# Specification and Verification of a Linear-Time Temporal Logic for Graph Transformation

Andrea Laretto[1], Fabio Gadducci[2], Davide Trotta[2]

1: Tallinn University of Technology, 2: University of Pisa

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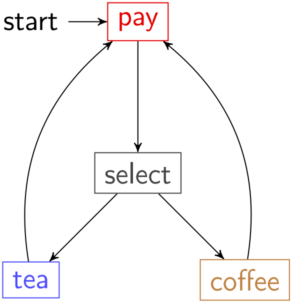


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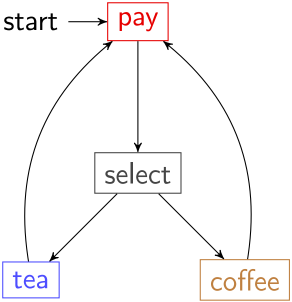


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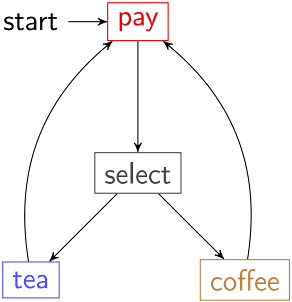


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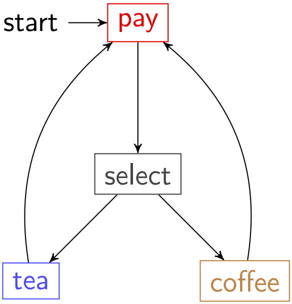


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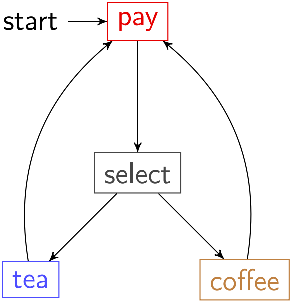


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3. Use a program to *check* that the *model* **satisfies** the formula

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- *Yes!* Using **counterpart models** and quantified temporal logics

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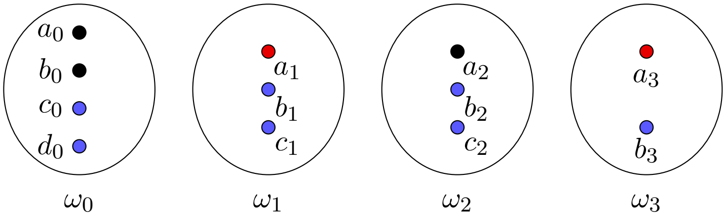

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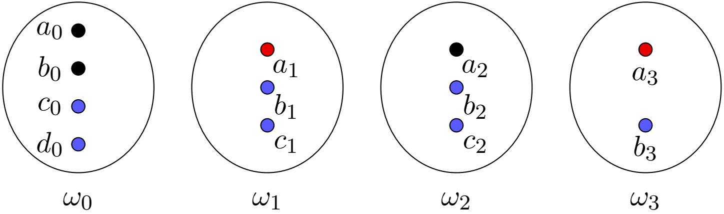


*How do we represent transitions?*

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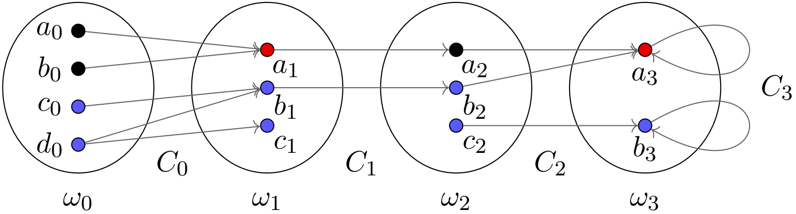

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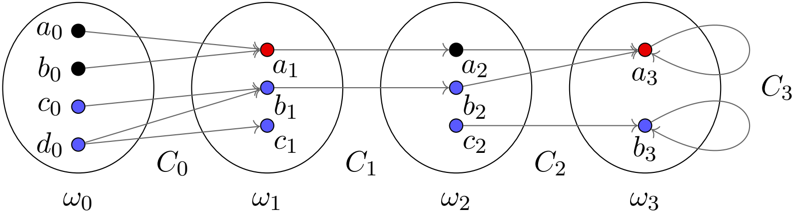


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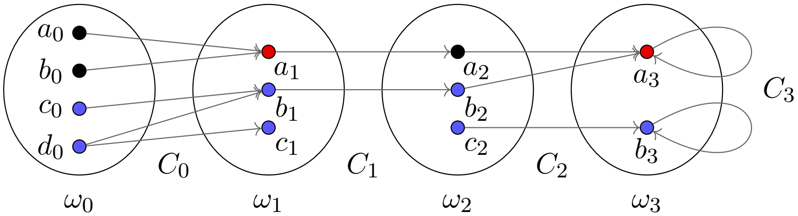


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- We call these sequences of worlds and relations **counterpart traces**

