

Beyond Simplicial Complexes: Modeling Phase Transitions and Emergence in Complex Systems Using Operads

Russi Chatterjee
ixaxaar@google.com

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Abstract

This paper presents an advanced algebraic approach to modeling phase transitions and emergent phenomena in complex systems using operads, a mathematical structure that extends beyond simplicial complexes. While simplicial complexes have provided valuable insights into higher-order interactions, they are limited by their rigid simplicial structure. We demonstrate how operads, with their flexible compositional structure and ability to represent directed higher-order interactions, can capture more nuanced aspects of complex systems dynamics. Our framework progresses from traditional networks to simplicial complexes and finally to operads, providing a unified perspective on phase transitions and emergence as compositional transformations. Using computational methods from higher category theory and algebraic topology, we analyze real-world systems and synthetic models to identify universal patterns in phase transitions that are inaccessible to simplicial methods. This approach offers profound new insights into the mechanisms underlying emergent behaviors in diverse complex systems.

1 Introduction

COMPLEX SYSTEMS are characterized by numerous interacting components that exhibit emergent behaviors and phase transitions not evident from the study of individual components [Mitchell, 2009]. Phase transitions in complex systems are marked by the emergence of sudden, qualitative changes where new collective properties arise from interconnected components. Understanding these transitions requires mathematical frameworks that can capture the multi-scale, hierarchical nature of complex systems.

1.1 Properties of Complex Systems

Complex systems exhibit two fundamental structural properties: near-decomposability and scale-invariance. *Near-decomposability*, first identified by Simon [1962], refers to systems that can be broken down into subsystems that interact weakly with each other while maintaining strong internal interactions. This property creates hierarchical organizations where subsystems are nested within larger systems. For instance, cells form tissues, which form organs, which form organisms—each level having its own internal dynamics while participating in broader systemic functions.

This hierarchical organization facilitates the emergence of new properties at different levels of abstraction, as local interactions lead to global phenomena. In biological systems, molecular interactions give rise to cellular behaviors, which in turn produce emergent patterns at the tissue or organism level [Battiston et al., 2020]. Similarly, in social systems, individual behaviors aggregate into collective phenomena such as consensus formation or cultural patterns [Castellano et al., 2009].

Complementing this property, *scale-invariance* describes how patterns exhibited across different scales of hierarchical organization are often self-similar, manifesting in power-law distributions and fractal-like structures [West, 2017]. Scale-invariant systems lack a characteristic scale—similar patterns repeat whether observed at micro or macro levels. Together, these properties create systems where local changes in compositional structure can cascade across scales to produce system-wide transformations—the essence of phase transitions in complex systems.

1.2 Evolution of Mathematical Models

The challenge of modeling complex systems has led to an evolution of increasingly sophisticated mathematical frameworks, each addressing limitations of its predecessors:

- **Network models** capture pairwise interactions between system components, providing a first approximation but missing higher-order effects.
- **Hypergraphs** extend networks to include higher-order interactions, allowing for the representation of multi-agent interactions and enabling the analysis of systems with more intricate connection patterns [Benson et al., 2016, Battiston et al., 2020].
- **Simplicial complexes** further extend these models to include higher-dimensional interactions, capturing simultaneous multi-agent interactions and enabling topological analysis [Battiston et al., 2020, Petri et al., 2014].

1.2.1 Networks

Network models have been widely deployed across diverse domains, from social systems to biological networks to technological infrastructures [Boccaletti et al., 2006, Newman, 2003]. By representing entities as nodes and their interactions as edges, networks reveal system properties such as connectivity patterns, robustness, and information flow [Barabási and Oltvai, 2004]. These models have successfully captured various dynamics, including synchronization in neural networks [Arenas et al., 2008], opinion formation in social networks [Castellano et al., 2009], cascading failures in infrastructure [Watts, 2002], percolation phenomena [Cohen and Havlin, 2010], epidemic spreading [Pastor-Satorras et al., 2015], and power grid failures [Dobson et al., 2007].

1.2.2 Hypergraphs

Hypergraphs generalize networks by introducing hyperedges that can connect multiple nodes simultaneously [Berge, 1984, Battiston et al., 2020]. Unlike standard networks where edges connect exactly two nodes, hyperedges map sets of source nodes to sets of target nodes. This structure better represents multi-agent interactions in systems like social networks [Zhou et al., 2007], biochemical pathways [Klamt et al., 2009], and technological infrastructures [Xu et al., 2013]. Hypergraphs have successfully modeled complex phenomena including social consensus formation across multiple influence layers [Neuhäuser et al., 2021], protein complex assembly and disassembly [Ramadan et al., 2020], financial contagion with directional interactions [Hüser and Kok, 2020], and coordinated neural firing patterns [Petri et al., 2014].

1.2.3 Simplicial Complexes

Simplicial complexes offer an even richer generalization by representing multi-dimensional interactions as geometric objects [Petri et al., 2014, Giusti et al., 2016, Sizemore et al., 2018]. A simplicial complex consists of simplices of varying dimensions—points (0-simplices), edges (1-simplices), filled triangles (2-simplices), solid tetrahedra (3-simplices), and higher-dimensional analogues—that capture interactions between corresponding numbers of nodes. This representation enables sophisticated topological analysis of complex systems, revealing structural features not visible through network or hypergraph representations. Researchers have successfully applied simplicial complexes to analyze brain functional networks [Petri et al., 2014], neural coding schemes [Giusti et al., 2016], and brain development [Sizemore et al., 2018].

