Directed equality with dinaturality

Andrea Laretto, Fosco Loregian, Niccolò Veltri

Tallinn University of Technology

5th Itaca Workshop, Padova

December 20th, 2024

Type theory and symmetric equality

- A very interesting aspect of Martin-Löf type theory/logic: *equality*.
- For simplicity, consider first order logic:

$$\frac{A \text{ type}}{[a : A, b : A]\ a = b \text{ prop}}$$

Type theory and symmetric equality

- A very interesting aspect of Martin-Löf type theory/logic: *equality*.
- For simplicity, consider first order logic:

$$\frac{A \text{ type}}{[a : A, b : A] \ a = b \text{ prop}}$$

$$\frac{}{[x : A] \ \Gamma \vdash x = x} \ (\text{refl})$$

Type theory and symmetric equality

- A very interesting aspect of Martin-Löf type theory/logic: *equality*.
- For simplicity, consider first order logic:

$$\frac{A \text{ type}}{[a:A, b:A]\ a = b \text{ prop}}$$

$$\frac{}{[x:A]\ \Gamma \vdash x = x}\ (\text{refl}) \qquad \frac{[z:A, x:C] \qquad \Gamma(z,z,x) \vdash P(z,z,x)}{[a:A, b:A, x:C]\ a = b,\ \Gamma(a,b,x) \vdash P(a,b,x)}\ (\text{J})$$

Type theory and symmetric equality

- A very interesting aspect of Martin-Löf type theory/logic: *equality*.
- For simplicity, consider first order logic:

$$\frac{A \text{ type}}{[a:A, b:A]\ a = b \text{ prop}}$$

$$\frac{}{[x:A]\ \Gamma \vdash x = x}\ (\text{refl}) \qquad \frac{[z:A, x:C] \qquad\qquad \Gamma(z,z,x) \vdash P(z,z,x)}{[a:A, b:A, x:C]\ a = b,\ \Gamma(a,b,x) \vdash P(a,b,x)}\ (\text{J})$$

- Transitivity of equality:

$$\cfrac{[z:A, c:A] \qquad\qquad \cfrac{}{z = c \vdash z = c}\ (\text{id})}{[a:A, b:A, c:A]\ a = b,\ b = c \vdash a = c}\ (\text{J})$$

Type theory and symmetric equality

- A very interesting aspect of Martin-Löf type theory/logic: *equality*.
- For simplicity, consider first order logic:

$$\dfrac{A \text{ type}}{[a : A, b : A]\ a = b \text{ prop}}$$

$$\dfrac{}{[x : A]\ \Gamma \vdash x = x}\ (\text{refl}) \qquad \dfrac{[z : A, x : C] \qquad \Gamma(z, z, x) \vdash P(z, z, x)}{[a : A, b : A, x : C]\ a = b,\ \Gamma(a, b, x) \vdash P(a, b, x)}\ (\text{J})$$

- Transitivity of equality:

$$\dfrac{\dfrac{}{[z : A, c : A] \qquad z = c \vdash z = c}\ (\text{id})}{[a : A, b : A, c : A]\ a = b,\ b = c \vdash a = c}\ (\text{J})$$

- Equality in MLTT is inherently symmetric:

$$\dfrac{\dfrac{}{[z : A] \qquad \top \vdash z = z}\ (\text{refl})}{[a : A, b : A]\ a = b \vdash b = a}\ (\text{J})$$

Models of MLTT

What can types be?

Martin-Löf type theory admits a sound interpretation in:

- Types as sets. [Martin-Löf 1971]
- Types as groupoids. [Hoffmann-Streicher 1999]
- Types as ∞-groupoids. [Voevodsky 2013], [van der Berg-Garner 2010]
- Types as **sSet**s, **Top**, cubical sets, ... [Awodey-Warren 2007], [CCHM 2007]
- Types as reflexive graphs for parametricity. [Atkey-Ghani-Johann 2007]

Models of MLTT

What can types be?

Martin-Löf type theory admits a sound interpretation in:

- Types as sets. [Martin-Löf 1971]
- Types as groupoids. [Hoffmann-Streicher 1999]
- Types as ∞-groupoids. [Voevodsky 2013], [van der Berg-Garner 2010]
- Types as **sSet**s, **Top**, cubical sets, ... [Awodey-Warren 2007], [CCHM 2007]
- Types as reflexive graphs for parametricity. [Atkey-Ghani-Johann 2007]

Intuition: types are groupoids, equalities are always-invertible morphisms.

Models of MLTT

What can types be?

Martin-Löf type theory admits a sound interpretation in:

- Types as sets. [Martin-Löf 1971]
- Types as groupoids. [Hoffmann-Streicher 1999]
- Types as ∞-groupoids. [Voevodsky 2013], [van der Berg-Garner 2010]
- Types as **sSet**s, **Top**, cubical sets, ... [Awodey-Warren 2007], [CCHM 2007]
- Types as reflexive graphs for parametricity. [Atkey-Ghani-Johann 2007]

Intuition: types are groupoids, equalities are always-invertible morphisms.

Some familiar structures are missing...

Models of MLTT

What can types be?

Martin-Löf type theory admits a sound interpretation in:

- Types as sets. [Martin-Löf 1971]
- Types as groupoids. [Hoffmann-Streicher 1999]
- Types as ∞-groupoids. [Voevodsky 2013], [van der Berg-Garner 2010]
- Types as **sSet**s, **Top**, cubical sets, ... [Awodey-Warren 2007], [CCHM 2007]
- Types as reflexive graphs for parametricity. [Atkey-Ghani-Johann 2007]

Intuition: types are groupoids, equalities are always-invertible morphisms.

Some familiar structures are missing... *what about categories and posets?!*

Motivation 1: Directed type theory

Martin-Löf type theory with refl/J *is intrinsically about symmetric equality.*
Directed type theory *is the generalization to "directed equality".*

Motivation 1: Directed type theory

Martin-Löf type theory with refl$/J$ *is intrinsically about symmetric equality.*
Directed type theory *is the generalization to "directed equality".*

The interpretation of directed type theory with *(1-)categories*:

$$\begin{aligned}
\text{Types} &\rightsquigarrow \text{Categories} \\
\text{Terms} &\rightsquigarrow \text{Functors} \\
\text{Relations} &\rightsquigarrow \text{Profunctors} \\
\text{Points of a type} &\rightsquigarrow \text{Objects of a category} \\
\text{Equalities } e : a = b &\rightsquigarrow \text{Morphisms } e : \hom(a, b) \\
\text{Equality types } =_A \colon A \times A \to \text{Type} &\rightsquigarrow \text{Hom types } \hom_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}
\end{aligned}$$

$\rightarrow$ Now types have a *polarity*, $\mathbb{C}$ and $\mathbb{C}^{\mathsf{op}}$, i.e., the opposite category.

Motivation 1: Directed type theory

Martin-Löf type theory with refl/J *is intrinsically about symmetric equality.*
Directed type theory *is the generalization to "directed equality".*

The interpretation of directed type theory with *(1-)categories*:

$$\begin{aligned}
\text{Types} &\rightsquigarrow \text{Categories} \\
\text{Terms} &\rightsquigarrow \text{Functors} \\
\text{Relations} &\rightsquigarrow \text{Profunctors} \\
\text{Points of a type} &\rightsquigarrow \text{Objects of a category} \\
\text{Equalities } e : a = b &\rightsquigarrow \text{Morphisms } e : \hom(a, b) \\
\text{Equality types } =_A : A \times A \to \text{Type} &\rightsquigarrow \text{Hom types } \hom_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}
\end{aligned}$$

$\rightarrow$ Now types have a *polarity*, $\mathbb{C}$ and $\mathbb{C}^{\mathsf{op}}$, i.e., the opposite category.
$\rightarrow$ Now equalities $e : \hom(a, b)$ have *directionality:* rewrites, trans., processes.

Motivation 1: Directed type theory

Martin-Löf type theory with refl$/J$ *is intrinsically about symmetric equality.*
Directed type theory *is the generalization to "directed equality".*

The interpretation of directed type theory with *(1-)categories*:

$$\begin{aligned}
\text{Types} &\rightsquigarrow \text{Categories} \\
\text{Terms} &\rightsquigarrow \text{Functors} \\
\text{Relations} &\rightsquigarrow \text{Profunctors} \\
\text{Points of a type} &\rightsquigarrow \text{Objects of a category} \\
\text{Equalities } e : a = b &\rightsquigarrow \text{Morphisms } e : \hom(a, b) \\
\text{Equality types } =_A : A \times A \to \text{Type} &\rightsquigarrow \text{Hom types } \hom_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}
\end{aligned}$$

$\rightarrow$ Now types have a *polarity*, $\mathbb{C}$ and $\mathbb{C}^{\mathsf{op}}$, i.e., the opposite category.
$\rightarrow$ Now equalities $e : \hom(a, b)$ have *directionality:* rewrites, trans., processes.

We want to find which syntactic restriction of MLTT
allow for types can be interpreted as categories.

Current approaches to directed type theory

- Semantically, refl should be $\mathsf{id}_c \in \hom_{\mathbb{C}}(c, c)$ for $c : \mathbb{C}$.

Current approaches to directed type theory

- Semantically, refl should be $\mathsf{id}_c \in \hom_{\mathbb{C}}(c, c)$ for $c : \mathbb{C}$.
- Transitivity of directed equality $\rightsquigarrow$ composition of morphisms in $\mathbb{C}$.

Current approaches to directed type theory

- Semantically, refl should be $\mathsf{id}_c \in \hom_{\mathbb{C}}(c,c)$ for $c : \mathbb{C}$.
- Transitivity of directed equality $\rightsquigarrow$ composition of morphisms in $\mathbb{C}$.

$$\dfrac{\dfrac{}{[z : \mathbb{C}^{\mathsf{op}}, c : \mathbb{C}] \quad \hom(z,c) \vdash \hom(z,c)}\ (\mathsf{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}, c : \mathbb{C}]\ \hom(a,b),\ \hom(b,c) \vdash \hom(a,c)}\ (J)$$

Current approaches to directed type theory

- Semantically, refl should be $\mathsf{id}_c \in \hom_{\mathbb{C}}(c,c)$ for $c : \mathbb{C}$.
- Transitivity of directed equality $\rightsquigarrow$ composition of morphisms in $\mathbb{C}$.

$$\dfrac{\dfrac{}{[z : \mathbb{C}^{\mathsf{op}}, c : \mathbb{C}] \quad \hom(z,c) \vdash \hom(z,c)}\ (\mathsf{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}, c : \mathbb{C}]\ \hom(a,b),\ \hom(b,c) \vdash \hom(a,c)}\ (J)$$

- However, directed type theory is not so straightforward:

$$\dfrac{a : \mathbb{C}}{\mathsf{refl}_a\text{...?} : \hom_{\mathbb{C}}(a,a)}$$

- *Problem:* rule is not functorial w.r.t. variance of $\hom_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$, since $a : \mathbb{C}$ appears both contravariantly and covariantly.

Current approaches to directed type theory

- Semantically, refl should be $\mathsf{id}_c \in \hom_{\mathbb{C}}(c,c)$ for $c : \mathbb{C}$.
- Transitivity of directed equality $\rightsquigarrow$ composition of morphisms in $\mathbb{C}$.

$$\dfrac{\dfrac{}{[z : \mathbb{C}^{\mathsf{op}}, c : \mathbb{C}] \quad \hom(z,c) \vdash \hom(z,c)}\ (\mathsf{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}, c : \mathbb{C}]\ \hom(a,b),\ \hom(\bar{b},c) \vdash \hom(a,c)}\ (J)$$

- However, directed type theory is not so straightforward:

$$\frac{a : \mathbb{C}}{\mathsf{refl}_a ...? : \hom_{\mathbb{C}}(a,a)} \quad \rightsquigarrow \quad \frac{a : \mathbb{C}^{\mathsf{core}}}{\mathsf{refl}_a : \hom(\mathsf{i}^{\mathsf{op}}(a), \mathsf{i}(a))}$$ [North 2018]

- *Problem:* rule is not functorial w.r.t. variance of $\hom_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$, since $a : \mathbb{C}$ appears both contravariantly and covariantly.
- Current approach to DTT in **Cat**: [North 2018], [Neumann 2024]
  $\to$ Use the maximal subgroupoid $\mathbb{C}^{\mathsf{core}}$ to collapse the two variances,

Current approaches to directed type theory

- Semantically, refl should be $\mathsf{id}_c \in \hom_{\mathbb{C}}(c, c)$ for $c : \mathbb{C}$.
- Transitivity of directed equality $\rightsquigarrow$ composition of morphisms in $\mathbb{C}$.

$$\dfrac{\dfrac{}{[z : \mathbb{C}^{\mathsf{op}}, c : \mathbb{C}] \quad \hom(z, c) \vdash \hom(z, c)}\ (\mathsf{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}, c : \mathbb{C}]\ \hom(a, b),\ \hom(\overline{b}, c) \vdash \hom(a, c)}\ (J)$$

- However, directed type theory is not so straightforward:

$$\dfrac{a : \mathbb{C}}{\mathsf{refl}_a \ldots ? : \hom_{\mathbb{C}}(a, a)} \quad \rightsquigarrow \quad \dfrac{a : \mathbb{C}^{\mathsf{core}}}{\mathsf{refl}_a : \hom(\mathsf{i}^{\mathsf{op}}(a), \mathsf{i}(a))}$$ [North 2018]

- *Problem:* rule is not functorial w.r.t. variance of $\hom_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$, since $a : \mathbb{C}$ appears both contravariantly and covariantly.
- Current approach to DTT in **Cat**: [North 2018], [Neumann 2024]
  $\to$ Use the maximal subgroupoid $\mathbb{C}^{\mathsf{core}}$ to collapse the two variances, since $(\mathbb{C}^{\mathsf{core}})^{\mathsf{op}} \cong \mathbb{C}^{\mathsf{core}}$, then use $\mathsf{i} : \mathbb{C}^{\mathsf{core}} \to \mathbb{C}$ and $\mathsf{i}^{\mathsf{op}} : \mathbb{C}^{\mathsf{core}} \to \mathbb{C}^{\mathsf{op}}$.

Current approaches to directed type theory

- Semantically, refl should be $\mathsf{id}_c \in \hom_{\mathbb{C}}(c,c)$ for $c : \mathbb{C}$.
- Transitivity of directed equality $\rightsquigarrow$ composition of morphisms in $\mathbb{C}$.

$$\dfrac{\dfrac{}{[z : \mathbb{C}^{\mathsf{op}}, c : \mathbb{C}] \quad \hom(z,c) \vdash \hom(z,c)}\ (\mathsf{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}, c : \mathbb{C}]\ \hom(a,b),\ \hom(\bar{b},c) \vdash \hom(a,c)}\ (J)$$

- However, directed type theory is not so straightforward:

$$\dfrac{a : \mathbb{C}}{\mathsf{refl}_a ...? : \hom_{\mathbb{C}}(a,a)} \quad \rightsquigarrow \quad \dfrac{a : \mathbb{C}^{\mathsf{core}}}{\mathsf{refl}_a : \hom(\mathsf{i}^{\mathsf{op}}(a), \mathsf{i}(a))}$$ [North 2018]

- *Problem:* rule is not functorial w.r.t. variance of $\hom_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$, since $a : \mathbb{C}$ appears both contravariantly and covariantly.
- Current approach to DTT in **Cat**: [North 2018], [Neumann 2024]
  $\to$ Use the maximal subgroupoid $\mathbb{C}^{\mathsf{core}}$ to collapse the two variances, since $(\mathbb{C}^{\mathsf{core}})^{\mathsf{op}} \cong \mathbb{C}^{\mathsf{core}}$, then use $\mathsf{i} : \mathbb{C}^{\mathsf{core}} \to \mathbb{C}$ and $\mathsf{i}^{\mathsf{op}} : \mathbb{C}^{\mathsf{core}} \to \mathbb{C}^{\mathsf{op}}$.
- Then a J-like rule is validated, but *again using groupoidal structure*.

Contribution 1 – Towards dinatural directed type theory

Is there a way to interpret a directed type theory in 1-categories without having to collapse categories back to the "undirected" groupoidal case?

Contribution 1 – Towards dinatural directed type theory

Is there a way to interpret a directed type theory in 1-categories without having to collapse categories back to the "undirected" groupoidal case?

Our approach: yes, by validating rules with dinatural transformations.

Contribution 1 – Towards dinatural directed type theory

Is there a way to interpret a directed type theory in 1-categories without having to collapse categories back to the "undirected" groupoidal case?

Our approach: yes, by validating rules with dinatural transformations.

- *Intuition:* dinatural transformations allow for the same variable to appear both covariantly and contravariantly $\to$ solving the issue with refl.

Contribution 1 – Towards dinatural directed type theory

Is there a way to interpret a directed type theory in 1-categories without having to collapse categories back to the "undirected" groupoidal case?

Our approach: yes, by validating rules with dinatural transformations.

- *Intuition:* dinatural transformations allow for the same variable to appear both covariantly and contravariantly $\rightarrow$ solving the issue with refl.
- We will see how dinaturality validates both
  $\rightarrow$ a refl rule (directed equality introduction),

Contribution 1 – Towards dinatural directed type theory

Is there a way to interpret a directed type theory in 1-categories without having to collapse categories back to the "undirected" groupoidal case?

Our approach: yes, by validating rules with dinatural transformations.

- *Intuition:* dinatural transformations allow for the same variable to appear both covariantly and contravariantly $\to$ solving the issue with refl.
- We will see how dinaturality validates both
  $\to$ a refl rule (directed equality introduction),
  $\to$ a J rule (directed equality elimination).

Contribution 1 – Towards dinatural directed type theory

Is there a way to interpret a directed type theory in 1-categories without having to collapse categories back to the "undirected" groupoidal case?

