Directed equality with dinaturality

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Type theory and symmetric equality

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- Equality in MLTT is inherently symmetric:

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Models of MLTT

What can types be?

Martin-Löf type theory admits a sound interpretation in:

- Types as sets. [Martin-Löf 1971]
- Types as groupoids. [Hoffmann-Streicher 1999]
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Some familiar structures are missing... <u>*what about categories and posets?!*</u>

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\text{Terms} &\rightsquigarrow \text{Functors} \\
\text{Relations} &\rightsquigarrow \text{Profunctors} \\
\text{Points of a type} &\rightsquigarrow \text{Objects of a category} \\
\text{Equalities } e : a = b &\rightsquigarrow \text{Morphisms } e : \hom(a, b) \\
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We want to find which syntactic restriction of MLTT *allow for types can be interpreted as categories.*

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- Then a J-like rule is validated, but *again using groupoidal structure*.

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  $\to$ a J rule (directed equality elimination).
- The directed J rule is extremely similar to the standard MLTT J rule, but with a syntactic restriction which does **not** allow for symmetry.
- This comes at the cost of compositionality:
  we have a cut rule only if variables appear *naturally* in a entailment.

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Comparison between symmetric equality and our directed equality rule:

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- Same terms provable in MLTT about equality: transitivity, transport, congruence, *but not symmetry!*
- Proving properties about them follows precisely the steps in MLTT. (e.g., we have the same computation rules.)

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- We then use the rules to give *simple proofs* of theorems in category theory, using dinaturality and viewing $\hom$ as directed equality.
- **Claim**: *(co)end calculus is the first-order instantiation of a directed type theory with quantifiers, with semantics in 1-categories*.

Motivation 2: Profunctors

Another story, now in CT:

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Relations $A \times B \to \{\top, \bot\}$ $\rightsquigarrow$	Profunctors $\mathbb{C}^{\mathsf{op}} \times \mathbb{D} \to$ **Set**
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Why do unitality and associativity hold?

Because of rules for equality in FOL. $\rightsquigarrow$ Because of <u>the coYoneda lemma</u>.

Because of rules for $\exists$ and $\wedge$ in FOL. $\rightsquigarrow$ Because of <u>properties of coends</u>.

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Relations $A \times B \to \{\top, \bot\}$	$\rightsquigarrow$	Profunctors $\mathbb{C}^{\mathsf{op}} \times \mathbb{D} \to \mathbf{Set}$
Existential quantifiers $\exists x$	$\rightsquigarrow$	Coends $\int^x$
Composition: $\exists b \in B.\ R(a,b) \wedge Q(b,c)$	$\rightsquigarrow$	$\int^{b:\mathbb{B}} R(a,b) \times Q(b,c)$
Conjunction of truth values	$\rightsquigarrow$	Cartesian products of **Set**s
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Why do unitality and associativity hold?

Because of rules for equality in FOL. $\rightsquigarrow$ Because of the coYoneda lemma.
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Another story, now in CT:

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- We use FOL in **Rel**: there is no *directed logic* for **Prof**.
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- A syntactic treatment of directed logic (with directed equality and its rules, and quantifiers), e.g., using a doctrinal approach.

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$$\mathsf{Eq}_{\mathbb{C}} := \exists_{\Delta_{\mathbb{C}}}(\top_{\mathbb{C}}) \quad \in \quad \mathcal{P}\mathsf{sh}(\mathbb{C} \times \mathbb{C})$$

$$\mathsf{Eq}_{\mathbb{C}} = (a, b) \mapsto \int^{x \in \mathbb{C}} \hom_{\mathbb{C}}(a, x) \times \hom_{\mathbb{C}}(b, x)$$

i.e., any two objects forming a cospan for some x should be equated.

Note: These conditions also fail in groupoids.

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- *Thus in particular "equality" should be the functor $\hom_{\mathbb{C}}$* [...].
- *The term which would take the place of $\Delta_{\mathbb{C}}$ in such a more enlightened theory of equality would be the forgetful functor* $\mathsf{Tw}(\mathbb{C}) \to \mathbb{C}^{\mathsf{op}} \times \mathbb{C}$. [...]
- *Of course to abstract from this example would require at least the addition of a functor $T \xrightarrow{\mathsf{op}} T$ to the structure of a [doctrine].*

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Background on dinaturality and (co)ends

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$$\begin{tikzcd}
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Theorem (Naturals are dinaturals [Dubuc and Street 1969])

A dinatural between functors which do not depend on $\mathbb{C}^{\mathsf{op}}$ is just a natural.

Background: Dinaturals don't always compose

- Dinaturals don't always compose: $\alpha : F \xrightarrow{\bullet\bullet} G$, $\beta : G \xrightarrow{\bullet\bullet} H$,

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- They always compose with naturals: $\alpha : F \xrightarrow{\bullet\bullet} G$, $\beta : G \longrightarrow H$,

$$
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- Kan extensions: $(F : \mathbb{C} \to \mathbb{D})$

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Logical interpretation of (co)end calculus

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- (Pointwise) right/left Kan extensions using ends/coends:

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(\mathsf{Ran}_F P)(a) & \cong & \displaystyle\int_{x:\mathbb{C}} & \hom_{\mathbb{D}}(a,F(x)) & \Rightarrow & P(x) \\
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- Consider a doctrine $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ with $\wedge, \Rightarrow, \forall_{A[\Gamma]}, \exists_{A[\Gamma]}, \mathsf{Eq}_A$.

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- Compare these to the formula to compute Kan extensions via (co)ends:

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▶ there is yet no decomposition of these properties for doctrines, e.g., coends as quantifiers/adjoints, or "having directed equality".

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- A logical proof that $\forall_f(P)$ is right adjoint to precomposition with f:

$$
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- There is yet no formal system to do this for the directed case;

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$$
\dfrac{\dfrac{\dfrac{[y:D] \qquad \Gamma(y) \vdash (\forall_f P)(y) := \forall x.(y = f(x) \Rightarrow P(x))}{[x:C, y:D] \qquad \Gamma(y) \vdash y = f(x) \Rightarrow P(x)}}{[x:C, y:D] \ y = f(x), \Gamma(y) \vdash P(x)}}{[x:C] \qquad \Gamma(f(x)) \vdash P(x)}
$$

- There is yet no formal system to do this for the directed case;
- We follow exactly this proof for Lan/Ran, using our rules for dinaturality.

