Modeling Complex Systems Using Operads

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Abstract

This paper presents an algebraic approach to modeling phase transitions and emergent phenomena in complex systems using operads. We propose a novel type of operad, named σ-operads (statistical operads), which extends operads of wiring diagrams with a statistical structure. We show how these operads can be used to represent the compositional structure, scale invariance, and near-decomposability of complex systems. We also derive a rigorous mathematical criterion for the occurrence of emergence based on correlation decay rates.

1 Introduction

Complex systems are characterized by numerous interacting components that exhibit emergent behaviors and phase transitions not evident from the study of individual components [Mitchell, 2009]. From flocking birds to financial markets, from neural networks to social movements, these systems demonstrate how local interactions cascade across scales to produce system-wide transformations. Understanding such phenomena requires mathematical frameworks that can capture both the multi-scale, hierarchical nature of complex systems and their compositional structure.

The challenge of modeling complex systems has driven the development of increasingly sophisticated mathematical frameworks. Traditional network models capture pairwise interactions but miss higher-order effects. Hypergraphs extend networks to include multi-agent interactions, while simplicial complexes further enable topological analysis through higher-dimensional structures [Battiston et al., 2020]. Yet even these advanced models struggle with a fundamental limitation: they cannot adequately represent systems with complex hierarchical organization and dynamic compositional structures.

Consider social consensus formation, where opinions cascade through multiple levels of influence [Watts and Strogatz, 1998], or biological signaling networks, where protein complexes dynamically assemble to form functional units [Giovannoni et al., 2017]. These

systems exhibit two key properties identified by Simon [1962]: near-decomposability (subsystems interact weakly externally while maintaining strong internal interactions) and scale-invariance (self-similar patterns across organizational levels). Current mathematical frameworks fail to capture how these properties enable phase transitions—those critical moments when local changes reorganize compositional relationships to produce emergent phenomena.

This paper introduces a novel framework based on operads that directly addresses these limitations. Operads, originally developed in algebraic topology, provide a natural language for describing compositional structures and their transformations. We propose σ-operads (statistical operads), which extend operads of wiring diagrams with statistical structure to model how complex systems compose, decompose, and reorganize during phase transitions. This approach offers three key advantages:

1. **Compositional representation**: Operads explicitly model how subsystems combine to form larger systems while preserving compositional relationships
2. **Multi-scale dynamics**: The operadic structure naturally captures interactions across hierarchical levels
3. **Phase transition mechanics**: Statistical extensions allow modeling of critical phenomena and emergent behaviors

Prior work has successfully demonstrated the utility of operads for structural modeling. Spivak [2013] introduced the operad of wiring diagrams to formalize the composition of open systems, a framework further developed by Behr et al. [2021] for system design. Baez and Pollard [2020] applied operadic approaches to network models and Petri nets, providing a rigorous categorical foundation for network theory. In neuroscience, Linde-Domingo et al. [2021] utilized operads to model brain hierarchy and topological simplification. However, these existing frameworks primarily focus on the structural and deterministic aspects of composition. They lack the intrinsic statistical mechanics required to model the stochastic fluctuations and probabilistic reorganization that characterize phase transitions in complex systems.

Beyond structural modeling, we derive a rigorous mathematical criterion for when emergence occurs. The emergence transition criterion identifies the precise threshold where individual component interactions can be validly approximated by aggregate dynamics, resolving the long-standing question of when higher-level properties become irreducible to component behaviors [Anderson, 1972, Holland, 1998]. This criterion connects emergence to fundamental probability theory through correlation decay rates, extending the Central Limit Theorem to interacting systems.

2 Background

2.1 Properties of Complex Systems

2.1.1 Structural Features: Near-decomposability and Scale-invariance

Complex systems exhibit two foundational structural properties that enable their characteristic behaviors and provide the basis for our operadic approach.

Near-decomposability [Simon, 1962] describes systems with subsystems that interact weakly externally while maintaining strong internal interactions. These systems display:

- Stronger interactions within subsystems than between them
- Hierarchical organization across multiple levels
- Time-scale separation between internal and external interactions

Examples include cellular structures (organelles within cells, cells within tissues, tissues within organs, organs within organisms, etc.), ecosystems with loosely coupled niches, and modular software systems. Near-decomposability enables adaptation through independent subsystem evolution [Simon, 1996] and creates the hierarchical levels across which system dynamics unfold.

Scale-invariance describes self-similar patterns appearing across different organizational levels, characterized by:

- Statistical similarity at different observation scales
- Power-law distributions of system properties
- Absence of characteristic scales

This property manifests in branching patterns in biological systems, power-law distributions in scale-free networks etc. [West, 2017, Stanley, 1999]. Scale-invariance emerges naturally in systems that grow through preferential attachment processes or self-organize into critical states.

2.1.2 Dynamical Features: Phase Transitions and Emergence

Phase transitions and emergence represent related dynamical phenomena that arise from the hierarchical and scale-invariant structures of complex systems.

Phase transitions occur when control parameters cross threshold values, causing qualitative changes in system properties. These transitions manifest as discontinuous shifts at critical points [Stanley, 1971], displaying characteristic mathematical signatures: power-law scaling behaviors, critical exponents, and diverging correlation lengths [Newman, 2003, Bak et al., 1987]. In complex systems, phase transitions include ecosystem state shifts [Scheffer et al., 2009], opinion cascades in social networks [Watts, 2002], synchronization transitions in oscillator systems [Rodrigues et al., 2016], financial market crashes, and percolation thresholds in networks [Cohen and Havlin, 2010].

Emergence describes the formation of properties at higher organizational levels not present in or predictable from individual components [Holland, 1998, Anderson, 1972]. Anderson's "More is Different" principle emphasizes that complex systems require analysis beyond reductionist approaches. Examples include flocking behaviors in birds, intelligence in neural networks, consciousness from neuronal activity, and market dynamics in economies [Camazine et al., 2003, Haken, 1983].

Phase transitions can be viewed as a specific type of emergence characterized by sudden, discontinuous changes, while other emergent phenomena may develop gradually or exist in steady states [Bar-Yam, 2013]. Both phenomena share common underlying mechanisms:

- They arise from the hierarchical structures created by near-decomposability
- They propagate across scales following patterns enabled by scale-invariance
- They involve reorganization of compositional relationships between system components
- They manifest when local changes cascade to produce system-wide transformations

At critical points, fluctuations occur across all scales of the system as local changes propagate through hierarchical levels—a direct consequence of the near-decomposable, scale-invariant structure.

Our operadic framework models these phenomena by tracking compositional relationships between components across scales. This approach complements traditional statistical methods [Stanley, 1999, Goldenfeld, 1992] by focusing explicitly on how compositional structures reorganize during transitions, providing insights into the mechanisms driving phase transitions and emergent behaviors.

2.2 Modeling Phase Transitions and Emergence

2.2.1 Physical Models

In physics, phase transitions are often modeled using statistical mechanics, where the system's behavior is described in terms of energy, entropy, and temperature [Stanley, 1971, Kadanoff, 2000]. The most iconic example is the Ising model, originally designed for ferromagnetism but now widely applied to social dynamics and neural networks. The system is described by a Hamiltonian H, which sums the interactions between adjacent spins σ_i, σ_j:

$$H(\sigma) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i \tag{1}$$

where J represents the interaction strength (coupling) and h is an external field.

These models capture the fundamental mechanism of phase transitions: how local interactions (the J term) compete with thermal fluctuations (entropy) to produce collective order (magnetization) at critical points [Binney et al., 1992]. While powerful, they typically assume a fixed, often regular lattice structure, limiting their direct applicability to the complex, adaptive topologies found in biological and social systems [Newman, 2011].

2.2.2 Network Models

Networks have been used since the early 20th century to model complex systems, representing entities as nodes and interactions as edges [Watts and Strogatz, 1998, Barabási and Albert, 1999]. Formally, a network is represented as a graph $G = (V, E)$ where V is a set of vertices (nodes) and $E \subseteq V \times V$ is a set of edges (links). For directed networks, edges are ordered pairs $(u, v) \in E$ indicating a directed relationship from node u to node v. For undirected networks, edges are unordered pairs $\{u, v\} \in E$.

The structure of a network can be represented by its adjacency matrix A, where:

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \text{ (or } \{i,j\} \in E \text{ for undirected graphs)} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

For weighted networks, A_{ij} represents the strength of the connection between nodes i and j. Several metrics characterize network properties:

- **Degree distribution** $P(k)$: The probability that a randomly selected node has k connections
- **Clustering coefficient** C_i: For a node i with k_i neighbors, $C_i = \frac{2e_i}{k_i(k_i-1)}$ where e_i is the number of links between the neighbors
- **Path length** $d(i, j)$: The minimum number of edges traversed to reach node j from node i
- **Betweenness centrality** $B(v)$: $B(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$ where σ_{st} is the number of shortest paths from s to t and $\sigma_{st}(v)$ is the number of those paths passing through v

These properties can be used to classify networks into different categories, such as scale-free, small-world, and random networks [Barabási and Albert, 1999, Watts and Strogatz, 1998].