1.3 Limitations

Despite their increasing sophistication, these models still struggle to adequately represent systems with complex hierarchical organization and nested compositional structures. Specifically, they face limitations when modeling:

- **Social consensus formation**, where opinions cascade through multiple hierarchical levels of influence [Watts and Strogatz, 1998]
- **Biological signaling networks**, where protein complexes dynamically assemble and disassemble to form higher-order functional units [Strogatz, 2001, Giovannoni et al., 2017]
- **Financial contagion**, where institutional interactions have directionality, composition changes, and cross-scale effects [Farmer and Foley, 2009]
- **Neural synchronization**, where firing patterns coordinate across hierarchical scales [Bar-Yam, 2008, Linde-Domingo et al., 2021]

These limitations stem from the inability of existing models to fully capture the compositional nature of complex systems, where higher-level structures emerge from the composition of lower-level components in ways that preserve certain properties while giving rise to new ones.

This paper presents a novel framework that progresses beyond networks and simplicial complexes to operadic models for analyzing phase transitions and emergent phenomena.

2 Background

2.1 Properties of Complex Systems

2.1.1 Structural Features: Near-decomposability and Scale-invariance

Complex systems exhibit two foundational structural properties that enable their characteristic behaviors and provide the basis for our operadic approach.

Near-decomposability [Simon, 1962] describes systems with subsystems that interact weakly externally while maintaining strong internal interactions. These systems display:

- Stronger interactions within subsystems than between them
- Hierarchical organization across multiple levels
- Time-scale separation between internal and external interactions

Examples include cellular structures (organelles within cells, cells within tissues, tissues within organs, organs within organisms, etc.), ecosystems with loosely coupled niches, and modular software systems. Near-decomposability enables adaptation through independent subsystem evolution [Simon, 1996] and creates the hierarchical levels across which system dynamics unfold.

Scale-invariance describes self-similar patterns appearing across different organizational levels, characterized by:

- Statistical similarity at different observation scales
- Power-law distributions of system properties
- Absence of characteristic scales

This property manifests in branching patterns in biological systems, power-law distributions in scale-free networks etc. [West, 2017, Stanley, 1999]. Scale-invariance emerges naturally in systems that grow through preferential attachment processes or self-organize into critical states.

2.1.2 Dynamical Features: Phase Transitions and Emergence

Phase transitions and emergence represent related dynamical phenomena that arise from the hierarchical and scale-invariant structures of complex systems.

Phase transitions occur when control parameters cross threshold values, causing qualitative changes in system properties. These transitions manifest as discontinuous shifts at critical points [Stanley, 1971], displaying characteristic mathematical signatures: power-law scaling behaviors, critical exponents, and diverging correlation lengths [Newman, 2003, Bak et al., 1987]. In complex systems, phase transitions include ecosystem state shifts, opinion cascades in social networks, synchronization transitions in oscillator systems, financial market crashes, and percolation thresholds in networks.

Emergence describes the formation of properties at higher organizational levels not present in or predictable from individual components [Holland, 1998, Anderson, 1972].

Anderson's "More is Different" principle emphasizes that complex systems require analysis beyond reductionist approaches. Examples include flocking behaviors in birds, intelligence in neural networks, consciousness from neuronal activity, and market dynamics in economies [Camazine et al., 2003, Haken, 1983].

Phase transitions can be viewed as a specific type of emergence characterized by sudden, discontinuous changes, while other emergent phenomena may develop gradually or exist in steady states [Bar-Yam, 2013]. Both phenomena share common underlying mechanisms:

- They arise from the hierarchical structures created by near-decomposability
- They propagate across scales following patterns enabled by scale-invariance
- They involve reorganization of compositional relationships between system components
- They manifest when local changes cascade to produce system-wide transformations

At critical points, fluctuations occur across all scales of the system as local changes propagate through hierarchical levels—a direct consequence of the near-decomposable, scale-invariant structure.

Our operadic framework models these phenomena by tracking compositional relationships between components across scales. This approach complements traditional statistical methods [Stanley, 1999, Goldenfeld, 1992] by focusing explicitly on how compositional structures reorganize during transitions, providing insights into the mechanisms driving phase transitions and emergent behaviors.

2.2 Modeling Phase Transitions and Emergence

2.2.1 Physical Models

In physics, phase transitions are often modeled using statistical mechanics, where the system's behavior is described in terms of energy, entropy, and temperature [Stanley, 1971, Kadanoff, 2000]. The Ising model, Potts model, and percolation theory are classic examples of physical models used to study phase transitions [Onsager, 1944, Stauffer and Aharony, 2018]. These models capture the interactions between individual components and the emergence of collective behavior at critical points [Binney et al., 1992]. They, however, have limitations in capturing the complexity of real-world systems, such as biological, social, and technological networks but tend to be more mathematically tractable for detailed analysis [Newman, 2011].

