NATIONAL UNIVERSITY OF SINGAPORE

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MA2001 Linear Algebra I **Tutorial 11**

EXERCISE 7.1

Determine whether each of the following are linear transformations. Write down the standard matrix for each of the linear transformations.

(a) $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y-x \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

(b) $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_2\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2^x \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

(c) $T_3 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T_3\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ 0 \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

(d) $T_4 : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T_4\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1 \\ y-x \\ y-z \end{pmatrix}$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

(e) $T_5 : \mathbb{R}^n \to \mathbb{R}$ such that $T_5(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{y}$ for $\boldsymbol{x} \in \mathbb{R}^n$, where $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ is a fixed vector in $\mathbb{R}^n$.

(f) $T_6 : \mathbb{R}^n \to \mathbb{R}$ such that $T_6(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{x}$ for $\boldsymbol{x} \in \mathbb{R}^n$.

In parts (e) and (f), $\mathbb{R}$ is regarded as $\mathbb{R}^1$.

A *linear transformation* is defined as a special function between Euclidean spaces. In particular, a linear transformation $f : \mathbb{R}^n \to \mathbb{R}$ is a function of the form

$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

Then $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if each component of T is a linear transformation $\mathbb{R}^n \to \mathbb{R}$. If the vectors are written in columns,

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{m1} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

If we set $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, and $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, then the linear transformation can be represented as

$$T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x} \in \mathbb{R}^n.$$

In other words, *linear transformations are precisely the multiplication with matrices.* The unique matrix $\boldsymbol{A}$ is called the *standard matrix* of the linear transformation of T. Note that the j^{th} column of $\boldsymbol{A}$ is $T(\boldsymbol{e}_j)$, where

$$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$$

is the standard basis for $\mathbb{R}^n$.

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* if and only if T is *linear*, in the sense that

(i) $T(\mathbf{0}) = \mathbf{0}$;

(ii) $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ for all $c \in \mathbb{R}$ and $\boldsymbol{v} \in \mathbb{R}^n$;

(iii) $T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.

Or simply

$$T(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k) = c_1T(\boldsymbol{v}_1) + c_2T(\boldsymbol{v}_2) + \cdots + c_kT(\boldsymbol{v}_k)$$

for any $c_1, c_2, \ldots, c_k \in \mathbb{R}$ and $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$.

The following criterion can be used to prove or disprove that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

(a) Suppose we want to show that that T *is* a linear transformation.

  (i) Show that each component of T has the form

$$a_1x_1 + \cdots + a_nx_n.$$

  (ii) Find an $m \times n$ matrix $\boldsymbol{A}$ such that $T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$.

(b) Suppose we want to show that T is *not* a linear transformation.

  (i) Show that $T(\mathbf{0}) \neq \mathbf{0}$; or

  (ii) Find $c \in \mathbb{R}$ and $\boldsymbol{v} \in \mathbb{R}^n$ such that $T(c\boldsymbol{v}) \neq cT(\boldsymbol{v})$; or

  (iii) Find $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ such that $T(\boldsymbol{u}+\boldsymbol{v}) \neq T(\boldsymbol{u}) + T(\boldsymbol{v})$.

7.1(a). $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y-x \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

Note that

$$x + y = 1x + 1y \quad \text{and} \quad y - x = (-1)x + 1y$$

are linear in x and y. Then T_1 is a linear transformation.

In order to get the standard matrix $\boldsymbol{A}$, we note that the 1st column of $\boldsymbol{A}$ is

$$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1+0 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and the 2nd column of $\boldsymbol{A}$ is

$$T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0+1 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

A direct way to obtain $\boldsymbol{A}$ is to rewrite the definition of T_1 in matrix form:

$$T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y-x \end{pmatrix} = \begin{pmatrix} 1x+1y \\ (-1)x+1y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Then T_1 is a linear transformation with standard matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

7.1(b). $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_2\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2^x \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

The 1st component of T_2 is 2^x, which does not seem to be linear. We may try to verify that T_2 is *not* linear.

A linear transformation must send the zero vector (in the domain) to the zero vector (in the codomain). However,

$$T_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2^0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So T_2 is not a linear transformation.

7.1(c). $T_3 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T_3\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ 0 \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

Each component of T_3 is linear:

$$x + y = 1x + 1y \quad \text{and} \quad 0 = 0x + 0y.$$

Then T_3 is a linear transformation. Moreover,

$$T_3\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1x+1y \\ 0x+0y \\ 0x+0y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Then T_3 is a linear transformation with standard matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

7.1(d). $T_4 : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T_4\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1 \\ y-x \\ y-z \end{pmatrix}$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

Note that the 1^{st} component of T_4 is a *constant*, which is not *linear*:

$$1 \neq ax + by + cz.$$

We may expect that T_4 is *not* a linear transformation. Indeed,

$$T_4\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0-0 \\ 0-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since T_4 does not send the zero vector to zero vector, it is not a linear transformation.

