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What condition we need to have for $Y$, s.t. $Y \simeq \Omega X$ ?

$Y$ ?? $Y = \Omega X \leftsquigarrow BY$

Thm

$Y$ conn. based. $Y \simeq \Omega X$ for some $X$ iff $Y$ is an $A_\infty$-space.

Def:

An $A_\infty$-structure on a space $X$ consist of an n-tuple of Maps

$$\begin{array}{ccccccccc} X = & E_1 & \subset & E_2 & \subset & \cdots & \subset & E_n \\ & \downarrow p_1 & & \downarrow p_2 & & \cdots & & \downarrow p_n \\ * = & B_1 & \subset & B_2 & \subset & \cdots & \subset & B_n \end{array}$$

$h(-,1) = id$

$h(-,0) = *$

s.t. $\pi_*(E_i, X) \simeq \pi_*(B_i)$, $\forall\ 1 \le i \le n$ $\quad 1 \le n \le \infty$

$X \to E_i \to B_i$ together with a contracting homotopy $h: CE_{n-1} \to E_n$

s.t. $h(CB_{i-1}) \subset B_i$, $\forall i$.

Associahedra ("stasheff operad")

cell complex $K_i$ ($\cong I^{i-2}$)

$\partial K_i = L_i \quad \frac{i(i-1)}{2} - 1$

$X_1 \cdots X_i$ Cell = Image $(K_r \times K_s)$ , $s = i+1-r$

$X_1 \cdots (X_k \cdots X_{k+i-1}) X_{i+k} \cdots X_i$ $\partial_k(r,s) : K_r \times K_s \to K_{r+s-1}$

Given a way

$X_1 \cdots (X_k X_{k+1} \cdots X_{k+r-1}) X_{k+r} \cdots X_i \rightsquigarrow$ cell $\subset L_i = \partial K_i$

$\cong$

$K_r \times K_s$ . $s = i+1-r$

Satisfying

$$\begin{tikzcd}
K_r \times K_s \times K_t \arrow[r, "1\times\partial(s,t)"] \arrow[d, "\partial(r,s)\times 1"'] & K_r \times K_{s+t-1} \arrow[d] \\
K_{r+s-1} \times K_t \arrow[r] & K_{r+s+t-2}
\end{tikzcd}$$

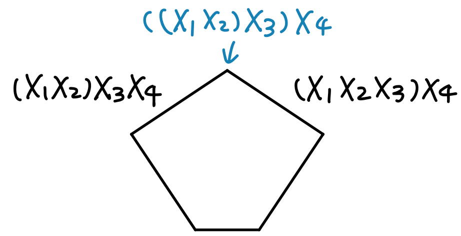
$$\begin{tikzcd}
K_r \times K_s \times K_t \arrow[r, "1\times T"] \arrow[d, "\partial(r,s)"] & K_r \times K_t \times K_s \arrow[r, "1\times\partial(t,s)"] & K_r \times K_{t+s-1} \arrow[d] \\
K_{r+s-1} \times K_t \arrow[rr] & & K_{r+t+s-2}
\end{tikzcd}$$

$i = 1$ ✓

$i = 2$ , $K_2 = *$

$i = 3$ , $K_3 = \frac{X_1(X_2X_3)}{(X_1X_2)X_3} = I$

$i=4$, $x_1x_2x_3x_4$, $\binom{4}{2}-1=5$




Thm $K_i \cong I^{i-2}$

$K_{n_1}\times K_{n_2}$ one pair

$K_{i_1}\times K_{i_2}$ two pairs

$\cdots$

An- form

$M_i : K_i \times X^i \to X \quad 2 \le i \le n, \qquad K_2 = *$

(1) $M_2(*, e, x) = M_2(*, x, e)$

(2) $\forall \rho \in K_r$, $\sigma \in K_s$, $(r+s=i+1)$, $\rho\times\sigma \subset L_i = \partial K_i$

$$M_i(\underbrace{\rho\times\sigma}_{\partial_k(r,s)(\rho,\sigma)}, x_1, \cdots, x_i) = M_r(\rho, x_1, \cdots, x_{k-1}, M_s(\sigma, x_1, \cdots, x_{r+s-1}), x_{k+s}, \cdots, x_s)$$

(3) $\tau = K_i$, $i>2$

$$M_i(\tau, x_1, \cdots, x_{j-1}, e, x_{j+1}, \cdots, x_i) = M_{i-1}(s_j(\tau), x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_i)$$

Thm

$X$ with An- structure $\Leftrightarrow$ $(X, M_i)$

Def:

An $A_n$-space is a space $X$ with $A_n$-form

or with $A_n$-structure

$2 \le n \le \infty$.

$$\begin{array}{lll} (X, M_i) \rightsquigarrow & \mathcal{E}_i = \coprod_{1 \le n \le i-1} K_{n+1} \times X^n/\sim & K_2 \times X^1 \amalg K_3 \times X^2 \\ \text{forgetting} & \downarrow & \downarrow \\ \text{first coordinate} & B_i = \coprod_{0 \le n \le i-2} K_{n+2} \times X^n/\sim & K_2 \times X^0 \amalg K_3 \times X^1 \end{array}$$

$$X - A_\infty \qquad \mathcal{E}_\infty = \coprod_{n \ge 1} K_{n-1} \times X^n/\sim$$

$$\downarrow$$

$$\text{"}BX\text{"} = B_\infty = \coprod_{n \ge 0} K_{n+2} \times X^n/\sim$$

$$X \to E_\infty \to B_\infty \rightsquigarrow X \xrightarrow{\simeq} \Omega B_\infty = \Omega BX$$

$$R = L_{i+1} \times X^i \cup K_{i+1} \times X \times \underbrace{X^{[i-1]}}_{\cap X^{i-1}} \subset K_{i+1} \times X^i$$

, one one coordinate $= e$

$$(K_{i+1} \times X^i, R) \to (\mathcal{E}_i, \mathcal{E}_{i-1})$$

$\alpha|_R$

...

$M_i : K_i \times X^i \to X$

$0$