2.2.2 Network Models

Networks have been used since the early 20th century to model complex systems, representing entities as nodes and interactions as edges [Watts and Strogatz, 1998, Barabási and Albert, 1999]. Formally, a network is represented as a graph $G = (V, E)$ where V is a set of vertices (nodes) and $E \subseteq V \times V$ is a set of edges (links). For directed networks, edges are ordered pairs $(u, v) \in E$ indicating a directed relationship from node u to node v . For undirected networks, edges are unordered pairs $\{u, v\} \in E$.

The structure of a network can be represented by its adjacency matrix A , where:

$$A_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \text{ (or } \{i, j\} \in E \text{ for undirected graphs)} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

For weighted networks, A_{ij} represents the strength of the connection between nodes i and j . Several metrics characterize network properties:

- **Degree distribution** $P(k)$: The probability that a randomly selected node has k connections
- **Clustering coefficient** C_i : For a node i with k_i neighbors, $C_i = \frac{2e_i}{k_i(k_i-1)}$ where e_i is the number of links between the neighbors
- **Path length** $d(i, j)$: The minimum number of edges traversed to reach node j from node i
- **Betweenness centrality** $B(v)$: $B(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$ where σ_{st} is the number of shortest paths from s to t and $\sigma_{st}(v)$ is the number of those paths passing through v

These properties can be used to classify networks into different categories, such as **scale-free**, **small-world**, and **random networks** [Barabási and Albert, 1999, Watts and Strogatz, 1998].

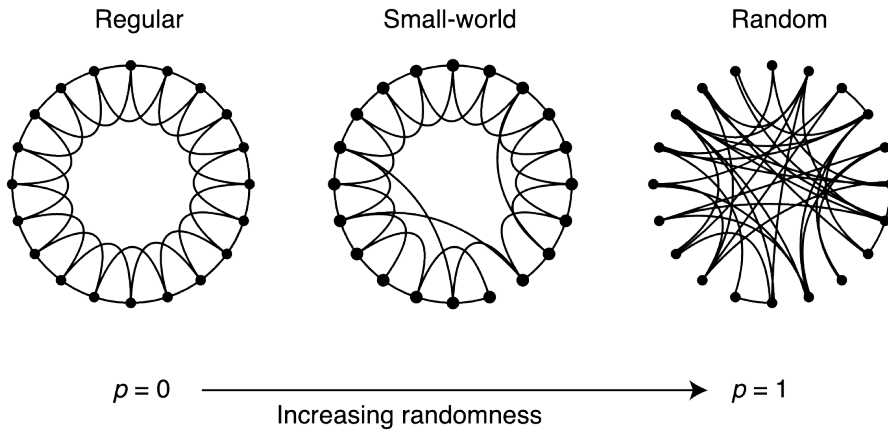


Figure 1: Small-world network model illustration showing the transition from regular to random networks. [Watts and Strogatz, 1998]. A regular network transitions to a small-world network by rewiring a fraction of the edges, leading to a significant reduction in the average path length while maintaining high clustering.

Critical phenomena in networks, such as phase transitions, often manifest through sudden changes in global network properties. For instance, the emergence of a giant connected component in random networks occurs at a critical probability $p_c = \frac{1}{N}$, where N is the number of nodes [Erdős and Rényi, 1960].

Network models have been successful in capturing the structure and dynamics of a wide range of systems, including social networks, biological networks, and technological

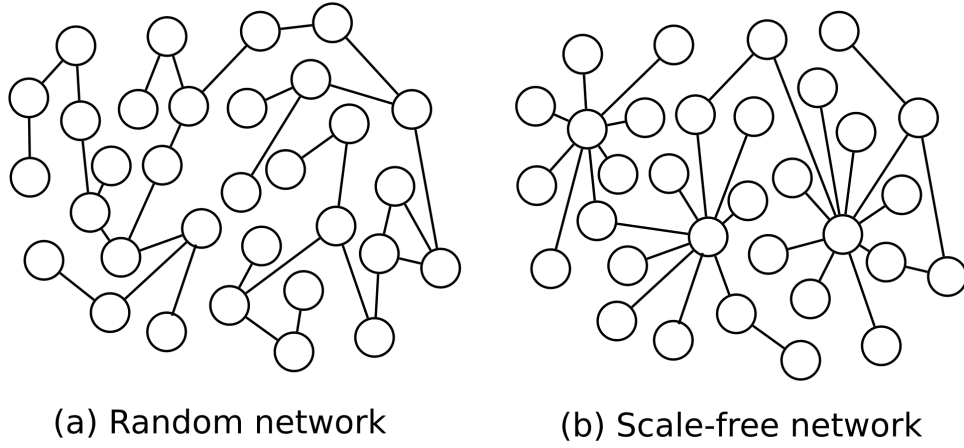


Figure 2: Visual comparison between random and scale-free networks. [Wikipedia contributors, 2023]. Notice the presence of hubs in the scale-free network, which are absent in the random network.