7.1(e). $T_5 : \mathbb{R}^n \to \mathbb{R}$ is defined by $T_5(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{y}$ for $\boldsymbol{x} \in \mathbb{R}^n$, where $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ is a fixed vector.

We write T_5 explicitly. Let $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$. Then

$$T_5\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

This is a linear transformation (note that $y_1, y_2, \ldots, y_n$ are constants):

$$y_1 x_1 + y_2 x_2 + \cdots + y_n x_n.$$

Note that this can be written in matrix form

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

It follows that T_5 is a linear transformation with standard matrix $\boldsymbol{A} = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix} = \boldsymbol{y}^{\mathrm{T}}$.

On the other hand, we have a direct argument. Recall that the dot product can be represented in terms of the matrix product; so

$$T_5(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}} \boldsymbol{x}.$$

Then T_5 is a linear transformation with standard matrix $\boldsymbol{A} = \boldsymbol{y}^{\mathrm{T}}$.

7.1(f). $T_6 : \mathbb{R}^n \to \mathbb{R}$ such that $T_6(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{x}$ for $\boldsymbol{x} \in \mathbb{R}^n$.

If we write $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, then

$$T_6\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1^2 + x_2^2 + \cdots + x_n^2.$$

This is *quadratic function*, which does not seem to be linear.

We now try to show that T_6 is not linear.

(i) $T_6(\mathbf{0}) = \mathbf{0} \cdot \mathbf{0} = 0$.

Although T_6 satisfies condition (i), we still need to verify conditions (ii) and / or (iii).

(ii) $T(c\boldsymbol{x}) = (c\boldsymbol{x}) \cdot (c\boldsymbol{x}) = c^2(\boldsymbol{x} \cdot \boldsymbol{x})$, while $cT(\boldsymbol{x}) = c(\boldsymbol{x} \cdot \boldsymbol{x})$. They look different!

So we can obtain a counterexample if $c \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^n$ are chosen so that

$$c^2(\boldsymbol{x} \cdot \boldsymbol{x}) \neq c(\boldsymbol{x} \cdot \boldsymbol{x}).$$

By (i) we shall not use $\boldsymbol{x} = \mathbf{0}$. If $\boldsymbol{x} \neq \mathbf{0}$, then $\boldsymbol{x} \cdot \boldsymbol{x} \neq 0$. Then

$$c^2(\boldsymbol{x} \cdot \boldsymbol{x}) \neq c(\boldsymbol{x} \cdot \boldsymbol{x}) \Leftrightarrow c^2 \neq c \Leftrightarrow c \neq 0 \text{ and } c \neq 1.$$

For instance, let $\boldsymbol{x} = \boldsymbol{e}_1 = (1, 0, \ldots, 0)$, and $c \neq 2$, then

$$T(2\boldsymbol{e}_1) = (2\boldsymbol{e}_1) \cdot (2\boldsymbol{e}_1) = 4(\boldsymbol{e}_1 \cdot \boldsymbol{e}_1) = 4,$$
$$2T(\boldsymbol{e}_1) = 2(\boldsymbol{e}_1 \cdot \boldsymbol{e}_1) = 2.$$

Since $T(2\boldsymbol{e}_1) \neq 2T(\boldsymbol{e}_1)$, we conclude that T_6 is not a linear transformation.

Exercise 7.2

For each of the following linear transformations,

(i) determine whether there is enough information for us to find the formula of T; and
(ii) find the formula and the standard matrix for T if possible.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that

$$T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = \begin{pmatrix}1\\3\\0\\1\end{pmatrix}, \quad T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}2\\2\\-1\\4\end{pmatrix} \quad \text{and} \quad T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}0\\4\\1\\6\end{pmatrix}.$$

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) = \begin{pmatrix}2\\0\end{pmatrix}, \quad T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}0\\2\end{pmatrix} \quad \text{and} \quad T\left(\begin{pmatrix}2\\0\end{pmatrix}\right) = \begin{pmatrix}2\\2\end{pmatrix}.$$

(f) $T : \mathbb{R}^3 \to \mathbb{R}$ such that

$$T\left(\begin{pmatrix}1\\-1\\0\end{pmatrix}\right) = -1, \quad T\left(\begin{pmatrix}0\\1\\-1\end{pmatrix}\right) = 1 \quad \text{and} \quad T\left(\begin{pmatrix}-1\\0\\1\end{pmatrix}\right) = 0.$$

Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then it is linear:

$$T(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k) = c_1T(\boldsymbol{v}_1) + c_2T(\boldsymbol{v}_2) + \cdots + c_kT(\boldsymbol{v}_k),$$

for $c_1, c_2, \ldots, c_k \in \mathbb{R}$ and $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$.

