

Presheaves on Orbit Category

Weinan Lin

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In the previous talk we have discussed the category of G -spaces $G\mathbf{Top}$. In this note, we will look at another model of $G\mathbf{Top}$, i.e. the category of presheaves on the orbit category that takes value in the category of spaces. Throughout this note, G is a finite group or a compact Lie group, and by saying the subgroups of G , we refer to only those closed subgroups.

For a group G , its orbit category \mathbf{Orb}_G is the category whose objects are orbits G/H , and morphisms G -equivariant maps. A presheaf on orbit category is a contravariant functor from the orbit category \mathbf{Orb}_G . Given a G -space X , take its H -fixed point for each subgroup H and the system of fixed point spaces $\{X^H, H \subset G\}$ forms a presheaf on orbit category. In the other direction, given a presheaf \mathcal{X} , evaluating at the orbit G/e with G action, we obtain a G -space $\mathcal{X}(G/e)$. This actually gives a pair of adjoint functors

$$\Theta : \mathcal{P}(\mathbf{Orb}_G) \xrightleftharpoons{\quad} G\mathbf{Top} : \Phi ,$$

where $\mathcal{P}(\mathbf{Orb}_G)$ denotes the category of presheaves on the orbit category that takes value in the category of spaces.

A. D. Elmendorf proved in [Elm83] that this adjoint pair induces equivalence between the homotopy categories. In fact, with the correct model category structures on both hand sides, this equivalence of homotopy categories can be seen in the model categorical level.

Theorem 0.1 (Elmendorf). *There is a Quillen equivalence.*

$$\Theta : \mathcal{P}(\mathbf{Orb}_G) \xrightleftharpoons{\quad} G\mathbf{Top} : \Phi$$

1 Model category

The model category theory is for doing homotopy theory. Quillen developed the definition of a model category to formalize the similarities between homotopy theory and homological algebra.

1.1 Weak factorization system and model structure

Definition 1.1. A weak factorization system (WFS) on a category \mathcal{C} is a pair \mathcal{L}, \mathcal{R} of classes of morphisms of \mathcal{C} such that

- Every morphism $f : X \rightarrow Y$ of \mathcal{C} may be factored as the composition of a morphism in \mathcal{L} followed by one in \mathcal{R}

$$f : X \xrightarrow{\in \mathcal{L}} Z \xrightarrow{\in \mathcal{R}} Y.$$

- The classes are closed under having the lifting property against each other:
 - \mathcal{L} is precisely the class of morphisms having the left lifting property against every morphism in \mathcal{R} ;
 - \mathcal{R} is precisely the class of morphisms having the right lifting property against every morphism in \mathcal{L} .

Definition 1.2. A model structure on a category \mathcal{C} is a choice of three distinguished classes of morphisms: cofibrations C , fibrations F and weak equivalences W , satisfying the following conditions:

- W contains all isomorphisms and is closed under two-out-of-three: given a composable pair of morphisms f, g , if two out of the three morphisms $f, g, g \circ f$ are in W , so is the third;
- $(C, F \cap W)$ and $(C \cap W, F)$ are two weak factorization systems on \mathcal{C} . $F \cap W$ is called acyclic fibrations and $C \cap W$ is called acyclic cofibrations.

When a category \mathcal{C} is complete and cocomplete category with a model structure, we call it a model category.

A key example which motivated the definition is the category of topological spaces.

Example 1.3. The category of topological spaces, **Top**, admits a standard model category structure with fibrations as Serre fibrations, equivalences as weak homotopy equivalences and cofibrations as the retracts of relative cell complexes.

In fact the model structure is cofibrantly generated in the sense that there are small sets of morphisms I and J which permit the small object argument such that I generates C and J generates $C \cap W$ by taking transfinite composition of pushouts of coproducts and taking retracts. The generating sets are

$$I = \{S^{n-1} \rightarrow D^n, n \geq 0\}, \quad J = \{D^n \rightarrow D^n \times I, n \geq 0\}.$$

Example 1.4. Cofibrantly generated model structures transfer along adjunctions. Use the cofibrantly generated model structure on **Top**, we obtain a cofibrantly generated model structure on $G\mathbf{Top}$ with generating sets

$$I_{G\mathbf{Top}} = \{G/H \times i\}_{H \subset G, i \in I}, \quad J_{G\mathbf{Top}} = \{G/H \times j\}_{H \subset G, j \in J}.$$

