

iWoAT Summer School on Chromatic Homotopy
Theory and Higher (Infinity-Categorical) Algebra

L4 : 1-category theory of ∞ -categories, II

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August 16th, 2022

Yoneda Lemma, 1-categorically:

Prop. For $F: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ in Cat and any $x \in \text{ob } \mathcal{C}$,

$$\phi_x: \text{Hom}_{\text{Fun}(\mathcal{C}^{\text{op}}, \text{Set})}(\text{Hom}_{\mathcal{C}}(-, x), F) \xrightarrow{\cong} F(x)$$

$$\eta \mapsto \eta(\text{id}_x)$$

$$\alpha_x = (g \mapsto g^* a = ag) \longleftrightarrow a: \varphi_x$$

This prop is proven effortlessly once we have produced a pointwise inverse ψ_x of ϕ_x 1-categorically b/c natural trans between ordinary cats can be completely recovered from its values on objects:

given $\alpha: f_0 \Rightarrow f_1$, s.t. $\alpha(c): f_0(c) \rightarrow f_1(c)$ is iso for every object c , we get $f_0 \Leftarrow f_1: \beta$ just by

$$\beta^{-1}(c) = \alpha(c)^{-1}.$$

There are at least two sources of difficulties when we generalize this proof to ∞ -cats as we saw in L2:

- ① $n=1$: an inverse of edges is NOT necessarily unique
- ② $n > 1$: one also needs to define the functor in higher dimensions!

It is remarkable that we have this "pointwise criterion for isomorphisms" in $\mathfrak{g}\text{-cats}$:

For $C \in \mathfrak{g}\text{-cat}$ and $X \in \mathbf{sSet}$,

an edge in $\text{Fun}(X, C)$ is an iso iff it is a pointwise isomorphism/equivalence.

- What is $\text{Fun}(X, C)$ and is it a $\mathfrak{g}\text{-cat}$?
- This result is a corollary of our big theorem today.

What we also observed 1-categorically:

$\mathcal{P}(C) = \text{Fun}(C^{\text{op}}, \mathbf{Set})$, the presheaf 1-cat on C .

$$\text{Hom}_{\mathcal{P}(C)}(\text{Hom}_C(-, x), \text{Hom}_C(-, y)) \xrightarrow{\cong} \text{Hom}_C(x, y)$$

$$\alpha_* = (g \mapsto g^* a = ag) \longleftrightarrow a$$

Cor. $C \rightarrow \mathcal{P}(C)$ is a full faithful embedding.

$$x \mapsto \text{Hom}_C(-, x)$$

↙ ∞ -cat of
 ∞ -groupoids!

L6, L9: $C \in \text{Cat}_{\infty}$, $\mathcal{P}(C) = \text{Fun}(C^{\text{op}}, \mathfrak{I}) \in \text{Cat}_{\infty}$

$\text{Hom-Sets} \rightsquigarrow \text{mapping spaces}$

$$\text{map}_e(-, -): C^{\text{op}} \times C^{\text{op}} \rightarrow \mathfrak{I}$$

Yoneda Embedding holds for $C \rightarrow \mathcal{P}(C)$.

Given $X, Y \in s\text{Set}$, the function complex $\text{Fun}(X, Y)$ is a simplicial set with

$$\text{Fun}(X, Y)_n = \text{Hom}(\Delta^n \times X, Y)$$

and $S : [m] \rightarrow [n]$ induces

$$\text{Hom}(S \times \text{id}_X, Y) : \text{Hom}(\Delta^n \times X, Y) \rightarrow \text{Hom}(\Delta^m \times X, Y).$$

Vertices are precisely the set of maps $X \rightarrow Y$ in $s\text{Set}$.
morphisms are nat. transformations defined in L2.

$\rightsquigarrow \text{Fun} : s\text{Set}^{\text{op}} \times s\text{Set} \rightarrow s\text{Set}$ is a functor.

$$\Rightarrow \text{Fun}(X \times Y, Z) \xrightarrow{\cong} \text{Fun}(X, \text{Fun}(Y, Z))$$

since $\text{Hom}(X \times Y, Z) \xrightarrow{\cong} \text{Hom}(X, \text{Fun}(Y, Z))$

$$\text{Fun}(X \times Y, Z)_n = \text{Hom}(\Delta^n \times X \times Y, Z)$$

$$\text{Fun}(X, \text{Fun}(Y, Z))_n = \text{Hom}(\Delta^n \times X, \text{Fun}(Y, Z))$$

+ Yoneda lemma

\rightarrow For any $C, D \in \text{Cat}$, $\text{Fun}(NC, ND) \approx N\text{Fun}(C, D)$

$$\begin{aligned} \text{Fun}(NC, ND)_n &= \text{Hom}(\Delta^n \times NC, ND) \\ &= \text{Hom}(N[\{n\}] \times NC, ND) \\ &= \text{Hom}(N(\{n\}) \times C, ND) \\ &= \text{Hom}_{\text{Cat}}(\{n\} \times C, D) \\ &= \text{Hom}_{\text{Cat}}(\{n\}, \text{Fun}(C, D)) \\ &= N\text{Fun}(C, D)_n \end{aligned}$$

Q: When is $\text{Fun}(X, Y) \in \infty\text{-Cat}$?

Lifting properties.

For a category, e.g. Set , which has all small colimits.

a weakly saturated class A is a class of morphisms

which contains all isos and is closed under cobase change, transfinite composition, (coproducts) & retracts.

- cobase change:

$$\begin{array}{ccc} A & \xrightarrow{g} & A' \\ f \downarrow & & \downarrow f' \\ B & \xrightarrow{r} & B' \end{array} \quad f \in A \Rightarrow f' \in A$$
- transfinite composition: for any ordinal λ (well-ordered set) and functor $X: \lambda \rightarrow \text{Set}$, if for every $i \in \lambda$ with $i \neq 0$,

$$\text{colim}_{j < i} X(j) \longrightarrow X(i)$$

is in A , $X(0) \longrightarrow \text{colim}_{j \in \lambda} X(j)$ is in A .

- retract:

$$\begin{array}{ccccc} & & \text{id} & & \\ & \nearrow & & \searrow & \\ A & \xrightarrow{\quad} & A' & \xrightarrow{\quad} & A \\ f \downarrow & & \downarrow g & & \downarrow f \\ B & \xrightarrow{\quad} & B' & \xrightarrow{\quad} & B \\ & \searrow & & \nearrow & \\ & & \text{id} & & \end{array} \quad g \in A \Rightarrow f \in A.$$

Given a class of maps S , its weak saturation \bar{S} is the smallest weakly saturated class containing S .