Our approach: yes, by validating rules with dinatural transformations.

- *Intuition:* dinatural transformations allow for the same variable to appear both covariantly and contravariantly $\rightarrow$ solving the issue with refl.
- We will see how dinaturality validates both
  $\rightarrow$ a refl rule (directed equality introduction),
  $\rightarrow$ a J rule (directed equality elimination).
- The directed J rule is extremely similar to the standard MLTT J rule,

Contribution 1 – Towards dinatural directed type theory

Is there a way to interpret a directed type theory in 1-categories without having to collapse categories back to the "undirected" groupoidal case?

Our approach: yes, by validating rules with dinatural transformations.

- *Intuition:* dinatural transformations allow for the same variable to appear both covariantly and contravariantly $\rightarrow$ solving the issue with refl.
- We will see how dinaturality validates both
  $\rightarrow$ a refl rule (directed equality introduction),
  $\rightarrow$ a J rule (directed equality elimination).
- The directed J rule is extremely similar to the standard MLTT J rule, but with a syntactic restriction which does **not** allow for symmetry.
- This comes at the cost of compositionality:
  we have a cut rule only if variables appear *naturally* in a entailment.

Contribution 1 – Directed type theory

Comparison between symmetric equality and our directed equality rule:

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \qquad \mathit{\Gamma}(z, z, x) \vdash P(z, z, x)}{[a : \mathbb{A}, b : \mathbb{A}, x : \mathbb{C}] \; a = b,\ \mathit{\Gamma}(a, b, x) \vdash P(a, b, x)} \text{ (eq)}$$

Contribution 1 – Directed type theory

Comparison between symmetric equality and our directed equality rule:

$$\dfrac{\dfrac{[z:\mathbb{A}, x:\mathbb{C}] \qquad \Gamma(z,z,x) \vdash P(z,z,x)}{[a:\mathbb{A}, b:\mathbb{A}, x:\mathbb{C}]\ a=b,\ \Gamma(a,b,x) \vdash P(a,b,x)}\ \textsf{(eq)}}{\dfrac{[z:\mathbb{A}, x:\mathbb{C}] \qquad \Gamma(\overline{z},z,\overline{x},x) \vdash P(\overline{z},z,\overline{x},x)}{[a:\mathbb{A}^{\mathsf{op}}, b:\mathbb{A}, x:\mathbb{C}]\ \hom(a,b),\ \Gamma(\overline{b},\overline{a},\overline{x},x) \vdash P(a,b,\overline{x},x)}\ (\hom)}$$

Contribution 1 – Directed type theory

Comparison between symmetric equality and our directed equality rule:

$$\dfrac{\dfrac{[z:\mathbb{A}, x:\mathbb{C}] \qquad \Gamma(z,z,x) \vdash P(z,z,x)}{[a:\mathbb{A}, b:\mathbb{A}, x:\mathbb{C}]\ a=b,\ \Gamma(a,b,x) \vdash P(a,b,x)}\ \textsf{(eq)}}{\dfrac{[z:\mathbb{A}, x:\mathbb{C}] \qquad \Gamma(\overline{z},z,\overline{x},x) \vdash P(\overline{z},z,\overline{x},x)}{[a:\mathbb{A}^{\mathsf{op}}, b:\mathbb{A}, x:\mathbb{C}]\ \hom(a,b),\ \Gamma(\overline{b},\overline{a},\overline{x},x) \vdash P(a,b,\overline{x},x)}\ (\hom)}$$

- **Idea 1:** if $\mathbb{A}$ is a type, $\mathbb{A}^{\mathsf{op}}$ is also a type. And $A^{\mathsf{opop}} \equiv A$ definitionally.
- **Idea 2:** variables have variance, *positive* ($a : \mathbb{A}$) or *negative* ($\overline{a} : \mathbb{A}^{\mathsf{op}}$).
- **Restriction:** a, b must appear only *positively* in P, and *negatively* in Γ.

Contribution 1 – Directed type theory

Comparison between symmetric equality and our directed equality rule:

$$\dfrac{[z : \mathbb{A}, x : \mathbb{C}] \qquad \Gamma(z, z, x) \vdash P(z, z, x)}{[a : \mathbb{A}, b : \mathbb{A}, x : \mathbb{C}]\ a = b,\ \Gamma(a, b, x) \vdash P(a, b, x)}\ \textsf{(eq)}$$

$$\dfrac{[z : \mathbb{A}, x : \mathbb{C}] \qquad \Gamma(\overline{z}, z, \overline{x}, x) \vdash P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ \hom(a, b),\ \Gamma(\overline{b}, \overline{a}, \overline{x}, x) \vdash P(a, b, \overline{x}, x)}\ (\hom)$$

- **Idea 1:** if $\mathbb{A}$ is a type, $\mathbb{A}^{\mathsf{op}}$ is also a type. And $A^{\mathsf{opop}} \equiv A$ definitionally.
- **Idea 2:** variables have variance, *positive* $(a : \mathbb{A})$ or *negative* $(\overline{a} : \mathbb{A}^{\mathsf{op}})$.
- **Restriction:** a, b must appear only *positively* in P, and *negatively* in Γ.
- (Again, dinaturality allows z to appear twice in $P(z, z, x)$.)

Contribution 1 – Directed type theory

Comparison between symmetric equality and our directed equality rule:

$$\dfrac{\dfrac{[z:\mathbb{A}, x:\mathbb{C}] \qquad \Gamma(z,z,x) \vdash P(z,z,x)}{[a:\mathbb{A}, b:\mathbb{A}, x:\mathbb{C}]\ a=b,\ \Gamma(a,b,x) \vdash P(a,b,x)}\ (\mathsf{eq})}{\dfrac{[z:\mathbb{A}, x:\mathbb{C}] \qquad \Gamma(\overline{z},z,\overline{x},x) \vdash P(\overline{z},z,\overline{x},x)}{[a:\mathbb{A}^{\mathsf{op}}, b:\mathbb{A}, x:\mathbb{C}]\ \hom(a,b),\ \Gamma(\overline{b},\overline{a},\overline{x},x) \vdash P(a,b,\overline{x},x)}\ (\hom)}$$

- **Idea 1:** if $\mathbb{A}$ is a type, $\mathbb{A}^{\mathsf{op}}$ is also a type. And $A^{\mathsf{opop}} \equiv A$ definitionally.
- **Idea 2:** variables have variance, *positive* $(a : \mathbb{A})$ or *negative* $(\overline{a} : \mathbb{A}^{\mathsf{op}})$.
- **Restriction:** a, b must appear only *positively* in P, and *negatively* in Γ.
- (Again, dinaturality allows z to appear twice in $P(z,z,x)$.)

- Same terms provable in MLTT about equality: transitivity, transport, congruence, *but not symmetry!*

Contribution 1 – Directed type theory

Comparison between symmetric equality and our directed equality rule:

$$\frac{\dfrac{[z:\mathbb{A}, x:\mathbb{C}] \qquad \Gamma(z,z,x) \vdash P(z,z,x)}{[a:\mathbb{A}, b:\mathbb{A}, x:\mathbb{C}]\ a=b,\ \Gamma(a,b,x) \vdash P(a,b,x)}\ \text{(eq)}}{\dfrac{[z:\mathbb{A}, x:\mathbb{C}] \qquad \Gamma(\overline{z},z,\overline{x},x) \vdash P(\overline{z},z,\overline{x},x)}{[a:\mathbb{A}^{\mathsf{op}}, b:\mathbb{A}, x:\mathbb{C}]\ \hom(a,b),\ \Gamma(\overline{b},\overline{a},\overline{x},x) \vdash P(a,b,\overline{x},x)}\ (\hom)}$$

- **Idea 1:** if $\mathbb{A}$ is a type, $\mathbb{A}^{\mathsf{op}}$ is also a type. And $A^{\mathsf{opop}} \equiv A$ definitionally.
- **Idea 2:** variables have variance, *positive* $(a : \mathbb{A})$ or *negative* $(\overline{a} : \mathbb{A}^{\mathsf{op}})$.
- **Restriction:** a, b must appear only *positively* in P, and *negatively* in Γ.
- (Again, dinaturality allows z to appear twice in $P(z, z, x)$.)

- Same terms provable in MLTT about equality: transitivity, transport, congruence, *but not symmetry!*
- Proving properties about them follows precisely the steps in MLTT. (e.g., we have the same computation rules.)

Contribution 2 – (Co)end calculus

Another important open question:

Contribution 2 – (Co)end calculus

Another important open question:

What should the quantifiers of directed type theory be?

Contribution 2 – (Co)end calculus

Another important open question:

What should the quantifiers of directed type theory be?

- Dinaturality again gives us a hint: *ends and coends*.

Contribution 2 – (Co)end calculus

Another important open question:

What should the quantifiers of directed type theory be?

- Dinaturality again gives us a hint: *ends and coends*.
- We present a series of "logical rules", inspired by the doctrinal approach, to view (co)ends as the *directed quantifiers* of DTT.

Contribution 2 – (Co)end calculus

Another important open question:

What should the quantifiers of directed type theory be?

- Dinaturality again gives us a hint: *ends and coends*.
- We present a series of "logical rules", inspired by the doctrinal approach, to view (co)ends as the *directed quantifiers* of DTT.
- We then use the rules to give *simple proofs* of theorems in category theory, using dinaturality and viewing $\hom$ as directed equality.

Contribution 2 – (Co)end calculus

Another important open question:

What should the quantifiers of directed type theory be?

- Dinaturality again gives us a hint: *ends and coends*.
- We present a series of "logical rules", inspired by the doctrinal approach, to view (co)ends as the *directed quantifiers* of DTT.
- ▶ We then use the rules to give *simple proofs* of theorems in category theory, using dinaturality and viewing $\hom$ as directed equality.
- **Claim**: *(co)end calculus is the first-order instantiation of a directed type theory with quantifiers, with semantics in 1-categories*.

Motivation 2: Profunctors

Another story, now in CT:

Rel	$\rightsquigarrow$	**Prof**
Sets	$\rightsquigarrow$	Categories
Relations $A \times B \to \{\top, \bot\}$	$\rightsquigarrow$	Profunctors $\mathbb{C}^{\mathsf{op}} \times \mathbb{D} \to$ **Set**
Existential quantifiers $\exists x$	$\rightsquigarrow$	Coends $\int^x$
Composition: $\exists b \in B.\ R(a,b) \wedge Q(b,c)$	$\rightsquigarrow$	$\int^{b:\mathbb{B}} R(a,b) \times Q(b,c)$
Conjunction of truth values	$\rightsquigarrow$	Cartesian products of **Set**s
Identity relation: $=_{\mathbb{C}}$	$\rightsquigarrow$	Identity profunctor: $\hom_{\mathbb{C}}$

Motivation 2: Profunctors

Another story, now in CT:

Rel	$\rightsquigarrow$	**Prof**
Sets	$\rightsquigarrow$	Categories
Relations $A \times B \to \{\top, \bot\}$	$\rightsquigarrow$	Profunctors $\mathbb{C}^{\mathsf{op}} \times \mathbb{D} \to$ **Set**
Existential quantifiers $\exists x$	$\rightsquigarrow$	Coends $\int^{x}$
Composition: $\exists b \in B.\ R(a,b) \wedge Q(b,c)$	$\rightsquigarrow$	$\displaystyle\int^{b:\mathbb{B}} R(a,b) \times Q(b,c)$
Conjunction of truth values	$\rightsquigarrow$	Cartesian products of **Set**s
Identity relation: $=_{\mathbb{C}}$	$\rightsquigarrow$	Identity profunctor: $\hom_{\mathbb{C}}$

Why do unitality and associativity hold?

Because of rules for equality in FOL. $\rightsquigarrow$ Because of <u>the coYoneda lemma</u>.

Because of rules for $\exists$ and $\wedge$ in FOL. $\rightsquigarrow$ Because of <u>properties of coends</u>.

Motivation 2: Profunctors

Another story, now in CT:

Rel	$\rightsquigarrow$	**Prof**
Sets	$\rightsquigarrow$	Categories
Relations $A \times B \to \{\top, \bot\}$	$\rightsquigarrow$	Profunctors $\mathbb{C}^{\mathsf{op}} \times \mathbb{D} \to$ **Set**
Existential quantifiers $\exists x$	$\rightsquigarrow$	Coends $\int^x$
Composition: $\exists b \in B.\ R(a,b) \wedge Q(b,c)$	$\rightsquigarrow$	$\int^{b:\mathbb{B}} R(a,b) \times Q(b,c)$
Conjunction of truth values	$\rightsquigarrow$	Cartesian products of **Set**s
Identity relation: $=_{\mathbb{C}}$	$\rightsquigarrow$	Identity profunctor: $\hom_{\mathbb{C}}$

Why do unitality and associativity hold?

Because of rules for equality in FOL. $\rightsquigarrow$ Because of the coYoneda lemma.
Because of rules for $\exists$ and $\wedge$ in FOL. $\rightsquigarrow$ Because of properties of coends.

- We use FOL in **Rel**: there is no *directed logic* for **Prof**.

Motivation 2: Profunctors

Another story, now in CT:

Rel $\rightsquigarrow$	**Prof**
Sets $\rightsquigarrow$	Categories
Relations $A \times B \to \{\top, \bot\}$ $\rightsquigarrow$	Profunctors $\mathbb{C}^{\mathsf{op}} \times \mathbb{D} \to$ **Set**
Existential quantifiers $\exists x$ $\rightsquigarrow$	Coends $\int^x$
Composition: $\exists b \in B.\ R(a,b) \wedge Q(b,c)$ $\rightsquigarrow$	$\int^{b:\mathbb{B}} R(a,b) \times Q(b,c)$
Conjunction of truth values $\rightsquigarrow$	Cartesian products of **Set**s
Identity relation: $=_{\mathbb{C}}$ $\rightsquigarrow$	Identity profunctor: $\hom_{\mathbb{C}}$

Why do unitality and associativity hold?

Because of rules for equality in FOL. $\rightsquigarrow$ Because of <u>the coYoneda lemma</u>.

Because of rules for $\exists$ and $\wedge$ in FOL. $\rightsquigarrow$ Because of <u>properties of coends</u>.

- We use FOL in **Rel**: there is no *directed logic* for **Prof**.
- A proper treatment of coends *as quantifiers*.

Motivation 2: Profunctors

Another story, now in CT:

Rel $\rightsquigarrow$	**Prof**
Sets $\rightsquigarrow$	Categories
Relations $A \times B \to \{\top, \bot\}$ $\rightsquigarrow$	Profunctors $\mathbb{C}^{\mathsf{op}} \times \mathbb{D} \to$ **Set**
Existential quantifiers $\exists x$ $\rightsquigarrow$	Coends $\int^x$
Composition: $\exists b \in B.\ R(a,b) \wedge Q(b,c)$ $\rightsquigarrow$	$\int^{b:\mathbb{B}} R(a,b) \times Q(b,c)$
Conjunction of truth values $\rightsquigarrow$	Cartesian products of **Set**s
Identity relation: $=_{\mathbb{C}}$ $\rightsquigarrow$	Identity profunctor: $\hom_{\mathbb{C}}$

Why do unitality and associativity hold?

Because of rules for equality in FOL. $\rightsquigarrow$ Because of the coYoneda lemma.
Because of rules for $\exists$ and $\wedge$ in FOL. $\rightsquigarrow$ Because of properties of coends.

- We use FOL in **Rel**: there is no *directed logic* for **Prof**.
- A proper treatment of coends *as quantifiers*.
- A syntactic treatment of directed logic (with directed equality and its rules, and quantifiers), e.g., using a doctrinal approach.

Motivation 3: Lawvere equality for categories

Can we interpret logic in ***Cat****, viewed as a category of contexts?*

Motivation 3: Lawvere equality for categories

*Can we interpret logic in **Cat**, viewed as a category of contexts?*

From type theory: why **Cat** is *not* a model for Martin-Löf type theory:

- **Cat** is *not* locally cartesian closed,
- **Cat** is *not* even regular (but it is 2-regular [Street CBS 1982])

Motivation 3: Lawvere equality for categories

*Can we interpret logic in **Cat**, viewed as a category of contexts?*

From type theory: why **Cat** is *not* a model for Martin-Löf type theory:

- **Cat** is *not* locally cartesian closed,
- **Cat** is *not* even regular (but it is 2-regular [Street CBS 1982])

From the doctrinal perspective: **Set** plays the role of *generalized truth values* in **Prof**, and presheaves are "generalized logical predicates", but:

Motivation 3: Lawvere equality for categories

*Can we interpret logic in **Cat**, viewed as a category of contexts?*

From type theory: why **Cat** is *not* a model for Martin-Löf type theory:

- **Cat** is *not* locally cartesian closed,
- **Cat** is *not* even regular (but it is 2-regular [Street CBS 1982])

From the doctrinal perspective: **Set** plays the role of *generalized truth values* in **Prof**, and presheaves are "generalized logical predicates", but:

- Equality in the presheaf hyperdoctrine

$$\mathcal{P}\mathrm{sh}(\mathbb{C}) := [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}] : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$$

  does **not** validate Frobenius and Beck-Chevalley: [Melliès-Zeilberger 2016]

Motivation 3: Lawvere equality for categories

*Can we interpret logic in **Cat**, viewed as a category of contexts?*

From type theory: why **Cat** is *not* a model for Martin-Löf type theory:

- **Cat** is *not* locally cartesian closed,
- **Cat** is *not* even regular (but it is 2-regular [Street CBS 1982])

From the doctrinal perspective: **Set** plays the role of *generalized truth values* in **Prof**, and presheaves are "generalized logical predicates", but:

- Equality in the presheaf hyperdoctrine

$$\mathcal{P}\mathsf{sh}(\mathbb{C}) := [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}] : \mathbf{Cat}^{\mathsf{op}} \to \mathbf{CAT}$$

does **not** validate Frobenius and Beck-Chevalley: [Melliès-Zeilberger 2016]

$$\mathsf{Eq}_{\mathbb{C}} := \exists_{\Delta_{\mathbb{C}}}(\top_{\mathbb{C}}) \quad \in \quad \mathcal{P}\mathsf{sh}(\mathbb{C} \times \mathbb{C})$$

$$\mathsf{Eq}_{\mathbb{C}} = (a,b) \mapsto \int^{x \in \mathbb{C}} \hom_{\mathbb{C}}(a,x) \times \hom_{\mathbb{C}}(b,x)$$

i.e., any two objects forming a cospan for some x should be equated.
Note: These conditions also fail in groupoids.