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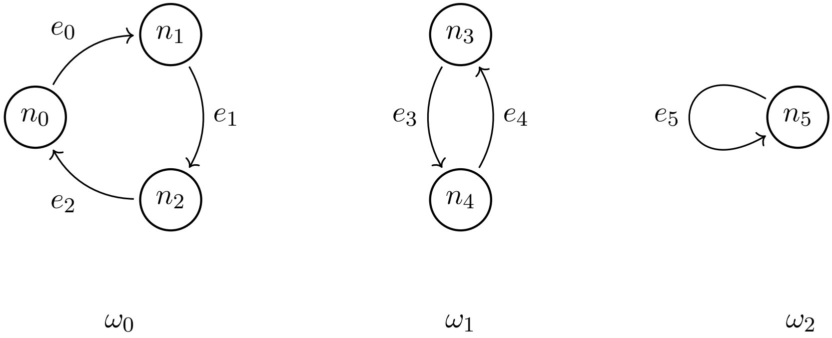

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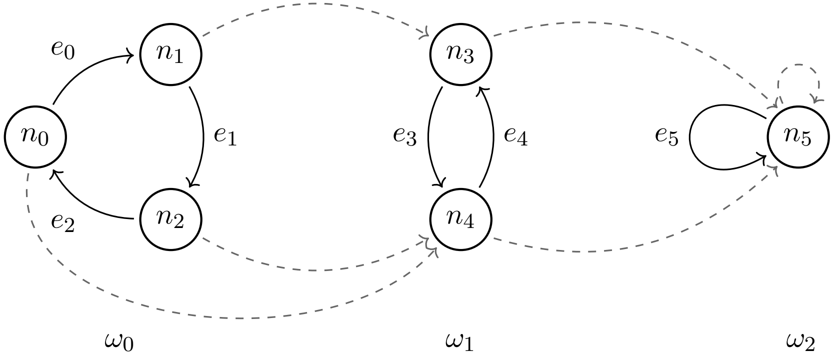

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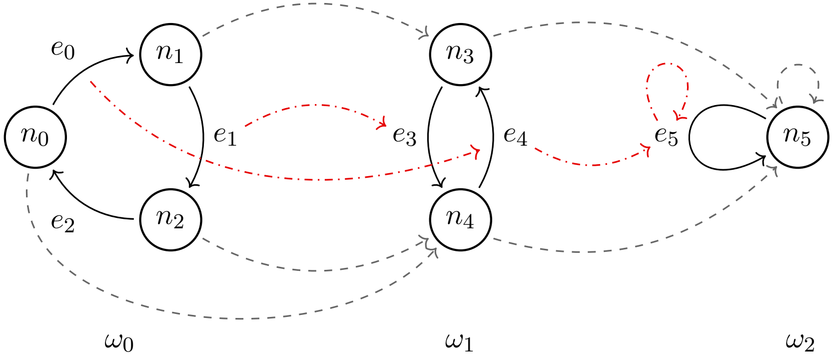

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# Example – Counterpart model

$$
\begin{tikzcd}
\omega_0 \arrow[loop left, "\mathsf{id}_{\omega_0}"] \arrow[r, "f"] & \omega_1 \arrow[r, "g"] \arrow[rr, bend left=50, "g\,;h"] & \omega_2 \arrow[r, bend left, "h'"] \arrow[r, bend right, "h"'] & \omega_3 \arrow[loop right, "k"]
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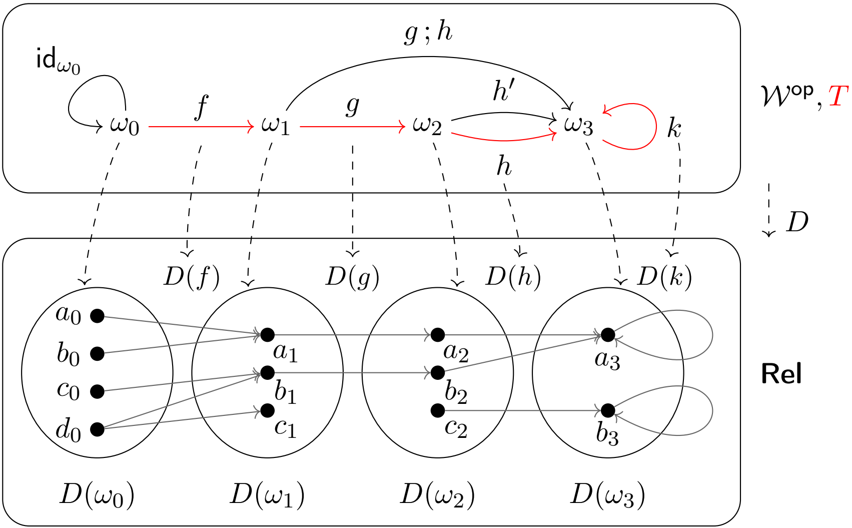

