Modeling Complex Systems Using Operads

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Abstract

This paper presents an advanced algebraic approach to modeling phase transitions and emergent phenomena in complex systems using operads, a mathematical structure that extends beyond simplicial complexes. While simplicial complexes have provided valuable insights into higher-order interactions, they are limited by their rigid simplicial structure. We demonstrate how operads, with their flexible compositional structure and ability to represent directed higher-order interactions, can capture more nuanced aspects of complex systems dynamics. Our framework progresses from traditional networks to simplicial complexes and finally to operads, providing a unified perspective on phase transitions and emergence as compositional transformations. Using computational methods from higher category theory and algebraic topology, we analyze real-world systems and synthetic models to identify universal patterns in phase transitions that are inaccessible to simplicial methods. This approach offers profound new insights into the mechanisms underlying emergent behaviors in diverse complex systems.

1 Introduction

Complex systems are characterized by numerous interacting components that exhibit emergent behaviors and phase transitions not evident from the study of individual components [Mitchell, 2009]. Phase transitions in complex systems are marked by the emergence of sudden, qualitative changes where new collective properties arise from interconnected components. Understanding these transitions requires mathematical frameworks that can capture the multi-scale, hierarchical nature of complex systems.

1.1 Properties of Complex Systems

Complex systems exhibit two fundamental structural properties: near-decomposability and scale-invariance. *Near-decomposability*, first identified by Simon [1962], refers to systems that can be broken down into subsystems that interact weakly with each other while maintaining strong internal interactions. This property creates hierarchical organizations where subsystems are nested within larger systems. For instance, cells form tissues, which form organs, which form organisms—each level having its own internal dynamics while participating in broader systemic functions.

This hierarchical organization facilitates the emergence of new properties at different levels of abstraction, as local interactions lead to global phenomena. In biological systems, molecular interactions give rise to cellular behaviors, which in turn produce emergent patterns at the tissue or organism level [Battiston et al., 2020]. Similarly, in social systems, individual behaviors aggregate into collective phenomena such as consensus formation or cultural patterns [Castellano et al., 2009].

Complementing this property, *scale-invariance* describes how patterns exhibited across different scales of hierarchical organization are often self-similar, manifesting in power-law distributions and fractal-like structures [West, 2017]. Scale-invariant systems lack a characteristic scale—similar patterns repeat whether observed at micro or macro levels. Together, these properties create systems where local changes in compositional structure can cascade across scales to produce system-wide transformations—the essence of phase transitions in complex systems.

1.2 Evolution of Mathematical Models

The challenge of modeling complex systems has led to an evolution of increasingly sophisticated mathematical frameworks, each addressing limitations of its predecessors:

- **Network models** capture pairwise interactions between system components, providing a first approximation but missing higher-order effects.
- **Hypergraphs** extend networks to include higher-order interactions, allowing for the representation of multi-agent interactions and enabling the analysis of systems with more intricate connection patterns [Benson et al., 2016, Battiston et al., 2020].
- **Simplicial complexes** further extend these models to include higher-dimensional interactions, capturing simultaneous multi-agent interactions and enabling topological analysis [Battiston et al., 2020, Petri et al., 2014].

1.2.1 Networks

Network models have been widely deployed across diverse domains, from social systems to biological networks to technological infrastructures [Boccaletti et al., 2006, Newman, 2003]. By representing entities as nodes and their interactions as edges, networks reveal system properties such as connectivity patterns, robustness, and information flow [Barabási and Oltvai, 2004]. These models have successfully captured various dynamics, including synchronization in neural networks [Arenas et al., 2008], opinion formation in social networks [Castellano et al., 2009], cascading failures in infrastructure [Watts, 2002], percolation phenomena [Cohen and Havlin, 2010], epidemic spreading [Pastor-Satorras et al., 2015], and power grid failures [Dobson et al., 2007].

1.2.2 Hypergraphs

Hypergraphs generalize networks by introducing hyperedges that can connect multiple nodes simultaneously [Berge, 1984, Battiston et al., 2020]. Unlike standard networks where edges connect exactly two nodes, hyperedges map sets of source nodes to sets of target nodes. This structure better represents multi-agent interactions in systems like social networks [Zhou et al., 2007], biochemical pathways [Klamt et al., 2009], and technological infrastructures [Xu et al., 2013]. Hypergraphs have successfully modeled complex phenomena including social consensus formation across multiple influence layers [Neuhäuser et al., 2021], protein complex assembly and disassembly [Ramadan et al., 2020], financial contagion with directional interactions [Hüser and Kok, 2020], and coordinated neural firing patterns [Petri et al., 2014].

1.2.3 Simplicial Complexes

Simplicial complexes offer an even richer generalization by representing multi-dimensional interactions as geometric objects [Petri et al., 2014, Giusti et al., 2016, Sizemore et al., 2018]. A simplicial complex consists of simplices of varying dimensions—points (0-simplices), edges (1-simplices), filled triangles (2-simplices), solid tetrahedra (3-simplices), and higher-dimensional analogues—that capture interactions between corresponding numbers of nodes. This representation enables sophisticated topological analysis of complex systems, revealing structural features not visible through network or hypergraph representations. Researchers have successfully applied simplicial complexes to analyze brain functional networks [Petri et al., 2014], neural coding schemes [Giusti et al., 2016], and brain development [Sizemore et al., 2018].

