

iWoAT Summer School on Chromatic Homotopy  
Theory and Higher (Infinity-Categorical) Algebra

L2 : 1-category theory of  $\infty$ -categories, I

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## Acknowledgement !

- I will follow the first half of the **lecture notes** by my advisor **Charles**, available at [faculty.math.illinois.edu/~rezk/quasicats.pdf](http://faculty.math.illinois.edu/~rezk/quasicats.pdf)

**Private conversations with Charles** is also a major reference for my talks.

- Special thanks to **David Gepner** for coordinating his talk with mine and offering to cover some of the more challenging topics that technically fall under my talks.
- I would like to thank also **Han Kong Jia**, **Ang Li**, **Guozhen Wang** for very helpful conversations during the preparation of my lectures.

Q: Why work in an  $\infty$ -setting?  $\infty - = (\infty, 1) -$

"A": It's the right place to do (generalized) homotopy theory using (generalized) topological spaces.

Solutions to a problem up to some equivalence are organized in the form of a groupoid.

e.g.  $X$  a top. space  $\rightsquigarrow \mathcal{F}$  sheaf on  $X$  valued in  
for  $U, V$  open in  $X$  groupoids

$$\begin{array}{ccc} \mathcal{F}(U \cup V) & \longrightarrow & \mathcal{F}(V) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U \cap V) \end{array} \quad \begin{matrix} \text{weak pullback} \\ \text{in groupoids} \end{matrix}$$

$\rightsquigarrow$  need category of groupoids & its homotopy theory.

Better:  $\infty$ -category of  $\infty$ -groupoids.

$\hookrightarrow$  higher groupoids : solve higher problems

"homotopy hypothesis"  $1 \leq n \leq \infty$

( $n$ -groupoids)  $\Leftrightarrow$  ( $n$ -truncated spaces)

$$n=1 \Rightarrow \pi_1(BG) \cong G, \quad \pi_{k>1}(BG) \cong 0$$

$\hookrightarrow$   $\infty$ -categorical: better colimits / limits (as opposed to  
say, derived functor in QMC)

$\rightsquigarrow$   $\infty$ -category of  $\infty$ -sheaves valued in  $\infty$ -groupoids ...

$\infty$ -groupoid =  $(\infty, 0)$  - category  
morphisms of  $\dim > 0$  are invertible

$\rightsquigarrow (\infty, 1)$  - category, which we will model by a  
quasi-category "qcat".

L2 & L4 : understanding qCat.

- In this model, a qcat is a particular kind of simplicial sets
- We will verify the homotopy hypothesis in this model.
- In general, qcats give a favorite model for  $\infty$ -cats; but there are settings where others are nicer.  
David will talk about one such model tomorrow.
- I will use "qcat" & " $\infty$ -cat" synonymously.

## Glossary

Cat = 1 - category of categories

Cat<sub>1</sub> =  $\infty$  - category of categories

qCat = 1 - category of  $\infty$  - categories ) Today !

sSet = 1 - category of simplicial sets

Cat <sub>$\infty$</sub>  =  $\infty$  - category of  $\infty$  - categories (David, L6 & L9)

## Simplicial operator category $\Delta$

obj:  $[n] := \{0 < 1 < 2 < \dots < n\}$  for  $n \geq 0$

mor:  $f: [m] \rightarrow [n]$  monotone  
"simplicial operators"

↪ face operators  $d^i = \langle 0, \dots, i-1, \hat{i}, i+1, \dots, n \rangle: [n-1] \rightarrow [n]$

degeneracy operators  $s^i = \langle 0, \dots, i-1, i, i, i+1, \dots, n \rangle: [n] \rightarrow [n+1]$

## Simplicial Set functor $X: \Delta^{\text{op}} \rightarrow \text{Set}$

→  $X_n := X[n]$ , the set of  $n$ -cells/n-simplices in  $X$ .

$a \in X_m$  is degenerate if  $a = bf$  for some

cell  $b \in X$  and  $f$  non-injective simplicial operator

$a \in X_m$  is nondegenerate otherwise.

↪  $X_n^{\text{deg}} = \{af \mid a \in X_k, f: [k] \rightarrow [n], k < n\}$

→  $sSet$  := 1-category of simplicial sets

mor: natural transformations of functors.

$\text{Set}_{\Delta}$  in kerodon, § 2.4.2

→ Standard  $n$ -simplex  $\Delta^n = \text{Hom}_{\Delta}(-, [n]) \in sSet$

$\Delta^m = \text{Hom}_{\Delta}([m], [n]) = \{f: [m] \rightarrow [n]\}$

Yoneda lemma  $\text{Hom}_{sSet}(\Delta^n, X) \rightarrow X_n$

$g \mapsto g(id_{[n]})$

→ initial obj: empty simplicial set  $[n] \mapsto \emptyset$

terminal obj:  $\Delta^0 = *$

→ pictures of  $\Delta^n = \text{Hom}_{\Delta}(-, [n])$

$$\Delta^m = \text{Hom}_{\Delta}([m], [n]) = \{ f: [m] \rightarrow [n] \}$$

$$\begin{matrix} \Delta^0: \\ <0> \end{matrix}$$

$$\begin{matrix} \Delta^1: \\ <0> \xrightarrow{\quad} <1> \end{matrix}$$

$$\begin{matrix} \Delta^2: \\ <0> \xrightarrow{\quad} <2> \\ \alpha \uparrow \quad \downarrow \\ <1> \end{matrix}$$

$$\begin{matrix} \Delta^3: \\ <0> \xrightarrow{\quad} <3> \\ & \nearrow \quad \searrow \\ <0> \xrightarrow{\quad} <2> \end{matrix}$$

only drawing non-deg. cells

$\text{sSet}$  has small (co)limits and computed degresswise.