Given morphisms $f: A \rightarrow B$, $g: X \rightarrow Y$, a lifting problem for (f, g) is a pair of morphisms (u, v) s.t. $vf = gu$. A lift is a morphism s with $sf = u$ and $gs = v$.

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ f \downarrow & \nearrow s & \downarrow g \\ B & \xrightarrow{v} & Y \end{array}$$

- We write $f \square g$ if every lifting problem admits a lift.

$$f \square g \Leftrightarrow \text{Hom}(B, X) \xrightarrow{\text{surjective}} \frac{\text{Hom}(A, X) \times \text{Hom}(B, Y)}{\text{Hom}(A, Y)}$$

- Also write $A \square B$ if $a \square b$ for all $a \in A$, $b \in B$.

$$\text{right complement } A^\square = \{g \mid a \square g \text{ for all } a \in A\}$$

$$\text{left complement } {}^\square A = \{f \mid f \square a \text{ for all } a \in A\}$$

↪ Prop. A^\square , ${}^\square A$ are weakly saturated.

$$\hookrightarrow A \subseteq B \Rightarrow A^\square \supseteq B^\square \quad {}^\square A \supseteq {}^\square B$$

$$\Rightarrow A \subseteq {}^\square(A^\square), \quad A \subseteq ({}^\square A)^\square$$

$$\Rightarrow A^\square \supseteq ({}^\square(A^\square))^\square, \quad A^\square \subseteq ({}^\square(A^\square))^\square$$

$$\Rightarrow A^\square = ({}^\square(A^\square))^\square, \quad {}^\square A = {}^\square(({}^\square A)^\square)$$

Small object argument. Let S be a set of morphisms in $sSet$. Every map f in $sSet$ admits a factorization

$$f = p_j \circ j \in \bar{S}, p \in S^\square.$$

$$\Rightarrow \bar{S} = \partial(S^\square)$$

That is, each S gives a weak factorization system

$$(\bar{S}, S^\square) \text{ for } sSet.$$

Rmk. This also works for \mathcal{P} the class of $0 \rightarrow P$ for P projective in an abelian category \mathcal{C} with enough projectives, where

P projective := LLP w.r.t. epimorphisms

\mathcal{P}^\square = class of epimorphisms

$\bar{\mathcal{P}}$ = monomorphisms with projective cokernel.

$$\text{InnHorn} := \{ A_j^n \subseteq \Delta^n \mid 0 < j < n, n \geq 2 \}$$

$\overline{\text{InnHorn}}$ = class of inner anodynes

$\text{InnFib} := \text{InnHorn}^\square =$ class of inner fibrations

$\Rightarrow (\overline{\text{InnHorn}}, \text{InnFib})$ is a weak factorization system.

$\hookrightarrow C \in \mathfrak{Cat} \Leftrightarrow C \rightarrow * \in \text{InnFib}$

\Leftrightarrow If $A \rightarrow B$ in $\overline{\text{InnHorn}}$, any map

$A \rightarrow C$ in $sSet$ has an extension to some $B \rightarrow C$.

→ Examples . . . $\Lambda_S^n = \bigcup_{i \in S} \Delta^{[n] \setminus i}$, $S \subseteq [n]$

$\Lambda_S^n \hookrightarrow \Delta^n$ generalized horn is in InnFib.

- $C' \subseteq C$ subcat is in InnFib.
- for $C \in \text{gCat}$, $D \in \text{Cat}$, any $(C \rightarrow D) \in \text{InnFib}$.

$$\begin{array}{ccc} \Lambda_j^n & \longrightarrow & C \\ \downarrow & \nearrow \exists ! & \downarrow \\ \Delta^n & \dashrightarrow & ND \\ & \exists ! & \end{array}$$

→ $f: X \rightarrow Y \in \overline{\text{InnFib}}$ $\Rightarrow hf: hX \rightarrow hY$ is an isomorphism.

$$\begin{array}{ccc} X & \longrightarrow & N(hX) \\ \downarrow & \nearrow \exists ! & \downarrow \\ Y & \xrightarrow{\quad} & N(hY) \end{array} \quad N(hf) \in \text{InnFib}$$

also used universal property of $h(-)$.

Boundary of Δ^n : $\partial \Delta^n = \bigcup_{j \in [n]} \Delta^{[n] \setminus j}$ union of codim 1 faces

$$(\partial \Delta^n)_k = \{ f: [k] \rightarrow [n] \mid f[k] \neq [n] \}$$

$\Rightarrow \partial \Delta^0 = \emptyset$, $\partial \Delta^n$ is the largest proper subcpx of Δ^n
(it does not contain $\text{id}_{[n]}$)

Define Cell := $\{ \partial \Delta^n \subseteq \Delta^n \mid n \geq 0 \}$. TrivFib := Cell \hookrightarrow

$\text{Cell} \subseteq \text{monomorphisms}$

where the RHS is weakly saturated.

$\Rightarrow \overline{\text{Cell}} \subseteq \text{monomorphisms}$ Converse?

Prop. $\overline{\text{Cell}} = \text{monomorphisms}$

Proof uses skeletal filtration.

k -skeleton $\text{sk}_k X$ of $X \in \text{sSet}$ is the subcpx

$$(\text{sk}_k X)_n = \bigcup_{0 \leq j \leq k} \{ f(y) \mid y \in X_j, f: [n] \rightarrow [j] \}$$

$$\rightarrow \text{sk}_{k-1} X \subseteq \text{sk}_k X,$$

$$X = \bigcup_k \text{sk}_k X,$$

$$\text{sk}_{n-1} \Delta^n = \partial \Delta^n.$$

Prop. Skeletal filtration:

for any $X \in \text{sSet}$, $A \subseteq X$ subcpx,

$$\bigsqcup_{a \in X_k^{\text{nd}} \setminus A_k^{\text{nd}}} \partial \Delta^k \longrightarrow A \cup \text{sk}_{k-1} X$$

$$\begin{array}{ccc}
 \downarrow & & \downarrow \\
 \bigsqcup_{a \in X_k^{\text{nd}} \setminus A_k^{\text{nd}}} \Delta^k & \xrightarrow{\quad\quad} & A \cup \text{sk}_k X
 \end{array} \tag{*}$$

Cor. $C \rightarrow * \in \text{TrivFib} \Rightarrow C \in \text{Kan}.$

Enriched lifting.

