Inverse problems for linear and non-linear elliptic equations

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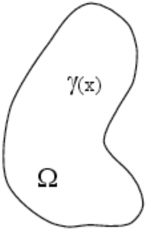
Outline of talk

1. Introduction
2. Cloaking in electrostatics model
3. Polyharmonic first order perturbation

Calderón problem



Consider a body $\Omega \subset \mathbb{R}^N$, $N \geq 2$ with conductivity $\gamma(x)$ where $x \in \Omega$. An electrical potential $u(x)$ causes the current $I(x) = \gamma(x)\nabla u(x)$.

The conductivity $\gamma(x)$ can be isotropic (scalar valued), or anisotropic (matrix valued).

If the current has no sources or sinks, we have

$$-div(\gamma(x)\nabla u(x)) = 0 \text{ in } \Omega.$$

Dirichlet-to-Neumann map

- Consider the following boundary value problem

$$-div(\gamma(x)\nabla u(x)) = 0 \text{ in } \Omega,$$
$$u = f \text{ on } \partial\Omega.$$

  $\gamma(x)$ = conductivity,
  f = voltage potential at $\partial\Omega$.

- Current flux at $\partial\Omega = (\nu \cdot \gamma\nabla u)\big|_{\partial\Omega}$ were ν is the unit outer normal.

  Information is encoded in map $\boxed{\Lambda_\gamma(f) = \nu \cdot \gamma\nabla u\big|_{\partial\Omega}}$

- Inverse problem

$$\text{Does } \Lambda_\gamma \text{ determine } \gamma?$$

  Λ_γ = Dirichlet-to-Neumann map

- **Different aspects of an inverse problem**
  - Uniqueness
  - Stability
  - Reconstruction techniques
  - Numerical implementation
  - Partial data problems
- Interior uniqueness: Does $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ imply $\gamma_1 = \gamma_2$ in Ω?
- Yes, if γ_i are scalar valued. First answered in 1987 for $N \geq 3$ by Sylvester and Uhlmann for C^2 conductivities.
- Mathematics of the problem for $N = 2$ is quite different.

Past work: Scalar case

- Uniqueness result was generalized for $N \geq 3$ by Brown and Torres in 1996 for $\gamma \in W^{\frac{3}{2},p}$ with $p > 2N$, Haberman and Tataru in 2013 for $\gamma \in C^1$ and Caro and Rogers in 2016 for $\gamma \in C^{0,1}$.
- For $N = 2$, the global uniqueness problem was first solved by Nachman in 1996 for $\gamma \in W^{2,p}$ for $p > 1$.
- This result was completely generalized by Astala and Päivärinta in 2005 for $\gamma \in L^{\infty}$.

Diffeomorphism: Anisotropic conductivities

- No for anisotropic conductivities. Choose a smooth diffeomorphism $\Phi : \Omega \to \Omega$ such that $\Phi(x) = x$ on $\partial\Omega$ and define

$$\Phi_* \gamma(y) = \frac{D\Phi(x)^T \gamma(x) D\Phi(x)}{|D\Phi|} \circ \Phi^{-1}(y),$$

  where $\Phi(x) = y$.
- $\Phi^* \gamma(y)$ is the push-forward of the conductivity γ by Φ. Since $\Phi = Id$ on $\partial\Omega$, we get

$$\Lambda_\gamma = \Lambda_{\Phi_* \gamma}.$$

- Such a restricted uniqueness was first shown in 1989 by Lee and Uhlmann for $N \geq 3$ for real analytic conductivities, and in 2006 by Astala, Lassas and Päivärinta for $N = 2$ for $\gamma \in L^\infty$.
- Uniqueness up to diffeomorphism holds only when we have upper and lower bounds for γ.

Outline of talk

Three key ideas

There are 3 key ideas associated with such a *non-uniqueness* result:

1. Cloaking
2. Approximate cloaking
3. Isotropic approximate cloaking

Cloaking

- Let $E \subset \Omega$ be fixed and let σ_c be a non negative matrix valued function defined on $\Omega \setminus E$. We say σ_c *cloaks* E if any extension of σ_c across E of following form,

$$\sigma_A(x) = \begin{cases} A(x) & x \in E, \\ \sigma_c(x) & x \in \Omega \setminus E, \end{cases}$$

  produces the same DN map as a uniform isotropic region, irrespective of the choice of $A(x)$.

