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Life-Cycle Economies and Aggregate Fluctuations

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Do the implications for business cycle issues change when we switch from studying infinitely-lived, representative-agent models to more sophisticated demographic structures with finitely lived agents? This article addresses that question by using a large, overlapping-generations model that is calibrated to U.S. demographic properties, microeconomic evidence, and National Income and Product Accounts. The finding is that the answers obtained are basically the same for the two kinds of models. The article also explores the relative volatility of hours across age groups, an issue that cannot be addressed by using the infinitely-lived, representative-agent abstraction.

1. INTRODUCTION

Since the work of Kydland and Prescott (1982), considerable success has been achieved in research by using general equilibrium models of the business cycle. This body of work tries to assess the contribution of different factors to aggregate fluctuations as well as study related issues. Common to all this work is the technological structure of the neoclassical growth model and the infinitely-lived representative-agent abstraction.[1]

In this article, I pose the following question: How do the cyclical properties of calibrated life-cycle models compare to the properties of the infinitely-lived model? In particular, do the main quantitative properties for business cycles of the models with infinitely-lived agents hold in a model with a more sophisticated demographic structure and finitely-lived agents? I address these questions by using an overlapping-generations model where agents live up to 75 periods. The model is calibrated to U.S. demographic properties, microeconomic evidence, and National Income and Product Accounts. By this I mean the following things: the demographic properties of the model correspond to those in the United States; the parameters that characterize preferences come from empirical microeconomic studies done in appropriate contexts; and the technology parameters are chosen so that, in the steady state, the model economy matches some growth relations found in the data.

In this model, agents engage in active asset trading and choose how much to work and how much to consume. This is done in a complete market setting, one where they can insure against all contingencies. An approximation of the model economies' equilibrium laws of motion is computed and used to generate a large set of samples. From this simulated data certain key statistics are calculated. The business cycle properties of the model are then compared to the data and to the findings of the existing real business cycle literature.

This article also explores the relative volatility of hours across age groups, an issue that clearly cannot be addressed by using the infinitely-lived, representative-agent abstraction.

1. The volume edited by Cooley (1995) includes surveys on the state of general equilibrium business cycle theory and many applications.

Specifically, I ask whether differences in relative volatility of hours across age groups can be accounted for by the interaction of agents who differ in age and have different life-cycle profiles of wages. This also permits a measure of the labour input to be constructed in the model, a measure that weighs hours from different types of workers differently. This issue is related to the well-known fact that unskilled workers have a higher volatility of hours worked than skilled workers do (see Clark and Summers (1981), Kydland (1984), Kydland and Prescott (1988), Ríos-Rull (1993), and Hansen (1993)).

The findings can be summarized as follows:

- In overlapping-generations models, fluctuations in the Solow residual can account for between 55% and 64% of the variance of output. This compares with the value of 62% found in Cooley and Prescott (1995).
- The relative volatilities of aggregate consumption and investment and the correlations of aggregate variables with output are in line with both the data and the representative-agent, real business cycle models.
- The model does fairly well in replicating the relative behaviour of hours worked by different age groups. Specifically, versions of the model calibrated to agents who live a large number of periods (16 to 90 years), who have relatively low risk aversion, and who have a high valuation of leisure generate an age profile of volatility of hours that is very similar to that in the data; the exception is the group between 25 and 44 years of age, which is underpredicted. These economies replicate the feature of the data that those measures of the labour input that put more weight on the hours of highly skilled agents have a volatility that is lower than aggregate unweighted hours.

This article also provides a methodological contribution. The class of stochastic economies whose quantitative properties can be explored is extended to include large overlapping-generations economies of the type used by Auerbach and Kotlikoff (1987) in deterministic settings.[2] I describe an operator and show how an equilibrium can be obtained as its fixed point. Computation of the equilibrium is achieved by successive applications of this operator.

Section 2 of the article describes the economy and defines its equilibrium. In Section 3 a certain operator is defined, successive applications of which are used to compute an approximation to the equilibrium. In Section 4 calibration is discussed, and a vector of parameters that constitute the baseline economy is described. Section 5 describes the steady-state implications of the parameterization and the implications of some alternatives. Section 6 reports the findings from a comparison of the statistical properties of the model with the data. Section 7 concludes. An appendix includes some proofs.

2. THE MODEL

The economy has overlapping generations of agents who live a maximum of I periods, with ages denoted by $i \in \mathscr{I} \equiv \{1, \ldots, I\}$. They can die earlier; the probability of surviving between age i and age $i+1$ is denoted by s_i, with $s_I = 0$. The unconditional probability of being alive at age i is $s^i = \prod_{j=1}^{i-1} s_j$. In the first period, the measure of newly born agents is

2. There are a few papers that compute the equilibrium in stochastic, overlapping-generations economies, such as Ni (1989), Baxter (1989), Labadie (1984), and Ríos-Rull (1993). However, the methods used in those papers are not easily extendable to economies where agents live more than two periods. Altig and Carlstrom (1991) use a stochastic version of Auerbach and Kotlikoff's model, but their "stochastic path" is not an equilibrium path.

normalized to 1. The population grows at a constant rate λ_μ, implying that the population of age 1 in period t is $\mu_1(t)=(1+\lambda_\mu)^t$, while that of age 2 is $\mu_2(t)=s_1(1+\lambda_\mu)^{t-1}$, and so on, up to $\mu_I(t)=s^{I-1}(1+\lambda_\mu)^{t+1-I}$ agents who are in the last period of life. This structure of the population is called the *stable population* and it has the property that the relative sizes of different cohorts do not change over time.

The technology of the model economy has neoclassical properties. There is a standard production function that is subject to constant, labour-augmenting productivity growth; the production function is also subject to multiplicative shocks given by $z_t f(K, (1+\lambda_y)^t N)$, where λ_y is the rate of growth of labour productivity, K is the capital input, and N is the labour input. The production function has the property that $\lim_{K\to\infty} f_1(K, (1+\lambda_y)^t N)=0$ for all t and $N>0$. Output can be used either for consumption in the same period that production takes place or for increasing the capital stock of the next period. Capital depreciates at rate δ. The multiplicative shock, $z_t \in Z \equiv \{z^1, z^2, \ldots, z^{n_z}\}$, follows a Markov process with a strictly positive transition matrix Γ, and it is observed at the beginning of the period. A typical history of shocks up to time t is denoted by $h_t \equiv \{z_0, z_1, \ldots, z_t\}$, with $h_{t+1}=\{h_t, z_{t+1}\}$. The set of possible histories up to time t, H_t, is a finite set, and $H \equiv \{H_t\}_{t=0}^{\infty}$ is a countable set. This suggests that commodities be indexed by history.[3]

Agents are endowed with one unit of time per period that can be allocated to leisure or to labour. An age i agent's unit of time can be transformed into one unit of leisure or ε_i units of labour input; $\varepsilon_i>0$ is an age-specific, exogenously given productivity parameter. Let $\{c_i(h_t), l_i(h_t)\}$ denote a consumption leisure pair for an age i agent in period t and after history h_t. The preferences of an agent born in period τ and after history h_τ are given by

$$E\{\textstyle\sum_{i\in\mathcal{I}} \beta^i s^i U(c_i(h_{\tau+i-1}), l_i(h_{\tau+i-1}))|h_\tau\}, \tag{1}$$

where $U(c, l)$ is a twice-continuously differentiable, strictly concave function such that $\lim_{c\to 0} U_1(c, l)=\infty$ for all $l>0$, and such that $\lim_{l\to 0} U_2(c, l)=\infty$ for all $c>0$. Note that the expectation takes into account the fact that agents might die before the last possible period of their lives. It does this by setting the value of the utility function to zero if the agent is dead. (After death agents obviously have no use for goods and no ability to work.)

In every period agents receive capital income, $a_i(h_t)R(h_t)$, where $a_i(h_t)$ is the amount of real assets that an age i agent holds in period t after history h_t, and $R(h_t)$ is the gross rate of return on the asset. Agents also receive labour income, $(1-l_i(h_t))\varepsilon_i W(h_t)$, where $W(h_t)$ is the real wage per efficient unit of labour in terms of the consumption good after history h_t. Agents use these funds either for consumption, $c_i(h_t)$, or for purchasing state-contingent claims, $b(h_t, z_{t+1})$. These claims pay one unit of the good at the beginning of the following period if the state is z_{t+1}; otherwise they pay zero. Since the claims pay at the beginning of the period, before production takes place, the goods delivered can be used as capital input in period $t+1$. Let $q(h_t, z_{t+1})$ be the price of such a security. If they are given the option of buying claims that pay only in the event of their survival, agents will place all their savings on these claims (since they have no use for goods after death). The fact that there is a large number of agents implies that there is no aggregate uncertainty regarding the size of the surviving cohort. Thus there could be firms that offer the following contract to all individuals of the same cohort i: one unit of security $b(h_t, z_{t+1})$ in this period exchanges for $1/s_i$ units of the capital good in the next period if the shock is z_{t+1}

3 Population size is exogenous and independent of the history of shocks; therefore I do not index cohort size by history, but only by period.

and if the agent survives; otherwise one unit of security exchanges for zero. This contract is actuarially fair, and agents will put all their securities in such contracts. These arrangements are termed *annuities contracts.* Agents are born with zero non-human wealth. Besides the sequence of budget constraints, there is also a zero-debt restriction in the last possible period of agents' lives. The budget constraints of an agent born in period τ and after history h_τ are written as follows:

$$a_1(h_\tau)=0, \tag{2}$$

$$a_i(h_{\tau+i-1})R(h_{\tau+i-1})+(1-l_i(h_{\tau+i-1}))\varepsilon_i W(h_{\tau+i-1})$$
$$=\sum_{z_{\tau+i}} b_i(h_{\tau+i-1}, z_{\tau+i})q(h_{\tau+i-1}, z_{\tau+i})+c_i(h_{\tau+i-1}), \quad \text{for } i=1,\ldots,I-1. \tag{3}$$

$$a_{i+1}(h_{\tau+i-1}, z_{\tau+i})=b_i(h_{\tau+i-1}, z_{\tau+i})/s_i, \quad \text{for } i=1,\ldots,I-1. \tag{4}$$

$$a_{I+1}(h_{\tau+I})\geqq 0. \tag{5}$$

The aggregate feasibility constraint is

$$K(h_t)+\sum_{i\in\mathcal{I}} \mu_i(t)c_i(h_t)\leqq z_t f(K(h_{t-1}), (1+\lambda_y)^t N(h_t))+(1-\delta)K(h_{t-1}), \tag{6}$$

where the aggregate inputs are

$$N(h_t)=\sum_{i\in\mathcal{I}} \mu_i(t)(1-l_i(h_t))\varepsilon_i \quad \text{and} \quad K(h_{t-1})=\sum_{i\in\mathcal{I}} \mu_i(t)a_i(h_t). \tag{7}$$

The initial conditions of this economy are an initial realization for the shock z_0 and a distribution of wealth across agents in each age group, $\{k_i\}_{i\in\mathcal{I}}$, with the property that aggregate capital is strictly positive, $K=\sum_{i\in\mathcal{I}} \mu_i(0)k_i>0$. The population structure used, with constant relative cohort size, allows us to ignore demographic variables as proper initial conditions, since their relative size remains constant over time.

