From last class,

We considered

$$y'' + by' + cy = 0$$

for $b = 0$.

$$\Rightarrow \quad y'' + cy = 0$$

$$\Rightarrow \quad \lambda^2 + c = 0$$

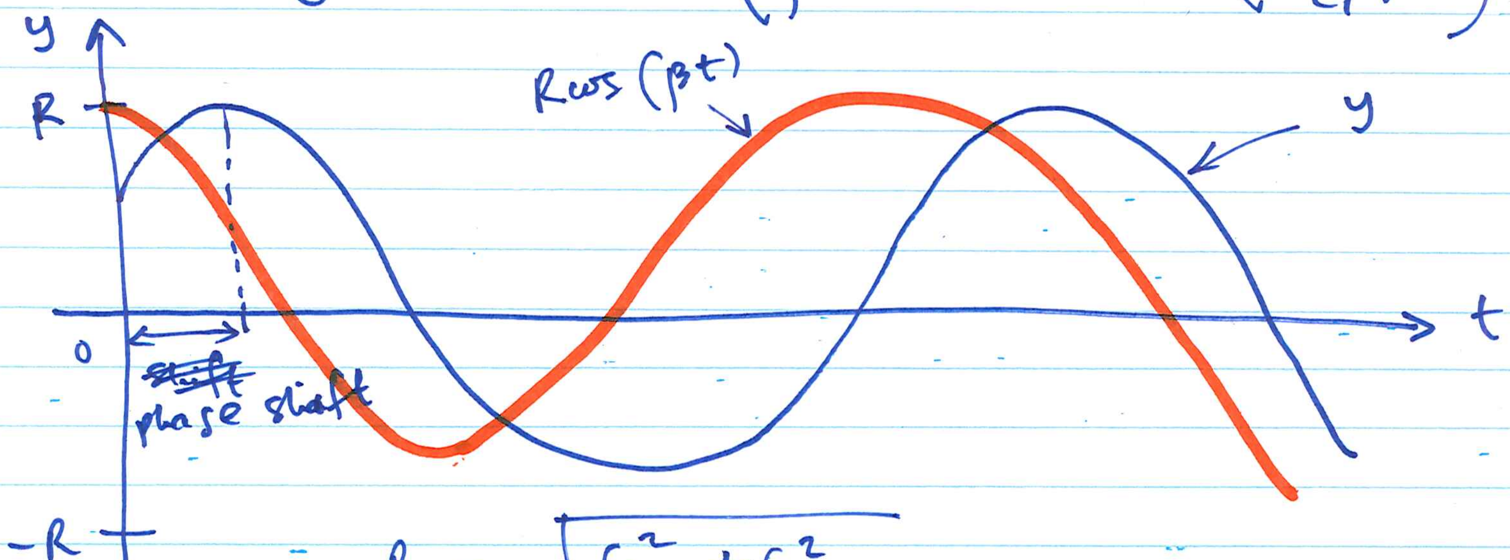
$\Rightarrow$ λ complex.

Let $\lambda = \pm i\beta$

$$y(t) = c_1 \cos(\beta t) + c_2 \sin(\beta t).$$

Then we wrote the solution in the form

$$y(t) = \sqrt{c_1^2 + c_2^2} \cos\left(\beta t - \arctan\left(\frac{c_2}{c_1}\right)\right)$$



$$R = \sqrt{c_1^2 + c_2^2}$$

To find the phase shift in the solution, we set

$$\beta t - \arctan\left(c_2/c_1\right) = 0$$

$$\Rightarrow t = \frac{1}{\beta} \arctan\left(c_2/c_1\right)$$

Natural frequency of undamped oscillator

Consider

$$y'' + \omega_0^2 y = 0$$

with general solution

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

$\Rightarrow$ ω_0 is the natural frequency of the oscillator.

Example:

① linear pendulum (without damping)

$$\theta'' + g/L\, \theta = 0$$

$\Rightarrow$ $\sqrt{g/L}$ is the natural frequency.

(ii) LRC circuit (without resistance)

$$L\frac{d^2\varphi}{dt^2} + \frac{1}{C}\varphi = 0$$

The natural frequency is $\frac{1}{\sqrt{LC}}$

(iii) Vibrating spring (undamped)

$$x'' + k/m\, x = 0$$

$\Rightarrow$ $\sqrt{k/m}$ is its natural frequency.

Non-homogeneous 2nd order ODEs

Given

$$y'' + by' + cy = g(t) \quad \text{——— (1)}$$

We can write this equation as a system of 1st order ODEs.

