The intersection form of spin 4-manifolds and Pin(2)-equivariant Mahowald invariants

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$$\begin{aligned} Q_N : H^2(N;\mathbb{Z}) \times H^2(N;\mathbb{Z}) &\longrightarrow \mathbb{Z}, \\ (a,b) &\longmapsto \langle a \cup b, [N] \rangle. \end{aligned}$$

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  2. The Kirby–Siebenmann invariant $ks(N) \in H^4(N;\mathbb{Z}/2) = \mathbb{Z}/2$.

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2. *Bilinear form Q: not even*
   $\implies$ *any* $(Q,\ \mathbb{Z}/2)$ *can be realized*

3. *Bilinear form Q: even*
   $\implies$ *only* $\left(Q,\ \frac{\mathrm{sign}(Q)}{8} \mod 2\right)$ *can be realized*

Smooth category

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  M smooth $\Longrightarrow ks(M) = 0$
- $+$ Freedman's theorem:

Theorem

Two closed simply connected smooth 4-manifolds are homeomorphic if and only if they have isomorphic intersection forms.

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Question (Botany Problem)

How many non-diffeomorphic 4-manifolds can realize Q?

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Completely answers the Geography Problem when Q is definite

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Theorem (Hasse–Minkowski)

1. *Q: not even*
   $Q \cong$ diagonal form with entries ± 1.
2. *Q: even*
   $Q \cong kE_8 \oplus q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for some $k \in \mathbb{Z}$ and $q \in \mathbb{N}$.

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Fact

Q: not even
Q can be realized by a connected sum of copies of $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$

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- By reversing the orientation of M, may assume $k \geq 0$

The $\frac{11}{8}$-Conjecture

Conjecture (version 1)

The form

$$2pE_8 \oplus q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

can be realized as the intersection form of a closed smooth spin 4-manifold if and only if $q \geq 3p$.

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  - K_3: $2E_8 \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
  - $S^2 \times S^2$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The $\frac{11}{8}$-Conjecture

The "only if" part can be reformulated as follows:

Conjecture (version 2)

Any closed smooth spin 4-manifold M must satisfy the inequality

$$b_2(M) \geq \frac{11}{8}|\operatorname{sign}(M)|,$$

where $b_2(M)$ and $\operatorname{sign}(M)$ are the second Betti number and the signature of M, respectively.

Progress on the $\frac{11}{8}$-Conjecture

- $p = 1$, assuming $H_1(M; \mathbb{Z})$ has no 2-torsions: Donaldson (anti-self-dual Yang–Mills equations)

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- Furuta's idea: combined Kronheimer's approach with "finite dimensional approximation"
  - Attacked the conjecture by using Pin(2)-equivariant stable homotopy theory

Furuta's $\frac{10}{8}$-Theorem

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Theorem (Furuta)

For $p \geq 1$, the bilinear form

$$2pE_8 \oplus q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is spin realizable only if $q \geq 2p + 1$.

Furuta's $\frac{10}{8}$-Theorem

Corollary (Furuta)

Any closed simply connected smooth spin 4-manifold M that is not homeomorphic to S^4 must satisfy the inequality

$$b_2(M) \geq \frac{10}{8}|\operatorname{sign}(M)| + 2.$$

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The inequality of manifolds with boundaries are proved by Manolescu, and Furuta–Li.

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Theorem (Hopkins–Lin–Shi–X.)

For $p \geq 2$, if the bilinear form $2pE_8 \oplus q\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is spin realizable, then

$$q \geq \begin{cases} 2p+2 & p \equiv 1,2,5,6 \pmod 8 \\ 2p+3 & p \equiv 3,4,7 \pmod 8 \\ 2p+4 & p \equiv 0 \pmod 8. \end{cases}$$

The limit is $\frac{10}{8} + 4$

Corollary (Hopkins–Lin–Shi–X.)