Definition 1. A competitive equilibrium is a set of stochastic processes for individual allocations, $\{c_i(h_t),\ l_i(h_t)\}_{i\in\mathcal{I}}$; wealth holdings, $\{a_i(h_t)\}_{i\in\mathcal{I}}$; real security holdings, $\{b_i(h_t, z_{t+1})\}_{i\in\mathcal{I}}^{z_{t+1}\in Z}$; aggregate inputs, $\{K(h_t),\ N(h_t)\}$; prices for the real securities, $\{q(h_t, z_{t+1})\}_{z_{t+1}\in Z}$; and prices for the factors of production, $\{R(h_t),\ W(h_t)\}$, for all $h_t\in H_t$, and all $H_t\in H$, all subject to the following: that the allocations are feasible, and that for all agents they maximize (1) subject to (2)–(5); that prices for the factors are the marginal productivities

$$R(h_t)=z_t f_1(K(h_{t-1}), (1+\lambda_y)^t N(h_t))+1-\delta, \tag{8}$$

$$W(h_t)=z_t f_2(K(h_{t-1}),(1+\lambda_y)^t N(h_t)); \tag{9}$$

and that the markets for real securities clear in this way:

$$\sum_{i\in\mathcal{I}} \mu_i(t)b_i(h_t, z_{t+1})=K(h_t), \quad \text{for all } h_t\in H_t, \text{ all } H_t\in H, \text{ all } z_{t+1}\in Z. \tag{10}$$

This last condition implies, as is standard, that aggregate capital is measurable with respect to the history up to one period before. Due to the existence of securities, however, the distribution of wealth is determined later, when the following period's shock is realized and contracts are honoured. In equilibrium the prices of the securities have to add up to one, because of a simple arbitrage argument.[4] Note that the equilibrium implications of

4. Implicit in the market specification is the existence of firms that intermediate in the market for securities. Their role is to purchase (sell) the good and sell (purchase) the securities. They store (at zero cost, since depreciation takes place during production) the good and then, when the next period's shock is observed, they honour the securities. Their problem in h_t is to $\max_y \sum_{z_{t+1}\in Z} q(h_t, z_{t+1})y-y$. This problem only has a solution if $\sum_{z_{t+1}\in Z} q(h_t, z_{t+1})=1$.

these securities are the same as for those securities that deliver units of the consumption good; their prices will differ, with the price of the latter being equal to $q(h_t, z_{t+1})R(h_{t+1})$.

Proposition 1. *A competitive equilibrium exists.*

Proof. See Appendix. ||

This economy has an interpretation that all trade takes place at time zero (see Wright (1987)). In order for this to happen, agents born under different histories are considered to be different agents. Under this interpretation, agents do not want to insure themselves against events that preceded their birth. Agents face two types of contingencies, the aggregate productivity shock z_t and individual death. The fact that agents have no use for goods if they are dead guarantees that the annuities markets described above suffice to perfectly insure against this contingency. The one-period-ahead securities allow agents to perfectly insure against the aggregate productivity shock (Arrow (1964)). Furthermore, if the price sequence belongs to the space of sequences that, if every component t is divided by $(1+\lambda_y)(1+\lambda_\mu))^t$, are absolutely summable, then the competitive equilibrium will also be an Arrow–Debreu (valuation) equilibrium, in the sense that the price system will be a linear function defined over the commodity space (see Debreu (1954)). This property prevents money from having a positive price, and it guarantees Pareto optimality of equilibrium allocations. Pareto optimality can be guaranteed if expected rates of return at every node are bigger than one (see Zilcha (1991)). This is not proven, but, as we will see later, the simulations indicate that this is the case for the model economies studied.

The economy described is not stationary. However, with preferences of the CRRA class, say $U(c, l) = ((c^\alpha l^{1-\alpha})^{1-\sigma})/(1-\sigma)$, the economy can be rewritten as stationary by applying standard transformations. Allocations, prices, and preferences are re-defined in the following way: re-dimension the size of each cohort by letting the newborn be of measure 1, while those of age i are of measure $s^i/(1+\lambda_\mu)^{i-1}$; re-define individual variables by dividing them by $(1-\lambda_y)^t$; divide aggregate variables by $((1-\lambda_y)(1+\lambda_\mu))^t$; and divide wages by $(1+\lambda_y)^t$. Regarding preferences, re-normalize the discount rate by letting $\beta + \beta(1-\lambda_y)^{\alpha(1-\sigma)}$. It is trivial, but cumbersome, to show that a competitive equilibrium in the original economy implies a stationary equilibrium in the normalized economy (in the sense that there exist finite bounds for the allocations that are common to every period). Conversely, equilibria in the normalized economy are stationary and imply non-stationary counterparts in the original economy.

In all the numerical exercises described below, the data from both the artificial economies and from the U.S. were transformed, first by taking logs, and then by applying the Hodrick–Prescott filter (see, for example, Cooley and Prescott (1995)). The deviations obtained by this procedure from a series x_t are the same as those obtained from $(1+\lambda_y)^t x_t$ with $\lambda_y > 0$. Henceforth I will look only at the transformed economy.

Although this standard definition of equilibrium is useful for proving existence and discussing optimality, it is not very convenient for computational purposes. For this reason, I define *recursive competitive equilibrium*, a construct that allows one to find equilibrium allocations without keeping track of the whole history of the economy. Allocations that satisfy the conditions of recursive equilibrium are also equilibria in the more standard sense (the reverse is not necessarily true, however, as the equilibrium path can be time-dependent).

Before proceeding, the state variables have to be defined both at the economy-wide level and at the agent's level. The state of the economy is characterized by the economy-wide shock $z \in Z$ and by the distribution of asset holdings by agents in each age group, $k = \{k_i\}_{i \in \mathcal{I}}$. The state variables for any given agent are its own asset holdings a and the economy-wide state $\{z, k\}$. Since all agents of the same age share the same strictly concave utility function and convex choice set, in equilibrium they make the same choices, which implies that $a_i = k_i$. Therefore, the aggregate laws of motion for the economy depend only on the decision rules of the agents and on the process for the shock; since this follows a Markov process, the current value of the shock and of the distribution of asset holdings is all the information needed to know tomorrow's distribution. The value functions involved in the definition are indexed by the age of the agent, and these functions measure expected remaining utility. As is customary in recursive formulations, primes are used to denote the next period's variables. Formally, the definition is as follows:

Definition 2. A recursive stochastic competitive equilibrium is a set of decision rules, $\{\{b_i(z, k, a, z')\}_{z' \in Z},\ l_i(z, k, a),\ c_i(z, k, a)\}_{i \in \mathcal{I}}$ and corresponding value functions, $\{v_i(z, k, a)\}_{i \in \mathcal{I}}$; functions for aggregate factors of production, $K(z, k)$ and $N(z, k)$, and for their prices, $R(z, k)$ and $W(z, k)$; a law of motion for the distribution of wealth, $\{g_i(z, k, z')\}_{i \in \mathcal{I}}^{z' \in Z}$; and a function for securities prices, $\{q(z, k, z')\}_{z' \in Z}$, such that the following conditions hold:

(a) The allocation is feasible, i.e. for all $z' \in Z$,

$$\begin{aligned}\textstyle\sum_{i \in \mathcal{I}} \mu_i(b_i(z, k, k_i, z') + c_i(z, k, k_i))\\ = zf(K(z, k), N(z, k)) + (1-\delta) \textstyle\sum_{i \in \mathcal{I}} \mu_i k_i.\end{aligned} \tag{11}$$

(b) Factor prices equal marginal productivities, i.e.

$$R(z, k) = zf_1(K(z, k), N(z, k)) + 1 - \delta, \tag{12}$$

$$W(z, k) = zf_2(K(z, k), N(z, k)). \tag{13}$$

(c) Given the law of motion for capital stocks, the price functions, and the transition for z, the decision rules for age i agents then solve their maximization problem, and the solutions generate the value functions, i.e. $\{\{b_i(z, k, a, z')\}_{z' \in Z}, l_i(z, k, a), c_i(z, k, a)\}$ solve

$$\begin{aligned} v_i(z, k, a) = \max\nolimits_{b(z'), l, c} U(c, l) + \beta E\{v_{i+1}(b(z'), g(z, k, z'), z') \mid z\}\\ \text{s.t. } aR(z, k) + (1-l)\varepsilon_i W(z, k) = c + \textstyle\sum_{z' \in Z} b(z') q(z, k, z').\end{aligned} \tag{14}$$

(d) The law of motion of asset holdings is generated by the decision rules of the agents, i.e.

$$\begin{aligned} &g_1(z, k, z') = 0,\\ &g_{i+1}(z, k, z') = b_i(z, k, k_i, z')/s_i \qquad z' \in Z, i = 1, \ldots, I-1.\end{aligned} \tag{15}$$

(e) Aggregate functions K and N are generated by aggregation of the decision rules of the individuals, i.e.

$$K(z, k) = \textstyle\sum_{i \in \mathcal{I}} \mu_i k_i, \tag{16}$$

$$N(z, k) = \textstyle\sum_{i \in \mathcal{I}} \mu_i (1 - l_i(z, k, k_i)) \varepsilon_i. \tag{17}$$

(f) There are no arbitrage opportunities:

$$\textstyle\sum_{z' \in Z} q(z, k, z') = 1. \tag{18}$$

Notice that the whole distribution of wealth is needed as a state variable. This is not obvious, since the role of the state variables is, from the point of view of individual agents, to help determine prices, which in turn depend only on the level of aggregate factors and on the shock. However, the labour-leisure choices will depend on how wealth is distributed across agents, and this makes the dependence of factor prices on the distribution explicit. More important, perhaps, is the fact that the state variables should also be sufficient statistics in the determination of future prices; this depends on the consumption-saving decisions of all agents, which in turn depend on the entire wealth distribution. Note also that there are no conditions that impose market clearing for securities. Since aggregate savings cannot depend on the next period's realization of the shock, the aggregate feasibility constraints already embody these market-clearing conditions.