Similarly, we can also put a cofibrantly generated model structure on $\mathcal{P}(\mathbf{Orb}_G)$ which has the generating sets:

$$\begin{aligned} I_{\mathcal{P}(\mathbf{Orb}_G)} &= \{\mathrm{Map}_G(G/H, -) \times i\}_{H \subset G, i \in I}, \\ J_{\mathcal{P}(\mathbf{Orb}_G)} &= \{\mathrm{Map}_G(G/H, -) \times j\}_{H \subset G, j \in J}. \end{aligned}$$

1.2 Homotopy category of model category

The homotopy category $Ho(\mathcal{C})$ of a model category \mathcal{C} is the localization of \mathcal{C} with respect to the class of weak equivalences

$$\mathcal{C} \rightarrow Ho(\mathcal{C}) = \mathcal{C}[W^{-1}],$$

so that the homotopy category has the universal property that the weak equivalences become actual isomorphisms.

Remark 1.5. This definition of homotopy category does not depend on the choice of fibrations and cofibrations. It only depends on the underlying category with weak equivalences (\mathcal{C}, W) . However, the model structure makes the homotopy category easier to handle. In fact, with a model structure, $Ho(\mathcal{C})$ is equivalent to the category whose objects are those which are both fibrant and cofibrant, and morphisms are the equivalence classes of morphisms under left homotopy. This definition of homotopy category avoids the set theory technical issues one may meet with while doing localization.

1.3 Quillen equivalences

Quillen equivalences are one convenient notion of morphisms between model categories.

Definition 1.6. For \mathcal{C} and \mathcal{D} two model categories, an adjoint pair

$$F : \mathcal{C} \xrightleftharpoons{\quad} \mathcal{D} : G$$

is a Quillen adjunction if the following equivalent conditions are satisfied:

1. F preserves cofibrations and acyclic cofibrations;
2. G preserves fibrations and acyclic fibrations;
3. F preserves cofibrations and G preserves fibrations;
4. F preserves acyclic cofibrations and G preserves acyclic fibrations.

Definition 1.7. The Quillen adjoint pair

$$F : \mathcal{C} \xrightleftharpoons{\quad} \mathcal{D} : G$$

is a Quillen equivalence, if for any cofibrant object $X \in \mathcal{C}$ and fibrant object $Y \in \mathcal{D}$, $FX \rightarrow Y$ is a weak equivalence iff the adjoint $X \rightarrow GY$ is a weak equivalence.

Proposition 1.8. *When (F, G) is an Quillen equivalence, they induces equivalence on homotopy categories, i.e. the derived functors (LF, RG) are equivalences of categories.*

2 Presheaves on orbit category

Definition 2.1. Orbit category associated to a group G , denoted by \mathbf{Orb}_G , is the category whose

- objects are G -orbits G/H ,
- morphisms are G -equivariant maps.

Example 2.2. Let $G\mathbf{Top}$ be the category of G -spaces and let $\mathcal{P}(\mathbf{Orb}_G)$ be the category of presheaves on the orbit category \mathbf{Orb}_G . Let X be a G -space. Consider the fixed point functor $X^{(-)}$ for every subgroup $H \subset G$. This defines a functor $\Phi : G\mathbf{Top} \rightarrow \mathcal{P}(\mathbf{Orb}_G)$. Φ has a left adjoint $\Theta : \mathcal{P}(\mathbf{Orb}_G) \rightarrow G\mathbf{Top}$ which sends a presheaf χ to $\chi(G/e)$, on which the G action is induced by the group action on orbit G/e .

In the previous example, we get a adjoint pair

$$\Theta : \mathcal{P}(\mathbf{Orb}_G) \xrightleftharpoons{\quad} G\mathbf{Top} : \Phi .$$

Consider the model structures on $G\mathbf{Top}$ and $\mathcal{P}(\mathbf{Orb}_G)$ which are inherited from the classical Quillen model structure on \mathbf{Top} . It is easy to check that this is a Quillen adjunction, and the next question is that if it is a Quillen equivalence. And Elmendorf's theorem gives an affirmative answer to this question.

Theorem 2.3. (*Elmendorf*)

There is a Quillen equivalence

$$\Theta : \mathcal{P}(\mathbf{Orb}_G) \xrightleftharpoons{\quad} G\mathbf{Top} : \Phi .$$

Elmendorf's original proof [Elm83] only showed these two categories have the same homotopy theory by constructing explicitly a functor $\Psi : \mathcal{P}(\mathbf{Orb}_G) \rightarrow G\mathbf{Top}$ and a natural transformation $\epsilon : \Phi\Psi \rightarrow id$ such that $\epsilon^H : (\Psi\chi)^H \rightarrow \chi(G/H)$ is a homotopy equivalence. We will also give another proof in the next section which proves the statement in the model category level.