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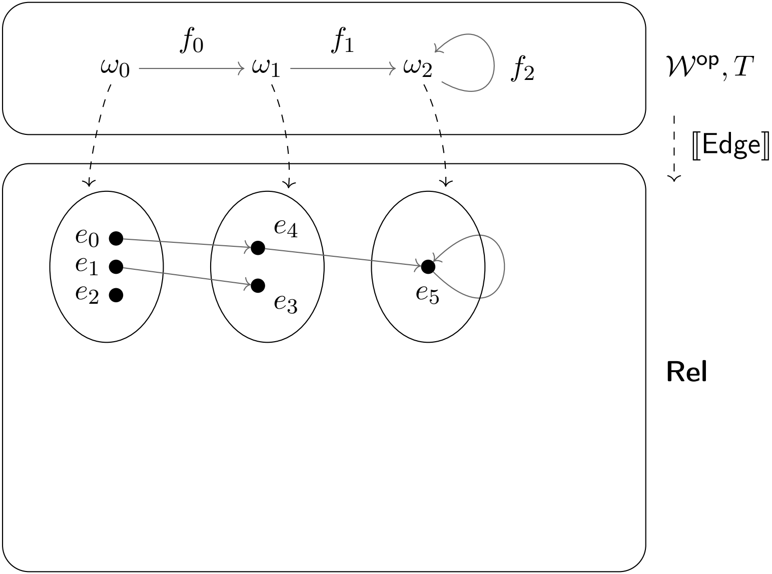

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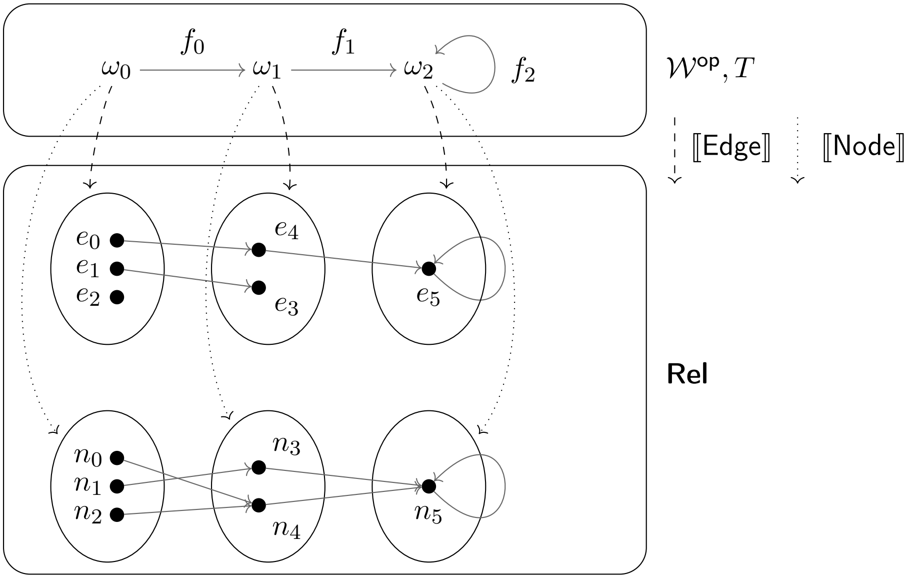

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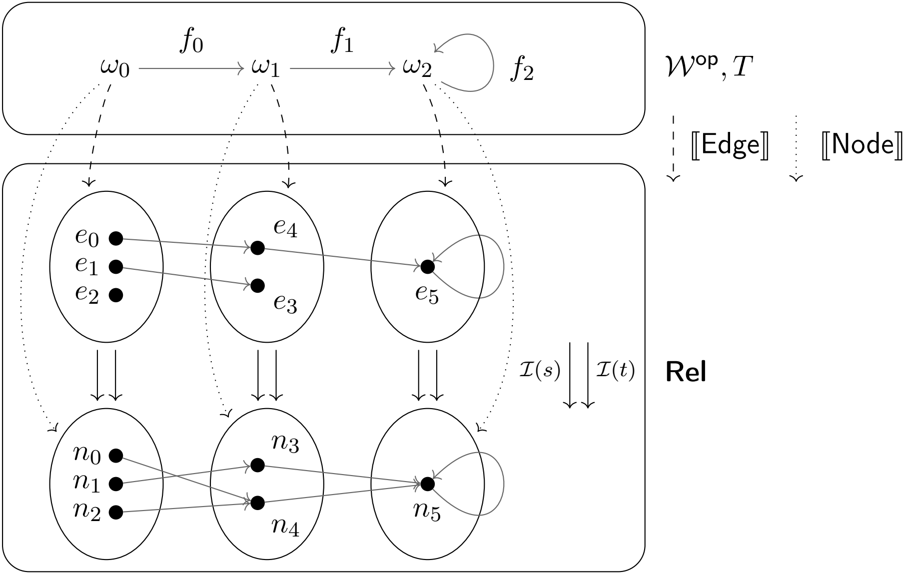

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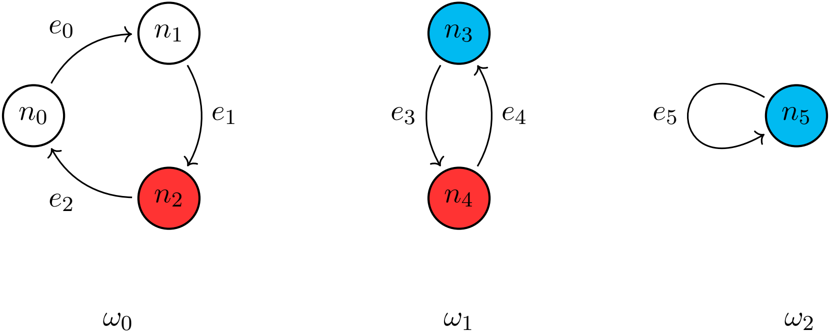
# Example – QLTL



