Modeling Complex Systems Using Operads

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Abstract

This paper presents an algebraic approach to modeling phase transitions and emergent phenomena in complex systems using operads. We propose a novel type of operad, named σ-operads (statistical operads), which extends operads of wiring diagrams with a statistical structure. We show how these operads can be used to represent the compositional structure, scale invariance, and near-decomposability of complex systems.

1 Introduction

Complex systems are characterized by numerous interacting components that exhibit emergent behaviors and phase transitions not evident from the study of individual components [Mitchell, 2009]. From flocking birds to financial markets, from neural networks to social movements, these systems demonstrate how local interactions cascade across scales to produce system-wide transformations. Understanding such phenomena requires mathematical frameworks that can capture both the multi-scale, hierarchical nature of complex systems and their compositional structure.

The challenge of modeling complex systems has driven the development of increasingly sophisticated mathematical frameworks. Traditional network models capture pairwise interactions but miss higher-order effects. Hypergraphs extend networks to include multi-agent interactions, while simplicial complexes further enable topological analysis through higher-dimensional structures [Battiston et al., 2020]. Yet even these advanced models struggle with a fundamental limitation: they cannot adequately represent systems with complex hierarchical organization and dynamic compositional structures.

Consider social consensus formation, where opinions cascade through multiple levels of influence [Watts and Strogatz, 1998], or biological signaling networks, where protein complexes dynamically assemble to form functional units [Giovannoni et al., 2017]. These systems exhibit two key properties identified by Simon [1962]: near-decomposability (subsystems interact weakly externally while maintaining strong internal interactions) and scale-invariance (self-similar patterns across organizational levels). Current mathematical frameworks fail to capture how these properties enable phase transitions—those critical moments when local changes reorganize compositional relationships to produce emergent phenomena.

This paper introduces a novel framework based on operads that directly addresses these limitations. Operads, originally developed in algebraic topology, provide a natural language for describing compositional structures and their transformations. We propose σ-operads (statistical operads), which extend operads of wiring diagrams with statistical structure to model how complex systems compose, decompose, and reorganize during phase transitions. This approach offers three key advantages:

1. **Compositional representation**: Operads explicitly model how subsystems combine to form larger systems while preserving compositional relationships

2. **Multi-scale dynamics**: The operadic structure naturally captures interactions across hierarchical levels

3. **Phase transition mechanics**: Statistical extensions allow modeling of critical phenomena and emergent behaviors

Prior work has successfully demonstrated the utility of operads for structural modeling. Spivak [2013] introduced the operad of wiring diagrams to formalize the composition of open systems, a framework further developed by Behr et al. [2021] for system design. Baez and Pollard [2020] applied operadic approaches to network models and Petri nets, providing a rigorous categorical foundation for network theory. In neuroscience, Linde-Domingo et al. [2021] utilized operads to model brain hierarchy and topological simplification. However, these existing frameworks primarily focus on the structural and deterministic aspects of composition. They lack the intrinsic statistical mechanics required to model the stochastic fluctuations and probabilistic reorganization that characterize phase transitions in complex systems.

By bridging category theory, statistical mechanics, and complex systems science, our framework provides new insights into how compositional reorganization drives phase transitions and emergence. We demonstrate applications to diverse systems including neural synchronization, social dynamics, and biological networks, showing how σ-operads reveal previously hidden mechanisms underlying complex system behaviors.

2 Background

2.1 Properties of Complex Systems

2.1.1 Structural Features: Near-decomposability and Scale-invariance

Complex systems exhibit two foundational structural properties that enable their characteristic behaviors and provide the basis for our operadic approach.

Near-decomposability [Simon, 1962] describes systems with subsystems that interact weakly externally while maintaining strong internal interactions. These systems display:

- Stronger interactions within subsystems than between them
- Hierarchical organization across multiple levels
- Time-scale separation between internal and external interactions

Examples include cellular structures (organelles within cells, cells within tissues, tissues within organs, organs within organisms, etc.), ecosystems with loosely coupled niches, and modular software systems. Near-decomposability enables adaptation through independent subsystem evolution [Simon, 1996] and creates the hierarchical levels across which system dynamics unfold.

Scale-invariance describes self-similar patterns appearing across different organizational levels, characterized by:

- Statistical similarity at different observation scales
- Power-law distributions of system properties
- Absence of characteristic scales

This property manifests in branching patterns in biological systems, power-law distributions in scale-free networks etc. [West, 2017, Stanley, 1999]. Scale-invariance emerges naturally in systems that grow through preferential attachment processes or self-organize into critical states.

2.1.2 Dynamical Features: Phase Transitions and Emergence

Phase transitions and emergence represent related dynamical phenomena that arise from the hierarchical and scale-invariant structures of complex systems.

Phase transitions occur when control parameters cross threshold values, causing qualitative changes in system properties. These transitions manifest as discontinuous shifts at critical points [Stanley, 1971], displaying characteristic mathematical signatures: power-law scaling behaviors, critical exponents, and diverging correlation lengths [Newman, 2003, Bak et al., 1987]. In complex systems, phase transitions include ecosystem state shifts, opinion cascades in social networks, synchronization transitions in oscillator systems, financial market crashes, and percolation thresholds in networks.

Emergence describes the formation of properties at higher organizational levels not present in or predictable from individual components [Holland, 1998, Anderson, 1972]. Anderson's "More is Different" principle emphasizes that complex systems require analysis beyond reductionist approaches. Examples include flocking behaviors in birds, intelligence in neural networks, consciousness from neuronal activity, and market dynamics in economies [Camazine et al., 2003, Haken, 1983].

Phase transitions can be viewed as a specific type of emergence characterized by sudden, discontinuous changes, while other emergent phenomena may develop gradually or exist in steady states [Bar-Yam, 2013]. Both phenomena share common underlying mechanisms:

- They arise from the hierarchical structures created by near-decomposability
- They propagate across scales following patterns enabled by scale-invariance

- They involve reorganization of compositional relationships between system components
- They manifest when local changes cascade to produce system-wide transformations

At critical points, fluctuations occur across all scales of the system as local changes propagate through hierarchical levels—a direct consequence of the near-decomposable, scale-invariant structure.

Our operadic framework models these phenomena by tracking compositional relationships between components across scales. This approach complements traditional statistical methods [Stanley, 1999, Goldenfeld, 1992] by focusing explicitly on how compositional structures reorganize during transitions, providing insights into the mechanisms driving phase transitions and emergent behaviors.

2.2 Modeling Phase Transitions and Emergence

2.2.1 Physical Models

In physics, phase transitions are often modeled using statistical mechanics, where the system's behavior is described in terms of energy, entropy, and temperature [Stanley, 1971, Kadanoff, 2000]. The most iconic example is the Ising model, originally designed for ferromagnetism but now widely applied to social dynamics and neural networks. The system is described by a Hamiltonian H, which sums the interactions between adjacent spins σ_i, σ_j:

$$H(\sigma) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i \tag{1}$$

where J represents the interaction strength (coupling) and h is an external field.