1.3 Limitations

Despite their increasing sophistication, these models still struggle to adequately represent systems with complex hierarchical organization and nested compositional structures. Specifically, they face limitations when modeling:

- **Social consensus formation**, where opinions cascade through multiple hierarchical levels of influence [Watts and Strogatz, 1998]
- **Biological signaling networks**, where protein complexes dynamically assemble and disassemble to form higher-order functional units [Strogatz, 2001, Giovannoni et al., 2017]
- **Financial contagion**, where institutional interactions have directionality, composition changes, and cross-scale effects [Farmer and Foley, 2009]
- **Neural synchronization**, where firing patterns coordinate across hierarchical scales [Bar-Yam, 2008, Linde-Domingo et al., 2021]

These limitations stem from the inability of existing models to fully capture the compositional nature of complex systems, where higher-level structures emerge from the composition of lower-level components in ways that preserve certain properties while giving rise to new ones.

This paper presents a novel framework that progresses beyond networks and simplicial complexes to operadic models for analyzing phase transitions and emergent phenomena.

2 Background

2.1 Properties of Complex Systems

2.1.1 Structural Features: Near-decomposability and Scale-invariance

Complex systems exhibit two foundational structural properties that enable their characteristic behaviors and provide the basis for our operadic approach.

Near-decomposability [Simon, 1962] describes systems with subsystems that interact weakly externally while maintaining strong internal interactions. These systems display:

- Stronger interactions within subsystems than between them
- Hierarchical organization across multiple levels
- Time-scale separation between internal and external interactions

Examples include cellular structures (organelles within cells, cells within tissues, tissues within organs, organs within organisms, etc.), ecosystems with loosely coupled niches, and modular software systems. Near-decomposability enables adaptation through independent subsystem evolution [Simon, 1996] and creates the hierarchical levels across which system dynamics unfold.

Scale-invariance describes self-similar patterns appearing across different organizational levels, characterized by:

- Statistical similarity at different observation scales
- Power-law distributions of system properties
- Absence of characteristic scales

This property manifests in branching patterns in biological systems, power-law distributions in scale-free networks etc. [West, 2017, Stanley, 1999]. Scale-invariance emerges naturally in systems that grow through preferential attachment processes or self-organize into critical states.

2.1.2 Dynamical Features: Phase Transitions and Emergence

Phase transitions and emergence represent related dynamical phenomena that arise from the hierarchical and scale-invariant structures of complex systems.

Phase transitions occur when control parameters cross threshold values, causing qualitative changes in system properties. These transitions manifest as discontinuous shifts at critical points [Stanley, 1971], displaying characteristic mathematical signatures: power-law scaling behaviors, critical exponents, and diverging correlation lengths [Newman, 2003, Bak et al., 1987]. In complex systems, phase transitions include ecosystem state shifts, opinion cascades in social networks, synchronization transitions in oscillator systems, financial market crashes, and percolation thresholds in networks.

Emergence describes the formation of properties at higher organizational levels not present in or predictable from individual components [Holland, 1998, Anderson, 1972].

Anderson's "More is Different" principle emphasizes that complex systems require analysis beyond reductionist approaches. Examples include flocking behaviors in birds, intelligence in neural networks, consciousness from neuronal activity, and market dynamics in economies [Camazine et al., 2003, Haken, 1983].

Phase transitions can be viewed as a specific type of emergence characterized by sudden, discontinuous changes, while other emergent phenomena may develop gradually or exist in steady states [Bar-Yam, 2013]. Both phenomena share common underlying mechanisms:

- They arise from the hierarchical structures created by near-decomposability
- They propagate across scales following patterns enabled by scale-invariance
- They involve reorganization of compositional relationships between system components
- They manifest when local changes cascade to produce system-wide transformations

At critical points, fluctuations occur across all scales of the system as local changes propagate through hierarchical levels—a direct consequence of the near-decomposable, scale-invariant structure.

Our operadic framework models these phenomena by tracking compositional relationships between components across scales. This approach complements traditional statistical methods [Stanley, 1999, Goldenfeld, 1992] by focusing explicitly on how compositional structures reorganize during transitions, providing insights into the mechanisms driving phase transitions and emergent behaviors.

2.2 Modeling Phase Transitions and Emergence

2.2.1 Physical Models

In physics, phase transitions are often modeled using statistical mechanics, where the system's behavior is described in terms of energy, entropy, and temperature [Stanley, 1971, Kadanoff, 2000]. The Ising model, Potts model, and percolation theory are classic examples of physical models used to study phase transitions [Onsager, 1944, Stauffer and Aharony, 2018]. These models capture the interactions between individual components and the emergence of collective behavior at critical points [Binney et al., 1992]. They, however, have limitations in capturing the complexity of real-world systems, such as biological, social, and technological networks but tend to be more mathematically tractable for detailed analysis [Newman, 2011].

2.2.2 Network Models

Networks have been used since the early 20th century to model complex systems, representing entities as nodes and interactions as edges [Watts and Strogatz, 1998, Barabási and Albert, 1999]. Formally, a network is represented as a graph $G = (V, E)$ where V is a set of vertices (nodes) and $E \subseteq V \times V$ is a set of edges (links). For directed networks, edges are ordered pairs $(u, v) \in E$ indicating a directed relationship from node u to node v. For undirected networks, edges are unordered pairs $\{u, v\} \in E$.