To see that a recursive equilibrium is also a competitive equilibrium, one can successively use the functions provided by the recursive equilibrium to construct stochastic processes for prices and quantities. It is straightforward, but tedious, to check that the constructed processes satisfy the definition of competitive equilibrium.

3. COMPUTATION

In this section I describe the computation procedure; readers who are less interested in this issue and more interested in the substantive results could skip to Section 4 with no loss of continuity.

Characterization of the equilibria requires at least a set of $n_z \times (I-1)$ functions to describe the law of motion of the distribution of wealth (one per age group older than the newborns, per possible value of the shock tomorrow) and describe a function for the aggregate labour input. The arguments of the functions are the state variables, which include the whole distribution of wealth, a vector of I variables, and the current shock. Since I am interested in business cycles, the maximum length of a period that can be considered is one year. If we want to calibrate I—the maximum number of years that an adult person lives—to the characteristics of the U.S. demographics, then we need at least $I \geqq 50$.

Given these dimensions, computation would be prohibitive without the imposition of linear decision rules. Despite great advances in computational techniques and reductions in the cost of hardware, linear quadratic methods are still the approximation of choice for a large number of researchers.[5] For these reasons, the computation of equilibrium laws of motion is based on linear approximations. This can be done in two different ways: by using a linear approximation to a set of Euler equations or by using quadratic objectives. I follow the second approach, given that it fits neatly with the notion of recursive equilibria. The procedure that I follow uses successive approximations. A description of the tasks involved in computing the equilibria will now be given.

First, the deterministic steady state of the economy is computed with the shocks set constant at their unconditional means. Solving for the steady state basically amounts to finding the solution to a single equation in one unknown. It is a fixed point in capital-labour ratios: given a capital-labour ratio, factor prices are determined, which then determine the

5. For a comparison of different methods, see Taylor and Uhlig (1990).

behaviour of agents that face those constant factor prices. Aggregation of the age-specific allocations provides both aggregate labour input and aggregate capital, and, hence, a new capital-labour ratio. In all the cases studied, these functions turned out to be monotone-decreasing, implying a unique steady state. This procedure is described in more detail by Auerbach and Kotlikoff (1987).

Next, a quadratic approximation around the state is made. This step produces a set of symmetric matrices $\{Q_{iz}\}_{i\in\mathscr{I}}^{z\in Z}$. These matrices approximate the utility function of agents of age i when the current shock is z; the budget constraint is used to eliminate consumption from the objective function. Factor prices are eliminated by using marginal productivities as functions of the distribution of wealth, aggregate labour, and the shock. Strict concavity of U guarantees strict concavity of the quadratic forms associated with the individual variables. The expression obtained is $U(c, l) \approx (1, k, a, N, q, b, l)Q_{iz}(1, k, a, N, q, b, l)^T$, with equality at steady states (where ^{T} denotes "transpose"). The aggregate variables k and N and the individual variables a and l are set at their steady-state values. The vector of state-contingent securities, b, is set, for every $z' \in Z$, to equal the steady-state wealth of the agent one period later times the survival probability, $a_{i+1}s_i$. The vector of security prices, q, is set to the conditional probabilities, $\Gamma(z'|z)$, of the next period's shock z', given current shock z. Only the first $n_z - 1$ values of the price vector q are used: the last value is restricted to be one minus the sum of the other $n_z - 1$ values (this imposes the no-arbitrage condition). The dimension of the Q_{iz} matrices is $(I + 3 + 2n_z)$. The approximation is a numerical implementation of Taylor's second-order approximation.

To compute equilibria I successively approximate the vector of value functions by using an operator, $\hat{T}$, until a fixed point is found. This operator is not necessarily well-defined, as it requires matrix inversions, operations that are not always possible. For this reason, I define another operator, T, that is well-behaved. However, computationally T is much more intensive than $\hat{T}$. In Lemma 1, I show how fixed points of T constitute recursive equilibria, and in Lemma 2, I show how fixed points of $\hat{T}$ imply fixed points of T, which guarantees that whenever we find a fixed point of $\hat{T}$, we have a recursive equilibrium. The operator T maps value, price, and quantity functions into value, price, and quantity functions, while $\hat{T}$ maps value functions into value functions and generates its own set of functions that describe the behaviour of equilibrium prices and quantities.

I start by defining T. Let $\mathscr{V} \equiv \{v_{i,z}\}_{i=1,..,I+1}^{z\in Z}$, where each $v_{i,z}$ is a symmetric matrix of dimension $(I+2)$ such that the quadratic forms that it generates are concave in the last argument (the space where the approximated value functions lie). Let $\mathscr{N} \equiv \{N_z\}_{z\in Z}$, where every N_z is a vector of dimension $(I+1)$ (the set of candidates for the aggregate labour input function). Let $\mathscr{Q} \equiv \{q_{z,z'}\}_{z\in Z}^{z'\in\{z_1,\ldots,z_{n_z-1}\}}$, where each $q_{z,z'}$ is a vector of dimension $(I+1)$ (the space of candidates for the equilibrium security prices) and let $q_z \equiv \{q_{z,z_1}, \ldots, q_{z,z_{n_z-1}}\}$. Let $\mathscr{G} = \{g_{z,z'}\}_{z,z'\in Z}$, where every $g_{z,z'}$ is a matrix of dimension $I \times (I+1)$ (the space of candidates for equilibrium laws of motion of the wealth distribution). T maps $\mathscr{V} \times \mathscr{N} \times \mathscr{Q} \times \mathscr{G}$ into itself. Note that we can think of elements $\{N_z, q_z, g_{z,z'}\}_{z,z'\in Z} \in \{\mathscr{N} \times \mathscr{Q} \times \mathscr{G}\}$ as candidates for the aggregate employment function, for the pricing function, and for the law of motion of assets. T is an operator that—given functions for continuation values, employment, prices, and laws of motion for assets—returns the following: the values implied by individual maximization, the employment and laws of motion of asset holdings that are generated by maximization and aggregation, and the prices that clear the securities markets. Before formally defining the operator, consider the following problem:

$$\max_{b,l} x_z Q_{iz} x_z^T + \beta s_i \sum_{z'\in Z} \Gamma(z'|z) y_{z,z'} v_{i+z'} (y_{z,z'})^T, \tag{19}$$

where $b \equiv \{b_{z'}\}_{z' \in Z}$, $x_z \equiv (1, k, a, N_z(^1_k), q_z(^1_k), b, l)$, and $y_{z,z'} \equiv (1, g_{z,z'}(^1_k), b_{z'})/s_i)$. This is the problem of an age i agent that values future utility according to value functions $\{v_{i+1,z'}\}_{z' \in Z}$, when the individual state is (z, k, a) and when employment and security prices are given by the linear functions N_z, q_z, while tomorrow's aggregate state is given by the linear function $g_{z,z'}$. The first-order conditions of (19) for all i and all z imply a set of linear equations that, when solved, can be written as $(I+2)$-dimensional vectors $\{\{b_{iz,z'}(1, k, a)\}_{z' \in Z}, l_{iz}(1, k, a)\}_{i \in \mathscr{I}}^{z \in Z}$. We write $b_{iz} \equiv \{b_{iz,z'}\}_{z' \in Z}$. Let $\{B_{iz,z'}, L_{iz}\}_{i \in \mathscr{I}}$ be the first $I+1$ components of $\{b_{iz,z'}, l_{iz}\}$, with the last component added to the element in the $i+1$ position. Finally, let

$$A_{iz} \equiv \begin{pmatrix} 1 & 0 & \cdots & & & & \\ 0 & I_I & 0 & N_z^T & q_z^T & b_{iz}^T & l_{iz}^T \\ 0 & 0 & 1 & 0 & \cdots & & \end{pmatrix} \quad \text{and} \quad C_{iz,z'} \equiv \begin{pmatrix} 1 & 0 & \cdots \\ 0 & g_{z,z'}^T & b_{iz,z'}^T/s_i \\ 0 & \cdots & \end{pmatrix}, \tag{20}$$

where I_I stands for the identity matrix of dimension I. Now we are ready to define the operator T:

$$(Tv_{iz})(v, N, q, g) \equiv A_{iz} Q_{iz} A_{iz}^T + \beta s_i \textstyle\sum_{z' \in Z} \Gamma(z'|z) C_{iz,z'} v_{i+1,z'} C_{iz,z'}^T, \qquad i \in \mathscr{I}, z \in Z, \tag{21}$$

$$(Tv_{I+1,z})(v, N, q, g) \equiv 0 \qquad z \in Z,$$

$$(TN_z)(v, N, q, g) \equiv \textstyle\sum_{i \in \mathscr{I}} \mu_i \varepsilon_i \{(1, 0, \ldots, 0) - L_{iz}\} \qquad z \in Z, \tag{22}$$

$$(Tq_{z,z'})(v, N, q, g) \equiv q_{z,z'} + \textstyle\sum_{i \in \mathscr{I}} \mu_i (B_{iz,z'} - B_{iz,z_{n_z}}) \qquad z \in Z, z' = z'_1, \ldots, z'_{n_z - 1}, \tag{23}$$

$$(Tg_{z,z'})(v, N, q, g) \equiv \begin{pmatrix} 0 \\ B_{1z,z'}/s_1 \\ \vdots \\ B_{I-1,z,z'}/s_{I-1} \end{pmatrix} \qquad z, z' \in Z, \tag{24}$$

where the left-hand sides of (20)–(24) describe the values that operator T assigns, component by component.

