NATIONAL UNIVERSITY OF SINGAPORE

CS2020 - Data Structures and Algorithms (Accelerated)

(Semester 2 AY2016/17)

Time Allowed: 2 Hour

Instructions

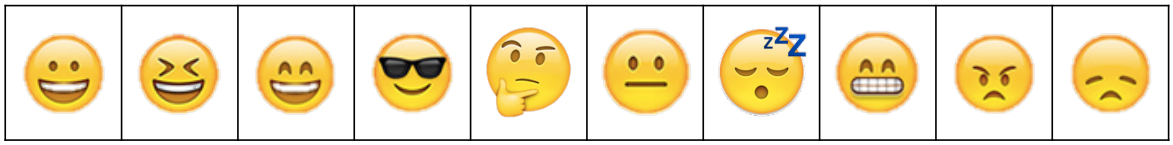
- Write your Student Number below, and on every page. Do not write your name.
- The assessment contains 7 multi-part problems. You have 120 minutes to earn 100 points.
- The assessment contains 28 pages, including the cover page and 5 pages of scratch paper.
- The assessment is closed book. You may bring two double-sided sheet of A4 paper to the assessment. You may not use a calculator, your mobile phone, or any other electronic device.
- Write your solutions in the space provided. If you need more space, please use the scratch paper.
- Show your work. Partial credit will be given. Please be neat.
- You may use (unchanged) any algorithm or data structure that we have studied in class, without restating it. If you want to modify the algorithm, you must explain exactly how your version works.
- Draw pictures frequently to illustrate your ideas.
- Good luck!

Student Number.: _____

Problem #	Name	Possible Points	Achieved Points
1	True/False	10	
2	Drawing Pictures	18	
3	Binary Search Redux	11	
4	The Last Trip	21	
5	Shortest Paths	20	
6	Levelling Up	20	
Total:		100	

Problem 0. Before we begin. [0 points]

Circle the image that best represents how you feel right now.



Problem 1. True/False [10 points]

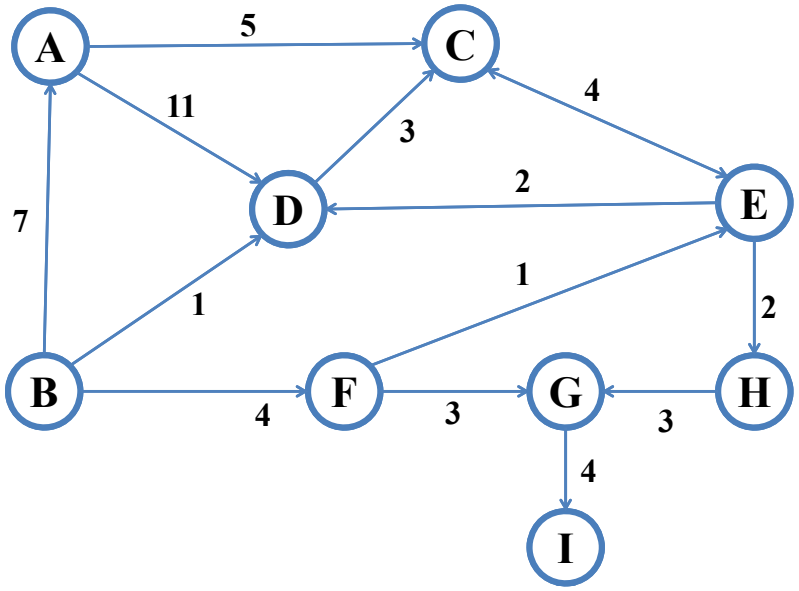
For each of the following statements, select **True** if the statement is always true, and select **False** otherwise. All graphs are connected and contain at least one node and at least one edge.

