# Counting differentials combinatorially

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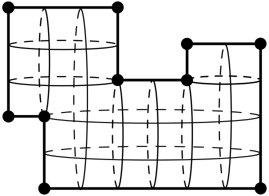
- **Delecroix, Goujard, Zograf, Zorich '21**: combinatorial formula for the volumes of principal strata of *quadratic* differentials, using cylinder decomposition of square-tiled surfaces; applications to large genus asymptotics, properties of random ST surfaces / random multicurves on hyperbolic surfaces.

# Cylinder decomposition of square-tiled surfaces

**DGZZ**: quadratic case.



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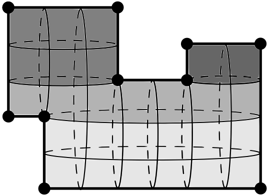

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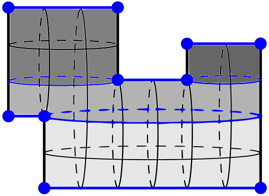

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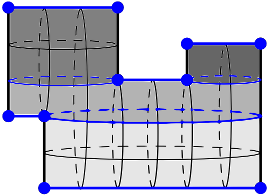

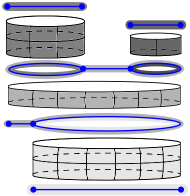

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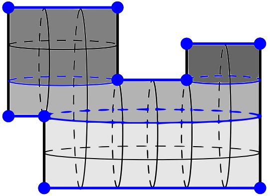

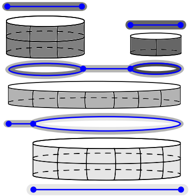

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Principal stratum ⇒ trivalent ribbon graphs (leaves also possible).

# Kontsevich polynomials

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**Theorem (Kontsevich, '92)**

The top-degree term of $\mathcal{N}_{g,n}$ is a homogeneous polynomial

$$\frac{1}{2^{5g-6+2n}} \sum_{d_1+\ldots+d_n=3g-3+n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^{n} \frac{L_i^{2d_i}}{d_i!},$$

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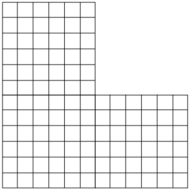
**Norbury '10**: $\mathcal{N}_{g,n}$ is a quasi-polynomial.

# Cylinder decomposition in the Abelian case



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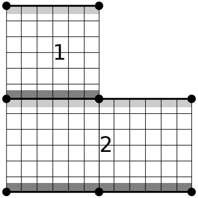

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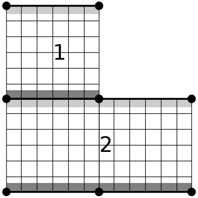


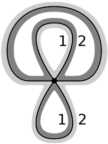

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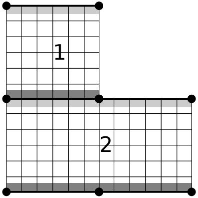


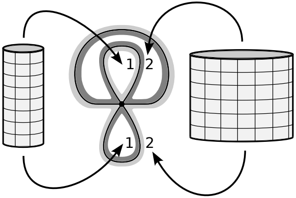


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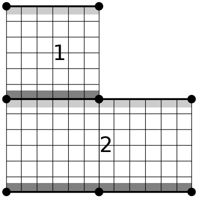


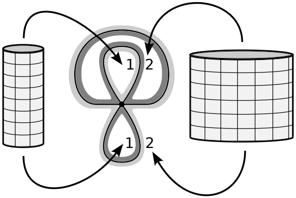


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Minimal stratum $\Rightarrow$ one-vertex graphs.

# Counting functions for one-vertex graphs

Let $\mathcal{P}_{g,(k,l)}(L_1,\ldots,L_k;L'_1,\ldots,L'_l)$ be the number of one-vertex face-bicolored ribbon graphs of genus $g$, with $k$ black and $l$ white boundaries of lengths $L_i, L'_j$.

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**Theorem (Y. '23)**

For any $g,k,l$, the top-degree term of $\mathcal{P}_{g,(k,l)}$ is a polynomial outside of the walls, whose coefficients count certain metric plane trees. Analogous statement is true on any intersection of walls.