The structure of a network can be represented by its adjacency matrix A, where:

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \text{ (or } \{i,j\} \in E \text{ for undirected graphs)} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For weighted networks, A_{ij} represents the strength of the connection between nodes i and j. Several metrics characterize network properties:

- **Degree distribution** $P(k)$: The probability that a randomly selected node has k connections
- **Clustering coefficient** C_i: For a node i with k_i neighbors, $C_i = \frac{2e_i}{k_i(k_i-1)}$ where e_i is the number of links between the neighbors
- **Path length** $d(i,j)$: The minimum number of edges traversed to reach node j from node i
- **Betweenness centrality** $B(v)$: $B(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$ where σ_{st} is the number of shortest paths from s to t and $\sigma_{st}(v)$ is the number of those paths passing through v

These properties can be used to classify networks into different categories, such as **scale-free**, **small-world**, and **random networks** [Barabási and Albert, 1999, Watts and Strogatz, 1998].

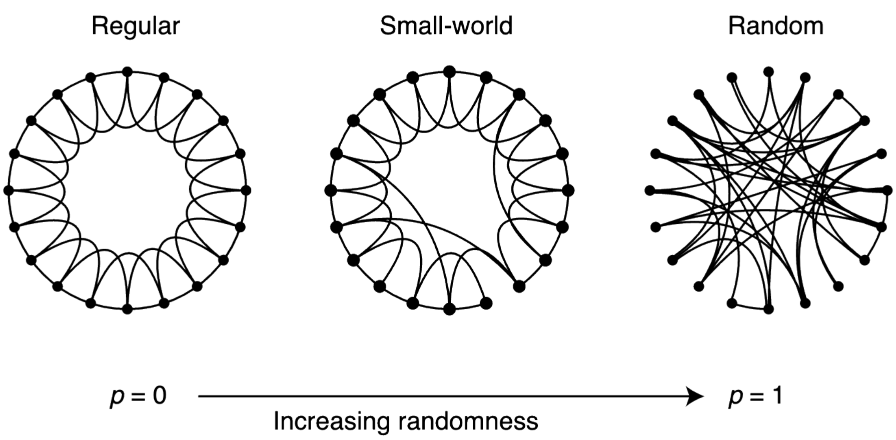


Figure 1: Small-world network model illustration showing the transition from regular to random networks. [Watts and Strogatz, 1998]. A regular network transitions to a small-world network by rewiring a fraction of the edges, leading to a significant reduction in the average path length while maintaining high clustering.

Critical phenomena in networks, such as phase transitions, often manifest through sudden changes in global network properties. For instance, the emergence of a giant connected component in random networks occurs at a critical probability $p_c = \frac{1}{N}$, where N is the number of nodes [Erdős and Rényi, 1960].

Network models have been successful in capturing the structure and dynamics of a wide range of systems, including social networks, biological networks, and technological

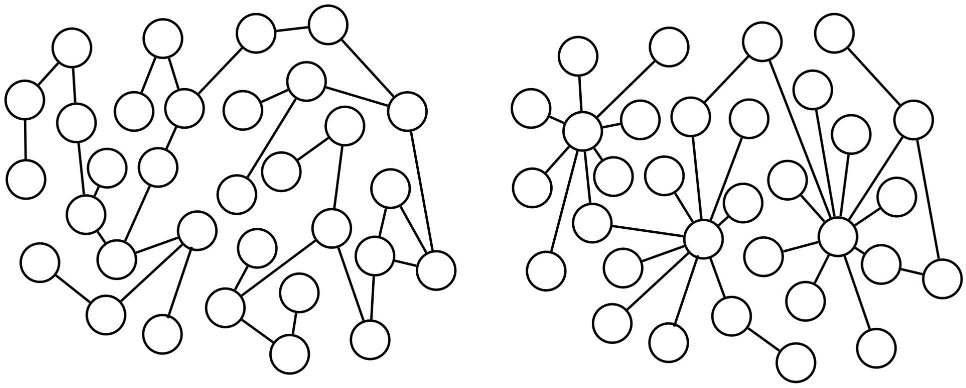


Figure 2: Visual comparison between random and scale-free networks. [Wikipedia contributors, 2023]. Notice the presence of hubs in the scale-free network, which are absent in the random network.

Network Type	**Degree Distribution**	**Clustering Coefficient**	**Average Path Length**
Random Networks	Poisson distribution $P(k) \sim \frac{\lambda^k e^{-\lambda}}{k!}$	Low ($C \sim \frac{p}{N}$)	Short ($L \sim \frac{\ln N}{\ln\langle k\rangle}$)
Regular Lattices	Constant degree	High (locally clustered)	Long ($L \sim N^{1/d}$)
Small-World Networks	Similar to random networks	High ($C \gg C_{random}$)	Short ($L \approx L_{random}$)
Scale-Free Networks	Power law $P(k) \sim k^{-\gamma}$	Hierarchical clustering	Very short (ultra-small world)
Hierarchical Networks	Power law	Hierarchical ($C(k) \sim k^{-1}$)	Short
Modular Networks	Varies	High within modules, low between modules	Long between modules, short within modules

