

Def. A category \mathcal{C} consists of

- collection of objects $X \in \mathcal{C}$
- for $X, Y \in \mathcal{C}$, set of morphisms $\mathcal{C}(X, Y)$ or $\text{Hom}_{\mathcal{C}}(X, Y)$
- identity $\text{id}_X \in \mathcal{C}(X, X)$
- composition. (unital + associative).

A functor $F: \mathcal{C} \rightarrow \mathcal{D}$, for $X \in \mathcal{C}$, $FX \in \mathcal{D}$,
for each $f: X \rightarrow Y$ $Ff: FX \rightarrow FY$,
 F should respect identities and composition.

A natural transformation between $F, G: \mathcal{C} \rightarrow \mathcal{D}$,

$\tau: F \Rightarrow G$, assigns for each $X \in \mathcal{C}$,

$\tau_X: FX \rightarrow GX$, in a natural way

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow Ff & \downarrow \tau_X \\ & & FX \xrightarrow{\tau_Y} GX \\ & & \downarrow Ff & \downarrow Gf \\ & & FY \xrightarrow{\tau_Y} GY \end{array}$$

Ex/. Category of Sets.

Category of groups

A morphism $f: X \rightarrow Y$ is an isomorphism if there exists $g: Y \rightarrow X$, $f \circ g = \text{id}_Y$ $g \circ f = \text{id}_X$.

Adjunctions

Limits & Colimits.

Def. $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$

$F \dashv G$ (F is left adjoint of G)

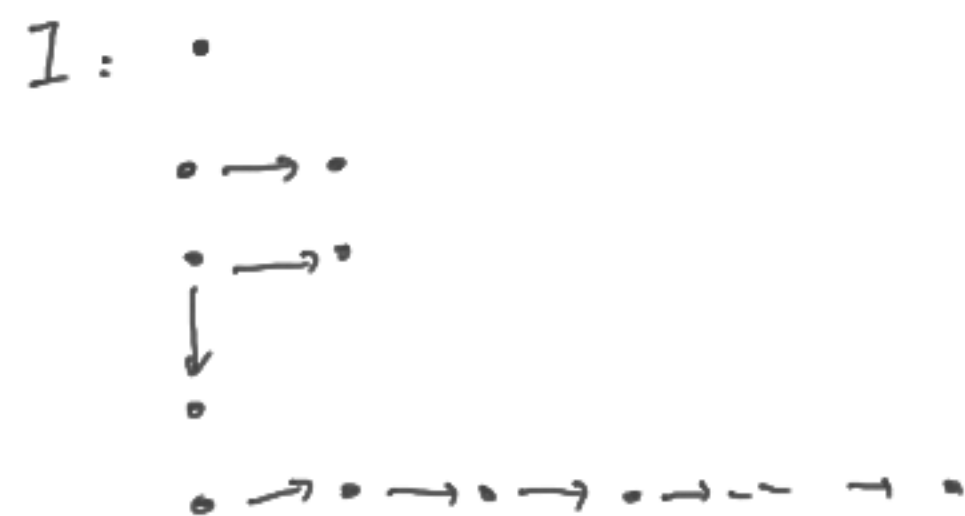
if there are natural bijections

$$\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$$

$\forall X \in \mathcal{C} \quad Y \in \mathcal{D}$.

A small category is a cat whose collection of
objs form a set.