Setting

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\text{Types} &\rightsquigarrow \text{Categories (possibly with } -^{\mathsf{op}}\text{)} \\
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\text{Terms} &\rightsquigarrow \text{Functors } \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D} \\
\text{Propositions} &\rightsquigarrow \text{Endoprofunctors } \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D} \\
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- Guiding intuition: we study the "doctrine" (in paracategories)

$$\begin{aligned}
&\mathsf{Dinats} : \mathbf{Cat}^{\mathsf{op}} \to \mathbf{PARACAT} \\
&\mathsf{Dinats}(\mathbb{C}) := [\mathbb{C}^{\mathsf{op}} \times \mathbb{C}, \mathbf{Set}]_{\mathsf{dinaturals}}
\end{aligned}$$

Directed type theory: notation

- Example of an entailment:

$$\frac{[x : \mathbb{C}, y : \mathbb{D}]\ P(\overline{x}, \overline{y}, x, y) \vdash \alpha : Q(\overline{x}, \overline{y}, x, y)}{\textit{Semantics}\text{: "}\alpha \text{ is a dinatural from } P \text{ to } Q : (\mathbb{C} \times \mathbb{D})^{\mathsf{op}} \times (\mathbb{C} \times \mathbb{D}) \to \mathbf{Set}\text{."}}$$

- **Crucial:** variables now can appear both as $x : \mathbb{C}$ or $\overline{x} : \mathbb{C}^{\mathsf{op}}$.
- We give names to assumptions p, q:

$$[x : \mathbb{C}]\ p : P(\overline{x}, x), q : Q(\overline{x}, x) \vdash h[p, q] : R(\overline{x}, x)$$

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Semantics: "α is a dinatural from P to $Q : (\mathbb{C} \times \mathbb{D})^{\mathsf{op}} \times (\mathbb{C} \times \mathbb{D}) \to \mathbf{Set}$."

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  prop. context $\to$ interpreted as pointwise product of presheaves in **Set**.
- The following dinaturals are the same:

$$\begin{aligned} [x : \mathbb{C}]\ F(\overline{x}) &\vdash \alpha\ \ : G(\overline{x}, x) \\ [x : \mathbb{C}^{\mathsf{op}}]\ F(x) &\vdash \alpha' : G(x, \overline{x}) \end{aligned}$$

Directed type theory: propositional rules – Products

- Dinaturals support propositional conjunction, using products in **Set**.

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Theorem (Product of dipresheaves)

*There is an isomorphism of sets natural in $\Gamma, P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to$ **Set**:*

$$\frac{[x : \mathbb{C}]\ \Gamma(\overline{x}, x) \vdash P(\overline{x}, x) \times Q(\overline{x}, x)}{[x : \mathbb{C}]\ \Gamma(\overline{x}, x) \vdash P(\overline{x}, x), \qquad [x : \mathbb{C}]\ \Gamma(\overline{x}, x) \vdash Q(\overline{x}, x).}$$

Bottom side: product of sets of dinaturals.
*Similarly, $\top_{\mathbb{C}} : \mathbb{C}^{\diamond} \to$ **Set** $:= (c, c') \mapsto \{*\}$ has a unique dinat $\Gamma \xrightarrow{\bullet\bullet} \top_{\mathbb{C}}$.*

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▶ We will show how to use our rules to justify the usual exponential.

Directed type theory: term rules – Reindexing

- Entailments/dinaturals can be reindexed by terms/difunctors:

Theorem (Reindexing with difunctors) ( )

Take a difunctor $F : \mathbb{C}^\diamond \to \mathbb{D}$ *and a dinat* $\alpha : P \overset{\bullet\bullet}{\longrightarrow} Q$ *for* $P, Q : \mathbb{D}^\diamond \to \mathbb{E}$.

$$\frac{[x : \mathbb{D}] \qquad\qquad P(\overline{x}, x) \vdash \alpha : Q(\overline{x}, x)}{[x : \mathbb{C}] \; P(F^{\mathsf{op}}(x, \overline{x}), F(\overline{x}, x)) \vdash F^*(\alpha) : Q(F^{\mathsf{op}}(x, \overline{x}), F(\overline{x}, x))} \text{ (reindex)}$$

defined by $F^*(\alpha)_x := \alpha_{F(x,x)}$.

Directed type theory with dinaturality – refl

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Theorem (Directed equality introduction) (👏)

There is a dinatural transformation $\mathsf{refl}_{\mathbb{C}} : \top \xrightarrow{\bullet\bullet} \hom$,

$$\frac{}{[x : \mathbb{C}]\ \top \vdash \mathsf{refl}_{\mathbb{C}} : \hom_{\mathbb{C}}(\overline{x}, x)}\ \text{(refl)}$$

where $\top$ *denotes the dipresheaf constant in* $\top_{\textbf{\textit{Set}}}$.

Proof. $\alpha_x(*) := \mathsf{id}_x$. *Dinaturality: for any* $f : a \to b$, $f\ ;\mathsf{id}_b = \mathsf{id}_a\ ; f$.

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- This is reflexivity of directed equality, via identities.

Directed type theory with dinaturality – Intermezzo

- Before introducing the directed J rule, we show its fundamental idea: the connection between naturality and dinaturality.

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Theorem (Characterization of dinaturals via naturality) (👏)

*There is an isomorphism, natural in $P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to$ **Set**, between the set of dinaturals $P \xrightarrow{\bullet\bullet} Q$ and certain natural transformations:*

$$\frac{[x : \mathbb{C}]\ P(\overline{x}, x) \xrightarrow{\bullet\bullet} Q(\overline{x}, x)}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}]\, \hom(a, b) \longrightarrow P^{\mathsf{op}}(b, a) \Rightarrow Q(a, b)}$$

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Proof.

($\Downarrow$) *Given $\alpha : P \xrightarrow{\bullet\bullet} Q$ and $f : \hom(a, b)$, the map $P(b, a) \to Q(a, b)$ is exactly the side of the hexagon in the definition of dinaturality. This is obtained via the functorial action of P, Q.*

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The isomorphism follows from (di)naturality of both maps.

- Directed equality elimination is just this result, uncurried.

