Solutions to Tutorial 9

MA1521 Calculus for Computing

1. (a) $$\int_0^b \int_0^a (x^2+y^2)\,dx\,dy = \int_0^b \left[\frac{1}{3}x^3 + xy^2\right]_{x=0}^{x=a} dy = \int_0^b \left(\frac{1}{3}a^3 + ay^2\right) dy$$
$$= \left[\frac{1}{3}a^3 y + \frac{1}{3}ay^3\right]_0^b = \frac{1}{3}a^3 b + \frac{1}{3}ab^3.$$

(b) $$\int_1^2 \int_0^1 \frac{xy}{\sqrt{4-x^2}}\,dx\,dy = \int_1^2 \left[-\frac{1}{2}y\left(2(4-x^2)^{1/2}\right)\right]_{x=0}^{x=1} dy$$
$$= \int_1^2 -y(3^{1/2}-4^{1/2})\,dy$$
$$= (2-\sqrt{3})\left[\frac{1}{2}y^2\right]_{y=1}^{y=2} = 3 - \frac{3}{2}\sqrt{3}.$$

(c) The region can be regarded as a Type I region

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le x, \quad 0 \le x \le 1\}.$$

$$\int_0^1 \int_0^x e^{x^2}\,dy\,dx = \int_0^1 \left[ye^{x^2}\right]_{y=0}^{y=x} dx = \int_0^1 xe^{x^2}\,dx$$
$$= \frac{1}{2}\left[e^{x^2}\right]_0^1 = \frac{1}{2}(e-1).$$

(d) The region can be regarded as a type I region with bottom boundary $y = x^2$ and top boundary $y = \sqrt{x}$.
Since the two curves intersect at $x = 0$ and $x = 1$, the left and right are bounded by $x = 0$ and $x = 1$ respectively. So

$$D = \{(x,y) \in \mathbb{R}^2 \mid x^2 \le y \le \sqrt{x}, \quad 0 \le x \le 1\}.$$

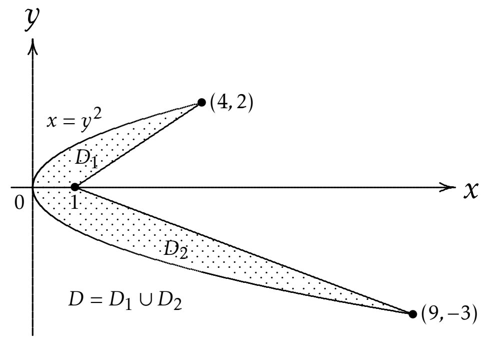
$$\int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)\,dy\,dx = \int_0^1 \left[xy + \frac{1}{2}y^2\right]_{y=x^2}^{y=\sqrt{x}} dx = \int_0^1 (x^{3/2} + \frac{1}{2}x - x^3 - \frac{1}{2}x^4)\,dx$$
$$= \left[\frac{1}{5}2x^{5/2} + \frac{1}{4}x^2 - \frac{1}{4}x^4 - \frac{1}{10}x^5\right]_0^1 = \frac{3}{10}.$$

2. The line joining $(1,0)$ and $(4,2)$ has equation

$$\frac{y-0}{x-1} = \frac{2-0}{4-1} = \frac{2}{3} \Leftrightarrow y = \frac{2}{3}x - \frac{2}{3} \Leftrightarrow x = \frac{3}{2}y + 1.$$

The line joining $(1,0)$ and $(9,-3)$ has equation

$$\frac{y-0}{x-1} = \frac{(-3)-0}{9-1} = -\frac{3}{8} \Leftrightarrow y = -\frac{3}{8}x + \frac{3}{8} \Leftrightarrow x = -\frac{8}{3}y + 1.$$



The region D is the union of D_1 and D_2, where

$$D_1 = \{(x, y) \in \mathbb{R}^2 \mid y^2 \le x \le \frac{3}{2}y + 1, \quad 0 \le y \le 2\},$$

$$D_2 = \{(x, y) \in \mathbb{R}^2 \mid y^2 \le x \le -\frac{8}{3}y + 1, \quad -3 \le y \le 0\}.$$

Hence the required answer is

$$\begin{aligned}
\iint_D x\, dA &= \iint_{D_1} x\, dA + \iint_{D_2} x\, dA \\
&= \int_0^2 \int_{y^2}^{(3y/2)+1} x\, dxdy + \int_{-3}^0 \int_{y^2}^{-(8y/3)+1} x\, dxdy \\
&= \frac{19}{5} + \frac{106}{5} = 25,
\end{aligned}$$

since

$$\int_0^2 \int_{y^2}^{(3y/2)+1} x\, dxdy = \int_0^2 \frac{1}{8}(9y^2 + 12y + 4 - 4y^4)\, dy = \frac{19}{5},$$

$$\int_{-3}^0 \int_{y^2}^{-(8y/3)+1} x\, dxdy = \int_{-3}^0 \frac{1}{18}(64y^2 - 48y + 9 - 9y^4)\, dy = \frac{106}{5}.$$