In particular, suppose $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is a basis for $\mathbb{R}^n$, then every $\boldsymbol{v} \in \mathbb{R}^n$ can be uniquely written as a linear combinations of vectors in S:

$$\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n, \qquad (c_1, c_2, \ldots, c_n) = (\boldsymbol{v})_S.$$

It follows from the linearity of T that

$$T(\boldsymbol{v}) = c_1T(\boldsymbol{v}_1) + c_2T(\boldsymbol{v}_2) + \cdots + c_kT(\boldsymbol{v}_k).$$

Hence, *a linear transformation T is uniquely determined by its images on a basis S.*

Moreover,

$$T(\boldsymbol{v}) = \begin{pmatrix}T(\boldsymbol{v}_1) & T(\boldsymbol{v}_2) & \cdots & T(\boldsymbol{v}_n)\end{pmatrix}\begin{pmatrix}c_1\\c_2\\\vdots\\c_n\end{pmatrix}.$$

Let $\boldsymbol{B} = \begin{pmatrix} T(\boldsymbol{v}_1) & T(\boldsymbol{v}_2) & \cdots & T(\boldsymbol{v}_n) \end{pmatrix}$. Then we can write

$$T(\boldsymbol{v}) = \boldsymbol{B}[\boldsymbol{v}]_S, \qquad \boldsymbol{v} \in \mathbb{R}^n.$$

In order to find the standard matrix $\boldsymbol{A}$ for T, i.e., the $m \times n$ matrix such that

$$T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}, \qquad \boldsymbol{v} \in \mathbb{R}^n,$$

we notice that if $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{pmatrix}$, then $[\boldsymbol{v}]_S$ is the *unique* solution to $\boldsymbol{P}\boldsymbol{x} = \boldsymbol{v}$. So

$$\boldsymbol{A}\boldsymbol{v} = T(\boldsymbol{v}) = \boldsymbol{B}[\boldsymbol{v}]_S = \boldsymbol{B}(\boldsymbol{P}^{-1}\boldsymbol{v}) = (\boldsymbol{B}\boldsymbol{P}^{-1})\boldsymbol{v}.$$

By the uniqueness of standard matrix, we obtain

$$\boldsymbol{A} = \boldsymbol{B}\boldsymbol{P}^{-1},$$

where

$$\boldsymbol{B} = \begin{pmatrix} T(\boldsymbol{v}_1) & T(\boldsymbol{v}_2) & \cdots & T(\boldsymbol{v}_n) \end{pmatrix} \quad \text{and} \quad \boldsymbol{P} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{pmatrix}.$$

7.2(a). $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that

$$T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = \begin{pmatrix}1\\3\\0\\1\end{pmatrix}, \quad T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}2\\2\\-1\\4\end{pmatrix} \quad \text{and} \quad T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}0\\4\\1\\6\end{pmatrix}.$$

Note that the images of T are given at

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix},$$

which form the standard basis. Then we can concluded immediately that T is uniquely determined from the given information. Moreover, the standard matrix is the matrix whose columns are the images $T(\boldsymbol{e}_1), T(\boldsymbol{e}_2), T(\boldsymbol{e}_3)$:

$$\begin{pmatrix}1 & 2 & 0\\3 & 2 & 4\\0 & -1 & 1\\1 & 4 & 6\end{pmatrix}$$

7.2(c). $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) = \begin{pmatrix}2\\0\end{pmatrix}, \quad T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}0\\2\end{pmatrix} \quad \text{and} \quad T\left(\begin{pmatrix}2\\0\end{pmatrix}\right) = \begin{pmatrix}2\\2\end{pmatrix}.$$

Note that $\dim(\mathbb{R}^2) = 2$. So T is uniquely determined by giving the images at 2 linearly independent vectors in $\mathbb{R}^2$. In particular, since

$$\left\{\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$$

is a basis for $\mathbb{R}^2$ (there are quite a number of ways showing it!), the linear transformation T is *uniquely* determined by

$$T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right)=\begin{pmatrix}2\\0\end{pmatrix} \quad\text{and}\quad T\left(\begin{pmatrix}1\\1\end{pmatrix}\right)=\begin{pmatrix}0\\2\end{pmatrix}.$$

How about the 3rd condition $T\left(\begin{pmatrix}2\\0\end{pmatrix}\right)=\begin{pmatrix}2\\2\end{pmatrix}$?