	True	**False**
G is a weighted, undirected graph, and u is a node in the graph. Edge (u, v) is the edge adjacent to u with the smallest weight. Then some minimum spanning tree of G contains edge (u, v).	☒	☐
You are given a hash table that resolves collisions with chaining. The table is of size m and contains n elements. Assume that the hash function satisfies the Simple Uniform Hashing Assumption. Then the expected time for a search operation is $O(1)$.	☐	☒
We will call the inner loop of Bellman-Ford (where each edge in the graph is relaxed exactly once) an "iteration." Assume you run Bellman-Ford on a graph G with no negative weight cycles. Then at the end of i iterations, every node within i hops of the source has an estimate equal to its correct distance from the source.	☐	☒
Given an array sorted from smallest to largest, we can build a min-heap (i.e., a heap where the smallest element is at the root) in time $\leq O(\log n)$.	☒	☐

	True	**False**
Assume you have a sorted array containing n items. Ten arbitrary items from the array are removed and replaced with ten new items. Then running InsertionSort on the resulting array will take $O(n)$ time.	☒	☐
Assume you have a graph with no positive weight cycles. Then you can find the longest path in the graph by negating all the edge weights and running Dijkstra's algorithm.	☐	☒
Assume you have an AVL tree, and let u be a node in the AVL tree of height h. Then, after every operation completes, the subtree rooted at node u contains at most 2^{h+1} nodes.	☒	☐
Every directed acyclic graph has *exactly* one valid topological ordering.	☐	☒
Given a set of 2D points P, the boundary of the convex hull of P always includes a point $p \in P$ with the maximum x-coordinate.	☒	☐
Given a complete graph K_n with n nodes of 2D positions with $\binom{n}{2}$ edges, if the weight of each edge is the distance between the two endpoints, we can compute the minimal spanning tree of K_n in $O(n \log n)$ expected time.	☒	☐

Problem 2. Drawing Pictures [18 points]

Problem 2.a. Find a topological ordering of the directed acyclic graph below using depth-first search (DFS). In the table below, for each node, fill in the order in which the DFS visits the node, and its position in the topological order. Assume the DFS begins at node *A*. *When there is more than once possible choice of which node to visit next, assume the DFS chooses the node with the earliest alphabetic id (e.g., X before Y before Z.).*



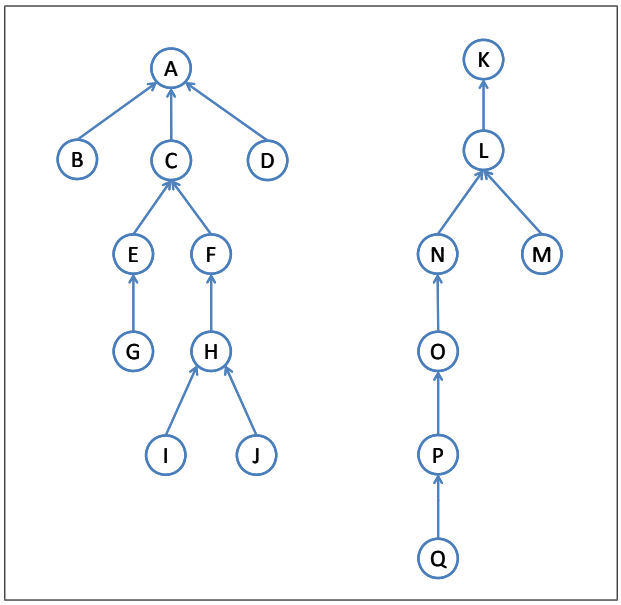
Visit order: (Please include every time a node is seen on the traversal, *not just the first time.*)

A	C	A	D	A	B	F	E	H	G	I	G	H	E	F	B

Topological order:

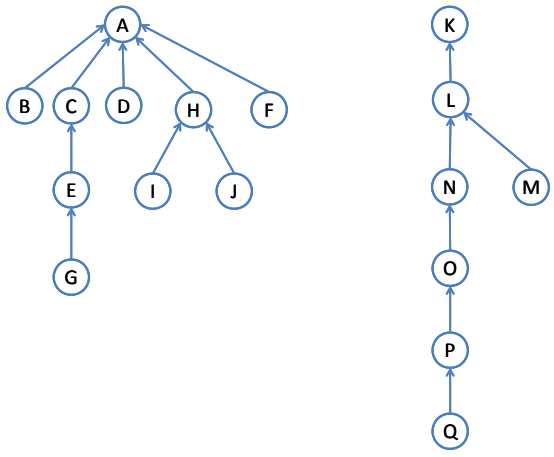
B	F	E	H	G	I	A	D	C

Problem 2.b. Below is a state of the Union-Find algorithm (for implementing the Disjoint Sets ADT), with 17 elements in two disjoint sets. Assume that Union-Find is implemented with Quick-Union and path compression. (The below state is actually impossible, but assume that it is really the state.)