Motivation 3: Lawvere on $\hom$ as equality

Lawvere commenting on the failure of Frobenius/Beck-Chevalley for $\mathcal{P}$sh:

[Lawvere 1970, Equality in Hyperdoctrines]

Motivation 3: Lawvere on $\hom$ as equality

Lawvere commenting on the failure of Frobenius/Beck-Chevalley for $\mathcal{P}$sh:

- [...] *This should not be taken as indicative of a lack of vitality of $\mathcal{P}$sh as hyperdoctrine, or even of a lack of a satisfactory theory of equality for it.*

[Lawvere 1970, Equality in Hyperdoctrines]

Motivation 3: Lawvere on $\hom$ as equality

Lawvere commenting on the failure of Frobenius/Beck-Chevalley for $\mathcal{P}$sh:

- [...] *This should not be taken as indicative of a lack of vitality of* $\mathcal{P}$*sh as hyperdoctrine, or even of a lack of a satisfactory theory of equality for it.*
- *Rather, it indicates that we have probably been too naive in defining equality in a manner too closely suggested by the classical conception.*

[Lawvere 1970, Equality in Hyperdoctrines]

Motivation 3: Lawvere on $\hom$ as equality

Lawvere commenting on the failure of Frobenius/Beck-Chevalley for $\mathcal{P}$sh:

- [...] *This should not be taken as indicative of a lack of vitality of* $\mathcal{P}sh$ *as hyperdoctrine, or even of a lack of a satisfactory theory of equality for it.*
- *Rather, it indicates that we have probably been too naive in defining equality in a manner too closely suggested by the classical conception.*
- *Equality should be the "graph" of the identity term. But present categorical conceptions indicate that the graph of a functor* $F : \mathbb{C} \to \mathbb{D}$ *should be* [...] *a binary attribute of* mixed *variance* $\mathcal{P}sh(\mathbb{C}^{\mathsf{op}} \times \mathbb{D})$.
- *Thus in particular "equality" should be the functor* $\hom_{\mathbb{C}}$ [...].

[Lawvere 1970, Equality in Hyperdoctrines]

Motivation 3: Lawvere on $\hom$ as equality

Lawvere commenting on the failure of Frobenius/Beck-Chevalley for $\mathcal{P}$sh:

- [...] *This should not be taken as indicative of a lack of vitality of* $\mathcal{P}$*sh as hyperdoctrine, or even of a lack of a satisfactory theory of equality for it.*
- *Rather, it indicates that we have probably been too naive in defining equality in a manner too closely suggested by the classical conception.*
- *Equality should be the "graph" of the identity term. But present categorical conceptions indicate that the graph of a functor* $F : \mathbb{C} \to \mathbb{D}$ *should be* [...] *a binary attribute of* mixed *variance* $\mathcal{P}sh(\mathbb{C}^{\mathsf{op}} \times \mathbb{D})$.
- *Thus in particular "equality" should be the functor* $\hom_{\mathbb{C}}$ [...].
- *The term which would take the place of* $\Delta_{\mathbb{C}}$ *in such a more enlightened theory of equality would be the forgetful functor* $\mathsf{Tw}(\mathbb{C}) \to \mathbb{C}^{\mathsf{op}} \times \mathbb{C}$. [...]
- *Of course to abstract from this example would require at least the addition of a functor* $T \xrightarrow{\mathsf{op}} T$ *to the structure of a [doctrine].*

[Lawvere 1970, Equality in Hyperdoctrines]

Background on dinaturality and (co)ends

Background: Dinatural transformations

- A **difunctor** from $\mathbb{C}$ to $\mathbb{D}$ is a functor $F, G : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$.

Background: Dinatural transformations

- A **difunctor** from $\mathbb{C}$ to $\mathbb{D}$ is a functor $F, G : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$.
- A **dipresheaf** on $\mathbb{C}$ is an endoprofunctor $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$.

Background: Dinatural transformations

- A **difunctor** from $\mathbb{C}$ to $\mathbb{D}$ is a functor $F, G : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$.
- A **dipresheaf** on $\mathbb{C}$ is an endoprofunctor $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$.
- Given difunctors $F, G : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$, a **dinatural transformation** $\alpha : F \xrightarrow{\bullet\bullet} G$ is a family of morphisms of $\mathbb{D}$ for each $x : \mathbb{C}$,

$$\alpha_x : F(x, x) \longrightarrow G(x, x)$$

Background: Dinatural transformations

- A **difunctor** from $\mathbb{C}$ to $\mathbb{D}$ is a functor $F, G : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$.
- A **dipresheaf** on $\mathbb{C}$ is an endoprofunctor $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$.
- Given difunctors $F, G : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$, a **dinatural transformation** $\alpha : F \xrightarrow{\bullet\bullet} G$ is a family of morphisms of $\mathbb{D}$ for each $x : \mathbb{C}$,

$$\alpha_x : F(x, x) \longrightarrow G(x, x)$$

such that for every $a, b : \mathbb{C}$ and $f : a \to b$ this hexagon commutes:

$$\begin{tikzcd}
 & F(b,b) \arrow[r, "\alpha_b"] & G(b,b) \arrow[dr, "G(f,\mathsf{id})"] & \\
F(b,a) \arrow[ur, "F(\mathsf{id},f)"] \arrow[dr, "F(f,\mathsf{id})"'] & & & G(a,b) \\
 & F(a,a) \arrow[r, "\alpha_a"] & G(a,a) \arrow[ur, "G(\mathsf{id},f)"'] &
\end{tikzcd}$$

Background: Dinatural transformations

- A **difunctor** from $\mathbb{C}$ to $\mathbb{D}$ is a functor $F, G : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$.
- A **dipresheaf** on $\mathbb{C}$ is an endoprofunctor $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$.
- Given difunctors $F, G : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$, a **dinatural transformation** $\alpha : F \xrightarrow{\bullet\bullet} G$ is a family of morphisms of $\mathbb{D}$ for each $x : \mathbb{C}$,

$$\alpha_x : F(x,x) \longrightarrow G(x,x)$$

such that for every $a, b : \mathbb{C}$ and $f : a \to b$ this hexagon commutes:

$$\begin{tikzcd}
 & F(b,b) \arrow[r, "\alpha_b"] & G(b,b) \arrow[dr, "G(f,\mathsf{id})"] & \\
F(b,a) \arrow[ur, "F(\mathsf{id},f)"] \arrow[dr, "F(f,\mathsf{id})"'] & & & G(a,b) \\
 & F(a,a) \arrow[r, "\alpha_a"] & G(a,a) \arrow[ur, "G(\mathsf{id},f)"'] &
\end{tikzcd}$$

Theorem (Naturals are dinaturals [Dubuc and Street 1969])

A dinatural between functors which do not depend on $\mathbb{C}^{\mathsf{op}}$ is just a natural.

Background: Dinaturals don't always compose

- Dinaturals don't always compose: $\alpha : F \xrightarrow{\bullet\bullet} G$, $\beta : G \xrightarrow{\bullet\bullet} H$,

$$
\begin{tikzcd}
 & F(b,b) \arrow[rr, "\alpha_b"] & & G(b,b) \arrow[rr, "\beta_b"] \arrow[dr, "G(f,\mathsf{id})"] & & H(b,b) \arrow[dr, "H(f,\mathsf{id})"] & \\
F(b,a) \arrow[ur, "F(\mathsf{id},f)"] \arrow[dr, "F(f,\mathsf{id})"'] & & G(b,a) \arrow[ur, "G(\mathsf{id},f)"] \arrow[dr, "G(f,\mathsf{id})"'] & & G(a,b) & & H(a,b) \\
 & F(a,a) \arrow[rr, "\alpha_a"'] & & G(a,a) \arrow[ur, "G(\mathsf{id},f)"'] \arrow[rr, "\beta_a"'] & & H(a,a) \arrow[ur, "H(\mathsf{id},f)"'] &
\end{tikzcd}
$$

Background: Dinaturals don't always compose

- Dinaturals don't always compose: $\alpha : F \xrightarrow{\bullet\bullet} G$, $\beta : G \xrightarrow{\bullet\bullet} H$,

$$\begin{tikzcd}
& F(b,b) \arrow[rr, "\alpha_b"] & & G(b,b) \arrow[rr, "\beta_b"] \arrow[dr, "G(f,\mathsf{id})"] & & H(b,b) \arrow[dr, "H(f,\mathsf{id})"] & \\
F(b,a) \arrow[ur, "F(\mathsf{id},f)"] \arrow[dr, "F(f,\mathsf{id})"'] & & G(b,a) \arrow[ur, "G(\mathsf{id},f)"] \arrow[dr, "G(f,\mathsf{id})"'] & & G(a,b) & & H(a,b) \\
& F(a,a) \arrow[rr, "\alpha_a"'] & & G(a,a) \arrow[rr, "\beta_a"'] \arrow[ur, "G(\mathsf{id},f)"'] & & H(a,a) \arrow[ur, "H(\mathsf{id},f)"'] &
\end{tikzcd}$$

- They always compose with naturals: $\alpha : F \xrightarrow{\bullet\bullet} G$, $\beta : G \longrightarrow H$,

$$\begin{tikzcd}
& F(b,b) \arrow[r, "\alpha_b"] & G(b,b) \arrow[rr, "\beta_{bb}"] \arrow[dr, "G(f,\mathsf{id})"] & & H(b,b) \arrow[dr, "H(f,\mathsf{id})"] & \\
F(b,a) \arrow[ur, "F(\mathsf{id},f)"] \arrow[dr, "F(f,\mathsf{id})"'] & & & G(a,b) \arrow[rr, "\beta_{ab}"] & & H(a,b) \\
& F(a,a) \arrow[r, "\alpha_a"'] & G(a,a) \arrow[rr, "\beta_{aa}"'] \arrow[ur, "G(\mathsf{id},f)"'] & & H(a,a) \arrow[ur, "H(\mathsf{id},f)"'] &
\end{tikzcd}$$

(Co)end calculus in one slide

- (Co)ends give elegant characterizations of theorems of category theory.

(Co)end calculus in one slide

- (Co)ends give elegant characterizations of theorems of category theory.
- Natural transformations as ends:

$$\mathsf{Nat}(F, G) \cong \int_{x:\mathbb{C}} \hom_{\mathbb{D}}(F(\overline{x}), G(x))$$

(Co)end calculus in one slide

- (Co)ends give elegant characterizations of theorems of category theory.
- Natural transformations as ends:

$$\mathsf{Nat}(F, G) \cong \int_{x:\mathbb{C}} \hom_{\mathbb{D}}(F(\overline{x}), G(x))$$

- Quantifier exchange
$(Q : (\mathbb{C}^{\mathsf{op}} \times \mathbb{C})$
$\times (\mathbb{D}^{\mathsf{op}} \times \mathbb{D}) \to \mathbf{Set})$:

$$\int_{x:\mathbb{C}} \int_{y:\mathbb{D}} Q(\overline{x}, x, \overline{y}, y)$$
$$\cong \int_{y:\mathbb{D}} \int_{x:\mathbb{C}} Q(\overline{x}, x, \overline{y}, y)$$
$$\cong \int_{(x,y):\mathbb{C}\times\mathbb{D}} Q(\overline{x}, x, \overline{y}, y)$$

(Co)end calculus in one slide

- (Co)ends give elegant characterizations of theorems of category theory.
- Natural transformations as ends:

$$\mathsf{Nat}(F, G) \cong \int_{x:\mathbb{C}} \hom_{\mathbb{D}}(F(\overline{x}), G(x))$$

- Quantifier exchange $(Q : (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \times (\mathbb{D}^{\mathsf{op}} \times \mathbb{D}) \to \textbf{Set})$:

$$\int_{x:\mathbb{C}} \int_{y:\mathbb{D}} Q(\overline{x}, x, \overline{y}, y)$$
$$\cong \int_{y:\mathbb{D}} \int_{x:\mathbb{C}} Q(\overline{x}, x, \overline{y}, y)$$
$$\cong \int_{(x,y):\mathbb{C}\times\mathbb{D}} Q(\overline{x}, x, \overline{y}, y)$$

- (Co)Yoneda lemma: $(P : \mathbb{C} \to \textbf{Set})$

$$P(a) \cong \int_{x:\mathbb{C}} \hom_{\mathbb{C}}(a, \overline{x}) \Rightarrow P(x)$$
$$P(a) \cong \int^{x:\mathbb{C}} \hom_{\mathbb{C}}(\overline{x}, a) \times P(x)$$

(Co)end calculus in one slide

- (Co)ends give elegant characterizations of theorems of category theory.
- Natural transformations as ends:

$$\mathsf{Nat}(F, G) \cong \int_{x:\mathbb{C}} \hom_{\mathbb{D}}(F(\overline{x}), G(x))$$

- Quantifier exchange $(Q : (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \times (\mathbb{D}^{\mathsf{op}} \times \mathbb{D}) \to \mathbf{Set})$:

$$\int_{x:\mathbb{C}} \int_{y:\mathbb{D}} Q(\overline{x}, x, \overline{y}, y)$$
$$\cong \int_{y:\mathbb{D}} \int_{x:\mathbb{C}} Q(\overline{x}, x, \overline{y}, y)$$
$$\cong \int_{(x,y):\mathbb{C}\times\mathbb{D}} Q(\overline{x}, x, \overline{y}, y)$$

- (Co)Yoneda lemma: $(P : \mathbb{C} \to \mathbf{Set})$

$$P(a) \cong \int_{x:\mathbb{C}} \hom_{\mathbb{C}}(a, \overline{x}) \Rightarrow P(x)$$
$$P(a) \cong \int^{x:\mathbb{C}} \hom_{\mathbb{C}}(\overline{x}, a) \times P(x)$$

- Kan extensions: $(F : \mathbb{C} \to \mathbb{D})$

$$(\mathsf{Ran}_F P)(a) \cong \int_{x:\mathbb{C}} \hom_{\mathbb{D}}(a, F(\overline{x})) \Rightarrow P(x)$$
$$(\mathsf{Lan}_F P)(a) \cong \int^{x:\mathbb{C}} \hom_{\mathbb{D}}(F(\overline{x}), a) \times P(x)$$

Logical interpretation of (co)end calculus

- (Co)Yoneda lemma:

$$\begin{array}{ccccccc} P(a) & \cong & \displaystyle\int_{x:\mathbb{C}} & \hom_{\mathbb{C}}(a, \overline{x}) & \Rightarrow & P(x) \\ P(a) & \Leftrightarrow & \forall(x : \mathbb{C}). & a =_{\mathbb{C}} x & \Rightarrow & P(x) \end{array}$$

$$\begin{array}{ccccccc} P(a) & \cong & \displaystyle\int^{x:\mathbb{C}} & \hom_{\mathbb{C}}(\overline{x}, a) & \times & P(x) \\ P(a) & \Leftrightarrow & \exists(x : \mathbb{C}). & x =_{\mathbb{C}} a & \wedge & P(x) \end{array}$$

Logical interpretation of (co)end calculus

- (Co)Yoneda lemma:

$$\begin{array}{lcccccc} P(a) & \cong & \int_{x:\mathbb{C}} & \hom_{\mathbb{C}}(a,\overline{x}) & \Rightarrow & P(x) \\ P(a) & \Leftrightarrow & \forall(x:\mathbb{C}). & a =_{\mathbb{C}} x & \Rightarrow & P(x) \end{array}$$

$$\begin{array}{lcccccc} P(a) & \cong & \int^{x:\mathbb{C}} & \hom_{\mathbb{C}}(\overline{x},a) & \times & P(x) \\ P(a) & \Leftrightarrow & \exists(x:\mathbb{C}). & x =_{\mathbb{C}} a & \wedge & P(x) \end{array}$$

- (Pointwise) right/left Kan extensions using ends/coends:

$$\begin{array}{lcccccc} (\mathsf{Ran}_F P)(a) & \cong & \int_{x:\mathbb{C}} & \hom_{\mathbb{D}}(a,F(x)) & \Rightarrow & P(x) \\ (\forall_f(P))(y) & \Leftrightarrow & \forall(x:C). & y =_D f(x) & \Rightarrow & P(x) \end{array}$$

$$\begin{array}{lcccccc} (\mathsf{Lan}_F P)(a) & \cong & \int^{x:\mathbb{C}} & \hom_{\mathbb{D}}(F(x),a) & \times & P(x) \\ (\exists_f(P))(y) & \Leftrightarrow & \exists(x:C). & f(x) =_D y & \wedge & P(x) \end{array}$$

Motivation 4: Computing adjoints to reindexings

- Consider a doctrine $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ with $\wedge, \Rightarrow, \forall_{A[\Gamma]}, \exists_{A[\Gamma]}, \mathsf{Eq}_A$.

Motivation 4: Computing adjoints to reindexings

- Consider a doctrine $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ with $\wedge, \Rightarrow, \forall_{A[\Gamma]}, \exists_{A[\Gamma]}, \mathsf{Eq}_A$.
- With just these assumptions, we can compute the adjoints $\forall_f, \exists_f : \mathcal{P}(C) \to \mathcal{P}(D)$ along any $f : C \to D$, not just projections.

