# IWOAT SUMMER SCHOOL 2025: COMPUTATIONS WITH THE ADAMS SPECTRAL SEQUENCE AND THE LAST KERVAIRE INVARIANT PROBLEM

ABSTRACT. In this summer school we will study the Adams spectral sequence, new methods in computing Adams differentials and its applications in resolving the Last Kervaire Invariant Problem [6]. We will study the $H\mathbb{F}_2$-synthetic spectra and the synthetic Adams spectral sequence. We will see how Adams differentials and extensions interplay with each other in the synthetic setting and how this feature can be used to formulate the Generalized Leibniz Rule and the Generalized Mahowald Trick. In the end, we will discuss the adhoc proof of the Last Kervaire Invariant Problem and its implications in the field of algebraic topology.
The summer school is aimed at senior undergraduate students, graduate students and young researchers in algebraic topology. The school will be held from July 1 to July 5, 2025.

**Prerequisites.** Stable homotopy group, homological algebra, spectral sequence.

**Day 1: Adams spectral sequence.** [1] [2] [7]

(1) (Alexander Waugh) Overview of the Kervaire invariant problem.
(2) (Ethan MacBrough) Cohomology, Steenrod algebra and Eilenberg-MacLane spectrum. Modules over Hopf algebras, tensor product and dual of $A$-modules ($H^*(X\wedge Y)$, $H^*(DX)$). Adams spectral sequence. Minimal resolutions and $\mathrm{Ext}^{\leq 1}(X)$. Leibniz rule and naturality. Proofs for $d_2h_i = h_0h_{i-1}^2$.
(3) (Shangjie Zhang) May spectral sequence. Elements generated by $h_i$ in $\mathrm{Ext}(S^0)$. Some massey products on the $E_2$-page of Adams (example: $\langle h_0, h_1, h_0\rangle = h_1^2$). Secondary steenrod operations and the Adams $d_2$-differential.
(4) (Shangjie Zhang) Problem session: Compute $\mathrm{Ext}(bo)$ using May spectral sequence or minimal resolution.

**Day 2: Extensions.** [10] [8] [6]

(1) (Zhen Gao) Massey products and Toda brackets. Convergence. Moss's theorem.
(2) (Yueshi Hou) Long exact sequences of Ext and $\pi_*$. Massey products $\langle h_i, x, y\rangle$ and $\mathrm{Ext}(Ch_i)$. Toda brackets $\langle \alpha_i, \beta, \gamma\rangle$ and $\pi_*(C\alpha_i)$. Matric brackets and finite CW spectra.
(3) (Manyi Guo) Extension problems. Extension spectral sequences. Crossings. Commutative diagrams and extension spectral sequences.
(4) (Yueshi Hou) Problem session: Find interesting Massey products and Toda brackets using the database of Adams spectral sequences and extension spectral sequences [5].

**Day 3: $H\mathbb{F}_2$-synthetic spectra.** [9] [4] [6]

(1) (Cheng Li) Overview of synthetic spectra. Adams filtration and $\lambda$-divisibility. Synthetic Adams spectral sequence. Rigidity. $E_\infty$-page of $X/\lambda^n$.
(2) (Yuwen Gu) Synthetic extension spectral sequences. Extensions along

$$\Sigma^{0,-n}\nu X/\lambda^{m-n} \xrightarrow{\lambda^n} \nu X/\lambda^m \xrightarrow{\rho_{n,m}} \nu X/\lambda^n \xrightarrow{\delta_{n,m}} \Sigma^{1,-n}\nu X/\lambda^{m-n}.$$

Classical Adams differentials and $d^\delta$-extensions. Extensions on $E_r$-page and crossings.
(3) (Yuwen Gu) Problem session: Define crossings of Adams differentials on $E_r$-page.

**Day 4: Generalized Leibniz Rule and Generalized Mahowald Trick.** [6]

(1) (Rixin Fang) Generalized Leibniz Rule.
(2) (Yuchen Wu) Generalized Mahowald Trick part 1. May's lemma on triangular categories with symmetric monoidal structure.
(3) (Yuchen Wu) Generalized Mahowald Trick part 2.
(4) (Zhonglin Wu) Problem session: Find applications of Generalized Leibniz Rule and Generalized Mahowald Trick in the database. Find applications using Lin's computer program.

**Day 5: The Last Kervaire Invariant Problem.** [3] [6]

(1) (Alexander Waugh) Quadratic construction. Dyer-Lashof operations and Nishida relations. Finite CW spectra and cell diagrams.
(2) (Yunze Lu) The inductive method of the Kervaire invariant problem.
(3) (Sihao Ma) The adhoc proof of the Last Kervaire Invariant Problem.

## References


[1] J. F. Adams. On the structure and applications of the Steenrod algebra. *Comment. Math. Helv.*, 32:180–214, 1958.

[2] J. F. Adams. On the non-existence of elements of Hopf invariant one. *Ann. of Math. (2)*, 72:20–104, 1960.

[3] M. G. Barratt, J. D. S. Jones, and M. E. Mahowald. The Kervaire invariant problem. In *Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982)*, volume 19 of *Contemp. Math.*, pages 9–22. Amer. Math. Soc., Providence, RI, 1983.

[4] Robert Burklund, Jeremy Hahn, and Andrew Senger. On the boundaries of highly connected, almost closed manifolds. *Acta Math.*, 231(2):205–344, 2023.

[5] Weinan Lin. Interactive plot of the Adams spectral sequence. `https://waynelin92.github.io/ss/kervaire-49.html` Github Webpage 2024.

[6] Weinan Lin, Guozhen Wang, and Zhouli Xu. On the Last Kervaire Invariant Problem. *arXiv Preprint, arXiv:2412.10879*, 2024.

[7] J. P. May. The cohomology of restricted Lie algebras and of Hopf algebras. *J. Algebra*, 3:123–146, 1966.

[8] J. Peter May. Matric Massey products. *J. Algebra*, 12:533–568, 1969.

[9] Piotr Pstragowski. Synthetic spectra and the cellular motivic category. *Invent. Math.*, 232(2):553–681, 2023.

[10] Hirosi Toda. *Composition methods in homotopy groups of spheres*, volume No. 49 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1962.