Network Type	Degree Distribution	Clustering Coefficient	Average Path Length
Random Networks	Poisson distribution $P(k) \sim \frac{\lambda^k e^{-\lambda}}{k!}$	Low ($C \sim \frac{p}{N}$)	Short ($L \sim \frac{\ln N}{\ln \langle k \rangle}$)
Regular Lattices	Constant degree	High (locally clustered)	Long ($L \sim N^{1/d}$)
Small-World Networks	Similar to random networks	High ($C \gg C_{random}$)	Short ($L \approx L_{random}$)
Scale-Free Networks	Power law $P(k) \sim k^{-\gamma}$	Hierarchical clustering	Very short (ultra-small world)
Hierarchical Networks	Power law	Hierarchical ($C(k) \sim k^{-1}$)	Short
Modular Networks	Varies	High within modules, low between modules	Long between modules, short within modules

Table 1: Comparison of Different Network Types and Their Characteristics. **Notation:**

$P(k)$ = probability that a randomly selected node has k connections (degree distribution)

k = node degree (number of connections)

$\langle k \rangle$ = average degree across the network

C = clustering coefficient (probability that two neighbors of a node are connected)

$C(k)$ = clustering coefficient for nodes with degree k

L = average shortest path length between any two nodes

N = total number of nodes in the network

p = probability of connection between any two nodes (in random networks)

d = dimension of the lattice (for regular networks)

γ = power-law exponent (typically $2 < \gamma < 3$ for scale-free networks)

networks [Newman, 2003, Albert and Barabási, 2002, Strogatz, 2001]. Networks are both a mathematically rigorous framework as well as intuitive and visually appealing, making them a popular choice for modeling complex systems [Newman, 2010]. Networks can capture the emergence of collective behavior through the study of network motifs, community structure, and dynamical processes on networks [Milo et al., 2002, Fortunato, 2010, Barrat et al., 2008], and has attracted significant attention in recent years [Barabási, 2016].

2.2.3 From Networks to Simplicial Complexes

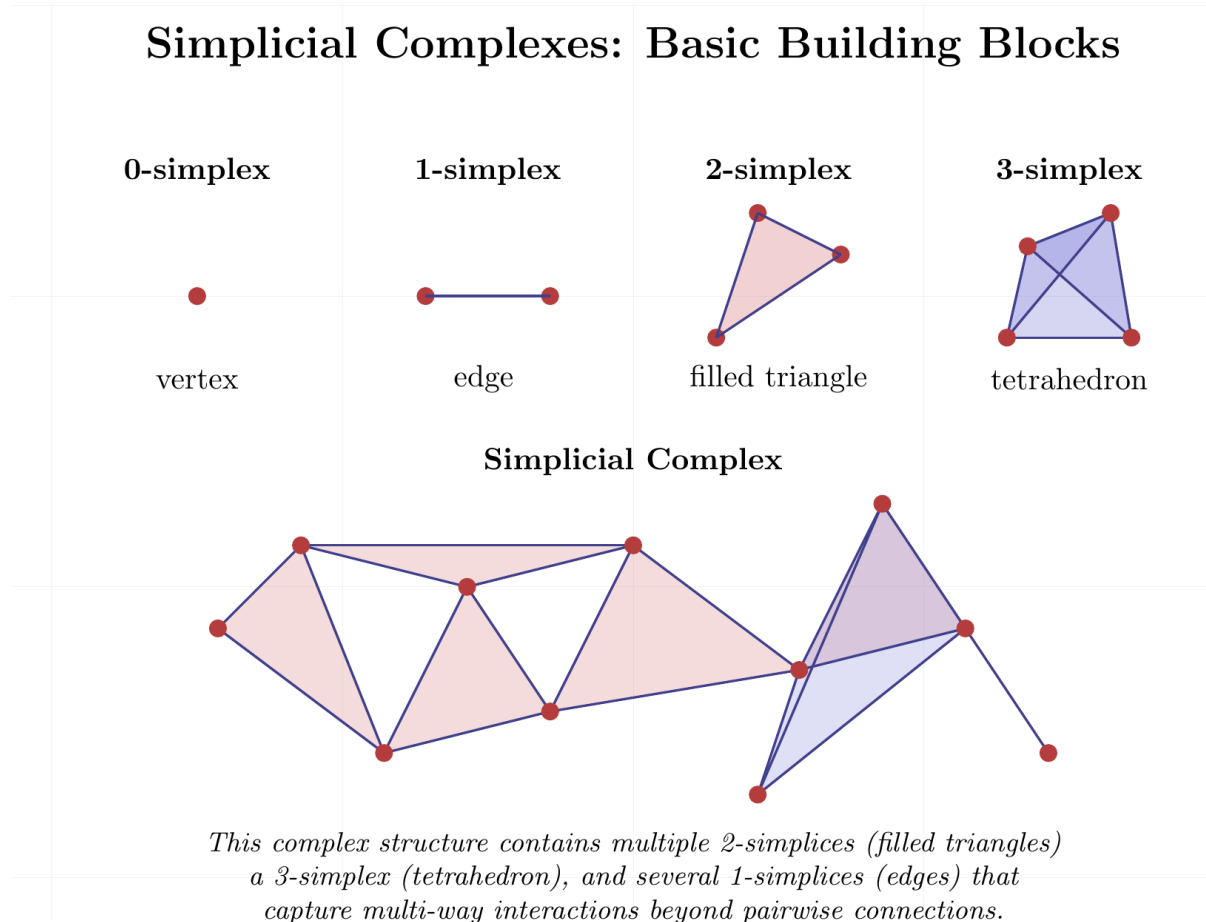


Figure 3: Visual representation of simplicial complexes. The top row shows individual simplices of different dimensions (0-simplex, 1-simplex, 2-simplex, and 3-simplex). The bottom part shows a more complex simplicial complex with multiple 2-simplices (filled triangles), a 3-simplex (tetrahedron), and connecting 1-simplices (edges) that capture multi-way interactions beyond pairwise connections.