Motivation 4: Computing adjoints to reindexings

- Consider a doctrine $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ with $\wedge, \Rightarrow, \forall_{A[\Gamma]}, \exists_{A[\Gamma]}, \mathsf{Eq}_A$.
- With just these assumptions, we can compute the adjoints $\forall_f, \exists_f : \mathcal{P}(C) \to \mathcal{P}(D)$ along any $f : C \to D$, not just projections.
- In the syntactic model, for $P : \mathcal{P}(C)$, $f : C \to D$:

$$(\forall_f(P))(y) := \forall x.(y =_D f(x) \Rightarrow P(x))$$
$$(\exists_f(P))(y) := \exists x.(f(x) =_D y \wedge P(x))$$

Motivation 4: Computing adjoints to reindexings

- Consider a doctrine $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ with $\wedge, \Rightarrow, \forall_{A[\Gamma]}, \exists_{A[\Gamma]}, \mathsf{Eq}_A$.
- With just these assumptions, we can compute the adjoints $\forall_f, \exists_f : \mathcal{P}(C) \to \mathcal{P}(D)$ along any $f : C \to D$, not just projections.
- In the syntactic model, for $P : \mathcal{P}(C)$, $f : C \to D$:

$$(\forall_f(P))(y) := \forall x.(y =_D f(x) \Rightarrow P(x))$$
$$(\exists_f(P))(y) := \exists x.(f(x) =_D y \wedge P(x))$$

- Compare these to the formula to compute Kan extensions via (co)ends:

$$(\mathsf{Ran}_F P)(y) \cong \int_{x:\mathbb{C}} \hom_{\mathbb{D}}(y, F(\overline{x})) \Rightarrow P(x)$$
$$(\mathsf{Lan}_F P)(y) \cong \int^{x:\mathbb{C}} \hom_{\mathbb{D}}(F(\overline{x}), y) \times P(x)$$

Motivation 4: Computing adjoints to reindexings

- Consider a doctrine $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ with $\wedge, \Rightarrow, \forall_{A[\Gamma]}, \exists_{A[\Gamma]}, \mathsf{Eq}_A$.
- With just these assumptions, we can compute the adjoints $\forall_f, \exists_f : \mathcal{P}(C) \to \mathcal{P}(D)$ along any $f : C \to D$, not just projections.
- In the syntactic model, for $P : \mathcal{P}(C)$, $f : C \to D$:

$$(\forall_f(P))(y) := \forall x.(y =_D f(x) \Rightarrow P(x))$$
$$(\exists_f(P))(y) := \exists x.(f(x) =_D y \wedge P(x))$$

- Compare these to the formula to compute Kan extensions via (co)ends:

$$(\mathsf{Ran}_F P)(y) \cong \int_{x:\mathbb{C}} \hom_{\mathbb{D}}(y, F(\overline{x})) \Rightarrow P(x)$$
$$(\mathsf{Lan}_F P)(y) \cong \int^{x:\mathbb{C}} \hom_{\mathbb{D}}(F(\overline{x}), y) \times P(x)$$

▶ there is yet no decomposition of these properties for doctrines, e.g., coends as quantifiers/adjoints, or "having directed equality".

Motivation 4: Computing Kan extensions with (co)ends

- A logical proof that $\forall_f(P)$ is right adjoint to precomposition with f:

$$
\begin{array}{ll}
[y : D] & \Gamma(y) \vdash (\forall_f P)(y) \\
 & \qquad := \forall x.(y = f(x) \Rightarrow P(x)) \\
\hline\hline
[x : C, y : D] & \Gamma(y) \vdash y = f(x) \Rightarrow P(x) \\
\hline\hline
[x : C, y : D] & y = f(x), \Gamma(y) \vdash P(x) \\
\hline\hline
[x : C] & \Gamma(f(x)) \vdash P(x)
\end{array}
$$

Motivation 4: Computing Kan extensions with (co)ends

- A logical proof that $\forall_f(P)$ is right adjoint to precomposition with f:

$$
\dfrac{\dfrac{\dfrac{[y:D] \qquad \Gamma(y) \vdash (\forall_f P)(y) := \forall x.(y = f(x) \Rightarrow P(x))}{[x:C, y:D] \qquad \Gamma(y) \vdash y = f(x) \Rightarrow P(x)}}{[x:C, y:D]\ y = f(x), \Gamma(y) \vdash P(x)}}{[x:C] \qquad \Gamma(f(x)) \vdash P(x)}
$$

▶ There is yet no formal system to do this for the directed case;

Motivation 4: Computing Kan extensions with (co)ends

- A logical proof that $\forall_f(P)$ is right adjoint to precomposition with f:

$$
\begin{array}{ll}
[y : D] & \Gamma(y) \vdash (\forall_f P)(y) \\
 & \qquad := \forall x.(y = f(x) \Rightarrow P(x)) \\
\hline\hline
[x : C, y : D] & \Gamma(y) \vdash y = f(x) \Rightarrow P(x) \\
\hline\hline
[x : C, y : D] & y = f(x), \Gamma(y) \vdash P(x) \\
\hline\hline
[x : C] & \Gamma(f(x)) \vdash P(x)
\end{array}
$$

- There is yet no formal system to do this for the directed case;
- We follow exactly this proof for Lan/Ran, using our rules for dinaturality.

Setting

We present the semantics for a *first-order* non-dependent directed type theory using dinaturality, where types are interpreted by categories, directed equality by $\hom$-functors, quantifiers by (co)ends.

Setting

We present the semantics for a *first-order* non-dependent directed type theory using dinaturality, where types are interpreted by categories, directed equality by $\hom$-functors, quantifiers by (co)ends.

We consider the following interpretation:

$$
\begin{aligned}
\text{Types} &\rightsquigarrow \text{Categories (possibly with } -^{\mathsf{op}}\text{)} \\
\text{Contexts} &\rightsquigarrow \text{Lists of categories} \\
\text{Terms} &\rightsquigarrow \text{Functors } \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D} \\
\text{Propositions} &\rightsquigarrow \text{Endoprofunctors } \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D} \\
\text{Entailments} &\rightsquigarrow \text{Dinatural transformations (not required to compose)} \\
\text{Directed equality} &\rightsquigarrow \hom\text{-functors } \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \textbf{Set}.
\end{aligned}
$$

Setting

We present the semantics for a *first-order* non-dependent directed type theory using dinaturality, where types are interpreted by categories, directed equality by $\hom$-functors, quantifiers by (co)ends.

We consider the following interpretation:

$$\begin{aligned}
\text{Types} &\rightsquigarrow \text{Categories (possibly with } -^{\mathsf{op}}) \\
\text{Contexts} &\rightsquigarrow \text{Lists of categories} \\
\text{Terms} &\rightsquigarrow \text{Functors } \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D} \\
\text{Propositions} &\rightsquigarrow \text{Endoprofunctors } \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D} \\
\text{Entailments} &\rightsquigarrow \text{Dinatural transformations (not required to compose)} \\
\text{Directed equality} &\rightsquigarrow \hom\text{-functors } \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}.
\end{aligned}$$

- **Warning:** we do not give a doctrinal presentation of this logic, precisely because dinaturals do not compose in general.

Setting

We present the semantics for a *first-order* non-dependent directed type theory using dinaturality, where types are interpreted by categories, directed equality by $\hom$-functors, quantifiers by (co)ends.

We consider the following interpretation:

$$\begin{aligned}
\text{Types} &\rightsquigarrow \text{Categories (possibly with } -^{\mathsf{op}}\text{)} \\
\text{Contexts} &\rightsquigarrow \text{Lists of categories} \\
\text{Terms} &\rightsquigarrow \text{Functors } \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D} \\
\text{Propositions} &\rightsquigarrow \text{Endoprofunctors } \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D} \\
\text{Entailments} &\rightsquigarrow \text{Dinatural transformations (not required to compose)} \\
\text{Directed equality} &\rightsquigarrow \hom\text{-functors } \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}.
\end{aligned}$$

- **Warning:** we do not give a doctrinal presentation of this logic, precisely because dinaturals do not compose in general.
- Guiding intuition: we study the "doctrine" (in paracategories)

$$\begin{aligned}
&\mathsf{Dinats} : \mathbf{Cat}^{\mathsf{op}} \to \mathbf{PARACAT} \\
&\mathsf{Dinats}(\mathbb{C}) := [\mathbb{C}^{\mathsf{op}} \times \mathbb{C}, \mathbf{Set}]_{\mathsf{dinaturals}}
\end{aligned}$$

Directed type theory: notation

- Example of an entailment:

$$\frac{[x:\mathbb{C}, y:\mathbb{D}]\ P(\overline{x},\overline{y},x,y) \vdash \alpha : Q(\overline{x},\overline{y},x,y)}{\textit{Semantics}\text{: "}\alpha\text{ is a dinatural from } P \text{ to } Q : (\mathbb{C}\times\mathbb{D})^{\mathsf{op}} \times (\mathbb{C}\times\mathbb{D}) \to \mathbf{Set}\text{."}}$$

- **Crucial:** variables now can appear both as $x : \mathbb{C}$ or $\overline{x} : \mathbb{C}^{\mathsf{op}}$.
- We give names to assumptions p, q:

$$[x:\mathbb{C}]\ p : P(\overline{x},x), q : Q(\overline{x},x) \vdash h[p,q] : R(\overline{x},x)$$

prop. context $\to$ interpreted as pointwise product of presheaves in **Set**.

Directed type theory: notation

- Example of an entailment:

$$\frac{[x : \mathbb{C}, y : \mathbb{D}]\ P(\overline{x}, \overline{y}, x, y) \vdash \alpha : Q(\overline{x}, \overline{y}, x, y)}{\textit{Semantics}\text{: "}\alpha \text{ is a dinatural from } P \text{ to } Q : (\mathbb{C} \times \mathbb{D})^{\mathsf{op}} \times (\mathbb{C} \times \mathbb{D}) \to \mathbf{Set}\text{."}}$$

- **Crucial:** variables now can appear both as $x : \mathbb{C}$ or $\overline{x} : \mathbb{C}^{\mathsf{op}}$.
- We give names to assumptions p, q:

$$[x : \mathbb{C}]\ p : P(\overline{x}, x), q : Q(\overline{x}, x) \vdash h[p, q] : R(\overline{x}, x)$$

  prop. context $\to$ interpreted as pointwise product of presheaves in **Set**.
- The following dinaturals are the same:

$$\begin{aligned} [x : \mathbb{C}]\ F(\overline{x}) &\vdash \alpha\ \ : G(\overline{x}, x) \\ [x : \mathbb{C}^{\mathsf{op}}]\ F(x) &\vdash \alpha' : G(x, \overline{x}) \end{aligned}$$

Directed type theory: propositional rules – Products

- Dinaturals support propositional conjunction, using products in **Set**.

Directed type theory: propositional rules – Products

- Dinaturals support propositional conjunction, using products in **Set**.

Theorem (Product of dipresheaves)

There is an isomorphism of sets natural in $\Gamma, P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \textbf{\textit{Set}}$:

$$\frac{[x : \mathbb{C}]\ \Gamma(\overline{x}, x) \vdash P(\overline{x}, x) \times Q(\overline{x}, x)}{[x : \mathbb{C}]\ \Gamma(\overline{x}, x) \vdash P(\overline{x}, x), \qquad [x : \mathbb{C}]\ \Gamma(\overline{x}, x) \vdash Q(\overline{x}, x).}$$

Bottom side: product of sets of dinaturals.
Similarly, $\top_{\mathbb{C}} : \mathbb{C}^{\diamond} \to \textbf{\textit{Set}} := (c, c') \mapsto \{*\}$ *has a unique dinat* $\Gamma \xrightarrow{\bullet\bullet} \top_{\mathbb{C}}$.

Directed type theory: propositional rules – Exponentials

- Contrary to naturals, “exponentials” of dipresheaves are *pointwise*:

Directed type theory: propositional rules – Exponentials

- Contrary to naturals, "exponentials" of dipresheaves are *pointwise*:

Theorem ("Exponential" of dipresheaves) ( )

There is an isomorphism of sets natural in $F, G, H : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \textbf{\textit{Set}}$*:*

$$\frac{[x:\mathbb{C}]\ F(\overline{x},x) \times G(\overline{x},x) \vdash H(\overline{x},x)}{\overline{\overline{[x:\mathbb{C}]\qquad\qquad G(\overline{x},x) \vdash F^{\mathsf{op}}(x,\overline{x}) \Rightarrow H(\overline{x},x)}}\ (\text{exp})$$

Proof. *Obvious by currying the families of morphisms.*

Directed type theory: propositional rules – Exponentials

- Contrary to naturals, "exponentials" of dipresheaves are *pointwise*:

Theorem ("Exponential" of dipresheaves)

There is an isomorphism of sets natural in $F, G, H : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$:

$$\begin{array}{ll} [x:\mathbb{C}] & F(\overline{x},x) \times G(\overline{x},x) \vdash H(\overline{x},x) \\ \hline\hline [x:\mathbb{C}] & \qquad G(\overline{x},x) \vdash F^{\mathsf{op}}(x,\overline{x}) \Rightarrow H(\overline{x},x) \end{array} \ (\mathsf{exp})$$

Proof. *Obvious by currying the families of morphisms.*

- *Why are exponentials of presheaves with naturals not pointwise?*

$$\begin{array}{ll} [x:\mathbb{C}] & F(x) \times G(x) \vdash H(x) \\ \hline\hline [x:\mathbb{C}] & \qquad G(x) \vdash F^{\mathsf{op}}(\overline{x}) \Rightarrow H(x) \end{array} \ (\mathsf{exp})$$

- Contrary to naturals, "exponentials" of dipresheaves are *pointwise*:

Theorem ("Exponential" of dipresheaves)

There is an isomorphism of sets natural in $F, G, H : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \textbf{\textit{Set}}$:

$$\begin{array}{c} [x : \mathbb{C}] \ F(\overline{x}, x) \times G(\overline{x}, x) \vdash H(\overline{x}, x) \\ \hline\hline [x : \mathbb{C}] \qquad\qquad G(\overline{x}, x) \vdash F^{\mathsf{op}}(x, \overline{x}) \Rightarrow H(\overline{x}, x) \end{array} \ (\mathsf{exp})$$

Proof. *Obvious by currying the families of morphisms.*

- *Why are exponentials of presheaves with naturals not pointwise?*

$$\begin{array}{c} [x : \mathbb{C}] \ F(x) \times G(x) \vdash H(x) \\ \hline\hline [x : \mathbb{C}] \qquad\qquad G(x) \vdash F^{\mathsf{op}}(\overline{x}) \Rightarrow H(x) \end{array} \ (\mathsf{exp})$$

but, the bottom family would be *dinatural* in x.

Directed type theory: propositional rules – Exponentials

- Contrary to naturals, "exponentials" of dipresheaves are *pointwise*:

Theorem ("Exponential" of dipresheaves)

*There is an isomorphism of sets natural in $F, G, H : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to$ **Set**:*

$$\begin{array}{l} [x : \mathbb{C}] \ F(\overline{x}, x) \times G(\overline{x}, x) \vdash H(\overline{x}, x) \\ \hline\hline [x : \mathbb{C}] \qquad\qquad G(\overline{x}, x) \vdash F^{\mathsf{op}}(x, \overline{x}) \Rightarrow H(\overline{x}, x) \end{array} \ (\mathsf{exp})$$

Proof. *Obvious by currying the families of morphisms.*

- *Why are exponentials of presheaves with naturals not pointwise?*

$$\begin{array}{l} [x : \mathbb{C}] \ F(x) \times G(x) \vdash H(x) \\ \hline\hline [x : \mathbb{C}] \qquad\qquad G(x) \vdash F^{\mathsf{op}}(\overline{x}) \Rightarrow H(x) \end{array} \ (\mathsf{exp})$$

but, the bottom family would be *dinatural* in x.

▶ We will show how to use our rules to justify the usual exponential.

Directed type theory: term rules – Reindexing

- Entailments/dinaturals can be reindexed by terms/difunctors:

Theorem (Reindexing with difunctors) ( )

Take a difunctor $F : \mathbb{C}^\diamond \to \mathbb{D}$ *and a dinat* $\alpha : P \xrightarrow{\bullet\bullet} Q$ *for* $P, Q : \mathbb{D}^\diamond \to \mathbb{E}$.

$$\frac{[x : \mathbb{D}] \qquad P(\overline{x}, x) \vdash \alpha : Q(\overline{x}, x)}{[x : \mathbb{C}] \;\; P(F^{\mathsf{op}}(x, \overline{x}), F(\overline{x}, x)) \vdash F^*(\alpha) : Q(F^{\mathsf{op}}(x, \overline{x}), F(\overline{x}, x))} \text{ (reindex)}$$

defined by $F^*(\alpha)_x := \alpha_{F(x,x)}$.

Directed type theory with dinaturality – refl

- Dinaturality allows us to solve the variance issue of refl: [North 2018]

Directed type theory with dinaturality – refl

- Dinaturality allows us to solve the variance issue of refl: [North 2018]

Theorem (Directed equality introduction)

There is a dinatural transformation $\mathsf{refl}_{\mathbb{C}} : \top \overset{\bullet\bullet}{\longrightarrow} \hom$,

$$\frac{}{[x : \mathbb{C}]\ \top \vdash \mathsf{refl}_{\mathbb{C}} : \hom_{\mathbb{C}}(\overline{x}, x)}\ (\mathsf{refl})$$

where $\top$ *denotes the dipresheaf constant in* $\top_{\textbf{Set}}$.
Proof. $\alpha_x(*) := \mathsf{id}_x$. *Dinaturality: for any* $f : a \to b$, $f \mathbin{;} \mathsf{id}_b = \mathsf{id}_a \mathbin{;} f$.

