Question 1 **[20 marks]**

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ a & b & c & d \end{pmatrix}$ be a 4×4 matrix where a, b, c, d are some real numbers.

(i) (4 marks) Find $\det \boldsymbol{A}$ and write down the condition in terms of a, b, c, d such that the homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$ has non-trivial solutions.

(ii) (4 marks) Let $S = \{(a, b, c, d) \mid \boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution$\}$. Is S a subspace of $\mathbb{R}^4$? Why?

(iii) (4 marks) Given $\operatorname{rank}\boldsymbol{A} = 3$, find the general solution of $\boldsymbol{Ax} = \boldsymbol{0}$. Show your working.

(iv) (4 marks) Given that $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\boldsymbol{A}$, find the condition satisfied by a, b, c, d.

(v) (4 marks) If a, b, c, d are all equal, find a basis for the column space of $\boldsymbol{A}$ in terms of a. Explain how you derive your answer.

Question 2a **[12 marks]**

Let $S = \{(1,1,2,0),(2,2,4,0),(0,0,1,3),(1,1,3,3),(1,1,1,-3)\}$ and $V = \text{span}(S)$.

(i) (4 marks) Find a basis S' for V such that $S' \subseteq S$ and write down $\dim V$.

(ii) (4 marks) Is $V = \text{span}\{(2,2,5,3),(2,2,3,-3),(1,1,0,-6)\}$? Justify your answer.

(iii) (4 marks) Let $W = \{(x,y,z,w) \mid x - y + z - w = 0\}$. Find $W \cap V$. Give your answer as a linear span.

Question 2b **[8 marks]**

Let W be a subspace of $\mathbb{R}^n$ and $W^\perp = \{\boldsymbol{w} \in \mathbb{R}^n \mid \boldsymbol{w} \cdot \boldsymbol{v} = 0 \text{ for all } \boldsymbol{v} \in W\}$.

(i) (2 marks) Show that $W \cap W^\perp = \{\mathbf{0}\}$.

(ii) (6 marks) Show that every vector $\boldsymbol{v} \in \mathbb{R}^n$ can be written uniquely as $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ where $\boldsymbol{v}_1 \in W$ and $\boldsymbol{v}_2 \in W^\perp$.

(You may assume in part (ii) that W and $W^\perp$ are associated to the row space and nullspace of certain matrix.)

Question 3a **[14 marks]**

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.

(i) (4 marks) Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where $\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{u}_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}$.

Show that S is an orthogonal basis for the column space V of $\boldsymbol{A}$.

(ii) (2 marks) Normalise S to get an orthonormal basis $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ for V.

(iii) (4 marks) Find the least squares solutions of $\boldsymbol{Ax} = \boldsymbol{b}$.

(iv) (4 marks) Extend the basis T in part (ii) to an orthonormal basis $T' = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ for $\mathbb{R}^4$ without using Gram-Schmidt.

Question 3b **[6 marks]**

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ be two orthonormal bases for a proper subspace V of $\mathbb{R}^n$.

Let $\boldsymbol{C} = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_m \end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_m \end{pmatrix}$ be matrices formed using the basis vectors of S and T as their columns respectively.

Determine whether the following are true or false. Justify your answers.

(i) (2 marks) $\boldsymbol{C}$ and $\boldsymbol{D}$ are orthogonal matrices.

(ii) (2 marks) If the reduced row echelon form of $(\boldsymbol{D} \mid \boldsymbol{C})$ is given by $(\boldsymbol{I} \mid \boldsymbol{P})$, then $\boldsymbol{P}$ is the transition matrix from S to T.

(iii) (2 marks) $\boldsymbol{C}^T\boldsymbol{D}$ is the transition matrix from T to S.

Question 4a **[12 marks]**

Let $\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$.

(i) (4 marks) Find the characteristic polynomial and all the eigenvalues of $\boldsymbol{C}$. Show your working.

(ii) (4 marks) Find a basis for each eigenspace of $\boldsymbol{C}$. Show your working.

(iii) (4 marks) Find a matrix $\boldsymbol{P}$ that orthogonally diagonalizes $\boldsymbol{C}$ and write down the corresponding diagonal matrix $\boldsymbol{D}$. Explain how your answers are derived.

Question 4b **[8 marks]**

Let $\boldsymbol{M}$ is an $n \times n$ matrix such that $\boldsymbol{M}^2 = \boldsymbol{M}$ and both 0 and 1 are eigenvalues of $\boldsymbol{M}$.

(i) (4 marks) Show that the column space of $\boldsymbol{M}$ is the eigenspace $\boldsymbol{E}_1$ associated to eigenvalue 1.

(ii) (4 marks) Show that $\boldsymbol{M}$ is diagonalizable

Question 5a **[12 marks]**

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix}, \; T\begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}2\\0\\4\end{pmatrix}, \; T\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}0\\2\\4\end{pmatrix}.$$

(i) (3 marks) Find the standard matrix of T. Show how your answer is derived.

(ii) (3 marks) Find the kernel of T. Give your answer as a linear span.

(iii) (3 marks) Find the largest possible subspace V of $\mathbb{R}^3$ such that every vector $\boldsymbol{v} \in V$ maps to itself under T. Explain how your answer is derived.

(iv) (3 marks) Are there any vector $\boldsymbol{v} \in \mathbb{R}^3$ such that $T(\boldsymbol{v}) = \begin{pmatrix}1\\2\\0\end{pmatrix}$? Justify your answer.

Question 5b **[8 marks]**

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and $R(T)$ is the range of T.
Denote $T^1 = T$ and $T^{k+1} = T \circ T^k$ for all (integers) $k \geq 1$.

(i) (2 marks) Show that $R(T^{k+1}) \subseteq R(T^k)$ for all $k \geq 1$.

(ii) (6 marks) Suppose T^m is the zero transformation for some $m > n$. Show that T^n must be the zero transformation. (Note that T itself need not be the zero transformation.)
<u>Hint</u>: Show that if $R(T^k) = R(T^{k+1})$ for some $k \geq 1$, then $R(T^k) = R(T^h)$ for all $h \geq k$.