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- $g=0$ case also in **Gendron, Tahar '22** and **Chen, Prado '23**;
- top-degree term on $\{L_i = L'_i,\ \forall i\} \Rightarrow$ cylinder contributions in $\mathcal{H}(2g-2)$.

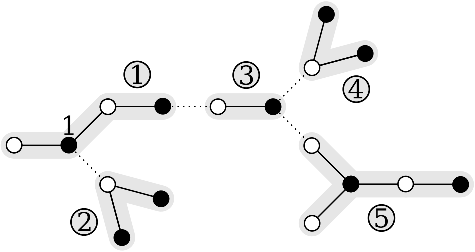
# Generic case

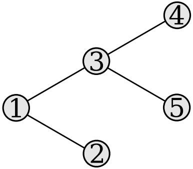
## Proposition

*Outside of the walls, the top-degree term of $\mathcal{P}_{g,(k,l)}(L,L')$ is equal to*

$$\frac{(k+l+2g-2)!}{2^{2g}} \cdot \sum_{\substack{b_1+\ldots+b_k+w_1+\ldots+w_l=g \\ b_i,w_i\geq 0}} \prod_{i=1}^{k} \frac{L_i^{2b_i}}{(2b_i+1)!} \cdot \prod_{j=1}^{l} \frac{{L'_j}^{2w_j}}{(2w_j+1)!}.$$

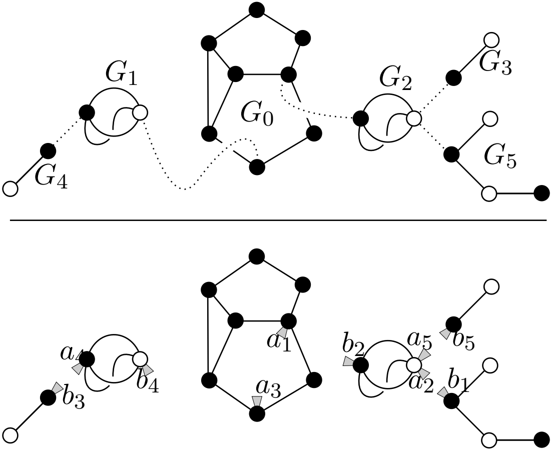
# Computing degeneration coefficients on the walls




$$(k+l-2)! = p^{b_1,\dots,b_n}_{w_1,\dots,w_n} + \sum_{t=2}^{n} \frac{(k+l-2)_{t-2}}{t!} \sum_{I_1,\dots I_t} \prod_{j=1}^{t} \left( \sum_{i\in I_j} (b_i + w_i) - 1 \right) p^{b_{I_j}}_{w_{I_j}},$$

# Degenerations in the quadratic case




A joining is admissible if and only if for every $i = 1, \ldots, m$ the following holds:

- if all of the descendants of $G_i$ have labels smaller then $i$, then the bridge joining $G_i$ to its parent is in the black corner of $G_i$;
- otherwise, the bridge joining $G_i$ to its parent and the bridge joining $G_i$ to the subtree containing the descendant of $G_i$ of maximal label are in the corners of $G_i$ of different colors.

# Cylinder contributions in $\mathcal{H}(2g-2)$

**Theorem (Y., 2023)**

*The contribution of $n$-cylinder square-tiled surfaces to the volume of $\mathcal{H}(2g-2)$ is equal to $\frac{2(2\pi)^{2g}}{(2g-1)!}a_{g,n}$, where $a_{g,n} \in \mathbb{Q}$, and whose generating function $\mathcal{C}(t,u) = 1 + \sum_{g\geq 1}\left(\sum_{n=1}^{g} a_{g,n}u^n\right)(2g-1)t^{2g}$ satisfies for all $g \geq 0$*

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- Faber-Pandharipande 2000: $\left(\frac{t/2}{\sin(t/2)}\right)^{u+1}$ is the generating function of Hodge integrals $\int_{\overline{\mathcal{M}}_{g,1}} \lambda_{g-i}\psi_1^{2g-2+i}$

# Cylinder contributions in spin components of $\mathcal{H}(2g-2)$

**Theorem (Y., 2024+)**

*The difference of the contributions of $n$-cylinder square-tiled surfaces to the volumes of even and odd spin subspaces of $\mathcal{H}(2g-2)$ is equal to $\frac{2(2\pi i)^{2g}}{(2g-1)!} d_{g,n}$, where $d_{g,n} \in \mathbb{Q}$, and whose generating function $\mathcal{D}(t,u) = 1 + \sum_{g\geq 1} \left(\sum_{n=1}^{g} d_{g,n} u^n\right)(2g-1)t^{2g}$ satisfies for all $k \geq 1$*

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$u = 1$ : formula from Chen, Möller, Sauvaget, Zagier.