Any closed simply connected smooth spin 4-manifold M that is not homeomorphic to S^4, $S^2 \times S^2$, or $K3$ must satisfy the inequality

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Furthermore, we show this is the **limit** of the current known approaches to the $\frac{11}{8}$-Conjecture

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- $\widetilde{SW} : \Gamma(S^+) \oplus i\Omega^1(M) \longrightarrow \Gamma(S^-) \oplus i\Omega^2_+(M) \oplus i\Omega^0(M)/\mathbb{R}$
- Sobolev completion $\Longrightarrow \widetilde{SW} : H_1 \longrightarrow H_2$ (Seiberg–Witten map)

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- In order to use homotopy theory, we want maps between finite dimensional spheres

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  3. When V_2 is large enough,
     $\widetilde{SW}_{\text{apr}}$ maps $S(V_1, R+1)$ to $V_2 \setminus \mathring{B}(V_2, \varepsilon)$

V_1
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$\widetilde{SW}_{\mathrm{apr}}$
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Proposition (Furuta)

If the intersection form of the manifold M is $2pE_8 \oplus q\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

then

$$V_1 - V_2 \cong p\mathbb{H} - q\widetilde{\mathbb{R}}$$

as virtual $\mathrm{Pin}(2)$*-representations.*

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The stable homotopy class of $\widetilde{SW}^{+}_{\mathrm{apr}}$ is called the Bauer–Furuta invariant $BF(M)$

$$\begin{tikzcd} S^{V_1} \arrow[dr, "\widetilde{SW}^{+}_{\mathrm{apr}}"] & \\ S^0 \arrow[u, hook] \arrow[r, hook] & S^{V_2} \end{tikzcd} \Longrightarrow \begin{tikzcd} S^{p\mathbb{H}} \arrow[dr, "BF(M)"] & \\ S^0 \arrow[u] \arrow[r] & S^{q\widetilde{\mathbb{R}}} \end{tikzcd}$$

Proposition (Furuta)

If the intersection form of the manifold M is $2pE_8 \oplus q\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

then

$$V_1 - V_2 \cong p\mathbb{H} - q\widetilde{\mathbb{R}}$$

as virtual $\mathrm{Pin}(2)$*-representations.*

The stable homotopy class of $\widetilde{SW}^{+}_{\mathrm{apr}}$ is called the Bauer–Furuta invariant $BF(M)$

Furuta–Mahowald class

Definition

For $p \geq 1$, a Furuta–Mahowald class of level-(p, q) is a stable map

$$\gamma : S^{p\mathbb{H}} \longrightarrow S^{q\widetilde{\mathbb{R}}}$$

that fits into the diagram

$$\begin{tikzcd}
S^{p\mathbb{H}} \arrow[dr, "\gamma"] & \\
S^0 \arrow[u, "a^p_{\mathbb{H}}"] \arrow[r, "a^q_{\widetilde{\mathbb{R}}}"'] & S^{q\widetilde{\mathbb{R}}}
\end{tikzcd}$$

- $a_{\mathbb{H}} : S^0 \longrightarrow S^{\mathbb{H}}$
- $a_{\widetilde{\mathbb{R}}} : S^0 \longrightarrow S^{\widetilde{\mathbb{R}}}$

Theorem (Furuta)

If the bilinear form $2pE_8 \oplus q\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *is spin realizable, then there exists a level-*(p, q) *Furuta–Mahowald class.*

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Theorem (Furuta)

A level-(p, q) *Furuta–Mahowald class exists* **only if** $q \geq 2p + 1$.

Question

What is the necessary and sufficient condition for the existence of a level-(p, q) Furuta–Mahowald class?

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- Subsequently, he made a conjecture

Jones' conjecture

Conjecture (Jones)

For $p \geq 2$, a level-(p, q) Furuta–Mahowald class exists **if and only if**

$$q \geq \begin{cases} 2p+2 & p \equiv 1 \pmod 4 \\ 2p+2 & p \equiv 2 \pmod 4 \\ 2p+3 & p \equiv 3 \pmod 4 \\ 2p+4 & p \equiv 0 \pmod 4. \end{cases}$$

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- Necessary condition: various progress has been made by Stolz, Schmidt and Minami
- Before our current work, the best result is given by Furuta–Kamitani

Theorem (Furuta–Kamitani)

For $p \geq 2$, a level-(p, q) Furuta–Mahowald class exists **only if**

$$q \geq \begin{cases} 2p+1 & p \equiv 1 \pmod 4 \\ 2p+2 & p \equiv 2 \pmod 4 \\ 2p+3 & p \equiv 3 \pmod 4. \\ 2p+3 & p \equiv 0 \pmod 4. \end{cases}$$

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- Much less is known about the sufficient condition
- So far, the best result is by Schmidt: constructed a Furuta–Mahowald class of level-$(5, 12)$
- We completely resolve this question

Main Theorem

Theorem (Hopkins–Lin–Shi–X.)

