arrow[dr, "R"] & \\
\mathbb{C} \arrow[ur, "L"] \arrow[rr, "J"'] & \arrow[u, phantom, "\dashv"] & \mathbb{X}
\end{tikzcd}$$

We say

L is a J-relative left adjoint to R

if there is a natural bijection

$$\begin{array}{c} \mathbb{X}(J(x), R(y)) \\ \hline\hline \mathbb{D}(L(x), \quad y \quad) \end{array}$$

Directed equality as relative left adjoint

Definition (Having directed equality)

A polarized doctrine $\mathcal{P} : \mathsf{ndp}(\mathbb{C})^{\mathsf{op}} \to \mathbf{Pos}$ has *directed equality* iff there is a $\mathcal{P}(\mathsf{wk}_A)$-relative left adjoint $\leq_A \times -$ to the functor $\mathcal{P}(\mathsf{contr}_A)$:

$$\begin{aligned} \mathcal{P}(\mathsf{contr}_A) &: \mathcal{P}(\Theta \times A \mid \Delta \mid \Gamma \times A) \to \mathcal{P}(\Theta \mid \Delta \times A \mid \Gamma) \\ \mathcal{P}(\mathsf{wk}_A) &: \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \to \mathcal{P}(\Theta \mid \Delta \times A \mid \Gamma) \end{aligned}$$

$$\begin{tikzcd} & \mathcal{P}(\Theta \times A \mid \Delta \mid \Gamma \times A) \arrow[dr, "\mathcal{P}(\mathsf{contr}_A)"] \arrow[d, phantom, "\dashv"] & \\ \mathcal{P}(\Theta \mid \Delta \mid \Gamma) \arrow[ur, "\leq_A \times -"] \arrow[rr, "\mathcal{P}(\mathsf{wk}_A)"'] & {} & \mathcal{P}(\Theta \mid \Delta \times A \mid \Gamma) \end{tikzcd}$$

$$\frac{[\Theta \mid \Delta \times A \mid \Gamma] \ \mathcal{P}(\mathsf{wk}_A)(\Phi) \leq \mathcal{P}(\mathsf{contr}_A)(\varphi)}{[\Theta \times A \mid \Delta \mid \Gamma \times A] \quad (\leq_A \times \Phi) \leq \varphi} \ (\leq)$$

...and Beck-Chevalley conditions.

Polarized exponentials

- Polarized exponentials are defined basically by following the syntax.

Definition (Polarized exponentials)

A polarized doctrine $\mathcal{P}$ *has polarized exponentials* iff it has conjunction $\wedge$ and there is a functor

$$\begin{aligned} - \Rightarrow - : & \mathcal{P}(N \times N' \mid \Delta \mid P \times P')^{\mathsf{op}} \\ & \times \mathcal{P}(\Theta \times N \times P' \mid \Delta \mid \Gamma \times P \times N') \\ & \to \mathcal{P}(\Theta \times P' \mid \Delta \times N \times P \mid \Gamma \times N') \end{aligned}$$

such that, for every $\Theta, \Delta, \Gamma, N, N', P, P' \in \mathbb{C}$,
for every $\Phi, \varphi \in \mathcal{P}(\Theta \times N \times P' \mid \Delta \mid \Gamma \times P \times N')$,
for every $\psi \in \mathcal{P}(N \times N' \mid \Delta \mid P \times P')$, the top holds iff the bottom holds:

$$\frac{\mathcal{P}(\pi_2, \mathsf{id}, \pi_2)(\mathcal{P}(\Uparrow^{\Delta}_{N',P'})(\psi)) \wedge \mathcal{P}(\Uparrow^{\Delta}_{N',P'})(\Phi) \leq \mathcal{P}(\Uparrow^{\Delta}_{N',P'})(\varphi)}{\mathcal{P}(\Uparrow^{\Delta}_{N,P})(\Phi) \leq \psi \Rightarrow \varphi}$$