## Subcomplexes

→ subcomplex  $A$  of  $X$  is just a subfunctr.

$A_n \subseteq X_n$  are closed under the action of simplicial operators.

- every subcomplex  $K$  of  $\Delta^n$  is a colimit:

$A :=$  poset of all nonempty  $S \subseteq [n]$  s.t.  $\Delta^S \subseteq K$

$\Rightarrow \text{colim}_{S \in A} \Delta^S \xrightarrow{\sim} K$  isomorphism.

- if  $K, L$  are subcomplexes of  $X \in \text{sSet}$ ,

$$K \cup L = \text{colim}(K \leftarrow K \cap L \rightarrow L), \quad A = \{K, L, K \cap L\}$$

pushout of simplicial sets.

→ (co)limits in  $s\text{Set}$

For any  $F: C \rightarrow \text{Set}$ ,  $C$  small cat, define " $\sim$ "  
on  $\coprod_{c \in \text{ob } C} F(c)$  by

$(c, x) \sim (c', x')$  if  $\exists \alpha: c \rightarrow c'$  in  $\text{mor } C$   
 w.r.t Symm. s.t.  $F(\alpha)(x) = x'$ .

Let  $X := \coprod_{c \in \text{ob } C} F(c) / \sim$ , where  $\approx$  is generated by  $\sim$ .

For  $c \in C$ , have  $\gamma_c: F(c) \rightarrow X$ ,  $x \mapsto [(c, x)]$

Then  $(X, \{\gamma_c\})$  is a colimit of  $F$ .

A colimit of  $J: C \rightarrow s\text{Set}$  in simplicial sets is  
computed degreewise, i.e.

$$\text{colim } J = \text{colim}_{c \in C} J(c) \Rightarrow (\text{colim } J)_n = \text{colim}_{c \in C} J(c)_n$$

When a colimit of functors to  $\text{Set}$  is more tractable...

Prop. Let  $A$  be a collection of subsets of a set  $S$ ,  
regarded as a poset under " $\subseteq$ ". If for any  
 $s \in S$ ,  $T, U \in A$  s.t.  $s \in T \cap U$ , there exists  
 $V \in A$  with  $s \in V \subseteq T \cap U$ . Then

$$\text{colim } T \longrightarrow \bigcup_{T \in A} T \text{ is a bijection.}$$

$$[(T, t)] \longmapsto t$$

Rmk.  $A$  satisfies a weaker closed under intersection  
condition.

For  $X, Y \in s\text{Set}$ ,

product " $\times$ " and coproduct " $\sqcup$ " are defined degree-wise

- $(X \times Y)_n = X_n \times Y_n ; X \times Y \in s\text{Set}$  (easy)
- $(X \sqcup Y)_n = X_n \sqcup Y_n ; X \sqcup Y \in s\text{Set}$ ? (yes.)

• Connected component  $\pi_0 X$  of  $X$

" $\approx$ " equivalence relation on  $\bigsqcup_{n \geq 0} X_n$  gen. by

$a \sim af \quad \forall \text{ all } n \geq 0, a \in X_n, f: [m] \rightarrow [n]$

$\pi_0 X = \{\text{equivalence classes } \sim \}$

$\hookrightarrow \pi_0: s\text{Set} \rightarrow \text{Set}$  is well-defined & monoidal  
w.r.t. " $\times$ ".

$\hookleftarrow \text{colim}_{\Delta^{\text{op}}} X \xrightarrow{\cong} \pi_0 X ;$

each connected component is a subcpX and  $\bigsqcup_{C \in \pi_0 X} C \xrightarrow{\cong} X$ ;

for any  $j, n$ ,  $\Delta_j^n$  and  $\Delta^n$  are connected, i.e.  $\pi_0$  is a singleton.

$\hookleftarrow$  A coproduct  $X = \bigsqcup_s X_s, X_s \in s\text{Set}$  is a simplicial set.

→ (Cartesian) product of  $X, Y \in \text{Set}$

$$(X \times Y)_n = X_n \times Y_n \quad \text{Cartesian product in Set}$$

$$f: [m] \rightarrow [n] \text{ acts by } (a, b) = (af, bf)$$

Example.  $X = \Delta^1 \times \Delta^1$

Let  $A, B$  be subcollection of cells in  $X$  s.t.

$$(f, g) \in \begin{cases} A & \text{if } f(i) \leq g(i), \text{ all } 0 \leq i \leq n \\ B & \text{if } f(i) \geq g(i), \text{ all } 0 \leq i \leq n \end{cases}$$

for simplicial operators  $f, g: [n] \rightarrow [1]$ ,

Then  $A \approx \Delta^2 \approx B$ ,  $A \cap B \approx \Delta^1$  as subcpxes of  $X$ .

$$\begin{array}{ccc} (0, 1) & \longrightarrow & (1, 1) \\ \uparrow & \nearrow A & \uparrow \\ (0, 0) & \longrightarrow & (1, 0) \end{array}$$

In fact,

$$\begin{array}{ccc} \Delta^1 & \xrightarrow{\langle 02 \rangle} & \Delta^2 \\ \downarrow \langle 02 \rangle & & \downarrow \langle 011 \rangle, \langle 001 \rangle \\ \Delta^2 & \xrightarrow{\lrcorner} & \Delta^1 \times \Delta^1 \\ \langle 001 \rangle, \langle 011 \rangle & & \end{array}$$

L4: This is a colimit in  $\infty\text{-catis}$ !