These models capture the fundamental mechanism of phase transitions: how local interactions (the J term) compete with thermal fluctuations (entropy) to produce collective order (magnetization) at critical points [Binney et al., 1992]. While powerful, they typically assume a fixed, often regular lattice structure, limiting their direct applicability to the complex, adaptive topologies found in biological and social systems [Newman, 2011].

2.2.2 Network Models

Networks have been used since the early 20th century to model complex systems, representing entities as nodes and interactions as edges [Watts and Strogatz, 1998, Barabási and Albert, 1999]. Formally, a network is represented as a graph $G = (V, E)$ where V is a set of vertices (nodes) and $E \subseteq V \times V$ is a set of edges (links). For directed networks, edges are ordered pairs $(u, v) \in E$ indicating a directed relationship from node u to node v. For undirected networks, edges are unordered pairs $\{u, v\} \in E$.

The structure of a network can be represented by its adjacency matrix A, where:

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \text{ (or } \{i,j\} \in E \text{ for undirected graphs)} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

For weighted networks, A_{ij} represents the strength of the connection between nodes i and j. Several metrics characterize network properties:

- **Degree distribution** $P(k)$: The probability that a randomly selected node has k connections
- **Clustering coefficient** C_i: For a node i with k_i neighbors, $C_i = \frac{2e_i}{k_i(k_i-1)}$ where e_i is the number of links between the neighbors
- **Path length** $d(i,j)$: The minimum number of edges traversed to reach node j from node i
- **Betweenness centrality** $B(v)$: $B(v) = \sum_{s\neq v\neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$ where σ_{st} is the number of shortest paths from s to t and $\sigma_{st}(v)$ is the number of those paths passing through v

These properties can be used to classify networks into different categories, such as scale-free, small-world, and random networks [Barabási and Albert, 1999, Watts and Strogatz, 1998].

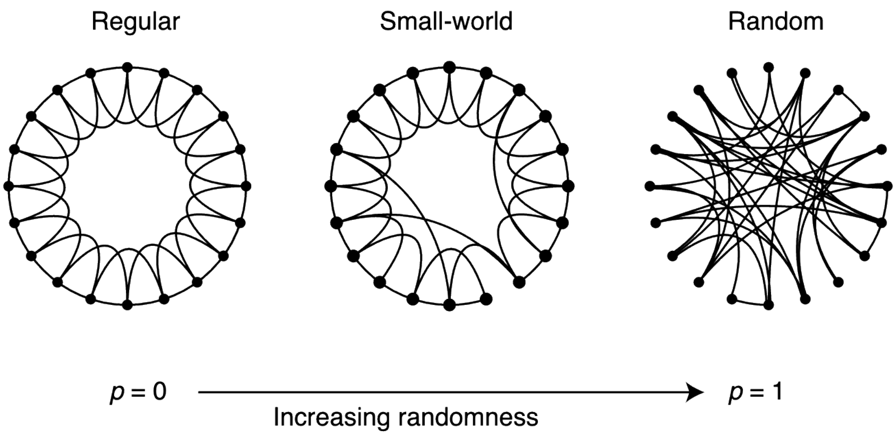


Figure 1: Small-world network model illustration showing the transition from regular to random networks. [Watts and Strogatz, 1998]. A regular network transitions to a small-world network by rewiring a fraction of the edges, leading to a significant reduction in the average path length while maintaining high clustering.

Critical phenomena in networks, such as phase transitions, often manifest through sudden changes in global network properties. For instance, the emergence of a giant connected component in random networks occurs at a critical probability $p_c = \frac{1}{N}$, where N is the number of nodes [Erdős and Rényi, 1960].

Network models have been successful in capturing the structure and dynamics of a wide range of systems, including social networks, biological networks, and technological networks [Newman, 2003, Albert and Barabási, 2002, Strogatz, 2001]. Networks are both a mathematically rigorous framework as well as intuitive and visually appealing, making them a popular choice for modeling complex systems [Newman, 2010]. Networks can capture the emergence of collective behavior through the study of network motifs, community structure, and dynamical processes on networks [Milo et al., 2002, Fortunato, 2010, Barrat et al., 2008], and has attracted significant attention in recent years [Barabási, 2016].

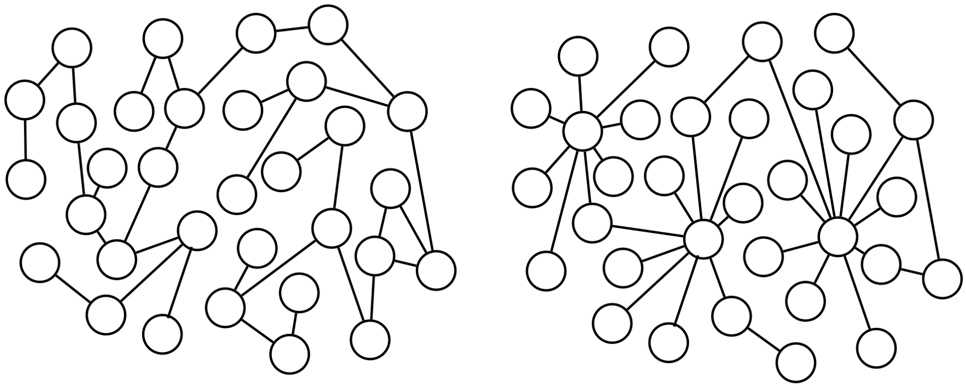


Figure 2: Visual comparison between random and scale-free networks. [Wikipedia contributors, 2023]. Notice the presence of hubs in the scale-free network, which are absent in the random network.

Network Type	**Degree Distribution**	**Clustering Coefficient**	**Average Path Length**
Random Networks	Poisson distribution $P(k) \sim \frac{\lambda^k e^{-\lambda}}{k!}$	Low ($C \sim \frac{p}{N}$)	Short ($L \sim \frac{\ln N}{\ln\langle k\rangle}$)
Regular Lattices	Constant degree	High (locally clustered)	Long ($L \sim N^{1/d}$)
Small-World Networks	Similar to random networks	High ($C \gg C_{random}$)	Short ($L \approx L_{random}$)
Scale-Free Networks	Power law $P(k) \sim k^{-\gamma}$	Hierarchical clustering	Very short (ultra-small world)
Hierarchical Networks	Power law	Hierarchical ($C(k) \sim k^{-1}$)	Short
Modular Networks	Varies	High within modules, low between modules	Long between modules, short within modules

Table 1: Comparison of Different Network Types and Their Characteristics. **Notation:**
$P(k)$ = probability that a randomly selected node has k connections (degree distribution)
k = node degree (number of connections)
$\langle k\rangle$ = average degree across the network
C = clustering coefficient (probability that two neighbors of a node are connected)
$C(k)$ = clustering coefficient for nodes with degree k
L = average shortest path length between any two nodes
N = total number of nodes in the network
p = probability of connection between any two nodes (in random networks)
d = dimension of the lattice (for regular networks)
γ = power-law exponent (typically $2 < \gamma < 3$ for scale-free networks)