- Open question: *can this be expressed as a (relative) adjunction?*

Polarized exponentials is property, not structure

Theorem

*In the presence of **ndv**, polarized exponentials are unique.*

Proof.

$$
\begin{array}{c|l}
[\quad N, N' \mid \Delta \mid \quad P, P']\ \psi\ \text{prop} & \\
[\Theta, N, P' \mid \Delta \mid \Gamma, P, N']\ \varphi\ \text{prop} & \\
\hline
[\Theta, P' \mid \Delta, N, P \mid \Gamma, N']\ \psi \Rightarrow \varphi \leq \psi \Rightarrow \varphi & \text{(hyp)} \\
\hline\hline
[\Theta, P' \mid \Delta, N, P \mid \Gamma, N']\ \mathcal{P}(\Uparrow^{\pm !}_{\Delta,N,P})(\varepsilon(\psi \Rightarrow \varphi)) \leq \psi \Rightarrow \varphi & ((\psi \Rightarrow \varphi) = \mathcal{P}(\Uparrow^{\pm !}_{\Delta,N,P})(\varepsilon(\psi \Rightarrow \varphi))) \\
\hline\hline
[\Theta, P' \mid \Delta, N, P \mid \Gamma, N']\ \mathcal{P}(\Uparrow^{\Delta}_{\Delta,N,P}\, ;\, (\pi_1 \mid !_{\Delta,N,P} \mid \pi_1))(\varepsilon(\psi \Rightarrow \varphi)) \leq \mathcal{P}(\Uparrow^{\pm !}_{\Delta})(\varepsilon(\psi)) \Rightarrow \mathcal{P}(\Uparrow^{\pm !}_{\Delta})(\varepsilon(\varphi)) & (\psi = \mathcal{P}(\Uparrow^{\pm !}_{\Delta})(\varepsilon(\psi))) \\
\hline\hline
[\Theta, P' \mid \Delta, N, P \mid \Gamma, N']\ \mathcal{P}(\Uparrow^{\Delta}_{N,P}\, ;\, \Uparrow^{\Delta}_{\Delta}\, ;\, (\pi_1 \mid !_{\Delta,N,P} \mid \pi_1))(\varepsilon(\psi \Rightarrow \varphi)) \leq \mathcal{P}(\Uparrow^{\pm !}_{\Delta})(\varepsilon(\psi)) \Rightarrow \mathcal{P}(\Uparrow^{\pm !}_{\Delta})(\varepsilon(\varphi)) & \\
\hline\hline
[\Theta, P' \mid \Delta, N, P \mid \Gamma, N']\ \mathcal{P}(\Uparrow^{\Delta}_{N,P})(\mathcal{P}(\Uparrow^{\Delta}_{\Delta}\, ;\, (\pi_1 \mid !_{\Delta,N,P} \mid \pi_1))(\varepsilon(\psi \Rightarrow \varphi))) \leq \mathcal{P}(\Uparrow^{\pm !}_{\Delta})(\varepsilon(\psi)) \Rightarrow \mathcal{P}(\Uparrow^{\pm !}_{\Delta})(\varepsilon(\varphi)) & \\
\hline\hline
[\Theta, P' \mid \Delta, N, P \mid \Gamma, N']\ \quad \cdots \quad \leq \mathcal{P}(\Uparrow^{\Delta}_{N',P'})(\mathcal{P}(\Uparrow^{\pm !}_{\Delta})(\varepsilon(\varphi))) & (\Rightarrow) \\
\hline\hline
[\Theta, P' \mid \Delta, N, P \mid \Gamma, N']\ \mathcal{P}(\Uparrow^{\Delta}_{N,P})(\mathcal{P}(\Uparrow^{\Delta}_{\Delta}\, ;\, (\pi_1 \mid !_{\Delta,N,P} \mid \pi_1))(\varepsilon(\psi \Rightarrow \varphi))) \leq \mathcal{P}(\Uparrow^{\pm !}_{\Delta})(\varepsilon(\psi)) \Rightarrow' \mathcal{P}(\Uparrow^{\pm !}_{\Delta})(\varepsilon(\varphi)) & (\Rightarrow') \\
\hline\hline
[\Theta, P' \mid \Delta, N, P \mid \Gamma, N']\ \mathcal{P}(\Uparrow^{\pm !}_{\Delta,N,P})(\varepsilon(\psi \Rightarrow \varphi)) \leq \mathcal{P}(\Uparrow^{\pm !}_{\Delta})(\varepsilon(\psi)) \Rightarrow' \mathcal{P}(\Uparrow^{\pm !}_{\Delta})(\varepsilon(\varphi)) & \\
\hline\hline
[\Theta, P' \mid \Delta, N, P \mid \Gamma, N']\ \psi \Rightarrow \varphi \leq \psi \Rightarrow' \varphi & \\
\end{array}
$$