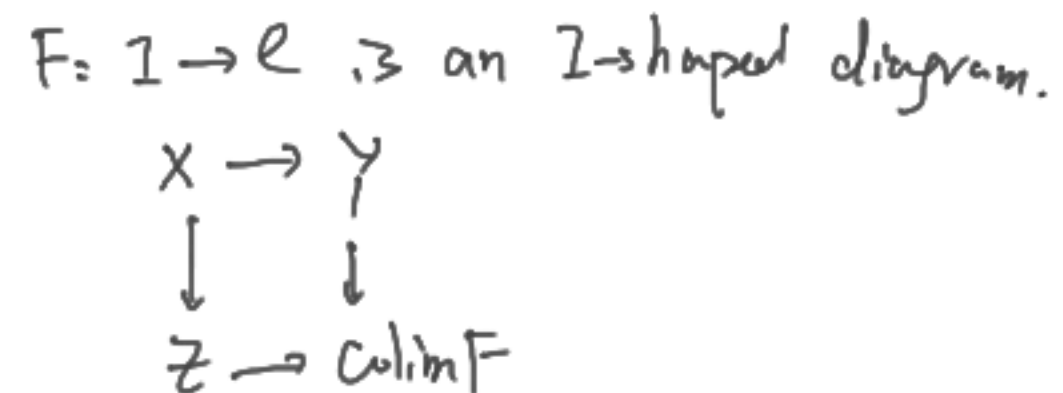
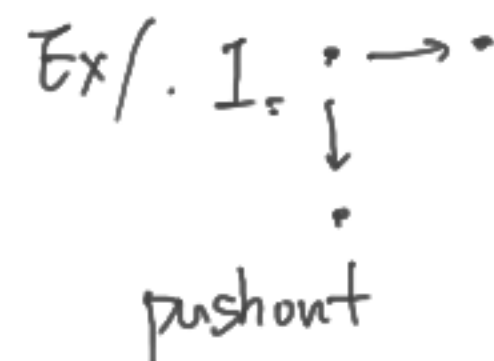
Let I be a small cat, \mathcal{C} be another category
the category I -shaped diagrams $\text{Fun}(I, \mathcal{C})$.



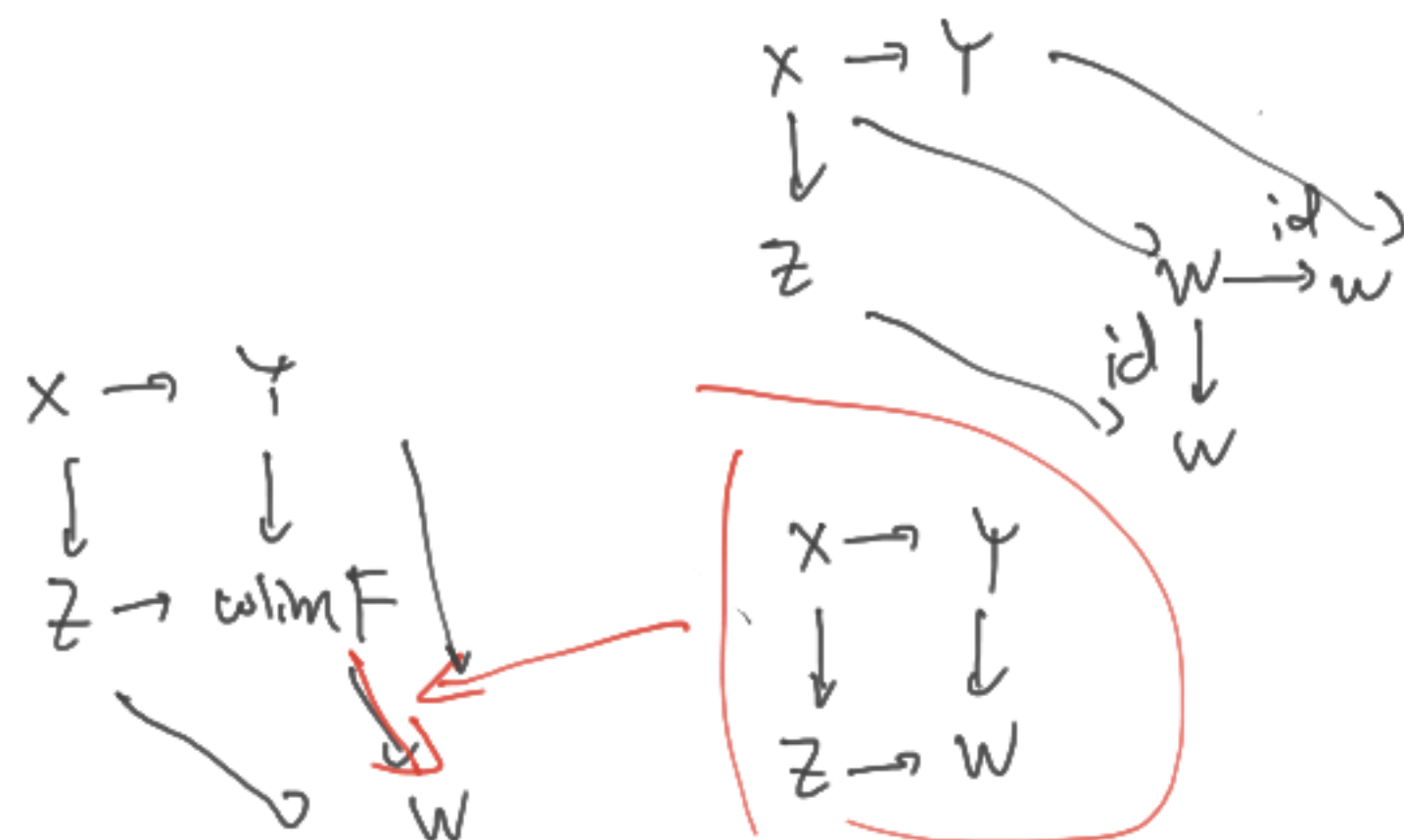
There is a diagonal functor $\Delta: \mathcal{C} \rightarrow \text{Fun}(I, \mathcal{C})$
 $X \mapsto \text{const } I\text{-shaped diagrams.}$

