Solutions to Tutorial 10

MA1521 Calculus for Computing

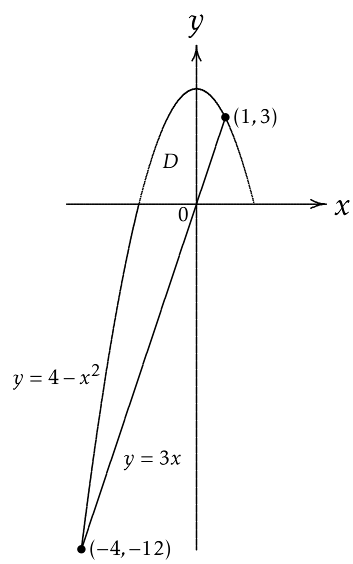
1. The volume is given by the double integral

$$V = \iint_D f(x,y)\,dA,$$

where D is the region bounded by the parabola $y = 4 - x^2$ and straight line $y = 3x$ and $f(x,y)$ is the function whose graph is the plane $x - z + 4 = 0$.

Writing the equation of the plane as $z = x + 4$, we get the function $f(x,y) = x + 4$.

A rough sketch of the region D is shown below:



D can be regarded as type I region

$$D = \{(x,y) \in \mathbb{R}^2 \mid 3x \le y \le 4 - x^2, -4 \le x \le 1\}.$$

(The two limits -4 and 1 of x are obtained by solving the two equations $y = 3x$ and $y = 4 - x^2$.)

Hence

$$V = \int_{-4}^{1}\int_{3x}^{4-x^2} (x+4)\,dy\,dx = \int_{-4}^{1} (x+4)(4 - x^2 - 3x)\,dx = \left[16x - 4x^2 - \frac{7}{3}x^3 - \frac{1}{4}x^4\right]_{-4}^{1} = \frac{625}{12}.$$

2. Let $z = \sqrt{2^2 - x^2 - y^2}$. Then
$z_x = -x(4 - x^2 - y^2)^{-1/2}$ and $z_y = -y(4 - x^2 - y^2)^{-1/2}$.
Substitute $z = 1$ into $x^2 + y^2 + z^2 = 4$ gives

$$x^2 + y^2 + 1 = 4 \quad \Rightarrow \quad x^2 + y^2 = 3$$

which is the equation of a circle of radius $\sqrt{3}$.
This means the plane $z = 1$ intersects the sphere at a circle of radius $\sqrt{3}$.
Hence the projected region R onto the xy-plane of the part of the sphere is a disk of radius $\sqrt{3}$. In polar coordinates, this is given by

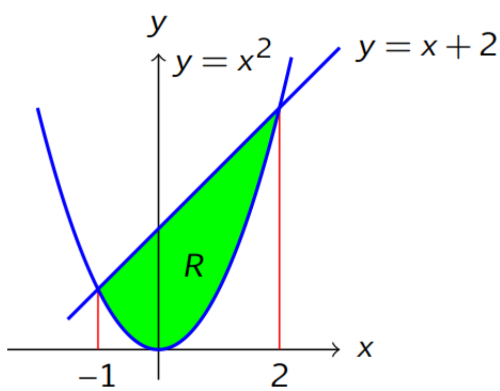
$$R = \{(r, \theta) \mid 0 \le r \le \sqrt{3}, \quad 0 \le \theta \le 2\pi\}.$$

Thus,

$$\begin{aligned}
A(S) &= \iint_R \sqrt{\frac{x^2+y^2}{4-x^2-y^2} + 1}\, dA = \int_0^{2\pi} \int_0^{\sqrt{3}} \left(\frac{r^2}{4-r^2} + 1\right)^{\frac{1}{2}} r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{3}} 2r(4-r^2)^{-\frac{1}{2}}\, dr\, d\theta = \int_0^{2\pi} d\theta \Big[-2(4-r^2)^{\frac{1}{2}}\Big]_{r=0}^{r=\sqrt{3}} \\
&= (2\pi)[-2(4-3)^{\frac{1}{2}} + 2(4)^{\frac{1}{2}}] = 4\pi.
\end{aligned}$$

3. $\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$. Therefore

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} = \sqrt{2}.$$



Note that D is given as a Type I region. Thus the surface area is given by

$$\int\int_D \sqrt{2} dx dy = \int_{-1}^{2} \left(\int_{x^2}^{x+2} \sqrt{2} dy \right) dx = \frac{9}{2}\sqrt{2}.$$

4. Write the equation of the saddle surface as $z = \frac{1}{a}x^2 - \frac{1}{a}y^2$. We have

$$z_x = \frac{2x}{a} \text{ and } z_y = \frac{-2y}{a}.$$

Let D denote the bounded circular region on the xy-plane bounded by the circle $x^2 + y^2 = a^2$.