Given $f: A \rightarrow B$, $g: K \rightarrow L$, $h: X \rightarrow Y$

in sets, the pushout-product $f \square g$ is the unique map fitting in

$$\begin{array}{ccc}
 A \times K & \xrightarrow{f_*} & A \times L \\
 f_* \downarrow & & \downarrow \\
 B \times K & \longrightarrow & A \times L \sqcup B \times K \\
 & & \xrightarrow{A \times K} \\
 & & f \square g \\
 & \swarrow & \searrow \\
 & B \times L &
 \end{array}$$

the pullback-hom h^Dg is the unique map fitting in

$$\begin{array}{ccccc}
 \text{Fun}(L, X) & & & & \\
 & \searrow h^Dg & & \swarrow g^* & \\
 & \text{Fun}(L, Y) \times \text{Fun}(K, X) & \longrightarrow & \text{Fun}(K, X) & \\
 & \text{Fun}(K, Y) \perp & & & \\
 & \downarrow & & & \downarrow h_* \\
 \text{Fun}(L, Y) & \xrightarrow{g^*} & \text{Fun}(K, Y) & &
 \end{array}$$

On vertices, h^Dg is $\text{Hom}(L, X) \rightarrow \text{Hom}(L, Y) \times \text{Hom}(K, X)$
 $\text{Hom}(K, Y)$
 $s \mapsto (hs, sg)$

$$\begin{array}{ccc}
 K & \xrightarrow{u} & X \\
 g \downarrow & \nearrow s & \downarrow h \\
 & \dashrightarrow & \\
 & v & Y
 \end{array}$$

Thus h^Dg is surjective on vertices
iff $g \otimes h$. So it is an "enriched"
lifting problem of $g \otimes h$.

Adjunction of lifting problems

Prop. $(f \square g) \square h \Leftrightarrow f \square (h^0 g)$.

$$\begin{array}{ccc}
 A \times L \sqcup B \times K & \xrightarrow{(h,v)} & X \\
 \downarrow A \times K & \nearrow s & \downarrow h \\
 B \times L & \xrightarrow{w} & Y
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \xrightarrow{\tilde{u}} & \text{Fun}(L, X) \\
 \downarrow f & \nearrow \tilde{s} & \downarrow h^0 g \\
 B & \xrightarrow{(\tilde{w}, \tilde{v})} & \text{Fun}(L, Y) \times \text{Fun}(K, X) \\
 & & \text{Fun}(K, Y)
 \end{array}$$

In particular, when $K = \emptyset$, $Y = *$, $f \square g = f \times L$

$$\begin{aligned}
 & (A \times L \xrightarrow{f \times L} B \times L) \square (X \longrightarrow *) \\
 \Leftrightarrow & (A \xrightarrow{f} B) \square (\text{Fun}(L, X) \longrightarrow *)
 \end{aligned}$$

→ Let $C := \text{Fun}([1], \text{sSet})$ be the arrow category of sSet .
 Then $(\square, \phi \subseteq \Delta^0)$ defines a symm. monoidal structure
 on C and $(-) \square g \rightarrow (-)^0 g$

Prop. For any set V of morphisms in sSet , $V\text{Fib} := V^\square$

Then $S \square T \subseteq \bar{u}$ implies $\bar{S} \square \bar{T} \subseteq \bar{u} = \bar{V}\text{Fib}$,

$$\text{Cor. } \bar{S} \square \text{UFib}^{\square \bar{T}} \Rightarrow \text{UFib}^{\square \bar{T}} \subseteq S\text{Fib}$$

$$\bar{T} \square \text{UFib}^{\square \bar{S}} \Rightarrow \text{UFib}^{\square \bar{S}} \subseteq S\text{Fib}$$

Prop. $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}} ; \quad \overline{\text{Cell}} \square \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$

$$\text{Cor. } \text{InnFib}^{\square \overline{\text{Cell}}} \subseteq \text{InnFib}$$

$$\text{InnFib}^{\square \overline{\text{InnHorn}}} \subseteq \text{TrivFib} ; \quad \text{TrivFib}^{\square \overline{\text{Cell}}} \subseteq \text{TrivFib}.$$

Thm. For $C \in \text{qCat}$, $L \in \text{sSet}$, $\text{Fun}(L, C) \in \text{qCat}$.

Proof.

$$\begin{array}{c} \overline{\text{InnHorn}} \sqcap \overline{\text{Cell}} \\ \Downarrow \\ (\Lambda_j^n \xrightarrow{f} \Delta^n) \sqcap (\emptyset \rightarrow L) \sqcap (C \rightarrow *) \\ \Leftrightarrow (\Lambda_j^n \xrightarrow{f} \Delta^n) \sqcap (\text{Fun}(L, C) \rightarrow *). \end{array}$$

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Restriction along $j: A \rightarrow B$ an inner anodyne or monomorphism induces trivial fibration or inner fibration of functor qcats:

$$j^* = (p: C \rightarrow *)^{\square j}: \text{Fun}(A, C) \rightarrow \text{Fun}(B, C).$$

Another characterization of a qcat.

We saw $\overline{\text{InnHorn}} \sqcap \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$; in fact,

$$\overline{\{\Lambda_1^2 \subseteq \Delta^2\} \sqcap \text{Cell}} = \overline{\text{InnHorn}}$$
 due to Joyal.

Cor. $C \in \text{sSet}$ is an ∞ -cat.

$$\Leftrightarrow f = (C \rightarrow *)^{\square \{\Lambda_1^2 \subseteq \Delta^2\}}: \text{Fun}(\Delta^2, C) \rightarrow \text{Fun}(\Lambda_1^2, C)$$

is a trivial fibration.

Proof. For all $n \geq 0$

$$\begin{aligned} & (\partial \Delta^n \rightarrow \Delta^n) \sqcap (\Lambda_1^2 \rightarrow \Delta^2) \sqcap (C \rightarrow *) \\ \Leftrightarrow & (\partial \Delta^n \rightarrow \Delta^n) \sqcap f \end{aligned}$$

Thus

$$\begin{aligned} & (C \rightarrow *) \in \text{InnFib} \\ \Leftrightarrow & (C \rightarrow *) \in (\{\Lambda_1^2 \subseteq \Delta^2\} \sqcap \text{Cell})^\square \\ \Leftrightarrow & f \in \text{TrivFib} = \text{Cell}^\square. \end{aligned}$$

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Composition revisited.