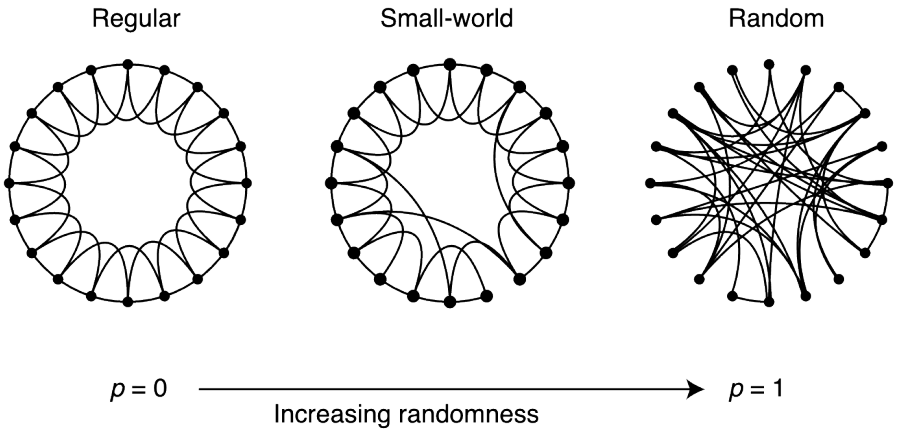


Figure 1: Small-world network model illustration showing the transition from regular to random networks. [Watts and Strogatz, 1998]. A regular network transitions to a small-world network by rewiring a fraction of the edges, leading to a significant reduction in the average path length while maintaining high clustering.

Critical phenomena in networks, such as phase transitions, often manifest through sudden changes in global network properties. For instance, the emergence of a giant connected component in random networks occurs at a critical probability $p_c = \frac{1}{N}$, where N is the number of nodes [Erdős and Rényi, 1960].

Network models have been successful in capturing the structure and dynamics of a wide range of systems, including social networks, biological networks, and technological networks [Newman, 2003, Albert and Barabási, 2002, Strogatz, 2001]. Networks are both a mathematically rigorous framework as well as intuitive and visually appealing, making them a popular choice for modeling complex systems [Newman, 2010]. Networks can capture the emergence of collective behavior through the study of network motifs, community structure, and dynamical processes on networks [Milo et al., 2002, Fortunato, 2010, Barrat et al., 2008], and has attracted significant attention in recent years [Barabási, 2016].

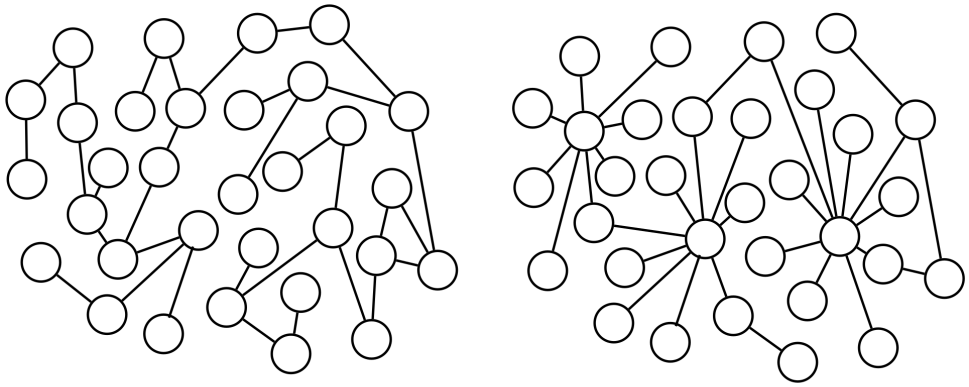


Figure 2: Visual comparison between random and scale-free networks. [Wikipedia contributors, 2023]. Notice the presence of hubs in the scale-free network, which are absent in the random network.

Network Type	**Degree Distribution**	**Clustering Coefficient**	**Average Path Length**
Random Networks	Poisson distribution $P(k) \sim \frac{\lambda^k e^{-\lambda}}{k!}$	Low ($C \sim \frac{p}{N}$)	Short ($L \sim \frac{\ln N}{\ln \langle k \rangle}$)
Regular Lattices	Constant degree	High (locally clustered)	Long ($L \sim N^{1/d}$)
Small-World Networks	Similar to random networks	High ($C \gg C_{random}$)	Short ($L \approx L_{random}$)
Scale-Free Networks	Power law $P(k) \sim k^{-\gamma}$	Hierarchical clustering	Very short (ultra-small world)
Hierarchical Networks	Power law	Hierarchical ($C(k) \sim k^{-1}$)	Short
Modular Networks	Varies	High within modules, low between modules	Long between modules, short within modules

Table 1: Comparison of Different Network Types and Their Characteristics. **Notation:**
$P(k)$ = probability that a randomly selected node has k connections (degree distribution)
k = node degree (number of connections)
$\langle k \rangle$ = average degree across the network
C = clustering coefficient (probability that two neighbors of a node are connected)
$C(k)$ = clustering coefficient for nodes with degree k
L = average shortest path length between any two nodes
N = total number of nodes in the network
p = probability of connection between any two nodes (in random networks)
d = dimension of the lattice (for regular networks)
γ = power-law exponent (typically $2 < \gamma < 3$ for scale-free networks)

2.2.3 From Networks to Simplicial Complexes

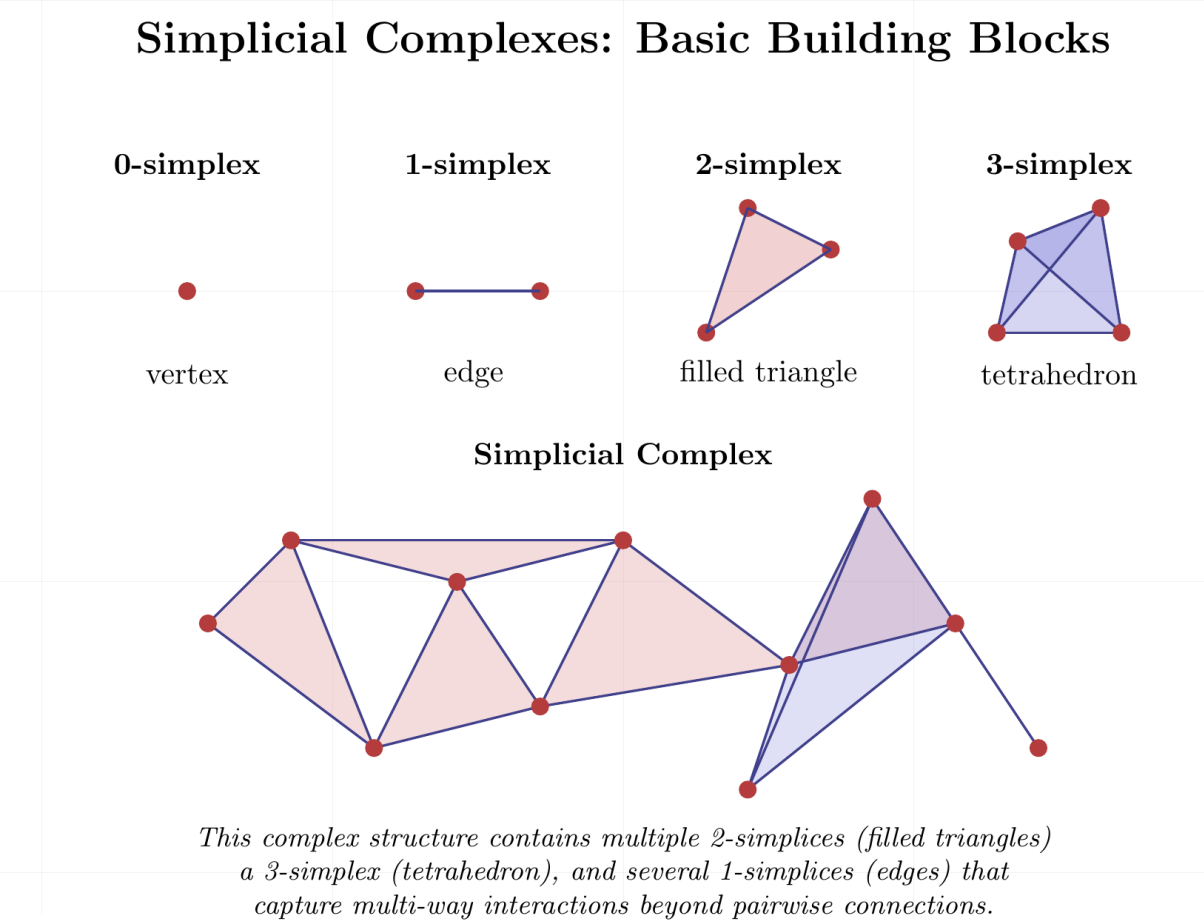


Figure 3: Visual representation of simplicial complexes. The top row shows individual simplices of different dimensions (0-simplex, 1-simplex, 2-simplex, and 3-simplex). The bottom part shows a more complex simplicial complex with multiple 2-simplices (filled triangles), a 3-simplex (tetrahedron), and connecting 1-simplices (edges) that capture multi-way interactions beyond pairwise connections.

Simplicial complexes generalize networks by incorporating higher-order interactions. Simplicial complexes can be thought of as triangles of various dimensions - vertices (0-simplices), edges (1-simplices), triangles (2-simplices), tetrahedra (3-simplices), and so on [Petri et al., 2014] connected together either via shared vertices, edges, or faces.