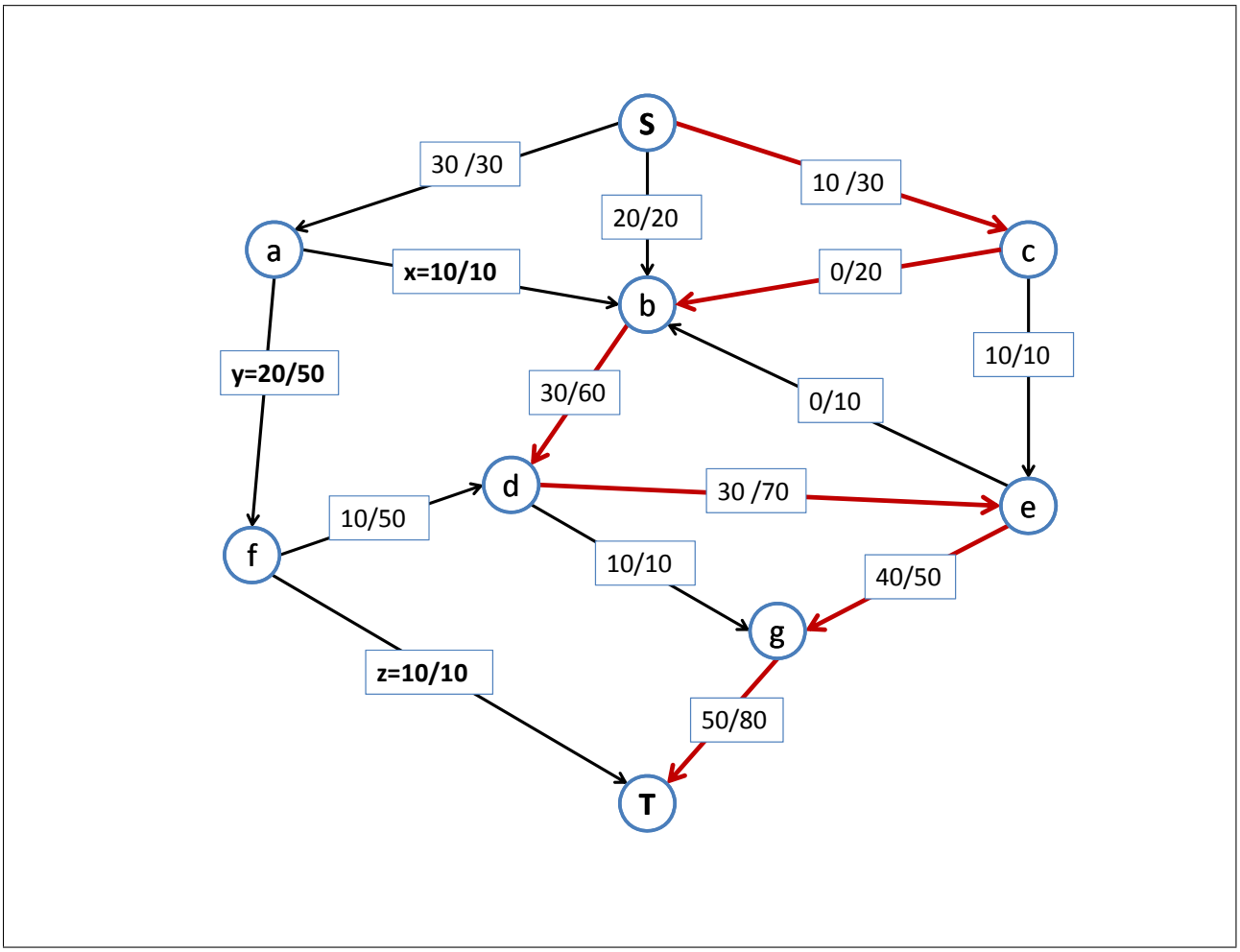
The program executes a `connected(K,H)` operation that determines whether K and H are connected using the Union-Find algorithm. (In this case, it should return the boolean **false**.) Draw (neatly) the state of the algorithm after the operation completes.

Solution:



Problem 2.c. Below is given a flow network with a source S and a target T. Each edge has a capacity, and each edge is assigned a non-negative flow. The label a/b for an edge indicates that the edge has flow a and capacity b. (For three of the edges, the flow is not given and is shown only as a variable x, y, or z.) The given flow respects all the requirements for a legal st-flow.