Simplicial complexes generalize networks by incorporating higher-order interactions. Simplicial complexes can be thought of as triangles of various dimensions - vertices (0-simplices), edges (1-simplices), triangles (2-simplices), tetrahedra (3-simplices), and so on [Petri et al., 2014] connected together either via shared vertices, edges, or faces.

Formally, a simplicial complex K on a vertex set V is a collection of subsets of V (called simplices) such that:

- For every vertex $v \in V$, $\{v\} \in K$ (0-simplex)

- If $\sigma \in K$ and $\tau \subset \sigma$, then $\tau \in K$ (closure property)

A k -simplex $\sigma = [v_0, v_1, \dots, v_k]$ represents an interaction between $k + 1$ vertices. For example:

- A 0-simplex is a vertex
- A 1-simplex is an edge (pairwise interaction)
- A 2-simplex is a filled triangle (three-way interaction)
- A 3-simplex is a solid tetrahedron (four-way interaction)
- and so on

Several researchers have successfully applied simplicial complexes to model complex systems. [Petri et al. \[2014\]](#) used simplicial complexes to analyze brain functional networks, revealing topological structures that correlate with cognitive states. [Giusti et al. \[2016\]](#) demonstrated how simplicial complexes can capture neural coding schemes beyond what traditional network models could represent. [Sizemore et al. \[2018\]](#) showed how clique topology in neural systems provides insights into brain development and function.

While simplicial complexes offer significant advantages over traditional networks, they have inherent limitations:

- They are *undirected*, with no natural way to represent asymmetric interactions
- *Temporal dynamics* are challenging to model in simplicial complexes

3 Methodology

3.1 Operads

A large class of mathematical theories consists of three ingredients:

1. A collection of objects.
2. A collection of morphisms between these objects.
3. A notion of composition of these morphisms.

The most well-known example of this pattern is arithmetic, where the objects are numbers, the morphisms are functions (addition, multiplication, etc.), and the composition is the usual function composition. All fields such as real numbers, complex numbers, and vector spaces can be described in this way.

As we go up the hierarchy of mathematics, we find more and more examples of this pattern. For example, in topology, the objects are topological spaces, the morphisms are continuous functions, and the composition is the usual function composition. Groups and family (magma, monoid, group, ring, field) are also examples of this pattern. Category theory is a generalization of this pattern, where the objects are categories, the morphisms are functors, and the composition is the usual functor composition.

Operads are a generalization of this pattern, where the objects are operations, the morphisms are operations of different arities, and the composition is a more general form

of function composition. Operads provide a framework for studying algebraic structures that arise in various areas of mathematics, including topology, algebra, and category theory.

Operads consists of:

1. A collection of operations of different arities.
2. A notion of composition of these operations.
3. The composition operations obey certain conditions - associativity and unitality.

Formal Definition

Consider a set \mathbb{X} , and an integer $n \in \mathbb{N}$.

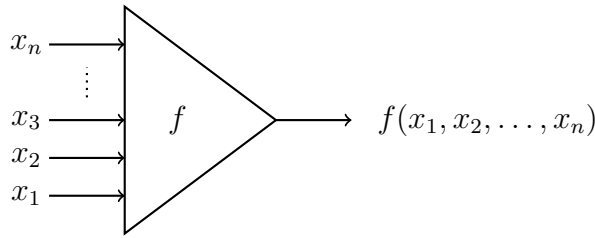
An Operad, \mathbb{P} , is defined as a set of n -ary operations, where each operation f has the signature $\mathbb{X}^n \rightarrow \mathbb{X}$:

$$\mathbb{P}(n) = \{f : \mathbb{X}^n \rightarrow \mathbb{X}\} \quad (2)$$

where \mathbb{X}^n is the cartesian product of \mathbb{X} with itself n times, i.e.

$$\mathbb{X}^n = \mathbb{X} \times \mathbb{X} \times \dots \times \mathbb{X} \quad (3)$$

i.e. all of these functions f take in n arguments from \mathbb{X} and return a single element from \mathbb{X} .