Formally, a simplicial complex K on a vertex set V is a collection of subsets of V (called simplices) such that:

- For every vertex $v \in V$, $\{v\} \in K$ (0-simplex)
- If $\sigma \in K$ and $\tau \subset \sigma$, then $\tau \in K$ (closure property)

A k-simplex $\sigma = [v_0, v_1, ..., v_k]$ represents an interaction between $k+1$ vertices. For example:

- A 0-simplex is a vertex
- A 1-simplex is an edge (pairwise interaction)

- A 2-simplex is a filled triangle (three-way interaction)
- A 3-simplex is a solid tetrahedron (four-way interaction)
- and so on

Several researchers have successfully applied simplicial complexes to model complex systems. Petri et al. [2014] used simplicial complexes to analyze brain functional networks, revealing topological structures that correlate with cognitive states. Giusti et al. [2016] demonstrated how simplicial complexes can capture neural coding schemes beyond what traditional network models could represent. Sizemore et al. [2018] showed how clique topology in neural systems provides insights into brain development and function.

The primary tool for analyzing these structures is homology, which characterizes the "holes" in the data at different dimensions. These are quantified by Betti numbers (β_k):

- β_0: Number of connected components
- β_1: Number of 1D holes (loops/cycles)
- β_2: Number of 2D holes (voids)

Techniques like Persistent Homology track how these topological features appear and disappear across different scales, offering a multi-scale view of the system's structure [Edelsbrunner and Harer, 2008].

While simplicial complexes offer significant advantages over traditional networks, they have inherent limitations:

- They are undirected, with no natural way to represent asymmetric interactions
- Temporal dynamics are challenging to model in simplicial complexes

3 Theory of Operads

3.1 Operads

A large class of mathematical theories consists of three ingredients:

1. A collection of objects.
2. A collection of morphisms between these objects.
3. A notion of composition of these morphisms.

The most well-known example of this pattern is arithmetic, where the objects are numbers, the morphisms are functions (addition, multiplication, etc.), and the composition is the usual function composition. All fields such as real numbers, complex numbers, and vector spaces can be described in this way.

As we go up the hierarchy of mathematics, we find more and more examples of this pattern. For example, in topology, the objects are topological spaces, the morphisms are continuous functions, and the composition is the usual function composition. Groups and family (magma, monoid, group, ring, field) are also examples of this pattern. Category

theory is a generalization of this pattern, where the objects are categories, the morphisms are functors, and the composition is the usual functor composition.

Operads are a generalization of this pattern, where the objects are operations, the morphisms are operations of different arities, and the composition is a more general form of function composition. Operads provide a framework for studying algebraic structures that arise in various areas of mathematics, including topology, algebra, and category theory.

Operads consists of:

1. A collection of operations of different arities.
2. A notion of composition of these operations.
3. The composition operations obey certain conditions - associativity and unitality.

3.1.1 Formal Definition

Consider a set $\mathbb{X}$, and an integer $n \in \mathbb{N}$.

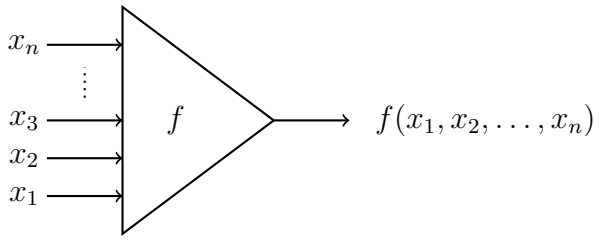
An Operad, $\mathbb{P}$, is defined as a set of n-ary operations, where each operation f has the signature $\mathbb{X}^n \to \mathbb{X}$:

$$\mathbb{P}(n) = \{f : \mathbb{X}^n \to \mathbb{X}\} \tag{3}$$

where $\mathbb{X}^n$ is the cartesian product of $\mathbb{X}$ with itself n times, i.e.

$$\mathbb{X}^n = \mathbb{X} \times \mathbb{X} \times \ldots \times \mathbb{X} \tag{4}$$

i.e. all of these functions f take in n arguments from $\mathbb{X}$ and return a single element from $\mathbb{X}$.



If we have a bunch of these sets of functions $\mathbb{P}(k_i)$ for each $k_i \in \mathbb{N}$, then we can define a composition operation $\circ$ for these operations as follows:

Let $f_i \in \mathbb{P}(k_i)$ be an operation that takes in k_i arguments from $\mathbb{X}$ and returns a single element from $\mathbb{X}$. We can take n numbers of such operations and use their outputs as inputs to another operation $f \in \mathbb{P}(n)$, which takes in n arguments from $\mathbb{X}$ and returns a single element from $\mathbb{X}$. The composition operation $\circ$ is defined as:

$$\mathbb{P}(n) \times (\mathbb{P}(k_1) \times \mathbb{P}(k_2) \times \ldots \times \mathbb{P}(k_n)) \to \mathbb{P}(k_1 + k_2 + \ldots + k_n) \tag{5}$$

$$f, (f_1, f_2, \ldots, f_n) \mapsto f \circ (f_1, f_2, \ldots, f_n) \tag{6}$$

where $f \circ (f_1, f_2, \ldots, f_n) \in \mathbb{P}(k_1 + k_2 + \ldots + k_n)$ is defined as the following diagram:

Associativity of this composition for Operads works as follows:

This composition operation $\circ$ satisfies the following properties:

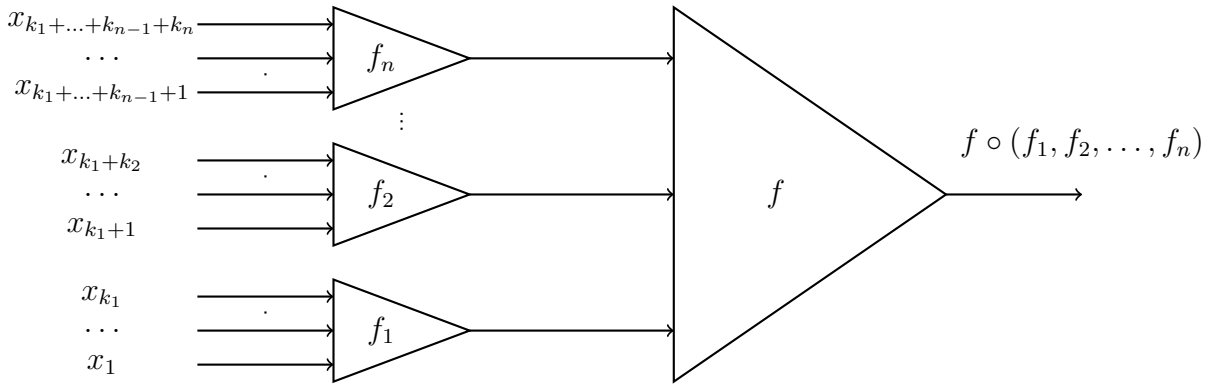


Figure 4: Operadic composition showing how multiple operations $f_1, f_2, \ldots, f_n$ with arities $k_1, k_2, \ldots, k_n$ can be composed with an operation f of arity n to form a new operation of arity $k_1 + k_2 + \ldots + k_n$.

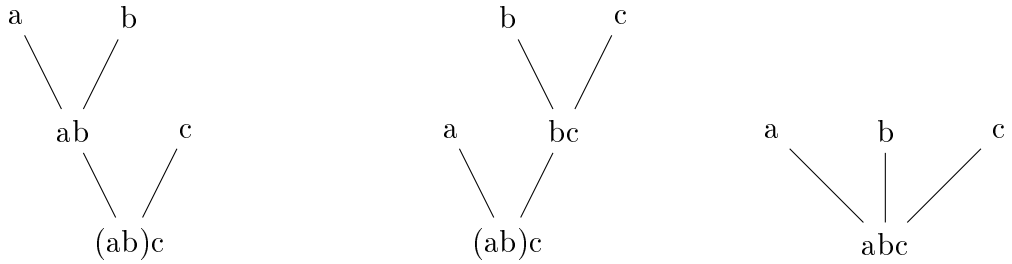


Figure 5: Associativity of operadic composition of arity 3

- **Associativity**: For all $f \in \mathbb{P}(n)$, $g \in \mathbb{P}(k_1)$, $h \in \mathbb{P}(k_2)$, and $i \in \mathbb{P}(k_3)$, we have:

$$f \circ (g \circ (h, i)) = (f \circ (g, h)) \circ i \tag{7}$$

- **Unitality**: For all $f \in \mathbb{P}(n)$, we have:

$$f \circ (\mathrm{id}_{k_1}, \mathrm{id}_{k_2}, \ldots, \mathrm{id}_{k_n}) = f \tag{8}$$

  where id_k is the identity function on $\mathbb{X}^k$.

Symmetry is not required for operads, but it can be added to form symmetric operads. The symmetry condition is:

$$f \circ (g_1, g_2, \ldots, g_n) = f \circ (g_{\sigma(1)}, g_{\sigma(2)}, \ldots, g_{\sigma(n)}) \tag{9}$$

where σ is a permutation of the set $\{1, 2, \ldots, n\}$ or $\sigma \in S_n$ and $g_{\sigma(i)}$ is the $\sigma(i)$-th element of the original sequence, i.e., the permutation σ permutes the order of operations used as inputs to f.