We saw in talk 1 that composition is unique up to homotopy:

$$[g] \circ [f] = [gf] \text{ is well-defined.}$$

Now we can do better.

- Every trivial fibration admits a section.

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & \nearrow s \text{ - } \dashrightarrow & \downarrow h \\ Y & \xrightarrow{\text{id}} & Y \end{array}$$

↙ spine

- $I^2 = \Delta_1^2 \rightarrow \Delta^2 \in \text{InnHorn}$ induces trivial fibration r_* in functor cats for which we pick a section s fitting in

$$\text{Fun}(\Delta_1^2, C) \xrightleftharpoons[r_*]{s} \text{Fun}(\Delta^2, C) \xrightarrow{r'} \text{Fun}(\Delta^{(0,2)}, C)$$

$\Rightarrow sr'$ is a composition functor.

↙ spine

- n-fold composition functor: $I^n \rightarrow \Delta^n \in \text{InnHorn}$ gives rise to

$$\text{Fun}([1], C) \times \text{Fun}([1], C) \times \cdots \times \text{Fun}([1], C)$$

C C C
 ||

$$\text{Fun}(I^n, C)$$

$$\begin{array}{c} r_* \uparrow \\ \Bigg\} s \\ \text{Fun}(\Delta^n, C) \xrightarrow{r'} \text{Fun}(\Delta^{(0,n)}, C) \end{array}$$

Rank: This definition is not unique since s is not. But we will see all such functors are "naturally isomorphic".

Categorical Equivalences "the correct notion of equivalences for qcats"

- A categorical inverse to $f: C \rightarrow D$ of qcats is a $g: D \rightarrow C$ s.t. gf is naturally isomorphic to 1_C and fg is naturally isomorphic to 1_D .
- A map $f: X \rightarrow Y$ in $sSet$ is a categorical equivalence if for every ∞ -cat E ,
 $f^*: \text{Fun}(Y, E) \rightarrow \text{Fun}(X, E)$ of qcats
admits a categorical inverse.

Prop. For $f: C \rightarrow D$ of qcats, f admits a categorical inverse iff it is a categorical equivalence.

Cor. For $f: X \rightarrow Y$ in $sSet$, TFAE:

- 1) f is a categorical equivalence;
- 2) for any qcat C ,
 $hf^*: h\text{Fun}(Y, C) \rightarrow h\text{Fun}(X, C)$
is an equivalence of 1-cats.
- 3) for any qcat C ,
 $\pi_0 \text{Fun}(Y, C)^\simeq \rightarrow \pi_0 \text{Fun}(X, C)^\simeq$
is a bijection.

Rank. Prop lets you reduce 3) \Rightarrow 1) to f between qcats.

3) is the definition Lurie uses for cat equiv in kerodon.

Prop. $\text{TrivFib} \subseteq \text{CatEq}$. Cor. $\overline{\text{InnFib}} \subseteq \text{CatEq}$.

Fix $p: X \rightarrow S \in \text{TrivFib}$.

① If $S = *$, p is a categorical equivalence exhibiting X as a contractible (π_0 singleton) Kan cpx.

② Recall for any $Y \in \text{sSet}$,

$$p_* = p^{\square(\emptyset \subset Y)}: \text{Fun}(Y, X) \longrightarrow \text{Fun}(Y, S)$$

is also a trivial fibration. (Same holds for InnFib .)

③ In the two pullbacks in sSets , vertical maps on the left

parametrizes sections of p $\text{Fun}_S(S, X) \rightarrow \text{Fun}(S, X)$

$$\begin{array}{ccc} \text{CatEq} & \xrightarrow{\quad} & p_* \in \text{TrivFib} \\ \downarrow & & \downarrow \\ \{\text{id}\} & \longrightarrow & \text{Fun}(S, S) \end{array}$$

$$sp \approx \text{id}_X \text{ in } \text{Fun}_S(X, X) \rightarrow \text{Fun}(X, X)$$

$$\begin{array}{ccc} \text{CatEq} & \xrightarrow{\quad} & p_* \in \text{TrivFib} \\ \downarrow & & \downarrow \\ \{p\} & \longrightarrow & \text{Fun}(X, S) \end{array}$$

are contractible ∞ -groupoids by ①.

$\Rightarrow p: X \rightarrow S \in \text{CatEq}$.

□

Rank : If p is a trivial fib between \mathfrak{scats} , any section is its categorical inverse and any two sections are nat. isomorphic, living in a contractible Kan cpx.

\Rightarrow Composition is unique up to a contractible space of choices.

Examples of cat. equiv.

F = free monoid on one generator g .

obj set is $\{g\}$, morphism set is $\{g^n \mid n \geq 0\}$.

$$\Rightarrow (NF)_d = \{(g^{m_1}, \dots, g^{m_d}) \mid m_i \geq 0\}$$

The map $\gamma: S^1 = \Delta^1 / \partial \Delta^1 \rightarrow NF$ is in $\overline{\text{InnHorn}}$.
 Simplicial circle $\xrightarrow{\sim} \overline{\langle 01 \rangle} \mapsto g$

(proof is an explicit computation using skeletal filtration.)

→ NF is the free monoid generated as a qCat by S^1 .

⇒ Enriched lifing + $N(-)$ monoidal gives

$\gamma^{xn}: (S^1)^{xn} \rightarrow N(F^{xn})$ is in $\overline{\text{InnHorn}} \subseteq \text{CatEq}$.

~> free comm monoid generated as a qCat.

Join

For $A, B \in \text{Cat}$, the join $A * B$ is the category

$$\text{ob}(A * B) = \text{ob}A \sqcup \text{ob}B$$

$$\text{mor}(A * B) = \text{mor}A \sqcup (\text{ob}A \times \text{ob}B) \sqcup \text{mor}B$$

with

$$\text{Hom}_{A * B}(x, y) = \begin{cases} \text{Hom}_{A * B}(x, y) & x, y \in \text{ob}A \\ \text{Hom}_{A * B}(x, y) & x, y \in \text{ob}B \\ \{\ast\} & x \in \text{ob}A \quad y \in \text{ob}B \\ \emptyset & x \in \text{ob}B \quad y \in \text{ob}A \end{cases}$$

left cone : $A^\diamond = [0] * A$ freely adjoins an initial obj.

right cone : $A^\diamond = A * [0]$ freely adjoins a terminal obj.