Horns For  $n \geq 1$ ,  $\Lambda_j^n \subseteq \Delta^n$  subject for  $0 \leq j \leq n$

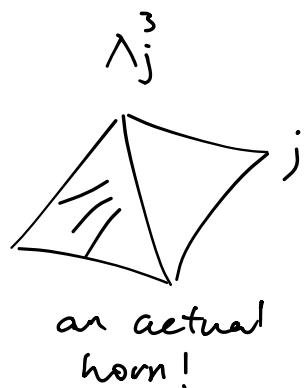
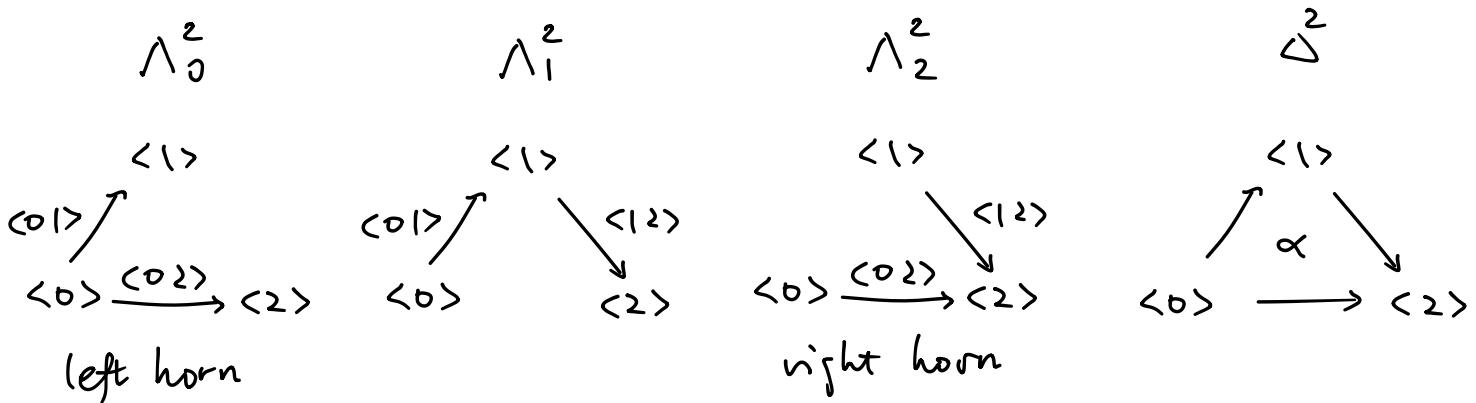
$$(\Lambda_j^n)_k = \{ f : [k] \rightarrow [n] \mid ([n] \setminus \{j\}) \not\subseteq f(k) \}$$

$$\Rightarrow \Lambda_j^n = \bigcup_{i \neq j} \Delta^{n \setminus \{i\}} \subseteq \Delta^n \quad 0 \leq j \leq n$$

is the subcpx obtained by removing the non-deg  $n$ -cell and the face (codim 1 cell) opposite of  $j$ .

$$\rightarrow \begin{array}{ll} j < n & \text{left horn} \\ 0 < j & \text{right horn} \end{array} \quad 0 < j < n \quad \text{inner horn}$$

$$\rightarrow \text{Pictures: } \Lambda_0^1 = \Delta^{\{0\}} \quad \Lambda_1^1 = \Delta^{\{1\}} \quad \Delta^1 : \\ \langle 0 \rangle \qquad \qquad \qquad \langle 1 \rangle \qquad \qquad \qquad \langle 0 \rangle \rightarrow \langle 1 \rangle$$



Nerve: • For any  $C \in \text{Cat}$ ,  $NC \in \text{SSet}$  s.t.

$$NC_n = \text{Hom}_{\text{Cat}}([n], C)$$

•  $F: C \rightarrow D$  functor,  $NF: NC \rightarrow ND$

$$(\alpha: [n] \rightarrow C) \mapsto (F\alpha: [n] \rightarrow D)$$

$\Rightarrow N(-): \text{Cat} \rightarrow \text{sSet}$

$\rightarrow N[m] = \Delta^m$  as sSet.

$\rightarrow NC_0 = \text{ob}(C)$ ,

$\rightarrow NC_1 = \text{mr}(C)$

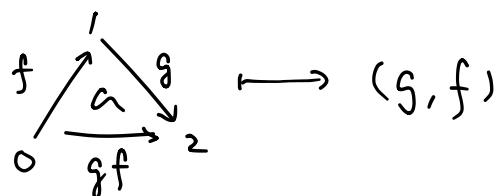
$\langle 0 \rangle^*, \langle 1 \rangle^*: NC_1 \rightarrow NC_0$ , mr  $\mapsto$  source, target

$\langle 00 \rangle^*: NC_0 \rightarrow NC_1$ ,  $x \mapsto \text{id}_x$

$\rightarrow NC_2 \approx \{\text{pairs of composable morphisms}\}$

$NC_2 \rightarrow NC_1 \times NC_1$

$a \mapsto (a_{01}, a_{12})$



## Characterization of Nerves.

Prop.  $X \in \text{sSet}$  is isomorphic to the nerve of some category iff for all  $n \geq 2$ , the following function

$$d_n : X_n \rightarrow \{(g_1, \dots, g_n) \in (X_1)^{x^n} \mid g_{i-1, \langle 1 \rangle} = g_i \langle 0 \rangle\}$$

$$a \mapsto (a_0, \dots, a_{n-1, n}) \quad \text{for all } 1 \leq i \leq n$$

$$\qquad \qquad \qquad \tilde{a}_{ij} = a \circ \langle i j \rangle$$

is a bijection.