3.1.2 Operads for Modeling

Operads have been applied to modeling systems from diverse fields. In physics, operads have been extensively applied to model quantum field theories, where they capture the

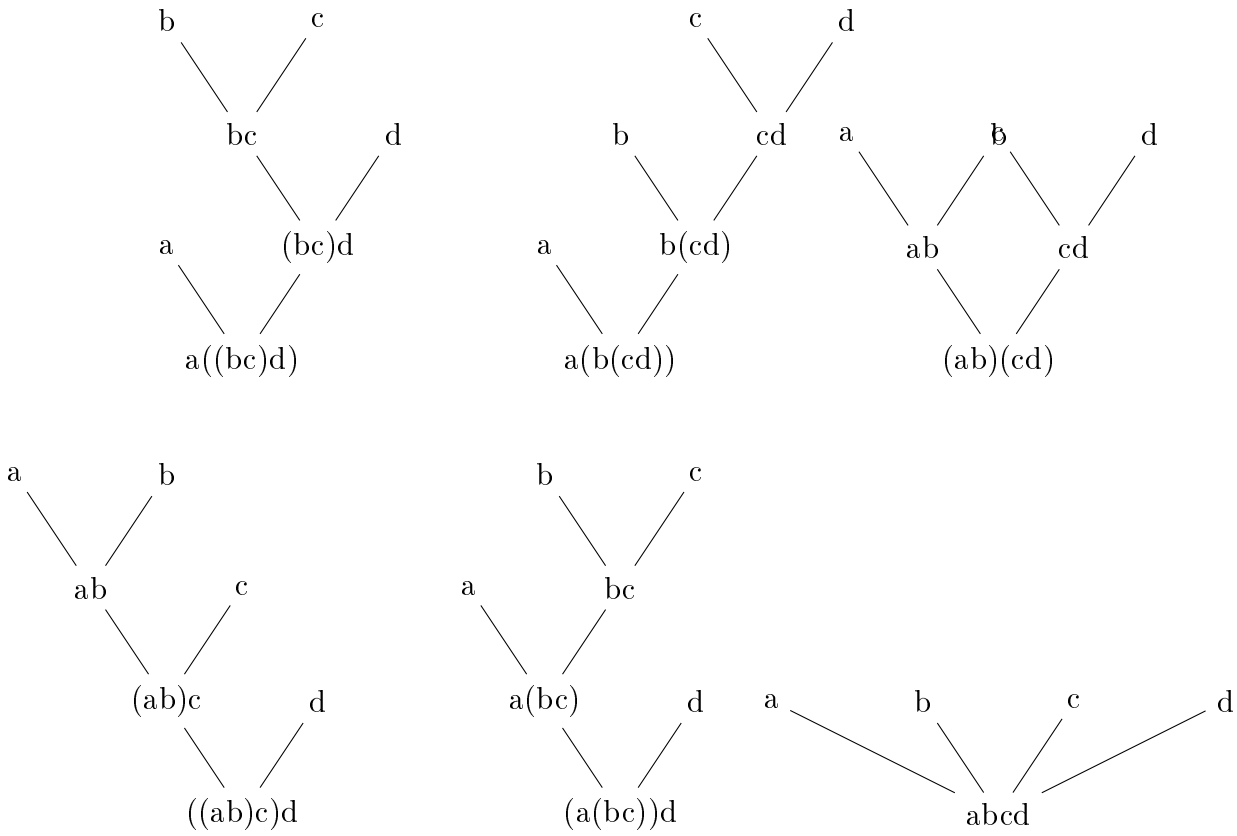


Figure 6: Associativity of operadic composition of arity 4

structure of Feynman diagrams and the composition of quantum interactions Baez and Dolan [1997]. Operads of wiring diagrams have been used to model electrical circuits petri nets and quantum circuits Spivak [2013], Baez and Pollard [2020]. Operads have also been employed in computer science to model SQL database query languages Spivak [2013], in systems design for modeling complex system design specification, analysis and synthesis Behr et al. [2021].

3.2 Operads of Wiring Diagrams

Wiring diagram operads provide a categorical framework for modeling directed compositional systems with explicit input-output interfaces Spivak [2013], Behr et al. [2021]. Unlike classical operads that focus purely on arity, wiring diagram operads encode both the connectivity structure and the directional flow of information through systems, making them particularly suited for modeling complex systems with hierarchical organization and modular decomposition.

3.2.1 Formal Definition

A wiring diagram operad $\mathcal{W}$ consists of graphical representations where operations are depicted as boxes with labeled input and output ports, connected by wires that carry typed information. Formally, we define:

Wiring Diagrams: A wiring diagram W over a finite set of types T is a directed

graph where:

- Vertices represent operations (boxes) with labeled input ports $\text{in}(v) \subseteq T$ and output ports $\text{out}(v) \subseteq T$
- Edges represent wires connecting output ports to input ports
- External inputs and outputs form the interface of the diagram

Operations: For a wiring diagram W with input interface $I \subseteq T$ and output interface $O \subseteq T$, we denote the set of operations as $\mathcal{W}(I; O)$. Each operation $f \in \mathcal{W}(I; O)$ represents a morphism:

$$f : \prod_{i \in I} X_i \to \prod_{o \in O} X_o \tag{10}$$

where X_t denotes the data type associated with type $t \in T$.

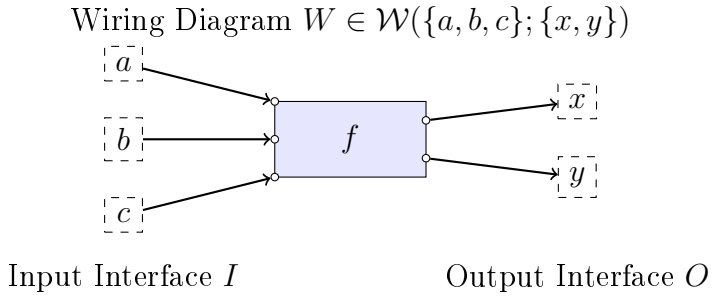


Figure 7: Basic wiring diagram showing an operation f with input interface $\{a, b, c\}$ and output interface $\{x, y\}$. Boxes represent operations, circles represent ports, and arrows represent typed wires.

3.2.2 Composition Structure

The composition operation in wiring diagram operads is defined through substitution and wire connecting. Given:

- A wiring diagram $f \in \mathcal{W}(I; O)$ with input interface I and output interface O
- Wiring diagrams $g_1 \in \mathcal{W}(I_1; O_1), g_2 \in \mathcal{W}(I_2; O_2), \ldots, g_k \in \mathcal{W}(I_k; O_k)$

The composition $f \circ (g_1, g_2, \ldots, g_k)$ is performed by:

1. **Interface Matching:** Ensuring output interfaces of g_i match corresponding input requirements in f
2. **Diagram Substitution:** Replacing designated boxes in f with the complete wiring diagrams g_i
3. **Wire Connection:** Connecting output wires of g_i to input wires of the corresponding positions in f

The resulting composition has input interface $I' = \bigcup_{i=1}^{k} I_i$ and output interface O:

$$f \circ (g_1, g_2, \ldots, g_k) \in \mathcal{W}\left(\bigcup_{i=1}^{k} I_i; O\right) \tag{11}$$

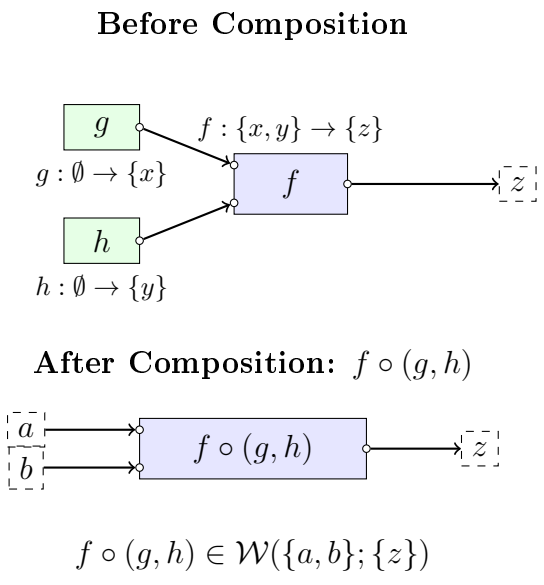


Figure 8: Composition of wiring diagrams showing how operations g and h are substituted into operation f. The top diagram shows the individual components, while the bottom shows the resulting composed operation.

3.2.3 Associativity and Unitality

Wiring diagram operads satisfy the fundamental operadic axioms:

Associativity: For compatible wiring diagrams, the composition operation is associative:

$$(f \circ g) \circ h = f \circ (g \circ h) \tag{12}$$

This corresponds to the fact that the order of substituting sub-diagrams does not affect the final connectivity structure.

Unitality: Identity wiring diagrams act as units under composition. For each type $t \in T$, there exists an identity operation $\mathrm{id}_t \in \mathcal{W}(\{t\}; \{t\})$ that simply connects its input directly to its output:

$$f \circ (\mathrm{id}_{t_1}, \mathrm{id}_{t_2}, \ldots, \mathrm{id}_{t_n}) = f \tag{13}$$

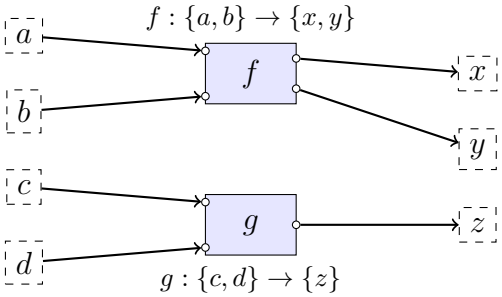
3.2.4 Categorical Properties

Wiring diagram operads form a symmetric monoidal category where:

- Objects are finite sets of types (interfaces)
- Morphisms are wiring diagrams between interfaces
- Composition is given by diagram substitution
- The monoidal product corresponds to parallel composition of diagrams
- Symmetry is given by wire permutation

This categorical structure enables the modeling of complex systems with multiple subsystems operating in parallel, hierarchical decomposition through nested composition, and modular design patterns where components can be independently developed and later integrated.

