Barr-Beck-Lurie th'm

Given an adjunction

$$F : \mathcal{C} \underset{\perp}{\rightleftarrows} \mathcal{D} : G$$

$$c \in \mathcal{C} \longleftrightarrow d \in \mathcal{D} + \text{extra data}$$

Ex ① (faithfully flat descent)

$A \xrightarrow{(*)} B$ a map of commutative rings.

$$\begin{array}{ccc} A\text{-Mod} & \xrightarrow{\otimes_A B} & B\text{-Mod} \\ \cup & & \cup \\ M & \longmapsto & N \end{array}$$

$\tau : B \otimes_A N \simeq N \otimes_A B$

satisfying "cocycle condition"

If $(*)$ is faithfully flat $\Rightarrow$ converse true

② Morita theory

R : commutative ring

P : R-module $\quad S = \mathrm{End}_R(P)$

$$\begin{array}{ccc} R\text{-Mod} & \xrightarrow{\mathrm{Hom}_R(P,-)} & \mathrm{Ab} \\ Q & \longmapsto & \mathrm{Hom}_R(P, Q) \circlearrowleft S \end{array}$$

When P is finitely generated projective generator $\Longrightarrow$ converse true

$R\text{-Mod} \simeq \mathrm{End}_R(P)\text{-Mod}$

I. Monads : monoids in the cat of endofunctors

$\mathcal{C}$: category

Def (Monad) $T \in \mathrm{End}(\mathcal{C})$

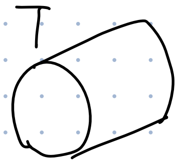
$\eta \in 1_{\mathcal{C}} \to T$

$\mu \in T^2 \to T$ satisfying

$$\begin{tikzcd} T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] \arrow[dr, phantom, "2"] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}, \quad \begin{tikzcd} T^2 \arrow[d, "\mu"'] \arrow[dr, phantom, "2"] & T \arrow[l, "\eta T"'] \arrow[dl, equal] \\ T & \end{tikzcd}, \quad \begin{tikzcd} T^2 \arrow[d, "\mu"'] \arrow[dr, phantom, "2"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\ T & \end{tikzcd}$$

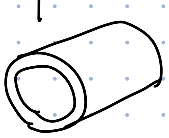
Ex ① burrito

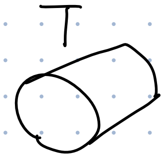
$1 \longrightarrow T$



stuffing

$T^2 \longrightarrow T$





Burritos for the Hungry Mathematician

Ed Morehouse

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Abstract

The advent of fast-casual Mexican-style dining establishments, such as Chipotle and Qdoba, has greatly improved the productivity of research mathematicians and theoretical computer scientists in recent years. Still, many experience confusion upon encountering burritos for the first time.

Numerous burrito tutorials (of varying quality) are to be found on the Internet. Some describe a burrito as the image of a crêpe under the action of the new-world functor. But such characterizations merely serve to reindex the confusion contravariantly. Others insist that the only way to really understand burritos is to eat many different kinds of burrito, until the common underlying concept becomes apparent.

It has been recently remarked by Yorgey [9] that a burrito can be regarded as an instance of a universally-understood concept, namely, that of monad. It is this characterization that we intend to explicate here. To wit, *a burrito is just a strong monad in the symmetric monoidal category of food, what's the problem?*

② from adjunctions

$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ with $F \dashv G$

unit $\eta : 1_{\mathcal{C}} \to GF$

counit $\epsilon : FG \to 1_{\mathcal{D}}$

$G \circ F \in \mathrm{End}(\mathcal{C})$ it defines a monad

$\mu : (G\circ F)(G\circ F) \xrightarrow{G\epsilon F} G\circ F$

$\eta : 1_{\mathcal{C}} \longrightarrow GF$

Exercise : check the diagrams ☺

II. Modules/Algebras over a monad

$\mathcal{C}$: category

(T, μ, η) : monad on $\mathcal{C}$

Def (module/algebra over T)