2.2.3 From Networks to Simplicial Complexes

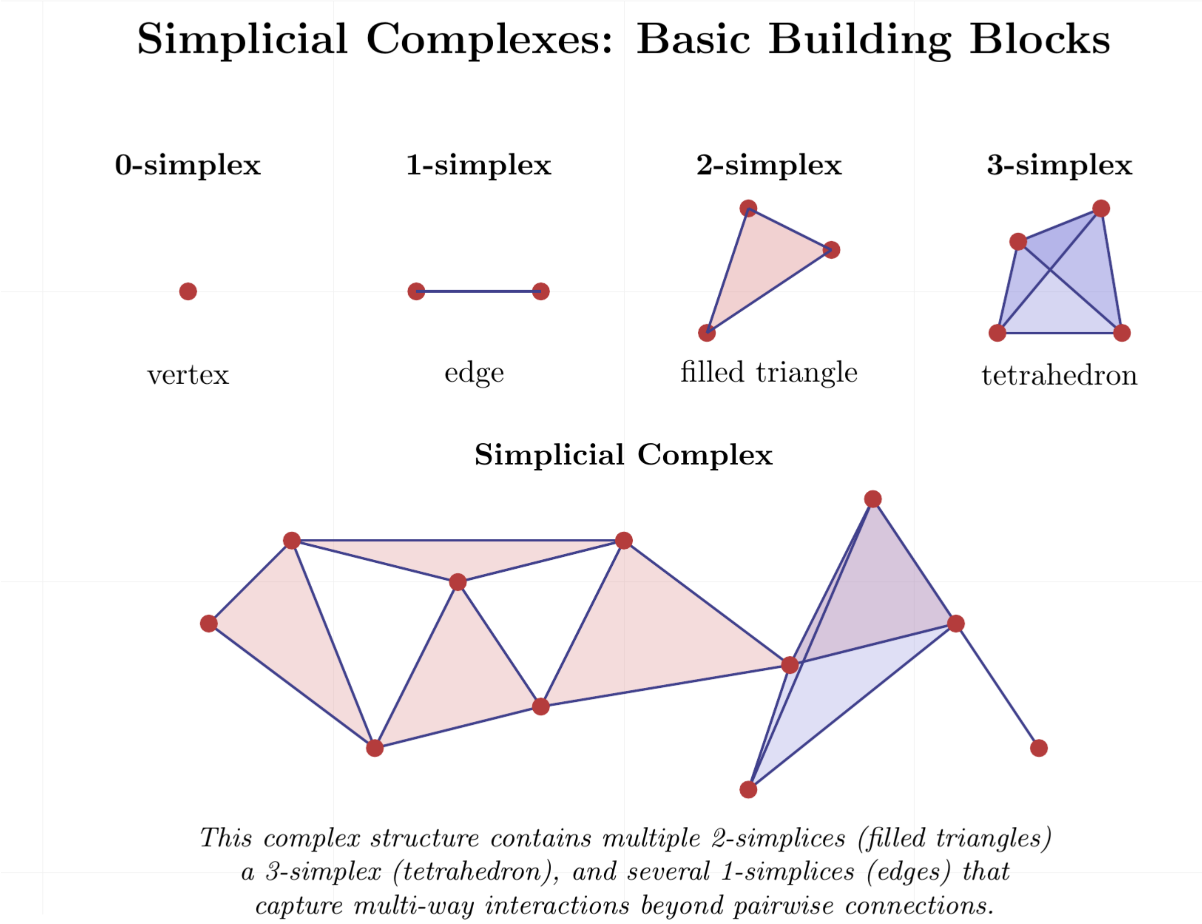


Figure 3: Visual representation of simplicial complexes. The top row shows individual simplices of different dimensions (0-simplex, 1-simplex, 2-simplex, and 3-simplex). The bottom part shows a more complex simplicial complex with multiple 2-simplices (filled triangles), a 3-simplex (tetrahedron), and connecting 1-simplices (edges) that capture multi-way interactions beyond pairwise connections.

Simplicial complexes generalize networks by incorporating higher-order interactions. Simplicial complexes can be thought of as triangles of various dimensions - vertices (0-simplices), edges (1-simplices), triangles (2-simplices), tetrahedra (3-simplices), and so on [Petri et al., 2014] connected together either via shared vertices, edges, or faces.

Formally, a simplicial complex K on a vertex set V is a collection of subsets of V (called simplices) such that:

- For every vertex $v \in V$, $\{v\} \in K$ (0-simplex)
- If $\sigma \in K$ and $\tau \subset \sigma$, then $\tau \in K$ (closure property)

A k-simplex $\sigma = [v_0, v_1, ..., v_k]$ represents an interaction between $k+1$ vertices. For example:

- A 0-simplex is a vertex
- A 1-simplex is an edge (pairwise interaction)

- A 2-simplex is a filled triangle (three-way interaction)
- A 3-simplex is a solid tetrahedron (four-way interaction)
- and so on

Several researchers have successfully applied simplicial complexes to model complex systems. Petri et al. [2014] used simplicial complexes to analyze brain functional networks, revealing topological structures that correlate with cognitive states. Giusti et al. [2016] demonstrated how simplicial complexes can capture neural coding schemes beyond what traditional network models could represent. Sizemore et al. [2018] showed how clique topology in neural systems provides insights into brain development and function.

The primary tool for analyzing these structures is homology, which characterizes the "holes" in the data at different dimensions. These are quantified by Betti numbers (β_k):

- β_0: Number of connected components
- β_1: Number of 1D holes (loops/cycles)
- β_2: Number of 2D holes (voids)

Techniques like Persistent Homology track how these topological features appear and disappear across different scales, offering a multi-scale view of the system's structure [Edelsbrunner and Harer, 2008].

While simplicial complexes offer significant advantages over traditional networks, they have inherent limitations:

- They are undirected, with no natural way to represent asymmetric interactions
- Temporal dynamics are challenging to model in simplicial complexes

3 Theory of Operads

3.1 Operads

A large class of mathematical theories consists of three ingredients:

1. A collection of objects.
2. A collection of morphisms between these objects.
3. A notion of composition of these morphisms.

The most well-known example of this pattern is arithmetic, where the objects are numbers, the morphisms are functions (addition, multiplication, etc.), and the composition is the usual function composition. All fields such as real numbers, complex numbers, and vector spaces can be described in this way.

As we go up the hierarchy of mathematics, we find more and more examples of this pattern. For example, in topology, the objects are topological spaces, the morphisms are continuous functions, and the composition is the usual function composition. Groups and family (magma, monoid, group, ring, field) are also examples of this pattern. Category theory is a generalization of this pattern, where the objects are categories, the morphisms are functors, and the composition is the usual functor composition.