For $p \geq 2$, a level-(p, q) Furuta–Mahowald class exists **if and only if**

$$q \geq \begin{cases} 2p+2 & p \equiv 1, 2, 5, 6 \pmod 8 \\ 2p+3 & p \equiv 3, 4, 7 \pmod 8 \\ 2p+4 & p \equiv 0 \pmod 8. \end{cases}$$

Comparison of known results

Minimal q such that a level-(p, q) Furuta–Mahowald class exists:

Jones' conjecture	Our theorem	Furuta–Kamitani	
$2p+2$	$2p+2$	$\geq 2p+1$	$p \equiv 1 \pmod 8$
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The limit is $\frac{10}{8} + 4$

Corollary (Hopkins–Lin–Shi–X.)

Any closed simply connected smooth spin 4-manifold M that is not homeomorphic to S^4, $S^2 \times S^2$, or $K3$ must satisfy the inequality

$$b_2(M) \geq \frac{10}{8}|\operatorname{sign}(M)| + 4.$$

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In the sense of classifying all Furuta–Mahowald classes of level-(p, q), this is the **limit**

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- $\pi_\bigstar^G S^0$: $RO(G)$-graded stable homotopy groups of spheres

Non-nilpotent elements in $\pi_\bigstar^G S^0$

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*The G-**equivariant Mahowald invariant of α with respect to β** is the following set of elements in $\pi_\bigstar^G S^0$:*

$$M_\beta^G(\alpha) = \{\gamma \,|\, \alpha = \gamma\beta^k,\ \alpha \text{ is not divisible by } \beta^{k+1}\}.$$

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$$S^0 \xrightarrow{\alpha} S^{-|\alpha|}$$

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S^{0} \arrow[u, "\beta"] \arrow[ur, "\alpha"] &
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S^{-k|\beta|} \arrow[dddrr, bend left, "\exists"] & & \\
\vdots \arrow[u, "\beta"] & & \\
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For all $q \geq 0$, the $RO(C_2)$-degree of $M^{C_2}_{a_\sigma}(a^q_\sigma)$ is zero.

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- Forget to the non-equivariant world $\Longrightarrow$ classical Mahowald invariant $M(\alpha)$

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For $q \geq 1$, the set $M(2^q)$ contains the first nonzero element of Adams filtration q in positive degree.

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Moreover, the following 4-periodic result holds:

$$|M_{a_\sigma}^{C_2}((\Phi^{C_2})^{-1}2^q)| = \begin{cases} (8k+1)\sigma & \text{if } q = 4k+1 \\ (8k+2)\sigma & \text{if } q = 4k+2 \\ (8k+3)\sigma & \text{if } q = 4k+3 \\ (8k+7)\sigma & \text{if } q = 4k+4. \end{cases}$$

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- Crabb, Schmidt, and Stolz studied the C_4-equivariant Mahowald invariant of powers of a_σ with respect to $a_{2\lambda}$

Theorem (Crabb, Schmidt, Stolz)

For $q \geq 1$, the following 8-periodic result holds:

$$|M^{C_4}_{a_{2\lambda}}(a^q_\sigma)| + q\sigma = \begin{cases} 8k\lambda & \text{if } q = 8k+1 \\ 8k\lambda & \text{if } q = 8k+2 \\ (8k+2)\lambda & \text{if } q = 8k+3 \\ (8k+2)\lambda & \text{if } q = 8k+4 \\ (8k+2)\lambda & \text{if } q = 8k+5 \\ (8k+4)\lambda & \text{if } q = 8k+6 \\ (8k+4)\lambda & \text{if } q = 8k+7 \\ (8k+4)\lambda & \text{if } q = 8k+8. \end{cases}$$

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- C_4 is a subgroup of Pin(2)
- Minami and Schmidt used this theorem to deduce the nonexistence of certain Furuta–Mahowald classes

Pin(2)-equivariant Mahowald invariant

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- To prove our main theorem, we analyze the Pin(2)-equivariant Mahowald invariants of powers of $a_{\widetilde{\mathbb{R}}}$ with respect to $a_{\mathbb{H}}$

Main Theorem

Theorem (Hopkins–Lin–Shi–X.)