$c \in \mathcal{C}$

$m : Tc \longrightarrow c$ satisfying

$$\begin{tikzcd} T^2c \arrow[r, "Tm"] \arrow[d, "\mu c"'] \arrow[dr, phantom, "\circlearrowleft"] & Tc \arrow[d, "m"] \\ Tc \arrow[r, "m"'] & c \end{tikzcd} \quad , \quad \begin{tikzcd} Tc \arrow[d, "m"'] & c \arrow[l, "\eta c"'] \arrow[dl, equal] \\ c & \end{tikzcd}$$

Ex ① (free module).

Tc is a module $/T$

$m: T^2c \xrightarrow{\mu} Tc$

② $Ab \underset{U}{\overset{\otimes R}{\rightleftarrows}} R\text{-mod}$

$T = U(-\otimes R) = \otimes R$ acts on Ab

$\Rightarrow$ $T\text{-mod} = R\text{-mod}$

③ In the adjunction setting

$F: \mathcal{C} \underset{}{\overset{\perp}{\rightleftarrows}} \mathcal{D} : G$

$\forall d \in \mathcal{D}$, $G(d)$ is a $\overset{T}{\overset{\shortparallel}{G \circ F}}$-module

$m: (GF)(Gd) \xrightarrow{G\varepsilon} G(d)$.

$\Rightarrow$ G factors $\mathrm{Mod}_T(\mathcal{C})$

$\mathcal{C} \xleftarrow{U} \mathrm{Mod}_T(\mathcal{C}) \xleftarrow{\bar{G}} \mathcal{D}$

Def G is monadic if $\bar{G}$ is eq.

Remark: Everything has dual version.

III. Barr-Beck th'm

$F \dashv D : G$

Beck thesis	Triples algebras and cohomologies.
Maclane	§ VI.7
Riehl	§ 5.6

Th'm

G is monadic if it satisfies

1) G is conservative. → $G(f)$ is an eq $\Rightarrow$ f is an eq.

2) (crude ver) $\mathcal{D}$ admits & G preserves reflexive coeq.

(precise ver) $\mathcal{D}$ admits & G preserves coeq of G-split pairs.

iff

Reflexive coequalize:

$$\begin{tikzcd} X_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X_0 \arrow[l, bend left=50, "s"] \arrow[r, "e"] & X_{-1} \end{tikzcd}$$

→ coequalizer

→ common section

Split coequalize:

$$\begin{tikzcd} X_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X_0 \arrow[l, bend left=50, "s"] \arrow[r, "e"] & X_{-1} \arrow[l, bend left=50, "t"] \end{tikzcd}$$

→ coeq

a split pair

$1_{X_0} = gs$ $\qquad et = 1_{X_{-1}}$

$fs = te$

G-split pair: a pair that is sent to a split pair by G

Ex In Morita theory

② Morita theory

R : commutative ring

P : R-module $\quad S = \mathrm{End}_R(P)$

$$\begin{array}{ccc} R\text{-Mod} & \xrightarrow{\mathrm{Hom}_R(P,-)} & \mathrm{Ab} \\ Q & \longmapsto & \mathrm{Hom}_R(P,Q) \circlearrowleft S \end{array}$$

When P is finitely generated projective generator $\Rightarrow$ converse true

$R\text{-Mod} \simeq \mathrm{End}_R(P)\text{-Mod}$

$$\begin{tikzcd} \mathrm{Ab} \arrow[r, shift left, "\otimes P"] \arrow[r, phantom, "\perp"] & R\text{-Mod} \arrow[l, shift left, "\mathrm{Hom}_R(P,-)"] \end{tikzcd}$$

① $\mathrm{Hom}_R(P,-)$ admits a left adjoint

② P is generator $\Rightarrow \mathrm{Hom}_R(P,-)$ is conservative

③ P is projective $\Rightarrow \mathrm{Hom}_R(P,-)$ preserves reflexive coeq

$R\text{-Mod} \simeq \mathrm{Mod}_T(\mathrm{Ab}) \underset{\text{identified}}{\overset{\text{can be}}{=}} S\text{-Mod}$

$T(M) = \mathrm{Hom}_R(P, P\otimes M)$

$T(\mathbb{Z}) = \mathrm{Hom}_R(P,P) = \mathrm{End}_R(P) = S$

IV higher categorical version $\boxed{\begin{matrix}\text{HA } \S 4.7 \\ \text{DAG } \S 4\end{matrix}}$

⓪ $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$.