Parallel Composition (Monoidal Product)



$$f \otimes g \in \mathcal{W}(\{a, b, c, d\}; \{x, y, z\})$$

Figure 9: Parallel composition in wiring diagram operads demonstrating the monoidal product structure. Two operations f and g operate independently in parallel, with their interfaces combined disjointly.

4 σ-operads

Modeling temporality is central to be able to capture the true essence of complex systems. Components of complex systems interact with one another and evolve across multiple time scales, leading to emergent phenomena that cannot be understood through static or purely structural models. However for modeling such systems, it is sufficient to model the participant components and their interactions as stochastic processes that evolve over time, rather than modeling time explicitly. This implies our model must capture sequences that represent the evolution of states and interactions over time, while preserving causal relationships. The different scales of time are captured by the hierarchical structure of operads, where different levels of the hierarchy can represent dynamics occurring at different time scales, with higher levels operating at slower time scales than lower ones.

We develop a novel mathematical framework, based on WD-operads, to model complex systems through temporal-causal relationships. Our approach enriches the established theory of WD-operads with temporal probability structures, creating operads that naturally capture sequential dynamics and causal interactions.

4.1 Temporal Probability Spaces

Let us first define the notion of temporal probability spaces, which will serve as an enriching category for our operads. We denote the category of temporal probability spaces as **TempProb**.

Each object of **TempProb** is a temporal probability space, $\Omega_{\mathcal{W}}$, shortened as Ω, where each object of the space represents a state of the system at a given time. Hence for each box of the WD-operad $\mathcal{W}$, the space $\Omega_{\mathcal{W}}$ contains the set of all possible execution paths or histories of the system component represented by that box. Additionally, to enable us to assign probabilities to subsets of these execution paths, we equip each temporal probability space with a sigma-algebra $\mathcal{F}$ of measurable events and a probability measure

$\mathbb{P}$ that assigns probabilities to these events.

Morphisms in **TempProb** are stochastic kernels that preserve the temporal structure of the spaces, ensuring that information flows consistently through time. A morphism between two temporal probability spaces represents how the probabilistic evolution of one system component conditions or influences the probability distribution over possible evolutions of another component over time. For example, a morphism $K : \Omega_{\mathcal{W}_1} \to \Omega_{\mathcal{W}_2}$ between two temporal probability spaces can be viewed as a stochastic kernel that maps each execution path in $\Omega_{\mathcal{W}_1}$ to a probability distribution over execution paths in $\Omega_{\mathcal{W}_2}$, while respecting the temporal ordering of events. At time t, given history up to time t in $\Omega_{\mathcal{W}_1}$, the morphism K tells us the conditional probability distribution over possible states at time t in $\Omega_{\mathcal{W}_2}$.

Formally, the category **TempProb** is defined as follows:

- **Objects**: Temporal probability spaces $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ where:
  - Ω is the sample space, representing the set of all possible system trajectories or execution paths.
  - $\mathcal{F}$ is a sigma-algebra on Ω - the collection of measurable events (subsets of trajectories) to which we can assign probabilities. Not every arbitrary subset of Ω is measurable; $\mathcal{F}$ specifies which subsets are "well-behaved" enough for probability theory.
  - $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration - an increasing family of sigma-algebras where $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$. This represents the accumulation of information: $\mathcal{F}_t$ captures everything we can know about the system's history up to time t.
  - $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure assigning a likelihood to each measurable event.
- **Morphisms**: Stochastic kernels $K : (\Omega_1, \mathcal{F}_1, \{\mathcal{F}_t^1\}) \to (\Omega_2, \mathcal{F}_2, \{\mathcal{F}_t^2\})$ that are adapted to the filtrations. These represent how one system component probabilistically influences another while respecting causality: at each time t, K maps histories in Ω_1 (measurable with respect to $\mathcal{F}_t^1$) to probability distributions over histories in Ω_2. Formally, K_t is $\mathcal{F}_t^1$-measurable for all t, ensuring the influence cannot depend on future information.
- **Composition**: Temporal probability spaces compose via a 2-step process:

  Step 1 - Monoidal product: For any two spaces, we first form the product:

  $$(\Omega_1, \mathcal{F}_1, \{\mathcal{F}_t^1\}, \mathbb{P}_1) \otimes (\Omega_2, \mathcal{F}_2, \{\mathcal{F}_t^2\}, \mathbb{P}_2) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{F}_t^1 \otimes \mathcal{F}_t^2, \mathbb{P}_1 \times \mathbb{P}_2)$$

  Step 2 - Dependency modification: For dependent systems, stochastic kernels $K : \Omega_1 \to \Omega_2$ modify the joint probability measure on the product space to create causal dependencies. Multiple dependencies compose via the Chapman-Kolmogorov equation [Kadanoff, 2000]:

  $$(L \circ K)(\omega_1, A) = \int_{\Omega_2} K(\omega_1, d\omega_2) L(\omega_2, A)$$

This category captures the essential features of complex systems: information accumulates through time (filtration), future states depend probabilistically on past states (stochastic kernels), temporal ordering enforces causality (adaptedness), and independent subsystems can evolve in parallel (monoidal structure).

4.1.1 Example: Neural Spike Chains

As a concrete example of **TempProb**, consider a simple system with two neurons arranged in sequence:

$$(input) \to [neuron_1] \to [neuron_2] \to (output)$$

Each neuron fires with a stochastic spike when its input exceeds a certain threshold, with some probability. For an input voltage of x volts, the sample space Ω contains all conceivable sequences, including acausal ones like backwards information flow. However, we focus on the σ-algebra $\mathcal{F}$ which contains only measurable events - causally valid sequences that respect temporal ordering.

Examples of events in $\mathcal{F}$ include:

- ω_0: $(input = x)$ only (initial state)
- ω_1: $(input = x) \to [neuron_1 \text{ fires at } t_1]$
- ω_2: $(input = x) \to [neuron_1 \text{ does not fire}]$
- ω_3: $(input = x) \leftarrow [neuron_1 \text{ fires at } t_1]$ (absurd)
- ω_4: $(input = x) \to [neuron_1 \text{ fires at } t_1] \to [neuron_2 \text{ does not fire}]$
- ω_5: $(input = x) \to [neuron_1 \text{ fires at } t_2] \leftarrow [neuron_2 \text{ fires at } t_1]$ (absurd)
- ω_6: $(input = x) \leftarrow [neuron_1 \text{ fires at } t_2] \to [neuron_2 \text{ fires at } t_1]$ (absurd)
- ω_7: $(input = x) \to [neuron_1 \text{ fires at } t_1] \to [neuron_2 \text{ fires at } t_2] \to (output)$
- etc.

Note that the internal dynamics of each neuron are abstracted away; we focus on observable spike events and their temporal ordering. The σ-algebra $\mathcal{F}$ excludes acausal sequences like $(input) \leftarrow [neuron_1] \leftarrow [neuron_2]$ - while these might exist in Ω, they are not measurable events in $\mathcal{F}$.

The filtration $\{\mathcal{F}_t\}_{t>0}$ captures information accumulation:

- $\mathcal{F}_{t_1}$: Events knowable up to t_1 (input + neuron1 state)
- $\mathcal{F}_{t_2}$: Events knowable up to t_2 (full system state, including output)

Each event has an associated probability:

$$\mathbb{P}(\omega_1) = P(\text{neuron}_1 \text{ fires} \mid \text{input} = x) \tag{14}$$

$$\mathbb{P}(\omega_7) = P(\text{neuron}_1 \text{ fires} \mid x) \cdot P(\text{neuron}_2 \text{ fires} \mid \text{neuron}_1 \text{ fired}) \tag{15}$$

4.1.2 Example: Composition of two spaces

Now, we show how temporal probability spaces compose. For simplicity, let's decompose our neural chain in the previous example into two separate TempProb objects and see how they combine.