Directed doctrines

Definition (Directed doctrine)

A directed doctrine is a polarized doctrine equipped with

- Directed equality $\leq_A$, polarized exponentials $\Rightarrow$,
- Polarized quantifiers $\forall^p x.\varphi, \exists^p x.\varphi$,
- Conjunction $\land$, terminals $\top$.

- **DDoctrine**: 2-category of directed doctrines (1-cells preserve everything)
- **Theory**: 1-category of theories (signature + axioms)

Theorem (Internal language correspondence)

Directed first order logic is the internal language of directed doctrines. There is a bijection up-to-isomorphism as follows:

$$\frac{\mathit{Syn}(\Sigma) \longrightarrow \mathcal{P} \ \textit{in}\ \text{DDoctrine}}{\overline{\Sigma \longrightarrow \mathit{Lang}(\mathcal{P}) \ \textit{in}\ \text{Theory}}}$$

Polarized doctrines to doctrines

- Directed doctrine $\to$ doctrine: precompose $\mathcal{P}$ with

$$\begin{aligned} \Downarrow &: \mathbb{C} \to \mathsf{ndp}(\mathbb{C}) \\ \Downarrow &:= C \mapsto (\top \mid C \mid \top) \end{aligned}$$

- Doctrine $\to$ directed doctrine: precompose $\mathcal{P}$ with

$$\begin{aligned} \Uparrow &: \mathsf{ndp}(\mathbb{C}) \to \mathbb{C} \\ \Uparrow &:= (\Theta \mid \Delta \mid \Gamma) \mapsto \Theta \times \Delta \times \Delta \times \Gamma \end{aligned}$$

  satisfying the *no-dinatural-variance* condition.
- Open question: *(2-)adjunctions DDoctrines* $\leftrightarrows$ *Doctrines?*

Conclusion and future work

We saw a simple extension to FOL with a model in preorders, with a notion of variance/polarity, polarized quantifiers, and directed equality characterized by a left relative adjunction to a diagonal-like reindexing.

Future work in order of decreasing importance:

1. Find "more geometric" models aside from preorders,
2. Adding op-types: internalize the swap between positive and negative contexts,
3. Completeness for preorders,
4. Investigate precisely 2-adjunctions for doctrines/directed doctrines,
5. Other examples: theory of Heyting algebras, rewriting logic, [Meseguer 2012] model checking via rewriting, modal extensions, etc. ...

More pressing issues for directed type theory:

1. Takeaway: polarized contexts + dinatural collapse + left relative adjunction.
2. This is a spinoff for the doctrinal and proof-irrelevant side of directedness.
3. Immediate future: *dinatural context extension* based on two-sided fibrations $\rightsquigarrow$ *towards dependent dinatural directed type theory.*

The $\int$.

Paper: *"Directed First-Order Logic"* (arXiv:2504.11225)
Website: iwilare.com

Thank you for the attention!