$\xrightarrow{\text{higher}}$ $M \to \Delta^1$ biCartesian

$M_0 \rightsquigarrow \mathcal{C}$ $\quad M_1 \rightsquigarrow \mathcal{D}$

coCart $\rightsquigarrow F$ $\quad$ Cart $\rightsquigarrow G$

① Monad $:= \{T^2 \to T,\ 1 \to T\ +$ finite diagrams$\}$

can't encode infinite coherence data

$\left(F \to FGF \to F \ \sim\ \mathrm{id} \Rightarrow \begin{tikzcd} T^2 \arrow[d] & T \arrow[l] \arrow[dl, equal] \\ T & \end{tikzcd} \right)$

$\xrightarrow{\text{higher}}$ a monad is an alg obj in $\mathrm{End}(\mathcal{C})$

② want G factors $\mathrm{Mod}_T(\mathcal{C})$

$\xrightarrow{\text{higher}}$ $G \in \mathrm{Fun}(\mathcal{D}, \mathcal{C})$

$\mathrm{End}(\mathcal{C})$

$G \circ F \in \{\mathrm{End}(\mathcal{C}) +$ an action on $G\}$ is classifying obj

$\Rightarrow G \in \mathrm{Mod}_T\mathrm{Fun}(\mathcal{D}, \mathcal{C}) \Rightarrow \bar{G}$

In DAG $\quad$ Adj data $\quad$ Adj data $\xrightarrow[\text{Kan}]{\text{trivial}}$ $\mathrm{Fun}^{\text{left adj}}(\mathcal{C}, \mathcal{D})$

$\rightsquigarrow \mathrm{End}(\mathcal{C}) \ni T$

③ Th'm conditions

(Crude version) $\quad$ reflexive coeq $(X_1 \rightleftarrows X_0) \to X_{-1}$

$\rightsquigarrow$ geometric realizations.

$$(\cdots \rightrightarrows X_2 \; \substack{\to\\\to\\\to\\\leftarrow\leftarrow} \; X_1 \; \substack{\leftarrow\\\to\\\to} \; X_0) \to X_{-1}$$

(precise version) split coeq $(X_1 \; \substack{\leftarrow\\\to\\\to} \; X_0) \; \substack{\leftarrow\\\to} \; X_{-1}$

$\rightsquigarrow$ split simplicial diagrams

Def $\Delta_{-\infty}$ obj $[-1] = \{\ \}$
$[0] = \{0\}$
$[1] = \{0 < 1\}$
$[2] = \{0 < 1 < 2\}$

morph : $[n] \to [m]$:
set of orderpreserving maps
$[n] \cup \{-\infty\} \longrightarrow [m] \cup \{-\infty\}$
sending $\{-\infty\}$ to $\{-\infty\}$.

Def A simplicial digram $N(\Delta^{op}) \to \mathcal{C}$
split if it extends to $N(\Delta^{op}_{-\infty})$.

Exercise $\Delta_{-\infty}$-diagrams give split coeq in ordinary category.

Thm (Bar-Beck-Lurie)

$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ adjunction

Then $\exists$ monad T s.t G factors

$$C \xleftarrow[U]{} \mathrm{Mod}_T(\mathcal{C}) \xleftarrow[\bar{G}]{} D$$

and $\bar{G}$ is an equivalence

if G is

1) conservative.

2) (Crude) D admits & G preserves geometric realization

(presise) ——— " ———

colim of G-split simplicial diagrams.