First component: $(\Omega^1, \mathcal{F}^1, \{\mathcal{F}_t^1\}, \mathbb{P}^1)$ represents (input) $\to$ [neuron$_1$] $\to$ (output) system:

- Ω^1 contains all conceivable sequences: input $\to$ neuron$_1$ $\to$ output
- $\mathcal{F}^1$ contains measurable events like: (*input*), (*input*) $\to$ [*neuron*$_1$ fires] $\to$ (*output*), (*input*) $\to$ [*neuron*$_1$ does not fire] with $\{\mathcal{F}_1^1\}$ capturing information at t_1
- $\mathbb{P}^1$ assigns probabilities to events in $\mathcal{F}_1^1$

Second component: $(\Omega^2, \mathcal{F}^2, \{\mathcal{F}_t^2\}, \mathbb{P}^2)$ represents a separate (input) $\to$ [neuron$_2$] $\to$ (output) system:

- Ω^2 contains all conceivable sequences: input $\to$ neuron$_2$ $\to$ output
- $\mathcal{F}^2$ contains measurable events like: (*input*), (*input*) $\to$ [*neuron*$_2$ fires] $\to$ (*output*), (*input*) $\to$ [*neuron*$_2$ does not fire] with $\{\mathcal{F}_1^2\}$ capturing information at t_1
- $\mathbb{P}^2$ assigns probabilities to events in $\mathcal{F}_1^2$

We compose temporal probability spaces in two steps:

Step 1 - Monoidal product: We first combine the systems using the monoidal structure:

$$(\Omega^1, \mathcal{F}^1, \{\mathcal{F}_t^1\}, \mathbb{P}^1) \otimes (\Omega^2, \mathcal{F}^2, \{\mathcal{F}_t^2\}, \mathbb{P}^2) = (\Omega^1 \times \Omega^2, \mathcal{F}^1 \otimes \mathcal{F}^2, \mathcal{F}_t^1 \otimes \mathcal{F}_t^2, \mathbb{P}^1 \times \mathbb{P}^2)$$

Step 2 - Add dependencies (if needed): For independent systems, we stop at Step 1. For dependent systems, we apply stochastic kernels to modify the probability measure on the product space. A kernel $K : \Omega^1 \to \Omega^2$ creates dependencies by changing the joint distribution:

$$\mathbb{P}(\text{dependent})(\omega_1, \omega_2) = \mathbb{P}^1(\omega_1) \cdot K(\omega_1, \omega_2)$$

where the Chapman-Kolmogorov equation ensures consistent composition of multiple dependencies.

4.1.3 Example: Temporal Integration with Delays

Consider a more complex scenario where *neuron*$_2$ receives two inputs: one from neuron$_1$ and a delayed external input *input*$_2$. Suppose *neuron*$_2$ has a memory window and can integrate signals arriving at different times. Lets model the second component from above again:

The temporal probability space $(\Omega^2, \mathcal{F}^2, \{\mathcal{F}_t^2\}, \mathbb{P}^2)$ for this system contains paths like:

- ω_1: (*input*$_1$), (*input*$_2$) $\to$ [*neuron*$_2$ fires] $\to$ *output*
- ω_2: (*input*$_1$), (*input*$_2$) $\to$ [*neuron*$_2$ does not fire]

The filtration captures the accumulation of temporal information:

- $\mathcal{F}^2_{t_1}$: Information about neuron_2 firing after $input_1$ has arrived
- $\mathcal{F}^2_{t_2}$: Information about neuron_2 firing after $input_2$ has arrived
- The probability depends on delay: $\mathbb{P}(\text{neuron}_2 \text{ fires} \mid \text{both inputs, delay } \delta)$ where $\delta = |t_2 - t_1|$

The insight here is that the temporal probability structure is fundamentally event-based rather than clock-time-based:

- **Events, not times**: The model tracks which events happened and their causal ordering, not precise timestamps. $neuron_2$ fires after $neuron_1$.
- **Delay as probability parameter**: The actual delay δ doesn't change the structure of Ω or $\mathcal{F}$ - it only affects $\mathbb{P}$, or the delay is a parameter to calculate the probability. Short delays might give $\mathbb{P}(\text{fire}) = 0.9$, long delays $\mathbb{P}(\text{fire}) = 0.1$.
- **Conditional dependencies**: What matters is whether neuron_1 fired (and when relative to other events), not the absolute timestamp. The filtration $\mathcal{F}_t$ captures what events have occurred rather than what time is it.

4.2 σ-operads: TempProb-Enriched WD-Operads

We now define σ-operads as WD-operads enriched in **TempProb**. Let $\mathcal{W}$ be the standard operad of wiring diagrams with interfaces (finite sets of typed ports) as objects and wiring diagrams as morphisms.

A σ-operad $\mathcal{W}_\sigma$ is the **TempProb**-enrichment of $\mathcal{W}$, where:

- **Objects**: Same as $\mathcal{W}$ - interfaces representing system boundaries.
- **Hom-objects**: $\mathcal{W}_\sigma(X, Y)$ is a temporal probability space rather than a set. Each wiring diagram now carries probabilistic temporal dynamics.
- **Composition**: Follows the 2-step TempProb composition: first monoidal product (for independent systems), then dependency modification via stochastic kernels (for connected ports).

4.2.1 Temporal Causality through Ports

In σ-operads, each port carries temporal probabilistic information:

- Each port p is associated with an adapted stochastic process $X_p = \{X_p(t)\}_{t \geq 0}$ representing the temporal evolution of information at that interface.
- Wires connecting ports encode conditional temporal dependence: if port p connects to port q, then $X_q(t)$ depends on the history $\{X_p(s)\}_{s \leq t}$. For the markovian case, this dependence is only on $X_p(t)$.
- The filtration structure ensures causality: information flows from past to future, never backwards in time, so even though Ω contains all trajectories, the filteration removes (or assigns zero probability to) any trajectories that violate causal ordering.

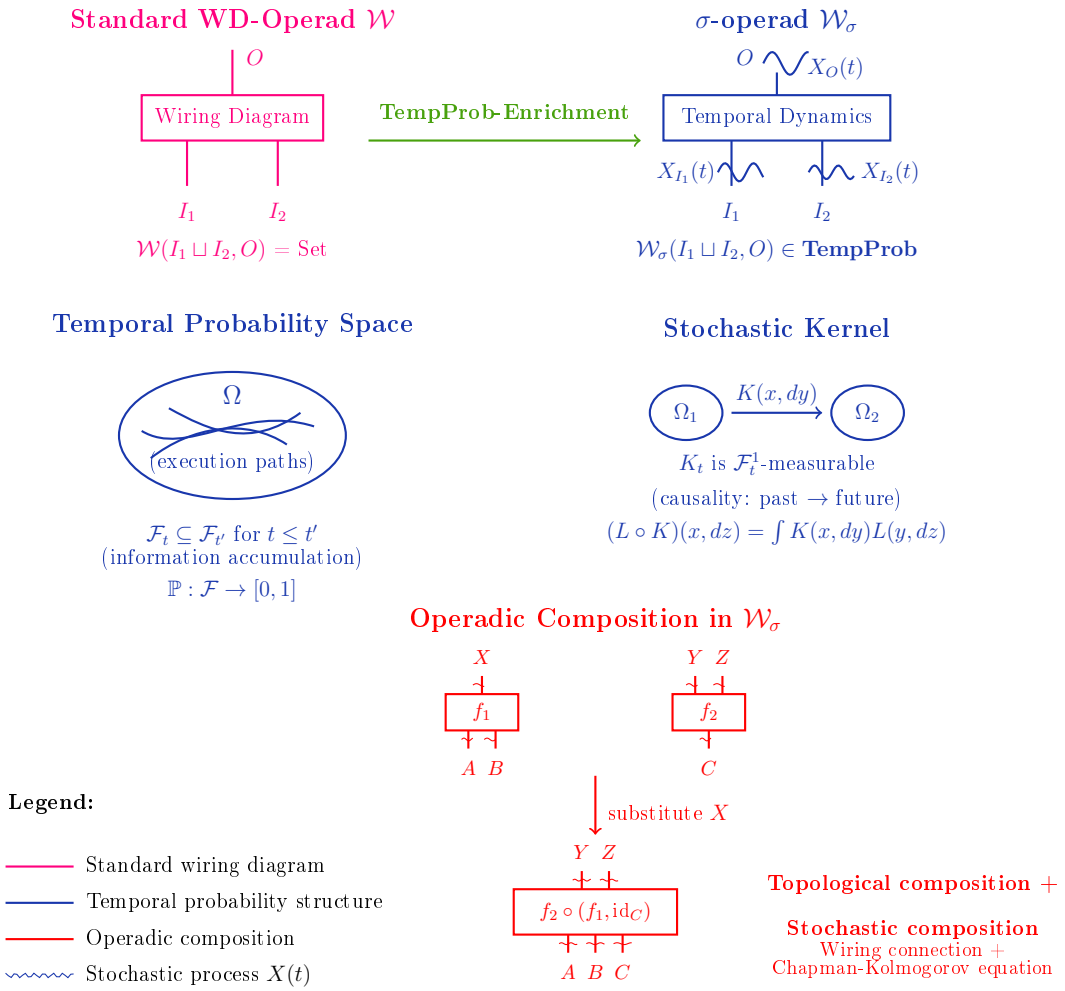


Figure 10: σ-operads as TempProb-enriched wiring diagrams. The top shows the enrichment process: standard wiring diagrams (left) become enriched with temporal probability structures (right), where each port carries stochastic processes $X(t)$ and hom-objects are temporal probability spaces. The middle illustrates the components: temporal probability spaces with filtrations and stochastic kernels preserving causality. The bottom demonstrates operadic composition, combining topological wiring with stochastic composition via the Chapman-Kolmogorov equation.

4.2.2 Composition of Temporal Morphisms

When composing morphisms in $\mathcal{W}_\sigma$, we combine topological composition of wiring diagrams with the 2-step TempProb composition:

1. **Topological Composition**: Standard wiring diagram composition connects outputs to inputs through interface matching, creating the overall system architecture.

2. **Step 1 - Monoidal Product**: Form the independent product of temporal probability spaces corresponding to all component systems: $(\Omega_1, \mathcal{F}_1, \{\mathcal{F}_t^1\}, \mathbb{P}_1) \otimes (\Omega_2, \mathcal{F}_2, \{\mathcal{F}_t^2\}, \mathbb{P}_2)$.

3. **Step 2 - Dependency via Wiring**: For each wire connecting output port p to input port q, apply the appropriate stochastic kernel $K_{p \to q}$ to model the temporal dependency. These kernels satisfy the Chapman-Kolmogorov equation and preserve causality through adapted filtrations.