Please answer all the questions on the next page regarding this flow network.



Question regarding the flow network on the previous page:

Notice that for three of the edges $((a,b),(a,f),(f,t))$ the flow is unspecified. Use the rest of the provided information to deduce the values of the flow on these edges:

$(a,b): \quad x \quad =$ 10

$(a,f): \quad y \quad =$ 20

$(f,T) \quad z \quad =$ 10

Give an *st*-augmenting path for the above flow network:

S	c	b	d	e	g	T			

What is the bottleneck capacity of the given augmenting path?

10

Problem 3. Binary Search Redux [11 points]

Software developer Espy Dittup has an idea for how to make binary search run even faster. A traditional binary search queries the middle element in an array, and then either recurses on the left or right half of the array. Espy wonders what will happen if we divide the array into *more* than two pieces. Espy proposes the following algorithm that divides the array into k pieces and searches for a value v:

- If the array has fewer than k elements, then do a linear search to look for the value v.
- Otherwise, choose k elements spaced evenly apart in the array. Call these elements set S.
- If any of the items in set S is equal to the value v you are looking for, then return true.
- Otherwise, let i be the index of the largest item in S that is $< v$, and let j be the index of the smallest item in S that is $> v$.
- Recursively search in the subarray $[i + 1, j - 1]$.

We will refer to this new algorithm as a k-ary Search.

Problem continued on next page.

Espy implements the k-ary search as follows:

```
1.  static boolean karySearch(int v, int[] A, int begin, int end, int k){
2.     int size = (end - begin + 1);
3.     if (size < k) {
4.         for (int i=begin; i<=end; i++) {
5.             if (A[i] == v) return true;
6.         }
7.         return false;
8.     }
9.     int lastProbe = 0;
10.    // Loop through probe points
11.    for (int i=1; i<k; i++) {
12.        int currentProbe = begin + (size*i)/k;
13.        if (A[currentProbe] == v) return true;
14.        else if (A[currentProbe] > v) {
15.            return karySearch(v, A, lastProbe, currentProbe, k);
16.        }
17.        lastProbe = currentProbe;
18.    }
19.
20.    return karySearch(v, A, lastProbe, end, k);
21. }
```

The code is designed to sort an array of integers, and it is designed to be used in the following manner, e.g., to search for the value 17 (where $k = 5$):

```
int[] A = new int[1000];
for (int i=0; i<1000; i++) A[i] = 2*i;
karySearch(17, A, 0, A.length-1, 5);
```

Problem 3.a. Unfortunately there is a bug in the code that will prevent it from working correctly for at least one value of k. Which of the following is a problem with this code?

☐ There is a compilation error.

☐ The program crashes (e.g., due to an invalid array access).

☒ The program does not terminate.

☐ The program returns an incorrect answer.

Problem 3.b.

Explain (briefly) what the bug is and how to fix it.

The problem is that when recursing into the segment (i, j), it sets $begin = i$ and $end = j$. If $k = 2$ and if $i = j + 1$, this will cause an infinite loop, since the base case for $k = 2$ only occurs when $i = j$, i.e., a segment contains < 2 elements. There are two possible ways to fix this. First, we could expand the base case, stopping the recursion when there are $\leq k$ elements. Second, we could adjust the recursion to use $begin = i$ and $end = j - 1$. (Be careful: if you do not change the base case, and you modify it to $begin = i + 1$ and $end = j - 1$ you will have the problem that $begin > end$ sometimes!).

Problem 3.c. Let $T(n, k)$ be the running time of the k-ary search on an array of n elements. What is the recurrence that describes the performance of this algorithm?

$$T(n, k) = T(n/k) + O(k)$$

Problem 3.d. What is the solution to the recurrence in the previous part? (Give your answer in terms of n and k.)

$$T(n) = O(k \log_k(n))$$

Problem 4. The Last Trip [21 points]

Problem 4.a. Explain briefly (in one to two sentences) the differences between the `Iterator` interface and the `Iterable` interface. When do you use one and when do you use the other?

Solution: The `Iterable` interface means that a class can be iterated, i.e., it means that there is an iterator available. The `Iterator` interface means that a class *is* an iterator, i.e., it can be used to iterate through some other collection.