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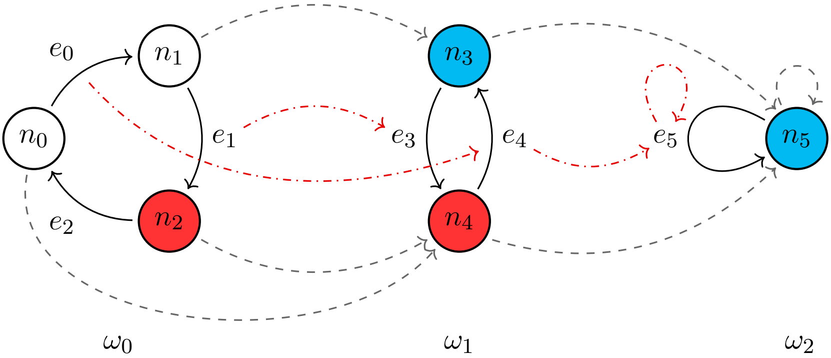

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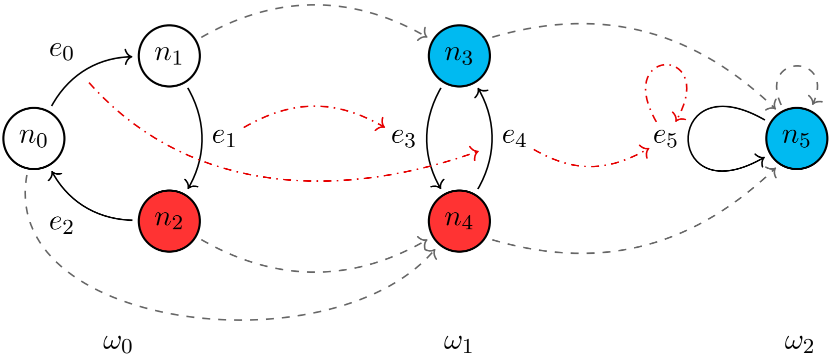


- $n_1 \vDash_{\omega_0} \mathsf{Next}(\mathsf{Blue}(x))$

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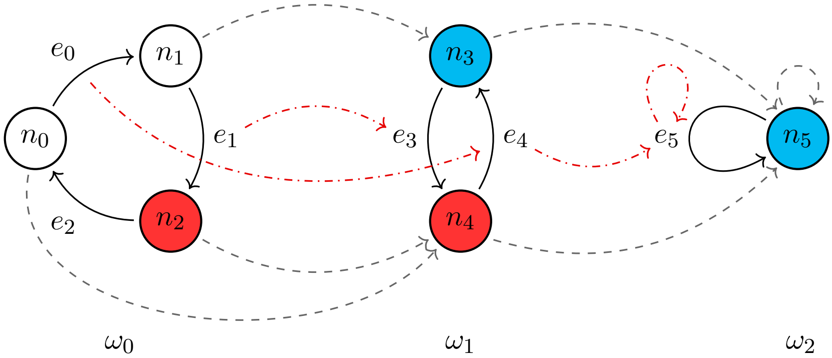


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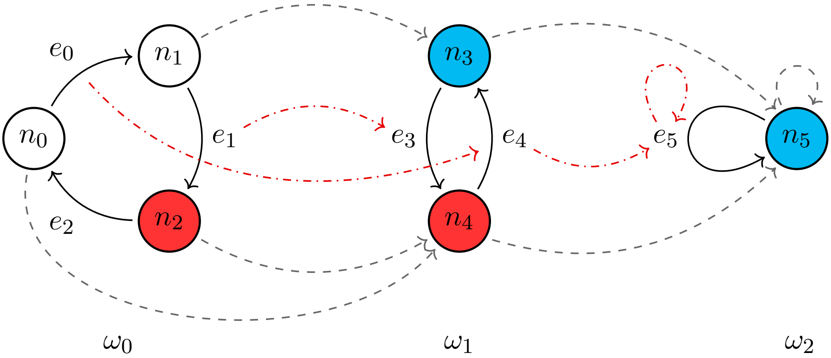


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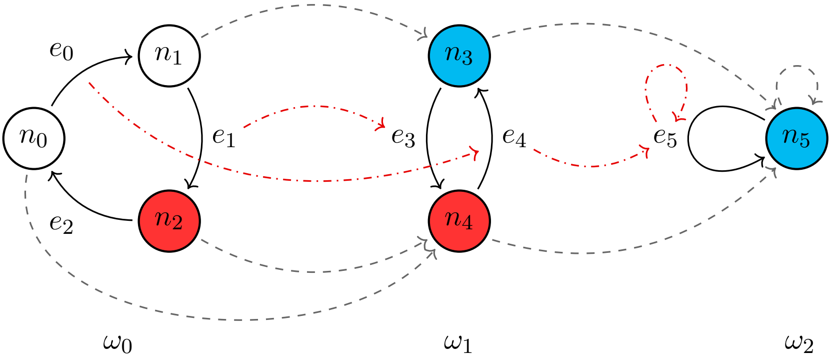