3. (a) The region of integration is the quarter of the circular region R centered at the origin with radius a in the first quadrant. In polar coordinates,

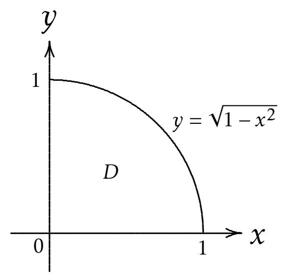
$$R = \{(r, \theta) \in \mathbb{R}^2 \mid 0 \le r \le a, \quad 0 \le \theta \le \pi/2\}$$

$$\begin{aligned}
\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{1}{1 + x^2 + y^2}\, dydx &= \int_0^{\pi/2} \int_0^a \frac{1}{1 + r^2} r\, drd\theta = \int_0^{\pi/2} d\theta \int_0^a \frac{r}{1 + r^2}\, dr \\
&= \frac{\pi}{2}\left[\frac{1}{2}\ln(1 + r^2)\right]_0^a = \frac{\pi}{4}\ln(1 + a^2).
\end{aligned}$$

(b) The region in Cartesian coordinates is given by

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le \sqrt{1-x^2}, \quad 0 \le x \le 1.\}$$

This is a type I region with x-axis as the bottom boundary and upper half of the unit circle as the upper boundary.
Since the range of x is from 0 to 1, the region D is the first quadrant of the unit disk.



In polar coordinates, this is given by

$$D = \{(r,\theta) \in \mathbb{R}^2 \mid 0 \le r \le 1, \quad 0 \le \theta \le \pi/2\}.$$

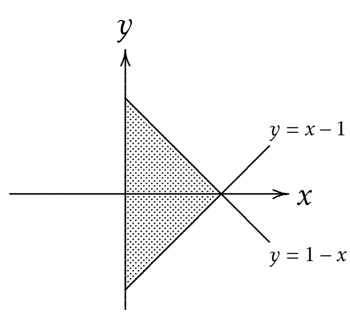
$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\, dy dx = \int_0^{\pi/2} \int_0^1 e^{r^2} r\, dr d\theta = \int_0^{\pi/2} d\theta \int_0^1 e^{r^2} r\, dr$$
$$= \frac{\pi}{2}\left[\frac{1}{2}e^{r^2}\right]_0^1 = \tfrac{1}{4}\pi(e-1).$$

4. (a) $\displaystyle\int_0^1 \int_0^{1-y} x\, dx dy + \int_{-1}^0 \int_0^{1+y} x\, dx dy = \int_0^1 \left[\frac{x^2}{2}\right]_0^{1-y} dy + \int_{-1}^0 \left[\frac{x^2}{2}\right]_0^{1+y} dy$

$$= \int_0^1 \frac{(1-y)^2}{2}\, dy + \int_{-1}^0 \frac{(1+y)^2}{2}\, dy = \left[-\frac{(1-y)^3}{6}\right]_0^1 + \left[\frac{(1+y)^3}{6}\right]_{-1}^0 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

Alternatively, as a type I region, we have

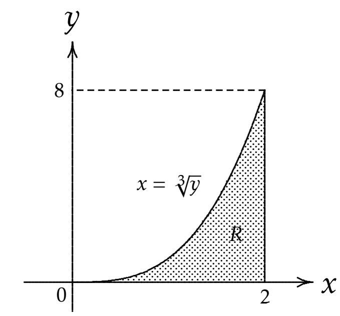
$$\int_0^1 \int_{x-1}^{1-x} x\, dy dx = \int_0^1 [xy]_{x-1}^{1-x}\, dx = \int_0^1 2x - 2x^2\, dx = \left[x^2 - \frac{2}{3}x^3\right]_0^1 = \frac{1}{3}.$$



(b) The type II region R is given by

$$R = \{(x, y) \in \mathbb{R}^2 \mid \sqrt[3]{y} \le x \le 2, \quad 0 \le y \le 8\}.$$

It is bounded on the left by the cubic curve $\sqrt[3]{y} = x$ and on the right by the vertical line $x = 2$.
Below it is bounded by the x-axis, and on top the left and right boundaries intersect at $y = 8$.