Directed type theory with dinaturality – refl

- Dinaturality allows us to solve the variance issue of refl: [North 2018]

Theorem (Directed equality introduction) (👏)

There is a dinatural transformation $\mathsf{refl}_{\mathbb{C}} : \top \xrightarrow{\bullet\bullet} \hom$,

$$\frac{}{[x : \mathbb{C}]\ \top \vdash \mathsf{refl}_{\mathbb{C}} : \hom_{\mathbb{C}}(\overline{x}, x)}\ (\mathsf{refl})$$

where $\top$ *denotes the dipresheaf constant in* $\top_{\mathbf{Set}}$.
Proof. $\alpha_x(*) := \mathsf{id}_x$. *Dinaturality: for any* $f : a \to b$, $f \,;\, \mathsf{id}_b = \mathsf{id}_a \,;\, f$.

- This is reflexivity of directed equality, via identities.

Directed type theory with dinaturality – Intermezzo

- Before introducing the directed J rule, we show its fundamental idea: the connection between naturality and dinaturality.

Directed type theory with dinaturality – Intermezzo

- Before introducing the directed J rule, we show its fundamental idea: the connection between naturality and dinaturality.

Theorem (Characterization of dinaturals via naturality) (👏)

*There is an isomorphism, natural in $P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to$ **Set**, between the set of dinaturals $P \xrightarrow{\bullet\bullet} Q$ and certain natural transformations:*

$$\frac{[x : \mathbb{C}]\ P(\overline{x}, x) \xrightarrow{\bullet\bullet} Q(\overline{x}, x)}{\overline{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}]\, \mathrm{hom}(a, b) \longrightarrow P^{\mathsf{op}}(b, a) \Rightarrow Q(a, b)}}$$

Directed type theory with dinaturality – Intermezzo

- Before introducing the directed J rule, we show its fundamental idea: the connection between naturality and dinaturality.

Theorem (Characterization of dinaturals via naturality) ( )

*There is an isomorphism, natural in $P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to$ **Set**, between the set of dinaturals $P \xrightarrow{\bullet\bullet} Q$ and certain natural transformations:*

$$\begin{array}{c} [x : \mathbb{C}]\ P(\overline{x}, x) \xrightarrow{\bullet\bullet} Q(\overline{x}, x) \\ \hline\hline [a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}]\, \hom(a, b) \longrightarrow P^{\mathsf{op}}(b, a) \Rightarrow Q(a, b) \end{array}$$

Proof.

($\Downarrow$) *Given $\alpha : P \xrightarrow{\bullet\bullet} Q$ and $f : \hom(a, b)$, the map $P(b, a) \to Q(a, b)$ is exactly the side of the hexagon in the definition of dinaturality. This is obtained via the functorial action of P, Q.*

Directed type theory with dinaturality – Intermezzo

- Before introducing the directed J rule, we show its fundamental idea: the connection between naturality and dinaturality.

Theorem (Characterization of dinaturals via naturality)

*There is an isomorphism, natural in $P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to$ **Set**, between the set of dinaturals $P \xrightarrow{\bullet\bullet} Q$ and certain natural transformations:*

$$\begin{array}{c} [x : \mathbb{C}]\ P(\overline{x}, x) \xrightarrow{\bullet\bullet} Q(\overline{x}, x) \\ \hline\hline [a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}]\, \hom(a, b) \longrightarrow P^{\mathsf{op}}(b, a) \Rightarrow Q(a, b) \end{array}$$

Proof.

($\Downarrow$) *Given $\alpha : P \xrightarrow{\bullet\bullet} Q$ and $f : \hom(a, b)$, the map $P(b, a) \to Q(a, b)$ is exactly the side of the hexagon in the definition of dinaturality. This is obtained via the functorial action of P, Q.*

($\Uparrow$) *Take $a = b$ and precompose with $\mathsf{id}_a \in \hom(a, a)$.*

Directed type theory with dinaturality – Intermezzo

- Before introducing the directed J rule, we show its fundamental idea: the connection between naturality and dinaturality.

Theorem (Characterization of dinaturals via naturality) ()

*There is an isomorphism, natural in $P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to$ **Set**, between the set of dinaturals $P \xrightarrow{\bullet\bullet} Q$ and certain natural transformations:*

$$\frac{[x : \mathbb{C}]\ P(\overline{x}, x) \xrightarrow{\bullet\bullet} Q(\overline{x}, x)}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}]\ \hom(a, b) \longrightarrow P^{\mathsf{op}}(b, a) \Rightarrow Q(a, b)}$$

Proof.

($\Downarrow$) *Given $\alpha : P \xrightarrow{\bullet\bullet} Q$ and $f : \hom(a, b)$, the map $P(b, a) \to Q(a, b)$ is exactly the side of the hexagon in the definition of dinaturality. This is obtained via the functorial action of P, Q.*

($\Uparrow$) *Take $a = b$ and precompose with $\mathrm{id}_a \in \hom(a, a)$.*

The isomorphism follows from (di)naturality of both maps.

Directed type theory with dinaturality – Directed J

- Directed equality elimination is just this result, uncurried.

Theorem (Directed equality elimination) ()

Take $\Gamma, P : (\mathbb{A}^{\mathsf{op}}) \times (\mathbb{A}) \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to \textbf{\textit{Set}}$.
Given a dinatural $h : \Gamma \xrightarrow{\bullet\bullet} P$, *there is a dinatural* $J(h)$ *as follows:*

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \qquad k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash h[k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ e : \hom(a, b), k : \Gamma(\overline{b}, \overline{a}, \overline{x}, x) \vdash J(h)[e, k] : P(a, b, \overline{x}, x)}\ (J)$$

Directed type theory with dinaturality – Directed J

- Directed equality elimination is just this result, uncurried.

Theorem (Directed equality elimination)

Take $\Gamma, P : (\mathbb{A}^{\mathsf{op}}) \times (\mathbb{A}) \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to \textbf{\textit{Set}}$.
Given a dinatural $h : \Gamma \xrightarrow{\bullet\bullet} P$, *there is a dinatural* $J(h)$ *as follows:*

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \qquad k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash h[k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}] \; e : \hom(a, b), k : \Gamma(\overline{b}, \overline{a}, \overline{x}, x) \vdash J(h)[e, k] : P(a, b, \overline{x}, x)} \; (J)$$

The dinatural $J(h)$ *satisfies the following "computation rule",*

$$J(h)_{zzx}[\mathsf{refl}_{\mathbb{A}z}, k] = h_{zx}[k]$$

for any object $z : \mathbb{A}, x : \mathbb{C}$ *and* $k \in \Gamma(z, z, x, x)$.

Directed type theory with dinaturality – Directed J

- Directed equality elimination is just this result, uncurried.

Theorem (Directed equality elimination) (👏)

*Take $\Gamma, P : (\mathbb{A}^{\mathsf{op}}) \times (\mathbb{A}) \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to$ **Set**.*
Given a dinatural $h : \Gamma \xrightarrow{\bullet\bullet} P$, there is a dinatural $J(h)$ as follows:

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \qquad k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash h[k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ e : \hom(a, b), k : \Gamma(\overline{b}, \overline{a}, \overline{x}, x) \vdash J(h)[e, k] : P(a, b, \overline{x}, x)} \ (J)$$

The dinatural $J(h)$ satisfies the following "computation rule",

$$J(h)_{zzx}[\mathsf{refl}_{\mathbb{A}z}, k] = h_{zx}[k]$$

for any object $z : \mathbb{A}, x : \mathbb{C}$ and $k \in \Gamma(z, z, x, x)$.

Proof. *Explicitly, the dinatural $J(h)$ is given by*

$$J(h)_{abx}[e, k] := (\Gamma(\mathsf{id}_b, e, \mathsf{id}_x, \mathsf{id}_x) \,;\, h_{bx} \,;\, P(e, \mathsf{id}_b, \mathsf{id}_x, \mathsf{id}_x))[k].$$

Computation clearly holds for $e = \mathsf{id}_z$, without dinaturality.

Directed type theory with dinaturality – Intuition for J

Intution for the directed J rule

- Whenever there are two positions $a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}$ in the conclusion P,

Intution for the directed J rule

- Whenever there are two positions $a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}$ in the conclusion P,
- and I have a directed equality in context $\hom_{\mathbb{A}}(a, b)$,

Directed type theory with dinaturality – Intuition for J

Intution for the directed J rule

- Whenever there are two positions $a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}$ in the conclusion P,
- and I have a directed equality in context $\hom_{\mathbb{A}}(a, b)$,

---

- it is enough to prove that P holds “on the diagonal”,
  where a, b are identified with the same dinatural variable $z : \mathbb{A}$.

Directed type theory with dinaturality – Intuition for J

Intution for the directed J rule

- Whenever there are two positions $a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}$ in the conclusion P,
- and I have a directed equality in context $\hom_{\mathbb{A}}(a, b)$,

---

- it is enough to prove that P holds "on the diagonal", where a, b are identified with the same dinatural variable $z : \mathbb{A}$.
- Moreover, a, b can be identified in context *only if they appear negatively*.

- With the directed J rule we can define maps+properties directed equality.
- Examples: *transitivity*, *congruence*, *transport* for directed equality.

Example (Composition in a category)

Transitivity of directed equality $\rightsquigarrow$ categories $\mathbb{C}$ have *composition maps*.

Directed type theory with dinaturality – Example

Example (Composition in a category)

Transitivity of directed equality $\rightsquigarrow$ categories $\mathbb{C}$ have *composition maps*. Composition is natural in $a : \mathbb{C}^{\mathsf{op}}, c : \mathbb{C}$ and dinatural in $b : \mathbb{C}$:

$$\dfrac{\dfrac{}{[z : \mathbb{C}, c : \mathbb{C}] \qquad \hom(\overline{z}, c) \vdash \mathsf{id} : \hom(\overline{z}, c)}\ (\mathsf{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}, c : \mathbb{C}]\ \hom(a, b),\ \hom(\bar{b}, c) \vdash J(\mathsf{id}) : \hom(a, c)}\ (J)$$

Directed type theory with dinaturality – Example

Example (Composition in a category)

Transitivity of directed equality $\rightsquigarrow$ categories $\mathbb{C}$ have *composition maps*. Composition is natural in $a : \mathbb{C}^{\mathsf{op}}, c : \mathbb{C}$ and dinatural in $b : \mathbb{C}$:

$$\dfrac{\dfrac{}{[z : \mathbb{C}, c : \mathbb{C}] \qquad \hom(\overline{z}, c) \vdash \mathsf{id} : \hom(\overline{z}, c)}\ (\mathsf{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}, c : \mathbb{C}]\ \hom(a, b),\ \hom(\overline{b}, c) \vdash J(\mathsf{id}) : \hom(a, c)}\ (J)$$

We contract $\hom(a, b)$. Rule (J) can be applied: a, b appear only negatively in ctx (a does not) and positively in conclusion ($\overline{b}$ does not).

Directed type theory with dinaturality – Example

Example (Composition in a category)

Transitivity of directed equality $\rightsquigarrow$ categories $\mathbb{C}$ have *composition maps*.
Composition is natural in $a : \mathbb{C}^{\mathsf{op}}, c : \mathbb{C}$ and dinatural in $b : \mathbb{C}$:

$$\dfrac{\dfrac{}{[z : \mathbb{C}, c : \mathbb{C}] \qquad \hom(\overline{z}, c) \vdash \mathsf{id} : \hom(\overline{z}, c)}\ (\mathsf{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}, c : \mathbb{C}]\ \hom(a, b),\ \hom(\overline{b}, c) \vdash J(\mathsf{id}) : \hom(a, c)}\ (J)$$

We contract $\hom(a, b)$. Rule (J) can be applied: a, b appear only negatively in ctx (a does not) and positively in conclusion ($\overline{b}$ does not).

The computation rule for $f\ ;g := J(\mathsf{id})[f, g]$ states unitality on the left.

Directed type theory with dinaturality – Example

Failure of symmetry for directed equality

These restrictions do *not* allow us to obtain directed equality is symmetric:

$$[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}]\ e : \hom(a, b) \nvdash \mathsf{sym} : \hom(\overline{b}, \overline{a})$$

$\hom(a, b)$ cannot be contracted: a, b must appear *positively* in conclusion.

Directed type theory – J as iso

- As in the classical case, the rule for directed equality is an isomorphism.

Theorem (J as isomorphism)

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \qquad k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash h[k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}] \; e : \hom(a, b), k : \Gamma(\overline{b}, \overline{a}, \overline{x}, x) \vdash J(h)[e, k] : P(a, b, \overline{x}, x)} \; (\hom)$$

Directed type theory – J as iso

- As in the classical case, the rule for directed equality is an isomorphism.

Theorem (J as isomorphism)

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \qquad k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash h[k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ e : \hom(a, b), k : \Gamma(\overline{b}, \overline{a}, \overline{x}, x) \vdash J(h)[e, k] : P(a, b, \overline{x}, x)} \text{(hom)}$$

Proof. *The inverse is* $J^{-1}(\alpha)_{zx}[k] := \alpha_{zzx}[\mathsf{refl}_{\mathbb{A}z}, k]$.

Directed type theory – J as iso

- As in the classical case, the rule for directed equality is an isomorphism.

Theorem (J as isomorphism)

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \qquad k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash h[k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}] \; e : \hom(a, b), k : \Gamma(\overline{b}, \overline{a}, \overline{x}, x) \vdash J(h)[e, k] : P(a, b, \overline{x}, x)} \;(\hom)$$

Proof. *The inverse is* $J^{-1}(\alpha)_{zx}[k] := \alpha_{zzx}[\mathsf{refl}_{\mathbb{A}z}, k]$.
The computation rule for $\hom$*-elimination is precisely* $J \,;\, J^{-1} = \mathsf{id}$.
On the other hand, $J^{-1} \,;\, J = \mathsf{id}$ *follows using (di)naturality.*

Directed type theory – dependent directed J

- Back to the transitivity example.
- *How do we prove associativity of transitivity and unitality on the right?*

Directed type theory – dependent directed J

- Back to the transitivity example.
- *How do we prove associativity of transitivity and unitality on the right?*
- Using the same method of MLTT: by *dependent* $\hom$ induction.

Directed type theory – dependent directed J

- Back to the transitivity example.
- *How do we prove associativity of transitivity and unitality on the right?*
- Using the same method of MLTT: by *dependent* $\hom$ induction.
- We work with *proof-relevant entailments*: we need a judgm. for $\alpha = \beta$.

Directed type theory – dependent directed J

- Back to the transitivity example.
- *How do we prove associativity of transitivity and unitality on the right?*
- Using the same method of MLTT: by *dependent* $\hom$ induction.
- We work with *proof-relevant entailments*: we need a judgm. for $\alpha = \beta$.

Theorem (Dependent J rule for judgemental equality) ()

*Take $\Gamma, P : (\mathbb{A}^{\mathsf{op}}) \times (\mathbb{A}) \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to$ **Set**. Given dinats α, β where J can be applied, then the equality above implies the one below:*

$$\frac{[z : \mathbb{A}, x : \mathbb{C}]\ k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash \alpha[\mathsf{refl}_z, k] = \beta[\mathsf{refl}_z, k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ e : \hom(a, b), k : \Gamma(\overline{a}, \overline{b}, \overline{x}, x) \vdash \alpha[e, k] = \beta[e, k] : P(a, b, \overline{x}, x)}\ (J\text{-eq})$$

Directed type theory – dependent directed J

- Back to the transitivity example.
- *How do we prove associativity of transitivity and unitality on the right?*
- Using the same method of MLTT: by *dependent* $\hom$ induction.
- We work with *proof-relevant entailments*: we need a judgm. for $\alpha = \beta$.

Theorem (Dependent J rule for judgemental equality) ()

*Take $\Gamma, P : (\mathbb{A}^{\mathsf{op}}) \times (\mathbb{A}) \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to$ **Set**. Given dinats α, β where J can be applied, then the equality above implies the one below:*

$$\frac{[z : \mathbb{A}, x : \mathbb{C}]\ k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash \alpha[\mathsf{refl}_z, k] = \beta[\mathsf{refl}_z, k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ e : \hom(a, b), k : \Gamma(\overline{a}, \overline{b}, \overline{x}, x) \vdash \alpha[e, k] = \beta[e, k] : P(a, b, \overline{x}, x)} \ (J\text{-eq})$$

i.e., $\forall k : P(z, z, x, x).\ \alpha_{zzx}[\mathsf{refl}_z, k] = \beta_{zzx}[\mathsf{refl}_z, k]$ implies $\alpha = \beta$.

Directed type theory – dependent directed J

- Back to the transitivity example.
- *How do we prove associativity of transitivity and unitality on the right?*
- Using the same method of MLTT: by *dependent* $\hom$ induction.
- We work with *proof-relevant entailments*: we need a judgm. for $\alpha = \beta$.

Theorem (Dependent J rule for judgemental equality) (👏)

*Take $\Gamma, P : (\mathbb{A}^{\mathsf{op}}) \times (\mathbb{A}) \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to$ **Set**. Given dinats α, β where J can be applied, then the equality above implies the one below:*

$$\frac{[z : \mathbb{A}, x : \mathbb{C}]\ k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash \alpha[\mathsf{refl}_z, k] = \beta[\mathsf{refl}_z, k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ e : \hom(a, b), k : \Gamma(\overline{a}, \overline{b}, \overline{x}, x) \vdash \alpha[e, k] = \beta[e, k] : P(a, b, \overline{x}, x)}\ (J\text{-eq})$$

i.e., $\forall k : P(z, z, x, x).\ \alpha_{zzx}[\mathsf{refl}_z, k] = \beta_{zzx}[\mathsf{refl}_z, k]$ *implies* $\alpha = \beta$.