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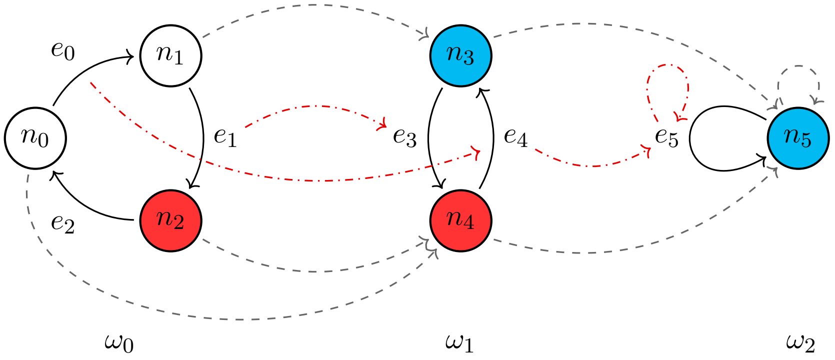


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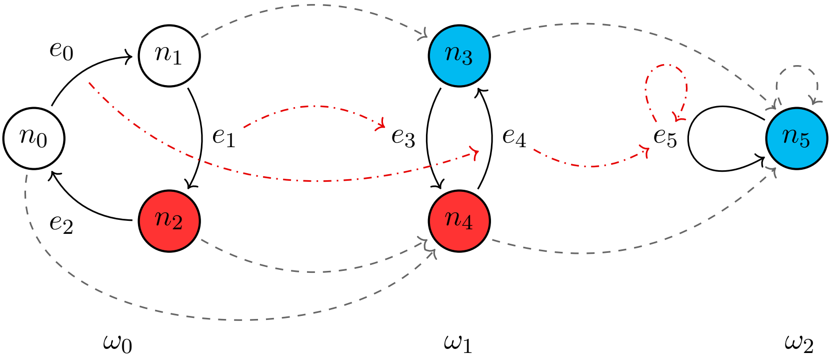


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- $(n_1, n_2) \vDash_{\omega_0}$ $(\neg(x = y))$ Until $(x = y)$

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\textbf{adjacent}(x, y) &:= \exists_{\mathsf{E}} e.((s(e) =_{\mathsf{N}} x \wedge t(e) =_{\mathsf{N}} y) \\
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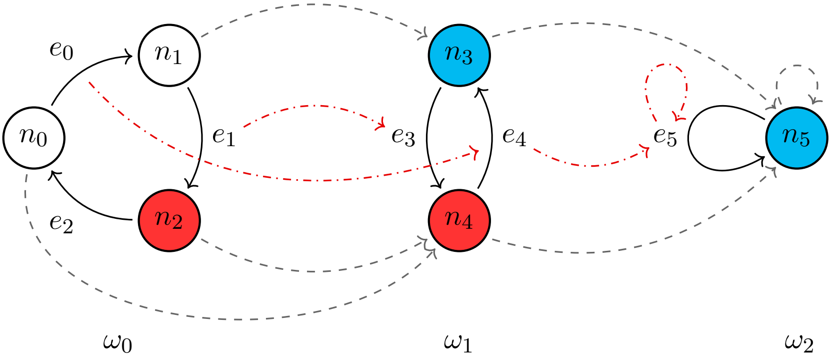
# Example – QLTL on graphs



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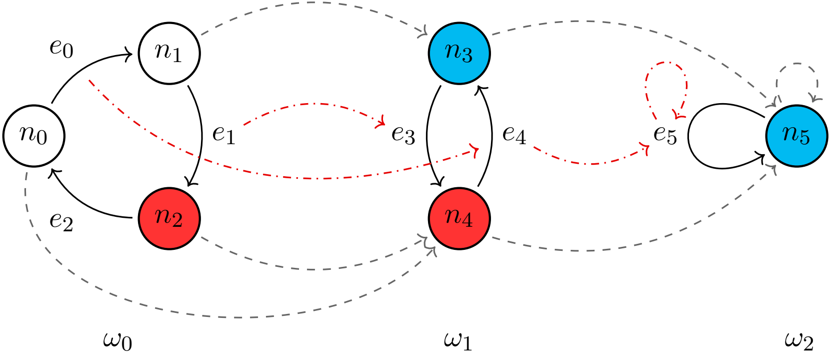


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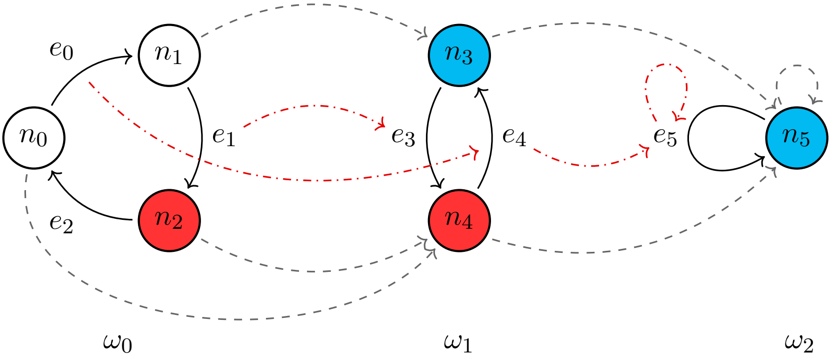