Ordered disjoint union

$$\sqcup : \Delta \times \Delta \longrightarrow \Delta, \quad [p] \sqcup [q] = [p+1+q] \\ (0, 1, \dots, p, \overset{\leftarrow}{0}, \dots, q)$$

We extend Δ to $\Delta_+ = \Delta^\diamond$ by adjoining $[-1] := \emptyset$

Thus $(\sqcup, [-1])$ is a monoidal structure on Δ_+ .

For each map $f : [p] \rightarrow [q_1] \sqcup [q_2]$ in Δ_+ , there's a unique decomposition $[p] = [p_1] \sqcup [p_2]$ s.t. f (uniquely) decomposes as

$$f = f_1 \sqcup f_2, \quad f_i : [p_i] \rightarrow [q_i].$$

For $X, Y \in \text{sSet}$, join $X * Y$ is the simplicial set with

$$(X * Y)_n := \bigsqcup_{[n] = [n_1] \sqcup [n_2]} X_{n_1} \times Y_{n_2}$$

over $[n_i] \in \text{ob } \Delta^+$, and we set $X_{-1} = * = Y_{-1}$

For $(x, y) \in X_{n_1} \times Y_{n_2} \subseteq (X * Y)_n$ and $f : [m] \rightarrow [n]$,

$$(x, y)f = (x f_1, y f_2),$$

where $f = f_1 \sqcup f_2$, $f_i : [m_i] \rightarrow [n_i]$ is the unique decomposition of f over $[n] = [n_1] \sqcup [n_2]$.

e.g. $(X * Y)_0 = X_0 \sqcup Y_0$

$$(X * Y)_1 = X_1 \sqcup (X_0 \times Y_0) \sqcup Y_1$$

$$(X * Y)_2 = X_2 \sqcup (X_1 \times Y_0) \sqcup (X_0 \times Y_1) \sqcup Y_2$$

$$\rightsquigarrow X \rightarrow X * Y \leftarrow Y \text{ subcpx}.$$

$\hookrightarrow (\star, \emptyset =: \Delta^{-1})$ is a monoidal structure on sSet and $X * -$, $- * X$ preserve pushout.

$$\hookrightarrow \Delta^p * \Delta^q = \Delta^{p+1+q}$$

$$\underline{\text{left cone}} : A^\diamond = \Delta^0 * A \quad \underline{\text{right cone}} : A^\diamond = A * \Delta^0$$

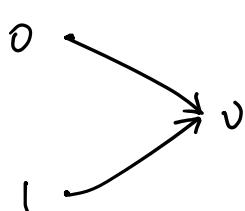
$$\rightsquigarrow N(A * B) = N(A) * N(B)$$

for $B = [0]$,

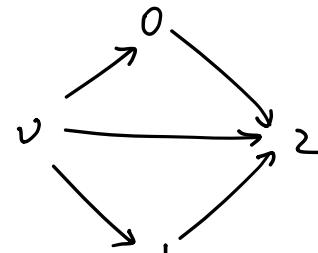
$$\Rightarrow N(A^\diamond) = (NA)^\diamond, \quad N(A^\diamond) = (NA)^\diamond$$

$$\text{Example : } (\partial \Delta^n)^\Delta = \Delta^0 * \partial \Delta^n \approx \Lambda_0^{n+1}$$

$$(\partial \Delta^n)^\Delta = \partial \Delta^n * \Delta^0 \approx \Lambda_{n+1}^{n+1}$$



$$(\partial \Delta^1)^\Delta = \Lambda_2^2$$



$$(\Lambda_2^2)^\Delta = \Delta^1 \times \Delta^1 = \Delta^2 \cup \Delta^2$$

Lemma. Maps $f: K \rightarrow X * Y$ of sets are in bijection with the triples

$$(\pi: K \rightarrow \Delta^1, f_{\{0\}}: K^{\{0\}} \rightarrow X, f_{\{1\}}: K^{\{1\}} \rightarrow Y)$$

$$\text{where } K^{\{j\}} = \pi^{-1}(\{j\}).$$

Prop. Join of qcats is a qcat.

Proof. Use the lemma and $\Lambda_j^n \rightarrow \Delta^1$ is either const at 0, or const at 1, or nonconst.

$\Rightarrow \Lambda_j^n \rightarrow X * Y$ factors as

$$\Lambda_j^n \rightarrow X * \Delta^{-1} \rightarrow X * Y$$

$$\text{or } \Lambda_j^n \rightarrow \Delta^{-1} * Y \rightarrow X * Y$$

$$\text{or } \Lambda_j^n \rightarrow \Delta^k * \Delta^{n-1-k} \rightarrow X * Y.$$

□

Slices

↙ coslice

For $p: S \rightarrow X$, $X_{p/}$ is the slice-under simplicial set
with

$$(X_{p/})_n = \text{Hom}_{\text{sSet}_S/} (S * \Delta^n, X)$$

For $q: T \rightarrow X$, $X_{/q}$ is the slice-over simplicial set
with

$$(X_{/q})_n = \text{Hom}_{\text{sSet}_T/} (\Delta^n * T, X)$$

Notices that $X * -$, $- * X$ do not preserve colimits

b/c $X * \phi = X \neq \phi$, $\phi * X = X \neq \phi$. But

Prop. For every $X \in \text{sSet}$,

$$X * -, - * X : \text{sSet} \longrightarrow \text{sSet}_{X/}$$

preserve colimits and they have right adjoints

$$(p: S \rightarrow X) \longmapsto X_{p/} : \text{sSet}_{S/} \longrightarrow \text{sSet}$$

$$(q: T \rightarrow X) \longmapsto X_{/q} : \text{sSet}_{T/} \longrightarrow \text{sSet}$$

In general, we have bijective correspondences, known as
the "join/slice adjunction"

$$\left\{ \begin{array}{ccc} S * \phi = S & \xrightarrow{p} & X \\ \downarrow & \nearrow & \\ S * K & \dashrightarrow & \end{array} \right\} \iff \{ K \dashrightarrow X_{p/} \}$$

$$\left\{ \begin{array}{ccc} \phi * T = T & \xrightarrow{q} & X \\ \downarrow & \nearrow & \\ K * T & \dashrightarrow & \end{array} \right\} \iff \{ K \dashrightarrow X_{/q} \}$$

