32. (All vectors in this question are written as column vectors.) Let $\boldsymbol{A}$ be an orthogonal matrix of order n and let $\boldsymbol{u}, \boldsymbol{v}$ be any two vectors in $\mathbb{R}^n$. Show that

(a) $\|\boldsymbol{u}\| = \|\boldsymbol{Au}\|$;

(b) $d(\boldsymbol{u}, \boldsymbol{v}) = d(\boldsymbol{Au}, \boldsymbol{Av})$; and

(c) the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is equal to the angle between $\boldsymbol{Au}$ and $\boldsymbol{Av}$.

a) $\|Au\|^2 = (Au)^T(Au) = u^T A^T A u = u^T u = \|u\|^2$

Since both $\|u\|$ and $\|Au\|$ are non negative, we have $\|Au\| = \|u\|$

b) $d(Au, Av) = \|Av - Au\| = \|A(u-v)\| = \|u-v\| = d(u,v)$

c) $(Au).(Av) = (Au)^T Av = u^T A^T Av = u^T v = u.v$

So the angle between u and v $= \cos^{-1}\left(\dfrac{u \cdot v}{\|u\|\|v\|}\right)$

$= \cos^{-1}\left(\dfrac{(Au)\cdot(Av)}{\|Au\|\|Av\|}\right)$

$=$ the angle between Au and Av

33. (All vectors in this question are written as column vectors.) Let $\boldsymbol{A}$ be an orthogonal matrix of order n and let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$.

(a) Show that $T = \{\boldsymbol{Au}_1, \boldsymbol{Au}_2, \ldots, \boldsymbol{Au}_n\}$ is a basis for $\mathbb{R}^n$.

(b) If S is orthogonal, show that T is orthogonal.

(c) If S is orthonormal, is T orthonormal?

a) Since A is invertible, T is linearly independent.

So T is a basis for $\mathbb{R}^n$

b) See 32

c) Yes

Exercise 6

1. For each of the following,

(i) find the characteristic equation of $\boldsymbol{A}$;

(ii) find all the eigenvalues of $\boldsymbol{A}$; and

(iii) find a basis for the eigenspace associated with each eigenvalues of $\boldsymbol{A}$.

(a) $\boldsymbol{A} = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$,

(b) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$,

(c) $\boldsymbol{A} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$,

(d) $\boldsymbol{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

(e) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$,

(f) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 9 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$,

(g) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$,

(h) $\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 1 \\ -2 & 2 & 1 \\ 2 & -1 & 0 \end{pmatrix}$,

(i) $\boldsymbol{A} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$,

(j) $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

h) $\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & 1 & 1 \\ -2 & \lambda-2 & 1 \\ 2 & -1 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda+1 & 1 & 1 \\ -2 & \lambda-2 & 1 \\ 0 & \lambda-3 & \lambda+1 \end{vmatrix}$

$= \lambda+1 \begin{vmatrix} \lambda-2 & 1 \\ \lambda-3 & \lambda+1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 1 \\ \lambda-3 & \lambda+1 \end{vmatrix}$

$(\lambda+1)(\lambda-1)^2 = 0$

$\lambda = -1, 1$

4. Let $\boldsymbol{A}$ be a square matrix such that $\boldsymbol{A}^2 = \boldsymbol{A}$.

(a) Show that if λ is an eigenvalue of $\boldsymbol{A}$, then $\lambda = 0$ or 1.

(b) Find all 2×2 matrices $\boldsymbol{A}$ such that $\boldsymbol{A}^2 = \boldsymbol{A}$ and $\boldsymbol{A}$ has eigenvalues 0 and 1.

a) Let x be a eigenvector of A associated with λ, ie $Ax = \lambda x$ and x is a non zero vector. Then

$$A^2 = A \Rightarrow A^2 x = Ax \Rightarrow \lambda^2 x = \lambda x \Rightarrow \lambda(\lambda - 1)x = 0$$

Since x is non zero, $\lambda = 0$ or 1

b) Since A has 2 distinct eigenvalues, it is diagonalizable. Let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

be an invertible matrix such that $P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} ad & -ab \\ cd & -cd \end{pmatrix}$$ Where $ad - bc \neq 0$

We can simplify the expression to $A = \begin{pmatrix} r & s \\ t & 1-r \end{pmatrix}$ where $st = r(1-r)$

8. Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$ and let $\boldsymbol{A}$ be an $n \times n$ matrix such that $\boldsymbol{A}\boldsymbol{u}_i = \boldsymbol{u}_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $\boldsymbol{A}\boldsymbol{u}_n = \boldsymbol{0}$. Show that the only eigenvalue of $\boldsymbol{A}$ is 0 and find all the eigenvectors of $\boldsymbol{A}$.

Note that for $i = 1, 2, \ldots, n$, $\boldsymbol{A}^n \boldsymbol{u}_i = \boldsymbol{A}^{n-1} \boldsymbol{u}_{i+1} = \cdots = \boldsymbol{A}^i \boldsymbol{u}_n = \boldsymbol{0}$.

Let $\boldsymbol{v} \in \mathbb{R}^n$ be an eigenvector of $\boldsymbol{A}$ associated with eigenvalue λ, i.e. $\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$. Since $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ is a basis for $\mathbb{R}^n$,

$$\boldsymbol{v} = c_1 \boldsymbol{u}_1 + c_2 \boldsymbol{u}_2 + \cdots + c_n \boldsymbol{u}_n$$

for some $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Then

$$\boldsymbol{A}^n \boldsymbol{v} = c_1 \boldsymbol{A}^n \boldsymbol{u}_1 + c_2 \boldsymbol{A}^n \boldsymbol{u}_2 + \cdots + c_n \boldsymbol{A}^n \boldsymbol{u}_n = \boldsymbol{0}.$$

From the proof of Question 6.3(a), $\boldsymbol{A}^n \boldsymbol{v} = \lambda^n \boldsymbol{v}$. Since $\boldsymbol{v} \neq \boldsymbol{0}$, $\lambda = 0$. Hence we have shown that $\boldsymbol{A}$ has only one eigenvalue 0.

As $\lambda = 0$, we get $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}$. Then

$$\boldsymbol{0} = \boldsymbol{A}\boldsymbol{v} = c_1 \boldsymbol{A}\boldsymbol{u}_1 + c_2 \boldsymbol{A}\boldsymbol{u}_2 + \cdots + c_n \boldsymbol{A}\boldsymbol{u}_n = c_1 \boldsymbol{u}_2 + c_2 \boldsymbol{u}_3 + \cdots + c_{n-1} \boldsymbol{u}_n.$$

Since $\boldsymbol{u}_2, \boldsymbol{u}_3, \ldots, \boldsymbol{u}_n$ are linearly independent, $c_1 = 0$, $c_2 = 0$, $\ldots$, $c_{n-1} = 0$, i.e. $\boldsymbol{v} = c_n \boldsymbol{u}_n$. Hence all eigenvectors of $\boldsymbol{A}$ are scalar multiples of $\boldsymbol{u}_n$.