Theorem (Directed equality elimination) ( )

Take $\Gamma, P : (\mathbb{A}^{\mathsf{op}}) \times (\mathbb{A}) \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to$ ***Set***.
Given a dinatural $h : \Gamma \xrightarrow{\bullet\bullet} P$*, there is a dinatural* $J(h)$ *as follows:*

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \qquad\qquad k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash h[k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}] \; e : \hom(a, b), k : \Gamma(\overline{b}, \overline{a}, \overline{x}, x) \vdash J(h)[e, k] : P(a, b, \overline{x}, x)} \; (J)$$

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The dinatural $J(h)$ *satisfies the following "computation rule",*

$$J(h)_{zzx}[\mathsf{refl}_{\mathbb{A}z}, k] = h_{zx}[k]$$

for any object $z : \mathbb{A}, x : \mathbb{C}$ *and* $k \in \Gamma(z, z, x, x)$.

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Proof. *Explicitly, the dinatural $J(h)$ is given by*

$$J(h)_{abx}[e, k] := (\Gamma(\mathsf{id}_b, e, \mathsf{id}_x, \mathsf{id}_x) \,;\, h_{bx} \,;\, P(e, \mathsf{id}_b, \mathsf{id}_x, \mathsf{id}_x))[k].$$

Computation clearly holds for $e = \mathsf{id}_z$, without dinaturality.

Directed type theory with dinaturality – Intuition for J

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- it is enough to prove that P holds "on the diagonal",
  where a, b are identified with the same dinatural variable $z : \mathbb{A}$.
- Moreover, a, b can be identified in context *only if they appear negatively*.

- With the directed J rule we can define maps+properties directed equality.
- Examples: *transitivity*, *congruence*, *transport* for directed equality.

Example (Composition in a category)

Transitivity of directed equality $\rightsquigarrow$ categories $\mathbb{C}$ have *composition maps*.

Directed type theory with dinaturality – Example

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$$\dfrac{\dfrac{}{[z : \mathbb{C}, c : \mathbb{C}] \qquad \hom(\overline{z}, c) \vdash \mathsf{id} : \hom(\overline{z}, c)}\ (\mathsf{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}, c : \mathbb{C}]\ \hom(a, b),\ \hom(\bar{b}, c) \vdash J(\mathsf{id}) : \hom(a, c)}\ (J)$$

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We contract $\hom(a, b)$. Rule (J) can be applied: a, b appear only negatively in ctx (a does not) and positively in conclusion ($\overline{b}$ does not).

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The computation rule for $f \mathbin{;} g := J(\mathsf{id})[f, g]$ states unitality on the left.

Failure of symmetry for directed equality

These restrictions do *not* allow us to obtain directed equality is symmetric:

$$[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}]\ e : \hom(a, b) \nvdash \mathsf{sym} : \hom(\overline{b}, \overline{a})$$

$\hom(a, b)$ cannot be contracted: a, b must appear *positively* in conclusion.

- As in the classical case, the rule for directed equality is an isomorphism.

Theorem (J as isomorphism)

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \quad k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash h[k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}] \; e : \hom(a, b), k : \Gamma(\overline{b}, \overline{a}, \overline{x}, x) \vdash J(h)[e, k] : P(a, b, \overline{x}, x)} \; (\hom)$$

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The computation rule for $\hom$*-elimination is precisely* $J \mathbin{;} J^{-1} = \mathsf{id}$.
On the other hand, $J^{-1} \mathbin{;} J = \mathsf{id}$ *follows using (di)naturality.*

- Back to the transitivity example.
- *How do we prove associativity of transitivity and unitality on the right?*

Directed type theory – dependent directed J

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Theorem (Dependent J rule for judgemental equality) ()

*Take $\Gamma, P : (\mathbb{A}^{\mathsf{op}}) \times (\mathbb{A}) \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to$ **Set**. Given dinats α, β where J can be applied, then the equality above implies the one below:*

$$\frac{[z : \mathbb{A}, x : \mathbb{C}]\ k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash \alpha[\mathsf{refl}_z, k] = \beta[\mathsf{refl}_z, k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ e : \hom(a, b), k : \Gamma(\overline{a}, \overline{b}, \overline{x}, x) \vdash \alpha[e, k] = \beta[e, k] : P(a, b, \overline{x}, x)} \quad (J\text{-eq})$$

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Proof. *By hypothesis, $J^{-1}(\alpha) = J^{-1}(\beta)$, simply apply J.*

Directed type theory – trans assoc/unit via dependent J

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To prove associativity, simply contract $f : \hom(a, b)$:

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- *Note!* These are exactly the steps in MLTT for transitivity of paths.

Directed type theory – congr for directed equality

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Example (Directed equality is a congruence)

Let $F : \mathbb{C} \to \mathbb{D}$ be a functor.

$$\dfrac{\dfrac{}{[z : \mathbb{C}]\ \top \vdash \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{z}), F(z))}\ \text{(reindex)+(refl)}}{[x : \mathbb{C}, y : \mathbb{C}]\ \hom_{\mathbb{C}}(\overline{x}, y) \vdash \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{x}), F(y))}\ (J)$$

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Take $\mathsf{map}_F[f] := J(F^*(\mathsf{refl}_{\mathbb{C}}))$. Computation rule: F *maps* refl *to* refl:

$$\dfrac{}{[z : \mathbb{C}]\ \top \vdash \mathsf{map}_F[\mathsf{refl}_z] = F^*(\mathsf{refl}_z) : \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{x}), F(x))}\ (J\text{-comp})$$

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Functoriality holds, since both top sides $=$ $\mathsf{map}_F[g]$ via computation rules:

$$\dfrac{\dfrac{}{[z : \mathbb{C}, c : \mathbb{C}]\qquad g : \hom(\overline{z}, c) \vdash \mathsf{map}_F[\mathsf{refl}_z \,;\, g] = \mathsf{refl}_{F(z)} \,;\, \mathsf{map}_F[g] : \hom(\overline{z}, d)}\ (J\text{-comp})}{[a : \mathbb{C}, b : \mathbb{C}, c : \mathbb{C}]\ f : \hom(\overline{a}, b), g : \hom(\overline{b}, c) \vdash \mathsf{map}_F[f \,;\, g] = \mathsf{map}_F[f] \,;\, \mathsf{map}_F[g] : \hom(\overline{a}, d)}\ (J\text{-eq})$$