For $q \geq 4$, the following 16-periodic result holds:

$$|M^{\mathrm{Pin}(2)}_{a_{\mathbb{H}}}(a^q_{\widetilde{\mathbb{R}}})| + q\widetilde{\mathbb{R}}$$

$$= \begin{cases} (8k-1)\mathbb{H} & \text{if } q = 16k+1 \\ (8k-1)\mathbb{H} & \text{if } q = 16k+2 \\ (8k-1)\mathbb{H} & \text{if } q = 16k+3 \\ (8k+1)\mathbb{H} & \text{if } q = 16k+4 \\ (8k+1)\mathbb{H} & \text{if } q = 16k+5 \\ (8k+2)\mathbb{H} & \text{if } q = 16k+6 \\ (8k+2)\mathbb{H} & \text{if } q = 16k+7 \\ (8k+2)\mathbb{H} & \text{if } q = 16k+8 \end{cases} \quad \begin{cases} (8k+3)\mathbb{H} & \text{if } q = 16k+9 \\ (8k+3)\mathbb{H} & \text{if } q = 16k+10 \\ (8k+4)\mathbb{H} & \text{if } q = 16k+11 \\ (8k+5)\mathbb{H} & \text{if } q = 16k+12 \\ (8k+5)\mathbb{H} & \text{if } q = 16k+13 \\ (8k+6)\mathbb{H} & \text{if } q = 16k+14 \\ (8k+6)\mathbb{H} & \text{if } q = 16k+15 \\ (8k+6)\mathbb{H} & \text{if } q = 16k+16. \end{cases}$$

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- Jones' conjecture would be true

Pin(2)-equivariant to non-equivariant

- C_2-action on $BS^1 = \mathbb{C}P^\infty$:
  $(z_1, z_2, z_3, z_4, \ldots, z_{2n-1}, z_{2n}) \longmapsto$
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- fiber bundle $\mathbb{R}P^2 \hookrightarrow B\,\mathrm{Pin}(2) \longrightarrow \mathbb{H}P^\infty$
  gives cell structures on $B\,\mathrm{Pin}(2)$ and $X(m)$.

The Mahowald Line

Consider the diagram

$$\begin{tikzcd} S^{p\mathbb{H}} \arrow[dr, dashed, "\exists"] & \\ S^0 \arrow[u, "a^p_{\mathbb{H}}"] \arrow[r, "a^q_{\widetilde{\mathbb{R}}}"] & S^{q\widetilde{\mathbb{R}}} \end{tikzcd}$$

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\iff X(q)^{4p-2-q} \longrightarrow S^0 \text{ is zero}
$$

$S^{3\mathbb{H}} \xrightarrow{\nexists} S^{8\widetilde{\mathbb{R}}}$
$S^{2\mathbb{H}} \xrightarrow{\exists} S^{8\widetilde{\mathbb{R}}}$

Lower bound

Classical Adams spectral sequence

Some relations in $\pi_* S^0$

- $\pi_4 = 0$
- $\pi_5 = 0$
- $\pi_{12} = 0$
- $\pi_{13} = 0$
- $\eta \cdot \pi_6 = 0$
- $\pi_8 \cdot \eta^2 = 0$

S^0

S^0

η^2
S^0

η^2
S^0

η^2
S^0

η^2
$\pi_8 \cdot \eta^2 = 0$
$\pi_{12} = 0$
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S^0

Now we start the induction

Inductive hypothesis
S^0

S^0

S^0

η^2

η^2
S^0

η^2
S^0

η^2
S^0

η^2
$\pi_8 \cdot \eta^2 = 0$
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S^0

η^2
S^0

η^2
S^0

η^2
S^0

Induction finished!
S^0

Intuition for a technical step

Another mini-movie

η

η

η^2

η^2

π_3

$\pi_4 = 0$

η

η

π_3

$\pi_4 = 0$

$\pi_5 = 0$

π_6

$\eta \cdot \pi_6 = 0$

η

η

π_3

$\pi_4 = 0$

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π_6

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Thank you!