Peter May 2023-08-14

Asking for our background firstly:

1) cohomology theories are represented by spectra

Weak Homotopy Equivalence

$A \subset X \to X/A$

$E^q(X/A) \to E^q(X) \to E^q(A)$

$E(\bigvee_i X_i) \cong \prod_i E^q(X_i)$

$$E^q(X) = [X, E_q]$$

$E^q(X) = E^{q+1}(\Sigma X)$, $E^q(X) = [X, E_q)$

$$\begin{array}{c} \cong \\ E^{q+1}(\Sigma X) = [\Sigma X, E_{q+1}] \cong [X, \Omega E_{q+1}] \end{array}$$

$[\Sigma X, Y] \cong [X, \Omega Y]$

Let $X = E_q$ & $E_q \xrightarrow{id} E_q$, we have $\underline{E_q \xrightarrow{WHE} \Omega E_{q+1}}$ Ω- Spectrum

$$\Rightarrow E^q(X) = \begin{cases} [X, E_q] , & q \geq 0 \\ [X, \Omega^{-q} E_0] , & q < 0 \end{cases}$$

$f_q : E_q \to E_q'$

$$\begin{tikzcd} E_q \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\curvearrowright"] & E_q' \arrow[d] \\ \Omega E_{q+1} \arrow[r, "\Omega f_{q+1}"'] & \Omega E_{q+1}' \end{tikzcd}$$

$E_q \xrightarrow[\cong]{\text{homeomorphism}} \Omega E_{q+1}$ is good.

Given a space X. what structure gives Y, s.t. $\Omega^n Y = X$?

$X \overset{?}{\simeq} \Omega BX$ $\qquad \Omega Y \times \Omega Y \to \Omega Y$ (Loop space has multiplication)

Which space has a classfying space

Stasheff $\quad A_\infty$ - Space

Boardman Vogt $\quad$ 1968 – 1974

Higher homotopy will be given in homotopy.

Segal 1969. 1974

$O(j),\ j \geq 0 \qquad O(j) \times \Sigma_j \to O(j)$

$$(x\sigma)\tau = x(\sigma\tau)$$

$$xe = x$$

$\gamma: O(k) \times O(j_1) \times \cdots \times O(j_k) \to O(j_+) \quad j_+ = j_1 + \cdots + j_k.$

$1 \in O(1)$

O-algebra X

$O(j) \times_{\Sigma_j} X^j \to X$

associate, unital

O is an E_∞-operad if $O(j)$ is contractible

$O(j) \times \Sigma_j \to O(j)$ is free

$x\sigma = x \Rightarrow \sigma = e.$

If X is connected, $\mathcal{O}$ is an E_∞-operad

X is $\mathcal{O}$-algebra, X is path-connected,

$\Rightarrow \exists$ Spectrum EX. s.t. $X \simeq (EX)_0$

A monad $\mathbb{C}$ on category $\mathcal{V}$

$\mathbb{C}: \mathcal{V} \to \mathcal{V}$. $\mu: \mathbb{C}\mathbb{C}X \to \mathbb{C}X$

$\eta: X \to \mathbb{C}X$.

$$\begin{tikzcd}
\mathbb{C}\mathbb{C}\mathbb{C} \arrow[r, "\mu"] \arrow[d, "\mathbb{C}\mu"'] & \mathbb{C}\mathbb{C} \arrow[d, "\mu"] \\
\mathbb{C}\mathbb{C} \arrow[r, "\mu"'] & \mathbb{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\eta"] \arrow[dr, equal] & \mathbb{C}\mathbb{C} \arrow[d, "\mu"] & \mathbb{C} \arrow[l, "\eta"'] \arrow[dl, equal] \\
& \mathbb{C} &
\end{tikzcd}$$

$(\Sigma, \Omega) = \mathcal{V}(\Sigma X, Y) \cong \mathcal{V}(X, \Omega Y)$

If $Y = \Sigma X$ & $\Sigma X \xrightarrow{id} \Sigma X$, $\eta: X \to \Sigma\Omega X$

$\varepsilon: \Sigma\Omega Y \to Y$

Monad:

$\Gamma = \Omega\Sigma$, $\Omega\Sigma\Omega\Sigma \xrightarrow{\mu = \Omega\varepsilon} \Omega\Sigma$

$I \xrightarrow{\eta} \Omega\Sigma$

$\Sigma: \mathcal{J} \to \mathcal{S}$

$\Omega: \mathcal{S} \to \mathcal{J}$

Γ-Algebra $\Gamma X \to X$

$Y \in \mathcal{S}$. ΩY, $\Gamma\Omega Y \to \Omega Y$. $\Omega\Sigma\Omega Y \xrightarrow{\Omega\varepsilon} \Omega Y$

$\Omega: \mathcal{S} \to \Gamma$-alg's

$B(\Sigma, \mathbb{C}, X) = |B_*(\Sigma, \mathbb{C}, X)|$, $B_q(\Sigma, \mathbb{C}, X) = \Sigma \mathbb{C}^q X$

X_q , $|X_*| = TX_n$

$\Sigma \mathbb{C} X \to \Sigma X$ If X is a $\mathbb{C}$-alg, $\mathbb{C}X \to X$

Given an operad C. $\mathbb{C}X = \coprod C(j) \times_{\Sigma_j} X^j / (\sim)$

$\mathbb{C}\Omega Y \xrightarrow{\theta} \Omega Y$

$$\begin{tikzcd}
\mathbb{C}X \arrow[r, "\mathbb{C}\eta"] \arrow[rr, bend right, "\alpha"'] & \mathbb{C}\Omega\Sigma X \arrow[r, "\theta"] & \Omega\Sigma X
\end{tikzcd}$$

$\Sigma \mathbb{C} X \xrightarrow[\beta]{} \Sigma X$

$$\begin{tikzcd}
X & B(\mathbb{C}, \mathbb{C}, X) \arrow[l] \arrow[r, "B(\alpha, id, id)"] \arrow[ddr, "\eta_{\mathbb{C}}"'] & B(\Omega\Sigma, \mathbb{C}, X) \arrow[d, dash] \\
& & \Omega B(\Sigma, \mathbb{C}, X) \\
& & \Omega_{\mathbb{C}} \underbrace{\Sigma_c B(\mathbb{C}, \mathbb{C}, X)}_{E_X} \arrow[u, phantom, "\cong" description]
\end{tikzcd}$$

$B(\mathbb{C}, \mathbb{C}, X) \simeq \overline{X}$

$\Omega_{\mathbb{C}} \Sigma_c \overline{X} = \Omega_{\mathbb{C}} \Sigma_c B(\mathbb{C}, \mathbb{C}, X)$

$$\begin{tikzcd}
\Omega : S \arrow[r, "St"] \arrow[dr] & \mathbb{C}[alg] \arrow[d] \\
& J
\end{tikzcd}$$

$(\Sigma^\infty, \Omega^\infty)$

connective spectra $(\pi_q E = 0, q < 0) \simeq$ grouplike $\mathbb{C}$-algebra