If Δ has a left adjoint, $\text{Fun}(I, \mathcal{C}) \rightarrow \mathcal{C}$
 $\varinjlim_I: \text{Fun}(I, \mathcal{C}) \rightarrow \mathcal{C}$.

Dually, a limit is a right adjoint to such a
diagonal functor.



Def of adj: $\varinjlim_I F$
 $\mathcal{C}(\varinjlim_I F, W) \cong \text{Fun}(I, \mathcal{C})(F, \Delta W)$.



Ex/. $I = \text{discrete category}$ (no non-trivial morphisms)
colimit: coproducts

$I = \bullet \bullet$



It is straightforward to check.

$$f_1 \simeq_L f_2, g_1 \simeq_L g_2$$

$$g_1 \circ f_1 \simeq_L g_2 \circ f_2.$$

\Rightarrow We can define Top : spaces + htpy classes of maps

An isomorphism in Top is called a htpy equivalence.

$$X \begin{matrix} \xrightarrow{f} \\ \xleftarrow{g} \end{matrix} Y \quad g \circ f \simeq \text{id}_X, f \circ g \simeq \text{id}_Y$$

Make up equivalence of categories.

$$C \begin{matrix} \xrightarrow{F} \\ \xleftarrow{G} \end{matrix} D \quad G \circ F \simeq \text{Id}_C \quad F \circ G \simeq \text{Id}_D$$

finite set \simeq finite ordinals

Def. For a space X , $\pi_0 X$ is the set of path components of X .

$$\pi_0 X := \{ \text{left htpy classes } * \rightarrow X \}$$

For $n \geq 1$, $\pi_n(X, x)$ is the group of left htpy classes of

maps $f: I^n \rightarrow X$, $f|_{\partial I^n} = x$.

this is preserved in htpy

group str: $f, g: I^n \rightarrow X \quad f \neq g, I^n \xrightarrow{\cong} I^n \cup_{I^{n-1}} I^n \xrightarrow{\text{fug}} X.$

$$[I^n] \rightarrow [I^n]$$

We say $f: X \rightarrow Y$ is a weak htpy equiv if $\pi_* f$ is an isomorphism. (RUGH).

Prop. Any htpy equiv is a weak htpy equiv \square .

Def: $S^n = \{ x_1^2 + \dots + x_{n+1}^2 = 1 \} \subseteq \mathbb{R}^{n+1}$

$$D^{n+1} = \{ x_1^2 + \dots + x_{n+1}^2 \leq 1 \} \subseteq \mathbb{R}^{n+1}$$

S^0 : 2 points. boundary $D^1 \cong I^1$.

$$D^0: * \quad S^{-1}: \emptyset.$$

Def. For $X \in \text{Top}$, n -cell attachment is the pushout of the following

$$\begin{array}{ccc} \coprod S^{n-1} & \xrightarrow{\text{attaching map}} & X \\ \downarrow \uparrow & & \downarrow \\ \coprod D^n & \longrightarrow & X \cup_{\coprod S^{n-1}} D^n. \end{array}$$

disjoint unions of $S^{n-1} \hookrightarrow D^n$

A relative CW complex $X \rightarrow Y$ is
a countable colimit of

$$X = X_0 \xrightarrow{0\text{-cell}} X_1 \xrightarrow{1\text{-cell}} X_2 \rightarrow \dots$$

sequential colimit.