Table 1: Comparison of Different Network Types and Their Characteristics. **Notation:**
$P(k)$ = probability that a randomly selected node has k connections (degree distribution)
k = node degree (number of connections)
$\langle k\rangle$ = average degree across the network
C = clustering coefficient (probability that two neighbors of a node are connected)
$C(k)$ = clustering coefficient for nodes with degree k
L = average shortest path length between any two nodes
N = total number of nodes in the network
p = probability of connection between any two nodes (in random networks)
d = dimension of the lattice (for regular networks)
γ = power-law exponent (typically $2 < \gamma < 3$ for scale-free networks)

networks [Newman, 2003, Albert and Barabási, 2002, Strogatz, 2001]. Networks are both a mathematically rigorous framework as well as intuitive and visually appealing, making them a popular choice for modeling complex systems [Newman, 2010]. Networks can capture the emergence of collective behavior through the study of network motifs, community structure, and dynamical processes on networks [Milo et al., 2002, Fortunato, 2010, Barrat et al., 2008], and has attracted significant attention in recent years [Barabási, 2016].

2.2.3 From Networks to Simplicial Complexes

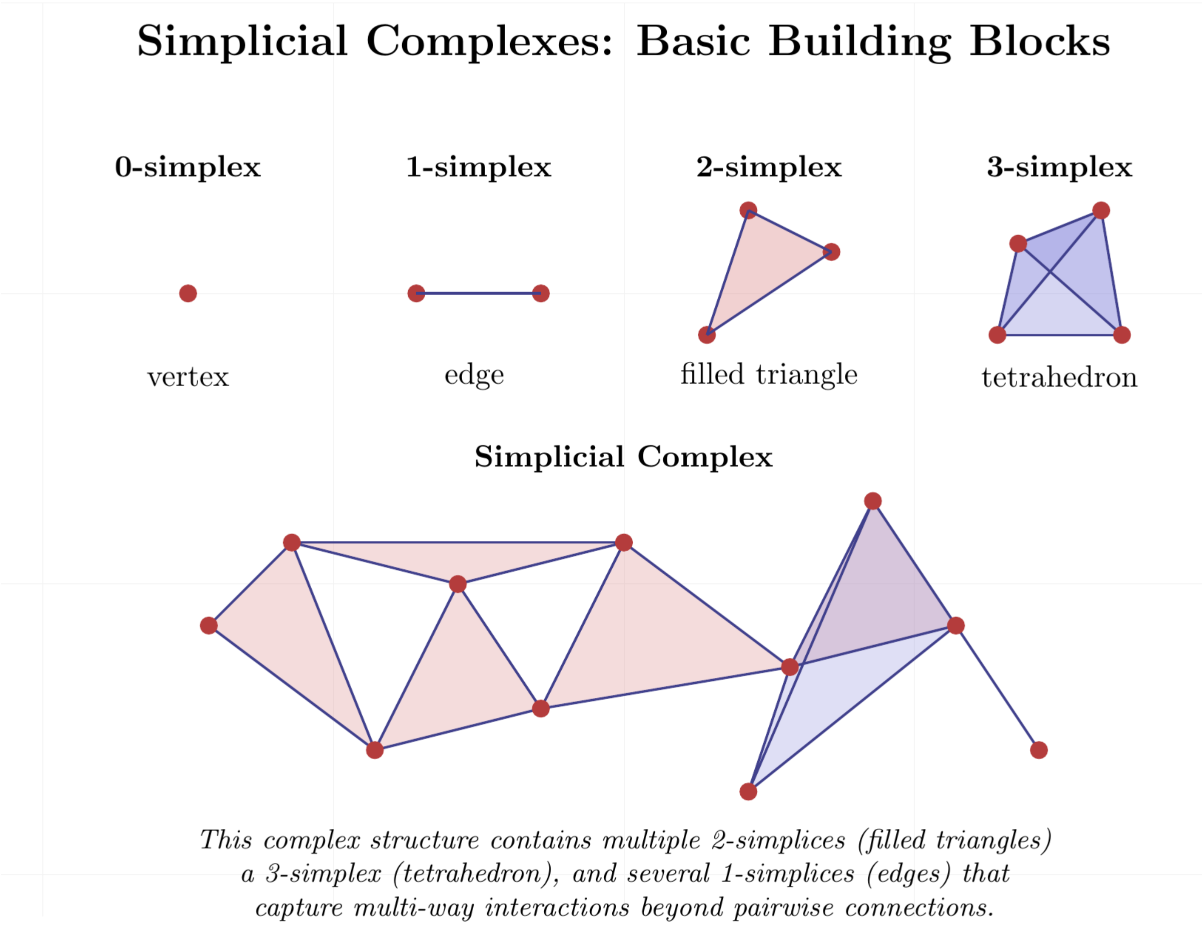


Figure 3: Visual representation of simplicial complexes. The top row shows individual simplices of different dimensions (0-simplex, 1-simplex, 2-simplex, and 3-simplex). The bottom part shows a more complex simplicial complex with multiple 2-simplices (filled triangles), a 3-simplex (tetrahedron), and connecting 1-simplices (edges) that capture multi-way interactions beyond pairwise connections.

Simplicial complexes generalize networks by incorporating higher-order interactions. Simplicial complexes can be thought of as triangles of various dimensions - vertices (0-simplices), edges (1-simplices), triangles (2-simplices), tetrahedra (3-simplices), and so on [Petri et al., 2014] connected together either via shared vertices, edges, or faces.