Problem 4.b. Explain briefly (in one to two sentences) the differences between the `HashMap` class and the `TreeMap` class. When do you use one and when do you use the other? What are the trade-offs?

Solution: A HashMap is built using a hash table and hence does not support efficient successor and predecessor queries. However, it supports faster inserts and lookups in $O(1)$ time, with high probability (depending on the hash function). A TreeMap is built using a tree and supports all operations in worst-case $O(\log n)$ time, including successor and predecessor queries.

Problem 4.c. Explain briefly (in one to two sentences) what is the `Comparable` interface and why would you use it?

Solution: The `Comparable` interface indicates that the class supports the `compareTo` method, which allows you to compare two elements of the class.

```
1.    public class LastTrip AAA ITravel {
2.          static int count = 0;
3.          LastTrip nextTrip = null; // Next trip that we will all take together!
4.          String name;
5.          ArrayList<BBB> listOfStuff;
6.
7.          LastTrip(String n, BBB x, BBB y, LastTrip next){
8.                listOfStuff = new CCC; // create new ArrayList
9.                listOfStuff.add(x);  listOfStuff.add(y);
10.               nextTrip = next;
11.               name = n;
12.         }
13.
14.         @Override // findDistance is part of the ITravel interface
15.         DDD int findDistance(){
16.               count++;
17.               int distance = 0;
18.               for (int i=0; i<listOfStuff.size(); i++) {
19.                     distance += listOfStuff.get(i);
20.               }
21.               return distance;
22.         }
23.
24.         public void printTripReport(){
25.               if (nextTrip != null) {
26.                     nextTrip.printTripReport();
27.               }
28.               System.out.println(findDistance());
29.         }
30.
31.         EEE LastTrip updateTrip(LastTrip trip, LastTrip newNext) {
32.               LastTrip temp = trip;
33.               trip = new LastTrip("empty", 0, 0, null);
34.               temp.nextTrip = newNext;
35.               return temp;
36.         }
37.
38.         public static void main(String[] args){
39.               LastTrip a = new LastTrip("a", 7, 10, null);
40.               LastTrip b = new LastTrip("b", 12, 15, a);
41.               LastTrip c = new LastTrip("c", 22, 32, a);
42.
43.               LastTrip d = updateTrip(a, c);
44.               c.nextTrip = b;
45.               d.nextTrip = null;
46.
47.               c.printTripReport();
48.         }
49.   }
```

Problem 4.d. In the `LastTrip` class on the previous page, important parts of the code have been replaced with the strings AAA, BBB, CCC, DDD, and EEE. Fill in below proper Java code to replace these placeholders.

Instructions	Line(s)	Placeholder		Answer
`ITravel` is an interface.	1	`AAA`	=	implements
	5, 7	`BBB`	=	Integer
	8	`CCC`	=	ArrayList ⟨ Integer ⟩ ()
Use some combination of: `public`, `private`, and `static`	15	`DDD`	=	public
Use either `public` or `public static`	31	`EEE`	=	public static

Problem 4.e. On executing the main method, at each of the indicated lines of the program, indicate the `name` of the trip and the name of the `nextTrip` (or null, if there is no nextTrip).

Line	Variable	`.name`	`.nextTrip.name`
line 42	a	a	null
line 42	b	b	a
line 42	c	c	a
line 46	a	a	null
line 46	b	b	a
line 46	c	c	b
line 46	d	a	null

Problem 4.f. What is printed out when the program completes? (You may not need all the lines provided.)

Output line 1 : 17

Output line 2 : 27

Output line 3 : 54

Output line 4 :

Output line 5 :

Problem 5. Shortest Paths [20 points]

You are given a weighted, connected, and undirected graph $G = (V, E)$ and a source S. For each edge (u, v), let $w(u, v)$ be the weight of the edge (i.e., the distance from u to v). Assume that all the edge weights are > 0. You can find some examples at the page after next.

Problem 5.a. Each node u in the graph is labelled with a value d_u. The graph expert, Prof. D. S. Tance, claims that these values represent the shortest distance from S to the node. For example, if node u has label d_u, then the shortest path from S to u had length d_u. You want to decide whether Tance is correct.

