Polynomial counting functions for metric ribbon graphs

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Variation 1

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Let $V_{g,n}^{m}(L)$ be the corresponding volume function.
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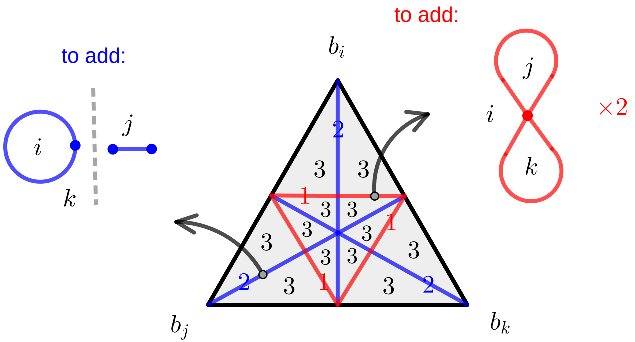
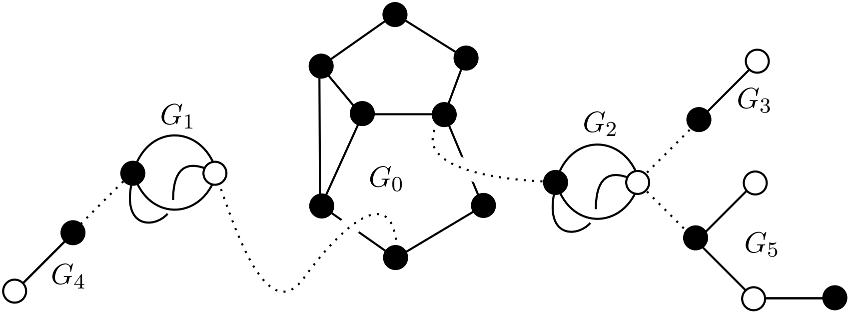


Variation 1

Theorem (Duryev-Goujard-Y. '25)

$V^m_{g,n}(L)$ is polynomial on any intersection of the hyperplanes, outside of lower-dimensional intersections.

Technology: count possible degenerations.



Variation 1

Application: recursion for volumes of odd strata of quadratic differentials (Duryev-Goujard-Y. '25).

Theorem 2.6. *For any odd composition $\underline{k} = (k_1, \ldots, k_r)$ of $4g-4$ with $r \geq 3$ and $k_i \geq -1$,*

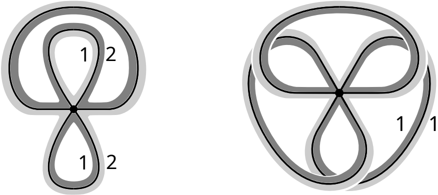
$$\overline{\mathrm{Vol}}(\mathcal{Q}(\underline{k})) = \mathrm{Vol}(\mathcal{Q}(\underline{k})) + \sum_{\substack{\underline{g}=(g_1,\ldots,g_r) \\ 0 \leq g_i \leq \frac{k_i+1}{4}}} \sum_{\substack{\underline{g}_i = \left(g_i^{(1)}, g_i^{(2)} \ldots\right) \\ |\underline{g}_i| = g_i,\ g_i^{(j)} > 0}} C_{\underline{g},\underline{g}_i} \cdot \mathrm{Vol}\left(\mathcal{Q}(\underline{k} - 4\underline{g}) \times \prod_{i,j} \mathcal{H}(2g_i^{(j)} - 2)\right),$$

where in the second sum, if $g_i = 0$, we set $\underline{g}_i$ to be the empty composition with $\ell(\underline{g}_i) = |\underline{g}_i| = 0$, and where the coefficients $C_{\underline{g},\underline{g}_i}$ are given by

$$C_{\underline{g},\underline{g}_i} = \prod_{i=1}^{r} (k_i - 4g_i + 2) \cdot \frac{1}{\ell(\underline{g}_i)! \cdot 2^{\ell(\underline{g}_i)}} \cdot \frac{k_i!!}{(k_i - 2\ell(\underline{g}_i) + 2)!!} \cdot \prod_{i,j} (2g_i^{(j)} - 1). \tag{8}$$

Variation 2

Consider face-bicolored MRGs with one-vertex .



Let $V^1_{g,(k,l)}(L; L')$ be the corresponding volume function.

Theorem (Y. '23)

For any g, k, l, the top-degree term of $V^1_{g,(k,l)}$ is a polynomial outside of a finite number of hyperplanes ("walls"), whose coefficients count certain metric plane trees. Analogous statement is true on any intersection of walls.

Variation 2

$$\frac{(k+l+2g-2)!}{2^{2g}} \cdot \sum_{\substack{b_1+\ldots+b_k+w_1+\ldots+w_l=g \\ b_i,w_i\geq 0}} \prod_{i=1}^{k} \frac{L_i^{2b_i}}{(2b_i+1)!} \cdot \prod_{j=1}^{l} \frac{{L'_j}^{2w_j}}{(2w_j+1)!}.$$

- outside of the walls: **Okounkov, Pandharipande '06**; genus 0: **Gendron, Tahar '22**; **Chen, Prado '23**.