Operads are a generalization of this pattern, where the objects are operations, the morphisms are operations of different arities, and the composition is a more general form of function composition. Operads provide a framework for studying algebraic structures that arise in various areas of mathematics, including topology, algebra, and category theory.

Operads consists of:

1. A collection of operations of different arities.
2. A notion of composition of these operations.
3. The composition operations obey certain conditions - associativity and unitality.

3.1.1 Formal Definition

Consider a set $\mathbb{X}$, and an integer $n \in \mathbb{N}$.

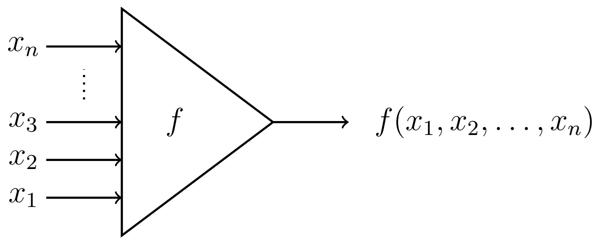
An Operad, $\mathbb{P}$, is defined as a set of n-ary operations, where each operation f has the signature $\mathbb{X}^n \to \mathbb{X}$:

$$\mathbb{P}(n) = \{f : \mathbb{X}^n \to \mathbb{X}\} \tag{3}$$

where $\mathbb{X}^n$ is the cartesian product of $\mathbb{X}$ with itself n times, i.e.

$$\mathbb{X}^n = \mathbb{X} \times \mathbb{X} \times \ldots \times \mathbb{X} \tag{4}$$

i.e. all of these functions f take in n arguments from $\mathbb{X}$ and return a single element from $\mathbb{X}$.



If we have a bunch of these sets of functions $\mathbb{P}(k_i)$ for each $k_i \in \mathbb{N}$, then we can define a composition operation $\circ$ for these operations as follows:

Let $f_i \in \mathbb{P}(k_i)$ be an operation that takes in k_i arguments from $\mathbb{X}$ and returns a single element from $\mathbb{X}$. We can take n numbers of such operations and use their outputs as inputs to another operation $f \in \mathbb{P}(n)$, which takes in n arguments from $\mathbb{X}$ and returns a single element from $\mathbb{X}$. The composition operation $\circ$ is defined as:

$$\mathbb{P}(n) \times (\mathbb{P}(k_1) \times \mathbb{P}(k_2) \times \ldots \times \mathbb{P}(k_n)) \to \mathbb{P}(k_1 + k_2 + \ldots + k_n) \tag{5}$$

$$f, (f_1, f_2, \ldots, f_n) \mapsto f \circ (f_1, f_2, \ldots, f_n) \tag{6}$$

where $f \circ (f_1, f_2, \ldots, f_n) \in \mathbb{P}(k_1 + k_2 + \ldots + k_n)$ is defined as the following diagram:

Associativity of this composition for Operads works as follows:

This composition operation $\circ$ satisfies the following properties:

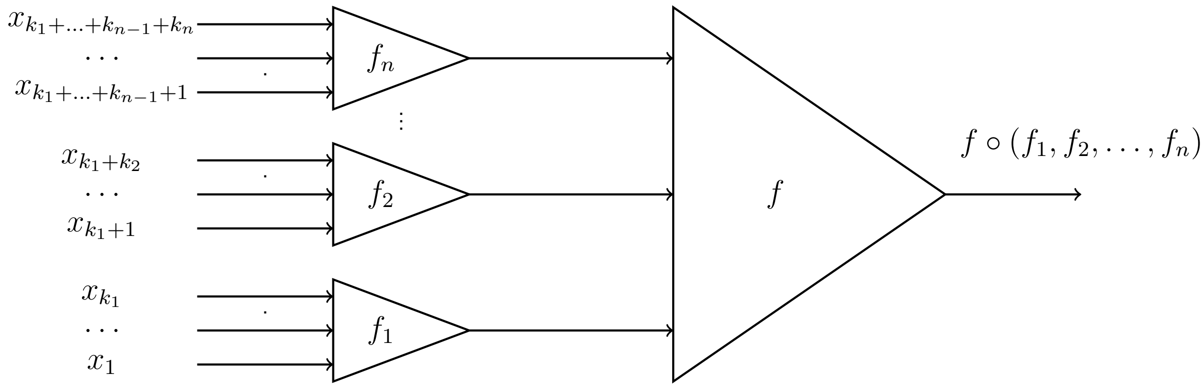


Figure 4: Operadic composition showing how multiple operations $f_1, f_2, \ldots, f_n$ with arities $k_1, k_2, \ldots, k_n$ can be composed with an operation f of arity n to form a new operation of arity $k_1 + k_2 + \ldots + k_n$.

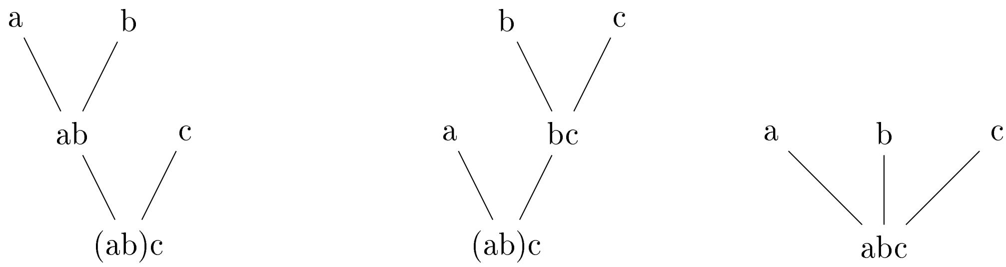


Figure 5: Associativity of operadic composition of arity 3

- **Associativity**: For all $f \in \mathbb{P}(n)$, $g \in \mathbb{P}(k_1)$, $h \in \mathbb{P}(k_2)$, and $i \in \mathbb{P}(k_3)$, we have:

$$f \circ (g \circ (h, i)) = (f \circ (g, h)) \circ i \tag{7}$$

- **Unitality**: For all $f \in \mathbb{P}(n)$, we have:

$$f \circ (\mathrm{id}_{k_1}, \mathrm{id}_{k_2}, \ldots, \mathrm{id}_{k_n}) = f \tag{8}$$

  where id_k is the identity function on $\mathbb{X}^k$.

Symmetry is not required for operads, but it can be added to form symmetric operads. The symmetry condition is:

$$f \circ (g_1, g_2, \ldots, g_n) = f \circ (g_{\sigma(1)}, g_{\sigma(2)}, \ldots, g_{\sigma(n)}) \tag{9}$$

where σ is a permutation of the set $\{1, 2, \ldots, n\}$ or $\sigma \in S_n$ and $g_{\sigma(i)}$ is the $\sigma(i)$-th element of the original sequence, i.e., the permutation σ permutes the order of operations used as inputs to f.