- Nerve preserves slices: for $p: A \rightarrow C$ in Cat ,
 $N(C_{p/}) \simeq NC_{Np/}, N(C_{/p}) \simeq NC_{/Np}$

- Functionality of slice**: given $T \xrightarrow{j} S \xrightarrow{P} X \xrightarrow{f} Y$

\rightsquigarrow

$$\begin{array}{ccc} X_{/p} & \longrightarrow & Y_{/fp} \\ \downarrow & & \downarrow \\ X_{/pj} & \longrightarrow & Y_{/fpj} \end{array} \quad \text{commutative square.}$$

$$\begin{array}{ccccc} T & \xrightarrow{j} & S & \xrightarrow{P} & X \xrightarrow{f} Y \\ \downarrow & & \downarrow & & \nearrow \tilde{u} \\ K \star T & \xrightarrow{K \star j} & K \star S & & \end{array} \quad \begin{array}{l} \text{slice-under} \\ \downarrow \\ T \star K, S \star K \end{array}$$

$u: K \rightarrow X_f$ restricting to $K \xrightarrow{u} X_f \rightarrow X_{pfj}$
 corresponds to \tilde{u} restricting to $P \tilde{u} (K \star j)$.

Letting $K = \Delta^n$ in the slice-over sets, we get assignments

on n -cells: id_X

$$\bullet \quad \emptyset \xrightarrow{j} S \xrightarrow{P} X \xrightarrow{f} X$$

$$\rightsquigarrow \quad X_{/p} \longrightarrow X$$

$$\tilde{x}: \Delta^n \star S \rightarrow X \mapsto \tilde{x} | \Delta^0 \star \emptyset$$

$$\bullet \quad \emptyset \xrightarrow{j} S \xrightarrow{P} X \xrightarrow{f} Y$$

$$\rightsquigarrow \quad X_f \longrightarrow Y_{/fp}$$

$$\tilde{x}: \Delta^n \star S \rightarrow X \mapsto f \tilde{x}: \Delta^n \star S \rightarrow Y$$

Slice categories of a qcat is a qcat.

Lifing properties of join/slices

- Given $i: A \rightarrow B$, $j: K \rightarrow L$, the pushout-join is $i \boxtimes j : (A * L) \coprod_{A * K} (B * K) \longrightarrow B * L$ induced by $(i * L, B * j)$.
- Note. \boxtimes is not symm since $*$ is not.
- For $0 \leq j \leq n$,

$$(\Lambda_j^n \subset \Delta^n) \boxtimes (\phi \subset \Delta^0) \approx (\Lambda_j^{n+1} \subset \Delta^{n+1})$$

$$(\phi \subset \Delta^0) \boxtimes (\Lambda_j^n \subset \Delta^n) \approx (\Lambda_{i+j}^{1+n} \subset \Delta^{1+n})$$

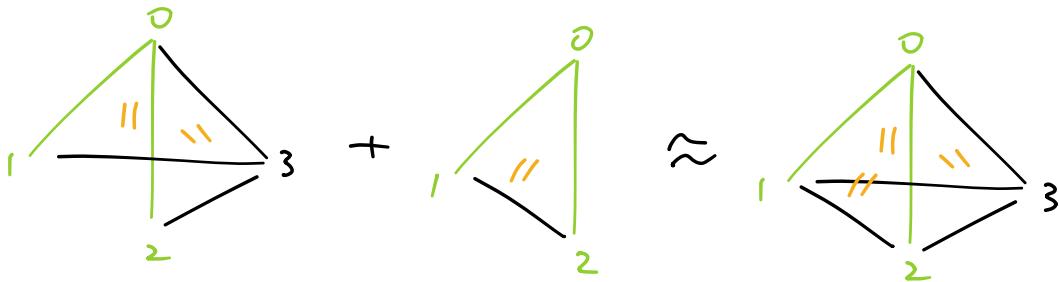
$$(\partial \Delta^n \subset \Delta^n) \boxtimes (\phi \subset \Delta^0) \approx (\partial \Delta^{n+1} \subset \Delta^{n+1})$$

$$(\phi \subset \Delta^0) \boxtimes (\partial \Delta^n \subset \Delta^n) \approx (\partial \Delta^{1+n} \subset \Delta^{1+n})$$

e.g. $(\Lambda_0^2 \subset \Delta^2) \boxtimes (\phi \subset \Delta^0)$ is the map

$$\Lambda_0^2 * \Delta^0 \sqcup \Delta^2 * \phi \approx \Lambda_0^3 \longrightarrow \Delta^3$$

$\Lambda_0^2 * \phi$



For general K ,

$$(\Lambda_j^n \subset \Delta^n) \boxtimes (\partial \Delta^k \subset \Delta^k) \approx (\Lambda_j^{n+1+k} \subset \Delta^{n+1+k})$$

$$(\partial \Delta^k \subset \Delta^k) \boxtimes (\Lambda_j^n \subset \Delta^n) \approx (\Lambda_{k+1+j}^{k+1+n} \subset \Delta^{k+1+n})$$

$$(\partial \Delta^n \subset \Delta^n) \boxtimes (\partial \Delta^k \subset \Delta^k) \approx (\partial \Delta^{n+1+k} \subset \Delta^{n+1+k})$$

Given $T \xrightarrow{i} S \xrightarrow{p} X \xrightarrow{f} Y$,

$$\begin{array}{ccc} X_{p/} & \longrightarrow & Y_{fp/} \\ \downarrow & & \downarrow \\ X_{pj/} & \longrightarrow & Y_{fpj/} \end{array}$$

commutes.