Proof. *By hypothesis, $J^{-1}(\alpha) = J^{-1}(\beta)$, simply apply J.*

Directed type theory – trans assoc/unit via dependent J

Example (Properties of composition)

Back to composition: we want to prove unitality and associativity.

Directed type theory – trans assoc/unit via dependent J

Example (Properties of composition)

Back to composition: we want to prove unitality and associativity.

$$\frac{}{[z : \mathbb{C}, c : \mathbb{C}]\ g : \hom(\overline{z}, c) \vdash \mathsf{refl}_z \mathbin{;} g = g : \hom(\overline{z}, c)}\ (J\text{-comp})$$

Directed type theory – trans assoc/unit via dependent J

Example (Properties of composition)

Back to composition: we want to prove unitality and associativity.

$$\frac{}{[z : \mathbb{C}, c : \mathbb{C}]\ g : \hom(\overline{z}, c) \vdash \mathsf{refl}_z \mathbin{;} g = g : \hom(\overline{z}, c)}\ (J\text{-comp})$$

Unitality on the right is shown by dependent $\hom$ induction:

$$\frac{\dfrac{}{[w : \mathbb{C}]\ \top \vdash \mathsf{refl}_w \mathbin{;} \mathsf{refl}_w = \mathsf{refl}_w : \hom(\overline{w}, w)}\ (J\text{-comp})}{[a : \mathbb{C}^{\mathsf{op}}, z : \mathbb{C}]\ f : \hom(a, z) \vdash f \mathbin{;} \mathsf{refl}_z = f : \hom(a, z)}\ (J\text{-eq})$$

Directed type theory – trans assoc/unit via dependent J

Example (Properties of composition)

Back to composition: we want to prove unitality and associativity.

$$\dfrac{}{[z:\mathbb{C}, c:\mathbb{C}]\ g:\hom(\overline{z},c) \vdash \mathsf{refl}_z \mathbin{;} g = g : \hom(\overline{z},c)}\ (J\text{-comp})$$

Unitality on the right is shown by dependent $\hom$ induction:

$$\dfrac{\dfrac{}{[w:\mathbb{C}]\ \top \vdash \mathsf{refl}_w \mathbin{;} \mathsf{refl}_w = \mathsf{refl}_w : \hom(\overline{w},w)}\ (J\text{-comp})}{[a:\mathbb{C}^{\mathsf{op}}, z:\mathbb{C}]\ f:\hom(a,z) \vdash f \mathbin{;} \mathsf{refl}_z = f : \hom(a,z)}\ (J\text{-eq})$$

To prove associativity, simply contract $f:\hom(a,b)$:

$$\dfrac{\dfrac{}{[z:\mathbb{C}, c:\mathbb{C}, d:\mathbb{C}]\quad g:\hom(\overline{z},c), h:\hom(\overline{c},d) \vdash \mathsf{refl}_z \mathbin{;} (g \mathbin{;} h) = (\mathsf{refl}_z \mathbin{;} g) \mathbin{;} h : \hom(\overline{z},d)}\ (J\text{-comp})}{[a:\mathbb{C}, b:\mathbb{C}, c:\mathbb{C}, d:\mathbb{C}]\ f:\hom(\overline{a},b), g:\hom(\overline{b},c), h:\hom(\overline{c},d) \vdash f \mathbin{;} (g \mathbin{;} h) = (f \mathbin{;} g) \mathbin{;} h : \hom(\overline{a},d)}\ (J\text{-eq})$$

where the top sequent $= g \mathbin{;} h$ by computation rules for comp $:= J(\mathsf{id})$.

Directed type theory – trans assoc/unit via dependent J

Example (Properties of composition)

Back to composition: we want to prove unitality and associativity.

$$\dfrac{}{[z:\mathbb{C}, c:\mathbb{C}]\ g:\hom(\overline{z},c) \vdash \mathsf{refl}_z \mathbin{;} g = g : \hom(\overline{z},c)}\ (J\text{-comp})$$

Unitality on the right is shown by dependent $\hom$ induction:

$$\dfrac{\dfrac{}{[w:\mathbb{C}]\ \top \vdash \mathsf{refl}_w \mathbin{;} \mathsf{refl}_w = \mathsf{refl}_w : \hom(\overline{w},w)}\ (J\text{-comp})}{[a:\mathbb{C}^{\mathsf{op}}, z:\mathbb{C}]\ f:\hom(a,z) \vdash f \mathbin{;} \mathsf{refl}_z = f : \hom(a,z)}\ (J\text{-eq})$$

To prove associativity, simply contract $f:\hom(a,b)$:

$$\dfrac{\dfrac{}{[z:\mathbb{C}, c:\mathbb{C}, d:\mathbb{C}]\ g:\hom(\overline{z},c), h:\hom(\overline{c},d) \vdash \mathsf{refl}_z \mathbin{;} (g \mathbin{;} h) = (\mathsf{refl}_z \mathbin{;} g) \mathbin{;} h : \hom(\overline{z},d)}\ (J\text{-comp})}{[a:\mathbb{C}, b:\mathbb{C}, c:\mathbb{C}, d:\mathbb{C}]\ f:\hom(\overline{a},b), g:\hom(\overline{b},c), h:\hom(\overline{c},d) \vdash f \mathbin{;} (g \mathbin{;} h) = (f \mathbin{;} g) \mathbin{;} h : \hom(\overline{a},d)}\ (J\text{-eq})$$

where the top sequent $= g \mathbin{;} h$ by computation rules for comp $:= J(\mathsf{id})$.

- *Note!* These are exactly the steps in MLTT for transitivity of paths.

Directed type theory – congr for directed equality

- Every term respects directed equality, i.e., it is a "congruence": semantically, this is just the functorial action of terms F on morphisms.

Directed type theory – congr for directed equality

- Every term respects directed equality, i.e., it is a "congruence": semantically, this is just the functorial action of terms F on morphisms.

Example (Directed equality is a congruence)

Let $F : \mathbb{C} \to \mathbb{D}$ be a functor.

$$\dfrac{\dfrac{}{[z : \mathbb{C}]\ \top \vdash \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{z}), F(z))}\ \text{(reindex)+(refl)}}{[x : \mathbb{C}, y : \mathbb{C}]\ \hom_{\mathbb{C}}(\overline{x}, y) \vdash \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{x}), F(y))}\ (J)$$

Directed type theory – congr for directed equality

- Every term respects directed equality, i.e., it is a "congruence": semantically, this is just the functorial action of terms F on morphisms.

Example (Directed equality is a congruence)

Let $F : \mathbb{C} \to \mathbb{D}$ be a functor.

$$\dfrac{\dfrac{}{[z : \mathbb{C}]\ \top \vdash \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{z}), F(z))}\ \text{(reindex)+(refl)}}{[x : \mathbb{C}, y : \mathbb{C}]\ \hom_{\mathbb{C}}(\overline{x}, y) \vdash \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{x}), F(y))}\ (J)$$

Take $\mathsf{map}_F[f] := J(F^*(\mathsf{refl}_{\mathbb{C}}))$. Computation rule: F *maps* refl *to* refl:

$$\dfrac{}{[z : \mathbb{C}]\ \top \vdash \mathsf{map}_F[\mathsf{refl}_z] = F^*(\mathsf{refl}_z) : \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{x}), F(x))}\ (J\text{-comp})$$

Directed type theory – congr for directed equality

- Every term respects directed equality, i.e., it is a "congruence": semantically, this is just the functorial action of terms F on morphisms.

Example (Directed equality is a congruence)

Let $F : \mathbb{C} \to \mathbb{D}$ be a functor.

$$\dfrac{\dfrac{}{[z : \mathbb{C}]\ \top \vdash \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{z}), F(z))}\ \text{(reindex)+(refl)}}{[x : \mathbb{C}, y : \mathbb{C}]\ \hom_{\mathbb{C}}(\overline{x}, y) \vdash \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{x}), F(y))}\ (J)$$

Take $\mathsf{map}_F[f] := J(F^*(\mathsf{refl}_{\mathbb{C}}))$. Computation rule: F *maps* refl *to* refl:

$$\dfrac{}{[z : \mathbb{C}]\ \top \vdash \mathsf{map}_F[\mathsf{refl}_z] = F^*(\mathsf{refl}_z) : \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{x}), F(x))}\ (J\text{-comp})$$

Functoriality holds, since both top sides $=$ $\mathsf{map}_F[g]$ via computation rules:

$$\dfrac{\dfrac{}{[z : \mathbb{C}, c : \mathbb{C}]\quad g : \hom(\overline{z}, c) \vdash \mathsf{map}_F[\mathsf{refl}_z \,;\, g] = \mathsf{refl}_{F(z)} \,;\, \mathsf{map}_F[g] : \hom(\overline{z}, d)}\ (J\text{-comp})}{[a : \mathbb{C}, b : \mathbb{C}, c : \mathbb{C}]\ f : \hom(\overline{a}, b), g : \hom(\overline{b}, c) \vdash \mathsf{map}_F[f \,;\, g] = \mathsf{map}_F[f] \,;\, \mathsf{map}_F[g] : \hom(\overline{a}, d)}\ (J\text{-eq})$$

Directed type theory – transport along directed equalities

- Transporting points of predicates (i.e., presheaves) along directed equalities is the functorial action of $P : \mathbb{C} \to \textbf{Set}$:

Directed type theory – transport along directed equalities

- Transporting points of predicates (i.e., presheaves) along directed equalities is the functorial action of $P : \mathbb{C} \to \mathbf{Set}$:

Example (Transporting along directed equality)

For any $P : \mathbb{C} \to \mathbf{Set}$:

$$\dfrac{\dfrac{}{[z : \mathbb{C}]\ P(z) \vdash P(z)}\ (\mathrm{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}]\ \ \hom(a, b), P(\overline{a}) \vdash P(b)}\ (J)$$

Directed type theory – transport along directed equalities

- Transporting points of predicates (i.e., presheaves) along directed equalities is the functorial action of $P : \mathbb{C} \to \mathbf{Set}$:

Example (Transporting along directed equality)

For any $P : \mathbb{C} \to \mathbf{Set}$:

$$\dfrac{\dfrac{}{[z : \mathbb{C}]\ P(z) \vdash P(z)}\ (\mathrm{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}]\ \ \hom(a, b), P(\overline{a}) \vdash P(b)}\ (J)$$

Computation rule for $\mathsf{subst}[f, k] := J(\mathsf{id})$:
"transporting a point of $P(a)$ along the path refl_z *is the identity"*:

$$\dfrac{}{[z : \mathbb{C}]\ k : P(z) \vdash \mathsf{subst}[\mathsf{refl}_z, k] = k : P(z)}\ (J\text{-comp})$$

Directed type theory – Groupoidal case

- When $\mathbb{A} \cong \mathbb{A}^{\mathsf{op}}$ is a groupoid, $\hom$ is the characterization of symmetric equality as left adjoint to reindexing on diagonals. (with Frobenius)

$$
\dfrac{
\dfrac{[z:\mathbb{A}, x:\mathbb{C}] \qquad \mathit{\Gamma}(\overline{z}, z, \overline{x}, x) \vdash P(\overline{z}, z, \overline{x}, x)}
{[a:\mathbb{A}^{\mathsf{op}}, b:\mathbb{A}, x:\mathbb{C}] \ \hom(a,b),\ \mathit{\Gamma}(\overline{b}, \overline{a}, \overline{x}, x) \vdash P(a, b, \overline{x}, x)}\ (\hom)
}{
\dfrac{[z:\mathbb{A}, x:\mathbb{C}] \qquad \mathit{\Gamma}(z, z, x) \vdash P(z, z, x)}
{[a:\mathbb{A}, b:\mathbb{A}, x:\mathbb{C}]\ a \cong b,\ \mathit{\Gamma}(b, a, x) \vdash P(a, b, x)}\ (\mathsf{eq})
}
$$

Directed type theory – Groupoidal case

- When $\mathbb{A} \cong \mathbb{A}^{\mathsf{op}}$ is a groupoid, $\hom$ is the characterization of symmetric equality as left adjoint to reindexing on diagonals. (with Frobenius)

$$\dfrac{\dfrac{[z:\mathbb{A}, x:\mathbb{C}] \quad \Gamma(\overline{z}, z, \overline{x}, x) \vdash P(\overline{z}, z, \overline{x}, x)}{[a:\mathbb{A}^{\mathsf{op}}, b:\mathbb{A}, x:\mathbb{C}] \ \hom(a,b),\ \Gamma(\overline{b}, \overline{a}, \overline{x}, x) \vdash P(a, b, \overline{x}, x)}\ (\hom)}{\dfrac{[z:\mathbb{A}, x:\mathbb{C}] \quad \Gamma(z, z, x) \vdash P(z, z, x)}{[a:\mathbb{A}, b:\mathbb{A}, x:\mathbb{C}]\ a \cong b,\ \Gamma(b, a, x) \vdash P(a, b, x)}\ (\mathsf{eq})}$$

- The *proof-relevant directness* of **Cat** seems to be the fundamental obstacle to a fully compositional theory of dinaturals:

Theorem (Dinaturals in groupoids compose) ( )

Given a groupoid $\mathbb{C} \cong \mathbb{C}^{\mathsf{op}}$ and any $\mathbb{D}$ for $F, G, H : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$, all dinaturals $\alpha : F \xrightarrow{\bullet\bullet} G$, $\beta : G \xrightarrow{\bullet\bullet} H$ compose.

Directed type theory – Groupoidal case

- When $\mathbb{A} \cong \mathbb{A}^{\mathsf{op}}$ is a groupoid, $\hom$ is the characterization of symmetric equality as left adjoint to reindexing on diagonals. (with Frobenius)

$$\frac{[z:\mathbb{A}, x:\mathbb{C}] \qquad \Gamma(\overline{z}, z, \overline{x}, x) \vdash P(\overline{z}, z, \overline{x}, x)}{[a:\mathbb{A}^{\mathsf{op}}, b:\mathbb{A}, x:\mathbb{C}]\ \ \hom(a,b),\ \Gamma(\overline{b}, \overline{a}, \overline{x}, x) \vdash P(a, b, \overline{x}, x)}\ (\hom)$$

$$\frac{[z:\mathbb{A}, x:\mathbb{C}] \qquad \Gamma(z, z, x) \vdash P(z, z, x)}{[a:\mathbb{A}, b:\mathbb{A}, x:\mathbb{C}]\ a \cong b,\ \Gamma(b, a, x) \vdash P(a, b, x)}\ (\mathsf{eq})$$

- The *proof-relevant directness* of **Cat** seems to be the fundamental obstacle to a fully compositional theory of dinaturals:

Theorem (Dinaturals in groupoids compose) ( )

Given a groupoid $\mathbb{C} \cong \mathbb{C}^{\mathsf{op}}$ and any $\mathbb{D}$ for $F, G, H : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$, all dinaturals $\alpha : F \xrightarrow{\bullet\bullet} G$, $\beta : G \xrightarrow{\bullet\bullet} H$ compose.

- *Can the directed rule also be characterized as an adjunction? Yes!...*

Directed equality as relative left adjoint

Symmetric equality is a left adjoint to identifying two variables together.

Directed equality as relative left adjoint

Symmetric equality is a left adjoint to identifying two variables together.

Idea: *Directed equality* is a *relative* left adjoint to identifying two *natural* variables together with a *dinatural* one.

Directed equality as relative left adjoint

Symmetric equality is a left adjoint to identifying two variables together.

Idea: *Directed equality* is a *relative* left adjoint to identifying two *natural* variables together with a *dinatural* one.

- The *relative* part imposes the syntactic restrictions of directed J: the relative functor is just reindexing along projections.

Directed equality as relative left adjoint

Symmetric equality is a left adjoint to identifying two variables together.

Idea: *Directed equality* is a *relative* left adjoint to identifying two *natural* variables together with a *dinatural* one.

- The *relative* part imposes the syntactic restrictions of directed J: the relative functor is just reindexing along projections.
- Unfortunately, for **Cat** we can only state this as a relative *para*-adjunction, because of non-compositionality of dinaturals.

Directed equality as relative left adjoint

Symmetric equality is a left adjoint to identifying two variables together.

Idea: *Directed equality* is a *relative* left adjoint to identifying two *natural* variables together with a *dinatural* one.

- The *relative* part imposes the syntactic restrictions of directed J: the relative functor is just reindexing along projections.
- Unfortunately, for **Cat** we can only state this as a relative *para*-adjunction, because of non-compositionality of dinaturals.
- *Para*- indicates that composition is partial: paracategories, parafunctors, para-adjunctions. [Hermida 2003]

Background: Relative adjunctions

Definition (Left relative adjunction [Arkor 2024])

Consider this situation of functors and categories:

$$\begin{tikzcd} & \mathbb{D} \arrow[dr, "R"] \arrow[d, phantom, "\dashv"] & \\ \mathbb{C} \arrow[ur, "L"] \arrow[rr, "J"'] & {} & \mathbb{X} \end{tikzcd}$$

We say that L *is the J-relative left adjoint to* R, written $L \dashv_J R$, if

$$\mathbb{D}(L(x), y) \cong \mathbb{X}(J(x), R(y))$$

is a bijection natural in both arguments $x : \mathbb{C}, y : \mathbb{D}$.