Cer. Maps to nerves are determined by edges.

## Maps between Nerves.

aka. why we identify a 1-cat with its nerve in sSet.

Thm.  $N: \text{Cat} \rightarrow \text{sSet}$  is fully faithful, i.e.

$$\text{Hom}_{\text{Cat}}(C, D) \approx \text{Hom}_{\text{sSet}}(NC, ND).$$

Proof: injectivity is clear since a functor (between cats) is determined by its value on objs & mors.

surjectivity: given  $g: NC \rightarrow ND$ ,  $g_0, g_1$  gives value of functor  $f$  on objs & mors;

The composition of two functors is unique by the previous prop:

Proof continued:  $f(id_x) = id_{f(x)}$  is  $g \langle oo \rangle^* = \langle oo \rangle^* g$

$$\begin{array}{ccc} NC_0 & \xrightarrow{\langle oo \rangle^*} & NC_1 \\ \downarrow g_0 & & \downarrow g_1 \\ ND_0 & \xrightarrow{\langle oo \rangle^*} & ND_1 \end{array} \quad \begin{array}{ccc} x & \longmapsto & id_x \\ \downarrow & & \downarrow \\ f(x) & \longmapsto & f(id_x) = id_{f(x)} \end{array}$$

$$f(x \rightarrow y \rightarrow z) = f(y \rightarrow z) f(x \rightarrow y)$$

$$\begin{array}{ccc} NC_2 & \xrightarrow{\langle oo \rangle^*} & NC_1 \\ \downarrow g_2 & & \downarrow g_1 \\ ND_2 & \xrightarrow{\langle oo \rangle^*} & ND_1 \end{array} \quad \begin{array}{ccc} \begin{array}{c} y \\ \nearrow a \\ x \longrightarrow z \end{array} & \longmapsto & \begin{array}{c} x \longrightarrow z \\ x \rightarrow y \rightarrow z \end{array} \\ \downarrow & & \downarrow \\ \begin{array}{c} x \xrightarrow{f(x)} y \xrightarrow{f(y)} z \\ \downarrow g(a) \\ x \xrightarrow{f(x)} g(a) \xrightarrow{f(g(a))} z \end{array} & \longmapsto & \begin{array}{c} f(x \rightarrow y \rightarrow z) \\ f(y \rightarrow z) f(x \rightarrow y) \end{array} \end{array}$$

let  $a$  be a 2-cell in  $NC_2$  s.t.  $a_{01} = x \rightarrow y$ ,  $a_{12} = y \rightarrow z$ .

Then by previous prop,  $a$  is uniquely determined by the composable pair  $(x \rightarrow y, y \rightarrow z)$ , and must have

$$a_{02} = x \rightarrow y \rightarrow z.$$

Similarly, the composable pair  $(f(x \rightarrow y), f(y \rightarrow z))$  uniquely determines a 2-cell in  $ND_2$ , which must be  $g(a)$  since

$$g(a)_{01} = f(x \rightarrow y), \quad g(a)_{12} = f(y \rightarrow z).$$

Thus

$$g(a)_{02} = f(y \rightarrow z) f(x \rightarrow y)$$

uniquely determines the composition rule of  $f$ .

Uniqueness of  $f$  follows from previous corollary.  $\square$

$\infty$ -cat / quasicat  $C$  is a simplicial set satisfying  
the inner horn extension property for all  $n \geq 2$  and  
 $0 \leq j \leq n$ .

$$\begin{array}{ccc} \Delta_j^n & \longrightarrow & C \\ \downarrow & & \nearrow \\ \Delta^n & \dashrightarrow & \exists \end{array} .$$

Prop. A simplicial set  $X$  is isomorphic to the nerve of a cat  
iff it has the unique inner horn extension property.

$$X_1 \stackrel{\delta}{\sim} \text{Hom}_{\text{sset}}(\Delta^n, X) \rightarrow \text{Hom}_{\text{sset}}(\Delta_j^n, X)$$

$$\text{So } N: \text{Cat} \longrightarrow \text{qCat} \subseteq \text{sset}$$

full

Remark:

- "Let  $C$  be an  $\infty$ -category, which is a category" = "let  $C$  be an  $\infty$ -category isomorphic to nerve of some cat".
- Composition in an  $\infty$ -cat cannot be defined the same as for a 1-cat.

Convention: For  $C$  an  $\infty$ -cat :  $C_0 = \text{objs}$ ,  $C_1 = \text{morphs}$  ;  
 for  $x \in C_0$ ,  $\text{id}_x = x \langle 00 \rangle \in C_1$  ;  
 for  $f \in C_1$ ,  $f \langle 0 \rangle = \text{source}$ ,  $f \langle 1 \rangle = \text{target}$ .