This composition naturally models how complex systems evolve: subsystems with their own temporal dynamics are connected through their interfaces, creating a larger system whose temporal evolution respects the causal flow through the network structure.

4.2.3 Example: Multi-Scale Epidemic Spread

Disease transmission shows complex multi-scale dynamics across three levels: individual immune responses → interpersonal contact networks → inter-community spread. σ-operads can model this at each scale.

Scale 1 - Individual immune system: Each person has immune state evolving via stochastic processes capturing T-cell responses, antibody production, etc. This forms the base temporal probability space modeling how individual susceptibility changes over time. At the scale of an individual, this can be modeled as a morphism taking $\{pathogen\_exposure\}$ to $\{immune\_response\}$ with temporal probability space

$$(\Omega^{ind}, \mathcal{F}^{ind}, \{\mathcal{F}_t^{ind}\}, \mathbb{P}^{ind})$$

capturing stochastic immune events (e.g. infection, recovery) over time.

Composition at this scale of the system is equivalent to composing immune subsystems and the pathogen's interactions - e.g. T-cell response f_T and antibody production f_A.

Step 1 - Monoidal product: Form independent product $(\Omega_T \times \Omega_A, \mathcal{F}_T \otimes \mathcal{F}_A, \mathcal{F}_t^T \otimes \mathcal{F}_t^A, \mathbb{P}_T \times \mathbb{P}_A)$. Here Ω_T contains all possible T-cell activation trajectories and Ω_A contains all antibody production trajectories.

Step 2 - Apply dependency kernel: T-cell activation affects antibody production via kernel $K_{T\to A}$:

$$K_{T\to A}(\omega_T, d\omega_A) = \exp\left(-\int_0^t \lambda(\text{T-cell state}(\omega_T, s))ds\right)\mathbb{P}_A(d\omega_A) + (\text{activated antibody measure})$$

where λ is the T-cell activation rate affecting antibody production.

Scale 2 - Individuals within a community: People move through shared spaces creating contact events. When persons i and j come within distance $d < \epsilon$, transmission probability can be modeled as say $p_{ij} = \phi_i(t) \cdot \phi_j(t) \cdot f(d)$ where $\phi_i(t)$ is infectiousness of person i at time t, $f(d)$ is a distance-dependent decay function which captures how transmission probability decreases with physical separation. The temporal probability space models individual movement patterns, meeting events, and resulting infection cascades within the community. This scale abstracts away individual immune details into a single susceptibility state $\phi_i(t)$ derived from Scale 1. Each individual i becomes a morphism taking $\{location\_trace^i\}$ to $\{infection\_state^i\}$ with temporal probability space

$$(\Omega^i, \mathcal{F}^i, \{\mathcal{F}_t^i\}, \mathbb{P}^i)$$

capturing stochastic contact and transmission events.

At this scale composition of movement patterns of persons i and j can proceed as follows:

Step 1 - Monoidal product: Form

$$(\Omega_i \times \Omega_j, \mathcal{F}_i \otimes \mathcal{F}_j, \mathcal{F}_t^i \otimes \mathcal{F}_t^j, \mathbb{P}_i \times \mathbb{P}_j)$$

where Ω_i contains all possible location trajectories for person i.

Step 2 - Apply contact kernel: When persons come within distance ϵ, transmission kernel $K_{i\to j}$ acts:

$$K_{i\to j}(\omega_i, d\omega_j) = \mathbb{P}_j(d\omega_j) \cdot \mathbb{I}[\text{dist}(\omega_i, \omega_j) > \epsilon] + \mathbb{P}_j^{\text{exposed}}(d\omega_j) \cdot \mathbb{I}[\text{dist}(\omega_i, \omega_j) \leq \epsilon]$$

where $\mathbb{P}_j^{\text{exposed}}$ has higher infection probability when persons are in contact.

Scale 3 - Community level: We further abstract contact networks into aggregate community infection states. Community A becomes a morphism taking $\{travelers\_in^A\}$ to $\{travelers\_out^A, infection\_state^A\}$. The temporal probability space

$$(\Omega^A, \mathcal{F}^A, \{\mathcal{F}_t^A\}, \mathbb{P}^A)$$

models community-wide infection events without specifying internal contact mechanisms - it captures the emergent dynamics from Scale 2 while abstracting away individual movements and contact patterns. The individual scale can be abstracted away into community-level infection rates, which along with travel patterns determine inter-community spread.

Composition at a community level can be thought of as inter-community travel and seeding of infections:

Step 1 - Monoidal product: Form

$$(\Omega_A \times \Omega_B, \mathcal{F}_A \otimes \mathcal{F}_B, \mathcal{F}_t^A \otimes \mathcal{F}_t^B, \mathbb{P}_A \times \mathbb{P}_B)$$

representing independent community evolution.

Step 2 - Apply travel kernel: Inter-community movement creates dependency via $K_{A\to B}$:

$$K_{A\to B}(\omega_A, d\omega_B) =$$

$$\int p(\text{travel rate}|\text{infection state}_A(\omega_A, t)) \cdot p(\text{seeding}|\text{travelers})\, \mu(d\text{travelers})\, \mathbb{P}_B^{\text{seeded}}(d\omega_B)$$

where the kernel integrates over all possible traveler states and their infection seeding effects.

Composing without abstraction

Each of these scales abstracts away lower-level details into higher-level stochastic processes, however, a higher scale can also be explicitly created out of a lower one without abstraction, by composing the temporal probability spaces directly.

For example, to create a community-level model (Scale 3) directly from individual contact networks (Scale 2) without abstraction, we can compose the temporal probability spaces of all individuals in the community:

$$(\Omega_{community}, \mathcal{F}_{community}, \{\mathcal{F}_t^{community}\}, \mathbb{P}_{community}) = \bigotimes_{i=1}^{N} (\Omega^i, \mathcal{F}^i, \{\mathcal{F}_t^i\}, \mathbb{P}^i)$$

where each $(\Omega^i, \mathcal{F}^i, \{\mathcal{F}_t^i\}, \mathbb{P}^i)$ represents individual i's movement and contact patterns from Scale 2.

The evolution of this detailed system requires composing all pairwise contact kernels $K_{i\to j}$ for every (i, j) pair according to the spatial contact network topology. For N individuals, this creates $O(N^2)$ contact kernels acting on the full product space.

To see how this works concretely, consider $N = 3$ individuals. The sample space becomes:

$$\Omega_{community} = \Omega^1 \times \Omega^2 \times \Omega^3$$

Each community trajectory $(\omega^1, \omega^2, \omega^3) \in \Omega_{community}$ contains the full location traces and infection histories of all three people. The system evolves through sequential contact kernel application:

$$\mathbb{P}_{final}(\omega^1, \omega^2, \omega^3) = \mathbb{P}^1(\omega^1) \times \mathbb{P}^2(\omega^2) \times \mathbb{P}^3(\omega^3) \tag{16}$$
$$\times\, k_{1\to 2}(\omega^1, \omega^2) \times k_{1\to 3}(\omega^1, \omega^3) \tag{17}$$
$$\times\, k_{2\to 1}(\omega^2, \omega^1) \times k_{2\to 3}(\omega^2, \omega^3) \tag{18}$$
$$\times\, k_{3\to 1}(\omega^3, \omega^1) \times k_{3\to 2}(\omega^3, \omega^2) \tag{19}$$

Each contact kernel $k_{i\to j}(\omega^i, \omega^j)$ models how person i's location trajectory and infection state affects person j's infection probability when they come within transmission distance ϵ.

Now, suppose we want to extract community-level information from this detailed model. We define aggregate observables like total infection count $I(t) = \phi_1(t) + \phi_2(t) + \phi_3(t)$ or infection rate $R(t) = \frac{1}{3}(I(t))$. The question becomes: can we derive a simpler kernel $K_{community}$ that operates directly on these aggregates without tracking individual trajectories?

For small systems like $N = 3$, individual variations matter significantly. However, as N increases, patterns emerge. With large N, individual variations start averaging out, and the aggregate infection rate $R(t) = \frac{1}{N}\sum_{i=1}^{N} \phi_i(t)$ becomes more predictable. The detailed kernel composition $\bigcirc_{i,j} K_{i\to j}$ acting on $\Omega_1 \times \ldots \times \Omega_{100}$ might be approximated by a simpler kernel acting on the aggregate space.

5 Results

The σ-operad framework naturally captures fundamental complex systems properties through its mathematical structure. We demonstrate how key features identified in complex systems literature emerge directly from operadic composition.

5.1 Built-in support for structural properties

The operadic structure provides native support for fundamental characteristics of complex systems.

5.1.1 Structural: hierarchy, modularity and near-decomposability

The σ-operad framework inherently captures the fundamental structural properties that define complex systems. **Near-decomposability** — systems with stronger internal than external interactions — emerges naturally from the operadic structure where each operation (box) defines a clear subsystem with well-defined input/output interfaces that enforce boundary conditions through composition rules. Strong internal interactions are captured by the temporal probability space within each operation, while weak external interactions are modeled by stochastic kernels between operations, creating the characteristic **interaction strength separation**. The framework supports **hierarchical organization** through nested composition where smaller operations combine into larger ones, with each level of composition corresponding to a distinct hierarchical level where higher-level operations operate on outputs of lower-level ones. Different hierarchical levels operate at

different temporal scales through the filtration structure $\{\mathcal{F}_t\}$, allowing higher-level operations to have coarser temporal discretization than lower-level ones, naturally capturing **multi-scale dynamics**. This **time-scale separation** is enforced by the 2-step composition process that first forms independent products then applies dependencies, ensuring that subsystem dynamics remain distinguishable across scales.