Lemma 1. *A fixed point of this operator, with associated decision rules, is a recursive equilibrium.*

Proof. See Appendix. ‖

Next I define the operator $\hat{T}$, which has the property that, given $v \in \mathscr{V}$, it will generically return elements of the same space $\mathscr{V}$, while effectively constructing objects that play the roles of laws of motion of asset holdings, and of pricing and employment functions. This operator is computationally much cheaper to implement than the previous one, although, since it requires matrix inversions, it is defined only generically.

To define $\hat{T}$, consider a version of problem (19), with $\hat{x} \equiv (1, k, a, N, q, b, l)$ and $\hat{y}_{z'} \equiv (1, k'_{z'}, b_{z'}/s_i)$, for some $v_{i+1,z'} \in v$ and some arbitrary $\{N, q, k'\} \equiv \{N, q, \{k'_{z'}\}_{z' \in Z}\}$:

$$\max\nolimits_{b,l} \hat{x} Q_{iz} \hat{x}^T + \beta s_i \textstyle\sum_{z' \in Z} \Gamma(z'|z) \hat{y}_{z'} v_{i+1,z'} \hat{y}_{z'}^T. \tag{25}$$

The first-order conditions of this problem generate a set of linear equations that, when solved, can be written as vectors, $\{\{\hat{b}_{iz,z'}(1, k, a, N, q, k')\}_{z'\in Z}, \hat{l}_{iz}(1, \mathrm{k}, \mathrm{a}, \mathrm{N}, \mathrm{q}, \mathrm{k}')\}$, for i and z of dimension $n_z+3+I(n_z+1)$. Now if we use the representative-agent condition which states that, in equilibrium, $a_i=k_i$, and if we aggregate over agents, we then obtain a function for aggregate labour input, $\hat{N}_z(1, k, N, q, k')$. Write the (n_z-1) market-clearing conditions of the securities market as $q_{z,z'}=q_{z,z'}+\sum_{i\in\mathcal{I}}\mu_i(k'_{z'}-k'_{z'_{n_z}})$, for all $z'\in\{z_1, \ldots, z_{n_z-1}\}$. As with T, we transform the specific securities demand functions, $\hat{b}_{iz,z'}$, into $\hat{B}_{iz,z'}$ by using the representative-agent assumption. We can write these $n_z\times I$ equations in matrix form:

$$\begin{pmatrix} N \\ q_{z'_1} \\ \vdots \\ q_{z'_{n_z-1}} \\ k'_{z'_1} \\ \vdots \\ k'_{z'_{n_z}} \end{pmatrix} = \begin{pmatrix} & & & & \hat{N}_z & & & & \\ 0 & 0 & 0 & 1 & \cdots & 0 & \mu & 0 & -\mu \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & \mu & -\mu \\ & & & & \hat{B}_{1z,z'_1}/s_1 & & & & \\ & & & & \vdots & & & & \\ & & & & \hat{B}_{I-1,z,z'_{n_z}}/s_{I-1} & & & & \end{pmatrix} \begin{pmatrix} 1 \\ k \\ N \\ q_{z'_1} \\ \vdots \\ q_{z'_{n_z-1}} \\ k'_{z'_1} \\ \vdots \\ k'_{z'_{n_z}} \end{pmatrix}. \tag{26}$$

The above matrix equation has to hold for every k; therefore it can be solved by matrix inversion, and we obtain a linear system that we denote $(N, q, k)^T=\hat{G}_z(v)\binom{1}{k}$. This is done for all z, and at this point $\hat{T}$ is obtained just by using $\hat{G}(v)\equiv\{\hat{G}_z(v)\}_{z\in Z}$ in (25). Regarding the relation between $\hat{T}$ and T, we then have following:

Lemma 2. *A fixed point of $\hat{T}$ implies a fixed point of T. That is, let $v^*\in\mathcal{V}$, $v^*=\hat{T}(v^*)$, and let $(N^*, q^*. g^*)=\hat{G}(v^*)$; then $(v^*, N^*, q^*, g^*)=T(v^*, N^*, q^*, g^*)$.*

Proof. See Appendix. ||

These two lemmas guarantee that a fixed point of $\hat{T}$ implies a recursive equilibrium. To find such a fixed point, I simply apply operator $\hat{T}$ successively, starting with the zero element of $\mathcal{V}$. In every single case, the iterations converged to a fixed point after the number of iterations is about $2I$. "Convergence" is taken to mean that further applications of the operator change the values of any element of the matrices involved by less than 10^{-8}.

4. PARAMETERIZATION OF THE MODEL

The standard calibration procedures in the representative-agent, real business cycle literature can be found in Cooley and Prescott (1995), while those in the overlapping-generations models are in Auerbach and Kotlikoff (1987). I start by describing what I call the baseline economy, whose parameterization follows the principles set by those works and is summarized in Table 1. I will also explore the quantitative implications of alternative calibrations.

With respect to the demographics, as in Auerbach and Kotlikoff (1987), agents are assumed to be born at age 20 and to die at age 74 at the latest. The rate of population

TABLE 1

Parameterization of the baseline economy targets

λ_μ	σ	α	β	λ_z	θ	δ
1·24%	4	1 011	0·33	1 83%	0 64	0·0541
z_1	z_2	Γ_{11}	Γ_{22}	ε	s	Retirement
0·976	1·024	0 907	0 907	Hansen (1993)	U S. 1985	65

growth is $\lambda_\mu = 1{\cdot}24\%$, the average rate of population growth for the U.S. over the last one hundred years. The probabilities of surviving at each age, $\{s_i\}_{i\in\mathcal{I}}$, are from the 1985 vital statistics of the U.S. population. The demographic structure becomes a stable population, one where the age distribution is the one that would be achieved if mortality rates remained at 1985 levels for a long period of time and where the rate of population growth remains constant at the chosen value.

4.1. *Preferences*

The utility function of an agent is

$$\frac{1}{1-\sigma}\sum_{i\in\mathcal{I}} \beta^i s^i (c_i^\alpha l_i^{1-\alpha})^{1-\sigma}.$$

The reasons for this choice are that this function implies that the level of leisure is independent of productivity, and that the parameters needed for its calibration, β, α, and σ, have been extensively studied in the literature.

We begin with the risk-aversion parameter, σ. As noted in Cooley and Prescott (1995), empirical work has not resulted in very precise estimates of this parameter. Auerbach and Kotlikoff (1987) set $\sigma=4$, whereas Cooley and Prescott (1995) set $\sigma=1$. In my baseline model I set $\sigma=4$, but I also examine other settings later.[6]

In infinitely-lived agent models, once capital's share of output, net of depreciation, is determined, there is a one-to-one mapping between α and the fraction of time devoted to market work in the steady state. This led Cooley and Prescott (1995) to set $\alpha=0{\cdot}33$, which results in households spending around 31% of their (discretionary) time endowment in market work. In the overlapping-generations model studied here, it is not so straightforward to choose a value for α. In the steady state the fraction of time devoted to market work is not constant across age groups. There are two reasons for this: first, the endowment of efficiency units varies with age, and second, the interest rate is not necessarily equal to the subjective rate-of-time preference. Of course, hours worked across people of different ages are not constant in the data, either. In the baseline model we set $\alpha=0{\cdot}33$. This results in agents devoting 32% of their time to market work during working years.

The discount factor, β, is set to 1·011, based on the empirical work of Hurd (1989), which employed a structure similar to that used here. It is common in the infinitely-lived agent model to set β so as to match a rate of return on capital. As we will see, the chosen value of β displays a satisfactory rate of return on capital. I also explore settings that are designed to match a particular value for the rate of return.

6 We should be careful when using estimates of this parameter from models without leisure. For example, a value of $\sigma=2$ in a utility function without leisure corresponds to $\sigma=4{\cdot}03$ when $\alpha=0{\cdot}33$ and when leisure is set to be constant.

Early retirement is imposed on agents at age 65. If they were left free to choose, agents would retire at age 70, even though in the last five years of work they would devote less than 10% of their time to market activities. The choices that they make are highly dependent on the efficiency units profile after 65, which, given the large self-selection problems involved, is very hard to measure. For this reason, I think that a retirement age of 65 is appropriate.

4.2. *Technology*

Unlike preferences, technology is basically standard. The production function is Cobb–Douglas:

$$f(K,(1+\lambda_y)^t N)=K^{1-\theta}((1+\lambda_y)^t N)^{\theta}.$$

The rate of growth of labour productivity is $\lambda_y = 1{\cdot}83\%$, which is the rate of growth of *per capita* consumption for the last one hundred years (see Mehra and Prescott (1985)).

The labour share is $\theta = 0{\cdot}64$. I follow the Cooley and Prescott (1995) procedure to allocate income to capital, except that I abstract completely from the government. This implies that $\theta = 0{\cdot}64$. Auerbach and Kotlikoff (1987) use a value of 0·75, but their economy is very different from the one in this article: they do not have depreciation.

The depreciation rate is set to $\delta = 0{\cdot}0541$. In the real business cycle literature, the parameters β and δ are chosen to deliver two conditions: a certain investment-output ratio and a certain capital-output ratio. In Cooley and Prescott (1995), after excluding both the stock of government capital and the income that it generates, the investment-output ratio becomes 0·252 and the capital-output ratio becomes 2·94. In the overlapping-generations model, the discount rate is not set to get a ratio, but is measured in the data. This implies that δ can be set to get one ratio, but not both. I choose δ so that the investment-output ratio is 0·25, which implies that $\delta = 0{\cdot}0541$.

Efficiency units are derived from the series by Hansen (1993) for median weekly earnings over the period 1979–1987, by sex and by various age groups. The series is aggregated into consistent age groups.[7] Then the average between the sexes is computed. Finally, to obtain a smooth series I linearly interpolate between the centre of the age brackets.[8] The profile for females is smoother than the one for males. Also, their participation rate is smaller. For this reason, I also used the profile for males—in this case without interpolating across age groups. Figure 1 depicts the baseline-smoothed case, while Figure 2 shows the males-only, unsmoothed profile.