3.1.2 Operads for Modeling

Operads have been applied to modeling systems from diverse fields. In physics, operads have been extensively applied to model quantum field theories, where they capture the

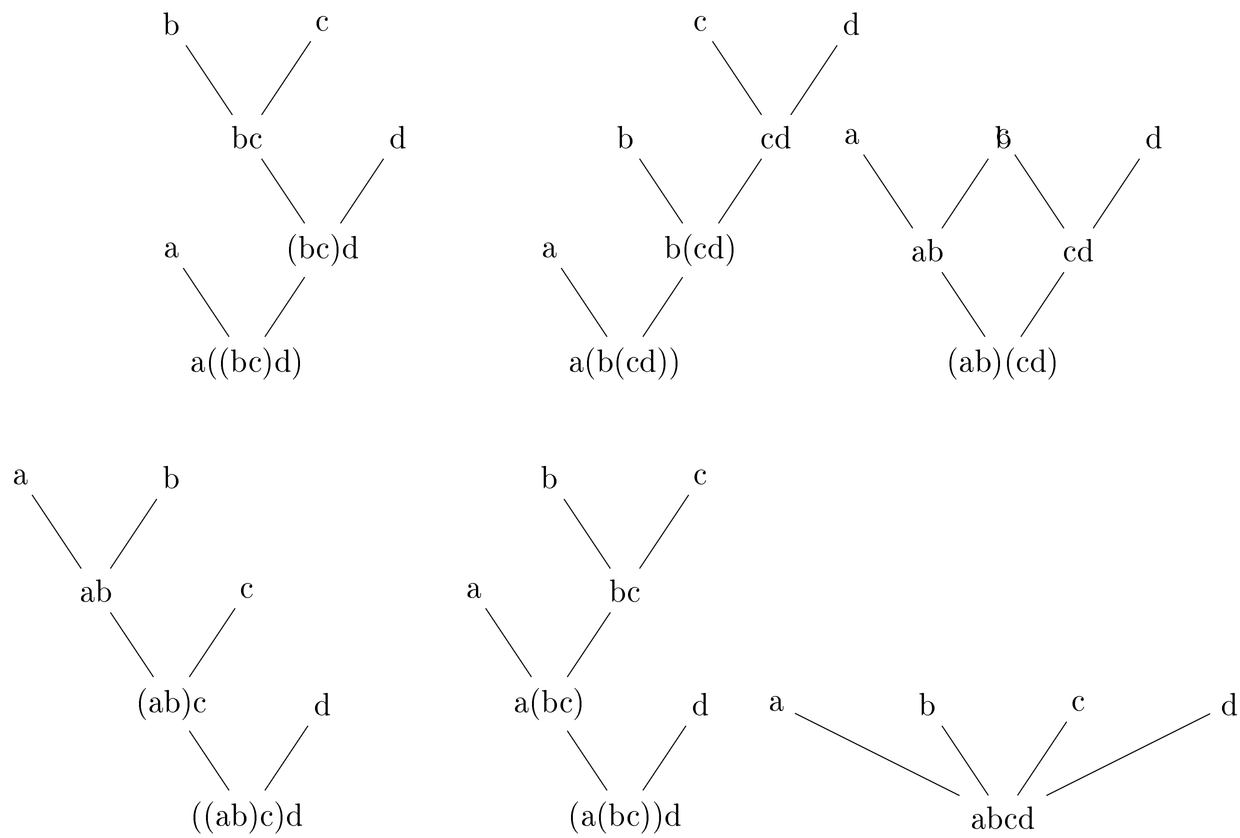


Figure 6: Associativity of operadic composition of arity 4

structure of Feynman diagrams and the composition of quantum interactions Baez and Dolan [1997]. Operads of wiring diagrams have been used to model electrical circuits petri nets and quantum circuits Spivak [2013], Baez and Pollard [2020]. Operads have also been employed in computer science to model SQL database query languages Spivak [2013], in systems design for modeling complex system design specification, analysis and synthesis Behr et al. [2021].

3.2 Operads of Wiring Diagrams

Wiring diagram operads provide a categorical framework for modeling directed compositional systems with explicit input-output interfaces Spivak [2013], Behr et al. [2021]. Unlike classical operads that focus purely on arity, wiring diagram operads encode both the connectivity structure and the directional flow of information through systems, making them particularly suited for modeling complex systems with hierarchical organization and modular decomposition.

3.2.1 Formal Definition

A wiring diagram operad $\mathcal{W}$ consists of graphical representations where operations are depicted as boxes with labeled input and output ports, connected by wires that carry typed information. Formally, we define:

Wiring Diagrams: A wiring diagram W over a finite set of types T is a directed graph where:

- Vertices represent operations (boxes) with labeled input ports $\mathrm{in}(v) \subseteq T$ and output ports $\mathrm{out}(v) \subseteq T$
- Edges represent wires connecting output ports to input ports
- External inputs and outputs form the interface of the diagram

Operations: For a wiring diagram W with input interface $I \subseteq T$ and output interface $O \subseteq T$, we denote the set of operations as $\mathcal{W}(I;O)$. Each operation $f \in \mathcal{W}(I;O)$ represents a morphism:

$$f : \prod_{i \in I} X_i \to \prod_{o \in O} X_o \tag{10}$$

where X_t denotes the data type associated with type $t \in T$.

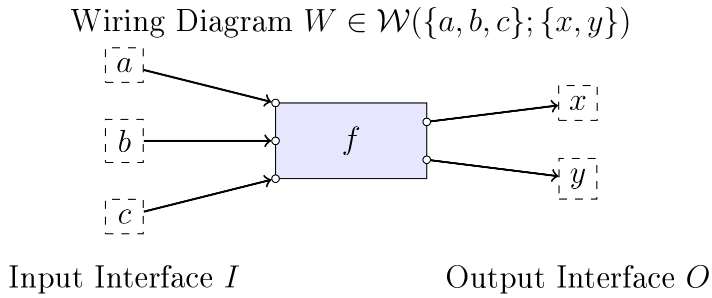


Figure 7: Basic wiring diagram showing an operation f with input interface $\{a, b, c\}$ and output interface $\{x, y\}$. Boxes represent operations, circles represent ports, and arrows represent typed wires.

