Problem Set 2
Intertemporal Choice

1. **Population Growth and Dynamic Inefficiency in the OLG Model.**

   Consider a Diamond (1965) OLG economy like the one in the handout `OLGModel`, assuming logarithmic utility and a Cobb-Douglas aggregate production function, except that population growth is not necessarily constant.

   a) Derive an equation for the evolution of capital per worker in this economy between periods t and $t+1$ assuming that population dynamics are given by $L_{t+1} = \Xi_{t+1} L_t$ where L is the population of young households and $\Xi \equiv (1+\xi)$ is the growth factor for the population of young people.

   b) Derive the steady-state level of k_t that the economy achieves if the rate of population growth is constant at $\Xi_t = \Xi \ \forall \ t$.

   Now suppose that the economy had been growing by this constant factor Ξ since the beginning of time, but all of a sudden at the beginning of period t everybody learns that henceforth and forever more, population will grow at a faster rate than before, $\hat{\Xi} > \Xi$.

   c) Will the new steady-state level of the capital stock per capita associated with $\Xi = \hat{\Xi}$ be larger or smaller than before? *Explain your answer.*

   d) (Use the programming language you are assigned to complete this exercise) Suppose in the OLG model, each period t corresponds to 30 years, so each generation lives for 60 years. Assume that the time preference factor per annum is 0.96 (Hint: this suggests $\beta = 0.96^{30}$), $\alpha = 0.33$. For convenience, the range of k_t and k_{t+1} is restricted from 0 to 0.05.

      i. Suppose population growth rate per annum is 0.01. Draw a diagram to show the $\mathrm{k}_{t+1}(k_t)$ curve. Indicate the steady state level of k_t on the graph (call it $\bar{k}_0$). (Hint: You've seen this kind of graph in the OLG lecture notes).

      ii. Suppose population growth rate per annum is 0.03. Using a different color, draw a diagram to show the $\mathrm{k}_{t+1}(k_t)$ curve. Indicate the steady state level of k_t on the graph (call it $\bar{k}_1$).

      iii. Draw a 45-degree line. Show the dynamic adjustment process for the capital stock from its old steady state $\bar{k}_0$ toward its new steady-state $\bar{k}_1$. You only need to consider the first period after the adjustment takes place.

   Does the graph support your answer to the previous part of this question?

   e) Define an index of aggregate consumption per person in period t as $\chi_t = c_{1,t} + c_{2,t}/\Xi$. First, explain why this is an appropriate measure of consumption per person (remember that lower-case variables are the upper-case version divided by L_t). Then derive a formula for the sustainable level of χ associated with a given level of k_t.

f) Show that a marginal increase in the population growth rate will never result in an increase in the steady-state level of consumption per capita, and explain in words why this result holds.

2. **Social Security and the Baby Boom.** Analysts often comment that the U.S. Social Security system is in trouble because of the large size of the 'baby boom' generation. Consider a society in which there is a single generation, born at date t, that is larger than both the generations before it (born in periods $t-1$ and earlier) and the generations after it (born in periods $t+1$ and later). Discuss what the generational accounting framework says about the effects on various generations from such a 'baby boom,' in a Pay As You Go social security system. (Specifically, compare a policy that keeps per-capita benefits constant across generations to a policy that keeps per-capita taxes constant across generations).

3. **The Value Function, the Envelope Theorem, and Dynamic Programming.** Consider a consumer who solves the following problem:

$$\max \mathrm{u}(c_1) + \beta \mathrm{u}(c_2) + \beta^2 \mathrm{u}(c_3)$$

subject to

$$\begin{aligned} m_{t+1} &= (m_t - c_t)\mathsf{R}_{t+1} \\ m_1 &\quad \text{is given exogenously} \\ m_4 &\geq 0 \\ \mathrm{u}(c) &= \log c \end{aligned}$$

a) (Easy). What is consumption in period 3 as a function of cash-on-hand (m) in period 3? What is the value function $\mathrm{v}_3(m_3)$?

b) Solve for consumption in period 2 as a function of m_2. Obtain an explicit analytical expression for the value function in period 2 as a function of assets in period 2. Verify that the Envelope theorem holds, i.e. that $\mathrm{v}_2'(m_2) = \mathrm{u}'(\mathrm{c}_2(m_2))$ where $\mathrm{c}_2(m_2)$ is the optimal period 2 consumption rule.

c) Solve for consumption in period 1 as a function of the initial endowment m_1. Note the point at which you can use the Envelope theorem. When you have obtained $\mathrm{c}_1(m_1)$, verify that the Envelope theorem holds in period 1.

References

Diamond, Peter A. (1965): "National Debt in a Neoclassical Growth Model," *American Economic Review*, 55, 1126–1150, http://www.jstor.org/stable/1809231.