For Z and Γ (the process on the technology shock), I assume that $Z = \{0{\cdot}976, 1{\cdot}024\}$ and that Γ is symmetric, with $\Gamma_{11} = 0{\cdot}907$. This process generates an unconditional standard deviation of 2·4% and an autocovariance of 0·814. These are exactly the same values generated at annual frequencies by a quarterly AR(1) process for the Solow residual with parameters of 0·95 and 0·763%, which are the parameters in Prescott (1986). It is important

7. The age brackets are not the same for males and for females. They are intertemporally aggregated so that the two sexes have the same age-bracket structure

8. Auerbach and Kotlikoff (1987) use estimates of Welch (1979) on the relation between experience and weekly earnings Specifically, they use the coefficients of experience and experience squared in a regression of the weekly earnings of white males who completed high school but did not attend college. The independent variables of the regression include cohort size, interaction with early career spline, exclusion rate due to nonwork, exclusion rate due to income imputation, unemployment rate, and a trend, but not tenure. With these estimates, the age profile of efficiency is very flat, with a ratio between the maximum and the minimum of 87% (with the minimum occurring at age 74) Compared with age 20, the maximum endowment of efficiency units is just 8·3% higher.

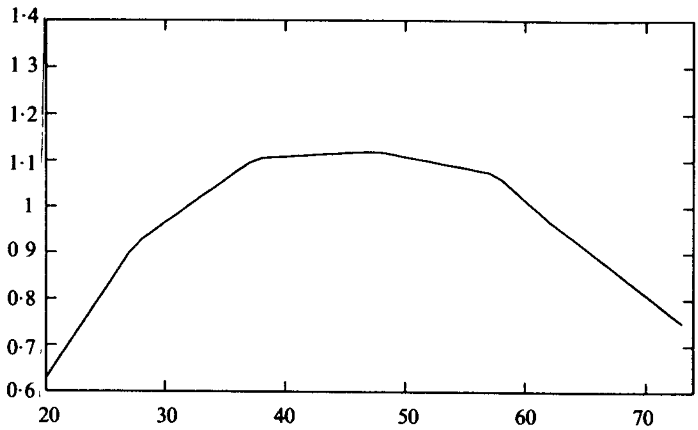

FIGURE 1
Profile of efficiency units: interpolated and averaged across males and females
Source. Hansen (1993), who used CPS data

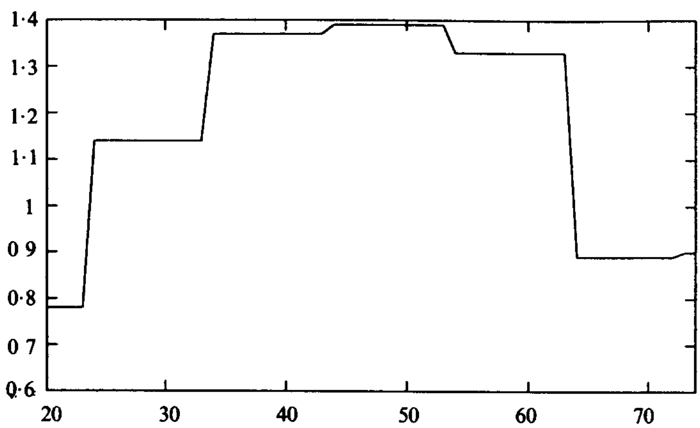

FIGURE 2
Unsmoothed profile of efficiency units for males
Source. Hansen (1993), who used CPS data

to note that in the overlapping-generations model, aggregate hours worked and the labour input do not coincide, and if we used Solow's procedure to obtain the residual, we would not obtain the original shocks. This could bias our findings. However, when comparing the Solow residual generated by the model economies with the realizations of the stochastic shock z_t, we obtain almost identical properties. For example, the proportional standard deviations of the unfiltered series are within 0·1% of each other. Furthermore, this difference is the same as the sample standard deviation of this statistic. The autocorrelations, on the other hand, are equal up to the third decimal point.

4.3. *Steady state under alternative calibration*

Table 2 shows the key growth observations to be matched, according to Cooley and Prescott (1995) (after adjusting their targets to abstract from the government). These are

TABLE 2

Cooley and Prescott's representative-agent calibration targets, adjusted to exclude the government

K/Y	Rate of Return	I/Y	NNP/Y	Fraction of time working
2 94	6 9%	0 252	0·858	0 31

the capital-output ratio (K/Y), the investment to output ratio (I/Y), the net national product to output ratio (NNP/Y), the rate of return, and the fraction of time spent working among those of working age. The first line of Table 3 reports these steady-state ratios for the baseline economy. This table shows that the ratios obtained in the baseline economy are very similar to those used in the real business cycle literature.[9] Figure 3 shows the steady-state profile of hours worked by age for the baseline economy and compares it with the average hours worked, by age, as reported in the Consumption Expenditure Survey of 1990. For the ages that the model considers, the model captures the increase in hours worked during the early twenties and the decline after the early fifties. However, the model economy overstates the hours worked by people under the age of 40. Some of the reasons for this are that the model completely abstracts from child rearing, which is a particularly time-intensive activity that is concentrated in people under 40, and from human capital accumulation.

The age profile is an important property, especially for understanding the relative volatility of hours worked vs. the labour input. For this reason, and in order to find how robust the findings are, other calibrations are explored. The age profile is a complicated

TABLE 3

Steady states of explored calibration procedures

	K/Y	Rate of Return	I/Y	NNP/Y	Fraction of time working
Agents aged 20–74					
Baseline economy	2·94	6·8%	0 250	0 841	0·324
$\alpha=0{\cdot}27$	3·16	6·0%	0 269	0·829	0 269
$\beta=0\ 97$	2 14	11·4%	0 182	0 884	0·306
$\theta=0{\cdot}67$	2·78	6 5%	0·236	0·850	0·328
$\delta=0{\cdot}08$	2 52	6·3%	0 280	0 798	0 329
$\sigma=1\ 6$	4 14	3 3%	0·352	0 776	0 360
$\sigma=1\ 6$ and $\beta=0{\cdot}975$	2 93	6·9%	0·249	0 842	0 325
Males Only	2 92	6 9%	0·248	0·842	0·323
Agents aged 16–90					
Baseline	3 12	6 1%	0 265	0·831	0 321
$\beta=1\ 005$	2 94	6 8%	0·250	0·841	0 317
$\beta=0\ 974$, $\sigma=1.6$	2·95	6·8%	0 251	0·840	0·316
$\beta=0\ 972$, $\sigma=1{\cdot}6$, $\alpha=0{\cdot}27$	2·94	6·8%	0·250	0·841	0·268

Note· The parameters not referred to explicitly are those of the baseline economy, except that the age of retirement in long-lived economies is set to 72.

9. This should not be taken as a statement that life-cycle savings account for all the wealth in the economy. There is a large literature on savings that tries to measure the relative importance of different saving motives (like altruism or precautionary savings), and there is a relative consensus that the life-cycle motive is, in itself, insufficient to account for all savings (see Kotlikoff (1989), especially Chapter 2). In this paper, the model completely abstracts from the role of the government, and if we introduced income taxes and social security, it is likely that the model would have generated a much lower capital-output ratio.

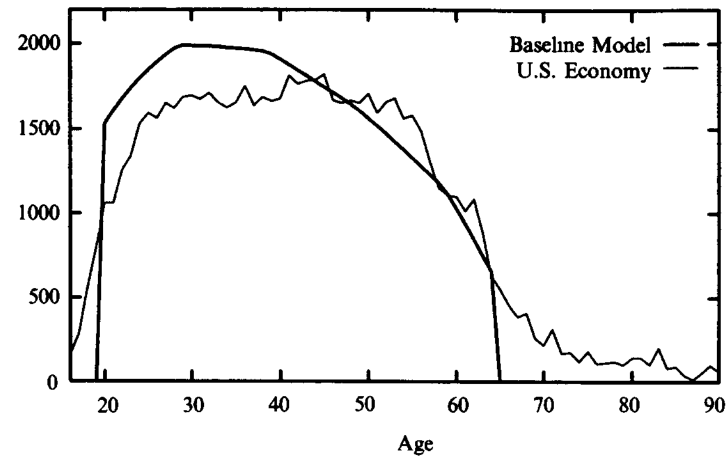

FIGURE 3

Hours worked by age in the baseline economy and in the U.S. (1990)
Source: Consumption Expenditure Survey, 1990

function of all the parameters of the model economy, and it cannot be affected without also affecting other targets of the calibration. I start by describing how the five calibration targets in Table 2 and the age profile of hours worked vary with alternative parameterizations.

Table 3 also shows the key growth ratios generated by a variety of departures from the baseline calibration. We can see that some of these departures matter more than others. Especially dramatic is the case of the discount rate: if β is set to a lower level, it generates a very large rate of return, a small capital-output ratio, and a very small investment to output ratio. A decrease in the parameter that governs the relative weight of consumption vs. leisure, α, from a value of 0·33 to a value of 0·27, reduces the steady-state fraction of time spent working, but it increases the steady-state capital-output ratio by 7%. An increase of the labour share parameter θ from 0·64 to 0·67 reduces the capital-output ratio by 5%, while increasing by about 1% the fraction of time spent working. Changing the depreciation rate to the value $\delta = 0{\cdot}08$ has a big effect on the capital-output ratio and on the investment to output ratio. A smaller risk-aversion parameter ($\sigma = 1{\cdot}6$) generates large changes in the steady state; among other things, it dramatically increases the capital-output ratio. The reason for this is that this parameter also governs the intertemporal elasticity of substitution, which affects the effective discount rate in a growing economy. For this reason, to better understand the role of a large intertemporal elasticity of substitution, I also explore an economy with $\sigma = 1{\cdot}6$, and with a lower discount rate, $\beta = 0{\cdot}975$, that generates the same capital output ratio as the baseline economy. Little effect is derived from changing the profile of efficiency units of labour to that of males only, without smoothing. Table 3 also shows the ratios of an economy with the age profile for males-only efficiency units. While some of these economies generate a satisfactory set of growth ratios, none of them produce an age profile of hours worked that is closer to the one in the data than the age profile displayed by the baseline economy.

For these reasons, I changed the calibration in another direction: the age-range of agents is now 16 to 90 rather than 20 to 74. Extending the life of agents requires the reparameterization of the economy in order to reproduce the growth ratios of Table 2. Therefore, I explored three alternative calibrations for economies with agents living a large number of periods that have satisfactory growth ratios. In these experiments with long-lived agents, the imposed age of retirement is greater than in the economies with shorter life spans. It is set to 72 years of age.[10] The calibration of these three economies with long-lived agents is like in the baseline except for the following parameters:

- The discount rate is $\beta = 1{\cdot}005$.
- The discount rate is $\beta = 0{\cdot}974$ and the coefficient of risk aversion is $\sigma = 1{\cdot}5$.
- The discount rate is $\beta = 0{\cdot}972$, the coefficient of risk aversion is $\sigma = 1{\cdot}5$, and the coefficient for consumption in the utility function is $\alpha = 0{\cdot}27$.