A CW complex (absolute). $\emptyset \rightarrow Y$.

A cell complex (arbitrary attachments).

Ex/. S^n . a 0-cell and a n-cell.

$$\begin{array}{ccc} S^{n-1} & \rightarrow & * \\ \downarrow \tau & & \downarrow \\ D^n & \rightarrow & S^n \end{array}$$

Prop. For any space X , there exists
a CW complex \tilde{X} w/ a weak htpy
equiv $\tilde{X} \rightarrow X$.
(could be made function)

Prop (Whitehead's Theorem). If $f: X \rightarrow Y$ is a
weak homotopy equivalence, then f is a h.e.

Serre fibrations, cofibrations.

Def. A map $p: E \rightarrow B$ is a Serre fibration,
if it has right lifting property for all maps
 $\{D^n \rightarrow D^n \times I\}_{n \geq 0}$

$$\text{if } \begin{array}{ccc} D^n & \rightarrow & E \\ \downarrow & & \downarrow p \\ D^n \times I & \rightarrow & B \end{array} \quad \text{then} \quad \begin{array}{ccc} D^n & \rightarrow & E \\ \downarrow & \exists \nearrow & \downarrow p \\ D^n \times I & \rightarrow & B \end{array}$$

i.e. $D^n \times I \rightarrow B$ could be lifted to $D^n \times I \rightarrow E$ if
one end of the htpy has a lift.

Ex/. A covering space $E \rightarrow B$ is a Serre fibration
(lift is unique there).

FACT. A Serre fibration has the RLP
against all $X \rightarrow X \times I$ if X is a
CW complex.

Prop. Let $f: X \rightarrow Y$ is a Serre fibration
and $y \in Y$ be a point of Y , define.

$$F_y := f^{-1}(y).$$

then there is an exact sequence for any $x \in F_y$,

$$\pi_*(F_y, x) \xrightarrow{i_*} \pi_*(X, x) \xrightarrow{f_*} \pi_*(Y, y).$$

Pf (Sketch): $F_y \rightarrow X \rightarrow Y$, so $\text{im}(i_*) \subseteq \ker(f_*)$

Suppose $[\alpha] \in \ker(f_*)$ represented by $\alpha: S^{n-1} \rightarrow X$.

Since $f \circ \alpha$ is homotopic to constant map.

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\alpha} & X \\ i \downarrow & & \downarrow f \\ D^n & \xrightarrow{F} & Y \end{array} \quad \left(\begin{array}{l} \text{extend } f \circ \alpha \text{ to a map} \\ F: D^n \rightarrow Y \end{array} \right).$$

Since $f(x) = y$, there is a homotopy.

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\alpha} & X \\ \downarrow & & \downarrow + \\ D^n & \xrightarrow{F} & Y \\ \downarrow & & \downarrow \text{id} \\ D^n & \xrightarrow{H} & Y \\ \downarrow & & \downarrow \\ * & \xrightarrow{\quad} & Y \end{array} \quad \begin{array}{l} D^n \text{ is contractible} \\ \text{so } F: D^n \rightarrow Y \text{ is} \\ \text{homotopic to constant} \\ \text{map} \end{array}$$

($H = F \simeq \text{const}_y$).

$X \rightarrow Y$ is a Serre fibration, so

we lift

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\alpha} & X \\ \downarrow & \nearrow & \downarrow \\ S^{n-1} \times I & & Y \\ \downarrow & \xrightarrow{H} & \\ D^n \times I & & Y \end{array}$$

this is a homotopy
of α to a map $\tilde{\alpha}$
which lies entirely
in F_y .
(the other end
of $D^n \times I$ is mapped
to y)

$\tilde{\alpha} \simeq \alpha$
 $\tilde{\alpha}: S^{n-1} \rightarrow F_y$, $[\alpha] = \text{im}[\tilde{\alpha}]$.
(to y)