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- Transporting points of predicates (i.e., presheaves) along directed equalities is the functorial action of $P : \mathbb{C} \to \mathbf{Set}$:

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Example (Transporting along directed equality)

For any $P : \mathbb{C} \to \mathbf{Set}$:

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Computation rule for $\mathsf{subst}[f,k] := J(\mathsf{id})$:
"transporting a point of $P(a)$ along the path refl_z is the identity":

$$\dfrac{}{[z : \mathbb{C}]\ k : P(z) \vdash \mathsf{subst}[\mathsf{refl}_z, k] = k : P(z)}\ (J\text{-comp})$$

Directed type theory – Groupoidal case

- When $\mathbb{A} \cong \mathbb{A}^{\mathsf{op}}$ is a groupoid, hom is the characterization of symmetric equality as left adjoint to reindexing on diagonals. (with Frobenius)

$$\frac{\dfrac{[z : \mathbb{A}, x : \mathbb{C}] \qquad\qquad \Gamma(\overline{z}, z, \overline{x}, x) \vdash P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}] \ \ \mathrm{hom}(a, b),\ \Gamma(\overline{b}, \overline{a}, \overline{x}, x) \vdash P(a, b, \overline{x}, x)}\ (\mathrm{hom})}{\dfrac{[z : \mathbb{A}, x : \mathbb{C}] \qquad\qquad \Gamma(z, z, x) \vdash P(z, z, x)}{[a : \mathbb{A}, b : \mathbb{A}, x : \mathbb{C}]\ a \cong b,\ \Gamma(b, a, x) \vdash P(a, b, x)}\ (\mathsf{eq})}$$

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- The *proof-relevant directness* of **Cat** seems to be the fundamental obstacle to a fully compositional theory of dinaturals:

Theorem (Dinaturals in groupoids compose) (👋)

Given a groupoid $\mathbb{C} \cong \mathbb{C}^{\mathsf{op}}$ *and any* $\mathbb{D}$ *for* $F, G, H : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$*, all dinaturals* $\alpha : F \xrightarrow{\bullet\bullet} G$, $\beta : G \xrightarrow{\bullet\bullet} H$ *compose.*

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- *Can the directed rule also be characterized as an adjunction? Yes!...*

Directed equality as relative left adjoint

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- Unfortunately, for **Cat** we can only state this as a relative *para*-adjunction, because of non-compositionality of dinaturals.
- *Para*- indicates that composition is partial: paracategories, parafunctors, para-adjunctions. [Hermida 2003]

Background: Relative adjunctions

Definition (Left relative adjunction [Arkor 2024])

Consider this situation of functors and categories:

$$\begin{tikzcd} & \mathbb{D} \arrow[dr, "R"] \arrow[d, phantom, "\dashv"] & \\ \mathbb{C} \arrow[ur, "L"] \arrow[rr, "J"'] & & \mathbb{X} \end{tikzcd}$$

We say that L *is the J-relative left adjoint to R*, written $L \dashv_J R$, if

$$\mathbb{D}(L(x), y) \cong \mathbb{X}(J(x), R(y))$$

is a bijection natural in both arguments $x : \mathbb{C}, y : \mathbb{D}$.

Directed equality as adjoint (1)

Theorem (Directed equality as relative left adjoint) ($\approx$)

- *Let* $[\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$ *the paracategory where morphisms are dinats natural in* $\mathbb{A}^{\mathsf{op}}, \mathbb{A}$ *and dinatural in* $\mathbb{C}$*, and* $[\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$ *as dinaturals.*

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- *Take the parafunctor* $\pi^{*}_{\mathbb{A}} : [\mathbb{C}^{\diamond}, \textbf{\textit{Set}}] \to [\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$ *defined in the intuitive way by precomposing with projections.*

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- *There is a dipresheaf* $\hom_{\mathbb{A}} \in [\mathbb{A}^{\mathsf{op}} \times \mathbb{A}, \textbf{\textit{Set}}]$ *such that the functor*

$$\hom_{\mathbb{A}} \times - : [\mathbb{C}^{\diamond}, \textbf{\textit{Set}}] \to [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$$

$$\hom_{\mathbb{A}} \times \Gamma := (\overline{a}, a, \overline{x}, x) \mapsto \hom(\overline{a}, a) \times \Gamma(\overline{x}, x),$$

$$(\hom_{\mathbb{A}} \times \alpha_x)_{abc} := \lambda(e \in \hom(a, b), k \in \Gamma(c, c)).(e, \alpha_c(k))$$

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determines a $\pi^*_{\mathbb{A}}$*-relative left adjoint to the functor*

$$\Delta_{\mathbb{A}} \times - : [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}] \to [\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$$
$$\Delta_{\mathbb{A}} \times P := P$$
$$(\Delta_{\mathbb{A}} \times \alpha_{abc})_{zx} := \alpha_{zzx}.$$

Directed equality as adjoint (2)

Theorem (Directed equality as relative left adjoint, cont.) (☺≈)

- *Thus the left relative adjointness situation*

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is as follows:

$$\begin{tikzcd}
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{[\mathbb{C}^\diamond, \textbf{\textit{Set}}]} \arrow[ur, "\hom_{\mathbb{A}} \times -"] \arrow[rr, "\pi^*_{\mathbb{A}}"'] & & {[\mathbb{A}^\diamond \times \mathbb{C}^\diamond, \textbf{\textit{Set}}]}
\end{tikzcd}$$

Directed equality as adjoint (2)

Theorem (Directed equality as relative left adjoint, cont.) (≈)

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$$\begin{tikzcd}
 & {[\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]} \arrow[dr, "\Delta_{\mathbb{A}} \times -"] & \\
{[\mathbb{C}^{\diamond}, \textbf{\textit{Set}}]} \arrow[ur, "\hom_{\mathbb{A}} \times -"] \arrow[rr, "\pi_{\mathbb{A}}^*"'] \arrow[rr, phantom, shift left=6, "\dashv"] & & {[\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]}
\end{tikzcd}$$

Proof. *The required isomorphism is the following:*

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \qquad\qquad \Gamma(\overline{x}, x) \vdash P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}] \;\; \hom_{\mathbb{A}}(a, b) \times \Gamma(\overline{x}, x) \vdash P(a, b, \overline{x}, x)} \text{(hom-rel-adj)}$$

which is an instance of directed J, where Γ is mute in $\overline{a} : \mathbb{A}, \overline{b} : \mathbb{A}^{\mathsf{op}}$.