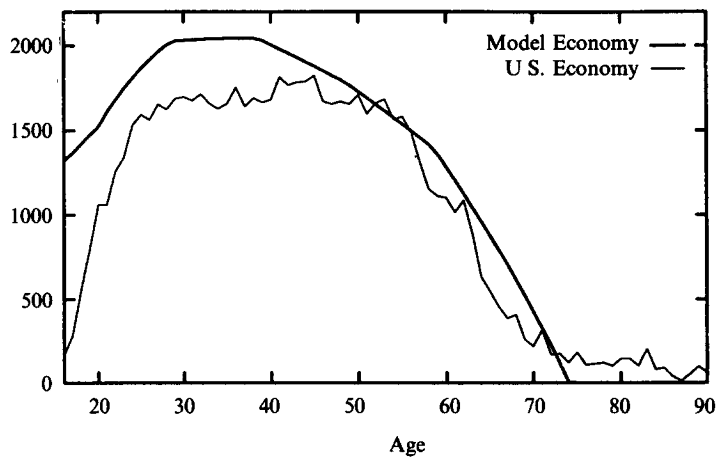

FIGURE 4

Yearly hours worked by age in a model with agents aged 16–90 and with the same parameters as in the baseline economy and in the U.S (1990)
Source: Consumption Expenditure Survey, 1990

Figures 4 to 6 show the associated age profiles of hours worked for the economies with long-lived agents that produce satisfactory growth ratios. Figure 4 shows hours worked in the data and in the long-lived model, for a value of $\beta = 1{\cdot}005$. We see that the model does particularly well for agents older than 40, but it overstates hours worked by younger people, especially among the youngest group, those between 16 and 25 years of age, when a large fraction of young people are involved in education. In the economy where the risk-aversion parameter is set at the low value of $\sigma = 1{\cdot}6$ (and $\beta = 0{\cdot}974$, to keep the steady-state capital-output ratio at the target value), the behaviour of hours by the youngest agents is a lot closer to the data (see Figure 5). However, in this version of the model, agents between the ages of 25 and 45 work much more than is shown in the data. Figure 6 shows the associated lifetime hours profile for the economy with $\sigma = 1{\cdot}6$, $\alpha = 0{\cdot}27$, and $\beta = 0{\cdot}972$. As we can see, the model still generates a few more hours worked for the agents in their thirties than is shown in the data, but it also has the property that agents of age 50 and higher work slightly less hours than they do in the U.S. economy.

10 The algorithm used to compute the equilibrium, linear quadratic approximations, cannot handle situations where certain constraints are binding sometimes but not always For this reason, even in the long-lived economies, an upper bound on the working age is always imposed at the simulation of the stochastic paths stage.

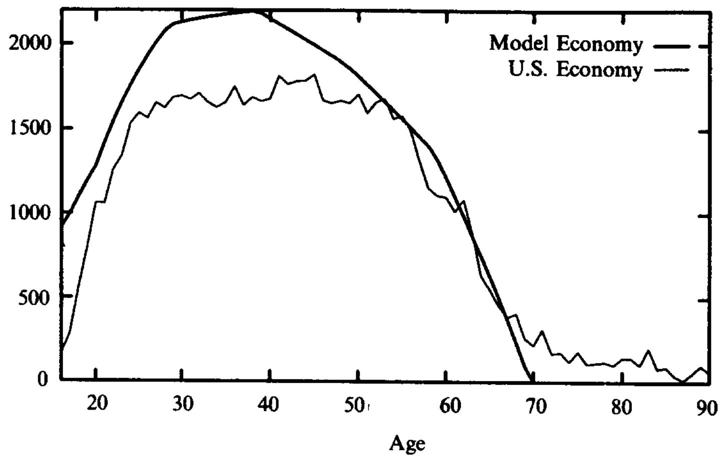

FIGURE 5
Yearly hours worked by age in a model with agents aged 16–90 and where $\sigma = 1{\cdot}6$ and $\beta = 0{\cdot}974$, and in the U.S (1990)
Source. Consumption Expenditure Survey, 1990

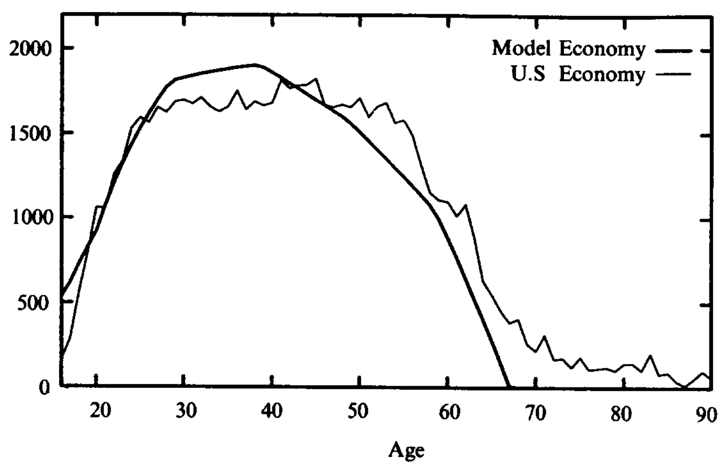

FIGURE 6
Yearly hours worked by age in a model with agents aged 16–90 and where $\sigma = 1{\cdot}6$, $\beta = 0{\cdot}972$ and $\alpha = 0{\cdot}27$ and in the U.S (1990)
Source Consumption Expenditure Survey, 1990

This last parametrization produces the overall pattern of hours worked across ages that is the closest to the data.[11]

5. FINDINGS

Table 4 reports some business cycle statistics for the U.S. economy. These include the standard deviations of GNP, consumption (including its durables, and non-durables plus

11. Bullard and Russell (1993) also calibrate the steady state of a similar economy. Their objective is to argue that the overlapping-generations model is a plausible model for monetary analysis, therefore taking the position that the life-cycle motive for savings suffices to generate the observed wealth to output ratio. This requires the rate of return in the economy without money to be smaller than productivity plus population

services), investment (including fixed private investment), total hours worked (as measured by the Bureau of Labor Statistics in the Current Population Survey), total hours worked by various age groups (below 25, 25–44, 45–64, and 65 and over), and the Hansen index of labour input, where hours of different age groups and sexes are weighted by their average hourly earnings (see Hansen (1993) for details). The table also reports the correlations of those variables with output, including two periods of leads and lags, which indicate whether the variables lead or lag the cycle. The data have been filtered, after taking logs, by using the Hodrick–Prescott filter. The data are in annual terms. For this reason, the value of the smoothing parameter of the Hodrick–Prescott filter (which weighs current deviations vs. one-period changes), is not set at the value of 1600 which is often used when filtering quarterly data. Following Backus and Kehoe (1988), a value of 100 is chosen.

TABLE 4

Cyclical behaviour of the U.S Economy deviations of key variables from trend 1956–1987

Variables x	St Dev %	Rel to Y	Cross-correlations of output with $x(t-2)$	$x(t-1)$	$x(t)$	$x(t+1)$	$x(t+2)$
Gross National Product	2·23	1·00	−0·10	0·50	1·00	0·51	−0·10
Total consumption	1·69	0·76	0·04	0·62	0·86	0·34	−0·14
Durable goods	5·94	2·66	0·17	0·63	0·80	0·14	−0·39
Non-durables and services	1·19	0·53	−0·02	0·55	0·82	0·44	−0·08
Investment							
Total expenditures	8·66	3·88	−0·03	0·46	0·87	0·14	−0·44
Fixed private	6·20	2·78	−0·08	0·49	0·89	0·28	−0·38
Total hours	1·88	0·84	−0·11	0·44	0·94	0·52	−0·22
Hours by 16–24	3·94	1·76	−0·15	0·33	0·82	0·44	−0·24
Hours by 25–44	1·83	0·82	−0·13	0·41	0·87	0·43	−0·32
Hours by 45–64	1·43	0·64	0·06	0·49	0·83	0·51	−0·03
Hours by 65+	3·22	1·44	−0·21	0·05	0·35	0·44	0·48
Labour input	1·72	0·77	−0·07	0·46	0·93	0·46	−0·28

Source Citibank Data Base, and Hansen (1993), who used CPS data.

Table 4 shows some of the familiar key business cycles features: pro-cyclicality of hours, consumption, and investment, and the fact that investment is a lot more volatile than consumption—around five times more volatile. In the calibration I have counted durables as part of investment, so we should exclude purchases of durables from consumption. If we look only at non-durables plus services, the ratio becomes 7·3.

Some other important properties of the data are that both consumption and investment slightly lead the cycle (the reason is that the other two components of output—government expenditures and net exports—lag it). Another property is that within two periods of difference, most of the correlations become slightly negative or zero.

growth. They choose a different calibration procedure than the one I follow The parameters that are key in generating different capital-output ratios (and rates of return) are the discount rate β, which they set to a high value of 1·04, based on a different (and more extreme) reading of Hurd (1989), and labour share θ, which they set to 0·75. In order to obtain such a large labour share, their calibration procedures abstract from the services from durables and assume that all proprietors' income can be imputed to the labour input. Other differences in their choices are $\alpha = 0{\cdot}2$, based on 24 (rather than 14) available hours per day, and a 100% labour participation rate They also use the lifetime efficiency units profile that Auerbach and Kotlikoff (1987) used, and a value of $\delta = 0{\cdot}1$. The key steady-state ratios generated by the parameters chosen by Bullard and Russell (1993) (except for the age-efficiency profile, where the profile in Hansen (1993) from CPS was used instead) are capital-output ratio, 3·05, rate of return, −1·8%, investment to output ratio, 0·351, Net National Product to output ratio, 0·695, and fraction of time working, 0·223. It is obvious from these ratios that their parameterization accomplishes reductions in the rate of return, but this is done at the expense of large differences with the calibration targets of Cooley and Prescott (1995).