Formally, a simplicial complex K on a vertex set V is a collection of subsets of V (called simplices) such that:

- For every vertex $v \in V$, $\{v\} \in K$ (0-simplex)

- If $\sigma \in K$ and $\tau \subset \sigma$, then $\tau \in K$ (closure property)

A k-simplex $\sigma = [v_0, v_1, ..., v_k]$ represents an interaction between $k+1$ vertices. For example:

- A 0-simplex is a vertex
- A 1-simplex is an edge (pairwise interaction)
- A 2-simplex is a filled triangle (three-way interaction)
- A 3-simplex is a solid tetrahedron (four-way interaction)
- and so on

Several researchers have successfully applied simplicial complexes to model complex systems. Petri et al. [2014] used simplicial complexes to analyze brain functional networks, revealing topological structures that correlate with cognitive states. Giusti et al. [2016] demonstrated how simplicial complexes can capture neural coding schemes beyond what traditional network models could represent. Sizemore et al. [2018] showed how clique topology in neural systems provides insights into brain development and function.

While simplicial complexes offer significant advantages over traditional networks, they have inherent limitations:

- They are *undirected*, with no natural way to represent asymmetric interactions
- *Temporal dynamics* are challenging to model in simplicial complexes

3 Theory of Operads

3.1 Operads

A large class of mathematical theories consists of three ingredients:

1. A collection of objects.
2. A collection of morphisms between these objects.
3. A notion of composition of these morphisms.

The most well-known example of this pattern is arithmetic, where the objects are numbers, the morphisms are functions (addition, multiplication, etc.), and the composition is the usual function composition. All fields such as real numbers, complex numbers, and vector spaces can be described in this way.

As we go up the hierarchy of mathematics, we find more and more examples of this pattern. For example, in topology, the objects are topological spaces, the morphisms are continuous functions, and the composition is the usual function composition. Groups and family (magma, monoid, group, ring, field) are also examples of this pattern. Category theory is a generalization of this pattern, where the objects are categories, the morphisms are functors, and the composition is the usual functor composition.

Operads are a generalization of this pattern, where the objects are operations, the morphisms are operations of different arities, and the composition is a more general form of function composition. Operads provide a framework for studying algebraic structures that arise in various areas of mathematics, including topology, algebra, and category theory.

Operads consists of:

1. A collection of operations of different arities.
2. A notion of composition of these operations.
3. The composition operations obey certain conditions - associativity and unitality.

3.1.1 Formal Definition

Consider a set $\mathbb{X}$, and an integer $n \in \mathbb{N}$.

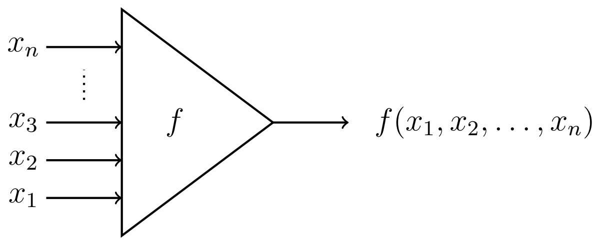
An Operad, $\mathbb{P}$, is defined as a set of n-ary operations, where each operation f has the signature $\mathbb{X}^n \to \mathbb{X}$:

$$\mathbb{P}(n) = \{f : \mathbb{X}^n \to \mathbb{X}\} \tag{2}$$

where $\mathbb{X}^n$ is the cartesian product of $\mathbb{X}$ with itself n times, i.e.

$$\mathbb{X}^n = \mathbb{X} \times \mathbb{X} \times \ldots \times \mathbb{X} \tag{3}$$

i.e. all of these functions f take in n arguments from $\mathbb{X}$ and return a single element from $\mathbb{X}$.



If we have a bunch of these sets of functions $\mathbb{P}(k_i)$ for each $k_i \in \mathbb{N}$, then we can define a composition operation $\circ$ for these operations as follows:

Let $f_i \in \mathbb{P}(k_i)$ be an operation that takes in k_i arguments from $\mathbb{X}$ and returns a single element from $\mathbb{X}$. We can take n numbers of such operations and use their outputs as inputs to another operation $f \in \mathbb{P}(n)$, which takes in n arguments from $\mathbb{X}$ and returns a single element from $\mathbb{X}$. The composition operation $\circ$ is defined as:

$$\mathbb{P}(n) \times (\mathbb{P}(k_1) \times \mathbb{P}(k_2) \times \ldots \times \mathbb{P}(k_n)) \to \mathbb{P}(k_1 + k_2 + \ldots + k_n) \tag{4}$$

$$f, (f_1, f_2, \ldots, f_n) \mapsto f \circ (f_1, f_2, \ldots, f_n) \tag{5}$$

where $f \circ (f_1, f_2, \ldots, f_n) \in \mathbb{P}(k_1 + k_2 + \ldots + k_n)$ is defined as the following diagram:
Associativity of this composition for Operads works as follows:
This composition operation $\circ$ satisfies the following properties:

- **Associativity**: For all $f \in \mathbb{P}(n)$, $g \in \mathbb{P}(k_1)$, $h \in \mathbb{P}(k_2)$, and $i \in \mathbb{P}(k_3)$, we have:

$$f \circ (g \circ (h, i)) = (f \circ (g, h)) \circ i \tag{6}$$

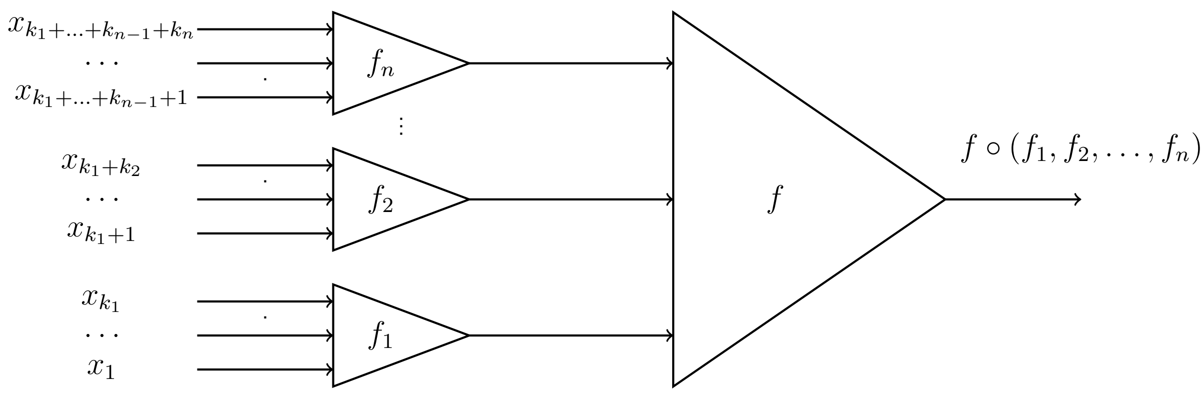


Figure 4: Operadic composition showing how multiple operations $f_1, f_2, \ldots, f_n$ with arities $k_1, k_2, \ldots, k_n$ can be composed with an operation f of arity n to form a new operation of arity $k_1 + k_2 + \ldots + k_n$.

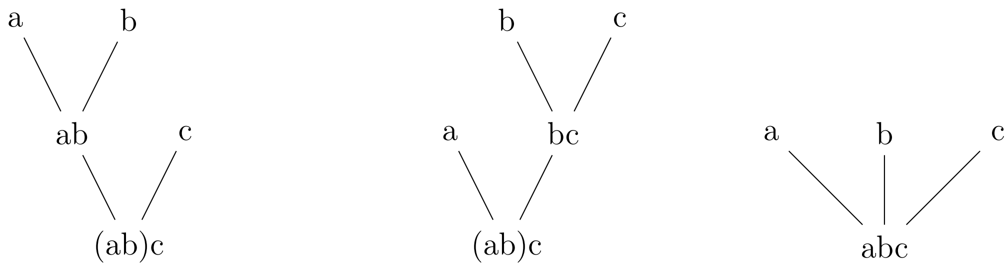


Figure 5: Associativity of operadic composition of arity 3

- **Unitality**: For all $f \in \mathbb{P}(n)$, we have:

$$f \circ (\mathrm{id}_{k_1}, \mathrm{id}_{k_2}, \ldots, \mathrm{id}_{k_n}) = f \tag{7}$$

  where id_k is the identity function on $\mathbb{X}^k$.

Symmetry is not required for operads, but it can be added to form symmetric operads. The symmetry condition is:

$$f \circ (g_1, g_2, \ldots, g_n) = f \circ (g_{\sigma(1)}, g_{\sigma(2)}, \ldots, g_{\sigma(n)}) \tag{8}$$

where σ is a permutation of the set $\{1, 2, \ldots, n\}$ or $\sigma \in S_n$ and $g_{\sigma(i)}$ is the $\sigma(i)$-th element of the original sequence, i.e., the permutation σ permutes the order of operations used as inputs to f.

3.1.2 Operads for Modeling

Operads have been applied to modeling systems from diverse fields. In **physics**, operads have been extensively applied to model quantum field theories, where they capture the structure of Feynman diagrams and the composition of quantum interactions Baez and Dolan [1997]. Operads of wiring diagrams have been used to model electrical circuits petri nets and quantum circuits Spivak [2013], Baez and Pollard [2020]. Operads have also been employed in **computer science** to model SQL database query languages Spivak