- Existence of such cloaks first shown by Greenleaf, Lassas and Uhlmann in 2003.

- A change-of-variable scheme is employed which essentially blows up a point to a region in space and is highly singular.

Approximate cloaking

- Consider a regularized change-of-variable scheme which blows a small ball to the region being cloaked.
- One can look at the asymptotic behavior as radius of the ball goes to 0 and recover the singular transform.
- For simplicity, let $\Omega = B_2$ and restrict our attention the case when $E = B_1$ needs to be nearly cloaked.
- Fix a small parameter $r > 0$.

Regular change of variables

- Consider $F^r : B_2 \to B_2$ such that

$$F^r(x) = \begin{cases} \frac{x}{r} & |x| \le r, \\ \left(\frac{2-2r}{2-r} + \frac{1}{2-r}|x|\right)\frac{x}{|x|} & r \le |x| \le 2. \end{cases}$$

- Consider

$$\sigma_A^r(x) = \begin{cases} A(x) & x \in B_1, \\ F_*^r 1 & x \in B_2 \setminus B_1. \end{cases}$$

- By approximate cloaking, we mean

$$|\langle \Lambda_{\sigma_A^r} f, g\rangle - \langle \Lambda_1 f, g\rangle| = \mathrm{o}(1)||f||_{H^{\frac{1}{2}}(\partial B_2)}||g||_{H^{\frac{1}{2}}(\partial B_2)},$$

where the $\mathrm{o}(1)$ term is independent of f and g.

Isotropic approximate cloaking

- The approximate cloaks using regular change of variables are anisotropic.
- In 2008, Greenleaf, Kurylev, Lassas and Uhlmann constructed isotropic and nonsingular parameters that give *approximate* cloaking to any desired degree of accuracy.
- They used the notion of H-convergence in linear settings to construct such isotropic cloaks.
- The H-limit of a sequence of isotropic cloaks need not be isotropic, and this key property allowed them to construct isotropic approximate cloaks.

Our work

1. We extend previously known results in electrostatics on cloaking, approximate cloaking and isotropic approximate cloaking to a *quasi-linear* operator.
2. This is achieved using the same techniques:
    - singular change of variables for cloaking,
    - regular change of variables for approximate cloaking &
    - H-convergence for isotropic approximate cloaking.

Basic Set up: quasi-linear equation

- Consider
$$\begin{aligned} -div(A(x,u(x))\nabla u(x)) &= 0 \text{ in } \Omega, \\ u &= f \text{ on } \partial\Omega, \end{aligned}$$
where $A(x,t) \in \mathcal{M}(\alpha, \beta, L; B_2 \times \mathbb{R})$.
- The above equation arises in modeling of thermal conductivity of the Earth's crust and heat conduction in composite materials.
- More specifically, steady state heat conduction in an inhomogeneous anisotropic nonlinear medium with Dirichlet boundary conditions is governed by the above equation.

Quasi-linear approximate cloaking: [Ghosh, I '18]

- Consider
$$\sigma^r_A(x,t) = \begin{cases} A(x,t) & (x,t) \in B_1 \times \mathbb{R}, \\ F^r_* 1 & (x,t) \in B_2 \setminus B_1 \times \mathbb{R}, \end{cases}$$
where $A(x,t) \in \mathcal{M}(\alpha, \beta, L; B_2 \times \mathbb{R})$.

- We show
$$|\langle \Lambda_{(F^r)^{-1}_* \sigma^r_A} f, g\rangle - \langle \Lambda_1 f, g\rangle| = \mathrm{o}(1) ||f||_{H^{\frac{1}{2}}(\partial B_2)} ||g||_{H^{\frac{1}{2}}(\partial B_2)},$$
where
$$(F^r)^{-1}_* \sigma^r_A = \begin{cases} (F^r)^{-1}_* A = \widetilde{A}^r(x,t) & (x,t) \in B_r \times \mathbb{R}, \\ 1 & (x,t) \in (B_2 \setminus B_r) \times \mathbb{R}. \end{cases}$$