5.1.2 Modularity and compositionality

Complex systems exhibit **modular organization** where components can be recombined in different configurations, a property that emerges directly from operadic composition principles. Each operation functions as a module with standardized interfaces (ports) that enable systematic construction of larger systems from smaller components through **operadic substitution**. The composition rules ensure that modules can only be connected when their interfaces match, preventing incompatible combinations while enabling flexible recombination of components. This **interface compatibility** creates a **compositional semantics** where the meaning (temporal dynamics) of a composite system is determined by the meanings of its components and their interaction patterns, following strict compositional principles that guarantee predictable system behavior from modular assembly.

5.1.3 Scale-invariance and fractal structure

Scale-invariance and **self-similar patterns** across scales emerge naturally from operadic self-composition, enabling the framework to capture the fractal-like organization characteristic of many complex systems. Operations can compose with themselves at different scales through **recursive application** $f \circ f \circ f \cdots$, creating hierarchical structures where the same interaction pattern repeats at multiple organizational levels. This self-similar composition generates the **recursive structures** observed in biological branching patterns, social network hierarchies, and economic market dynamics. The temporal probability spaces can exhibit **scale-free behavior** when stochastic kernels preserve **power-law distributions** during composition—if the initial distribution follows $\mathbb{P}(X) \sim X^{-\alpha}$, appropriately chosen kernels maintain this scaling relationship through successive compositions. This mathematical property enables the framework to model systems that lack **characteristic scales**, such as neural networks with scale-free connectivity or ecosystems with power-law species distributions. The operadic structure naturally supports **recursive definitions** where complex systems are defined in terms of smaller versions of themselves, generating the self-similar hierarchies that exhibit **statistical invariance** across observation scales.

5.2 Emergence

Unlike the structural properties examined above, emergence represents a fundamentally different challenge: identifying when and how higher-level properties arise that cannot be reduced to or predicted from individual component behaviors. The σ-operad framework provides a novel mathematical criterion for emergence based on the transition between explicit individual modeling and aggregate approximations.

5.2.1 The emergence transition criterion

Recall from our epidemic example the transition from explicit individual modeling (say N=3 where individual variations matter) to valid aggregate approximations (say N=100,000 where statistical behavior emerges). This transition point provides a precise mathematical criterion for emergence. We propose that emergence occurs when the composition of all individual stochastic kernels can be approximated by simpler aggregate kernels within controlled error bounds:

$$\|K_{macro} - \text{Proj}\left(\bigcirc_{i,j} K_{i\to j}\right)\| < \epsilon$$

where K_{macro} operates on aggregate observables (like total infection rate $R(t) = \frac{1}{N}\sum_{i=1}^{N}\phi_i(t)$) and Proj is the projection operator that maps individual-level dynamics to aggregate quantities.

Below the emergence threshold (small N) individual variations dominate and the approximation fails. The system behavior depends critically on specific individual properties, for example, a single super-spreader can dramatically alter global dynamics. No meaningful aggregate description exists. Above the emergence threshold (large N), individual variations become statistically insignificant compared to collective behavior. The aggregate observables follow predictable dynamics that are irreducible to individual properties where knowing everything about individuals still requires the aggregate calculation to predict system behavior.

5.2.2 Correlation decay and measure concentration

The transition criterion connects to fundamental results in probability theory [Binney et al., 1992], but extends beyond standard applications through the network structure of interacting stochastic kernels. The key insight here is that emergence depends on how **correlations between individuals decay with system size**. The Central Limit Theorem tells us that for independent variables, sample averages converge with variance decreasing as $1/N$ (so standard deviation decreases as $1/\sqrt{N}$). However, in complex systems with correlated components, if pairwise correlations decay as $\text{Corr}(\phi_i, \phi_j) \sim N^{-\alpha}$ with $\alpha > 1/2$, then aggregate observables concentrate around their means faster than the standard CLT rate, and the macro-approximation becomes valid. However, if $\alpha \leq 1/2$, correlations persist at all scales and emergence fails—individual fluctuations continue to affect global behavior. The network topology of stochastic kernels determines the correlation decay rate. Dense interaction networks (where each individual influences many others) tend to have slower correlation decay, requiring larger N for emergence. Sparse networks with local interactions facilitate emergence at smaller scales.

This mathematical criterion reveals why certain macro descriptions work so well despite computational irreducibility claims [Bar-Yam, 2013]. While individual trajectories may be computationally irreducible and requiring full simulation, aggregate statistics become predictable above the emergence threshold. The macro-kernel K_{macro} provides algorithmic shortcuts for predicting collective behavior even when individual behavior remains unpredictable. The resolution lies in **measure concentration** [Stanley, 1999]. Above the emergence threshold, the vast majority of individual configurations yield similar aggregate behavior. The typical behavior becomes predictable through K_{macro} even though specific individual trajectories remain computationally complex. Emergence occurs precisely when the information needed to predict aggregate dynamics becomes dramatically

smaller than the information in individual states. The projection Proj represents massive information compression that preserves predictive power for macro-observables while discarding individual details.

The criterion also reveals why certain properties can only be understood at the aggregate level and cannot be reduced to individual component properties. Consider the epidemic example above the emergence threshold. The infection rate dynamics $\dot{R}(t)$ depends on aggregate contact patterns, travel flows, and population mixing—properties that emerge from individual interactions but cannot be computed from individual properties alone. Even with complete knowledge of each person's immune state $\phi_i(t)$, predicting $\dot{R}(t)$ requires the full kernel composition $\bigcirc_{i,j} K_{i\to j}$, which is computationally equivalent to solving the aggregate problem. Above the emergence threshold, the system exhibits properties that have no individual-level analogues. Herd immunity thresholds, critical vaccination rates, and epidemic phase transitions [Pastor-Satorras et al., 2015] are collective properties that emerge from the statistical structure of interactions but cannot be defined for individual agents. These represent genuinely new system properties.

Emergent properties acquire causal power in the sense that macro-states influence future macro-evolution through K_{macro} in ways that cannot be reduced to sums of individual influences. The community infection state affects travel patterns, which affect other communities—a causal chain that operates at the emergent level.

6 Discussion

We have introduced σ-operads as a mathematical framework that enriches wiring diagrams with temporal probability structures to model complex systems. Our results demonstrate both the framework's natural ability to capture fundamental complex systems properties and its potential to provide novel theoretical insights.

6.1 Theoretical Contributions

Our primary theoretical contribution is the emergence transition criterion, which provides a novel rigorous mathematical definition of when emergence occurs in complex systems. The criterion $\|K_{macro} - \text{Proj}\,(\bigcirc_{i,j} K_{i\to j})\| < \epsilon$ identifies the precise point where aggregate approximations become valid and irreducible macro-properties appear.

This criterion resolves long-standing debates about emergence by connecting it to fundamental probability theory through correlation decay rates. When correlations decay as $N^{-\alpha}$ with $\alpha > 1/2$, systems exhibit super-CLT concentration that enables predictable aggregate behavior even when individual trajectories remain computationally irreducible. Unlike existing approaches that either focus purely on structure (networks, simplicial complexes) or dynamics (statistical mechanics models), σ-operads naturally integrate both through the enrichment of compositional structure with temporal probability. This provides a unified foundation for modeling hierarchy, modularity, scale-invariance, and emergence within a single mathematical framework.

6.2 Comparison with Existing Frameworks

Network models: While networks excel at capturing pairwise interactions and topological properties, they struggle with temporal dynamics and multi-scale composition.

σ-operads extend networks by adding temporal probability structure and natural composition operations that enable hierarchical modeling.

Statistical mechanics: Physical models like the Ising model provide precise mathematical treatment of phase transitions but assume fixed interaction topologies. σ-operads support adaptive, compositional architectures while maintaining rigorous probabilistic foundations.

Category theory approaches: Previous category-theoretic models of complex systems (like Baez-Fong's network theory) focus on structural composition but lack temporal dynamics. Our temporal probability enrichment adds the missing dynamical component while preserving compositional structure.

6.3 Limitations and Open Questions

Computational tractability: While we identified mathematical criteria for emergence, computing the exact transition thresholds requires solving large-scale kernel compositions, which may be computationally intensive for realistic systems. Developing efficient approximation methods remains an open challenge.

Empirical validation: Our framework makes testable predictions about emergence thresholds (e.g., specific values of N where aggregate descriptions become valid), but validating these predictions requires extensive empirical studies across different domains.

Scale boundary universality: We demonstrated emergence criteria for specific examples, but whether universal scaling laws exist across different types of complex systems (biological, social, technological) remains an open question.

6.4 Future Directions

The emergence transition criterion could guide intervention design in complex systems by identifying optimal scales for action. For epidemics, it suggests when individual-level vs. population-level interventions will be most effective. Developing efficient algorithms for computing kernel compositions and emergence thresholds could enable practical application to large-scale systems. Machine learning approaches might approximate the projection operators for specific system types. While our framework captures emergence through correlation decay, extending it to model phase transitions with critical exponents and universality classes could bridge statistical physics and complex systems theory.

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