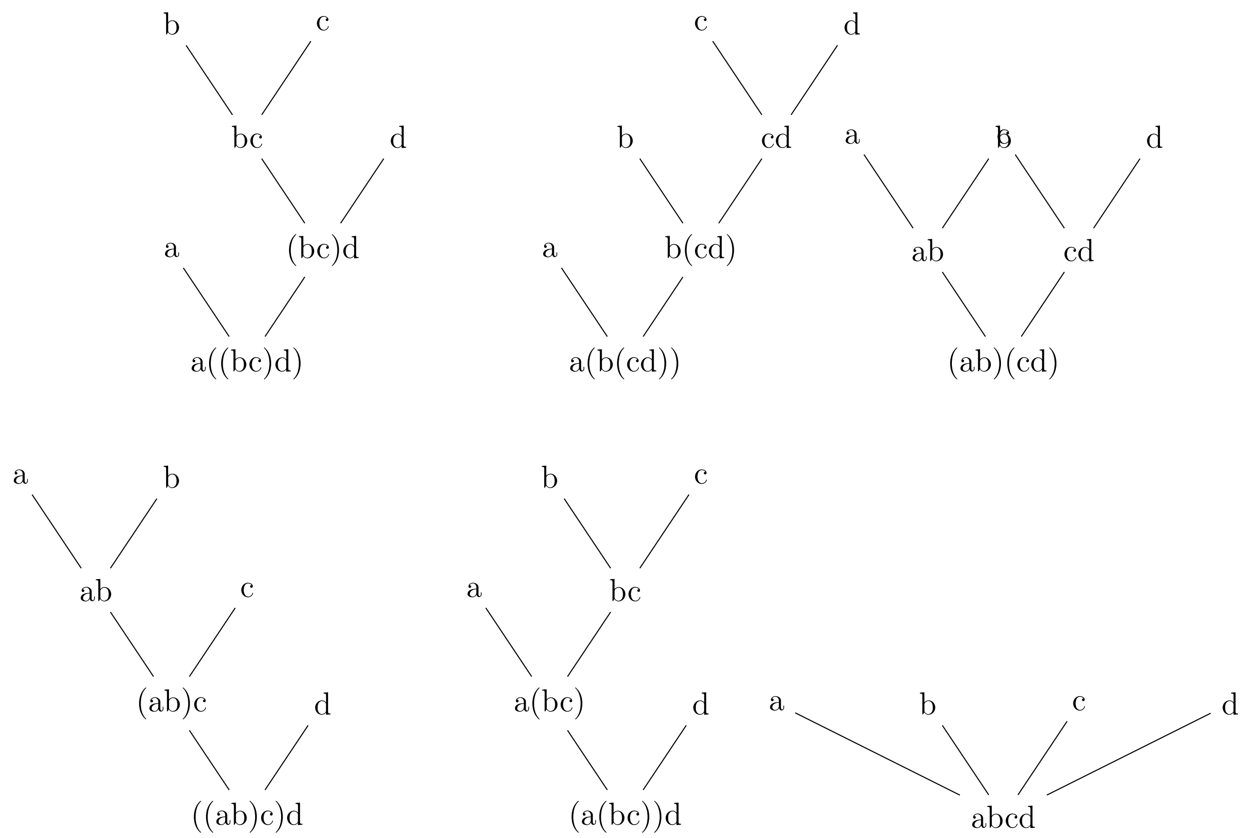


Figure 6: Associativity of operadic composition of arity 4

[2013], in **systems design** for modeling complex system design specification, analysis and synthesis **????**.

3.2 Operads of Wiring Diagrams

Wiring diagram operads provide a categorical framework for modeling directed compositional systems with explicit input-output interfaces Spivak [2013], Behr et al. [2021]. Unlike classical operads that focus purely on arity, wiring diagram operads encode both the connectivity structure and the directional flow of information through systems, making them particularly suited for modeling complex systems with hierarchical organization and modular decomposition.

3.2.1 Formal Definition

A wiring diagram operad $\mathcal{W}$ consists of graphical representations where operations are depicted as boxes with labeled input and output ports, connected by wires that carry typed information. Formally, we define:

Wiring Diagrams: A wiring diagram W over a finite set of types T is a directed graph where:

- Vertices represent operations (boxes) with labeled input ports $\mathrm{in}(v) \subseteq T$ and output ports $\mathrm{out}(v) \subseteq T$

- Edges represent wires connecting output ports to input ports
- External inputs and outputs form the interface of the diagram

Operations: For a wiring diagram W with input interface $I \subseteq T$ and output interface $O \subseteq T$, we denote the set of operations as $\mathcal{W}(I; O)$. Each operation $f \in \mathcal{W}(I; O)$ represents a morphism:

$$f : \prod_{i \in I} X_i \to \prod_{o \in O} X_o \tag{9}$$

where X_t denotes the data type associated with type $t \in T$.

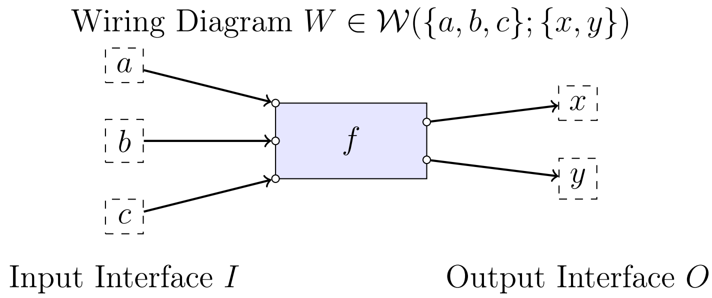


Figure 7: Basic wiring diagram showing an operation f with input interface $\{a, b, c\}$ and output interface $\{x, y\}$. Boxes represent operations, circles represent ports, and arrows represent typed wires.

3.2.2 Composition Structure

The composition operation in wiring diagram operads is defined through **substitution** and **wire connecting**. Given:

- A wiring diagram $f \in \mathcal{W}(I; O)$ with input interface I and output interface O
- Wiring diagrams $g_1 \in \mathcal{W}(I_1; O_1), g_2 \in \mathcal{W}(I_2; O_2), \ldots, g_k \in \mathcal{W}(I_k; O_k)$

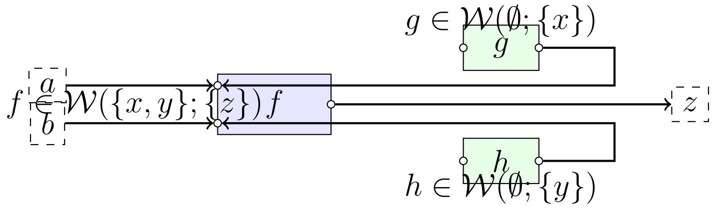
The composition $f \circ (g_1, g_2, \ldots, g_k)$ is performed by:

1. **Interface Matching:** Ensuring output interfaces of g_i match corresponding input requirements in f
2. **Diagram Substitution:** Replacing designated boxes in f with the complete wiring diagrams g_i
3. **Wire Connection:** Connecting output wires of g_i to input wires of the corresponding positions in f

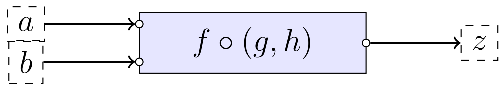
The resulting composition has input interface $I' = \bigcup_{i=1}^{k} I_i$ and output interface O:

$$f \circ (g_1, g_2, \ldots, g_k) \in \mathcal{W}\left(\bigcup_{i=1}^{k} I_i; O\right) \tag{10}$$

Before Composition



After Composition: $f \circ (g, h)$



$$f \circ (g, h) \in \mathcal{W}(\{a, b\}; \{z\})$$

Figure 8: Composition of wiring diagrams showing how operations g and h are substituted into operation f. The top diagram shows the individual components, while the bottom shows the resulting composed operation.

3.2.3 Associativity and Unitality

Wiring diagram operads satisfy the fundamental operadic axioms:

Associativity: For compatible wiring diagrams, the composition operation is associative:

$$(f \circ g) \circ h = f \circ (g \circ h) \tag{11}$$

This corresponds to the fact that the order of substituting sub-diagrams does not affect the final connectivity structure.

Unitality: Identity wiring diagrams act as units under composition. For each type $t \in T$, there exists an identity operation $\mathrm{id}_t \in \mathcal{W}(\{t\}; \{t\})$ that simply connects its input directly to its output:

$$f \circ (\mathrm{id}_{t_1}, \mathrm{id}_{t_2}, \dots, \mathrm{id}_{t_n}) = f \tag{12}$$

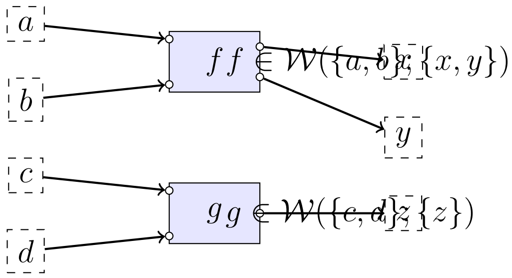
3.2.4 Categorical Properties

Wiring diagram operads form a **symmetric monoidal category** where:

- Objects are finite sets of types (interfaces)
- Morphisms are wiring diagrams between interfaces
- Composition is given by diagram substitution
- The monoidal product corresponds to parallel composition of diagrams
- Symmetry is given by wire permutation

This categorical structure enables the modeling of complex systems with multiple subsystems operating in parallel, hierarchical decomposition through nested composition, and modular design patterns where components can be independently developed and later integrated.

Parallel Composition (Monoidal Product)



$$f \otimes g \in \mathcal{W}(\{a, b, c, d\}; \{x, y, z\})$$

Figure 9: Parallel composition in wiring diagram operads demonstrating the monoidal product structure. Two operations f and g operate independently in parallel, with their interfaces combined disjointly.

4 T-operads for modeling Complex Systems

In order to model complex systems using T-operads, we first look at how to model their structural hallmarks, i.e. near-decomposability including hierarchcal structures and modularity followed by self-similarity and scale invariance of the structures.

4.1 Modeling Near-decomposability

Modeling neardecomposability requires us to model:

- **Subsystems**: The system can be decomposed into subsystems that interact with each other, and such interactions are specific to the subsystems.
- **Interactions**: The interactions within subsystems are stronger than those between subsystems.
- **Time-scale separation**: Different levels of the hierarchical structure operate at different time scales.

4.1.1 Modeling Subsystems

A subsystem is a part of a system that can be studied independently, but also interacts with other subsystems. For example, in a biological system, a cell can be considered a subsystem that interacts with other cells to form tissues. In a social system, an individual can be considered a subsystem that interacts with other individuals to form groups or communities. Each individual or cell has its own internal dynamics, but also interacts with other individuals or cells in a way that is specific to the subsystem. This means that the interactions within a subsystem are stronger than those between subsystems. For example, in a social system, an individual may have strong ties to their family or close friends, but weaker ties to acquaintances or strangers. Similarly, in a biological system, a cell may have strong interactions with other cells in its tissue, but weaker interactions with cells in other tissues. The members of a subsystem are often more similar to each other than to members of other subsystems, which can lead to the emergence of new properties at different levels of the hierarchy. For example, in a biological system, cells in a tissue may have similar functions and properties, while cells in different tissues may have different functions and properties. In a social system, individuals in a community may share similar beliefs or behaviors, while individuals in different communities may have different beliefs or behaviors.

If we were to distill the features of a subsystem into features of a T-operad, we would have the following:

- **Compositional hierarchy**:
  - **Compositional structure**: The subsystem can be represented as a T-operad, where the objects are the members of the subsystem and the morphisms are the interactions between them.
  - **Hierarchical structure**: The T-operad can be decomposed into smaller T-operads, each representing a subsystem. This allows us to model the hierarchical structure of the system.
  - **Modularity**: The T-operad can be composed with other T-operads to form larger T-operads, allowing us to model the modularity of the system.

5 Results

6 Discussion

7 Conclusion

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