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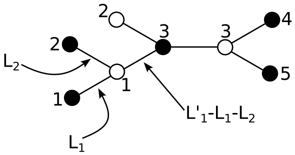

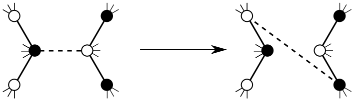

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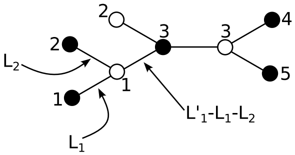

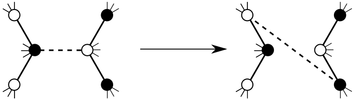

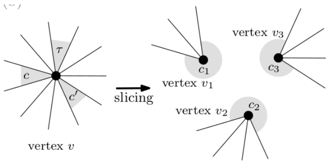

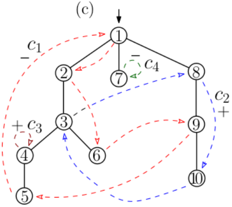

Variation 2

Application: distribution of the number of cylinders in random square-tiled surfaces in strata of abelian differentials with one zero.

Theorem (Y., 2023)

The contribution of n-cylinder square-tiled surfaces to the volume of $\mathcal{H}(2g-2)$ is equal to $\frac{2(2\pi)^{2g}}{(2g-1)!}a_{g,n}$, where $a_{g,n} \in \mathbb{Q}$, and whose generating function $\mathcal{C}(t,u) = 1 + \sum_{g\geq 1}\left(\sum_{n=1}^{g} a_{g,n}u^n\right)(2g-1)t^{2g}$ satisfies for all $g \geq 0$

$$\frac{1}{(2g)!}[t^{2g}]\mathcal{C}(t,u)^{2g} = [t^{2g}]\left(\frac{t/2}{\sin(t/2)}\right)^{u}.$$

- setting $u=1$ we recover the result of Sauvaget.
- Faber-Pandharipande 2000: $\left(\frac{t/2}{\sin(t/2)}\right)^{u+1}$ is the generating function of Hodge integrals $\int_{\overline{\mathcal{M}}_{g,1}} \lambda_{g-i}\psi_1^{2g-2+i}$

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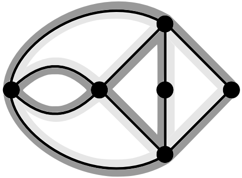

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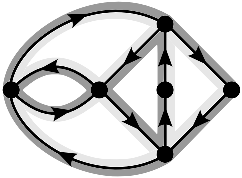

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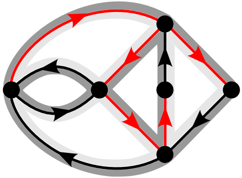

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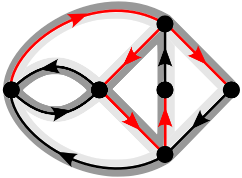

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For $g = 0$, any $k, l \geq 1$ and any vertex degree profile m, the *weighted* counting function $\widetilde{V}^m_{0,(k,l)}$ is polynomial (outside of some hyperplanes) equal to

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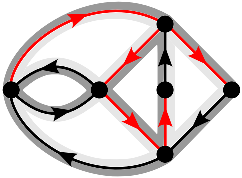

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Also true for $g > 0$...

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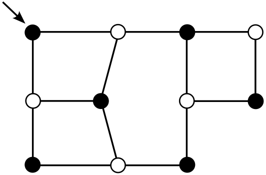

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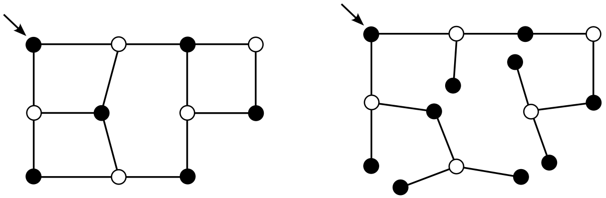

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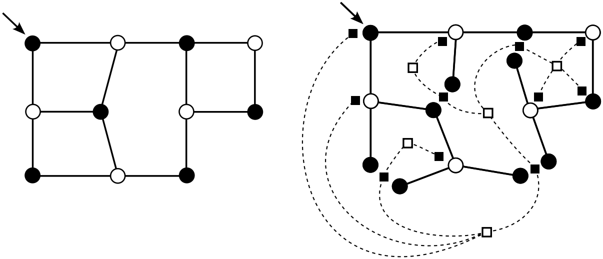

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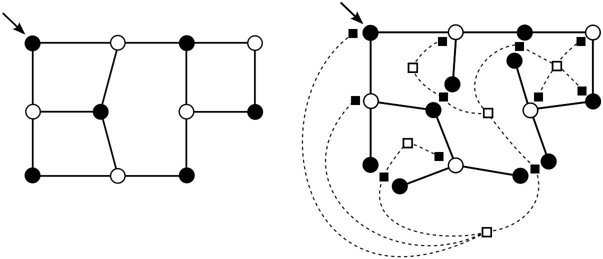

$\rightsquigarrow$ control of both the metric and the combinatorics of a tree.

Back to Kontsevich polynomials

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- Weighted version:

$$\sum_G \frac{V_G(L)}{|\operatorname{Aut}(G)|} \cdot w_d(G) = \operatorname{const} \cdot \left(\int_{\mathcal{M}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) \cdot L_1^{2d_1} \cdots L_n^{2d_n},$$

where $w_d(G)$ is the number of orientations of G with fixed indegrees.