The pullback slices are

$$f^{\boxtimes p}: X_{p/} \longrightarrow X_{pj/} \times_{Y_{fp/}} Y_{fpj/}, \quad f^{j \boxtimes p}: X_{jp/} \longrightarrow X_{pj/} \times_{Y_{fp/}} Y_{fpj/}$$

Prop. $(i \boxtimes j) \square f \Leftrightarrow i \square (f^{\boxtimes g})^j$ for all $g: L \longrightarrow X$
 $\Leftrightarrow j \square (f^{i \boxtimes p})^g$ for all $p: B \longrightarrow X$

Proof for first \Leftrightarrow :

$$\begin{array}{ccccc} & q & & & \\ & \swarrow & & \searrow & \\ \emptyset * L & \longrightarrow & A * L \sqcup B * K & \xrightarrow{(u,v)} & X \\ & & A * K & & \\ & i \boxtimes j & \downarrow & \nearrow s & \downarrow f \\ & B * L & \xrightarrow{w} & Y & \end{array}$$

$$\begin{array}{ccccc} A & \xrightarrow{\tilde{u}} & X_{/g} & & \\ i \downarrow & \nearrow \tilde{s} & \downarrow f^{\boxtimes g} & & \\ B & \xrightarrow{(\tilde{v}, \tilde{w})} & X_{/gj} \times_{Y_{/hg}} Y_{/hj} & & \end{array} .$$

□

Prop. $\overline{\text{RHorn}} \boxtimes \overline{\text{Cell}} \subseteq \overline{\text{InHorn}} ; \quad \overline{\text{Cell}} \boxtimes \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$
 $\overline{\text{Cell}} \boxtimes \overline{\text{LHorn}} \subseteq \overline{\text{InHorn}} ;$

Cor. Join preserves monomorphisms.

For $i: A \rightarrow B$ in $\overline{\text{Cell}}$, factor $S * i$ as

$$S * A \rightarrow S * A \cup \emptyset * B \xrightarrow{(\phi \subseteq S) * i} S * B$$

$\underbrace{\quad}_{\emptyset * A}$

cobase change of a mono \Rightarrow mono. \square

Cor. Slice preserves trivial fibrations.

$$X_{p/} \xrightarrow{f \boxtimes_p (\phi \subseteq S)} X \times_Y Y_{fp/} \xrightarrow{\text{base change of trivial fib}} Y_{fp/}$$

for $X \xrightarrow{f} Y$ trivial fib. \square

Cor. Given $T \xrightarrow{j} S \xrightarrow{P} C$ with $C \in \text{qCat}$ (so $f: C \rightarrow *$ is an inner fibration.) We have pullback slices

$$\ell = f \boxtimes_P j: C_{p/} \longrightarrow C_{pj/}, \quad r = f^j \boxtimes_P: C_{/p} \longrightarrow C_{pj}$$

1) $j \in \overline{\text{Cell}} \Rightarrow \ell \in L\tilde{\text{fib}}, r \in R\tilde{\text{fib}}$

1') $T = \emptyset, (\ell: C_{p/} \longrightarrow C) \in L\tilde{\text{fib}} \subseteq \text{Triv}\tilde{\text{fib}}$

$(r: C_{/p} \longrightarrow C) \in R\tilde{\text{fib}} \subseteq \text{Triv}\tilde{\text{fib}}$

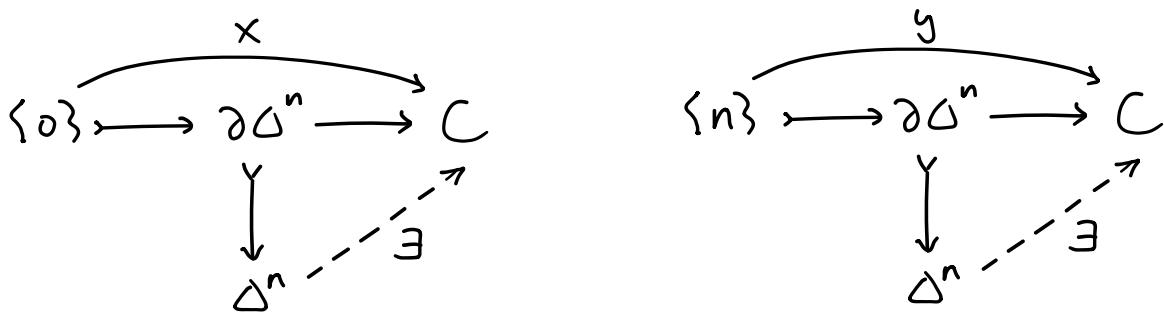
$$\begin{array}{ccc} \Lambda_j^n & \longrightarrow & C_{p/} \\ \downarrow & \nearrow \exists & \downarrow \in \text{Triv}\tilde{\text{fib}} \\ \Delta^n & \dashrightarrow & C \\ & \exists & \end{array}$$

\Rightarrow Slices of a qcat is a qcat!

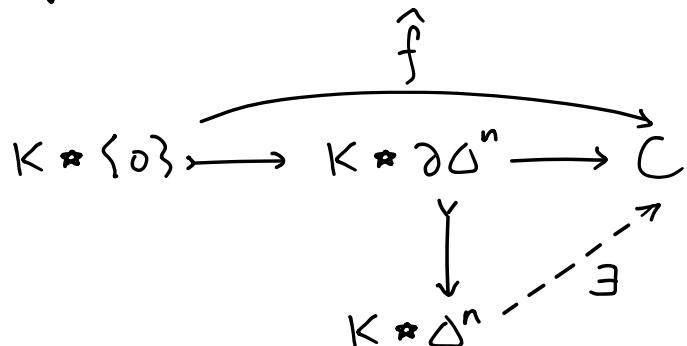
Colimits & Limits

An initial object of a gCat C is an $x \in C_0$ s.t. every $f : \partial\Delta^n \rightarrow C$ ($n \geq 1$) with $f|\{\circ\} = x$ admits an extension to $f' : \Delta^n \rightarrow C$.

A terminal object is an initial object in C^{op} .



Given $f : K \rightarrow C$ for C a gCat, a colimit of f is an initial object of the slice gCat C_f . That is, a colimit $\hat{f} : K^{\triangleright} \rightarrow C$ extends f and lift exists in every diagram (for $n \geq 1$) of the form



\hat{f} is also referred to as a colimit cone of f , with cone point $\hat{f}|_{K * \{\circ\}}$.

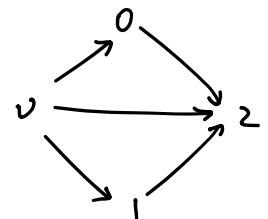
Similarly, a limit of f is a terminal object of C_f .

Example. ① For $K = \emptyset$, $f: \emptyset \rightarrow C$ for C a gCat.

$\Rightarrow C_{/\emptyset} = C$ and a colimit of f is precisely an initial object of C .

② $K = \Delta_2^2$; $K^\Delta \approx \Delta^1 \times \Delta^1 \rightarrow C$

is a pullback diagram in C .