Directed equality as adjoint (2)

Theorem (Directed equality as relative left adjoint, cont.) (👏≈)

- *Thus the left relative adjointness situation*

$$\hom_{\mathbb{A}} \times - \dashv_{\pi^*_{\mathbb{A}}} \Delta_{\mathbb{A}} \times -$$

is as follows:

$$\begin{tikzcd}
 & {[\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]} \arrow[dr, "\Delta_{\mathbb{A}} \times -"] \arrow[d, phantom, "\dashv"] & \\
{[\mathbb{C}^{\diamond}, \textbf{\textit{Set}}]} \arrow[ur, "\hom_{\mathbb{A}} \times -"] \arrow[rr, "\pi^*_{\mathbb{A}}"'] & & {[\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]}
\end{tikzcd}$$

Proof. *The required isomorphism is the following:*

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which is an instance of directed J*, where* Γ *is mute in* $\overline{a} : \mathbb{A}, \overline{b} : \mathbb{A}^{\mathsf{op}}$.
$(\text{hom-rel-adj}) \Leftrightarrow (\text{hom})$*: pick* $P := \Gamma^{\mathsf{op}}(b, a, x, \overline{x}) \Rightarrow P(a, b, \overline{x}, x)$*, use* (exp).

(Co)end calculus via dinaturality

- *What are the quantifiers of directed type theory?*
- Dinaturality allows us to view (co)ends as “adjoints” to weakening:

Theorem (Ends and coends as quantifiers) ( )

Take $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \textbf{\textit{Set}}$, *and the functor precomposing with projections*

$$\pi^*_{\mathbb{A}}(P) : \mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond} \to \textbf{\textit{Set}}$$

$$\pi^*_{\mathbb{A}}(P) := (\overline{a}, a, \overline{x}, x) \mapsto P(\overline{x}, x),$$

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There are isos of sets of dinats, natural in $P : \mathbb{C}^{\diamond} \to \textbf{\textit{Set}}, Q : \mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond} \to \textbf{\textit{Set}}$*:*

$$\begin{array}{c} [a : \mathbb{A}, x : \mathbb{C}]\ P(\overline{x}, x) \vdash Q(\overline{a}, a, \overline{x}, x) \\ \hline\hline [x : \mathbb{C}]\ P(\overline{x}, x) \vdash \displaystyle\int_{a:\mathbb{A}} Q(\overline{a}, a, \overline{x}, x) \end{array} \ \text{(end)}$$

$$\begin{array}{c} [x : \mathbb{C}]\ \displaystyle\int^{a:\mathbb{A}} Q(\overline{a}, a, \overline{x}, x) \vdash P(\overline{x}, x) \\ \hline\hline [a : \mathbb{A}, x : \mathbb{C}]\ \qquad Q(\overline{a}, a, \overline{x}, x) \vdash P(\overline{x}, x) \end{array} \ \text{(coend)}$$

(Co)ends as quantifiers

Theorem (Beck-Chevalley and Frobenius condition for (co)ends)

(Co)ends satisfy a Beck-Chevalley condition: for $F : \mathbb{C}^\diamond \to \mathbb{D}$ *there is a strict isomorphism in the (large) functor category* $[[\mathbb{A}^\diamond \times \mathbb{D}^\diamond, \textbf{\textit{Set}}], [\mathbb{D}^\diamond, \textbf{\textit{Set}}]]$

$$\int_{\mathbb{A}[\mathbb{D}]} ; F^* \cong (\mathsf{id}_{\mathbb{A}^\diamond} \times F)^* ; \int_{\mathbb{A}[\mathbb{C}]}$$

where $\int_{\mathbb{A}[\mathbb{C}]}, \int^{\mathbb{A}[\mathbb{C}]} : [\mathbb{A}^\diamond \times \mathbb{C}^\diamond, \textbf{\textit{Set}}] \to [\mathbb{C}^\diamond, \textbf{\textit{Set}}]$ *are parametric (co)ends, and* $F^* : [\mathbb{D}^\diamond, \textbf{\textit{Set}}] \to [\mathbb{C}^\diamond, \textbf{\textit{Set}}]$ *is precomposition with* $F^\diamond$*.*

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Moreover, a Frobenius condition for coends holds: there is an isomorphism natural in $\Gamma : \mathbb{A}^\diamond \times \mathbb{C}^\diamond \to \boldsymbol{Set}, P : \mathbb{C}^\diamond \to \boldsymbol{Set}$,

$$\int^{\mathbb{A}[\mathbb{C}]} (\pi^*_{\mathbb{A}[\mathbb{C}]}(P) \wedge \Gamma) \cong \pi^*_{\mathbb{A}[\mathbb{C}]}(P) \wedge \int^{\mathbb{A}[\mathbb{C}]}(\Gamma),$$

where $- \wedge - : [\mathbb{C}, \boldsymbol{Set}] \times [\mathbb{C}, \boldsymbol{Set}] \to [\mathbb{C}, \boldsymbol{Set}]$ *is the pointwise product.*
Proof. *Beck-Chevalley is easy. For Frobenius, we can use logical rules to mirror the argument in [Jacobs 1999, 1.9.12(i)] with exponentials.*

(Co)end calculus using our rules

- Using dinaturals as semantics, we can prove useful theorems “logically”.

Rules for (co)ends as quantifiers + directed equality for proofs of:

- (Co)Yoneda,
- Adjointess of Kan extensions via (co)ends,
- Presheaves are closed under exponentials,
- Associativity of composition of profunctors,
- Right lifts in profunctors,
- (Co)ends preserve limits,
- Adjointness of (co)ends in natural transformations,
- Characterization of dinaturals as certain ends,
- Frobenius property of (co)ends using exponentials.