Since $S=\left\{\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$, there is a unique way to write $\begin{pmatrix}2\\0\end{pmatrix}$ as a linear combination of vectors in S. Then by the linearity of T, we can *evaluate* the image of T at $\begin{pmatrix}2\\0\end{pmatrix}$.

There are two cases:

(i) If the evaluation coincides with the 3rd condition $T\left(\begin{pmatrix}1\\1\end{pmatrix}\right)=\begin{pmatrix}0\\2\end{pmatrix}$, then the 3rd condition is *redundant*, because it can be derived from the other two conditions.

(ii) If the evaluation is different from the 3rd condition $T\left(\begin{pmatrix}1\\1\end{pmatrix}\right)=\begin{pmatrix}0\\2\end{pmatrix}$, then the 3rd condition *contradicts* the other two conditions. Hence, the linear T does not exist.

For this question, it is easy to write

$$\begin{pmatrix}2\\0\end{pmatrix}=1\begin{pmatrix}1\\-1\end{pmatrix}+1\begin{pmatrix}1\\1\end{pmatrix}.$$

Then from the 1st and the 2nd conditions,

$$T\left(\begin{pmatrix}2\\0\end{pmatrix}\right)=T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right)+T\left(\begin{pmatrix}1\\1\end{pmatrix}\right)=\begin{pmatrix}2\\0\end{pmatrix}+\begin{pmatrix}0\\2\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix},$$

which agrees with the 3rd condition. Therefore, the 3rd condition is redundant and T is uniquely determined by

$$T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right)=\begin{pmatrix}2\\0\end{pmatrix} \quad\text{and}\quad T\left(\begin{pmatrix}1\\1\end{pmatrix}\right)=\begin{pmatrix}0\\2\end{pmatrix}.$$

Let $\boldsymbol{P}=\begin{pmatrix}1&1\\-1&1\end{pmatrix}$ and $\boldsymbol{B}=\begin{pmatrix}2&0\\0&2\end{pmatrix}$. Then the standard matrix for T is

$$\boldsymbol{A}=\boldsymbol{B}\boldsymbol{P}^{-1}=\begin{pmatrix}1&-1\\1&1\end{pmatrix}.$$

7.2(f). $T:\mathbb{R}^3\to\mathbb{R}$ such that

$$T\left(\begin{pmatrix}1\\-1\\0\end{pmatrix}\right)=-1,\quad T\left(\begin{pmatrix}0\\1\\-1\end{pmatrix}\right)=1 \quad\text{and}\quad T\left(\begin{pmatrix}-1\\0\\1\end{pmatrix}\right)=0.$$

Note that $\dim(\mathbb{R}^3) = 3$, and there are 3 conditions given for the linear transformation T. It is sufficient to determine T if the three vectors:

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

form a basis for $\mathbb{R}^3$. However, we have a few ways to show that they do not form a basis. (How many ways can you use, and what are they?) For instance,

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, we do *not* have enough information to determine T.

Specifically, since $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ are linearly independent, they form a basis for

$$V = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Then the 1^{st} and the 2^{nd} conditions uniquely determine T on V, and we can use them to evaluate the image of $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ under T:

$$T\left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right) = -T\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right) - T\left(\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right) = -(-1) - 1 = 0,$$

which agrees with the 3^{rd} condition.

We conclude that: there is no contradiction in the given conditions; but the given conditions can only uniquely determine T on

$$V = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

In order to uniquely determine T (on $\mathbb{R}^3$), we need any value of T at some vector $\boldsymbol{v} \in \mathbb{R}^3 \setminus V$.

Exercise 7.7

Let $\boldsymbol{n}$ be a unit vector in $\mathbb{R}^n$. Define $P : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$P(\boldsymbol{x}) = \boldsymbol{x} - (\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} \quad \text{for } \boldsymbol{x} \in \mathbb{R}^n.$$

(a) Show that P is a linear transformation and find the standard matrix P.

(b) Prove that $P \circ P = P$.

7.7(a). In order to prove that T is a linear transformation, we first write $(\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n}$ explicitly.