Let $y_1 = y$, $\Rightarrow$ $y_1' = y' = y_2$

$y_2 = y'$ $\Rightarrow$ $y_2' = y''$

from (1), $y'' = -by' - cy + g(t)$

$$y'' = -by_2 - cy_1 + g(t)$$

$\therefore$ We have

$$y_1' = y_2$$
$$y_2' = -cy_1 - by_2 + g(t)$$

In matrix form,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix} \quad \text{—— (2)}$$

To solve the 2nd order forced ODE in (1), we can solve the system in equation (2) using variation of parameters.

But the problem with this method is evaluating the integrals involved for some functions.

The method of undetermined coefficient is more efficient for this type of problems.

Method of undetermined coefficient.

Given $y'' + by' + cy = g(t)$

We know that the solution is

$$y(t) = y_H + y_p .$$

~~Where~~ where y_H is the homogeneous solution and y_p is particular solution.

<u>Example:</u> Solve $y'' - 3y' - 4y = t^2$ —— (1)

For y_H, we solve

$$y'' - 3y' - 4y = 0$$

$$\Rightarrow \quad \lambda^2 - 3\lambda - 4 = 0$$

$$\Rightarrow \quad \lambda - 4\lambda + \lambda - 4 = 0$$

$$\lambda_1 = 4, \quad \lambda_2 = -1$$

$$\therefore \quad y_H = c_1 e^{4t} + c_2 e^{-t} .$$

For y_p,

Guess: $y_p = At^2 + Bt + C$

put y_p into (1)

$$y_p' = 2At + B$$

$$y_p'' = 2A$$

$$\Rightarrow \quad 2A - 3(2At + B) - 4(At^2 + Bt + C) = t^2$$

$$-4At^2 + (-6A - 4B)t + (2A - 3B - 4C) = t^2$$

comparing coefficients of powers of t,

for t^2: $\quad -4A = 1$

$$\Rightarrow A = -\frac{1}{4}$$

For t: $\quad -6A - 4B = 0$

$$-6\left(-\frac{1}{4}\right) - 4B = 0$$

$$4B = \frac{3}{2}$$

$$B = \frac{3}{8}$$

For constants:

$$2A - 3B - 4C = 0$$

$$2\left(-\tfrac{1}{4}\right) - 3\left(\tfrac{3}{8}\right) = 4C$$

$$\Rightarrow \quad 4C = -\tfrac{1}{2} - \tfrac{9}{8}$$

$$C = -\tfrac{13}{32}$$

$$\therefore \; y_p = -\frac{1}{4}t^2 + \tfrac{3}{8}t - \tfrac{13}{32}$$

$$y(t) = y_H + y_p$$

$$y(t) = c_1 e^{4t} + c_2 e^{-t} - \tfrac{1}{4}t^2 + \tfrac{3}{8}t - \tfrac{13}{32}$$

Example: Solve $\quad y'' - 2y' - 3y = 3e^{2t}$

$$y = y_H + y_p$$

We know how to get y_H,

$$y_H = c_1 e^{3t} + c_2 e^{-t}$$

For the particular solution,

Guess: $\quad y_p = Ae^{2t}$

$$y_p' = 2Ae^{2t} \quad , \; y_p'' = 4Ae^{2t}$$

put y_p into

$$y_p'' - 2y_p' - 3y_p = 3e^{2t}$$

$$\Rightarrow \quad 4Ae^{2t} - 2\left(2Ae^{2t}\right) - 3Ae^{2t} = 3e^{2t}$$

$$(4A - 4A - 3A)e^{2t} = 3e^{2t}$$

$$\Rightarrow \quad -3A = 3$$

$$A = -1$$

$$\therefore y(t) = c_1e^{3t} + c_2e^{-t} - 1e^{2t}$$

Let us consider the generic example:

$$y'' + by' + cy = e^{\alpha t} \qquad (*1)$$

Let us guess y_p to be

$$y_p = Ae^{\alpha t}$$

$$\Rightarrow \quad y_p' = \alpha Ae^{\alpha t}$$

$$y_p'' = \alpha^2 Ae^{\alpha t}$$

put y_p into *1

$$\alpha^2 A e^{\alpha t} + b\alpha A e^{\alpha t} + cA e^{\alpha t} = e^{\alpha t}$$

$$\left(\alpha^2 A + b\alpha A + cA\right) e^{\alpha t} = e^{\alpha t}$$

$$\Rightarrow A\left(\alpha^2 + b\alpha + c\right) = 1$$

$$A = \frac{1}{\alpha^2 + b\alpha + c}$$