Sketch of proof

- If $A \in \mathcal{M}(\alpha, \beta, L; B_1 \times \mathbb{R})$, then $\widetilde{A}^r(x,t) \in \mathcal{M}(\frac{\alpha}{r^{N-2}}, \frac{\beta}{r^{N-2}}, L; B_r \times \mathbb{R})$.
- This implies that for $r \ll 1, (F^r)_*^{-1}\sigma_A^r \in \mathcal{M}(1, \frac{\beta}{r^{N-2}}, L; B_r \times \mathbb{R})$.
- This gives us

$$|\langle \Lambda_{(F^r)_*^{-1}\sigma_A^r} f, g\rangle - \langle \Lambda_1 f, g\rangle| \leq C r^{-\frac{N}{2}+2} ||f||_{H^{\frac{1}{2}}(\partial B_2)} ||g||_{H^{\frac{1}{2}}(\partial B_2)}.$$

- Pass to the limit as $r \to 0$ for perfect cloaking.

Homogenization set up

- The approximate cloaks discussed earlier are anisotropic. We will now construct *isotropic* approximate cloaks.

- Set up: Let $A(x, y, t) = [a_{ij}(x, y, t)] \in \mathcal{M}(\alpha, \beta, L; \Omega \times Y \times \mathbb{R})$ be such that

  $y \mapsto a_{ij}(x, y, t)$ are $Y = [0, 1]^N$-periodic functions for a.e $(x, t) \in \Omega \times \mathbb{R}$.

  Let
$$A^\epsilon(x, t) = \left[a_{ij}(x, \frac{x}{\epsilon}, t)\right], \quad (x, t) \in \Omega \times \mathbb{R}.$$

Quasi-linear H-convergence: [Ghosh, I '18]

Theorem

Let A^ϵ and A^* belong to $\mathcal{M}(\alpha, \beta, L; \Omega \times \mathbb{R})$. We say $A^\epsilon \xrightarrow{H} A^*$, if up to a subsequence, the corresponding solutions $\{u^\epsilon\}$ to

$$\begin{aligned} -div\left(A^\epsilon(x, u^\epsilon)\nabla u^\epsilon(x)\right) &= 0 \text{ in } \Omega, \\ u^\epsilon &= f \in H^{\frac{1}{2}}(\partial\Omega), \end{aligned}$$

are such that

$$\begin{aligned} u^\epsilon \rightharpoonup u \quad &\text{weakly in } H^1(\Omega) \text{ and} \\ A^\epsilon(x, u^\epsilon)\nabla u^\epsilon \rightharpoonup A^*(x, u)\nabla u \quad &\text{weakly in } L^2(\Omega)^N, \end{aligned}$$

where $u \in H^1(\Omega)$ uniquely solves

$$\begin{aligned} -div\left(A^*(x, u(x))\nabla u(x)\right) &= 0 \text{ in } \Omega, \\ u &= f \text{ on } \partial\Omega. \end{aligned}$$

H-limit

The homogenized conductivity $A^*(x,t)$

$$A^*(x,t) = [a^*_{ij}(x,t)] \in \mathcal{M}(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{L}; \Omega \times \mathbb{R}),$$

can be defined by its entries as

$$a^*_{kl}(x,t) = \int_Y a_{ij}(x,y,t) \frac{\partial}{\partial y_i}(\chi_k(x,y,t) + y_k) \frac{\partial}{\partial y_j}(\chi_l(x,y,t) + y_l) dy,$$

where for each canonical basis vector $e_k \in \mathbb{R}^N$, $\chi_k(x,y,t)$ satisfy the following cell-problem for almost every $(x,t) \in \Omega \times \mathbb{R}$:

$$\begin{aligned} - div_y \; A(x,y,t)(\nabla_y \chi_k(x,y,t) + e_k) = 0 \quad \text{in } \mathbb{R}^N, \\ y \to \chi_k(x,y,t) \quad \text{is } Y\text{-periodic for all } (x,t) \in \Omega \times \mathbb{R}. \end{aligned}$$

Isotropic approximate cloaking

- Our isotropic cloaks take the form

$$A^{\epsilon}(x,t) = \sigma\left(x, \frac{|x|}{\epsilon}, t\right) I_{N\times N}, \quad (x,t) \in \Omega \times \mathbb{R}.$$

- Temporarily fix $R > 1$ and introduce a new parameter $\eta > 0$.