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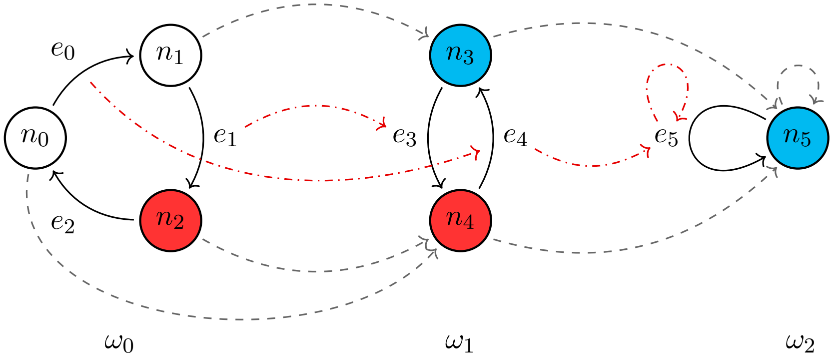


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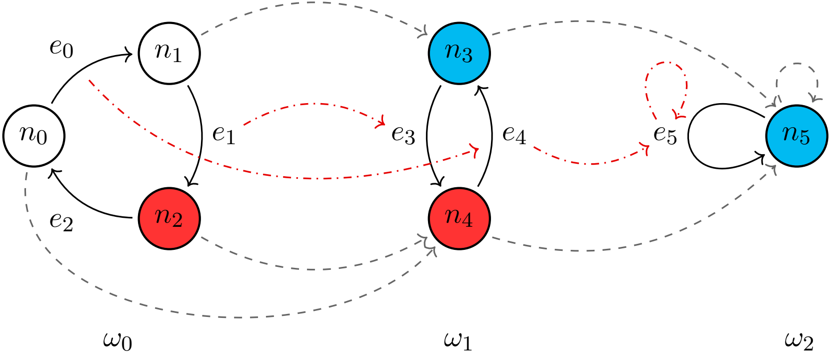


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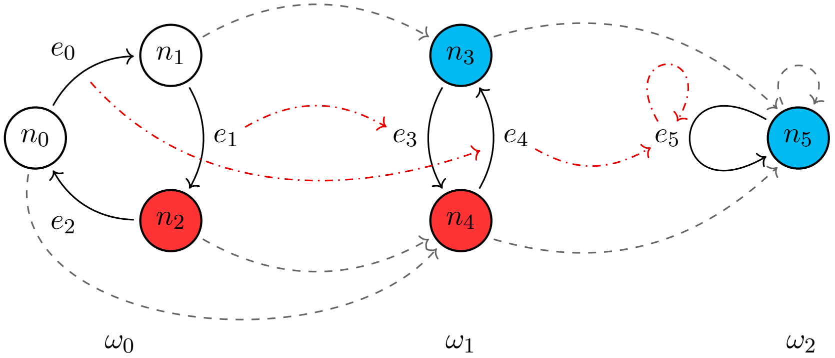


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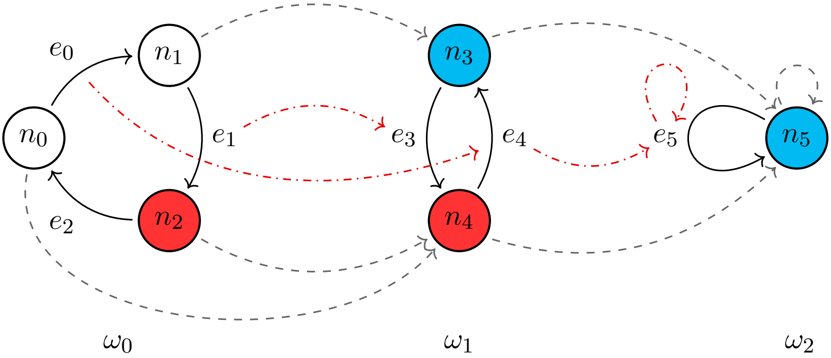


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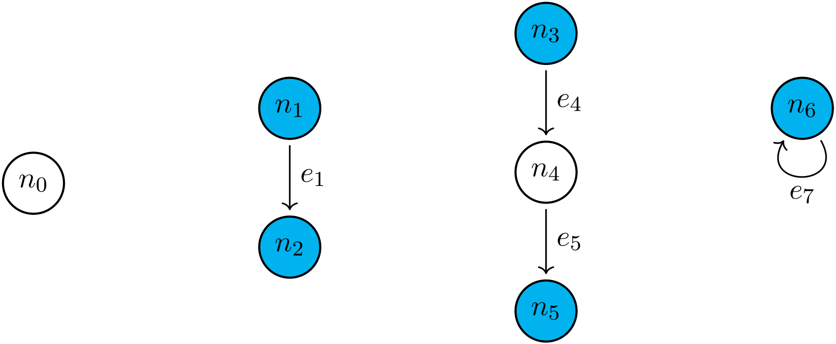
- Become particularly useful to treat duplicating relations