The standard deviation of aggregate, unweighted hours is 84% of the standard deviation of output, while that of Hansen's measure of the labour input is 76% of that of output. Both series are highly correlated with output. Regarding the age distribution of the volatility of hours, the hours of the youngest workers show the highest volatility, followed by the people over 65, then by the 25–44 group, and, finally, by the 45–64 group. This is related to the well-known property that hours are more volatile for unskilled workers than for skilled workers (see Clark and Summers (1981), Kydland (1984), Kydland and Prescott (1988) and Ríos-Rull (1993)).

5.1. *The baseline economy*

Table 5 reports these same statistics for the baseline economy. The statistics are the averages of 50 realizations of 32 periods each. Each realization is computed 300 periods after departing from the steady state. The variables of the model economies have also been logged and filtered in exactly the same way as were the data for the U.S. economy.[12]

TABLE 5

Cyclical behaviour of the baseline economy: agents live from 20 to 74 years of age and retire at 65
Deviations of key variables from trend, 32 observations

Variables x	St Dev %	Rel to Y	$x(t-2)$	$x(t-1)$	$x(t)$	$x(t+1)$	$x(t+2)$
Output	1·645	1 00	0·109	0 468	1 00	0 476	0 106
	(0 006)		(0·274)	(0 221)	(0·000)	(0·232)	(0·291)
Aggregate consumption	0·743	0 45	0 004	0 378	0·969	0 581	0 261
	(0 003)		(0·287)	(0·231)	(0 016)	(0·210)	(0·275)
Aggregate investment	4 450	2 71	0 157	0 497	0·985	0 402	0 010
	(0·017)		(0 260)	(0 205)	(0 056)	(0·227)	(0 266)
Total hours	0 592	0 36	0 148	0·479	0·945	0 340	−0·050
	(0·002)		(0·291)	(0 284)	(0 258)	(0 280)	(0 277)
Hours by 20–24	0 575	0 35	0·145	0 480	0·952	0·341	−0·042
	(0 002)		(0 282)	(0 263)	(0 212)	(0 245)	(0 262)
Hours by 25–44	0·473	0 29	0 165	0 492	0 955	0 353	−0·037
	(0 002)		(0 245)	(0·225)	(0 186)	(0·251)	(0 266)
Hours by 45–64	0 828	0 50	0 155	0 486	0·950	0 343	−0·046
	(0 003)		(0·274)	(0 261)	(0·221)	(0 259)	(0·269)
Labour input	0·589	0 36	0 148	0 480	0 946	0 341	−0·049
	(0·002)		(0 291)	(0 283)	(0·256)	(0 278)	(0 276)

As Table 5 shows, the artificial economy also generates the key business cycle facts. All variables move together (they are pro-cyclical). Output is less volatile than investment, but more volatile than consumption. The phase shift is such that the correlations of aggregate flow variables with lead and lagged output are basically zero within two periods (the large standard deviations of the sample statistics of the model economy put the values of the U.S. phase shift well within the margins of sampling error). Investment leads the cycle and consumption lags it; the two variables cannot simultaneously lead the cycle.

The variance of output in the baseline economy is 55% of the variance of output in the U.S. economy. This can be compared to Cooley and Prescott (1995), who find that variations of the Solow residual account for 62% of the variance of output, and to Kydland and Prescott (1991), who account for 70% of output volatility in a model where the labour input can be adjusted by changes in the extensive margin as well as the intensive margin.

12. In the other experiments conducted, the realizations for the exogenous shocks were identical to those used for the baseline economy.

The volatility of investment is six times larger than that of consumption, which is smaller than the ratio of 7·3 found in the data.

In the baseline economy, hours are only about 31% as volatile as in the U.S. data. By contrast, the representative-agent model of Cooley and Prescott (1995) implies that hours are about 50% as volatile as the data. The behaviour of the Hansen measure of the labour input is almost identical to that of aggregate hours, resulting in a standard deviation that is 34% of the labour input in the data. The similarities between the behaviour of the two series in the baseline economy are due to the fact that the age group with the highest volatility of hours is the group with agents aged 45–64, followed by the group with agents aged 25–44, which is the opposite of what is shown in the data.

Overall, we see that, with the parametrization chosen for the baseline economy, the fluctuations obtained have many of the features that characterize the business cycle. The baseline economy indicates that fluctuations in the Solow residual account for a smaller fraction of output fluctuations than do standard, representative-agent models. Furthermore, the model economy fails to produce a volatility pattern of variation of hours across age groups that resembles the data. Consequently, the baseline economy does not produce a measure of the labour input that is less volatile than aggregate, unweighted hours.

In the next subsection I explore whether these properties of the baseline economy are sensitive to changes in parameter values.

5.2. *Alternative parameterizations*

I will now explore various alternative parameterizations of the model. Table 6 shows the standard deviations of the variables studied for a variety of economies whose parameterizations differ from the baseline economy.[13]

TABLE 6

Key cyclical statistics of economies with alternative calibration

Economy	Percentage standard deviation of								
	Y	C	I	Total hours	Hours <24	Hours 25–44	Hours 45–64	Hours 65+	Labour input
U S. economy*	2·23	1·19	8·66	1·88	3·94	1 83	1·43	3·22	1·72
Agents aged 20–74									
Baseline economy	1 64	0 74	4·45	0·59	0·58	0·47	0·83	—	0·59
$\alpha=0{\cdot}27$	1 70	0·75	4 38	0·68	0·68	0·54	0·96	—	0·69
$\sigma=1.6, \beta=0\ 975$	1·73	0 72	5 03	0 73	0 90	0 57	0 98	—	0·72
Agents aged 16–90									
$\beta=1.005$	1 68	0·76	4·56	0 62	0·67	0·46	0 69	2 90	0·61
$\beta=0\ 974, \sigma=1{\cdot}6$	1·73	0·71	5 00	0·76	1 08	0 53	0·78	6 42	0·72
$\beta=0\ 972, \sigma=1\ 6$ $\alpha=0{\cdot}27,$	1·78	0 73	5 20	0 85	1·42	0·60	0 94	7 61	0 80

Notes. All parameters not explicitly referred to are those of the baseline economy reported in Table 1. In the long-lived economies, agents retire at 68.

The statistics reported for the U S are for consumption of nondurables and services and for total investment expenditures.

13. Some of them resulted in no changes from the properties of the baseline economy, and, therefore, their properties are not reported. They include changes of labour share to $\theta=0\ 67$ and changing the age profile of efficient units to unsmoothed males-only, as depicted in Figure 2. Setting the depreciation parameter to $\delta=0{\cdot}08$ reduces the steady-state capital-output ratio, rate of return, and net national product to output ratio, it also results in a slightly lower volatility of output and hours worked and in a relatively large reduction in the volatility of investment, while that of consumption shows a very small increase

If we use a higher valuation of leisure vs. consumption ($\alpha = 0{\cdot}27$, rather than the baseline value of 0·33), then the volatility of hours worked increases by about 12%. The volatility of hours worked by all age groups increases in roughly the same proportion.

A reduction in the coefficient of risk aversion (with associated reduction in the discount rate to obtain steady-state ratios that are in line with the targets) increases the standard deviation of output by 5·5%, and that of hours worked by 23%. This increase is proportionally bigger in the youngest group, making the behaviour of hours worked across the different age groups in the low risk-aversion economy closer to the data than is the behaviour of hours worked in the baseline economy. The higher standard deviation of output translates into a much higher standard deviation of investment as the volatility of consumption becomes a little smaller than that in the baseline economy. The ratio between the volatility of investment and that of consumption becomes 7. This feature resembles the data more than the corresponding feature of the baseline economy does.

The study of economies where agents live a long number of periods allows for the study of hours worked by the group that is over 65 years of age. Recall that we study three versions of the long-lived economies that differ in risk-aversion, in the valuation of leisure, and in the discount rate, and that the three economies satisfy the calibration targets. In the model economy with the high coefficient of risk-aversion (which is the same as the one in the baseline economy), we see that the volatility of aggregate variables is quite similar to that of the baseline economy. The only differences arise in the age distribution of the volatility of hours worked. In this economy the volatility of the youngest is higher than in the baseline economy, while that of the middle age is smaller, and that of the group between 25 and 44 remains basically unchanged. The volatility of the group above 65 is very high, almost as high as in the data. Overall, this age distribution of volatility is much more in line with the data than is the age distribution of the baseline economy.

Regarding the economy where agents live a large number of periods and have relatively low risk-aversion, we see that it also has a high value for the standard deviation of output, the same as the economy where agents live up to age 74 (and it has the same low risk-aversion). However, the age profile of the volatility of hours worked is different. Now the volatility of the group over 65 years of age is higher than that in the data, and the volatility among those below 25 years of age is also very high—higher than that of any of the middle groups, as in the data. This is the first economy for which the Hansen measure of labour input is less volatile than the measure of aggregate unweighted hours (about 95%).

The last economy that I looked at—the one that produces a steady-state age profile that is the closest to the one in the data—has agents who live from age 16 to age 90, a coefficient of risk-aversion, σ, of 1·6, and a parameter that weighs consumption vs. leisure, α, of 0·27. This economy has the highest volatility of output among the model economies studied: its variance is 64% of the one observed in the data. The ratio of the volatilities of investment and consumption is 7·1, close to that in the data. The volatility of hours worked is 45% of the one observed in the data. The match of the age profile of the volatility of hours is similar to the one in the data, except for the relative behaviour of those in the 25–44 and 45–64 age groups, which is reversed, and for the excessive volatility of the over-65 group. The volatility of Hansen's measure of labour input is 94% of that of aggregate unweighted hours, while it is 91% in the data. Among the model economies studied here, this is the one that performs best: first, its output is the most volatile and the volatility of investment relative to the volatility of consumption is also the highest (this property makes it the closest to both the Cooley and Prescott (1995) economy and the U.S. data); second, it generates the most satisfactory age profile of steady-state hours worked; and, third, it generates a pattern of volatility of hours across age groups and, consequently, of Hansen's measure of the labour input that resembles the data the most.