Give the most efficient algorithm to decide whether the labels are correct, i.e., return **true** if and only if the labels represent the lengths of the shortest paths from S:

First, we need to find a shortest path tree. Identify all the edges (u, v) that are "tight," i.e., $d_u + w(u, v) = d_v$, and direct them from the smallest estimate to the larger. Find a spanning tree T rooted at the source that uses only these identified edges. If no such tree T exists, the labels are not correct. Otherwise, if the labels are correct, then T should be a shortest path tree. At the very least, we know that the label at each node represents the distance of *some* path to the source, and so each label is $\geq$ the real distance. The above procedure takes $O(E)$ time as it requires looking at each edge to see if it is tight, and doing a BFS (or DFS) of the identified edges.
Next, to check with T is a real shortest path tree, run one iteration of Bellman-Ford on the entire graph, relaxing all the edges in the graph exactly once, treating the labels as the current estimate. If any estimate changes, then they do not represent shortest paths.
If the labels were shortest paths, then there would be no change. Conversely, assume some node u had a label that was not correct. Look at the shortest path from the source to u. We know that the label at u is $\geq$ the real distance (because of T), so when we relax all the edges on the shortest path from s to u, at least one of the estimates will change. This ensures that the algorithm is correct.
This requires examining each edge once, and so takes time $O(|E|)$.

Assume the graph has n nodes and m edges. What is the running time of your algorithm:

`Time` : $O(m)$

Problem 5.b. Prof. D. S. Tance loves finding shortest paths, and has now given each node u a label representing the shortest distance from *every* node in the graph. Each node u in the graph has a label $d_u[v]$ representing the shortest path from v to u, for every node v. Assume that these labels are correct. (You may assume that Tance calculated these labels using an All-Pairs-Shortest-Path algorithm.)

Some of the edges in the graph are useless, in that they do not contribute to *any* shortest path. For every edge (u, v), we say that it is *useless* if it is not part of a shortest path from some node x to another node y. For simplicity, assume that for every pair of nodes x, y there is a single unique shortest path between x and y.

Give the most efficient algorithm you can to find *all* the useless edges in the graph:

For every pair of nodes (u, v): compare $d_u[v]$ to $w(u, v)$. Notice $d_u[v] \leq w(u, v)$, since the shortest path is always $\leq$ the direct path from u to v. If $d_u[v] = w(u, v)$, then the edge (u, v) is useful; keep it. Otherwise, if $d_u[v] < w(u, v)$, then the edge is useless; add it to the set of useless edges.

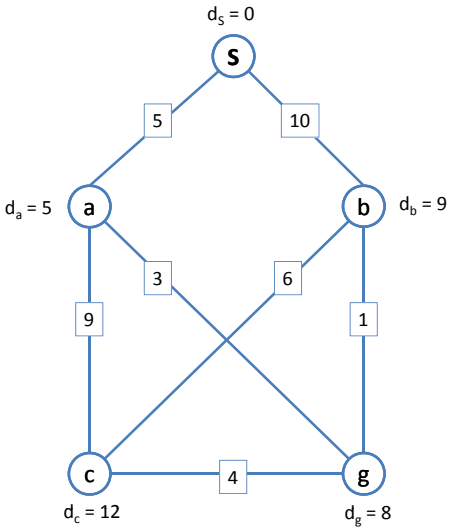
If we keep the edge, it is clearly part of the (unique) shortest path from u to v, and so clearly we have to keep it. Assume we decide not to keep the edge (u, v), and yet it is part of the shortest path from s to t. Assume (without loss of generality) that the path from s to t visits u first before visiting v. Since we deleted the edge (u, v), we know that there is a shorter path from u to v than $w(u, v)$. Thus, we can re-route the path from s to t to go first from s to u (as before), then from u to v on the shortest path from u to v, and then from v to t (as before). This path is strictly shorter, contradicting our claim that the edge (u, v) is on the shortest path from s to t. Thus if we delete edge (u, v), then edge (u, v) really is useless.

The running time of this algorithm simply involves examining each edge exactly once. Thus the running time of the algorithm is $O(|E|)$.

