Problem Sheet 10: The master equation approach in quantum physics: Numerical simulations of Lindblad equation (22 January 2019)

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Exercise 22. [6p] Population and coherence dynamics in a two-site system.

We consider a two-site system where site k is characterized by its ground and excited states, $|g_k\rangle$ and $|e_k\rangle$, with the energy-level Ω

$$H = \frac{\Omega}{2}\sigma_z^{(1)} + \frac{\Omega}{2}\sigma_z^{(2)} + J(\sigma_+^{(1)}\sigma_-^{(2)} + \sigma_+^{(2)}\sigma_-^{(1)}), \tag{1}$$

where $\sigma_z^{(k)} = |e_k\rangle\langle e_k| - |g_k\rangle\langle g_k|$, $\sigma_+^{(k)} = |e_k\rangle\langle g_k|$ and $\sigma_-^{(k)} = |g_k\rangle\langle e_k|$. J describes the coupling between sites, which induces a population transfer and generates inter-site coherences. We consider Markovian noise described by the Lindbladd equation in the form

$$\frac{d\rho(t)}{dt} = -i[H,\rho(t)] + \sum_{k=1}^{2}\frac{\gamma}{4}\left(\sigma_z^{(k)}\rho(t)\sigma_z^{(k)} - \frac{1}{2}\left\{\left(\sigma_z^{(k)}\right)^2, \rho(t)\right\}\right), \tag{2}$$

where γ denotes a local dephasing rate. We assume that an excitation is initially localized at the first site, $\rho(0) = |e_1, g_2\rangle\langle e_1, g_2|$.

(a) [3p] Compute the dynamics of the population of the first site, defined by $P_1(t) = \sum_{j=g,e}\langle e_1, j_2|\rho(t)|e_1, j_2\rangle$, and inter-site coherence, $C_{12}(t) = \langle e_1, g_2|\rho(t)|g_1, e_2\rangle$, in the absence of the noise. Consider $J = 10$ and $\gamma = 0$ and perform simulations up to $t \approx 1$.

(b) [3p] Compute the population and coherence dynamics in the presence of the noise. Consider $J = 10$ and $\gamma \in \{1, 10, 100, 1000\}$ and check how the oscillatory features in system dynamics are affected by the noise.

hint: Note that the Hamiltonian and Markovian noise do not change the number of excitations within the system. This means that it is sufficient to consider the single excitation subspace spanned by $|1\rangle = |e_1, g_2\rangle$ and $|2\rangle = |g_1, e_2\rangle$, where the Hamiltonian and noise operators are reduced to

$$H' = \Omega|1\rangle\langle 1| + \Omega|2\rangle\langle 2| + J(|1\rangle\langle 2| + |2\rangle\langle 1|), \tag{3}$$

$$\frac{d\rho(t)}{dt} = -i[H',\rho(t)] + \sum_{k=1}^{2}\gamma\left(|k\rangle\langle k|\rho(t)|k\rangle\langle k| - \frac{1}{2}\{|k\rangle\langle k|, \rho(t)\}\right), \tag{4}$$

where the initial state is reduced to $\rho(0) = |1\rangle\langle 1|$. Note that when the two sites have the same energy-level Ω, the value of Ω does not affect the time evolution of the reduced system density matrix $\rho(t)$.

Exercise 23. [6p] Classical and quantum random walk.

We consider a linear chain consisting of 101 sites where the Hamiltonian and noise operators in the single excitation subspace are described by

$$H' = \sum_{k=1}^{101}\Omega|k\rangle\langle k| + \sum_{k=1}^{100}J(|k\rangle\langle k+1| + |k+1\rangle\langle k|), \tag{5}$$

$$\frac{d\rho(t)}{dt} = -i[H',\rho(t)] + \sum_{k=1}^{101}\gamma\left(|k\rangle\langle k|\rho(t)|k\rangle\langle k| - \frac{1}{2}\{|k\rangle\langle k|, \rho(t)\}\right), \tag{6}$$

(a) [3p] We assume that an excitation is initially localized at the 51-th site, namely at the center of the chain. Compute the time evolution of the mean square displacement $\sqrt{\langle x^2(t)\rangle}$ from the initial site, defined by $\langle x^2(t)\rangle = \sum_{k=1}^{101} P_k(t)x_k^2 = \sum_{k=1}^{101} P_k(t)(k-51)^2$ where $x_k = (k-51)$ quantifies the degree of displacement and $P_k(t) = \langle k|\rho(t)|k\rangle$ denotes the population of site k. Show that in the absence of the noise, the mean square displacement linearly increases in time, $\sqrt{\langle x^2(t)\rangle} \propto t$. Consider $J = 10$ and $\gamma = 0$ and perform simulations up to $t \approx 1$. Note that for $t \gg 1$, the excitation can be reflected at the boundaries of the chain.

(b) [3p] Compute the mean square displacement in the presence of the noise. Consider $J = 10$ and $\gamma \in \{1, 10, 100, 1000\}$. Show that as the dephasing rate γ increases, the mean square displacement starts to show a quadratic behavior, $\sqrt{\langle x^2(t)\rangle} \propto \sqrt{t}$. This implies that the quantum speed-up in transport, $\sqrt{\langle x^2(t)\rangle} \propto t$, is suppressed by the noise and starts to show the classical behavior, $\sqrt{\langle x^2(t)\rangle} \propto \sqrt{t}$, for sufficiently high dephasing rates $\gamma \gg J$.