The behaviour of this last economy regarding the relative volatility of hours worked across age groups shows that a profile similar to the one in the data—where the persons that are either very young or very old have a much higher volatility of hours worked than those in the middle age groups—can be generated in a model economy where there are no cyclical changes in the relative efficiency of the various age groups. The key to understanding this result lies in the life-cycle nature of the model. In models with infinitely-lived agents and Cobb–Douglas preferences, permanent differences in the efficiency of agents do not generate differences in the behaviour of hours worked.[14] Not all life-cycle economies, however, generate the same age profile of volatility of hours. The specific properties of this profile depend on:

- The age profile of efficiency units. The age profile is crucial in determining the periods in which there is a comparative advantage of working longer hours. If agents were equally efficient at all ages, then the age profile of hours worked would have been monotonically decreasing, since the interest rate is lower than the discount rate. In this case the associated age profile of the volatility of hours worked would have been very different from the one observed in the data.
- The parameter σ, which determines the intertemporal elasticity of both consumption and leisure in Cobb–Douglas utility functions. The lower the value of σ, the higher the intertemporal elasticity of substitution, and a high elasticity of substitution is necessary to induce high volatility of hours for both the young and the old. In Table 6 we can see that in economies with low σ the volatility of hours worked by the middle age groups relative to that of the youngest and the oldest age groups is lower than in economies with high values of σ.

6. CONCLUSION

In this paper I have explored the business cycle behaviour of large, stochastic, overlapping-generation economies that are calibrated with the aid of demographic, microeconomic, and National Income and Product Accounts measurements. I have found that the behaviour of these economies is similar to the behaviour found in standard, representative-agent, real business cycle models. In particular, a model economy where agents live from ages 16 to 90, with a relatively low risk-aversion parameter and with an attitude toward the consumption-leisure trade-off that matches the average age profile of hours worked, generates a set of business cycle statistics that is almost identical to the one found in Cooley and Prescott (1995), which has been taken to be the standard description of the findings of real business cycle theory. Furthermore, overlapping-generations economies can be used to study properties of the age distribution of various variables. In this respect, the aforementioned economy generates an age profile of the volatility of hours worked that is very similar to the one in the data (with the exception of the 25–44 age group). Associated with this feature, the model also produces the familiar fact that the volatility of hours worked of the individuals with low skills is higher than the one of individuals with high skills.

The approach used to model this class of economies and to solve the equilibria has applications that go beyond business cycle issues. These include attempting to answer the following questions: What are the long-term implications for savings and interest rates of

14. For this to hold, all agents have to have the same non-human wealth

the current ageing of the population? Can large current account deficits be generated by differences in the age structure of the population? Also, in this class of economies, there is active trade of securities, as agents insure themselves against risks. This permits the study of issues related to the volume of trade, the risk premia, and the quantitative importance of markets for securities that are contingent on the aggregate state of the economy.

APPENDIX

Proof of Proposition 1, *Existence of Equilibrium*

The argument follows that of Balasko, Cass, and Shell (1980)

1 Let $x \equiv \{c_i(h_t), l_i(h_t), a_i(h_t), \{b_i(h_t, z_{t+1})\}_{z_{t+1} \in Z}\}_{i \in \mathcal{I}, h_t \in H}$ be an allocation Define $p^c(h_t) \equiv \prod_{\tau=0}^{t} q(h_\tau, z_{\tau+1}) R(h_\tau)$ as the time-zero value of a unit of the good in state h_t, while $p^l(h_t) \equiv p^c(h_t) W(h_t)$ is the price of one efficiency unit of labour The constraints faced by agents in the model that are presented in the text are clearly the same as those faced by an agent that has a single budget constraint, with prices given by $\boldsymbol{p} \equiv \{p(h_t)\}_{h_t \in H} \equiv \{p^c(h_t), p^l(h_t)\}_{h_t \in H}$, as long as the agent is indexed not only by his date of birth, but also by the history of shocks up to his birth date (see Wright (1987)). Therefore, it suffices to show the existence of equilibria for the newly defined prices.

2. Define an *h_t-equilibrium* as a pair of feasible allocations and strictly positive prices, for every node in H, that clear all markets up to h_t, denote this as $\{\boldsymbol{x}^{h_t}, \boldsymbol{p}^{h_t}\}$ Standard arguments yield the existence of an h_t-equilibrium for all h_t. By construction, if $\{\boldsymbol{x}^{h_{t+1}}, \boldsymbol{p}^{h_{t+1}}\}$ is an h_{t+1}-equilibrium, then it is also an h_t-equilibrium. Let $\{\boldsymbol{x}^{h_\tau}, \boldsymbol{p}^{h_\tau}\}_{h_\tau \in H}$ be a sequence of h_τ-equilibria. Then there exists a subsequence of nodes with corresponding h_ξ-equilibria $\{\boldsymbol{x}^{h_\xi}, \boldsymbol{p}^{h_\xi}\}$ such that $\lim_{\xi \to \infty} \boldsymbol{x}^{h_\xi} = \boldsymbol{x}^*$

3 Note that the economy in this paper is irreducible, since every generation is endowed with positive amounts of efficiency units of labour in every period, and K_t is bounded away from zero in every period. This guarantees the equivalence of compensated and competitive equilibrium (see Balasko *et al* (1980)), which can be used to establish that for every h_t, $\{\boldsymbol{x}^*, \boldsymbol{p}^{*,h_t}\}$ is an h_t-equilibrium, where $\boldsymbol{p}^{*\,h_t}$ consists of the pointwise limits of the first h_{t+1} elements of the subsequence h_ξ that yielded $\boldsymbol{x}^*$.

4. We want to show that every h_t-equilibrium $(\boldsymbol{x}^*, \boldsymbol{p}^{h_t})$ can be rewritten with renormalized prices $\hat{p}^{h_t}$ so that $\|\hat{p}^{h_t}(h_0)\| = 1$ and $\|\hat{p}^{h_t}(h_\tau)\| \leqq B(h_\tau)$ for all $\tau \leqq t$ To see this, note first that $\|p^{h_t}(h_0)\| > 0$ Next, suppose that the bounds did not exist; then there would exist a sequence of h_t-equilibria, $\{x_\xi^{h_t}, p_\xi^{h_t}\}_{\xi=1}^{\infty}$, $\|p_\xi^{h_t}(h_0)\| > 0$, with the property that $\lim_{\xi \to \infty} \|p_\xi^{h_t}(h_\tau)\| / \|p^{h_t}\xi(h_0)\| = \infty$. Renormalize the ξ sequence so that $\|p_\xi^{h_t}(h_0), \ldots, p_\xi^{h_t}(h_{t+1})\| = 1$ Then take a subsequence so that these first elements converge pointwise to some $\{p^\infty(h_0), \ldots, p^\infty(h_{t+1})\}$. As was argued before, $\boldsymbol{p}^\infty \equiv \{p^\infty(h_0), \ldots, p^\infty(h_{t+1}), p(h_{t+2}), \ldots\}$, with $\|p^\infty(h_0)\| > 0$, is an h_t-equilibrium for any strictly positive continuation prices $\{p(h_{t+2}), \ldots\}$. But this contradicts the limit behavior of $\{\boldsymbol{p}^{h_t \xi}\}_{\xi=1}^{\infty}$, so the proposed bounds exist

5 By now it can be argued that $\boldsymbol{p}^{*\,h_t}$ has a convergent subsequence with limit $\boldsymbol{p}^*$. By construction, it is immediate that the pair $\{\boldsymbol{x}^*, \boldsymbol{p}^*\}$ is a competitive equilibrium. ‖

Lemma 1. *A fixed point of this operator, with associated decision rules, is a recursive equilibrium*

Proof. We check the conditions of the definition Conditions (b) and (f) are directly imposed by substitution in the approximation (21) guarantees (c), while (24) implies (d) Aggregation of the labour input (e) is guaranteed by (22). Finally, and in a less obvious way, notice that a fixed point of T requires that $\sum_{i \in \mathcal{I}} \mu_i B_{iz}^{z'} = \sum_{i \in \mathcal{I}} \mu_i B_{iz}^{z_{n_z}}$ for all $z' \in Z$. Both sides of these equations give us aggregate capital in the next period as a function of the current state. Condition (23) prevents the next period's shock from affecting aggregate capital, but this is exactly condition (a) from the definition of recursive equilibria ‖

Lemma 2. *A fixed point of $\hat{T}$ implies a fixed point of T. That is, let $v^* \in \mathcal{V}$, $v^* = \hat{T}(v^*)$, and let $(N^*, q^*, g^*) = \hat{G}(v^*)$; then $(v^*, N^*, q^*, g^*) = T(v^*, N^*, q^*, g^*)$*

Proof. Compare the decision rules from (19)—$\{b_{iz}, l_{iz}\}$, functions of $\{1, k, a)$—with those from (25)—$\{\hat{b}_{iz}, \hat{l}_{iz}\}$, functions of $(1, k, a, N, q, k')$· the latter, evaluated at $N = N_z^*\binom{1}{k}$, $q = q_z^*\binom{1}{k}$, $k' = g_z^*\binom{1}{k} \equiv \{g_{z'z}^*\binom{1}{k}\}_{z' \in Z}$, equal the former. Next, aggregation of labour in both sets of decision rules implies that $\hat{N}_z(1, k, N_z^*\binom{1}{k}, q_z^*\binom{1}{k}, g_z^*)) = N_z\binom{1}{k}$ and $\hat{B}_z(1, k, N_z^*\binom{1}{k}, q_z^*\binom{1}{k}, g_z^*\binom{1}{k})) = B_z\binom{1}{k}$. Finally, write the $n_z - 1$ market-clearing conditions of $\hat{T}$ as

$$(0, 0, 0, \ldots, 1, \ldots, \mu, 0, -\mu)(1, k, N_z^*\binom{1}{k}, q_z^*\binom{1}{k}, g_z^*\binom{1}{k}))^T \equiv \phi_z^{z'}(1, k, N_z^*\binom{1}{k}, q_z^*\binom{1}{k}, g_z^*\binom{1}{k}))^T.$$

But this is exactly the definition of (Tq_z). So we can write all this in a compact way as

$$\begin{pmatrix} N^*_z \\ q^*_z \\ g^*_z \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix}=\begin{pmatrix} \hat{N}_z \\ \hat{\phi}_z \\ \hat{B}_z/s \end{pmatrix}\begin{pmatrix} 1 & & 0 \\ 0 & & 1 \\ & N^*_z & \\ & q^*_z & \\ & g^*_z & \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix}=\begin{pmatrix} N_z \\ (Tq_z) \\ B_z/s \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix}=\begin{pmatrix} (TN^*_z) \\ (Tq^*_z) \\ (Tg^*_z) \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix}, \tag{27}$$

where the first equality arises from algebraic manipulation of equation (26) and the last equality arises from the definition of T. ||

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