Yoneda lemma: $(P, \mathit{\Gamma} : \mathbb{C} \to \mathbf{Set})$

$$
\begin{array}{cl}
[a : \mathbb{C}]\ \mathit{\Gamma}(a) \vdash \displaystyle\int_{x:\mathbb{C}} \hom_{\mathbb{C}}^{\mathsf{op}}(a, \overline{x}) \Rightarrow P(x) & \\
\hline\hline
[a : \mathbb{C}, x : \mathbb{C}]\ \mathit{\Gamma}(a) \vdash \hom_{\mathbb{C}}^{\mathsf{op}}(a, \overline{x}) \Rightarrow P(x) & (\mathsf{end}) \\
\hline\hline
[a : \mathbb{C}]\ \ \hom_{\mathbb{C}}(\overline{a}, x) \times \mathit{\Gamma}(a) \vdash P(x) & (\mathsf{exp}) \\
\hline\hline
[z : \mathbb{C}]\ \mathit{\Gamma}(z) \vdash P(z) & (\mathrm{hom})
\end{array}
$$

(Co)end calculus with dinaturality (1)

Yoneda lemma: $(P, \Gamma : \mathbb{C} \to \mathbf{Set})$

$$\dfrac{\dfrac{\dfrac{[a : \mathbb{C}]\ \Gamma(a) \vdash \int_{x:\mathbb{C}} \hom^{\mathsf{op}}_{\mathbb{C}}(a, \overline{x}) \Rightarrow P(x)}{[a : \mathbb{C}, x : \mathbb{C}]\ \Gamma(a) \vdash \hom^{\mathsf{op}}_{\mathbb{C}}(a, \overline{x}) \Rightarrow P(x)}\ \text{(end)}}{[a : \mathbb{C}]\ \ \hom_{\mathbb{C}}(\overline{a}, x) \times \Gamma(a) \vdash P(x)}\ \text{(exp)}}{[z : \mathbb{C}]\ \Gamma(z) \vdash P(z)}\ (\mathrm{hom})$$

CoYoneda lemma:

$$\dfrac{\dfrac{[a : \mathbb{C}]\ \ \int^{x:\mathbb{C}} \hom_{\mathbb{C}}(\overline{x}, a) \times P(x) \vdash \Gamma(a)}{[a : \mathbb{C}, x : \mathbb{C}]\ \ \hom_{\mathbb{C}}(\overline{a}, x) \times P(a) \vdash \Gamma(x)}\ \text{(coend)}}{[z : \mathbb{C}]\ P(z) \vdash \Gamma(z)}\ (\mathrm{hom})$$

(Co)end calculus with dinaturality (2)

Presheaves are cartesian closed: $(\mathit{\Gamma}, A, B : \mathbb{C} \to \mathbf{Set})$

$$\begin{aligned} [x : \mathbb{C}]\ \mathit{\Gamma}(x) \vdash (A \Rightarrow B)(x) & \\ &:= \mathsf{Nat}(\hom_{\mathbb{C}}(x, -) \times A, B) \\ &\cong \int_{y:\mathbb{C}} \hom^{\mathsf{op}}_{\mathbb{C}}(x, \overline{y}) \times A^{\mathsf{op}}(\overline{y}) \Rightarrow B(y) \end{aligned}$$

$$\begin{array}{c} \hline\hline \end{array} \ \text{(end)}$$

$$[x : \mathbb{C}, y : \mathbb{C}]\ \mathit{\Gamma}(x) \vdash \hom^{\mathsf{op}}_{\mathbb{C}}(x, \overline{y}) \times A^{\mathsf{op}}(\overline{y}) \Rightarrow B(y)$$

$$\begin{array}{c} \hline\hline \end{array} \ \text{(exp)}$$

$$[x : \mathbb{C}, y : \mathbb{C}]\ A(y) \times \hom_{\mathbb{C}}(\overline{x}, y) \times \mathit{\Gamma}(x) \vdash B(y)$$

$$\begin{array}{c} \hline\hline \end{array} \ \text{(coend+frob.)}$$

$$[y : \mathbb{C}]\ A(y) \times \left(\int^{x:\mathbb{C}} \hom_{\mathbb{C}}(\overline{x}, y) \times \mathit{\Gamma}(x) \right) \vdash B(y)$$

$$\begin{array}{c} \hline\hline \end{array} \ \text{(coYoneda)}$$

$$[y : \mathbb{C}]\ A(y) \times \mathit{\Gamma}(y) \vdash B(y)$$

(Co)end calculus with dinaturality (3)

Right Kan extensions via ends are right adjoints to precomposition with $F : \mathbb{C} \to \mathbb{D}$ ($P : \mathbb{C} \to \textbf{Set}, \mathit{\Gamma} : \mathbb{D} \to \textbf{Set}$):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{r} [y : \mathbb{D}]\ \mathit{\Gamma}(y) \vdash (\mathsf{Ran}_F P)(y) \\ := \displaystyle\int_{x:\mathbb{C}} \hom^{\mathsf{op}}_{\mathbb{D}}(y, F^{\mathsf{op}}(\overline{x})) \Rightarrow P(x) \end{array}}{[x : \mathbb{C}, y : \mathbb{D}]\ \mathit{\Gamma}(y) \vdash \hom^{\mathsf{op}}_{\mathbb{D}}(y, F^{\mathsf{op}}(\overline{x})) \Rightarrow P(x)}\ (\text{end})}{[x : \mathbb{C}, y : \mathbb{D}]\ \hom_{\mathbb{D}}(\overline{y}, F(x)) \times \mathit{\Gamma}(y) \vdash P(x)}\ (\text{exp})}{[x : \mathbb{C}]\ \displaystyle\int^{y:\mathbb{D}} \hom_{\mathbb{D}}(\overline{y}, F(x)) \times \mathit{\Gamma}(y) \vdash P(x)}\ (\text{coend})}{[x : \mathbb{C}]\ \mathit{\Gamma}(F(x)) \vdash P(x)}\ (\text{coYoneda})
$$

(Co)end calculus with dinaturality (4)