Then the required surface area is given by

$$\begin{aligned}
S &= \int\int_D \sqrt{1 + z_x^2 + z_y^2}\,dxdy \\
&= \tfrac{1}{a}\int\int_D \sqrt{a^2 + 4x^2 + 4y^2}\,dxdy \\
&= \tfrac{1}{a}\int_0^{2\pi}\int_0^a \sqrt{a^2 + 4r^2}\,rdrd\theta \\
&= \tfrac{2\pi}{a}\int_0^a \sqrt{a^2 + 4r^2}\,d\left(\tfrac{a^2+4r^2}{8}\right) \\
&= \tfrac{\pi}{6a}\left[\left(a^2 + 4r^2\right)^{\frac{3}{2}}\right]_0^a \\
&= \tfrac{\pi a^2}{6}\left(5^{\frac{3}{2}} - 1\right).
\end{aligned}$$

5. (a) Suppose such a function $f(x,y)$ exists. Then $f_x = 4x^3y - \frac{1}{1+x^2} + e^y$ and $f_y = x^4 + xe^y + x$. Since f has continuous second order partial derivatives, we have by Clairaut's theorem that $f_{xy} = f_{yx}$. That is $\frac{\partial}{\partial y}(4x^3y - \frac{1}{1+x^2} + e^y) = \frac{\partial}{\partial x}(x^4 + xe^y + x)$. Thus $4x^3 + e^y = 4x^3 + e^y + 1 \Leftrightarrow 0 = 1$, which is a contradiction. Consequently, such a function f does not exist.

   (b) Let $f(x,y)$ be a function such that $\nabla f = (4x^3y + y + e^y)\mathbf{i} + (x^4 + xe^y + x + y)\mathbf{j}$. Then $f_x = 4x^3y + y + e^y$ – (1) and $f_y = x^4 + xe^y + x + y$ – (2). Integrating equation (1) with respect to x, we have $f(x,y) = x^4y + xy + xe^y + g(y)$, where $g(y)$ is a function of y only. Thus $f_y = x^4 + x + xe^y + g'(y)$. Comparing with (2), we obtain $g'(y) = y$. Thus $g(y) = \frac{1}{2}y^2 + C$, where C is a constant. Consequently, $f(x,y) = x^4y + xy + xe^y + \frac{1}{2}y^2 + C$.

6. (a) $y' = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$. Integrating, we have $y = \ln\left|\frac{x}{x+1}\right| + C$.

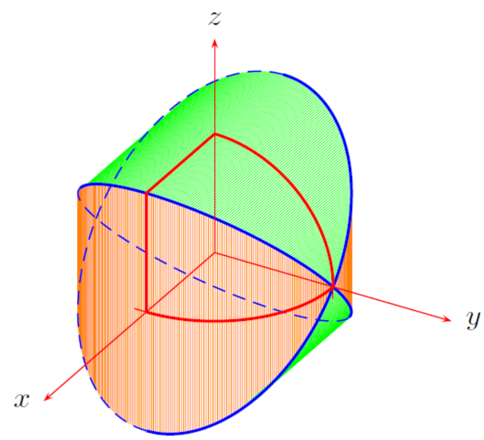
   (b) $\frac{dy}{dx} = e^x e^{-3y}$ so that $e^{3y}dy = e^x dx$. Integrating, we have $\frac{1}{3}e^{3y} = e^x + C$.

   (c) Rewrite the equation $(1+y)y' + (1-2x)y^2 = 0$ as $\frac{1+y}{y^2}dy = (2x-1)dx$. Integrating, we have $\ln|y| - \frac{1}{y} = x^2 - x + C$. $y = 0$ is also a solution.

Solutions to Further Exercises

1. $$\iint_D \frac{1}{(1+x^2+y^2)^{3/2}}\,dA = \int_0^{\pi/2}\int_0^4 \frac{1}{(1+r^2)^{3/2}}\,r\,dr\,d\theta$$
$$= \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^4 (1+r^2)^{-3/2}\,r\,dr\right)$$
$$= \frac{\pi}{2}\left[-(1+r^2)^{-1/2}\right]_0^4 = \frac{\pi}{2}\left(1-\frac{1}{\sqrt{17}}\right).$$

2. By symmetry, the desired volume V is 8 times the volume V_1 in the first octant.



$x^2+y^2=r^2, \quad y^2+z^2=r^2$

$$V_1 = \int_0^r \int_0^{\sqrt{r^2-y^2}} \sqrt{r^2-y^2}\,dx\,dy = \int_0^r \left[x\sqrt{r^2-y^2}\right]_{x=0}^{x=\sqrt{r^2-y^2}} dy$$
$$= \int_0^r (r^2-y^2)\,dy = \left[r^2y-\frac{1}{3}y^3\right]_0^r = \frac{2}{3}r^3.$$

Therefore, $V = \frac{16}{3}r^3$.

3. Let $z = x+2y$. Then $y' = \frac{1}{2}(z'-1)$. [$'$ denotes differentiation with respect to x.] Thus the differential equation becomes $(z-1)+\frac{3z}{2}(z'-1)=0$. That is $3zz' = z+2$, or equivalently, $\int 3-6(z+2)^{-1}\,dz = \int dx$, provided $z+2\neq 0$. From this, we get $3z-6\ln|z+2| = x+c$, or $x+3y+c = 3\ln|x+2y+2|$. If $z+2=0$, then $x+2y+2=0$ is also a solution by direct verification.