CS1231S: Discrete Structures

Tutorial #11: Graph II and Tree

(Week 13: 7 – 11 November 2022)

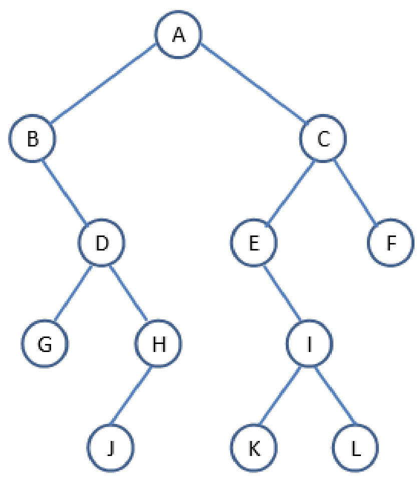
I. Discussion Questions

You are strongly encouraged to discuss D1 – D3 on Canvas or QnA. No answers will be provided.

D1. For any simple connected graph with n $(n > 0)$ vertices, what is the minimum and maximum number of edges the graph may have?

D2. (AY2016/17 Semester 1 Exam Question)
How many simple graphs on 3 vertices are there?
In general, how many simple graphs on n $(n > 1)$ vertices are there?

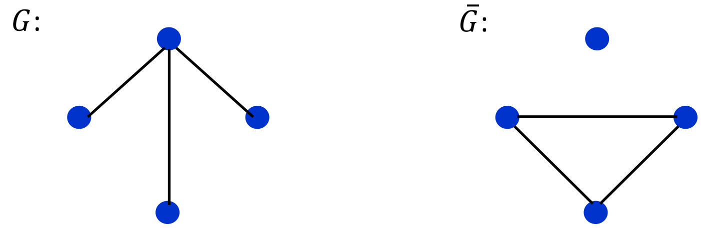
D3. Given the following binary tree, write the pre-order, in-order, and post-order traversals of its vertices.



II. Definitions

Definition 1. If G is a simple graph, the *complement* of G, denoted $\bar{G}$, is obtained as follows: the vertex set of $\bar{G}$ is identical to the vertex set of G. However, two distinct vertices v and w of $\bar{G}$ are connected by an edge if and only if v and w are not connected by an edge in G.

The figure below shows a graph G and its complement $\bar{G}$.



A graph G and its complement $\bar{G}$.

Definition 2. A *self-complementary* graph is isomorphic with its complement.

Definition 3. A simple circuit (cycle) of length three is called a *triangle*.

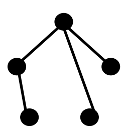
You can freely use the following lemma (without proof). The proof is left as an optional exercise.

Lemma 10.5.5. Let G be a simple, undirected graph. Then if there are two distinct paths from a vertex v to a different vertex w, then G contains a cycle (and hence G is cyclic).

III. Tutorial Questions

1. (a) For the following graph G, draw its complement graph $\bar{G}$.

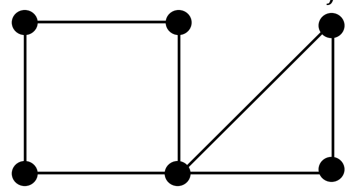
G:



(b) Consider simple graphs on n vertices. Draw all self-complementary graphs with n vertices (for $n = 3, 4, 5, 6$), or justify why there are none.

2. (AY2016/17 Semester 1 Exam Question)
Let G be a simple graph with n vertices where every vertex has degree at least $\left\lfloor \frac{n}{2} \right\rfloor$. Prove that G is connected.

3. Consider the graph G given below. How many spanning trees of G are there?



4. (a) Draw all non-isomorphic trees with n nodes, $n = 1,2,3,4$.

   (b) For each non-isomorphic tree in part (a), how many isomorphic copies are there?

5. (a) Let $G = (V, E)$ be a simple, undirected graph. Prove that if G is connected, then $|E| \geq |V| - 1$.

   (b) Is the converse true?

6. (a) Let $G = (V, E)$ be a simple, undirected graph. Prove that if G is acyclic, then $|E| \leq |V| - 1$.

   (b) Is the converse true?

7. Let $G = (V, E)$ be a simple, undirected graph. Prove that if G is a tree if and only if there is exactly one path between every pair of vertices.

8. (a) Draw all possible binary trees with 3 vertices X, Y and Z with in-order traversal: $X\ Y\ Z$.

   (b) Draw all possible binary trees with 4 vertices A, B, C and D with in-order traversal: $A\ B\ C\ D$.

9. (a) A binary tree T_1 has 9 nodes. The in-order and pre-order traversals of T_1 are given below. Draw the tree T_1 and give its post-order traversal.

   In-order: E A C K F H D B G
   Pre-order: F A E K C D H G B

   (b) A binary tree T_2 has 9 nodes. The in-order and post-order traversals of T_2 are given below. Draw the tree T_2 and give its pre-order traversal.

   In-order: D B F E A G C H K
   Post-order: D F E B G K H C A

10. (*Modified from* AY2016/17 Semester 1 Exam Question)
The figure below shows a graph where the vertices are Pokestops. Using either Kruskal's algorithm or Prim's algorithm, find its minimum spanning tree (MST). If you use Prim's algorithm, you must start with the top-most vertex.

Indicate the order of the edges inserted into the MST in your answer.

[OPTIONAL, for the FUN of it] In addition to (but not in place of), you can also use Guan's algorithm from the optional notes. The one that repeatedly removes the longest edge in *any* cycle.