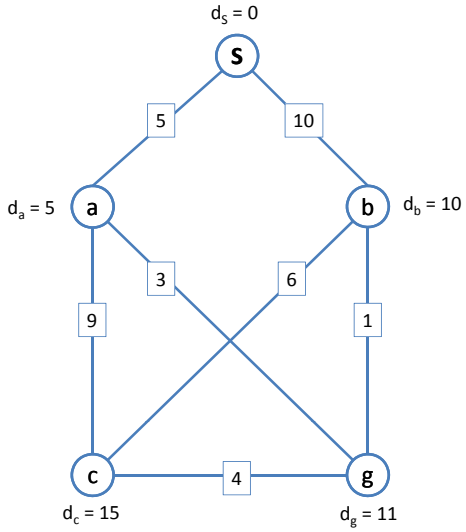
Assume the graph has n nodes and m edges. What is the running time of your algorithm:

`Time` : $O(m)$

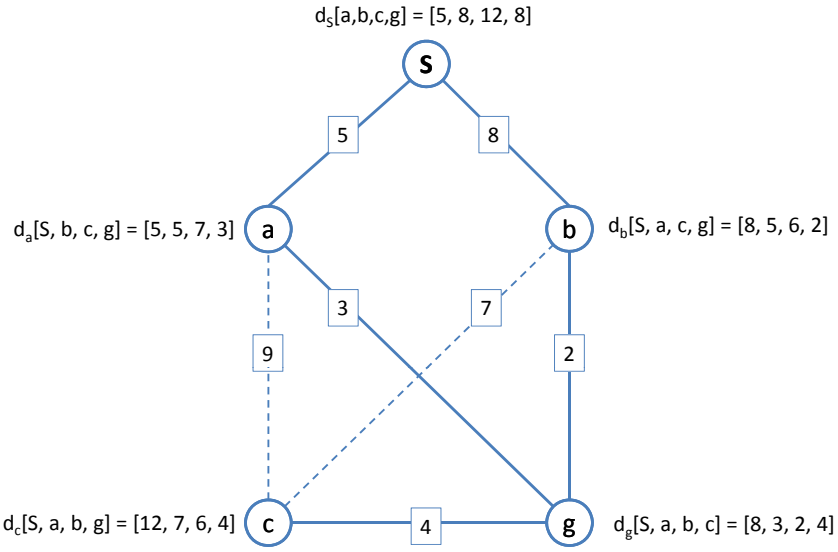
Examples for Problem 6:



Part a: Each of the distances from the source S is correctly labelled. For example, the distance from S to node b is 9 (via the path (S, a, g, b)). For part (a), the algorithm should return true.



Part a: The distances from the source S are not correctly labelled. For example, the distance from S to node b is 9 (via the path (S, a, g, b)), not 10 (as labelled). For part (a), the algorithm should return false.



Part b: The solid edges are useful, while the dotted edges are useless. For example, the edge (S, a) is useful, because it is on the shortest path from S to a. The edge (a, c) is not useful because it is not part of any shortest path. For part (b), the algorithm should return the edges (a, c) and (b, c).

Problem 6. Levelling Up [20 points]

The *CS2020: The Last Voyage* is my favorite game. In this game, we have a maze consisting of rooms. We start at the entrance to the maze and the goal is to reach the exit. However, not everyone can enter each room in the maze. Each player has a *level*, and each room has a minimum level requirement: you cannot enter the room unless you have reached the minimum level.

Assume that the maze is given to you in the form of an undirected, connected graph $G = (V, E)$ where the nodes represent rooms and the edges represent doors between the rooms. Assume there are n nodes and m edges. For each node u, you are given $level(u)$, i.e., the minimum required level for that room. There is a designated entrance s and a designated exit d. Assume the graph is provided as an adjacency list.

Inside the maze, there are a few special relics that will raise your level for the remainder of the maze. When you use a relic, it will add R levels to your current level. This will enable you to enter rooms that are many levels higher than would otherwise have been possible. Assume that there are at most k such relics and all of them raise your level by R. (Do not assume that k is necessarily small.) While inside the maze, you can only use one relic during the entire game because once you have found and used one relic, the others will be vaporized.

Your goal in this problem is to give an algorithm to find the minimum level a player needs on entering the maze in order to reach the exit. The running time of your algorithm should depend only on n, m, and k. It should *not* depend on R or the maximum level number.

In this example, $R = 20$. You need to have initial level 40 in order to exit the maze. This is sufficient to reach either the relic in the bottom right corner or the top left corner, which boosts you to level 60, enabling you to exit the maze.