- More specifically,

$$\sigma_n(x,t) := \left[1 + a^1_{R_n,\eta_n}(x,t)\zeta_1\left(\frac{r}{\varepsilon_n}\right) - a^2_{R_n,\eta_n}(x,t)\zeta_2\left(\frac{r}{\varepsilon_n}\right)\right]^2, \quad r = |x|,$$

where a_1, a_2, ζ_1, ζ_2 are chosen to satisfy Lipschitz condition in t variable and periodicity in r variable.

Concluding steps

- We first let $\epsilon_n \to 0$ (approximate isotropic $\to$ approximate anisotropic).
- Then let $\eta_n \to 0$ and finally $R_n \searrow 1$
(approximate anisotropic $\to$ cloaking).
- We get the desired strong convergence of the DN maps,

$$||(\Lambda_{\sigma_n} - \Lambda_1)(f)||_{H^{\frac{1}{2}}(\partial\Omega)} \to 0 \text{ as } n \to \infty.$$

- This finishes the proof.

Outline of talk

Preliminaries

- Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a bounded open set with C^∞ boundary.
- Consider the operator $\mathcal{H}_{A,q} = -\Delta + A \cdot D + q$, where A is a vector-valued potential and q is a scalar valued potential.
- If 0 is not a Dirichlet eigenvalue of $\mathcal{H}_{A,q}$, then we can define the DN map $\Lambda^{\mathcal{H}}_{A,q}$.
- Research question Are A and q uniquely determined by $\Lambda^{\mathcal{H}}_{A,q}$?
- No. When $A \neq 0$, there is a natural obstruction to uniqueness.

Counter example

- Consider the magnetic Schrödinger operator

$$\mathcal{M}_{A,q} = \sum_{j=1}^{N} (-i\frac{\partial}{\partial x_j} + A_j(x))^2 + q(x).$$

- Let $A' = A + \nabla g$, where $g = \frac{\partial g}{\partial N} = 0$ on $\partial\Omega$.

- $\Lambda^{\mathcal{M}}_{A',q} = \Lambda^{\mathcal{M}}_{A,q}$

- Research question: Is this the only obstruction to uniqueness?

- Yes. In 1993, Sun proved such a restricted uniqueness result for $A \in W^{2,\infty}(\mathbb{R}^N) \cap \mathcal{E}'(\bar{\Omega})$ and $q \in L^\infty(\Omega)$.

Polyharmonic operator

- Surprisingly, for the polyharmonic operator

$$\mathcal{L}_{A,q} = (-\Delta)^m + A \cdot D + q, \quad m \geq 2,$$

  one can uniquely recover both A and q.

- This result was first proved by Krupchyk, Lassas and Uhlmann in 2014, for $A \in W^{1,\infty}(\mathbb{R}^N) \cap \mathcal{E}'(\bar{\Omega})$ and $q \in L^\infty(\Omega)$.

- Higher order polyharmonic operators occur in areas of physics and geometry such as
  - Kirchoff plate equation in the theory of elasticity.
  - Paneitz-Branson operator in conformal geometry.

Definition of A and q

- Let first order perturbation A be in $W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N) \cap \mathcal{E}'(\bar{\Omega})$, where

$$\begin{cases} p' \in [2N/m, \infty) & \text{if} \quad m < N, \\ p' \in (2, \infty) & \text{if} \quad m = N \quad \text{or} \quad m = N+2, \\ p' \in [2, \infty) & \text{otherwise.} \end{cases} \tag{A}$$

- For a fixed δ with $0 < \delta < \frac{1}{2}$, let the zeroth order perturbation q be in $W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N) \cap \mathcal{E}'(\bar{\Omega})$, where

$$\begin{cases} r' \in [2N/(m-2\delta), \infty), & \text{if} \quad m < N, \\ r' \in [2N/(m-2\delta), \infty), & \text{if} \quad m = N, \\ r' \in [2, \infty), & \text{if} \quad m \geq N+1. \end{cases} \tag{q}$$

Main Result

Theorem (Assylbekov-I '17)

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a bounded open set with C^∞ boundary, and let $m \geq 2$ be an integer. Let $0 < \delta < 1/2$. Suppose that A_1, A_2 satisfy (A) *and q_1, q_2 satisfy* (q) *and 0 is not in the spectrum of $\mathcal{L}_{A_1,q_1}$ and $\mathcal{L}_{A_2,q_2}$.*

If $\Lambda^{\mathcal{L}}_{A_1,q_1} = \Lambda^{\mathcal{L}}_{A_2,q_2}$, then $A_1 = A_2$ and $q_1 = q_2$.