If we have a bunch of these sets of functions $\mathbb{P}(k_i)$ for each $k_i \in \mathbb{N}$, then we can define a composition operation \circ for these operations as follows:

Let $f_i \in \mathbb{P}(k_i)$ be an operation that takes in k_i arguments from \mathbb{X} and returns a single element from \mathbb{X} . We can take n numbers of such operations and use their outputs as inputs to another operation $f \in \mathbb{P}(n)$, which takes in n arguments from \mathbb{X} and returns a single element from \mathbb{X} . The composition operation \circ is defined as:

$$\mathbb{P}(n) \times (\mathbb{P}(k_1) \times \mathbb{P}(k_2) \times \dots \times \mathbb{P}(k_n)) \rightarrow \mathbb{P}(k_1 + k_2 + \dots + k_n) \quad (4)$$

$$f, (f_1, f_2, \dots, f_n) \mapsto f \circ (f_1, f_2, \dots, f_n) \quad (5)$$

where $f \circ (f_1, f_2, \dots, f_n) \in \mathbb{P}(k_1 + k_2 + \dots + k_n)$ is defined as the following diagram:

Associativity of this composition for Operads works as follows:

This composition operation \circ satisfies the following properties:

- **Associativity:** For all $f \in \mathbb{P}(n)$, $g \in \mathbb{P}(k_1)$, $h \in \mathbb{P}(k_2)$, and $i \in \mathbb{P}(k_3)$, we have:

$$f \circ (g \circ (h, i)) = (f \circ (g, h)) \circ i \quad (6)$$

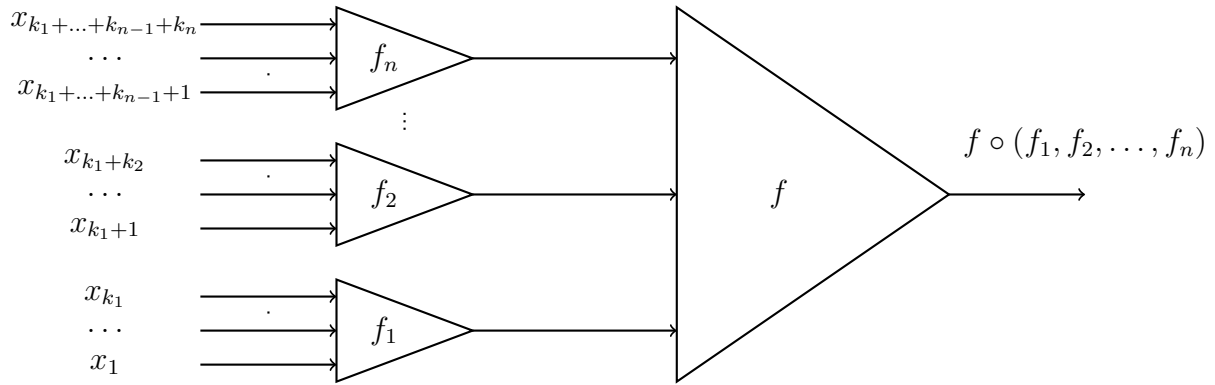


Figure 4: Operadic composition showing how multiple operations f_1, f_2, \dots, f_n with arities k_1, k_2, \dots, k_n can be composed with an operation f of arity n to form a new operation of arity $k_1 + k_2 + \dots + k_n$.

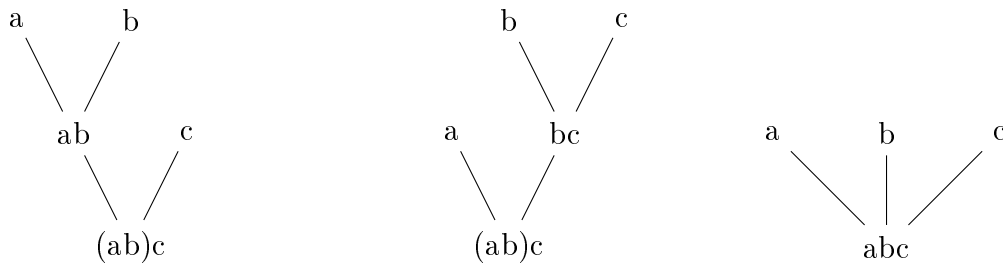


Figure 5: Associativity of operadic composition of arity 3

- **Unitality:** For all $f \in \mathbb{P}(n)$, we have:

$$f \circ (\text{id}_{k_1}, \text{id}_{k_2}, \dots, \text{id}_{k_n}) = f \quad (7)$$

where id_k is the identity function on \mathbb{X}^k .

Symmetry is not required for operads, but it can be added to form symmetric operads. The symmetry condition is:

$$f \circ (g_1, g_2, \dots, g_n) = f \circ (g_{\sigma(1)}, g_{\sigma(2)}, \dots, g_{\sigma(n)}) \quad (8)$$

where σ is a permutation of the set $\{1, 2, \dots, n\}$ or $\sigma \in S_n$ and $g_{\sigma(i)}$ is the $\sigma(i)$ -th element of the original sequence, i.e., the permutation σ permutes the order of operations used as inputs to f .

3.1.1 T-operads

T-operads, or typed operads, extend the concept of operads by introducing a typing system, making them particularly suitable for modeling complex systems with heterogeneous components and interactions. Unlike standard operads where all operations have the same domain and codomain, T-operads allow for typed inputs and outputs, enabling more precise representation of systems with different types of entities and their interactions.