3.2.2 Composition Structure

The composition operation in wiring diagram operads is defined through substitution and wire connecting. Given:

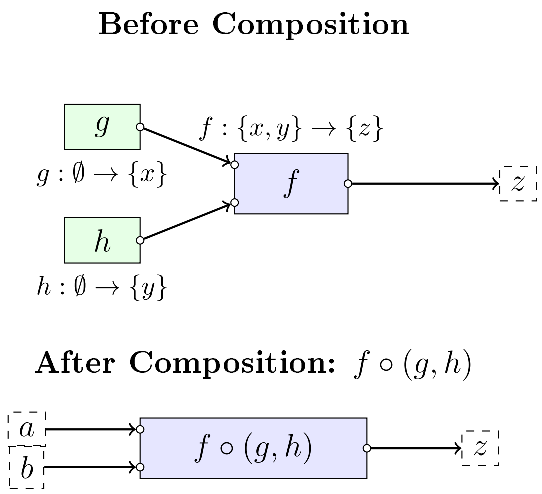
- A wiring diagram $f \in \mathcal{W}(I;O)$ with input interface I and output interface O
- Wiring diagrams $g_1 \in \mathcal{W}(I_1;O_1), g_2 \in \mathcal{W}(I_2;O_2), \ldots, g_k \in \mathcal{W}(I_k;O_k)$

The composition $f \circ (g_1, g_2, \ldots, g_k)$ is performed by:

1. **Interface Matching:** Ensuring output interfaces of g_i match corresponding input requirements in f
2. **Diagram Substitution:** Replacing designated boxes in f with the complete wiring diagrams g_i
3. **Wire Connection:** Connecting output wires of g_i to input wires of the corresponding positions in f

The resulting composition has input interface $I' = \bigcup_{i=1}^{k} I_i$ and output interface O:

$$f \circ (g_1, g_2, \ldots, g_k) \in \mathcal{W}\left(\bigcup_{i=1}^{k} I_i; O\right) \tag{11}$$



$$f \circ (g, h) \in \mathcal{W}(\{a, b\}; \{z\})$$

Figure 8: Composition of wiring diagrams showing how operations g and h are substituted into operation f. The top diagram shows the individual components, while the bottom shows the resulting composed operation.

3.2.3 Associativity and Unitality

Wiring diagram operads satisfy the fundamental operadic axioms:

Associativity: For compatible wiring diagrams, the composition operation is associative:

$$(f \circ g) \circ h = f \circ (g \circ h) \tag{12}$$

This corresponds to the fact that the order of substituting sub-diagrams does not affect the final connectivity structure.

Unitality: Identity wiring diagrams act as units under composition. For each type $t \in T$, there exists an identity operation $\mathrm{id}_t \in \mathcal{W}(\{t\}; \{t\})$ that simply connects its input directly to its output:

$$f \circ (\mathrm{id}_{t_1}, \mathrm{id}_{t_2}, \ldots, \mathrm{id}_{t_n}) = f \tag{13}$$

3.2.4 Categorical Properties

Wiring diagram operads form a symmetric monoidal category where:

- Objects are finite sets of types (interfaces)
- Morphisms are wiring diagrams between interfaces
- Composition is given by diagram substitution
- The monoidal product corresponds to parallel composition of diagrams
- Symmetry is given by wire permutation

This categorical structure enables the modeling of complex systems with multiple subsystems operating in parallel, hierarchical decomposition through nested composition, and modular design patterns where components can be independently developed and later integrated.

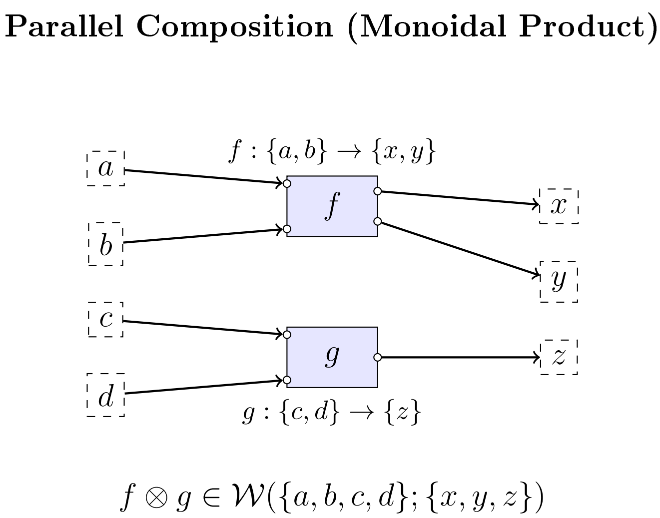


Figure 9: Parallel composition in wiring diagram operads demonstrating the monoidal product structure. Two operations f and g operate independently in parallel, with their interfaces combined disjointly.

4 T-operads for modeling Complex Systems

We develop a novel mathematical framework to model complex systems that exhibit both structural connectivity and internal statistical dynamics. Our approach extends the traditional operadic framework of wiring diagrams by enriching it with network structures, specifically Bayesian networks, to capture the probabilistic relationships inherent in such systems.

4.1 Network-Enriched Wiring Diagram Operads

To model complex systems that exhibit both structural connectivity and internal statistical dynamics, we cannot rely on standard wiring diagram operads alone. We require a framework that enriches the topological structure of wiring diagrams with an additional layer of information — specifically, a network structure that can encode causal and probabilistic dependencies. We propose a tiered construction: first defining a general network-enriched operad, and then instantiating it with Bayesian networks to capture statistical mechanics. A thing to take specific note of is that we posit that modeling causality is all we need to model complex adaptive systems, without the need to explicity model time.

4.1.1 The Network-Enriched WD-Operad

Let $\mathcal{W}$ be the operad of wiring diagrams, where objects are interfaces (finite sets of typed ports) and morphisms are wiring diagrams. We define an **Network-Enriched WD-Operad**, denoted by $\mathcal{E}$, as an extension of $\mathcal{S}$.

[Network-Enriched WD-Operad] The *Network-Enriched WD-Operad* $\mathcal{S}_{Net}$ is defined as follows:

- **Objects:** The objects of $\mathcal{S}_{Net}$ are the same as those of $\mathcal{S}$, i.e., interfaces consisting of finite sets of typed ports.

- **Morphisms:** For interfaces X and Y, a morphism in $\mathcal{S}_{Net}(X, Y)$ is a pair (D, G), where $D \in \mathcal{S}(X, Y)$ is a wiring diagram and G is a graph structure (the "network") whose nodes correspond to the ports of D.
- **Composition:** The composition of morphisms in $\mathcal{S}_{Net}$ is defined by combining the wiring substitution of $\mathcal{S}$ with the gluing of the corresponding graph structures.
- **Symmetry:** The symmetric group actions on the input interfaces extend naturally to the network structures, ensuring that permutations of inputs correspond to relabelings of the nodes in the graph.

A morphism in $\mathcal{S}(X, Y)$ is a pair (D, G), where:

- $D \in \mathcal{S}(X, Y)$ is a standard wiring diagram defining the physical or logical connectivity.
- $G = (V, E)$ is a graph structure (the "network") whose nodes V are mapped to the ports of D.

The composition law in $\mathcal{S}_{Net}$ must respect both the wiring substitution and the graph structure. If we compose $(D_{out}, G_{out}) \circ ((D_{in_1}, G_{in_1}), \ldots, (D_{in_k}, G_{in_k}))$, the resulting graph G_{new} is formed by the union of the graphs, identifying nodes that correspond to connected ports (the "gluing" step).