Key steps

We use the following two key steps in the proof:

- Carleman estimates
- Construction of Complex Geometric Optics (CGO) solutions to $\mathcal{L}_{A,q}$.

Carleman estimates

- Carleman estimates : For $0 < h \ll 1$, we have

$$||u||_{H^{m/2}_{scl}(\mathbb{R}^N)} \lesssim \frac{1}{h^m} ||e^{\phi/h}(h^{2m}\mathcal{L}_{A,q})e^{-\phi/h}u||_{H^{-3m/2}_{scl}(\mathbb{R}^N)}$$

for all $u \in C_0^\infty(\Omega)$.

- Here, ϕ is the so-called Limiting Carleman Weight for $-h^2\Delta$ and $H^s_{scl}(\mathbb{R}^N)$ are weighted Sobolev spaces.

- The two key ingredients in the proof of the Carleman estimates are:

  - Estimates for the Laplacian due to Salo and Tzou with a gain of two derivatives and

  - Continuity of multiplication between two Sobolev spaces.

CGO solutions

Proposition

Let $\zeta \in \mathbb{C}^N$ and $h > 0$ be such that $\zeta \cdot \zeta = 0$, $\zeta = \zeta_0 + \zeta_1$ with ζ_0 independent of h and $\zeta_1 = \mathcal{O}(h)$ as $h \to 0$. For all $h > 0$ small enough, there exists $u(x, \zeta; h) \in H^{m/2}(\Omega)$ solving $\mathcal{L}_{A,q} u = 0$, and of the form

$$u(x, \zeta; h) = e^{\frac{ix \cdot \zeta}{h}} (a(x, \zeta_0) + h^{m/2} r(x, \zeta; h)),$$

where $a(\cdot, \zeta_0) \in C^\infty(\overline{\Omega})$ satisfies

$$(\zeta_0 \cdot \nabla)^2 a = 0 \quad in \quad \Omega,$$

and the correction term r is such that $||r||_{H^{m/2}_{scl}(\Omega)} = \mathcal{O}(1)$ as $h \to 0$.

Concluding steps

- Construct $u_2(\cdot, \zeta_2; h)$ and$v(\cdot, \zeta_1; h)$ in $H^{\frac{m}{2}}(\Omega)$ solving $\mathcal{L}_{A_2,q_2} u_2 = 0$ and $\mathcal{L}^*_{A_1,q_1} v = 0$ in Ω and plug them in to the following integral identity,

$$\int_\Omega ((A_2 - A_1) \cdot Du_2)\bar{v}\, dx + \int_\Omega (q_2 - q_1) u_2 \bar{v}\, dx = 0.$$

- First let $h \to 0$ and obtain $A_1 = A_2$.
- Now plug in $A_1 = A_2$ and $a_1 = a_2 = 1$ into the integral identity, and let $h \to 0$ to get $q_1 = q_2$, which finishes the proof.
- Remark: Obtain conditions on A and q by working backwards from construction of CGO solutions.

Summary

- We looked at the injectivity of the DN map for two different PDEs, one quasi-linear and one linear.
- For the quasi-linear PDE, the DN map is not injective. Uniqueness up to diffeomorphism fails in certain cases and this gives a recipe for *cloaking*.
- We used the same ideas as in the linear settings, to give a scheme for cloaking, approximate cloaking and isotropic approximate cloaking in quasi-linear settings.
- For the linear poly-harmonic PDE, the DN map is injective and goal of our work was to prove injectivity of the DN map for rough A and q.
- Our main contribution is Carleman estimates and construction of CGO solutions with proper decay of remainder term.