Problem 6.a. Explain, succinctly in **two to three sentences**, the main idea of how you plan to solve the problem.

Solution: Build a new graph with weighted edges where the weight is equal to the maximum level of the adjacent nodes. Build a second new graph where the weight of each edge is reduced by R and connected the two graphs at nodes with relics. Then find an MST and return the maximum weight on the path on the MST between the start and the exit (taking the minimum of the two graphs).

Problem 6.b. Draw a picture that illustrates your idea.

Problem 6.c. Assume that the graph has n nodes, m edges, and k relics. What is the time complexity of your algorithm? Use asymptotic (big-O) notation.

Time complexity : $O(m \log n)$

Solution: The algorithm will run in $O(m \log n)$ expected time.

Problem 6.d. Give the complete details of your algorithm. Explain why you achieve the performance claimed in the previous part.

Solution: First, for every edge (u, v) in G, assign it a weight $w(u, v)$ that is equal to the maximum of $level(u)$ and $level(v)$. Notice that the maximum weight on any path in G is equivalent to the maximum required level on the corresponding path in the maze, and vice versa. (Beware: this is not true for the sum of the weights, i.e., for shortest paths.) This takes time $O(m)$.

Next, make 2 copies of the initial graph: G_0 and G_1. Graph G_0 is identical to G. Graph G_1 is identical to G except that we reduce the weight of every edge by R. For each node u that contains a relic, add an edge (u_0, u_1) connecting u in graph G_0 with u in graph G_1; assign the edge weight equivalent to $level(u)$, the minimum required level for u. Notice that the resulting graph has $O(n)$ nodes and $O(m)$ edges. Building this graph takes time $O(n+m)$.

Then, run an MST algorithm to find the minimum spanning tree of the graph. This takes time $O(m \log n)$.

Finally, do a DFS on the MST, labelling each node with the minimum level needed to reach that node. (You can reach the start with only level zero. Starting at a labelled node u, you label a child v of u with the maximum of the label at u and the edge weight from u to v.) Then you check the level for the exit node in each of the $k + 1$ copies of the graph and return the minimum value. This final step takes time $O(n)$.

Thus the total running time is $O(m \log n)$.

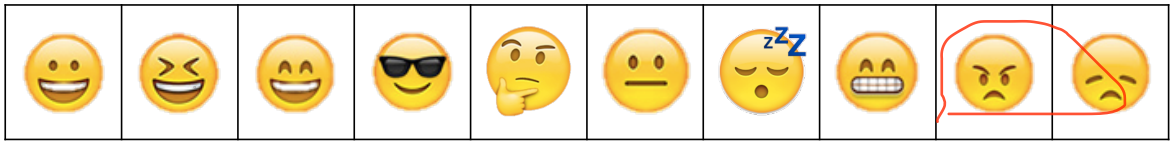
Why does this work? Notice that any legal (simple) path in the new graph (with two copies) is also a legal path in the original graph: the only way to move to graph G_1 is via a relic and hence your level has been boosted by R, making the moves in G_1 legal. (You can also argue that any good path will not return from graph G_1 to graph G_0, since you can always follow the same edges in G_1.) So the maximum edge weight on a path in the new graph corresponds to the maximum required level in the original maze. Similarly, the best path in the original maze can be mapped to a legal path in the new graph. So it suffices to find the path from the start to the finish in the new graph with the smallest maximum weight.

And it turns out that such a path will necessarily be on the minimum spanning tree. Assume not (for the sake of contradiction): there is a better path from the start to the exit that uses edges not on the MST. Then this alternative path, when combined with the MST, will create a cycle. We know that the heaviest edge on that cycle is *not* on the MST, hence it must be on the better path. But then we can modify the path to follow the MST edges on the cycle instead of the original path without making the path any worse. We can continue to do this until there are no more cycles, i.e., the path follows only MST edges. Thus we conclude that there exists a path from the start to the exist that has minimum maximum weight and only uses MST edges.

You can achieve the same results without building the doubled graph. For example, you can find the MST, and then find the minimum maximum weight path from the start to each relic, and also from the exit to each relic. Then, by looking at each of the possible paths, you can choose the best one.

Problem 7. After you are done. [0 points]

Circle the image that best represents how you feel right now.



Scratch Paper

Scratch Paper

Scratch Paper

Scratch Paper

Scratch Paper

End of Paper