# Example – Duplicating relations



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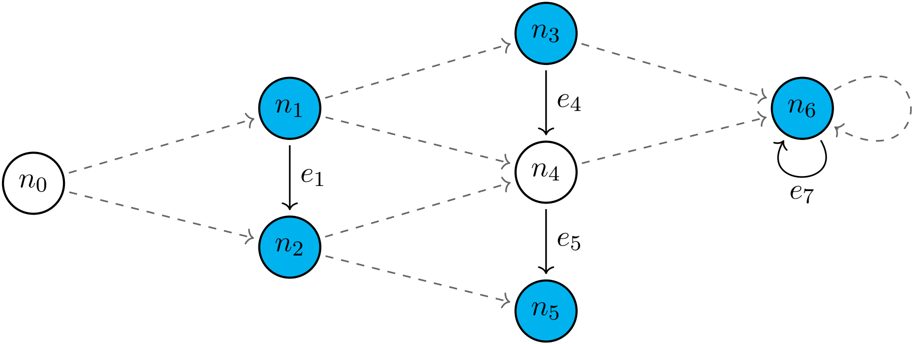

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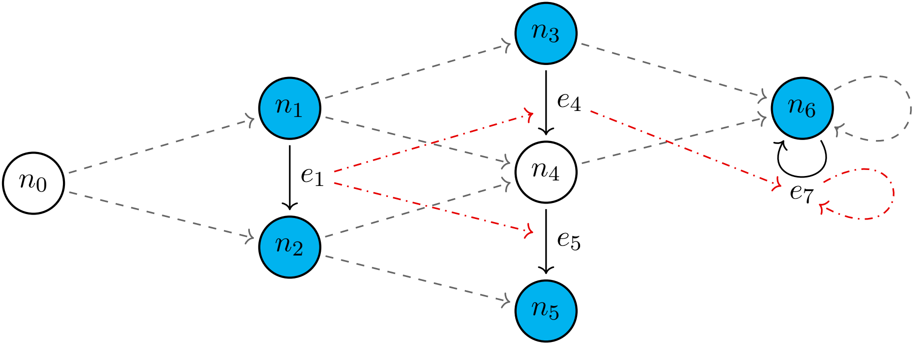

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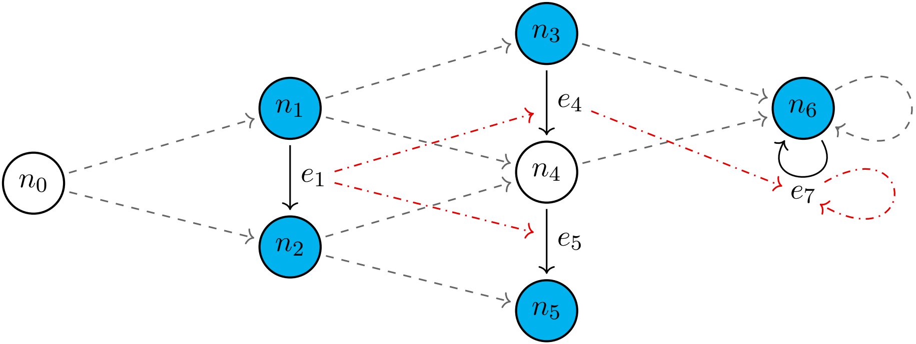


- $n_0 \vDash_{\omega_0} \mathsf{NextF}(\mathsf{Blue}(x))$

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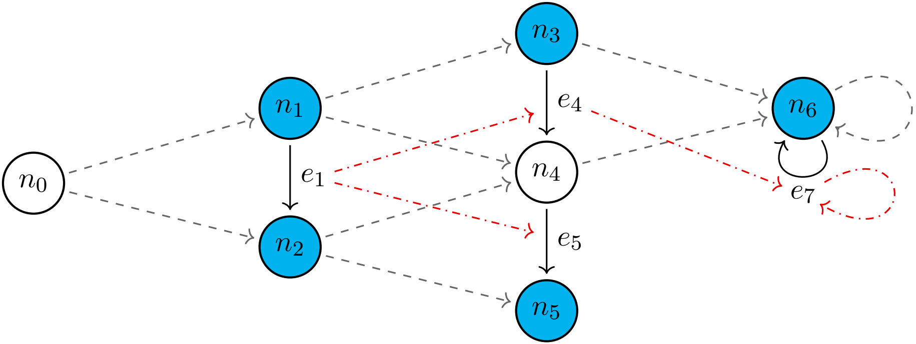


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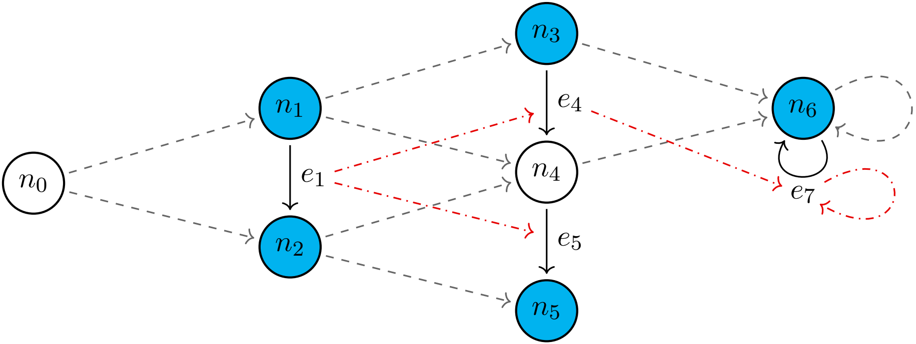


