Last day

* System of D.E with complex eigenvalues and eigenvectors.

Consider

$$\vec{X}'(t) = A \vec{X}(t)$$

where A is a real matrix with complex eigenvalues λ and $\bar{\lambda}$, and corresponding eigenvectors $\vec{x}$ and $\bar{\vec{x}}$.

The general solution (in complex form)

$$\vec{X}(t) = c_1 e^{\lambda t} \vec{x} + c_2 e^{\bar{\lambda} t} \bar{\vec{x}}$$

and the real-valued general solution is given by

$$\vec{X}(t) = \alpha_1 \, \text{Re}\left(e^{\lambda t} \vec{x} \right) + \alpha_2 \, \text{Im}\left(e^{\lambda t} \vec{x} \right)$$
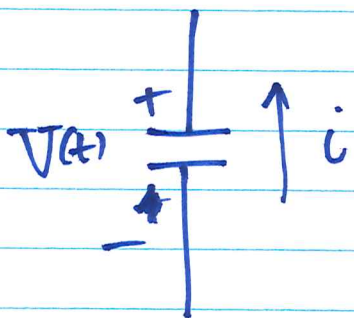
LCR Circuits

L - Inductor

C - Capacitor

R - resistor

* Capacitors

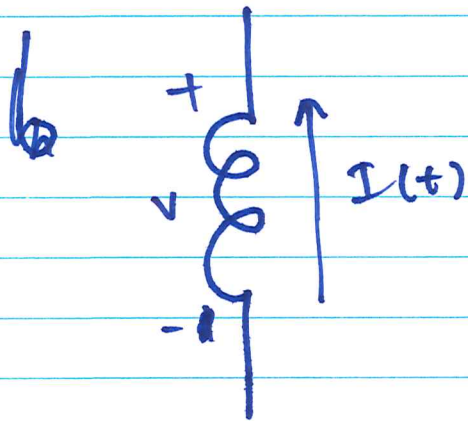$$V(t) \;\;\; +\!\!\!=\!\!\!\!= \;\; \uparrow i \;\; -$$

- it acts as voltage source with voltage $v(t)$.

- the change in voltage across the capacitor is proportional to the current through it.

$$\Rightarrow \quad \frac{dv(t)}{dt} = -\frac{i}{C}$$

where C is the capacitance of the capacitor.

* Inductor

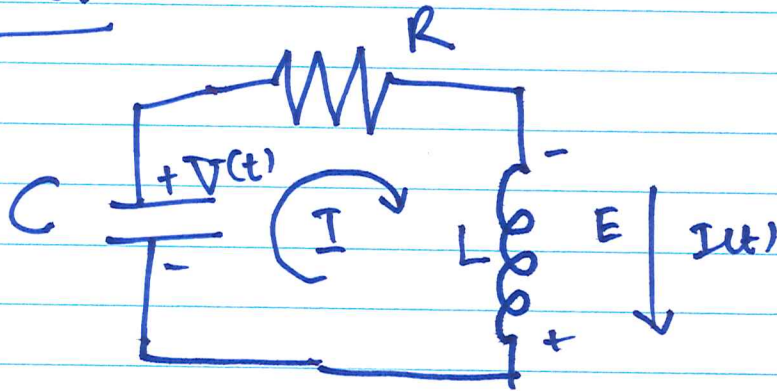* acts as a current source with current $I(t)$

* the change in current is proportional to
the voltage v across it

$$\Rightarrow \quad \frac{dI(t)}{dt} = -\frac{v}{L}$$

where L is the inductance of the inductor.

We want to determine how the sources of
voltage and current change over time. To do
this, we need to know the current through the
capacitors and the voltage across the inductor.
We use these information to set up system of
D.E equations # which will be solved to
determine the time evolution of $I(t)$ and $V(t)$.

Example: Consider the LCR circuit

$$IR - V - E = 0$$

Derive a system of differential equations for the $V(t)$ and $I(t)$ and determine how they change over time.