3 Proof of Elmendorf's theorem

We provide two proofs here. One is by Piacenza and uses model category (see [May82, VI.6]). The other is Elmendorf's original proof, by constructing explicitly the functor Ψ using bar construction (see [May82, V.3]).

3.1 Sketch proof using model category

For a Quillen equivalence, we need model category structures on both categories. The model structure we use on $\mathcal{P}(\mathbf{Orb}_G)$ and $G\mathbf{Top}$ are inherited by the classical model structure on the category of spaces, i.e. the weak equivalences and fibrations are defined level-wise, and the cofibrations are defined by lifting property.

Definition 3.1. There is a model structure on $\mathcal{P}(\mathbf{Orb}_G)$ where $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a weak equivalence (resp. fibration) if

$$f(G/H) : \mathcal{X}(G/H) \rightarrow \mathcal{Y}(G/H)$$

is a weak equivalence (resp. fibration) in \mathbf{Top} for every $G/H \in \mathbf{Orb}_G$, and $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a cofibration if it has left lifting property with respect to all acyclic fibrations.

Definition 3.2. There is a model structure on $G\mathbf{Top}$, where $f : X \rightarrow Y$ is a weak equivalence (resp. fibration) if $f^H : X^H \rightarrow Y^H$ is a weak equivalence (resp. fibration) in \mathbf{Top} for every subgroup $H \subset G$. $f : X \rightarrow Y$ is a cofibration if it has left lifting property with respect to all acyclic fibrations.

Under these model structures, we are able to identify the cofibrant objects in $\mathcal{P}(\mathbf{Orb}_G)$: cofibrant objects are retracts of cellular objects, and cellular objects are generated under pushout along inclusions and direct colimits, by

$$\{\mathrm{Map}_G(G/H, -) \times X \mid H \subset G, X \text{ is a cell in } \mathbf{Top}\}$$

A key observation is that $(-)^H$ preserves retracts, relevant pushouts, and direct colimits. Therefore, by checking it on the generating objects, we obtain the following lemma:

Lemma 3.3. *If $\mathcal{X} \in \mathcal{P}(\mathbf{Orb}_G)$ is cofibrant, then the unit of the adjunction*

$$\eta : \mathcal{X} \rightarrow \Phi\Theta(\mathcal{X})$$

is an isomorphism.

Proof. It's sufficient to check when $\mathcal{X} = \mathrm{Map}_G(G/H, -) \times \mathcal{Y}$. And we have that

$$\mathcal{X}(G/K) = \mathrm{Map}_G(G/H, G/K) \times \mathcal{Y} \cong (G/H)^K \times \mathcal{Y} \cong (\mathcal{X}(G/e))^K.$$

□

The lemma leads to the following proof of the main theorem.

Proof of Elmendorf's theorem. Given $f : \Theta(\mathcal{X}) \rightarrow Y$, we have that

$$\mathcal{X}(G/H) \xrightarrow{\eta_H} (\mathcal{X}(G/e))^H \xrightarrow{f^H} Y^H.$$

By the previous lemma, η_H is a weak equivalence. The 2-out-of-3 axiom then shows that f^H is a weak equivalence iff the composition $\mathcal{X}(G/H) \rightarrow Y^H$ is. Since weak equivalences on $\mathcal{P}(\mathbf{Orb}_G)$ and $G\mathbf{Top}$ are defined level-wise, $f : \Theta(\mathcal{X}) \rightarrow Y$ is a weak equivalence iff its adjoint $\mathcal{X} \rightarrow \Phi(Y)$ is. □

3.2 Construction using bar construction

Since (Θ, Φ) is a Quillen equivalence, it induces an equivalence of homotopy categories, i.e. the derived functors $(L\Theta, R\Phi)$ are equivalences of homotopy categories. In the original proof in [Elm83], the left derived functor $\Psi = L\Theta$ is constructed explicitly using bar construction.

Definition 3.4. Let \mathcal{C} be a category and let \mathcal{D} be a category equipped with a Cartesian product \times . Let F be a contravariant functor $\mathcal{C} \rightarrow \mathcal{D}$ and let G be a covariant functor $\mathcal{C} \rightarrow \mathcal{D}$. The construction $B_*(F, \mathcal{C}, G)$ defines a simplicial \mathcal{D} object whose n -th level is

$$\coprod_{c_n \rightarrow \dots \rightarrow c_0} F(c_0) \times G(c_n).$$

The face maps are defined by composition and the degeneracies by inserting the identity map.