4.1.2 Compatibility Conditions (Network Enrichment)

For the enrichment to be meaningful, the graph G must be compatible with the wiring diagram D. We enforce the following *compatibility conditions*:

1. **Node Mapping**: There exists a bijection between the set of vertices $V(G)$ and the set of ports in D.
2. **Edge Consistency**: If there is a wire in D connecting port u to port v, the graph G must allow for a relationship between the corresponding nodes (e.g., a directed edge $u \to v$ if G is directed).
3. **Acyclicity**: Since wiring diagrams represent feed-forward composition, the union of graph structures during composition must not introduce cycles if the underlying system is to remain causal.

4.1.3 Symmetry

The symmetry actions in $\mathcal{S}_{Net}$ extend those in $\mathcal{S}$. A permutation $\sigma \in \Sigma_k$ acting on the input interfaces of a morphism (D, G) induces a relabeling of the corresponding nodes in G. The enriched operad must satisfy the condition that the statistical or causal semantics encoded by G remain invariant under such permutations, ensuring that the operadic structure respects the symmetry of input ordering.

4.1.4 Algebra over Network-Enriched Operads

To give concrete meaning to our Statistical Wiring Diagrams, we must define an *algebra* over the $\mathcal{SWD}$ operad. In operad theory, an algebra is a map that assigns semantic objects to the abstract types and concrete operations to the abstract morphisms, preserving the composition structure.

Formally, an algebra $\mathcal{A}$ over the operad $\mathcal{SWD}$ is a strict symmetric monoidal functor from the category underlying $\mathcal{SWD}$ to a target symmetric monoidal category of semantics, typically the category of Markov kernels, denoted **Mark**.

- **Objects:** For every type t in our interface set, the algebra assigns a measurable space $\mathcal{A}(t) = (X_t, \Sigma_t)$. A composite interface $I = \{t_1, \ldots, t_k\}$ is mapped to the product space $\mathcal{A}(I) = \prod_i X_{t_i}$.
- **Morphisms:** For every morphism $F = (D, \mathcal{B})$ in $\mathcal{SWD}(I, O)$, the algebra assigns a Markov kernel:

  $$\mathcal{A}(F) : \mathcal{A}(I) \to \mathcal{A}(O)$$

  This kernel $K(x, dy)$ represents the conditional probability distribution $P(\text{outputs} \mid \text{inputs})$ encoded by the Bayesian network $\mathcal{B}$.

The defining feature of this algebra is the preservation of composition. If $F = G \circ (H_1, \ldots, H_k)$ is a composite morphism in $\mathcal{SWD}$, then the corresponding kernel must be the composition of the constituent kernels:

$$\mathcal{A}(F) = \mathcal{A}(G) \circ (\mathcal{A}(H_1) \otimes \cdots \otimes \mathcal{A}(H_k))$$

In the category **Mark**, this composition corresponds to the Chapman-Kolmogorov equation (integration over intermediate states). Thus, the abstract "gluing" of Bayesian networks in the operad perfectly mirrors the chaining of conditional probabilities in the semantic domain. This guarantees that our topological construction of complex systems yields valid, computable statistical models.

4.2 Bayesian Networks as a Candidate Algebra

Having defined the general structure of $\mathcal{S}_{Net}$, we now select a specific candidate for the network layer to model statistical mechanics: **Bayesian Networks**.

We define the *Statistical Wiring Diagram (SWD) Operad* as the specific instance of $\mathcal{S}_{Net}$ where the enriching graphs are Bayesian Networks equipped with conditional probability distributions.

4.2.1 Formal Derivation

A morphism in the SWD operad is a pair $(D, \mathcal{B})$, derived as follows:

1. **Topological Base**: We start with the wiring diagram D.
2. **Probabilistic Enrichment**: The graph G becomes the Directed Acyclic Graph (DAG) of a Bayesian Network $\mathcal{B}$.

- **Nodes**: Each port p in D is associated with a random variable X_p.
- **Edges**: Directed edges in $\mathcal{B}$ represent conditional dependencies.
- **Parameters**: Each node X_p is equipped with a Conditional Probability Distribution (CPD) $P(X_p|\text{Pa}(X_p))$, where $\text{Pa}(X_p)$ are the parents of X_p in the graph.

4.2.2 Composition of Statistical Morphisms

The composition of two statistical morphisms $(D_2, \mathcal{B}_2) \circ (D_1, \mathcal{B}_1)$ follows the logic of the enriched operad but adds the probabilistic algebra:

- **Graph Gluing**: We form the union of the DAGs. If an output port of D_1 connects to an input port of D_2, the corresponding random variables are identified ($X_{out} \equiv X_{in}$), fusing the networks.
- **Factorization**: The joint probability distribution of the composed system is the product of the individual factors. This preserves the Markov property: the global system's probability distribution factorizes over the structure of the composed wiring diagram.

This construction proves that SWDs are not an ad-hoc definition but a rigorous algebra over the category of network-enriched wiring diagrams.

4.2.3 Symmetry and Exchangeability

A critical feature of operads is their symmetry—the ability to permute the order of inputs without changing the operation's outcome (up to isomorphism). For $\mathcal{SWD}$, this imposes a strict consistency condition on the statistical structure.

Let $\sigma \in \Sigma_k$ be a permutation of the k input interfaces of a morphism F. The action of σ on the wiring diagram D_F is topological: it physically swaps the input bundles. For the associated Bayesian network $\mathcal{B}_F$, this action corresponds to a relabeling of the random variables associated with the input ports.

The **Symmetry Condition** for SWD operads requires that the probabilistic semantics are invariant under this relabeling. Formally, if $\mathbf{X}_{in} = (X_1, \ldots, X_k)$ are the input variable sets, then for any permutation σ:

$$P_F(\text{outputs} \mid X_{\sigma(1)}, \ldots, X_{\sigma(k)}) \cong P_F(\text{outputs} \mid X_1, \ldots, X_k)$$

where $\cong$ denotes equality of distributions under the coordinate transformation induced by σ. This ensures that the statistical mechanics of the system depends only on the causal connectivity (the wiring), not on the arbitrary labeling of the input ports. In statistical terms, this is a form of *exchangeability* conditional on the wiring structure.

4.3 Modeling Near-decomposability

Modeling near-decomposability requires us to model:

- **Subsystems**: The system can be decomposed into subsystems that interact with each other, and such interactions are specific to the subsystems.

- **Interactions**: The interactions within subsystems are stronger than those between subsystems.
- **Time-scale separation**: Different levels of the hierarchical structure operate at different time scales.

4.3.1 Modeling Subsystems

A subsystem is a part of a system that can be studied independently, but also interacts with other subsystems. For example:

- In a biological system, a cell can be considered a subsystem that interacts with other cells to form tissues.
- In a social system, an individual can be considered a subsystem that interacts with other individuals to form groups or communities.

Each individual or cell has its own internal dynamics, but also interacts with other individuals or cells in a way that is specific to the subsystem. This means that the interactions within a subsystem are stronger than those between subsystems. For example:

- In a social system, an individual may have strong ties to their family or close friends, but weaker ties to acquaintances or strangers.
- In a biological system, a cell may have strong interactions with other cells in its tissue, but weaker interactions with cells in other tissues.