Formal Definition

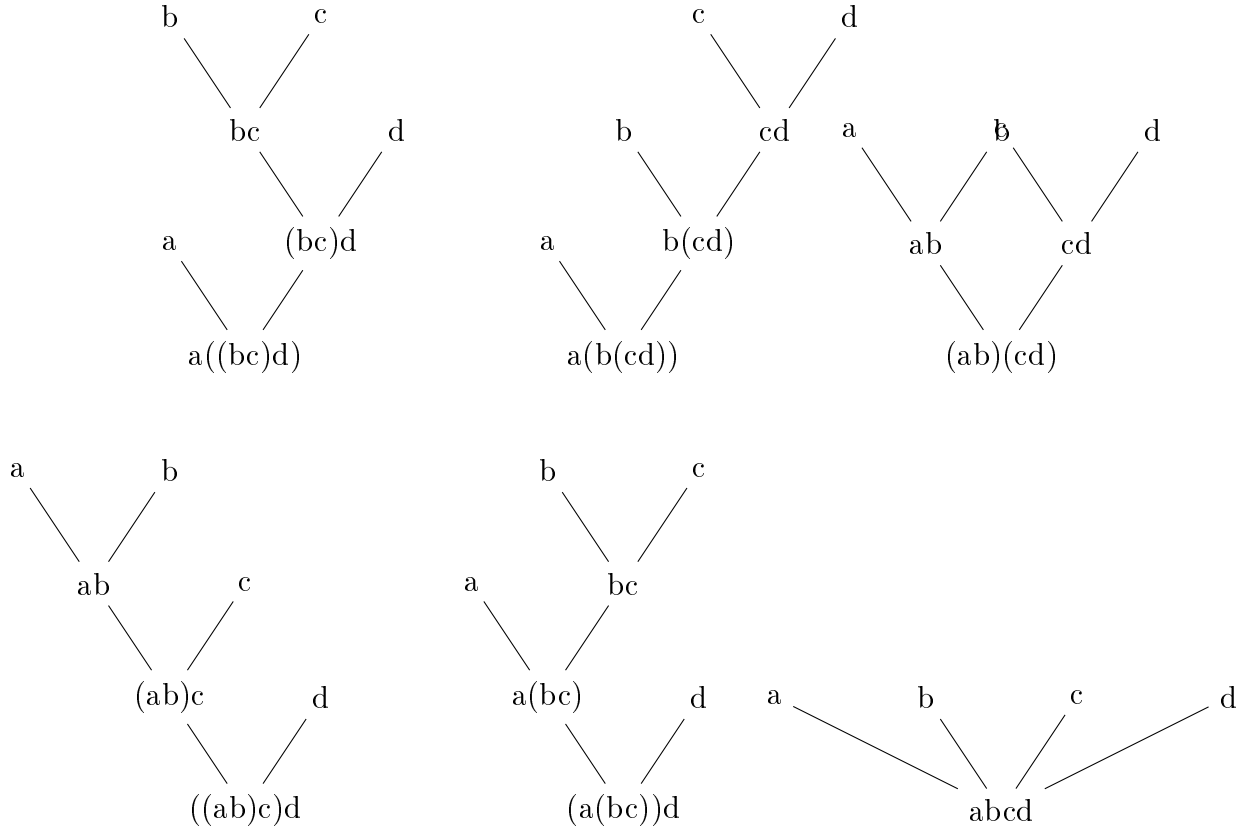


Figure 6: Associativity of operadic composition of arity 4

A T-operad consists of:

1. A set of types $T = \{t_1, t_2, \dots, t_m\}$
2. For each sequence of input types $\mathbf{s} = (s_1, s_2, \dots, s_n)$ where each $s_i \in T$, and an output type $t \in T$, a set of operations $\mathbb{P}(\mathbf{s}; t)$
3. A composition operation that respects types

More formally, let T be a set of types. A T-operad \mathbb{P} assigns to each sequence of input types $\mathbf{s} = (s_1, s_2, \dots, s_n)$ and output type t a set $\mathbb{P}(\mathbf{s}; t)$ of operations that take inputs of types s_1, s_2, \dots, s_n and produce an output of type t :

$$\mathbb{P}(\mathbf{s}; t) = \{f : s_1 \times s_2 \times \dots \times s_n \rightarrow t\} \quad (9)$$

The composition operation must respect the typing constraints. Given operations:

- $f \in \mathbb{P}((t_1, t_2, \dots, t_n); t)$
- $g_1 \in \mathbb{P}(\mathbf{s}_1; t_1)$
- $g_2 \in \mathbb{P}(\mathbf{s}_2; t_2)$
- ...
- $g_n \in \mathbb{P}(\mathbf{s}_n; t_n)$

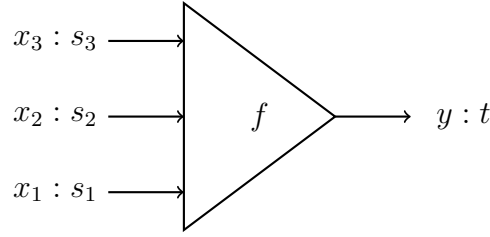


Figure 7: A typed operation $f \in \mathbb{P}((s_1, s_2, s_3); t)$ taking three inputs of potentially different types and producing an output of type t .