When \mathcal{D} is tensored over **Top**, we can take the geometric realization of this simplicial object, and we obtain the bar construction $|B_*(F, \mathcal{C}, G)|$.

By some general theory of bar construction, when G is a functor corepresented by $c \in \mathcal{C}$, i.e. $G = G_c = \text{Map}_{\mathcal{C}}(c, -)$, there is a natural homotopy equivalence obtained by composing and apply F :

$$|B_*(F, \mathcal{C}, G)| \rightarrow F(c).$$

Now we apply the above facts in our case. Let \mathcal{C} be \mathbf{Orb}_G , F be X and let G be the functor $J : \mathbf{Orb}_G \rightarrow G\mathbf{Top}$ which realizes each orbit G/H to the corresponding G -space G/H . Since G acts on J in a way compatible with the face and degeneracy maps, the bar construction gives a G -space.

Define $\Psi : \mathcal{P}(\mathbf{Orb}_G) \rightarrow G\mathbf{Top}$ to be

$$\Psi(X) := |B_*(X, \mathbf{Orb}_G, J)|.$$

Bar construction commutes with fixed point functors. Therefore we have

$$(\Psi(X))^H \cong |B_*(X, \mathbf{Orb}_G, J^H)| \xrightarrow{\sim} X(G/H).$$

In other words, Ψ gives a homotopy inverse of Φ .

References

- [May82] J. P. May *Equivariant homotopy and cohomology theory*
- [Elm83] A. D. Elmendorf. *Systems of fixed point sets*. Transactions of the American Mathematical Society, 277(1):275–284, 1983. pdf
- [B] A. Blumberg *Lecture notes* pdf
- [HTT] J. Lurie *Higher topos theory* pdf

- [HA] J. Lurie *Higher algebra* [pdf](#)
- [SHGH] J. F. Adams. Stable Homotopy and Generalised Homology.
- [Ad] J. F. Adams. Prerequisites (on equivariant stable homotopy) for Carlsson's lecture.
- [Ati] M. F. Atiyah. K-theory and reality.
- [HILS] F. R. Cohen, T. J. Lada, J. P. May. The Homology of Iterated Loop Spaces.
- [Dug] An Atiyah-Hirzebruch spectral sequence for KR-theory.
- [HoA] S. I. Gelfand, Y. I. Manin. Homological Algebra.
- [SHT] P. G. Goerss, J. F. Jardine. Simplicial Homotopy Theory.
- [Gr] J. P. C. Greenlees. Equivariant connective K-theory for compact Lie groups.
- [GM] B. Guillou, J. P. May. Models of G -spectra as presheaves of spectra.
- [HHR] M. A. Hill, M. J. Hopkins, D. C. Ravenel. On the nonexistence of elements of Kervaire invariant one.
- [ESHT] M. A. Hill, M. J. Hopkins, D. C. Ravenel. Equivariant Stable Homotopy and the Kervaire Invariant One Problem.
- [MMSS] M. A. Mandell, J. P. May, S. Schwede, B. Shipley. Model categories of diagram spectra.
- [GILS] J. P. May. The Geometry of Iterated Loop Spaces.
- [EHCT] J. P. May. Equivariant Homotopy and Cohomology Theory.
- [CCAT] J. P. May. A Concise Course in Algebraic Topology.
- [NS] T. Nikolaus, P. Scholze. On topological cyclic homology.
- [Oli] R. Oliver. A proof of the Conner conjecture.
- [RV] H. Reich, M. Varisco. On the Adams isomorphism for equivariant orthogonal spectra.
- [Rob] M. Robalo. K-theory and the bridge from motives to noncommutative motives.
- [GHT] S. Schwede. Global Homotopy Theory.
- [Sch] S. Schwede. Lecture notes on equivariant stable homotopy theory.
- [May09] May, J. P. "What precisely are E_∞ ring spaces and E_∞ ring spectra?." *Geometry & Topology Monographs* 16 (2009): 215-282.

- [MM] Mandell, Michael A., and J. Peter May. Equivariant orthogonal spectra and S -modules. American Mathematical Soc., 2002.
- [Day] Day, Brian. "On closed categories of functors." In Reports of the Midwest Category Seminar IV, pp. 1-38. Springer, Berlin, Heidelberg, 1970.
- [Seg] Segal, Graeme. "Categories and cohomology theories." Topology 13, no. 3 (1974): 293-312.