Let $\boldsymbol{n} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ and $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$. Then

$$(\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n} = (c_1x_1 + c_2x_2 + \cdots + c_nx_n)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} c_1(c_1x_1 + c_2x_2 + \cdots + c_nx_n) \\ c_2(c_1x_1 + c_2x_2 + \cdots + c_nx_n) \\ \vdots \\ c_n(c_1x_1 + c_2x_2 + \cdots + c_nx_n) \end{pmatrix}$$

$$= \begin{pmatrix} c_1c_1 & c_1c_2 & \cdots & c_1c_n \\ c_2c_1 & c_2c_2 & \cdots & c_2c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_nc_1 & c_nc_2 & \cdots & c_nc_n \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \boldsymbol{A}\boldsymbol{x},$$

where

$$\boldsymbol{A} = \begin{pmatrix} c_1c_1 & c_1c_2 & \cdots & c_1c_n \\ c_2c_1 & c_2c_2 & \cdots & c_2c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_nc_1 & c_nc_2 & \cdots & c_nc_n \end{pmatrix}.$$

Then

$$T(\boldsymbol{x}) = \boldsymbol{x} - (\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n} = \boldsymbol{I}\boldsymbol{x} - \boldsymbol{A}\boldsymbol{x} = (\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x}, \quad \boldsymbol{x} \in \mathbb{R}^n.$$

It follows that T is a linear transformation with standard matrix $\boldsymbol{I} - \boldsymbol{A}$.

The matrix $\boldsymbol{A}$ is obviously obtained from $\boldsymbol{n}$. Let us try to express it in terms of $\boldsymbol{n}$.

Note that $\boldsymbol{A}$ is a square matrix of order n whose (i,j)-entry is c_ic_j. Then $\boldsymbol{A}$ can only be obtained from the product of an $n \times 1$ matrix and a $1 \times n$ matrix. Since $\boldsymbol{n}$ is $n \times 1$ (a column vector), $\boldsymbol{n}^{\mathrm{T}}$ is a $1 \times n$ matrix. Perhaps we can try

$$\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}\begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix} = \begin{pmatrix} c_1c_1 & c_1c_2 & \cdots & c_1c_n \\ c_2c_1 & c_2c_2 & \cdots & c_2c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_nc_1 & c_nc_2 & \cdots & c_nc_n \end{pmatrix} = \boldsymbol{A}.$$

Hence, the standard matrix for T is

$$\boldsymbol{I} - \boldsymbol{A} = \boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}.$$

Will it be possible if we can derive $(\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n} = \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}\boldsymbol{x}$ directly?

Note that the right-hand $\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}\boldsymbol{x}$ is the product of $n \times 1$, $1 \times n$, and $n \times 1$ matrices. We may rewrite the left-hand side in terms of multiplication too.

(i) The dot product $\boldsymbol{u} \cdot \boldsymbol{v}$ is clearly $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v}$ or $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u}$.

(ii) Let $c \in \mathbb{R}$ and $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$. Then the scalar product $c\boldsymbol{v} = \begin{pmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{pmatrix}$. But this is *not* in matrix multiplication — c is a 1×1 matrix and $\boldsymbol{v}$ is an $n \times 1$ matrix; so $c\boldsymbol{v}$ is not well-defined as matrix multiplication.

On the other hand, $\boldsymbol{v}c$ is well-defined as matrix multiplication:

$$\boldsymbol{v}c = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \begin{pmatrix} c \end{pmatrix} = \begin{pmatrix} v_1 c \\ v_2 c \\ \vdots \\ v_n c \end{pmatrix} = c \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = c\boldsymbol{v}.$$

In other words, $c\boldsymbol{v} = \boldsymbol{v}c$, while the left-hand is scalar product and the right-hand side is matrix product.

Now we are ready to write

$$(\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{x}) = \boldsymbol{n}(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{x}) = (\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})\boldsymbol{x}.$$

Hence,

$$T(\boldsymbol{x}) = \boldsymbol{x} - (\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} = \boldsymbol{x} - (\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})\boldsymbol{x} = (\boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})\boldsymbol{x},$$

which shows that T is a linear transformation with standard matrix $\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}$.

7.7(b). Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ be linear transformations with standard matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. Then their composite $S \circ T$, defined by

$$S \circ T(\boldsymbol{x}) = S(T(\boldsymbol{x})), \qquad \boldsymbol{x} \in \mathbb{R}^n.$$

is again a linear transformation $S \circ T : \mathbb{R}^n \to \mathbb{R}^p$ with standard matrix $\boldsymbol{B}\boldsymbol{A}$.

By **7.7(a)**, $P : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation with standard matrix $\boldsymbol{B} = \boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}$. Then $P \circ P$ is a linear transformation with standard matrix

$$\begin{aligned}
\boldsymbol{B}\boldsymbol{B} &= (\boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})(\boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}) \\
&= \boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} - (\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})(\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}) \\
&= \boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} + \boldsymbol{n}(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{n})\boldsymbol{n}^{\mathrm{T}} \\
&= \boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} + \boldsymbol{n}1\boldsymbol{n}^{\mathrm{T}} \\
&= \boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} + \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} \\
&= \boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} = \boldsymbol{B}.
\end{aligned}$$

Since $P \circ P$ and P have the same standard matrix, they are the same linear transformation:

$$P \circ P = P.$$

EXERCISE 7.10

A linear operator T on $\mathbb{R}^n$ (i.e., a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$) is called an *isometry* if $\|T(\boldsymbol{u})\| = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \mathbb{R}^n$.