Directed equality as adjoint (1)

Theorem (Directed equality as relative left adjoint) ($\approx$)

- *Let* $[\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$ *the paracategory where morphisms are dinats natural in* $\mathbb{A}^{\mathsf{op}}, \mathbb{A}$ *and dinatural in* $\mathbb{C}$*, and* $[\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$ *as dinaturals.*

Directed equality as adjoint (1)

Theorem (Directed equality as relative left adjoint) ($\approx$)

- *Let* $[\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$ *the paracategory where morphisms are dinats natural in* $\mathbb{A}^{\mathsf{op}}, \mathbb{A}$ *and dinatural in* $\mathbb{C}$*, and* $[\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$ *as dinaturals.*
- *Take the parafunctor* $\pi^*_{\mathbb{A}} : [\mathbb{C}^{\diamond}, \textbf{\textit{Set}}] \to [\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$ *defined in the intuitive way by precomposing with projections.*

Directed equality as adjoint (1)

Theorem (Directed equality as relative left adjoint) ($\approx$)

- *Let* $[\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$ *the paracategory where morphisms are dinats natural in* $\mathbb{A}^{\mathsf{op}}, \mathbb{A}$ *and dinatural in* $\mathbb{C}$*, and* $[\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$ *as dinaturals.*
- *Take the parafunctor* $\pi_{\mathbb{A}}^{*} : [\mathbb{C}^{\diamond}, \textbf{\textit{Set}}] \to [\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$ *defined in the intuitive way by precomposing with projections.*
- *There is a dipresheaf* $\hom_{\mathbb{A}} \in [\mathbb{A}^{\mathsf{op}} \times \mathbb{A}, \textbf{\textit{Set}}]$ *such that the functor*

$$\hom_{\mathbb{A}} \times - : [\mathbb{C}^{\diamond}, \textbf{\textit{Set}}] \to [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$$

$$\hom_{\mathbb{A}} \times \Gamma := (\overline{a}, a, \overline{x}, x) \mapsto \hom(\overline{a}, a) \times \Gamma(\overline{x}, x),$$

$$(\hom_{\mathbb{A}} \times \alpha_x)_{abc} := \lambda(e \in \hom(a, b), k \in \Gamma(c, c)).(e, \alpha_c(k))$$

Directed equality as adjoint (1)

Theorem (Directed equality as relative left adjoint) ($\approx$)

- *Let* $[\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$ *the paracategory where morphisms are dinats natural in* $\mathbb{A}^{\mathsf{op}}, \mathbb{A}$ *and dinatural in* $\mathbb{C}$*, and* $[\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$ *as dinaturals.*
- *Take the parafunctor* $\pi^*_{\mathbb{A}} : [\mathbb{C}^{\diamond}, \textbf{\textit{Set}}] \to [\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$ *defined in the intuitive way by precomposing with projections.*
- *There is a dipresheaf* $\hom_{\mathbb{A}} \in [\mathbb{A}^{\mathsf{op}} \times \mathbb{A}, \textbf{\textit{Set}}]$ *such that the functor*

$$\hom_{\mathbb{A}} \times - : [\mathbb{C}^{\diamond}, \textbf{\textit{Set}}] \to [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$$

$$\hom_{\mathbb{A}} \times \Gamma := (\overline{a}, a, \overline{x}, x) \mapsto \hom(\overline{a}, a) \times \Gamma(\overline{x}, x),$$

$$(\hom_{\mathbb{A}} \times \alpha_x)_{abc} := \lambda(e \in \hom(a, b), k \in \Gamma(c, c)).(e, \alpha_c(k))$$

determines a $\pi^*_{\mathbb{A}}$*-relative left adjoint to the functor*

$$\Delta_{\mathbb{A}} \times - : [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}] \to [\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]$$

$$\Delta_{\mathbb{A}} \times P := P$$

$$(\Delta_{\mathbb{A}} \times \alpha_{abc})_{zx} := \alpha_{zzx}.$$

Directed equality as adjoint (2)

Theorem (Directed equality as relative left adjoint, cont.) (≈)

- *Thus the left relative adjointness situation*

$$\hom_{\mathbb{A}} \times - \dashv_{\pi^*_{\mathbb{A}}} \Delta_{\mathbb{A}} \times -$$

is as follows:

$$\begin{tikzcd}
 & {[\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]} \arrow[dr, "\Delta_{\mathbb{A}} \times -"] \arrow[d, phantom, "\dashv"] & \\
{[\mathbb{C}^{\diamond}, \textbf{\textit{Set}}]} \arrow[ur, "\hom_{\mathbb{A}} \times -"] \arrow[rr, "\pi^*_{\mathbb{A}}"'] & & {[\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]}
\end{tikzcd}$$

Directed equality as adjoint (2)

Theorem (Directed equality as relative left adjoint, cont.) (✊≈)

- *Thus the left relative adjointness situation*

$$\hom_{\mathbb{A}} \times - \dashv_{\pi^*_{\mathbb{A}}} \Delta_{\mathbb{A}} \times -$$

is as follows:

$$\begin{tikzcd}
 & {[\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]} \arrow[dr, "\Delta_{\mathbb{A}} \times -"] \arrow[d, phantom, "\dashv"] & \\
{[\mathbb{C}^{\diamond}, \textbf{\textit{Set}}]} \arrow[ur, "\hom_{\mathbb{A}} \times -"] \arrow[rr, "\pi^*_{\mathbb{A}}"'] & {} & {[\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]}
\end{tikzcd}$$

Proof. *The required isomorphism is the following:*

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \quad \Gamma(\overline{x}, x) \vdash P(\overline{z}, z, \overline{x}, x)}{\overline{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}] \quad \hom_{\mathbb{A}}(a, b) \times \Gamma(\overline{x}, x) \vdash P(a, b, \overline{x}, x)}} \; (\hom\text{-rel-adj})$$

which is an instance of directed J, where Γ is mute in $\overline{a} : \mathbb{A}, \overline{b} : \mathbb{A}^{\mathsf{op}}$.

Directed equality as adjoint (2)

Theorem (Directed equality as relative left adjoint, cont.) (≈)

- *Thus the left relative adjointness situation*

$$\hom_{\mathbb{A}} \times - \dashv_{\pi^*_{\mathbb{A}}} \Delta_{\mathbb{A}} \times -$$

is as follows:

$$\begin{tikzcd}
 & {[\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]} \arrow[dr, "\Delta_{\mathbb{A}} \times -"] \arrow[d, phantom, "\dashv"] & \\
{[\mathbb{C}^{\diamond}, \textbf{\textit{Set}}]} \arrow[ur, "\hom_{\mathbb{A}} \times -"] \arrow[rr, "\pi^*_{\mathbb{A}}"'] & {} & {[\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \textbf{\textit{Set}}]}
\end{tikzcd}$$

Proof. *The required isomorphism is the following:*

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \qquad \Gamma(\overline{x}, x) \vdash P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}] \;\; \hom_{\mathbb{A}}(a, b) \times \Gamma(\overline{x}, x) \vdash P(a, b, \overline{x}, x)} \; (\text{hom-rel-adj})$$

which is an instance of directed J, where Γ is mute in $\overline{a} : \mathbb{A}, \overline{b} : \mathbb{A}^{\mathsf{op}}$.
$(\text{hom-rel-adj}) \Leftrightarrow (\text{hom})$*: pick* $P := \Gamma^{\mathsf{op}}(b, a, x, \overline{x}) \Rightarrow P(a, b, \overline{x}, x)$, *use* (exp).

(Co)end calculus via dinaturality

- *What are the quantifiers of directed type theory?*
- Dinaturality allows us to view (co)ends as "adjoints" to weakening:

Theorem (Ends and coends as quantifiers) (👏)

Take $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \textbf{\textit{Set}}$, *and the functor precomposing with projections*

$$\pi^*_{\mathbb{A}}(P) : \mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond} \to \textbf{\textit{Set}}$$
$$\pi^*_{\mathbb{A}}(P) := (\overline{a}, a, \overline{x}, x) \mapsto P(\overline{x}, x),$$

(Co)end calculus via dinaturality

- *What are the quantifiers of directed type theory?*
- Dinaturality allows us to view (co)ends as "adjoints" to weakening:

Theorem (Ends and coends as quantifiers) ( )

Take $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \textbf{\textit{Set}}$, *and the functor precomposing with projections*

$$\pi^*_{\mathbb{A}}(P) : \mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond} \to \textbf{\textit{Set}}$$

$$\pi^*_{\mathbb{A}}(P) := (\overline{a}, a, \overline{x}, x) \mapsto P(\overline{x}, x),$$

There are isos of sets of dinats, natural in $P : \mathbb{C}^{\diamond} \to \textbf{\textit{Set}}, Q : \mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond} \to \textbf{\textit{Set}}$:

$$\frac{[a : \mathbb{A}, x : \mathbb{C}]\ P(\overline{x}, x) \vdash Q(\overline{a}, a, \overline{x}, x)}{\overline{\overline{[x : \mathbb{C}]\ P(\overline{x}, x) \vdash \int_{a:\mathbb{A}} Q(\overline{a}, a, \overline{x}, x)}}\ \text{(end)}$$

$$\frac{[x : \mathbb{C}]\ \int^{a:\mathbb{A}} Q(\overline{a}, a, \overline{x}, x) \vdash P(\overline{x}, x)}{\overline{\overline{[a : \mathbb{A}, x : \mathbb{C}]\qquad Q(\overline{a}, a, \overline{x}, x) \vdash P(\overline{x}, x)}}\ \text{(coend)}$$

(Co)ends as quantifiers

Theorem (Beck-Chevalley and Frobenius condition for (co)ends)

(Co)ends satisfy a Beck-Chevalley condition: for $F : \mathbb{C}^\diamond \to \mathbb{D}$ *there is a strict isomorphism in the (large) functor category* $[[\mathbb{A}^\diamond \times \mathbb{D}^\diamond, \textbf{Set}],[\mathbb{D}^\diamond, \textbf{Set}]]$

$$\int_{\mathbb{A}[\mathbb{D}]} ; F^* \cong (\mathsf{id}_{\mathbb{A}^\diamond} \times F)^* ; \int_{\mathbb{A}[\mathbb{C}]}$$

where $\int_{\mathbb{A}[\mathbb{C}]}, \int^{\mathbb{A}[\mathbb{C}]} : [\mathbb{A}^\diamond \times \mathbb{C}^\diamond, \textbf{Set}] \to [\mathbb{C}^\diamond, \textbf{Set}]$ *are parametric (co)ends, and* $F^* : [\mathbb{D}^\diamond, \textbf{Set}] \to [\mathbb{C}^\diamond, \textbf{Set}]$ *is precomposition with* $F^\diamond$.

(Co)ends as quantifiers

Theorem (Beck-Chevalley and Frobenius condition for (co)ends)

(Co)ends satisfy a Beck-Chevalley condition: for $F : \mathbb{C}^\diamond \to \mathbb{D}$ there is a strict isomorphism in the (large) functor category $[[\mathbb{A}^\diamond \times \mathbb{D}^\diamond, \textbf{\textit{Set}}],[\mathbb{D}^\diamond, \textbf{\textit{Set}}]]$

$$\int_{\mathbb{A}[\mathbb{D}]} ; F^* \cong (\mathsf{id}_{\mathbb{A}^\diamond} \times F)^* ; \int_{\mathbb{A}[\mathbb{C}]}$$

where $\int_{\mathbb{A}[\mathbb{C}]}, \int^{\mathbb{A}[\mathbb{C}]} : [\mathbb{A}^\diamond \times \mathbb{C}^\diamond, \textbf{\textit{Set}}] \to [\mathbb{C}^\diamond, \textbf{\textit{Set}}]$ *are parametric (co)ends, and* $F^* : [\mathbb{D}^\diamond, \textbf{\textit{Set}}] \to [\mathbb{C}^\diamond, \textbf{\textit{Set}}]$ *is precomposition with* $F^\diamond$*.*
Moreover, a Frobenius condition for coends holds: there is an isomorphism natural in $\Gamma : \mathbb{A}^\diamond \times \mathbb{C}^\diamond \to \textbf{\textit{Set}}, P : \mathbb{C}^\diamond \to \textbf{\textit{Set}}$*,*

$$\int^{\mathbb{A}[\mathbb{C}]} (\pi^*_{\mathbb{A}[\mathbb{C}]}(P) \wedge \Gamma) \cong \pi^*_{\mathbb{A}[\mathbb{C}]}(P) \wedge \int^{\mathbb{A}[\mathbb{C}]} (\Gamma),$$

where $- \wedge - : [\mathbb{C}, \textbf{\textit{Set}}] \times [\mathbb{C}, \textbf{\textit{Set}}] \to [\mathbb{C}, \textbf{\textit{Set}}]$ *is the pointwise product.*
Proof. *Beck-Chevalley is easy. For Frobenius, we can use logical rules to mirror the argument in [Jacobs 1999, 1.9.12(i)] with exponentials.*

(Co)end calculus using our rules

- Using dinaturals as semantics, we can prove useful theorems “logically”.

Rules for (co)ends as quantifiers + directed equality for proofs of:

- (Co)Yoneda,
- Adjointess of Kan extensions via (co)ends,
- Presheaves are closed under exponentials,
- Associativity of composition of profunctors,
- Right lifts in profunctors,
- (Co)ends preserve limits,
- Adjointness of (co)ends in natural transformations,
- Characterization of dinaturals as certain ends,
- Frobenius property of (co)ends using exponentials.

(Co)end calculus with dinaturality (1)

Yoneda lemma: $(P, \Gamma : \mathbb{C} \to \mathbf{Set})$

$$\dfrac{\dfrac{\dfrac{[a : \mathbb{C}]\ \Gamma(a) \vdash \int_{x:\mathbb{C}} \hom_{\mathbb{C}}^{\mathsf{op}}(a, \overline{x}) \Rightarrow P(x)}{[a : \mathbb{C}, x : \mathbb{C}]\ \Gamma(a) \vdash \hom_{\mathbb{C}}^{\mathsf{op}}(a, \overline{x}) \Rightarrow P(x)}\ \text{(end)}}{[a : \mathbb{C}]\ \ \hom_{\mathbb{C}}(\overline{a}, x) \times \Gamma(a) \vdash P(x)}\ \text{(exp)}}{[z : \mathbb{C}]\ \Gamma(z) \vdash P(z)}\ (\mathrm{hom})$$

(Co)end calculus with dinaturality (1)

Yoneda lemma: $(P, \Gamma : \mathbb{C} \to \mathbf{Set})$

$$
\begin{array}{c}
[a : \mathbb{C}]\ \Gamma(a) \vdash \int_{x:\mathbb{C}} \hom^{\mathsf{op}}_{\mathbb{C}}(a, \overline{x}) \Rightarrow P(x) \\
\hline\hline
[a : \mathbb{C}, x : \mathbb{C}]\ \Gamma(a) \vdash \hom^{\mathsf{op}}_{\mathbb{C}}(a, \overline{x}) \Rightarrow P(x) \\
\hline\hline
[a : \mathbb{C}]\ \ \hom_{\mathbb{C}}(\overline{a}, x) \times \Gamma(a) \vdash P(x) \\
\hline\hline
[z : \mathbb{C}]\ \Gamma(z) \vdash P(z)
\end{array}
\begin{array}{l}
\\ \text{(end)} \\ \\ \text{(exp)} \\ \\ \text{(hom)} \\
\end{array}
$$

CoYoneda lemma:

$$
\begin{array}{c}
[a : \mathbb{C}]\ \ \int^{x:\mathbb{C}} \hom_{\mathbb{C}}(\overline{x}, a) \times P(x) \vdash \Gamma(a) \\
\hline\hline
[a : \mathbb{C}, x : \mathbb{C}]\ \ \hom_{\mathbb{C}}(\overline{a}, x) \times P(a) \vdash \Gamma(x) \\
\hline\hline
[z : \mathbb{C}]\ P(z) \vdash \Gamma(z)
\end{array}
\begin{array}{l}
\\ \text{(coend)} \\ \\ \text{(hom)} \\
\end{array}
$$

(Co)end calculus with dinaturality (2)

Presheaves are cartesian closed: $(\mathit{\Gamma}, A, B : \mathbb{C} \to \textbf{Set})$

$$
\begin{aligned}
[x : \mathbb{C}]\ \mathit{\Gamma}(x) \vdash (A \Rightarrow B)(x) &:= \mathsf{Nat}(\hom_{\mathbb{C}}(x, -) \times A, B) \\
&\cong \int_{y:\mathbb{C}} \hom_{\mathbb{C}}^{\mathsf{op}}(x, \overline{y}) \times A^{\mathsf{op}}(\overline{y}) \Rightarrow B(y)
\end{aligned}
$$

$$
\begin{array}{c}
\underline{\underline{\qquad [x : \mathbb{C}]\ \mathit{\Gamma}(x) \vdash \int_{y:\mathbb{C}} \hom_{\mathbb{C}}^{\mathsf{op}}(x, \overline{y}) \times A^{\mathsf{op}}(\overline{y}) \Rightarrow B(y) \qquad}}\ \text{(end)} \\
\underline{\underline{\qquad [x : \mathbb{C}, y : \mathbb{C}]\ \mathit{\Gamma}(x) \vdash \hom_{\mathbb{C}}^{\mathsf{op}}(x, \overline{y}) \times A^{\mathsf{op}}(\overline{y}) \Rightarrow B(y) \qquad}}\ \text{(exp)} \\
\underline{\underline{\qquad [x : \mathbb{C}, y : \mathbb{C}]\ A(y) \times \hom_{\mathbb{C}}(\overline{x}, y) \times \mathit{\Gamma}(x) \vdash B(y) \qquad}}\ \text{(coend+frob.)} \\
\underline{\underline{\qquad [y : \mathbb{C}]\ A(y) \times \left(\int^{x:\mathbb{C}} \hom_{\mathbb{C}}(\overline{x}, y) \times \mathit{\Gamma}(x) \right) \vdash B(y) \qquad}}\ \text{(coYoneda)} \\
[y : \mathbb{C}]\ A(y) \times \mathit{\Gamma}(y) \vdash B(y)
\end{array}
$$

(Co)end calculus with dinaturality (3)