The members of a subsystem are often more similar to each other than to members of other subsystems, which can lead to the emergence of new properties at different levels of the hierarchy. For example:

- In a social system, members of a community may share similar beliefs or behaviors that differ from those of other communities.
- In a biological system, cells in a tissue may have similar functions and properties that differ from those of cells in other tissues.

If we were to distill the features of a subsystem into features of a σ-operad, we would have the following mapping:

- **Operations as Subsystems:** Each subsystem corresponds to an operation (a box) in the wiring diagram. Its input and output ports define its interaction interface.
- **Morphisms as Interactions:** The interactions between components within or between subsystems are represented by the morphisms (wires and internal operations) of the σ-operad.
- **Composition as Hierarchy and Modularity:** The operadic composition (substitution of operations) naturally models hierarchical organization and modularity, allowing smaller σ-operads to combine into larger ones.
- **Internal Dynamics as Statistical State:** The specific internal dynamics of a subsystem are captured by the stochastic state space attached to its operation, enabling the modeling of probabilistic behaviors and fluctuations.

5 Results

6 Discussion

7 Conclusion

References

Réka Albert and Albert-László Barabási. Statistical mechanics of complex networks. *Reviews of Modern Physics*, 74(1):47, 2002.

Philip W Anderson. More is different. *Science*, 177(4047):393–396, 1972.

John C Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *Journal of Mathematical Physics*, 36(11):6073–6105, 1997.

John C Baez and Blake S Pollard. Network models from petri nets with catalysts. *Compositionality*, 2:4, 2020.

Per Bak, Chao Tang, and Kurt Wiesenfeld. Self-organized criticality: An explanation of the 1/f noise. *Physical Review Letters*, 59(4):381, 1987.

Yaneer Bar-Yam. Computability and emergence in self-organizing systems. *Complexity*, 18(5):10–15, 2013.

Albert-László Barabási. *Network science*. Cambridge University Press, 2016.

Albert-László Barabási and Réka Albert. Emergence of scaling in random networks. *Science*, 286(5439):509–512, 1999.

Alain Barrat, Marc Barthelemy, and Alessandro Vespignani. *Dynamical processes on complex networks*. Cambridge University Press, 2008.

Federico Battiston, Giulia Cencetti, Iacopo Iacopini, Vito Latora, Maxime Lucas, Alice Patania, Jean-Gabriel Young, and Giovanni Petri. Networks beyond pairwise interactions: structure and dynamics. *Physics Reports*, 874:1–92, 2020.

Nicolas Behr, Jules Hedges, and Pawel Sobocinski. Operad of wiring diagrams: Compositional techniques for system design. In *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–13, 2021.

James J Binney, Nigel J Dowrick, Andrew J Fisher, and Mark Newman. *The theory of critical phenomena: an introduction to the renormalization group*. Oxford University Press, 1992.

Scott Camazine, Jean-Louis Deneubourg, Nigel R Franks, James Sneyd, Guy Theraula, and Eric Bonabeau. *Self-organization in biological systems*. Princeton University Press, 2003.

Herbert Edelsbrunner and John Harer. Persistent homology—a survey. *Contemporary mathematics*, 453:257–282, 2008.

Paul Erdős and Alfréd Rényi. On the evolution of random graphs. *Publications of the Mathematical Institute of the Hungarian Academy of Sciences*, 5:17–61, 1960.

Santo Fortunato. Community detection in graphs. *Physics Reports*, 486(3-5):75–174, 2010.

Matteo Giovannoni, Andreas Klein, Moritz Kreienkamp, and Gunter M Schütz. Dynamic density functional theory with hydrodynamic interactions and fluctuations. *The Journal of chemical physics*, 146(22):224102, 2017.

Chad Giusti, Robert Ghrist, and Danielle S Bassett. Two's company, three (or more) is a simplex. *Journal of computational neuroscience*, 41:1–14, 2016.

Nigel Goldenfeld. *Lectures on phase transitions and the renormalization group*. CRC Press, 1992.

Hermann Haken. *Synergetics: An introduction. Nonequilibrium phase transitions and self-organization in physics, chemistry and biology*. Springer, 1983.

John H Holland. *Emergence: From chaos to order*. Perseus Books, 1998.

Leo P Kadanoff. *Statistical physics: statics, dynamics and renormalization*. World Scientific, 2000.

Javier Linde-Domingo, Eugenia Cheng, Inga Kurthen, and Marc W Howard. An operad for brain hierarchy: topological simplification of neural activity via sequence learning. *arXiv preprint arXiv:2110.02673*, 2021.

Ron Milo, Shai Shen-Orr, Shalev Itzkovitz, Nadav Kashtan, Dmitri Chklovskii, and Uri Alon. Network motifs: simple building blocks of complex networks. *Science*, 298(5594): 824–827, 2002.

Melanie Mitchell. *Complexity: A guided tour*. Oxford University Press, 2009.

Mark Newman. *Networks: an introduction*. Oxford University Press, 2010.

Mark EJ Newman. The structure and function of complex networks. *SIAM review*, 45 (2):167–256, 2003.

Mark EJ Newman. Structure and function of complex networks. *SIAM Review*, 45(2): 167–256, 2011.

Giovanni Petri, Paul Expert, Federico Turkheimer, Robin Carhart-Harris, David Nutt, Peter J Hellyer, and Francesco Vaccarino. Homological scaffolds of brain functional networks. *Journal of The Royal Society Interface*, 11(101):20140873, 2014.

Herbert A Simon. The architecture of complexity. *Proceedings of the American Philosophical Society*, 106(6):467–482, 1962.

Herbert A Simon. *The sciences of the artificial*. MIT press, Cambridge, MA, 3 edition, 1996.

Ann E Sizemore, Jennifer E Phillips-Cremins, Robert Ghrist, and Danielle S Bassett. The importance of the whole: topological data analysis for the network neuroscientist. *Network Neuroscience*, 3(3):656–673, 2018.

David I Spivak. The operad of wiring diagrams: formalizing a graphical language for databases, recursion, and plug-and-play circuits. *arXiv preprint arXiv:1305.0297*, 2013.

H Eugene Stanley. *Phase transitions and critical phenomena*. Clarendon Press, Oxford, 1971.

H Eugene Stanley. Scaling, universality, and renormalization: Three pillars of modern critical phenomena. *Reviews of Modern Physics*, 71(2):S358, 1999.

Steven H Strogatz. Exploring complex networks. *Nature*, 410(6825):268–276, 2001.

Duncan J Watts and Steven H Strogatz. Collective dynamics of 'small-world' networks. *Nature*, 393(6684):440–442, 1998.

Geoffrey B West. *Scale: The universal laws of growth, innovation, sustainability, and the pace of life in organisms, cities, economies, and companies*. Penguin, New York, 2017.

Wikipedia contributors. Scale-free network. Wikipedia, The Free Encyclopedia, 2023. URL https://en.wikipedia.org/wiki/Scale-free_network. Accessed: [insert access date here].