Proof of prop via:

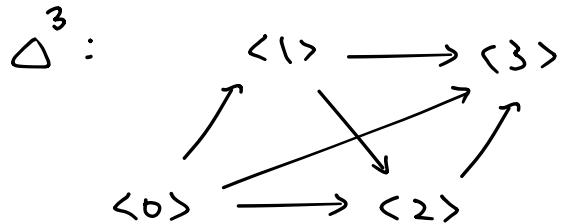
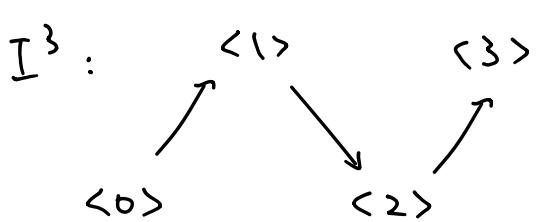
$$\text{Hom}_{\text{Set}}(\Delta^n, X) \xrightarrow{\delta} X_n \rightarrow \text{Hom}_{\text{Set}}(\Lambda_j^n, X) \xrightarrow{r} \text{Hom}_{\text{Set}}(I^n, X)$$

"only if":

- $r$  is injective: straightforward.

- composite is bijective:

$I^n$  = "spine" = largest chain of edges in  $\Delta^n$



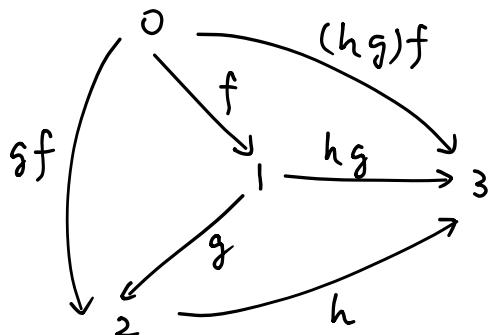
"if":

- composition rule on  $X_1$

$$\begin{array}{ccc} \Lambda_1^2 & \xrightarrow{(f,g)} & X \\ \downarrow & \nearrow & \\ \Delta_2 & \dashv \exists! u & \end{array}$$

$$u_{02} = gf$$

- composition is unital if we let one of  $f, g$  to be id.
- composition is associative:



$\Lambda_1^3 \rightarrow X$  extends  
uniquely to  $v: \Delta^3 \rightarrow X$   
with  $v_{<023>} =$  witnessing  
 $h(gf) = (hg)f$

→ can define  $\alpha_n: X_n \rightarrow NC_n$  by restricting to  $I_n$

→ bijective: obvious for  $n \leq 1$  + induction.  $\square$

## Opposite of an $\infty$ -cat.

Given  $C \in \text{Cat}$ ,  $C^{\text{op}}$ :  $\text{obj } C^{\text{op}} = \text{obj } C$ ,

$$\text{Hom}_{C^{\text{op}}}(x, y) = \text{Hom}_C(y, x)$$

For  $X \in \text{sSet}$ ,  $X^{\text{op}} = X^{\circ \text{op}} \rightarrow (\Delta^n)^{\text{op}} \approx \Delta^n$ ,  $(\Lambda_j^n)^{\text{op}} \approx \Lambda_{n-j}^n$

For  $X \in \infty\text{-Cat}$ ,  $X^{\text{op}} \in \infty\text{-Cat}$ ;  $(NC)^{\text{op}} = N(C^{\text{op}})$

$$\text{op} \circ \text{op} = \text{id} \Rightarrow (X^{\text{op}})^{\text{op}} = X$$

Functor between  $\infty$ -cats is just a functor between simplicial sets.

Nat trans  $h: f_0 \Rightarrow f_1$  between  $f_0, f_1: C \rightarrow D$  in  $\text{qCat}$ , is the map

$$h: C \times N[1] = C \times \Delta^1 \rightarrow D \text{ in sSet}$$

$$\text{s.t. } h|_{C \times \{i\}} = f_i.$$

Subcats of an  $\infty$ -cat. A subcategory  $C' \subseteq C$  for  $C \in \text{qCat}$  is a subcategory if for  $n \geq 2$ ,  $0 < k < n$ , every  $f: \Delta^n \rightarrow C$  s.t.  $f(\Lambda_k^n) \subseteq C'$  satisfies  $f(\Delta^k) \subseteq C'$ .

It is a full subcat if for all  $n$  and all  $a \in C_n$ ,  
 $a \in C'_n \Leftrightarrow a_i \in C'_0$  for all  $0 \leq i \leq n$ .

Warning: If  $C'$ ,  $C$  are  $\infty$ -cats and  $C' \subseteq C$  is a subcategory, i.e. a subfunctor,  $C'$  need **not** be a subcat of  $C$  as  $\infty$ -cats.

$\text{Cat}_1 := \infty\text{-category of small 1-cats}.$

$(\text{Cat}_1)_n := \{ (C_i, F_{ij}, \zeta_{ijk}) \mid$

(0) for each  $i \in [n]$ ,  $C_i$  is a small category.

i.e. obj's / 0-cells : small cats

(1) for each  $i \leq j$  in  $[n]$ ,  $F_{ij} : C_i \rightarrow C_j$  is a functor.

i.e. 1-morphisms / 1-cells : functors

(2) 2-cells are nat isos  $\zeta_{012} : F_{02} \Rightarrow F_{12}F_{01}$ .

(2') for each  $i \leq j \leq k$  in  $[n]$ ,  $\zeta_{ijk} : F_{ik} \Rightarrow F_{jk} \circ F_{ij}$  is a nat isomorphism of functors  $C_i \rightarrow C_k$

s.t. • for each  $i \in [n]$ ,  $F_{ii}$  is  $\text{id}_{C_i}$ .