Solution

We know that

$$\frac{dV(t)}{dt} = -\frac{I(t)}{C}$$

and

$$\frac{dI(t)}{dt} = -\frac{E}{L}$$

$$V + E = IR \quad \Rightarrow \quad E = IR - V$$

$$\frac{dI}{dt} = -\frac{(IR - V)}{L} = \frac{V}{L} - \frac{IR}{L}$$

∴ our system of equations is

$$\frac{dV}{dt} = -I/C$$

$$\frac{dI}{dt} = \frac{V}{L} - \frac{IR}{L}$$

In matrix form,

$$\begin{pmatrix} V' \\ I' \end{pmatrix} = \begin{pmatrix} 0 & -1/C \\ \frac{1}{L} & -R/L \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}$$

Let $A = \begin{pmatrix} 0 & -1/C \\ \frac{1}{L} & -R/L \end{pmatrix}$

Let λ be an eigenvalue of A,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & -1/C \\ \dfrac{1}{L} & -\dfrac{R}{L} - \lambda \end{vmatrix} = 0$$

$$-\lambda\left(-\frac{R}{L} - \lambda\right) + \frac{1}{LC} = 0$$

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

using the quadratic formula,

$$\lambda_{1,2} = \frac{-\dfrac{R}{L} \pm \sqrt{\dfrac{R^2}{L^2} - \dfrac{4}{LC}}}{2}$$

$$= -\frac{R}{2L} \pm \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}$$

Case I : $R^2 - \dfrac{4L}{C} > 0$

$\Rightarrow$ λ_1 and λ_2 are real numbers.

$\Rightarrow$ $R^2 > \dfrac{4L}{C}$ $\Rightarrow$ $R > 2\sqrt{\dfrac{L}{C}}$

$\Rightarrow$ $\dfrac{R}{2}\sqrt{\dfrac{C}{L}} > 1$.

If R, C, and L satisfy the relation

$$\frac{R}{2} \cdot \sqrt{\frac{C}{L}} \overset{?}{\gtrless} 1 \qquad \qquad \textcircled{1}$$

then the eigenvalues of A are real numbers.

Case II: $R^2 - \frac{4L}{C} < 0$

$\Rightarrow \lambda_1$ and λ_2 are complex numbers.

$$R^2 - \frac{4L}{C} < 0$$

$$\Rightarrow R^2 < \frac{4L}{C}$$

$$R < 2\sqrt{\frac{L}{C}} \qquad \qquad \textcircled{2}$$

$$\Rightarrow \frac{R}{2}\sqrt{\frac{C}{L}} < 1$$
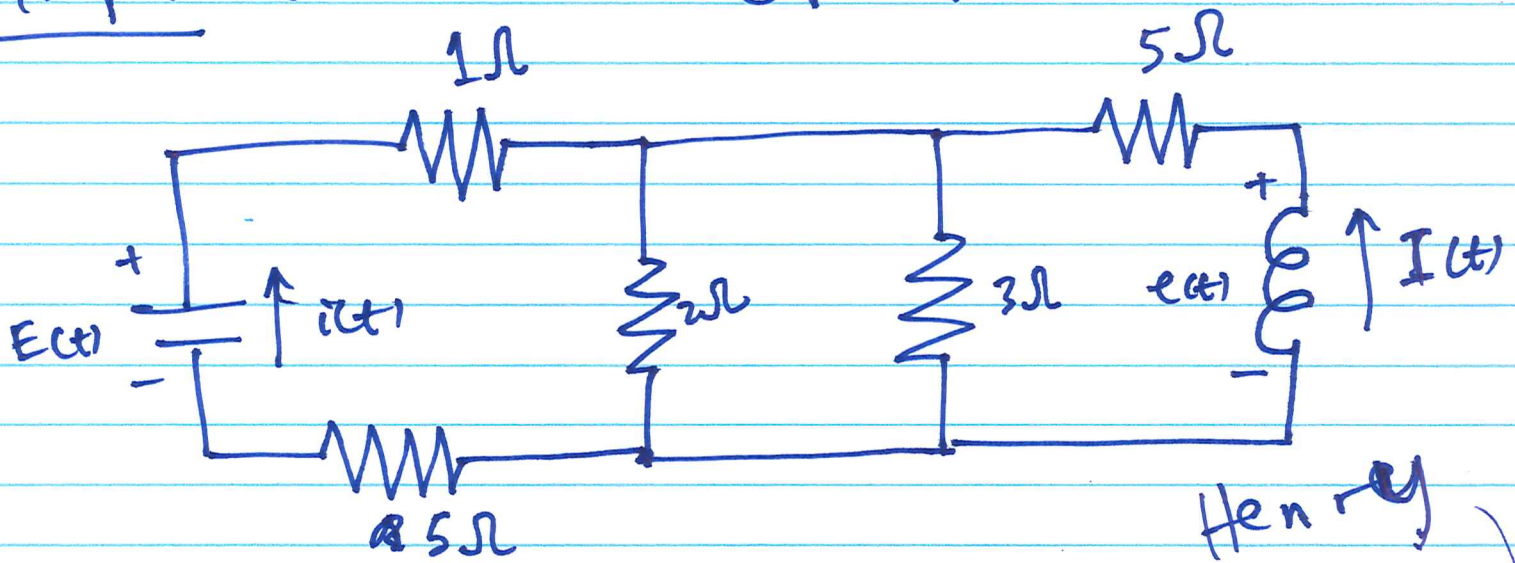
∴ If R, C, and L satisfy this relation, (2)

