Def. A <u>category</u> $\mathcal{C}$ consists of

- collection of objects $X \in \mathcal{C}$
- for $X, Y \in \mathcal{C}$, set of morphisms $\mathcal{C}(X,Y)$ or $\mathrm{Hom}_{\mathcal{C}}(X,Y)$
- identity $id_X \in \mathcal{C}(X,X)$
- composition. (unital + associative).

A <u>functor</u>. $F: \mathcal{C} \to \mathcal{D}$, for $X \in \mathcal{C}$, $FX \in \mathcal{D}$,
for each $f: X \to Y$ $Ff: FX \to FY$.
F should respect identities and composition.

A <u>natural transformation</u> between $F, G: \mathcal{C} \to \mathcal{D}$.
$\tau: F \Rightarrow G$, assigns for each $X \in \mathcal{C}$,
$\tau_X: FX \to GX$. in a <u>natural way</u>

$$X \xrightarrow{f} Y \qquad \begin{tikzcd} FX \arrow[r, "\tau_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\tau_Y"'] & GY \end{tikzcd}.$$

Ex/. Category of sets.
Category of groups

A morphism $f: X \to Y$ is an <u>isomorphism</u>. if there exists $g: Y \to X$, $f \circ g = id_Y$ $g \circ f = id_X$.

Adjunctions

Limits & Colimits.

Def. $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$

$F \dashv G$ (F is left adjoint of G)

if there are natural bijections
$$\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$$
$\forall X \in \mathcal{C}$ $Y \in \mathcal{D}$.

A small category is a cat whose collection of objs form a set.

Let I be a small cat, $\mathcal{C}$ be another category, the category I-shaped diagrams $\mathrm{Fun}(I,\mathcal{C})$.

I: $\bullet$

$\bullet \to \bullet$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

$\bullet \to \bullet \to \bullet \to \bullet \to \cdots \to \bullet$

There is a diagonal functor $\mathcal{C} \xrightarrow{\Delta} \mathrm{Fun}(I,\mathcal{C})$

$X \longmapsto$ const I-shaped diagrams.

If Δ has a left adjoint, $\mathrm{Fun}(I,\mathcal{C}) \to \mathcal{C}$

$\underrightarrow{\mathrm{colim}}_I : F \to \mathcal{C}$.

Dually, a limit is a right adjoint to such a diagonal functor.

Ex/. I:
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$
pushout

$F: I \to \mathcal{C}$ is an I-shaped diagram.
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & \mathrm{colim}\,F \end{tikzcd}$$

$\underrightarrow{\mathrm{colim}}_I F$

Def of adj:
$$\mathcal{C}(\underrightarrow{\mathrm{colim}}_I F, W) \cong \underline{\mathrm{Fun}(I,\mathcal{C})(F, \Delta W)}.$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[drr] & Y \arrow[rr, bend left] & & W \\ Z \arrow[drr] & & W \arrow[r, "id"] \arrow[d, "id"] & W \\ & & W & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left] & \\ Z \arrow[r] \arrow[drr, bend right] & \mathrm{colim}\,F \arrow[dr, dashed] & \\ & & W \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

Ex/. I = discrete category (no non-trivial morphisms)

colimit: coproducts

$I = \bullet \quad \bullet$

$$\begin{tikzcd} X \arrow[dr] \arrow[ddr, bend right] & & Y \arrow[dl] \arrow[ddl, bend left] \\ & X \amalg Y \arrow[d, dashed] & \\ & Z & \end{tikzcd}$$

Let Top be the category of CGWH spaces

(compactly generated weak Hausdorff).

nice spaces, including all CW cplxes

w/ morphisms continuous maps btwn spaces.

- Top is bicomplete (it has all small colimits & limits).

Ex/. $X, Y, Z \in \mathrm{Top}$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d] \\ Z \arrow[r] & Y \sqcup_X Z \end{tikzcd}$$

$Y \sqcup_X Z = Y \sqcup Z / f(x) \sim g(x),\ x \in X.$

as a set

In particular, if Z is a point, f is an inclusion

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & Y/X \end{tikzcd}$$

Y/X ← cofiber of $X \hookrightarrow Y$

<u>make up</u> : $*$. terminal obj of Top ($X \xrightarrow{\exists !} *$)

ϕ initial obj of Top ($\phi \xrightarrow{\exists !} X$)

The reasons why we restrict to Top_{CGWH}

① . $Y \in \mathrm{Top}_{CGWH}$

$\mathrm{Top}(X \times Y, Z) \cong \mathrm{Top}(X, Z^Y)$. ← Cartesian closed.

$- \mapsto - \times Y \qquad Z \mapsto Z^Y$

mapping space

w/ compact open topology

② . smash product associative ...

<u>Def</u>. Let $I = [0,1]$ be the interval.

a (left) homotopy between two maps $f, g : X \to Y$

$$\begin{tikzcd} X \arrow[d, "i_0"'] \arrow[dr, "f"] & \\ X \times I \arrow[r, "H"] & Y \\ X \arrow[u, "i_1"] \arrow[ur, "g"'] & \end{tikzcd}$$

is a map $H : X \times I \to Y$. s.t. the diagram commutes.

$X \times I$: the cylinder over X

It is straightforward to check.

$f_1 \simeq_l f_2$, $g_1 \simeq_l g_2$

$g_1 \circ f_1 \simeq_l g_2 \circ f_2$.

$\Rightarrow$ We can define Toph: spaces + htpy classes of maps

An isomorphism in Toph is called a htpy equivalence.

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y \qquad g\circ f \simeq id_X, \quad f\circ g \simeq id_Y$$

Make up. equivalence of categories.