• for each  $i \leq k$  in  $[n]$ ,

$\zeta_{iik} : F_{ik} \Rightarrow F_{ik} \circ \text{id}_{C_i}$ ,  $\zeta_{ikk} : F_{ik} \Rightarrow \text{id}_{C_k} F_{ik}$

are identity nat transformation of  $F_{ik}$ .

• for each  $i \leq j \leq k$  in  $[n]$ , the diagram commutes

$$\begin{array}{ccc} F_{il} & \xrightarrow{\zeta_{ije}} & F_{jl} F_{ij} \\ \zeta_{ile} \Downarrow & \curvearrowright & \Downarrow \zeta_{jke} F_{ij} \\ F_{ke} F_{ik} & \xrightarrow[F_{ke} \zeta_{ijk}]{} & F_{ke} F_{jk} F_{ij} \end{array}$$

$$\zeta_{jke} F_{ij} \circ \zeta_{ije} = F_{ke} \zeta_{ijk} \circ \zeta_{ile} \text{ as nat isos. } \}$$

$\delta : [m] \rightarrow [n]$  acts by

$$\delta(C_i, F_{ij}, \zeta_{ijk}) = (C_{\delta(i)}, F_{\delta(i)\delta(j)}, \zeta_{\delta(i)\delta(j)\delta(k)})$$

$\text{Cat}_1 \in \text{gCat}$ :

- Given  $\Lambda_1^2 \rightarrow \text{Cat}_1$ ,

$$\begin{array}{ccc} & C_1 & \\ F_{01} \nearrow & & \searrow F_{12} \\ C_0 & & C_2 \end{array}$$

we can extend it to  $\zeta_{012} = \text{id}_{F_{12}} \circ F_{01}$ ,  $F_{02} = F_{12} \circ F_{01}$ .

But this filler is **not** unique:

- 2-cells in  $\text{Cat}_1$  are nat isos  $\zeta_{012} : F_{02} \Rightarrow F_{12} F_{01}$ .

So we just need  $F_{02}$  to be nat isomorphic to  $F_{12} \circ F_{01}$ .

- Extensions for higher inner horns are **unique**.

filler for  $\Lambda_1^3$  is the commutative diagram with the filled-in face  $\zeta_{023}$ .

$$\begin{array}{ccc} & C_1 & \longrightarrow C_3 \\ C_0 \nearrow & \swarrow & \nearrow \\ & C_2 & \end{array}$$

$$\begin{array}{ccccc} F_{03} & \xrightarrow{\zeta_{013}} & F_{13} & F_{01} & \\ \zeta_{023} \Downarrow & \curvearrowright & & & \Downarrow \zeta_{123} F_{01} \\ F_{23} F_{02} & \xrightarrow{F_{23} \zeta_{012}} & F_{23} F_{12} F_{01} & & \end{array}$$

Fact:  $N\text{Cat}$  is isomorphic to the subcpx of  $\text{Cat}_1$  consisting of cells  $(C_i, F_{ij}, \zeta_{ijk})$  s.t.  $F_{ik} = F_{jk} \circ F_{ij}$ .

So  $N\text{Cat}$  is not a subcategory of  $\text{Cat}_1$  since it misses all nat isos  $F_{02} \Rightarrow F_{12} \circ F_{01}$  which are not identity nor trans.

Fundamental category for  $X \in \text{sSet}$  consists of :

- 1) a category  $hX$
- 2)  $\alpha: X \rightarrow N(hX)$  in  $\text{sSet}$  s.t. for any  $C \in \text{Cat}$   
 $\alpha^*: \text{Hom}(N(hX), NC) \rightarrow \text{Hom}(X, NC)$   
 is a bijection.

Prop: Every simplicial set has a fundamental category.

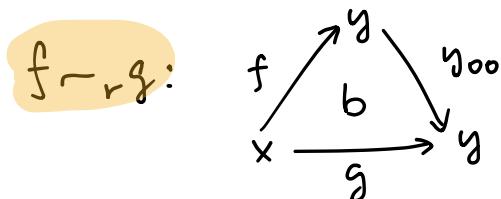
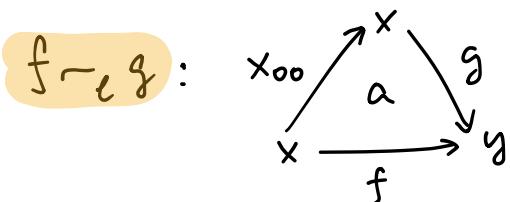
$$\Rightarrow h: \text{sSet} \xrightarrow{\perp} \text{Cat} : N$$

homotopy relation on morphisms  $C_1$  for  $C \in \text{qCat}$ .

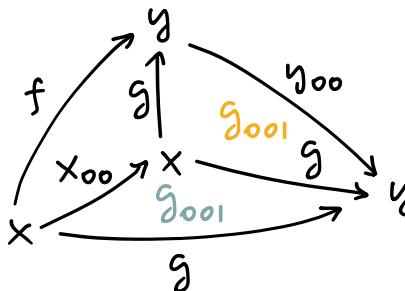
$\sim_l, \sim_r$  on  $\text{hom}_C(x, y) = \{f \in C_1 : f_0 = x, f_1 = y\}$

$f \sim_l g \Leftrightarrow \exists a \in C_2, a_{01} = 1_x, a_{02} = f, a_{12} = g$

$f \sim_r g \Leftrightarrow \exists b \in C_2, b_{12} = 1_x, b_{01} = f, b_{02} = g$



- $g \sim_l g, g \sim_r g$
- $f \sim_l g \Rightarrow f \sim_r g$   
 $\Delta^{(f_0 2 3)}$



$$\Delta^3 \rightarrow C$$

- $f \sim_l g, g \sim_l h \Rightarrow f \sim_l h$

↪  $\sim_l$  and  $\sim_r$  are equivalent to each other and are an equivalence relation on  $\text{hom}_C(x, y)$ .