The composition $f \circ (g_1, g_2, \dots, g_n)$ is defined only when the output type of each g_i matches the corresponding input type required by f . The resulting operation has input types given by the concatenation of all input types of the g_i operations and output type t :

$$f \circ (g_1, g_2, \dots, g_n) \in \mathbb{P}(s_1 \cdot s_2 \cdot \dots \cdot s_n; t) \quad (10)$$

where \cdot denotes concatenation of type sequences.

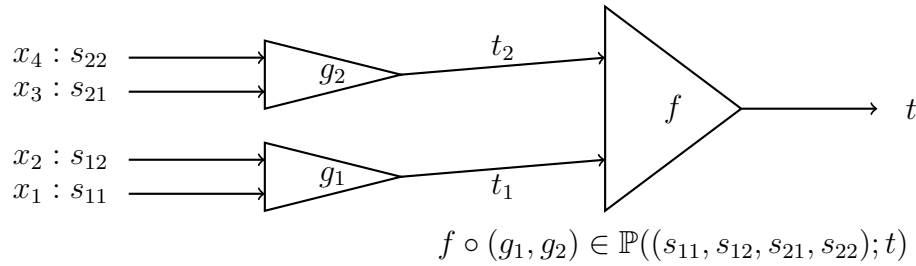


Figure 8: Typed operadic composition showing how the output types of g_1 and g_2 must match the input types required by f .

Applications of T-operads to Complex Systems

T-operads provide several key advantages for modeling complex systems:

1. **Heterogeneous Systems:** T-operads can model systems with different types of components, such as cells of different types in biological tissues, different species in ecological networks, or various agent types in economic systems.
2. **Type-Restricted Interactions:** They capture the reality that not all elements can interact with all others—proteins can only bind to compatible partners, economic agents have specific roles, and neural signals have specific receptor requirements.
3. **Phase Transitions:** Changes in the type structure of a T-operad can model fundamental shifts in system behavior. For example, the differentiation of stem cells involves changes in the types of operations available as cells specialize, and phase transitions in social systems correspond to changes in the roles individuals can play.
4. **Hierarchical Organization:** T-operads naturally represent hierarchical systems where higher-level entities emerge from compositions of lower-level entities, potentially with type transformations occurring during composition.

In biological systems, T-operads can represent protein interactions where specific protein types can only bind with compatible partners, forming complexes that then have new binding capabilities. In neural networks, T-operads can model how different neuron types (excitatory, inhibitory) combine to form functional circuits with specific input-output behaviors. In social systems, T-operads can represent how individuals with different expertise (types) combine to form organizations with emergent capabilities.

Detecting Phase Transitions with T-operads

Phase transitions in complex systems can be identified through changes in the type structure of T-operads:

1. **Type Emergence:** New types appear in the system, indicating new functionalities or states have emerged.
2. **Compositional Reorganization:** Changes in which operation types can compose with each other, signaling a restructuring of system interactions.
3. **Type Collapse:** Multiple types merge into a single type, indicating a loss of differentiation or specialization.
4. **Connectivity Changes:** Alterations in the graph of possible type compositions, which can indicate critical transitions in system behavior.

For example, in a financial system modeled with T-operads, a phase transition from stability to crisis might be detected by:

- New operation types emerging (representing novel, risky financial instruments)
- Changes in which institution types can interact with which financial product types
- Collapse of previously distinct risk categories into correlated failure modes

These changes in the T-operad structure provide early warning signals for system-wide transitions that may not be detectable using traditional network or simplicial complex approaches.

T-operads thus provide a powerful framework for modeling complex systems where the types of interactions and entities matter, enabling us to detect and analyze phase transitions that involve changes in the system's compositional structure and typing relationships.

4 Results

5 Discussion

6 Conclusion

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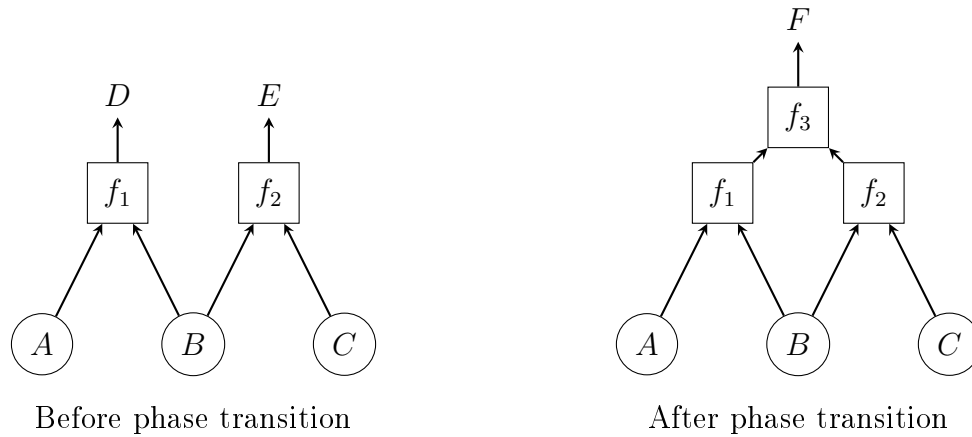


Figure 9: Phase transition in a T-operad: Before the transition, operations f_1 and f_2 produce different types. After the transition, a new composition capability emerges with operation f_3 that can take outputs from both f_1 and f_2 to produce a new type F .

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