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \qquad G\circ F \cong Id_{\mathcal{C}} \quad F\circ G \cong Id_{\mathcal{D}}.$$

finite set $\simeq$ finite ordinals

Def. For a space X, $\pi_0 X$ is the set of path component of X.

$$\pi_0 X := \{\text{left htpy classes } * \to X\}$$

$x \in X$.

For $n \geq 1$, $\pi_n(X, x)$ is the group of left htpy classes of maps $f : I^n \to X$, $f|_{\partial I^n} = x$. this is preserved in htpy

group str: $f, g : I^n \to X$ $\quad f * g$, $\quad I^n \cong I^n \cup_{I^{n-1}} I^n \xrightarrow{f \cup g} X$. $\quad \boxed{I^n} \to \boxed{I^n \mid I}$

We say $f : X \to Y$ is a weak htpy equiv if $\pi_* f$ is an isomorphism. (ROUGH).

Prop. Any htpy equiv is a weak htpy equiv $\square$.

Def. $S^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subseteq \mathbb{R}^{n+1}$

$D^{n+1} = \{x_1^2 + \cdots + x_{n+1}^2 \leq 1\} \subseteq \mathbb{R}^{n+1}$.

S^0: 2 points. boundary $D^1 \cong I^1$.

D^0: $*$ $\quad S^{-1}$: $\emptyset$.

Def. For $X \in Top$, n-cell attachment is the pushout of the following

$$\begin{array}{ccc} \coprod S^{n-1} & \xrightarrow{\text{attaching map}} & X \\ \downarrow & & \downarrow \\ \coprod D^n & \longrightarrow & X \cup_{\coprod S^{n-1}} D^n \end{array}$$

disjoint unions of $S^{n-1} \hookrightarrow D^n$

A <u>relative CW complex</u> $X \to Y$ is a countable colimit of

$$\begin{tikzcd} X = X_0 \arrow[r, "0\text{-cell}"] \arrow[drr] & X_1 \arrow[r, "1\text{-cell}"] \arrow[dr] & X_2 \arrow[r] \arrow[d] & \cdots \\ & & Y & \end{tikzcd}$$

sequential colimit.

A CW complex (absolute). $\emptyset \to Y$.

A cell complex (arbitrary attachments).

Ex/. S^n. a 0-cell and a n-cell.

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & * \arrow[d] \\ D^n \arrow[r] & S^n \end{tikzcd}$$

<u>Prop</u>. For any space X, there exists a CW complex $\widetilde{X}$ w/ a weak htpy equiv $\widetilde{X} \to X$.

(could be made functor)

<u>Prop</u> (Whitehead's Theorem). If $f: X \to Y$ is a weak homotopy equivalence, then f is a h.e.

Serre fibrations, cofibrations.

<u>Def</u>. A map $p: E \to B$ is a <u>Serre fibration</u>, if it has right lifting property for all maps $\{D^n \to D^n \times I\}_{n \geq 0}$

if
$$\begin{tikzcd} D^n \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r] & B \end{tikzcd}$$
then
$$\begin{tikzcd} D^n \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r] \arrow[ur, dashed, "\exists"] & B \end{tikzcd}$$

i.e. $D^n \times I \to B$ could be lifted to $D^n \times I \to E$ if one end of the htpy has a lift.

Ex/. A covering space $E \to B$ is a Serre fibration (lift is unique there).

FACT. A Serre fibration has the RLP against all $X \to X\times I$ if X is a CW complex.

Prop. Let $f: X \to Y$ is a Serre fibration and $y \in Y$ be a point of Y, define.

$F_y := f^{-1}(y)$.

then there is an exact sequence for any $x \in F_y$,

$$\pi_*(F_y, x) \xrightarrow{i_*} \pi_*(X, x) \xrightarrow{f_*} \pi_*(Y, y).$$

Pf (Sketch). $F_y \to X \to Y$, so $\mathrm{im}(i_*) \subseteq \ker(f_*)$

Suppose $[\alpha] \in \ker(f_*)$ represented by $\alpha: S^{n-1} \to X$.

Since $f\circ\alpha$ is homotopic to constant map.

$$\begin{tikzcd} S^{n-1} \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\ D^n \arrow[r, "F"] & Y \end{tikzcd}$$

(extend $f\circ\alpha$ to a map $F: D^n \to Y$).

Since $f(x) = y$, there is a homotopy.

$$\begin{tikzcd} & S^{n-1} \arrow[r, "\alpha"] \arrow[d] & X \arrow[d, "f"] \\ & D^n \arrow[r, "F"] \arrow[d] & Y \arrow[dd, "id"] \\ D^n \arrow[r] \arrow[d] & D^n \times I \arrow[dr, "H"] & \\ * \arrow[rr] & & Y \end{tikzcd}$$

D^n is contractible so $F: D^n \to Y$ is homotopic to constant map

($H: F \simeq const_y$).

$X \to Y$ is a Serre fibration, so we lift

$$\begin{tikzcd} S^{n-1} \arrow[r, "\alpha"] \arrow[d] & X \arrow[dd] \\ S^{n-1}\times I \arrow[ur, dashed] \arrow[d] & \\ D^n\times I \arrow[r, "H"] & Y \end{tikzcd}$$

this is a homotopy of α to a map $\tilde{\alpha}$ which lies entirely in F_y.

(the other end of $D^n\times I$ is mapped to y)

$\tilde{\alpha} \simeq \alpha$

$\tilde{\alpha}: S^{n-1} \to F_y$, $[\alpha] = \mathrm{im}[\tilde{\alpha}]$.