Composition (on both sides) in **Prof** has a right adjoint (right lifts).
($P : \mathbb{C}^{\mathsf{op}} \times \mathbb{A} \to \mathbf{Set}, Q : \mathbb{A}^{\mathsf{op}} \times \mathbb{D} \to \mathbf{Set}$, $\mathit{\Gamma} : \mathbb{C}^{\mathsf{op}} \times \mathbb{D} \to \mathbf{Set}$):

$$
\dfrac{\dfrac{\dfrac{\dfrac{[x : \mathbb{C}^{\mathsf{op}}, z : \mathbb{D}] \qquad (P \mathbin{;} -)(Q)(x,z) := \displaystyle\int^{y:\mathbb{A}} P(x,y) \times Q(\overline{y},z) \vdash \mathit{\Gamma}(x,z)}{[x : \mathbb{C}^{\mathsf{op}}, y : \mathbb{A}, z : \mathbb{D}]\ P(x,y) \times Q(\overline{y},z) \vdash \mathit{\Gamma}(x,z)}\ (\text{coend})}{[x : \mathbb{C}^{\mathsf{op}}, y : \mathbb{A}, z : \mathbb{D}]\ Q(\overline{y},z) \vdash P^{\mathsf{op}}(\overline{x},\overline{y}) \Rightarrow \mathit{\Gamma}(x,z)}\ (\text{exp})}{[y : \mathbb{A}, z : \mathbb{D}]\ Q(\overline{y},z) \vdash \displaystyle\int_{x:\mathbb{C}} P^{\mathsf{op}}(\overline{x},\overline{y}) \Rightarrow \mathit{\Gamma}(x,z)}\ (\text{end})}{[y : \mathbb{A}^{\mathsf{op}}, z : \mathbb{D}]\ Q(y,z) \vdash \displaystyle\int_{x:\mathbb{C}} P^{\mathsf{op}}(\overline{x},y) \Rightarrow \mathit{\Gamma}(x,z)}\ (\cong)
$$

$$:= \mathsf{Rift}_P(\mathit{\Gamma})(y,z)$$

(Co)end calculus with dinaturality (5)

Fubini for ends (Γ : **Set**, $P : (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \times (\mathbb{D}^{\mathsf{op}} \times \mathbb{D}) \to$ **Set**)

$$
\cfrac{\cfrac{\cfrac{\cfrac{[\,]\ \Gamma \vdash \displaystyle\int_{x:\mathbb{C}} \int_{y:\mathbb{D}} P(\overline{x}, x, \overline{y}, y)}{[x : \mathbb{C}]\ \Gamma \vdash \displaystyle\int_{y:\mathbb{D}} P(\overline{x}, x, \overline{y}, y)}\ \text{(end)}}{[x : \mathbb{C}, y : \mathbb{D}]\ \Gamma \vdash P(\overline{x}, x, \overline{y}, y)}\ \text{(end)}}{[y : \mathbb{D}, x : \mathbb{C}]\ \Gamma \vdash P(\overline{x}, x, \overline{y}, y)}\ \text{(structural property)}}{\cfrac{[y : \mathbb{D}]\ \Gamma \vdash \displaystyle\int_{x:\mathbb{C}} P(\overline{x}, x, \overline{y}, y)}{[\,]\ \Gamma \vdash \displaystyle\int_{y:\mathbb{D}} \int_{x:\mathbb{C}} P(\overline{x}, x, \overline{y}, y)}\ \text{(end)}}\ \text{(end)}
$$

(Co)end calculus with dinaturality (6)

Composition of profunctors is associative: ($\Gamma : \mathbb{A}^{\mathsf{op}} \times \mathbb{D} \to \mathbf{Set}$, $P : \mathbb{A}^{\mathsf{op}} \times \mathbb{B} \to \mathbf{Set}, Q : \mathbb{B}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}, R : \mathbb{C}^{\mathsf{op}} \times \mathbb{D} \to \mathbf{Set}$)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[a : \mathbb{A}, d : \mathbb{D}] \ \int^{b:\mathbb{B}} P(\overline{a}, b) \times \left(\int^{c:\mathbb{C}} Q(\overline{b}, c) \times R(\overline{c}, d)\right) \vdash \Gamma(\overline{a}, d)}{[a : \mathbb{A}, b : \mathbb{B}, d : \mathbb{D}] \ P(\overline{a}, b) \times \left(\int^{c:\mathbb{C}} Q(\overline{b}, c) \times R(\overline{c}, d)\right) \vdash \Gamma(\overline{a}, d)} \text{(coend)}}{[a : \mathbb{A}, b : \mathbb{B}, c : \mathbb{C}, d : \mathbb{D}] \ P(\overline{a}, b) \times (Q(\overline{b}, c) \times R(\overline{c}, d)) \vdash \Gamma(\overline{a}, d)} \text{(coend)}}{[a : \mathbb{A}, b : \mathbb{B}, c : \mathbb{C}, d : \mathbb{D}] \ (P(\overline{a}, b) \times Q(\overline{b}, c)) \times R(\overline{c}, d) \vdash \Gamma(\overline{a}, d)} \text{(struct. prop.)}}{[a : \mathbb{A}, c : \mathbb{C}, d : \mathbb{D}] \ \left(\int^{b:\mathbb{B}} P(\overline{a}, b) \times Q(\overline{b}, c)\right) \times R(\overline{c}, d) \vdash \Gamma(\overline{a}, d)} \text{(coend)}}{[a : \mathbb{A}, d : \mathbb{D}] \ \int^{c:\mathbb{C}} \left(\int^{b:\mathbb{B}} P(\overline{a}, b) \times Q(\overline{b}, c)\right) \times R(\overline{c}, d) \vdash \Gamma(\overline{a}, d)} \text{(coend)}$$

Conclusion and future work

We have seen how dinaturality allows us to give a semantic interpretation towards a directed type theory in ***Cat*** *with quantifiers, where directed equality is given by* hom*-functors and quantifiers by (co)ends.*

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   - This can be used to characterize *the directed logic of posets*.
   - The left relative adjunction characterizes directed equality in **Pos**.
   - Axiomatize this logic of directed equality using *directed doctrines*.
3. Long-term future: now that types are categories,
   - Internalize semantics of type theory inside type theory (e.g., dQIIT).
   - Naturality for free, since it can be proven internally.

The $\int$.

Paper: *"Directed equality with dinaturality"* (arXiv:2409.10237)
Agda code: github.com/iwilare/dinaturality

Thank you for the attention!