Right Kan extensions via ends are right adjoints to precomposition with $F : \mathbb{C} \to \mathbb{D}$ ($P : \mathbb{C} \to \textbf{Set}, \mathit{\Gamma} : \mathbb{D} \to \textbf{Set}$):

$$
\begin{array}{cl}
[y : \mathbb{D}]\ \mathit{\Gamma}(y) \vdash (\mathsf{Ran}_F P)(y) := \displaystyle\int_{x:\mathbb{C}} \hom^{\mathsf{op}}_{\mathbb{D}}(y, F^{\mathsf{op}}(\overline{x})) \Rightarrow P(x) & \\
\hline\hline
[x : \mathbb{C}, y : \mathbb{D}]\ \mathit{\Gamma}(y) \vdash \hom^{\mathsf{op}}_{\mathbb{D}}(y, F^{\mathsf{op}}(\overline{x})) \Rightarrow P(x) & \text{(end)} \\
\hline\hline
[x : \mathbb{C}, y : \mathbb{D}]\ \hom_{\mathbb{D}}(\overline{y}, F(x)) \times \mathit{\Gamma}(y) \vdash P(x) & \text{(exp)} \\
\hline\hline
[x : \mathbb{C}]\ \displaystyle\int^{y:\mathbb{D}} \hom_{\mathbb{D}}(\overline{y}, F(x)) \times \mathit{\Gamma}(y) \vdash P(x) & \text{(coend)} \\
\hline\hline
[x : \mathbb{C}]\ \mathit{\Gamma}(F(x)) \vdash P(x) & \text{(coYoneda)}
\end{array}
$$

(Co)end calculus with dinaturality (4)

Composition (on both sides) in **Prof** has a right adjoint (right lifts).
$(P : \mathbb{C}^{\mathsf{op}} \times \mathbb{A} \to \mathbf{Set}, Q : \mathbb{A}^{\mathsf{op}} \times \mathbb{D} \to \mathbf{Set},\ \mathit{\Gamma} : \mathbb{C}^{\mathsf{op}} \times \mathbb{D} \to \mathbf{Set})$:

$$\dfrac{\dfrac{\dfrac{\dfrac{[x : \mathbb{C}^{\mathsf{op}}, z : \mathbb{D}] \quad (P \mathbin{;} -)(Q)(x,z) := \displaystyle\int^{y:\mathbb{A}} P(x,y) \times Q(\overline{y}, z) \vdash \mathit{\Gamma}(x,z)}{[x : \mathbb{C}^{\mathsf{op}}, y : \mathbb{A}, z : \mathbb{D}]\ P(x,y) \times Q(\overline{y}, z) \vdash \mathit{\Gamma}(x,z)}\ \text{(coend)}}{[x : \mathbb{C}^{\mathsf{op}}, y : \mathbb{A}, z : \mathbb{D}]\ Q(\overline{y}, z) \vdash P^{\mathsf{op}}(\overline{x}, \overline{y}) \Rightarrow \mathit{\Gamma}(x,z)}\ \text{(exp)}}{[y : \mathbb{A}, z : \mathbb{D}]\ Q(\overline{y}, z) \vdash \displaystyle\int_{x:\mathbb{C}} P^{\mathsf{op}}(\overline{x}, \overline{y}) \Rightarrow \mathit{\Gamma}(x,z)}\ \text{(end)}}{[y : \mathbb{A}^{\mathsf{op}}, z : \mathbb{D}]\ Q(y, z) \vdash \displaystyle\int_{x:\mathbb{C}} P^{\mathsf{op}}(\overline{x}, y) \Rightarrow \mathit{\Gamma}(x,z)}\ (\cong)$$

$$:= \mathsf{Rift}_P(\mathit{\Gamma})(y,z)$$

(Co)end calculus with dinaturality (5)

Fubini for ends (Γ : **Set**, $P : (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \times (\mathbb{D}^{\mathsf{op}} \times \mathbb{D}) \to$ **Set**)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\,]\ \Gamma \vdash \int_{x:\mathbb{C}} \int_{y:\mathbb{D}} P(\overline{x}, x, \overline{y}, y)}{[x : \mathbb{C}]\ \Gamma \vdash \int_{y:\mathbb{D}} P(\overline{x}, x, \overline{y}, y)}\ \text{(end)}}{[x : \mathbb{C}, y : \mathbb{D}]\ \Gamma \vdash P(\overline{x}, x, \overline{y}, y)}\ \text{(end)}}{[y : \mathbb{D}, x : \mathbb{C}]\ \Gamma \vdash P(\overline{x}, x, \overline{y}, y)}\ \text{(structural property)}}{[y : \mathbb{D}]\ \Gamma \vdash \int_{x:\mathbb{C}} P(\overline{x}, x, \overline{y}, y)}\ \text{(end)}}{[\,]\ \Gamma \vdash \int_{y:\mathbb{D}} \int_{x:\mathbb{C}} P(\overline{x}, x, \overline{y}, y)}\ \text{(end)}
$$

(Co)end calculus with dinaturality (6)

Composition of profunctors is associative: ($\Gamma : \mathbb{A}^{\mathsf{op}} \times \mathbb{D} \to \mathbf{Set}$, $P : \mathbb{A}^{\mathsf{op}} \times \mathbb{B} \to \mathbf{Set}, Q : \mathbb{B}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}, R : \mathbb{C}^{\mathsf{op}} \times \mathbb{D} \to \mathbf{Set}$)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[a : \mathbb{A}, d : \mathbb{D}] \ \int^{b:\mathbb{B}} P(\overline{a}, b) \times \left(\int^{c:\mathbb{C}} Q(\overline{b}, c) \times R(\overline{c}, d) \right) \vdash \Gamma(\overline{a}, d)}{[a : \mathbb{A}, b : \mathbb{B}, d : \mathbb{D}] \ P(\overline{a}, b) \times \left(\int^{c:\mathbb{C}} Q(\overline{b}, c) \times R(\overline{c}, d) \right) \vdash \Gamma(\overline{a}, d)} \text{(coend)}}{[a : \mathbb{A}, b : \mathbb{B}, c : \mathbb{C}, d : \mathbb{D}] \ P(\overline{a}, b) \times (Q(\overline{b}, c) \times R(\overline{c}, d)) \vdash \Gamma(\overline{a}, d)} \text{(coend)}}{[a : \mathbb{A}, b : \mathbb{B}, c : \mathbb{C}, d : \mathbb{D}] \ (P(\overline{a}, b) \times Q(\overline{b}, c)) \times R(\overline{c}, d) \vdash \Gamma(\overline{a}, d)} \text{(struct. prop.)}}{[a : \mathbb{A}, c : \mathbb{C}, d : \mathbb{D}] \ \left(\int^{b:\mathbb{B}} P(\overline{a}, b) \times Q(\overline{b}, c) \right) \times R(\overline{c}, d) \vdash \Gamma(\overline{a}, d)} \text{(coend)}}{[a : \mathbb{A}, d : \mathbb{D}] \ \int^{c:\mathbb{C}} \left(\int^{b:\mathbb{B}} P(\overline{a}, b) \times Q(\overline{b}, c) \right) \times R(\overline{c}, d) \vdash \Gamma(\overline{a}, d)} \text{(coend)}
$$

Conclusion and future work

We have seen how dinaturality allows us to give a semantic interpretation towards a directed type theory in ***Cat*** *with quantifiers, where directed equality is given by* $\hom$*-functors and quantifiers by (co)ends.*

Conclusion and future work

*We have seen how dinaturality allows us to give a semantic interpretation towards a directed type theory in **Cat** with quantifiers, where directed equality is given by* $\hom$*-functors and quantifiers by (co)ends.*

Future work:

1. Big piece missing from the story: compositionality of dinaturals.

Conclusion and future work

*We have seen how dinaturality allows us to give a semantic interpretation towards a directed type theory in **Cat** with quantifiers, where directed equality is given by* $\hom$*-functors and quantifiers by (co)ends.*

Future work:

1. Big piece missing from the story: compositionality of dinaturals.
   - *Claim:* non-compositionality is intrinsic to **Cat**, like failure of UIP.

Conclusion and future work

*We have seen how dinaturality allows us to give a semantic interpretation towards a directed type theory in **Cat** with quantifiers, where directed equality is given by* $\hom$*-functors and quantifiers by (co)ends.*

Future work:

1. Big piece missing from the story: compositionality of dinaturals.
   - *Claim:* non-compositionality is intrinsic to **Cat**, like failure of UIP.
   - Find suitable structures axiomatizing composition of dinaturality (e.g., operads/multicategories but with explicit variances of variables.).

Conclusion and future work

We have seen how dinaturality allows us to give a semantic interpretation towards a directed type theory in ***Cat*** *with quantifiers, where directed equality is given by* $\hom$*-functors and quantifiers by (co)ends.*

Future work:

1. Big piece missing from the story: compositionality of dinaturals.
   - *Claim:* non-compositionality is intrinsic to **Cat**, like failure of UIP.
   - Find suitable structures axiomatizing composition of dinaturality (e.g., operads/multicategories but with explicit variances of variables.).
2. Everything shown so far can be done in **Pos**, where all dinats compose.

Conclusion and future work

*We have seen how dinaturality allows us to give a semantic interpretation towards a directed type theory in **Cat** with quantifiers, where directed equality is given by* $\hom$*-functors and quantifiers by (co)ends.*

Future work:

1. Big piece missing from the story: compositionality of dinaturals.
   - *Claim:* non-compositionality is intrinsic to **Cat**, like failure of UIP.
   - Find suitable structures axiomatizing composition of dinaturality (e.g., operads/multicategories but with explicit variances of variables.).
2. Everything shown so far can be done in **Pos**, where all dinats compose.
   - This can be used to characterize *the directed logic of posets*.

Conclusion and future work

*We have seen how dinaturality allows us to give a semantic interpretation towards a directed type theory in **Cat** with quantifiers, where directed equality is given by* $\hom$*-functors and quantifiers by (co)ends.*

Future work:

1. Big piece missing from the story: compositionality of dinaturals.
   - *Claim:* non-compositionality is intrinsic to **Cat**, like failure of UIP.
   - Find suitable structures axiomatizing composition of dinaturality (e.g., operads/multicategories but with explicit variances of variables.).
2. Everything shown so far can be done in **Pos**, where all dinats compose.
   - This can be used to characterize *the directed logic of posets*.
   - The left relative adjunction characterizes directed equality in **Pos**.
   - Axiomatize this logic of directed equality using *directed doctrines*.

Conclusion and future work

*We have seen how dinaturality allows us to give a semantic interpretation towards a directed type theory in **Cat** with quantifiers, where directed equality is given by* hom*-functors and quantifiers by (co)ends.*

Future work:

1. Big piece missing from the story: compositionality of dinaturals.
   - *Claim:* non-compositionality is intrinsic to **Cat**, like failure of UIP.
   - Find suitable structures axiomatizing composition of dinaturality (e.g., operads/multicategories but with explicit variances of variables.).
2. Everything shown so far can be done in **Pos**, where all dinats compose.
   - This can be used to characterize *the directed logic of posets*.
   - The left relative adjunction characterizes directed equality in **Pos**.
   - Axiomatize this logic of directed equality using *directed doctrines*.
3. Long-term future: now that types are categories,
   - Internalize semantics of type theory inside type theory (e.g., dQIIT).
   - Naturality for free, since it can be proven internally.

The $\int$.

Paper: *"Directed equality with dinaturality"* (`arXiv:2409.10237`)
Agda code: `github.com/iwilare/dinaturality`

Thank you for the attention!

Directed doctrine (WIP)

- Take a category $\mathbb{C}$ with binary products and terminal.
- The category $\mathbb{C}^{N\Delta P}$ is defined as follows:
  - Objects: $\mathbb{C}_0 \times \mathbb{C}_0 \times \mathbb{C}_0$
  - Morphisms $(N, \Delta, P) \to (N', \Delta', P')$: triples (n, p, d) where
    - $n : N \times \Delta \to N'$
    - $p : P \times \Delta \to P'$
    - $d : \Delta \to \Delta'$
  - Identities: $(\pi_1, \mathsf{id}_\Delta, \pi_1)$
- **Definition**: a *directed doctrine* is a pseudofunctor $\mathcal{P} : \mathbb{C}^{N\Delta P} \to \mathbf{Pos}$.
- $\mathcal{P}$ has **products** when each fiber has products.
- $\mathcal{P}$ has **exponentials** when certain left/right-relative adjunctions hold.
- $\mathcal{P}$ has **directed equality** when a certain left relative adjunction holds.
- $\mathcal{P}$ has **involutions** when $\mathbb{C}$ has a functor $-^{\mathsf{op}} : \mathbb{C} \to \mathbb{C}$ s.t. $\mathsf{op} \mathbin{;} \mathsf{op} = \mathsf{id}_{\mathbb{C}}$ and an iso $\mathcal{P}(P, \Delta, N) \cong \mathcal{P}(N^{\mathsf{op}}, \Delta^{\mathsf{op}}, N^{\mathsf{op}})$ natural in P, Δ, N.

Directed doctrines - relative adj. for exponentials (WIP)

- Downgrade to dinaturality:

$$\frac{[a :^{+} \mathbb{C}, \mathbb{x} : \Gamma] \;\; A(a, \mathbb{x}) \times \Gamma(a, \mathbb{x}) \vdash B(a, \mathbb{x})}{[a :^{\Delta} \mathbb{C}, \mathbb{x} : \Gamma] \uparrow_a^{\Delta} \Gamma(a, \mathbb{x}) \vdash A^{\mathsf{op}}(\overline{a}, \mathbb{x}) \Rightarrow \uparrow_a^{\Delta} B(a, \mathbb{x})}$$

$$\frac{[a :^{-} \mathbb{C}, \mathbb{x} : \Gamma] \;\; A(\overline{a}, \mathbb{x}) \times \Gamma(\overline{a}, \mathbb{x}) \vdash B(\overline{a}, \mathbb{x})}{[a :^{\Delta} \mathbb{C}, \mathbb{x} : \Gamma] \uparrow_a^{\Delta} \Gamma(\overline{a}, \mathbb{x}) \vdash A^{\mathsf{op}}(a, \mathbb{x}) \Rightarrow \uparrow_a^{\Delta} B(\overline{a}, \mathbb{x})}$$

- Dinatural to positive:

$$\frac{[a :^{\Delta} \mathbb{C}, \mathbb{x} : \Gamma] \;\; A(\overline{a}, \mathbb{x}) \times \uparrow_a^{\Delta} \Gamma(a, \mathbb{x}) \vdash \uparrow_a^{\Delta} B(a, \mathbb{x})}{[a :^{+} \mathbb{C}, \mathbb{x} : \Gamma] \;\; \Gamma(a, \mathbb{x}) \vdash A^{\mathsf{op}}(a, \mathbb{x}) \Rightarrow B(a, \mathbb{x})}$$

- Inversion:

$$\frac{[a :^{+} \mathbb{C}, \mathbb{x} : \Gamma] \;\; A(a, \mathbb{x}) \times \Gamma(\mathbb{x}) \vdash B(\mathbb{x})}{[a :^{-} \mathbb{C}, \mathbb{x} : \Gamma] \;\; \pi_a^{-} \Gamma(\mathbb{x}) \vdash A^{\mathsf{op}}(\overline{a}, \mathbb{x}) \Rightarrow \pi_a^{-} B(\mathbb{x})}$$

$$\frac{[a :^{-} \mathbb{C}, \mathbb{x} : \Gamma] \;\; A(\overline{a}, \mathbb{x}) \times \Gamma(\mathbb{x}) \vdash B(\mathbb{x})}{[a :^{+} \mathbb{C}, \mathbb{x} : \Gamma] \;\; \pi_a^{+} \Gamma(\mathbb{x}) \vdash A^{\mathsf{op}}(a, \mathbb{x}) \Rightarrow \pi_a^{+} B(\mathbb{x})}$$

- Dinatural to negative:

$$\frac{[a :^{\Delta} \mathbb{C}, \mathbb{x} : \Gamma] \;\; A(a, \mathbb{x}) \times \uparrow_a^{\Delta} \Gamma(\overline{a}, \mathbb{x}) \vdash \uparrow_a^{\Delta} B(\overline{a}, \mathbb{x})}{[a :^{-} \mathbb{C}, \mathbb{x} : \Gamma] \;\; \Gamma(\overline{a}, \mathbb{x}) \vdash A^{\mathsf{op}}(\overline{a}, \mathbb{x}) \Rightarrow B(\overline{a}, \mathbb{x})}$$

Homotopical interpretation of dinaturality

We have maps both ways between these entailments:

$$\frac{[\,] \qquad \top \vdash P}{[x : C]\ x = x \vdash P}$$

but in MLTT they are not isomorphic.

In the directed case, we do not even have both maps!

$$\frac{[\,] \qquad\qquad \top \vdash P}{[x : \mathbb{C}]\ \ \hom(\overline{x}, x) \vdash P}$$

We only have a map from top to bottom.

Related works

- **[Caccamo-Winskel 2001]**: axiomatic system to manipulate (co)ends; quantifier exchange is postulated, no type theory presented.
- **[Licata 2011]**: a model in **Cat** with directed equality defined judgementally and not propositionally.
- **[Nuyts 2015]**: a preliminary system of contexts variances is given, with no formal syntax nor models.
- **[North 2018]**: dependent type theory, but uses groupoid structure to type the refl rule. We use dinaturality precisely to avoid this problem.
- **[Riehl-Shulman 2017]**: a synthetic theory of $(\infty, 1)$-categories with a directed interval type, no model in **Cat**.
- **[New-Licata 2022]**: a DTT with models in virtual equipments, with directed equality and quantifiers, but very different syntax w.r.t. MLTT
- **[Ahrens 2023]**: judgemental structure for directed equality reminiscent of a bicategorical model.
- **[Neumann 2024]**: groupoids are used in contexts rather than in the type of the conclusion.