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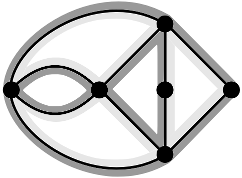

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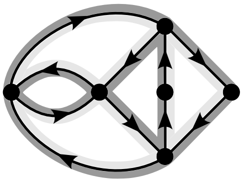

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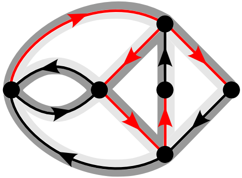

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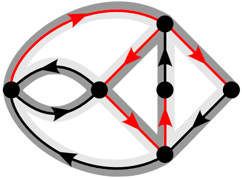

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For $g = 0$, any $k, l \geq 1$ and any vertex degree profile $d$, the *weighted* counting function $\widetilde{\mathcal{P}}^{d}_{0,(k,l)}$ has a polynomial top-degree term, equal to

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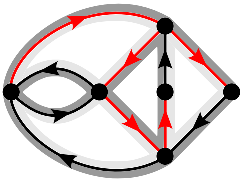

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Also true for $g > 0$...

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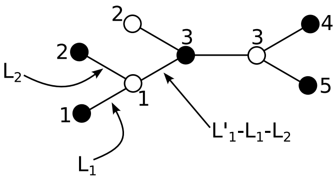

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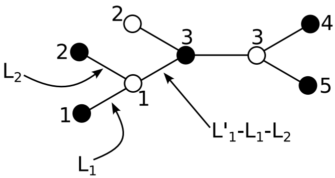
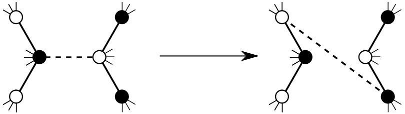
- when traversing a wall, we "loose" and "gain" an equal number of trees.



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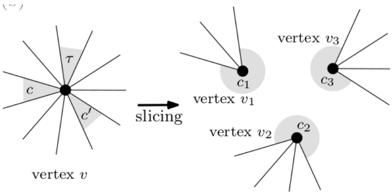


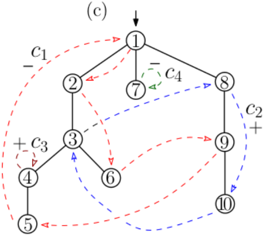

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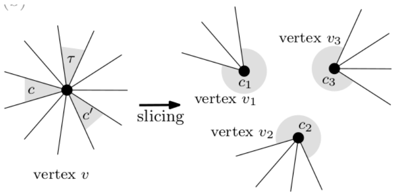
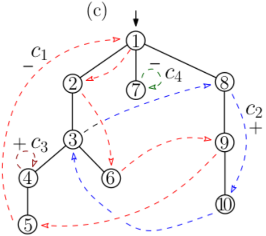
- allows to control the metric! □



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Invariant? $\rightarrow$ parity?

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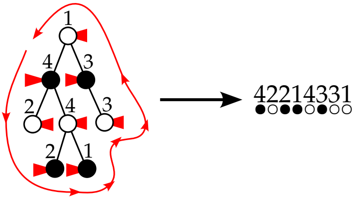
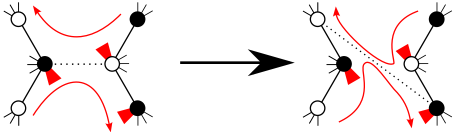
$\Rightarrow$ Prefix-postfix traversal!



# Flipping edges cleverly

For any edge, there is a unique way to flip it so that the prefix-postfix sequence does not change!