$V(t)$ and $I(t)$ will oscillate over time.

The general solution of the system is given by

$$\begin{pmatrix} V(t) \\ I(t) \end{pmatrix} = c_1 \vec{X_1} e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \vec{X_2}$$

and If we are given R, L, C, and the initial current and voltage, then we can find the constants c_1 and c_2.

Example: consider the circuit below

The capacitance of the capacitor is $C = 2$ Farad and the inductance of the inductor is $L = 2$ Henry

(a) Derive a system of equation for the circuit.

(i.e for $I(t)$ and $E(t)$)
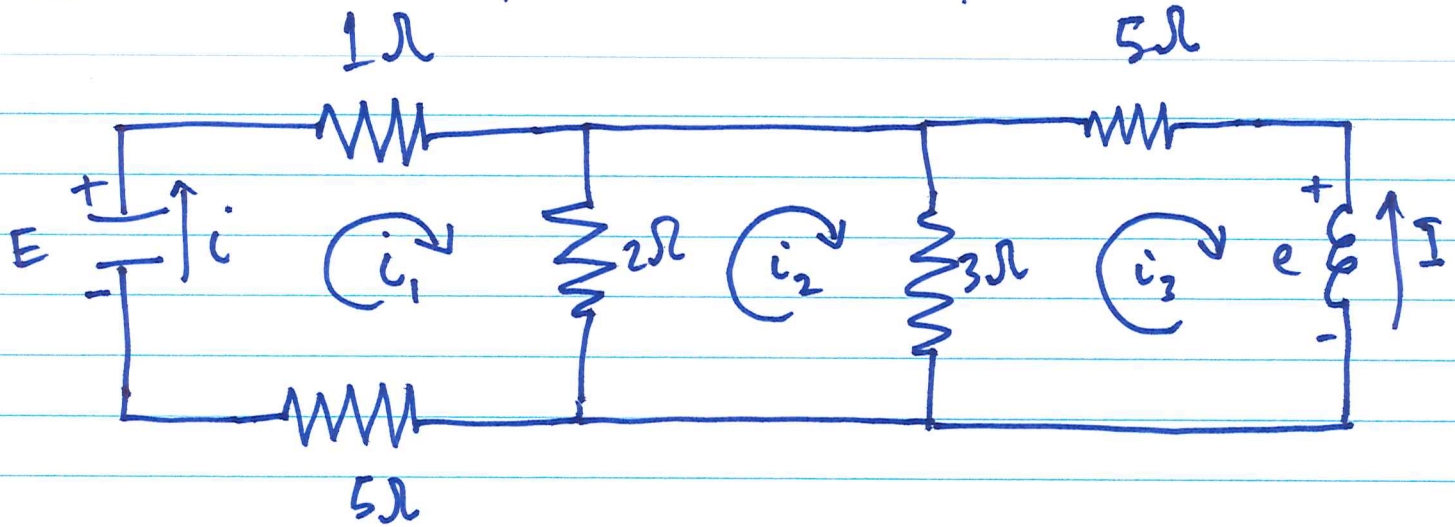
(b) Will $E(t)$ and $I(t)$ oscillate in time?

Recall,

$$\frac{dE(t)}{dt} = -\frac{i(t)}{C} \quad \text{and} \quad \frac{dI}{dt} = -\frac{e(t)}{L}$$

but $C = 2$ and $L = 2$,

$$\therefore \frac{dE}{dt} = -\frac{i}{2} \quad \text{and} \quad \frac{dI}{dt} = -\frac{e}{2}$$

Let us use loop current technique to find i and e.