(a) If T is an isometry on $\mathbb{R}^n$, show that $T(\boldsymbol{u}) \cdot T(\boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.

(b) Let $\boldsymbol{A}$ be the standard matrix for a linear operator T. Show that T is an isometry if and only if $\boldsymbol{A}$ is an orthogonal matrix.

(c) Find all isometries on $\mathbb{R}^2$.

By definition, a linear operator T is an isometry if it preserves the *norm*.

7.10(a). We have seen that norm is defined using the dot product:

$$\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v}.$$

Consequently, suppose that a linear operator T preserves the dot product, i.e.,

$$T(\boldsymbol{u}) \cdot T(\boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v} \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n.$$

In particular, let $\boldsymbol{u} = \boldsymbol{v}$, then

$$\|T(\boldsymbol{u})\|^2 = T(\boldsymbol{u}) \cdot T(\boldsymbol{u}) = \boldsymbol{u} \cdot \boldsymbol{u} = \|\boldsymbol{u}\|^2,$$

i.e., $\|T(\boldsymbol{u})\| = \|\boldsymbol{u}\|$. So T also preserves the norm, and it is an isometry.

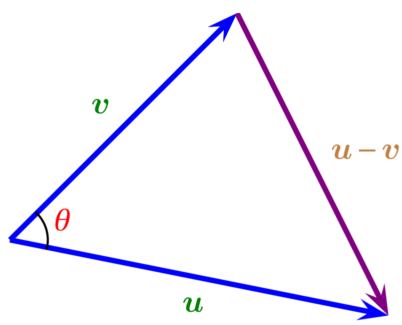
However, this question asks for the *converse*: If a linear operator preserves the norm, then it also preserves the dot product.

The question is: How to construct dot product from the norm?

Well, we may be too familiar with the formula $\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v}$, but forget the motivation of defining dot product.

Consider nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Then $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{u} - \boldsymbol{v}$ form the sides of a triangle. Let θ be the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. Then by Law of Cosine:

$$\|\boldsymbol{u} - \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - 2\|\boldsymbol{u}\|\,\|\boldsymbol{v}\| \cos\theta.$$



Hence,

$$\cos\theta = \frac{\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2}{2\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|}.$$

If $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$ are (nonzero) vectors in $\mathbb{R}^2$, then

$$\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2 = (u_1^2 + u_2^2) + (v_1^2 + v_2^2) - [(u_1 - v_1)^2 + (u_2 - v_2)^2] = 2(u_1 v_1 + u_2 v_2).$$

This gives us the motivation to define the dot product

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_1 v_1 + u_2 v_2.$$

In other words, we intend to define

$$\boldsymbol{u}\cdot\boldsymbol{v} = \frac{1}{2}(\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - (\|\boldsymbol{u}-\boldsymbol{v}\|)^2, \qquad \boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^n.$$

This identity can be verified easily:

$$\begin{aligned}
\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2 &= \boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{v}\cdot\boldsymbol{v} - (\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v}) \\
&= \boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{v}\cdot\boldsymbol{v} - (\boldsymbol{u}\cdot\boldsymbol{u} - \boldsymbol{u}\cdot\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{u} + \boldsymbol{v}\cdot\boldsymbol{v}) \\
&= 2(\boldsymbol{u}\cdot\boldsymbol{v}).
\end{aligned}$$

Suppose the linear operator that preserves the norm: $\|T(\boldsymbol{u})\| = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \mathbb{R}^n$. Then

$$\begin{aligned}
2T(\boldsymbol{u})\cdot T(\boldsymbol{v}) &= \|T(\boldsymbol{u})\|^2 + \|T(\boldsymbol{v})\|^2 - \|T(\boldsymbol{u}) - T(\boldsymbol{v})\|^2 \\
&= \|T(\boldsymbol{u})\|^2 + \|T(\boldsymbol{v})\|^2 - \|T(\boldsymbol{u}-\boldsymbol{v})\|^2 \\
&= \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2 \\
&= 2\boldsymbol{u}\cdot\boldsymbol{v},
\end{aligned}$$

that is, $T(\boldsymbol{u})\cdot T(\boldsymbol{v}) = \boldsymbol{u}\cdot\boldsymbol{v}$ for all $\boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^n$.

