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What condition we need to have for Y, s.t. $Y \simeq \Omega X$?

Y ?? $Y = \Omega X \Leftarrow B Y$

Thm

Y conn. based. $Y \simeq \Omega X$ for some X iff Y is an A_∞-space.

Def:

An A_∞-structure on a space X consist of an n-tuple of Maps

$$X = E_1 \subset E_2 \subset \cdots \subset E_n$$
$$\downarrow P_1 \quad \downarrow P_2 \quad \cdots \quad \downarrow P_n$$
$$* = B_1 \subset B_2 \subset \cdots \subset B_n$$

s.t. $\pi_*(E_i, X) \simeq \pi_*(B_i)$, $\forall \, 1 \le i \le n$ $1 \le n \le \infty$

$h(-, 1) = id$

$h(-, 0) = *$

$\uparrow$

$X \to E_i \to B_i$ together with a contracting homotopy $h: CE_{n-1} \to E_n$

s.t. $h(CB_{i-1}) \subset B_i$, $\forall i$.

Associahedra ("stasheff operad")

cell complex K_i ($\simeq I^{i-2}$)

$\partial K_i = L_i \qquad \frac{i(i-1)}{2} - 1$

$$X_1 \quad \cdots \quad X_i \qquad\qquad \text{Cell} = \text{Image}\,(k_r \times k_s)\,,\quad S = i+1-r$$

$$X_1 \cdots (X_k \cdots X_{k+i-1})\, X_{i+k} \cdots X_i \qquad \partial_k(r,s): k_r \times k_s \to k_{r+s-1}$$

Given a way

$$X_1 \cdots (X_k\, X_{k+1} \cdots X_{k+r-1})\, X_{k+r} \cdots X_i \;\rightsquigarrow\; \text{cell} \subset L_i = \partial k_i$$

$$\|$$

$$k_r \times k_s\,.\quad S = i+1-r$$

Satisfying

$$
\begin{array}{ccc}
k_r \times k_s \times k_t & \xrightarrow{\;1 \times \partial(s,t)\;} & k_r \times k_{s+t-1} \\
\Big\downarrow{\scriptstyle \partial(r,s)\times 1} & & \Big\downarrow \\
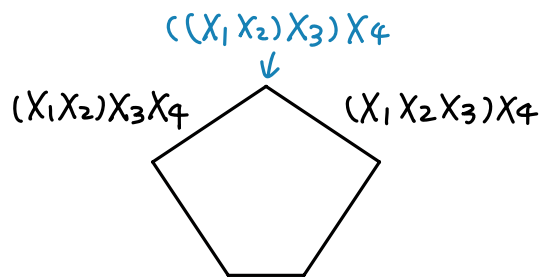k_{r+s-1} \times k_t & \xrightarrow{\hspace{2cm}} & k_{r+s+t-2}
\end{array}
$$

$$
\begin{array}{ccccc}
k_r \times k_s \times k_t & \xrightarrow{\;1\times T\;} & k_r \times k_t \times k_s & \xrightarrow{\;1 \times \partial(t,s)\;} & k_r \times k_{t+s-1} \\
\Big\downarrow{\scriptstyle \partial(r,s)} & & & & \Big\downarrow \\
k_{r+s-1} \times k_t & & \xrightarrow{\hspace{2cm}} & & k_{r+t+s-2}
\end{array}
$$

$$i = 1 \quad \checkmark$$

$$i = 2,\quad k_2 = *$$

$$i = 3,\quad k_3 = \dfrac{X_1(X_2 X_3)}{(X_1 X_2) X_3} = \mathbb{I}$$

$i=4$, $X_1 X_2 X_3 X_4$, $\binom{4}{2} - 1 = 5$

$$((X_1 X_2) X_3) X_4$$

$(X_1 X_2) X_3 X_4$ $\qquad$ $(X_1 X_2 X_3) X_4$

$(X_1 X_2) X_3 X_4$ $\qquad\qquad$ $(X_1 X_2 X_3) X_4$

Thm $\quad K_i \cong I^{i-2}$

$K_{n_1} \times K_{n_2}$ $\qquad$ one pair

$K_{i_1} \times K_{i_2}$ $\qquad$ two pairs

$\cdots$

A_n- form

$M_i : K_i \times X^i \to X \quad 2 \leq i \leq n, \qquad K_2 = *$

(1) $M_2(*, e, X) = M_2(*, X, e)$

(2) $\forall \rho \in K_r$, $\sigma \in K_s$. $(r+s = i+1)$, $\rho \times \sigma \subset L_i = \partial K_i$

$\quad M_i(\rho \times \sigma, X_1, \cdots, X_i) = M_r(\rho, X_1, \cdots, X_{K-1}, M_s(\sigma, X_1, \cdots; X_{r+s-1}), X_{K+s}, \cdots, X_s)$

$\qquad\uparrow$
$\quad \partial_K (r, s)(\rho, \sigma)$

(3) $\tau = K_i$, $i > 2$

$\quad M_i(\tau, X_1, \cdots, X_{j-1}, e, X_{j+1}, \cdots; X_i) = M_{i-1}(S_j(\tau), X_1, \cdots, X_{j-1}, X_{j+1}, \cdots, X_i)$

Thm

X with A_n- structure $\Leftrightarrow$ (X, M_i)

Def:

An A_n-space is a space X with A_n-form

or with A_n-structure

$2 \le n \le \infty$.

$(X, M_i) \rightsquigarrow \mathcal{E}_i = \coprod_{1 \le n \le i-1} K_{n+1} \times X^n / \sim$ $\qquad K_2 \times X^1 \amalg K_3 \times X^2$

forgetting $\qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$

first coordinate $B_i = \coprod_{0 \le n \le i-2} K_{n+2} \times X^n / \sim$ $\qquad K_2 \times X^0 \amalg K_3 \times X^1$

$X - A_\infty \qquad \mathcal{E}_\infty = \coprod_{n \ge 1} K_{n-1} \times X^n / \sim$

$\downarrow$

"BX" $= B_\infty = \coprod_{n \ge 0} K_{n+2} \times X^n / \sim$

$X \to E_\infty \to B_\infty \quad \rightsquigarrow \quad X \xrightarrow{\sim} \Omega B_\infty = \Omega BX$

$R = L_{i+1} \times X^i \cup K_{i+1} \times X \times \underbrace{X^{[i-1]}}_{\cap X^{i-1}} \subset K_{i+1} \times X^i$

X^{i-1}, one one coordinate $= e$

$(K_{i+1} \times X^i, R) \to (\mathcal{E}_i. \mathcal{E}_{i-1})$

$\alpha|_R$

$\cdots$

$M_i : K_i \times X^i \to X$

0

$K_3 \times K_1.$

$(X_1 X_2 X_3) \cdot X_4$

$(X_1(X_2 X_3)) X_4$

$((X_1 X_2) X_3) X_4$

$(X_1 X_2) X_3 X_4$

$(X_1 X_2)(X_3 X_4)$

$X_1 X_2 (X_3 X_4)$

$X_1(X_2(X_3 X_4))$

$X_1(X)$

$((X_1(X_2)) X_3) X_4$

$(X \times X_1 X_2 X_3)$

$(X_1(X_2)(X_3)) X_4$