From loop 1,

$$5i_1 + i_1 + 2(i_1 - i_2) = E$$

$$8i_1 - 2i_2 = E$$

but $i = i_1$

$$\Rightarrow \quad 8i - 2i_2 = E \quad \underline{\hspace{3cm}} \quad (1)$$

From loop 2,

$$2(i_2 - i_1) + 3(i_2 - i_3) = 0$$

$$-2i_1 + 5i_2 - 3i_3 = 0$$

$$\Rightarrow \quad -2i + 5i_2 - 3i_3 = 0 \quad \underline{\hspace{1.5cm}} \quad (2)$$

from loop 3,

$$3(i_3 - i_2) + 5i_3 = -e$$

$$-3i_2 + 8i_3 = -e$$

but $i_3 = -I$

$$\Rightarrow \quad -3i_2 + e = 8I \quad \text{————} \quad ③$$

use $i_3 = -I$ in ②,

$$-2i + 5i_2 = -3I$$

∴ our system of equations is

$$8i - 2i_2 \qquad = E$$

$$-2i + 5i_2 \qquad = -3I$$

$$\qquad -3i_2 + e = 8I$$

our unknowns are i, i_2, and e.

the augmented matrix is

$$\begin{bmatrix} 8 & -2 & 0 & \bigg| & E \\ -2 & 5 & 0 & \bigg| & -3I \\ 0 & -3 & 1 & \bigg| & 8I \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -\frac{1}{4} & 0 & \bigg| & E/8 \\ 0 & 1 & 0 & \bigg| & -\frac{2}{3}I + E/18 \\ 0 & 0 & 1 & \bigg| & 6I + E/6 \end{bmatrix}$$

$$\Rightarrow \quad e = 6I + E/6 \quad \underline{\hspace{4cm}} \quad \circledast$$

$$i_2 = -\frac{2}{3}I + \frac{E}{18}$$

$$i - \frac{1}{4}i_2 = E/8$$

$$i = \frac{E}{8} + \frac{1}{4}i_2 = \frac{E}{8} + \frac{1}{4}\left(-\frac{2}{3}I + \frac{E}{18}\right)$$

$$i = \frac{5E}{36} - \frac{1}{6}I \quad \underline{\hspace{4cm}} \quad (\ast\ast)$$

Recall that our system of differential equations
is

$$\frac{dE}{dt} = -i/2 \quad \text{and} \quad \frac{dI}{dt} = -e/2$$

Substituting (*) and (**), we have

$$\frac{dE}{dt} = -\frac{5}{72}E + \frac{1}{12}I$$

$$\frac{dI}{dt} = \frac{-E}{12} - 3I$$

In matrix form,

$$\begin{pmatrix} E' \\ I' \end{pmatrix} = \begin{pmatrix} -5/72 & 1/12 \\ -1/12 & -3 \end{pmatrix} \begin{pmatrix} E \\ I \end{pmatrix}$$

Let

$$A = \begin{pmatrix} -5/72 & 1/12 \\ -1/12 & -3 \end{pmatrix}$$

If λ is an eigenvalue of A,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\frac{5}{72} - \lambda & \frac{1}{12} \\ -\frac{1}{12} & -3 - \lambda \end{vmatrix} = 0$$

$$(-3 - \lambda)\left(-\frac{5}{72} - \lambda\right) + \frac{1}{144} = 0$$

$$\lambda^2 + \frac{221}{72}\lambda + \frac{31}{144} = 0$$

$$\lambda_{1,2} = \frac{-\frac{221}{72} \pm \sqrt{\left(\frac{221}{72}\right)^2 - \left(\frac{31}{144}\right)4}}{2}$$

$$\lambda_{1,2} = \frac{-221}{2(72)} \pm \frac{1}{2}\sqrt{\frac{(221)^2}{72^2} - \frac{31}{36}}$$

λ_1, and λ_2 are negative real numbers, therefore, the solution of the system will not oscillate.

⇒ Since $\lambda_1, \lambda_2 < 0$, $E(t)$ and $I(t)$ will decay as $t \longrightarrow \infty$.