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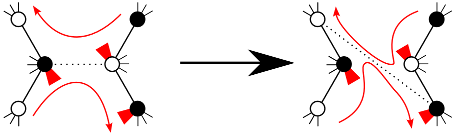

**Proposition (Y., '24)**

*For any $k, l \geq 1$, any permutation of vertex labels $\pi$, and any point $(L; L')$ outside of the walls, there is exactly 1 tree with prefix-postfix sequence $\pi$ and contributing at $(L; L')$.*

# Application 1: many-vertex face-bicolored graphs

*Idea of proof*: **Bernardi '07**: bijection between plane maps with a distinguished spanning tree and pairs of plane trees, via explosions of vertices.

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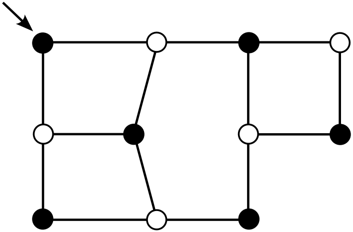

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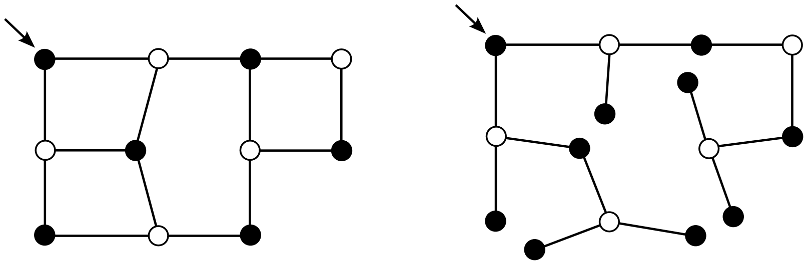

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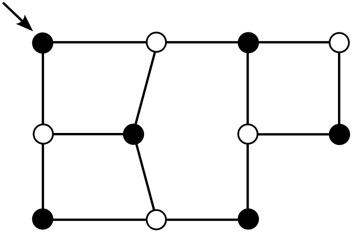

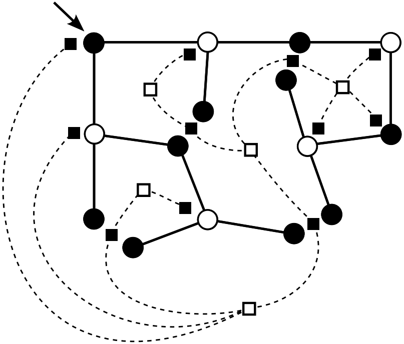

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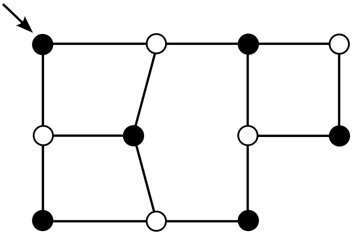

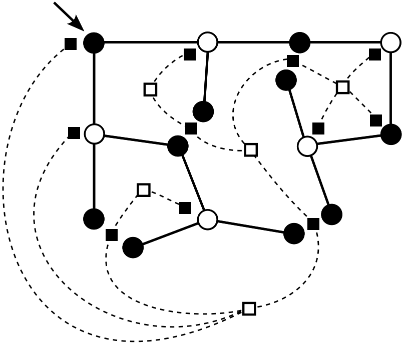

Prefix-postfix sequences allow us to control both the metric and the combinatorics of a tree (positions of first/last visit corners of vertices)! □

# Application 2: triangulations!

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*Observation*: any tree contributes for values of $(L; L')$ in a *simplicial* cone

$$cone\left(e_i + e'_j : \bullet\!\!\overset{i}{-}\!\!\circ^{j} \text{ is an edge}\right) \subset (\mathbb{R}_+^k \times \mathbb{R}_+^l) \cap \left\{\sum L_i = \sum L'_j\right\},$$