For  $f \in \text{hom}_C(x, y)$ ,  $g \in \text{hom}_C(y, z)$ ,  $h \in \text{hom}_C(x, z)$ ,  
 $h$  is a composite of  $(g, f)$  if  $\exists$  2-cell  $a \in C_2$   
 s.t.  $a_{01} = f$ ,  $a_{12} = g$ ,  $a_{02} = h$ .

Write  $[g] \circ [f]$  for the equivalence class containing  
 a composite of  $(g, f)$ .

→ It is well-defined, unital & associative.

→  $[h] = [g] \circ [f]$  iff  $\exists u \in C_2$  s.t.  $u|\Delta^{(0,1)} = f$ ,  $u|\Delta^{(1,2)} = g$ ,  
 and  $u|\Delta^{(0,2)} = h$ . So every morphism in  $[h]$  can be interpreted  
 as a composite of  $g$  with  $f$ .

Homotopy category  $hC$  for  $C \in \mathbf{qCat}$  has  $\text{obj}(hC) = C_0$

and morphism sets  $\text{hom}_{hC}(x, y) := \text{hom}_C(x, y)/\approx$

→  $\pi: C \rightarrow N(hC)$  identity on  $C_0 = N(hC)_0$ , and  
 Send  $f \in C_1 \mapsto [f]$ ; in particular, for an  $n$ -cell  $a$ ,  
 $\pi(a) \in N(hC)_n$  with  $\pi(a)_{i-1,i} = \pi(a_{i-1,i})$ .

Prop. For  $C \in \mathbf{qCat}$ ,  $hC$  is its fundamental category:  
 any  $\phi: C \rightarrow ND$  factors uniquely thru  $\pi$ .

Proof.  $\psi: N(hC) \rightarrow ND$  uniquely determined by values  
 on vertices and edges:

$$x \in C_0, \quad \psi(x) = \phi(x)$$

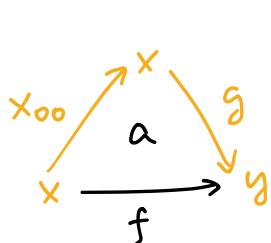
$[f] \in hC_1$ ,  $\psi([f]) = \phi(f)$  is well-defined;

these assignments are forced by  $\pi$  since it is identity  
 on vertices and surjective on edges.  $\square$

$$\Rightarrow h: \infty\text{-Cat} \xleftarrow{\perp} \mathbf{Cat} : N$$

Observation: Let  $C' \subseteq C$  be a subcat of  $\infty$ -cats.

If  $[f] = [g] \in \text{hom}_{hC}(x, y)$ , then  $f \in C'_1 \Leftrightarrow g \in C'_1$



$$\begin{aligned} g \in C'_1 &\Rightarrow \Delta^1 \subseteq C'_1 \\ &\Rightarrow \Delta^2 \subseteq C'_1 \\ &\Rightarrow f \in C'_1 \end{aligned}$$

Prop + observation give the following **bijective correspondence**

for  $C \in q\text{Cat}$ ,

$$\{(\text{full}) \text{ subcat of } C\} \longleftrightarrow \{(\text{full}) \text{ subcat of } hC\}$$

Backward: given  $D' \subseteq hC$ , and  $f: \Delta^n \rightarrow C$  whose restriction to  $\Delta_j^n$  factors thru  $\tilde{D}'$ ,

$$\begin{array}{ccccc} \Delta_j^n & \longrightarrow & \tilde{D}' & \longrightarrow & C \\ \downarrow & \nearrow \exists! & \downarrow & \nearrow f & \downarrow \pi \\ \Delta^n & \dashrightarrow & ND' & \longrightarrow & N(hC) \\ \exists! & & & & \end{array}$$

$\Rightarrow$  for  $D \in \text{Cat}$

$$\{(\text{full}) \text{ subcat of } ND\} \longleftrightarrow \{(\text{full}) \text{ subcat of } D\}.$$

For  $C \in \infty\text{-Cat}$ ,  $f: x \rightarrow y \in C_1$  is an equivalence / isomorphism if its image in  $hC$  is an isomorphism, i.e. if there exists  $g: y \rightarrow x$  s.t.  $[g] \circ [f] = [1_x]$  and  $[f] \circ [g] = [1_y]$  — in this case,  $g$  is an inverse of  $f$ .

- ↪ Inverses of a morphism in an  $\infty$ -cat are not necessarily unique, b/c they are defined to be unique up to htpy.
- ↪ A functor  $F: C \rightarrow D$  of  $\infty$ -cats sends equivalences to equivalences.

A Kan Complex is an  $\infty$ -cat  $K$  with extension property for all horns, i.e. for all  $0 \leq j \leq n$ ,  $n \geq 1$

$$\text{Hom}(\Delta^n, K) \longrightarrow \text{Hom}(\Lambda_j^n, K) \text{ is surjective}$$

An  $\infty$ -groupoid is an  $\infty$ -cat  $C$  s.t.  $hC$  is a groupoid, i.e. an  $\infty$ -cat in which every morphism is an equivalence.

Prop. Every Kan cpx  $K$  is an  $\infty$ -groupoid.

