Question 4 [15 marks]

Determine whether each of the following statements is true or false. You need to **justify** your answers.

(a) Let $\boldsymbol{A}, \boldsymbol{B}$ be $m \times n$ and $n \times m$ matrices respectively. If $m > n$, then $\boldsymbol{AB}$ is singular.

(b) Let $\boldsymbol{A}, \boldsymbol{B}$ be $m \times n$ and $n \times m$ matrices respectively. If $m < n$, then $\boldsymbol{AB}$ is singular.

(c) Let $\boldsymbol{A}$ be a square matrix. If $\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{A} = \boldsymbol{0}$, then $\boldsymbol{A}$ is singular.

(d) Let $\boldsymbol{A}$ be a square matrix. If $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{I}$ and $\det(\boldsymbol{A}) < 0$, then $\boldsymbol{A} + \boldsymbol{I}$ is singular.

(e) Let $\boldsymbol{A}$ be a square matrix. If $\mathbf{adj}(\mathbf{adj}(\mathbf{adj}(\boldsymbol{A}))) = \boldsymbol{0}$, then $\boldsymbol{A}$ is singular.

(a) True. Let $A' = \left[\begin{array}{c:c} A & 0 \end{array}\right]$ and $B' = \left[\begin{array}{c} B \\ \hdashline 0 \end{array}\right]$

s.t. A' and B' are $m \times m$ matrices.

Note that $AB = A'B'$.

Since A' has a 0-column (or equivalently, B' has a 0-row), $\det(AB) = \det(A'B') = \det(A')\det(B') = 0$

$\Rightarrow AB$ is singular.

(b) False. For example:

$A = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$AB = [2]$. It is invertible.

(c) False. Example:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(d) True.

$$\det(AA^T) = \det(A)\det(A^T) = \det(A)^2 = 1$$

$$\Rightarrow \det(A) = -1.$$

$$A + I = A + AA^T = A(I + A^T)$$

so $\det(A+I) = \det(A)\det(A^T+I)$

Note that $\det(A^T+I) = \det(A+I)$ because $A^T+I = (A+I)^T$

Thus $\det(A+I) = -\det(A+I)$

$$\Rightarrow \det(A+I) = 0.$$

P.S. I think this part is a little tricky & challenging as we have not learnt about eigenvalues. Don't worry too much about it!

(e) True. If A is inv. then so is $\operatorname{adj}(A)$, $\operatorname{adj}(\operatorname{adj}(A))$, $\operatorname{adj}(\operatorname{adj}(\operatorname{adj}(A)))$. $\Rightarrow \det(\operatorname{adj}(\operatorname{adj}(\operatorname{adj}(A)))) \neq 0$

$\Rightarrow$ contradiction. So A is singular.