Directed doctrine (WIP)

- Take a category $\mathbb{C}$ with binary products and terminal.
- The category $\mathbb{C}^{N\Delta P}$ is defined as follows:
  - Objects: $\mathbb{C}_0 \times \mathbb{C}_0 \times \mathbb{C}_0$
  - Morphisms $(N, \Delta, P) \to (N', \Delta', P')$: triples (n, p, d) where
    - $n : N \times \Delta \to N'$
    - $p : P \times \Delta \to P'$
    - $d : \Delta \to \Delta'$
  - Identities: $(\pi_1, \mathsf{id}_\Delta, \pi_1)$
- **Definition**: a *directed doctrine* is a pseudofunctor $\mathcal{P} : \mathbb{C}^{N\Delta P} \to \mathbf{Pos}$.
- $\mathcal{P}$ has **products** when each fiber has products.
- $\mathcal{P}$ has **exponentials** when certain left/right-relative adjunctions hold.
- $\mathcal{P}$ has **directed equality** when a certain left relative adjunction holds.
- $\mathcal{P}$ has **involutions** when $\mathbb{C}$ has a functor $-^{\mathsf{op}} : \mathbb{C} \to \mathbb{C}$ s.t. $\mathsf{op} \,;\mathsf{op} = \mathsf{id}_{\mathbb{C}}$ and an iso $\mathcal{P}(P, \Delta, N) \cong \mathcal{P}(N^{\mathsf{op}}, \Delta^{\mathsf{op}}, N^{\mathsf{op}})$ natural in P, Δ, N.

Directed doctrines - relative adj. for exponentials (WIP)

- Downgrade to dinaturality:

$$\begin{array}{c} [a :^{+} \mathbb{C}, \mathbb{x} : \Gamma] \;\; A(a,\mathbb{x}) \times \Gamma(a,\mathbb{x}) \vdash B(a,\mathbb{x}) \\ \hline\hline [a :^{\Delta} \mathbb{C}, \mathbb{x} : \Gamma] \uparrow_a^{\Delta} \Gamma(a,\mathbb{x}) \vdash A^{\mathsf{op}}(\overline{a},\mathbb{x}) \Rightarrow \uparrow_a^{\Delta} B(a,\mathbb{x}) \end{array}$$

$$\begin{array}{c} [a :^{-} \mathbb{C}, \mathbb{x} : \Gamma] \;\; A(\overline{a},\mathbb{x}) \times \Gamma(\overline{a},\mathbb{x}) \vdash B(\overline{a},\mathbb{x}) \\ \hline\hline [a :^{\Delta} \mathbb{C}, \mathbb{x} : \Gamma] \uparrow_a^{\Delta} \Gamma(\overline{a},\mathbb{x}) \vdash A^{\mathsf{op}}(a,\mathbb{x}) \Rightarrow \uparrow_a^{\Delta} B(\overline{a},\mathbb{x}) \end{array}$$

- Dinatural to positive:

$$\begin{array}{c} [a :^{\Delta} \mathbb{C}, \mathbb{x} : \Gamma] \;\; A(\overline{a},\mathbb{x}) \times \uparrow_a^{\Delta} \Gamma(a,\mathbb{x}) \vdash \uparrow_a^{\Delta} B(a,\mathbb{x}) \\ \hline\hline [a :^{+} \mathbb{C}, \mathbb{x} : \Gamma] \;\; \Gamma(a,\mathbb{x}) \vdash A^{\mathsf{op}}(a,\mathbb{x}) \Rightarrow B(a,\mathbb{x}) \end{array}$$

- Inversion:

$$\begin{array}{c} [a :^{+} \mathbb{C}, \mathbb{x} : \Gamma] \;\; A(a,\mathbb{x}) \times \Gamma(\mathbb{x}) \vdash B(\mathbb{x}) \\ \hline\hline [a :^{-} \mathbb{C}, \mathbb{x} : \Gamma] \;\; \pi_a^{-}\Gamma(\mathbb{x}) \vdash A^{\mathsf{op}}(\overline{a},\mathbb{x}) \Rightarrow \pi_a^{-}B(\mathbb{x}) \end{array}$$

$$\begin{array}{c} [a :^{-} \mathbb{C}, \mathbb{x} : \Gamma] \;\; A(\overline{a},\mathbb{x}) \times \Gamma(\mathbb{x}) \vdash B(\mathbb{x}) \\ \hline\hline [a :^{+} \mathbb{C}, \mathbb{x} : \Gamma] \;\; \pi_a^{+}\Gamma(\mathbb{x}) \vdash A^{\mathsf{op}}(a,\mathbb{x}) \Rightarrow \pi_a^{+}B(\mathbb{x}) \end{array}$$

- Dinatural to negative:

$$\begin{array}{c} [a :^{\Delta} \mathbb{C}, \mathbb{x} : \Gamma] \;\; A(a,\mathbb{x}) \times \uparrow_a^{\Delta} \Gamma(\overline{a},\mathbb{x}) \vdash \uparrow_a^{\Delta} B(\overline{a},\mathbb{x}) \\ \hline\hline [a :^{-} \mathbb{C}, \mathbb{x} : \Gamma] \;\; \Gamma(\overline{a},\mathbb{x}) \vdash A^{\mathsf{op}}(\overline{a},\mathbb{x}) \Rightarrow B(\overline{a},\mathbb{x}) \end{array}$$

Homotopical interpretation of dinaturality

We have maps both ways between these entailments:

$$\frac{[\,]\qquad \top \vdash P}{[x : C]\ x = x \vdash P}$$

but in MLTT they are not isomorphic.

In the directed case, we do not even have both maps!

$$\frac{[\,]\qquad\qquad \top \vdash P}{[x : \mathbb{C}]\ \ \hom(\overline{x}, x) \vdash P}$$

We only have a map from top to bottom.

Related works

- **[Caccamo-Winskel 2001]**: axiomatic system to manipulate (co)ends; quantifier exchange is postulated, no type theory presented.
- **[Licata 2011]**: a model in **Cat** with directed equality defined judgementally and not propositionally.
- **[Nuyts 2015]**: a preliminary system of contexts variances is given, with no formal syntax nor models.
- **[North 2018]**: dependent type theory, but uses groupoid structure to type the refl rule. We use dinaturality precisely to avoid this problem.
- **[Riehl-Shulman 2017]**: a synthetic theory of $(\infty, 1)$-categories with a directed interval type, no model in **Cat**.
- **[New-Licata 2022]**: a DTT with models in virtual equipments, with directed equality and quantifiers, but very different syntax w.r.t. MLTT
- **[Ahrens 2023]**: judgemental structure for directed equality reminiscent of a bicategorical model.
- **[Neumann 2024]**: groupoids are used in contexts rather than in the type of the conclusion.