Proof. For any  $f: x \rightarrow y$  in  $K_1$ , extension along  $\Delta_0^2 \xrightarrow{a} \Delta^2$

$$\begin{array}{ccc} & y & \\ f \swarrow & \downarrow \tilde{a} & \\ x & \xrightarrow{x_{00}} & x \end{array}$$

with  $a|_{\Delta^{2,1,3}} = f$  gives a postinverse.

Every edge in  $K$  has a postinverse.

$\Rightarrow K$  is an  $\infty$ -groupoid.  $\square$

Rmk. The converse holds and is due to Joyal!

For  $C \in qCat$ , its core  $C^\simeq$  is the subcpx consisting of cells all of whose edges are equivalences, i.e.

$$\begin{array}{ccc} C^\simeq & \longrightarrow & C \\ \downarrow & & \downarrow \pi \\ (N(hC))^\simeq & \longrightarrow & N(hC) \end{array}$$

pullback in sSet

$$\rightarrow (ND)^\simeq \approx N(D^\simeq).$$

Prop.  $C \in \infty\text{-Cat} \Rightarrow C^\simeq$  is a subcat and an  $\infty$ -groupoid.

Every subcpx of  $C$  which is an  $\infty$ -groupoid is contained in  $C^\simeq$ .

$\rightarrow (hC)^\simeq \approx h(C^\simeq)$ .  $C^\simeq$  is the subcat with all equivalences.

## Singular complex of a space

The topological n-simplex is

$$\Delta_{\text{top}}^n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum x_i = 1, x_i \geq 0\}.$$

$$\rightsquigarrow \Delta_{\text{top}} : \Delta \rightarrow \text{Top}, \quad \Delta_{\text{top}}[n] = \Delta_{\text{top}}^n$$

For  $f : [m] \rightarrow [n]$ ,

$$\Delta_{\text{top}} f : (x_0, \dots, x_m) \mapsto (y_0, \dots, y_n) . \quad y_j = \sum_{f(i)=j} x_i$$

For  $T \in \text{Top}$ , the singular complex  $\text{Sing } T \in \text{sSet}$  with

$$(\text{Sing } T)_n = \text{Hom}_{\text{Top}}(\Delta_{\text{top}}^n, T)$$

The topological horn is

$$(\Lambda_j^n)_{\text{top}} := \{x \in \Delta_{\text{top}}^n : \exists i \in [n] \setminus j \text{ s.t. } x_i = 0\} \subseteq \Delta_{\text{top}}^n.$$

There exists a continuous retraction  $\Delta_{\text{top}}^n \rightarrow (\Lambda_j^n)_{\text{top}}$   
giving rise to, via Yoneda,

$$\text{Hom}(\Delta_{\text{top}}^n, \text{Sing } T) \rightarrow \text{Hom}(\Lambda_j^n, \text{Sing } T)$$

which is surjective for every horn.

$\Rightarrow \text{Sing } T$  is a Kan complex and thus an  $\infty$ -groupoid.

$$h\text{Sing } T \approx \pi_1 T,$$

$$\pi_0 \text{Sing } T = \{\text{path components of } T\}$$

The  $k$ -skeleton  $Sk_k X$  of  $X \in sSet$  is the subcpx

$$(Sk_k X)_n = \bigcup_{0 \leq j \leq k} \{ \text{yf } | y \in X_j, f: [n] \rightarrow [j] \}$$

$$\rightarrow Sk_{k-1} X \subseteq Sk_k X, \quad X = \bigcup_k Sk_k X, \quad Sk_{n-1} \Delta^n = \partial \Delta^n.$$

Prop. Skeletal filtration: For any  $X \in sSet$ ,

$$\begin{array}{ccc} \bigsqcup_{a \in X_k^{nd}} \partial \Delta^k & \longrightarrow & Sk_{k-1} X \\ \downarrow & & \downarrow \\ \bigsqcup_{a \in X_k^{nd}} \Delta^k & \longrightarrow & Sk_k X \end{array}$$

Geometric realization  $|| - ||: sSet \rightarrow \bar{\text{Top}}$

$$||X|| := \text{Ceq} \left[ \bigsqcup_{\substack{f: [m] \rightarrow [n]}} X_n \times \Delta_{\text{top}}^m \xrightarrow[\substack{(x, f(t)) \\ (f(x), t)}} \bigsqcup_{[P]} X^P \times \Delta_{\text{top}}^P \right]$$

- $|| - ||: sSet \xrightarrow{\perp} \bar{\text{Top}}: \text{Sing}(-)$ ,
- $||\Delta^n|| = \Delta_{\text{top}}^n, \quad ||\partial \Delta^n|| = \partial \Delta_{\text{top}}^n$
- For a group  $G$ ,  $B(-) := ||N(-)||$  the classifying space.

$$\begin{array}{ccc} \bigsqcup_{a \in X_k^{nd}} \partial \Delta_{\text{top}}^k & \longrightarrow & ||Sk_{k-1} X|| \\ \downarrow & & \downarrow \\ \bigsqcup_{a \in X_k^{nd}} \Delta_{\text{top}}^k & \longrightarrow & ||Sk_k X|| \end{array}$$

Thus,  $||X|| = \bigcup_k ||Sk_k X||$  is a CW-cpx whose cells are in bijection of non-deg. cells in  $X$ .

Thank you for listening !

Next time : join , slices ; (co)limits in  $\mathfrak{Cat}$  ;

Joyal lifting .