As a conclusion, *a linear operator preserves the norm if and only if it preserves the dot product.*

7.10(b). Let $\boldsymbol{A}$ be the standard matrix for the linear operator T, i.e., $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{R}^n$.

Since we are more familiar with matrices, we first assume that $\boldsymbol{A}$ is an orthogonal matrix. To prove that T is an isometry we shall show that $\|T(\boldsymbol{v})\| = \|\boldsymbol{v}\|$ for all $\boldsymbol{v} \in \mathbb{R}^n$, i.e.,

$$\|\boldsymbol{A}\boldsymbol{v}\| = \|\boldsymbol{v}\| \quad \text{for all } \boldsymbol{v} \in \mathbb{R}^n.$$

This was proved in the previous tutorial (Exercise 5.32):

$$\|\boldsymbol{A}\boldsymbol{v}\|^2 = (\boldsymbol{A}\boldsymbol{v})^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{v}) = \boldsymbol{v}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})\boldsymbol{v} = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{v} = \|\boldsymbol{v}\|^2;$$

so $\|\boldsymbol{A}\boldsymbol{v}\| = \|\boldsymbol{v}\|$.

Conversely, suppose that T is an isometry. By definition, it preserves the norm:

$$\|T(\boldsymbol{v})\| = \|\boldsymbol{v}\|.$$

By **7.10(a)** it also preserves the dot product:

$$T(\boldsymbol{u}) \cdot T(\boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v}.$$

There are several equivalent conditions for the standard matrix $\boldsymbol{A}$ being orthogonal. But we shall use one related to *vectors*, because the given relations are on vectors.

The standard matrix $\boldsymbol{A}$ is obtained such that the j^{th} column is $T(\boldsymbol{e}_j)$, where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is the standard basis for $\mathbb{R}^n$. Hence, $\boldsymbol{A}$ is an orthogonal matrix if and only if its columns

$$\{T(\boldsymbol{e}_1), T(\boldsymbol{e}_2), \ldots, T(\boldsymbol{e}_n)\}$$

form an orthonormal basis for $\mathbb{R}^n$.

(i) Since T preserves the norm:

$$\|T(\boldsymbol{e}_i)\| = \|\boldsymbol{e}_i\| = 1, \qquad i = 1, \ldots, n.$$

In other words, each column of $\boldsymbol{A}$ is a unit vector.

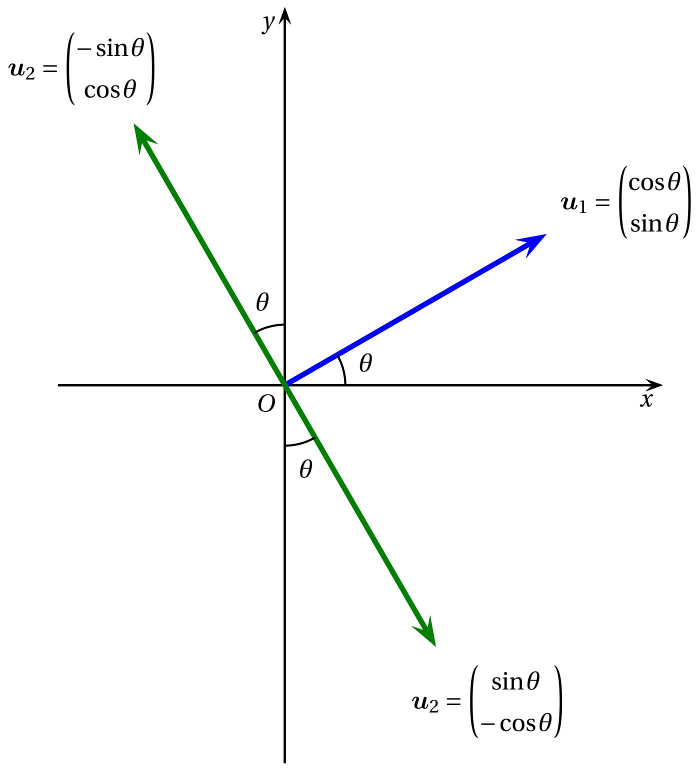
(ii) Since T preserves the dot product:

$$T(\boldsymbol{e}_i) \cdot T(\boldsymbol{e}_j) = \boldsymbol{e}_i \cdot \boldsymbol{e}_j = 0 \quad \text{for } i \neq j.$$

In other words, the columns of $\boldsymbol{A}$ are orthogonal.

Therefore, the columns of $\boldsymbol{A}$ form an orthonormal basis for $\mathbb{R}^n$. So $\boldsymbol{A}$ is an orthogonal matrix.