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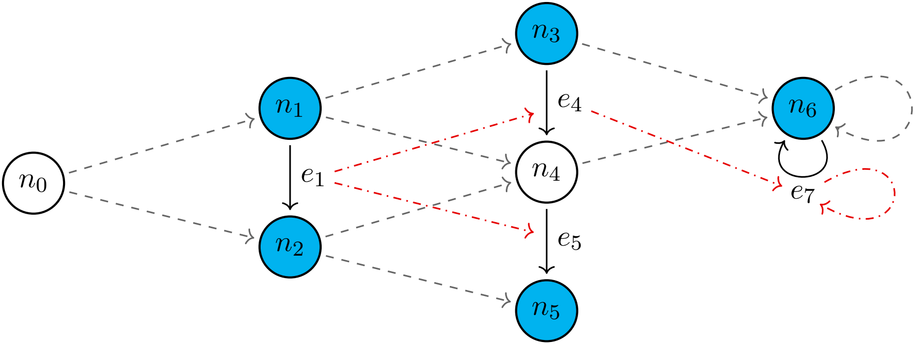


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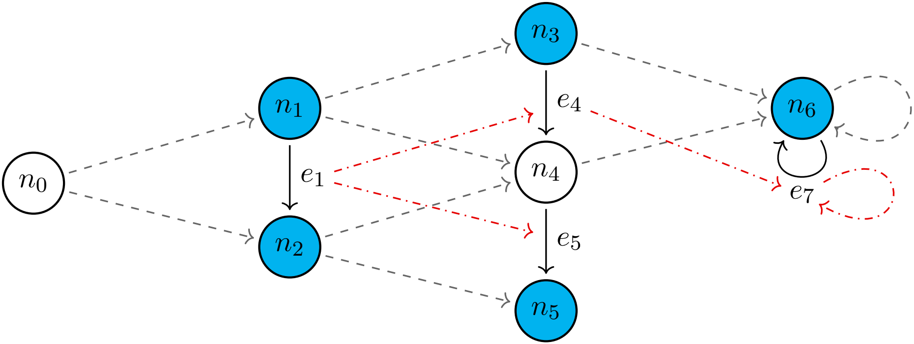


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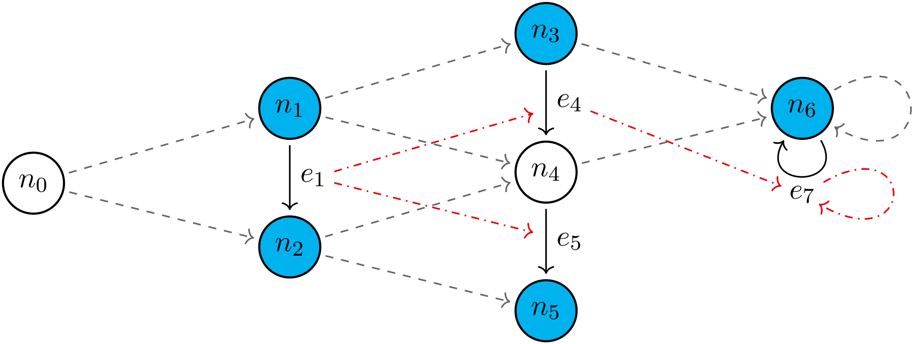


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$\Rightarrow$ (But they can be mostly recovered in the case of partial functions.)

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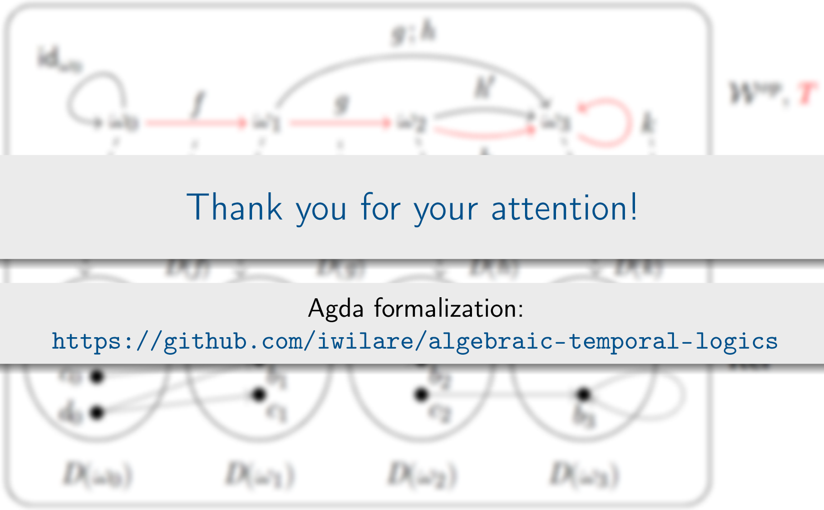

# Thank you for your attention!

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