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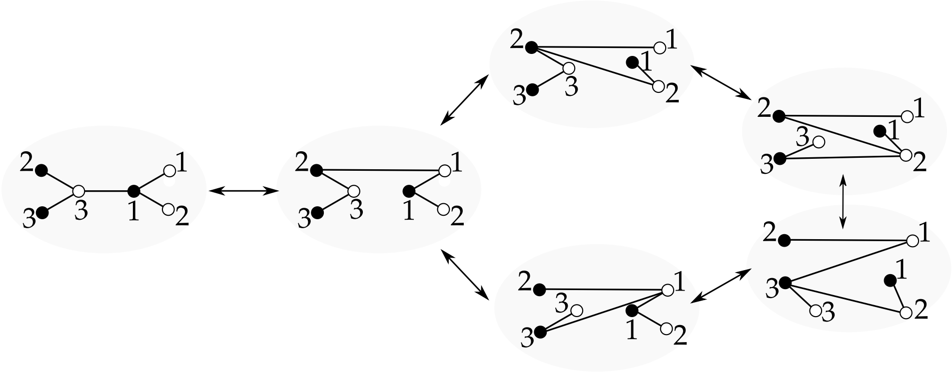
**Theorem (Y., '24)**

*For any $k, l \geq 1$ and any permutation $\pi$ of the vertex labels, the simplices corresponding to plane trees with prefix-postfix sequence $\pi$ form a triangulation of $\Delta_k \times \Delta_l$.*

# Application 2: triangulations!

Example for $k = l = 3$:

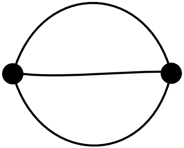


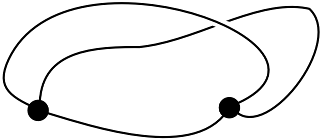
# Work in progress:

- parity of ribbon graphs of genus $g > 0$;
- weighted counting functions for $g > 0$;
- study Kontsevich polynomials combinatorially;
- study connection with intersection theory;
- study obtained triangulations of $\Delta_k \times \Delta_l$.

# Ribbon graphs (= combinatorial maps)

A ribbon graph is a graph (loops and multiple edges allowed) with a circular ordering of (half-)edges incident to every vertex.





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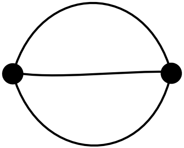

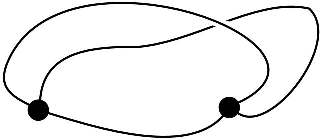

Equivalently, cellular embedding of a graph into a surface.

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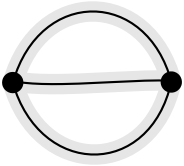

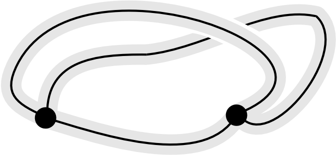

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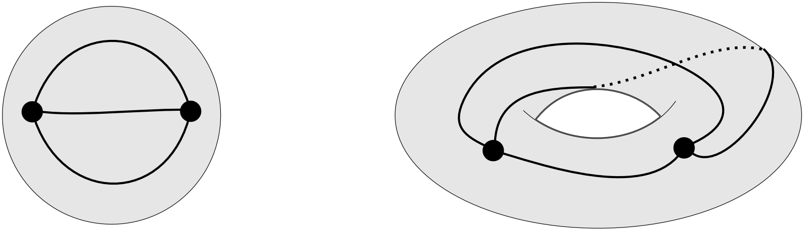

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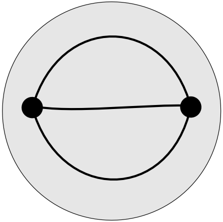

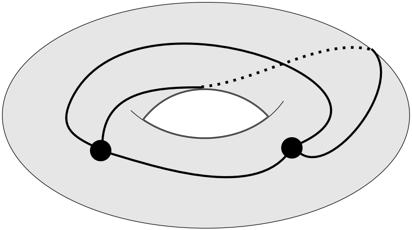

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Euler's formula: $|V(G)| - |E(G)| + |F(G)| = 2 - 2g$.