Prop. $x \in C_0$ is initial $\Leftrightarrow (C_{/x} \rightarrow C) \in \text{TrivFib}$

$x \in C_0$ is terminal $\Leftrightarrow (C_{/x} \rightarrow C) \in \text{TrivFib}$.

Proof. For all $n \geq 0$,

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{f} & C_{/x} \\ \downarrow & \nearrow & \downarrow \pi \\ \Delta^n & \xrightarrow{g} & C \end{array}$$

$$\begin{array}{ccc} \Delta^0 * \phi & \xrightarrow{\phi * \Delta^n \sqcup \Delta^0 * \partial\Delta^n} & C \\ & \xrightarrow{\phi * \partial\Delta^n} & \\ & i \otimes j \downarrow & \\ & \Delta^0 * \Delta^n & \end{array}$$

$\xrightarrow{(\tilde{f}, g)}$

where $(i: \phi \subset \Delta^0) \boxtimes (j: \partial\Delta^n \subset \Delta^n) \approx (\partial\Delta^{1+n} \subset \Delta^{1+n})$.

$$\begin{array}{ccc} \{0\} & \xrightarrow{x} & \partial\Delta^{1+n} \rightarrow C \\ \Leftrightarrow & & \downarrow \\ & & \Delta^{1+n} \dashrightarrow \exists \end{array}$$

□

Cor. Let $\tilde{f}: K^\Delta \rightarrow C$ for C a gCat and write $f := \tilde{f}|_K$

then \tilde{f} is a colimit iff $(C_{/\tilde{f}} \rightarrow C_{/f}) \in \text{TrivFib}$;

$\tilde{f}: K^\Delta \rightarrow C$ is a limit iff $(C_{/\tilde{f}} \rightarrow C_{/f}) \in \text{TrivFib}$.

Proof. $C_{/\tilde{f}} = (C_{/f})_{\tilde{f}}$, where $\tilde{f} \in (C_{/f})_0$. □

Prop. For $C \in \text{Cat}$, let C^{init} and C^{term} denote respectively the full subcat spanned by initial objs & terminal objs. Then each of C^{init} and C^{term} are either empty or categorically equivalent to Δ^0 . That is, initial and terminal objs are unique to unique isomorphism.

Proof. • $C^{\text{term}} = ((C^{\text{op}})^{\text{init}})^{\text{op}}$.

• For $n \geq 1$, $(\partial\Delta^n)_0 = (\Delta^n)_0 \neq \emptyset$.

For any $\partial\Delta^n \xrightarrow{f} C^{\text{init}}$,

$$\begin{array}{ccccc} & f|_{\{\partial\}} & & & \\ \swarrow & & \searrow & & \\ \{\partial\} & \longrightarrow & \partial\Delta^n & \xrightarrow{f} & C^{\text{init}} \longrightarrow C \\ & \downarrow & \exists \dashrightarrow & \dashrightarrow & \exists \dashrightarrow \\ & \Delta^n & & & \end{array}$$

If $C^{\text{init}} \neq \emptyset$, extn exists also for $n=0$.

Thus $(C^{\text{init}} \longrightarrow *) \in \text{TrivFib}$, so C^{init} is a contractible Kan Cpx. \square

Cor. colimits and limits are unique.

Rmk. Need Joyal lifting for some other properties of initial & terminal objs.

Joyal extension & lifting theorem !!!

Joyal extension theorem. Let $C \in \mathfrak{g}\text{Cat}$ and $f \in C_1$. TFAE.

- 1) The edge represented by f is an isomorphism.
- 2) Every $a: \Delta_0^n \rightarrow C$ with $n \geq 2$ s.t. $f = a|_{\Delta^{[0,1]}_0}$ admits an extension to a map $\Delta^n \rightarrow C$
- 3) Every $b: \Delta_n^n \rightarrow C$ with $n \geq 2$ s.t. $f = b|_{\Delta^{[n-1,n]}_n}$ admits an extension to a map $\Delta^n \rightarrow C$

Special case of :

Joyal lifting theorem. Let $p: C \rightarrow D$ be an inner fibration between $\mathfrak{g}\text{Cats}$ and $f \in C_1$ with $p(f)$ being an isomorphism in D . TFAE.

- 1) The edge f is an isomorphism in C .
- 2) For all $n \geq 2$, lift exists in any

$$\begin{array}{ccccc} & f & & & \\ \Delta^{[0,1]}_0 & \longrightarrow & \Delta_0^n & \longrightarrow & C \\ \downarrow & & \exists \dashrightarrow & \downarrow p & \\ \Delta^n & \longrightarrow & D & & \end{array} .$$

- 3) For all $n \geq 2$, lift exists in any

$$\begin{array}{ccccc} & f & & & \\ \Delta^{[n-1,n]}_n & \longrightarrow & \Delta_n^n & \longrightarrow & C \\ \downarrow & & \exists \dashrightarrow & \downarrow p & \\ \Delta^n & \longrightarrow & D & & \end{array} .$$

Corollaries of Joyal's theorems!

Exe. Every simplicial set has extension property against 1-dim'l horns $\Delta_j^1 \rightarrow \Delta^1$.

Theorem. An ∞ -groupoid is a Kan cpx.

Prop. If $f: x \rightarrow y$ edge in $C \in \text{qCat}$ is an isomorphism,

$$\begin{array}{ccc} C/x & \xleftarrow{\quad\pi\quad} & C/f & \xrightarrow{\quad\pi\quad} & C/y \\ \text{TrivFib} & & & & \text{TrivFib by Joyal} \\ \text{basic lifting along} \\ \Delta^0 & \xrightarrow{\quad\pi\quad} & C & \xrightarrow{\quad f \quad} & C \\ \text{Lhorn} & & & & \end{array}$$

Both are in CatEquiv and so C/x and C/y are categorically equivalent.

Note: The composition functor can be thought of realizing

$$(C \xrightarrow{g} x) \mapsto (C \xrightarrow{gf} y)$$

$$\begin{matrix} \uparrow \\ C/x \end{matrix} \qquad \qquad \begin{matrix} \uparrow \\ C/y \end{matrix}$$

Thm. Invariance of initial / terminal obj.

Let $f \in C_1$ for C a qcat and $\tilde{f} \in (C_{x/})_0$ correspond to f .

\tilde{f} initial in $C_{x/} \Leftrightarrow f$ is an isomorphism.

Thank you for listening !