7.10(c). Let T be an isometry on $\mathbb{R}^2$. Then its standard matrix $\boldsymbol{A}$ is an orthogonal matrix. So the classification of all isometries on $\mathbb{R}^2$ is equivalent to the classification of all orthogonal matrices of order 2. Write $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{pmatrix}$. Then $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ is an orthonormal basis for $\mathbb{R}^2$.



Since $\boldsymbol{u}_1$ is a unit vector, let θ be the angle from the positive direction of the x-axis to $\boldsymbol{u}_1$. Then we can write $\boldsymbol{u}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$.

Since $\boldsymbol{u}_2$ is a unit vector that is orthogonal to $\boldsymbol{u}_1$, it is obtained from $\boldsymbol{u}_1$ by a rotation of $90°$ (or $\pi/2$ radians) about the origin O.

(i) If the rotation is anticlockwise, then

$$\boldsymbol{u}_2 = \begin{pmatrix} \cos(\theta+\pi/2) \\ \sin(\theta+\pi/2) \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

(ii) If the rotation is clockwise, then

$$\boldsymbol{u}_2 = \begin{pmatrix} \cos(\theta-\pi/2) \\ \sin(\theta-\pi/2) \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}.$$

Therefore, $\boldsymbol{A}$ has the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

Exercise 7.13

In each of the following parts, use the given information to find the nullity of the linear transformation T.

(a) $T : \mathbb{R}^4 \to \mathbb{R}^6$ has rank 2.

(b) The range of $T : \mathbb{R}^6 \to \mathbb{R}^4$ is $\mathbb{R}^4$.

(c) The reduced row-echelon form of the standard matrix for $T : \mathbb{R}^6 \to \mathbb{R}^6$ has 4 nonzero rows.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. It is associated two subspaces:

(i) The *range* of T is the collection of all images under T:

$$\mathrm{R}(T) = \{T(\boldsymbol{v}) \mid \boldsymbol{v} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

Its dimension is called the *rank* of T.

(ii) The *kernel* of T is the collection of all vectors mapped to $\mathbf{0}$ by T:

$$\mathrm{Ker}(T) = \{\boldsymbol{v} \mid T(\boldsymbol{v}) = \mathbf{0}\} \subseteq \mathbb{R}^n.$$

Its dimension is called the *nullity* of T.

Let $\boldsymbol{A}$ be the standard matrix for T. Then

(i) $\mathrm{R}(T)$ is the column space of $\boldsymbol{A}$, and $\mathrm{rank}(T) = \mathrm{rank}(\boldsymbol{A})$.

(ii) $\mathrm{Ker}(T)$ is the nullspace of $\boldsymbol{A}$, and $\mathrm{nullity}(T) = \mathrm{nullity}(\boldsymbol{A})$.

Moreover, *dimension theorem* for linear transformation states that

$$\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(\text{domain of } T).$$

7.13(a). $T : \mathbb{R}^4 \to \mathbb{R}^6$ has rank 2.

By dimension theorem,

$$\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(\mathbb{R}^4) = 4.$$

It is given that $\mathrm{rank}(T) = 2$, then $\mathrm{nullity}(T) = 4 - \mathrm{rank}(T) = 4 - 2 = 2$.

7.13(b). The range of $T : \mathbb{R}^6 \to \mathbb{R}^4$ is $\mathbb{R}^4$.

By dimension theorem,

$$\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(\mathbb{R}^6) = 6.$$

It is given that $\mathrm{R}(T) = \mathbb{R}^4$; so

$$\mathrm{rank}(T) = \dim \mathrm{R}(T) = \dim(\mathbb{R}^4) = 4.$$

Then $\mathrm{nullity}(T) = 6 - \mathrm{rank}(T) = 6 - 4 = 2$.

7.13(c). The reduced row-echelon form of the standard matrix for $T : \mathbb{R}^6 \to \mathbb{R}^6$ has 4 nonzero rows.

The same as **7.13(b)**, by dimension theorem

$$\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(\mathbb{R}^6) = 6.$$

It is given that the reduced row-echelon form $\boldsymbol{R}$ of the standard matrix $\boldsymbol{A}$ has 4 nonzero rows. Note that the nonzero rows of $\boldsymbol{R}$ form a basis for the row space of $\boldsymbol{R}$, as well as for the row space of $\boldsymbol{A}$. Then

$$\operatorname{rank}(\boldsymbol{A}) = 4,$$

which equals $\operatorname{rank}(T)$. Hence, $\operatorname{nullity}(T) = 6 - \operatorname{rank}(T) = 6 - 4 = 2$.