*Faces = boundary components.*

# Families of ribbon graphs

$$\mathcal{RG}_{g,n} := \left\{ \begin{array}{c} \text{genus } g \text{ ribbon graphs with} \\ n \textit{ labeled} \text{ boundary components} \end{array} \right\}$$

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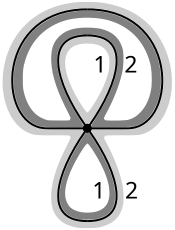

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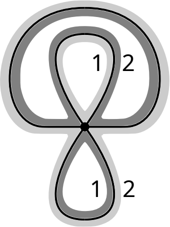


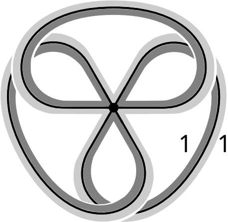

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$$\mathcal{P}_G(L_1, \ldots, L_n) = \# \left\{ \begin{array}{c} \textit{integer} \text{ metrics on } G \text{ with perimeter of} \\ \text{the } i\text{-th boundary component equal to } L_i \end{array} \right\}$$

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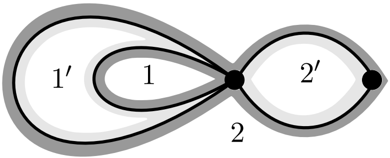
For $G \in \mathcal{RG}_{g,(k,l)}$ and $L_1, \ldots, L_k, L'_1, \ldots, L'_l \in \mathbb{Z}$, define analogously

$$\mathcal{P}_G(L_1, \ldots, L_k; L'_1, \ldots, L'_l)$$

# Computing a counting function



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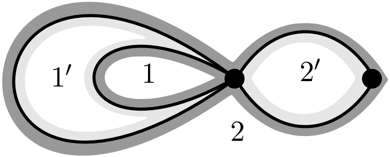


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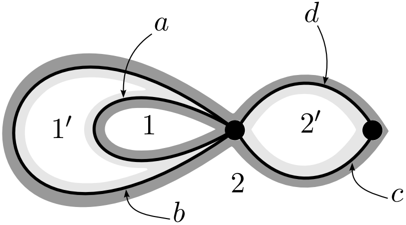


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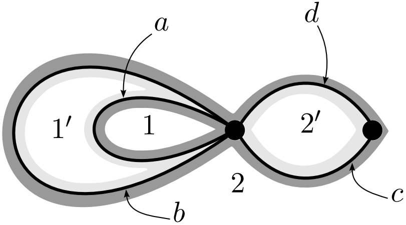


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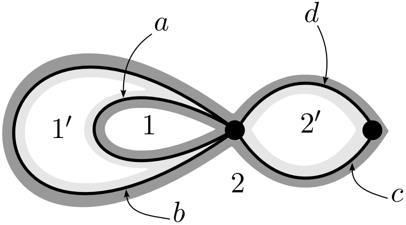


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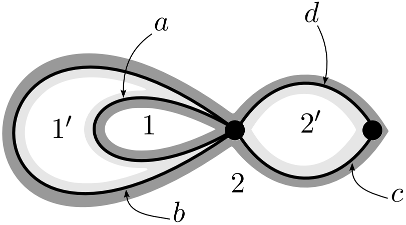


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# Counting functions

Let $\mathcal{PS}_{k,l}$ be the polyhedral subdivision of $\mathbb{R}^k \times \mathbb{R}^l$ generated by the hyperplanes ("walls") of the form $\sum_{i\in I} L_i = \sum_{j\in J} L'_j,\ I \subset \{1,\dots,k\},\ J \subset \{1,\dots,l\}$.

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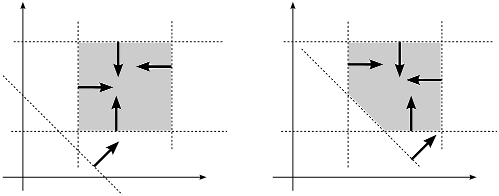
Idea of proof: counting integer points in a deforming polytope.



# Counting functions

Introduce the counting functions for families of ribbon graphs:

$$\mathcal{P}_{g,n}(L) = \sum_{G \in \mathcal{RG}_{g,n}} \frac{1}{|\operatorname{Aut}(G)|} \cdot \mathcal{P}_G(L),$$

$$\mathcal{P}_{g,(k,l)}(L, L') = \sum_{G \in \mathcal{RG}_{g,(k,l)}} \frac{1}{|\operatorname{Aut}(G)|} \cdot \mathcal{P}_G(L, L').$$

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*A priori* piecewise (quasi-)polynomials...