Converting to type I region, the lower boundary is $y = 0$, the top boundary is the cubic curve $y = x^3$.
On the left, these two boundaries intersect at $x = 0$ and on the right, it is bounded by $x = 2$.
So the region is given by

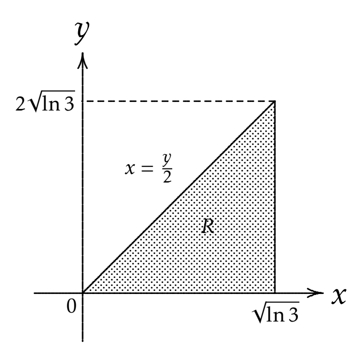
$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le x^3, \quad 0 \le x \le 2\}.$$

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx dy = \int_0^2 \int_0^{x^3} e^{x^4}\, dy dx = \int_0^2 e^{x^4} [y]_{y=0}^{y=x^3}\, dx = \int_0^2 x^3 e^{x^4}\, dx$$
$$= \left[\frac{1}{4} e^{x^4}\right]_0^2 = \frac{1}{4}(e^{16} - 1).$$

(c) The type II region R is given by

$$R = \{(x, y) \in \mathbb{R}^2 \mid y/2 \le x \le \sqrt{\ln 3}, \quad 0 \le y \le 2\sqrt{\ln 3}\}.$$

It is bounded on the left by the straight line $x = y/2$ and on the right by the vertical line $x = \sqrt{\ln 3}$.
Below it is bounded by the x-axis, and on top the left and right boundaries intersect at $y = 2\sqrt{\ln 3}$.



Converting to type I region, the lower boundary is $y = 0$, the top boundary is the line $y = 2x$.
On the left, these two boundaries intersect at $x = 0$ and on the right, it is bounded by $x = \sqrt{\ln 3}$.
So the region is given by

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 2x, \quad 0 \le x \le \sqrt{\ln 3}\}.$$

$$\begin{aligned}\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\, dx dy = \int_0^{\sqrt{\ln 3}} \int_0^{2x} e^{x^2}\, dy dx = \int_0^{\sqrt{\ln 3}} e^{x^2} [y]_{y=0}^{y=2x}\, dx = \int_0^{\sqrt{\ln 3}} 2xe^{x^2}\, dx \\ = \left[e^{x^2}\right]_0^{\sqrt{\ln 3}} = e^{\ln 3} - 1 = 2.\end{aligned}$$

Solutions to Further Exercises

1. (a) $$\begin{aligned}\int_0^1 \int_0^{\sqrt{x}} \frac{2y}{x^2+1}\, dy dx &= \int_0^1 \left[\frac{y^2}{x^2+1}\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 \frac{x}{x^2+1}\, dx \\ &= \frac{1}{2}\left[\ln|x^2+1|\right]_0^1 = \frac{1}{2}\ln 2.\end{aligned}$$

   (b) $$\begin{aligned}\int_0^1 \int_0^{x^2} x\cos y\, dy dx &= \int_0^1 [x\sin y]_{y=0}^{y=x^2}\, dx = \int_0^1 x\sin(x^2)\, dx \\ &= -\frac{1}{2}\left[\cos(x^2)\right]_0^1 = \frac{1}{2}(1-\cos(1)).\end{aligned}$$

   (c) $$\begin{aligned}\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x-y)\, dy dx &= \int_{-2}^2 \left[2xy - \frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx = \int_{-2}^2 4x\sqrt{4-x^2}\, dx \\ &= \left[-\frac{4}{3}(4-x^2)^{3/2}\right]_{-2}^2 = 0.\end{aligned}$$

   Or observe that $4x\sqrt{4-x^2}$ is an odd function, so $\int_{-2}^2 4x\sqrt{4-x^2}, dx = 0$.
   We may also use polar coordinates. Let $x = r\cos\theta$, $y = r\sin\theta$. Then $dA = r dr d\theta$. Thus,

   $$\begin{aligned}\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x-y)\, dy dx &= \int_0^2 \int_0^{2\pi} (2r\cos\theta - r\sin\theta) r\, d\theta dr \\ &= \int_0^2 \left[2r^2\sin\theta + r^2\cos\theta\right]_0^{2\pi} dr = 0.\end{aligned}$$

2. We need to find the volume of the solid lying under the surface $z = 1 + (x-1)^2 + 4y^2$ and above the rectangle $R = [0,3]\times[0,2]$ in the xy-plane.

$$
\begin{aligned}
V &= \int_0^3 \int_0^2 [1+(x-1)^2+4y^2]\,dy\,dx = \int_0^3 \left[y+(x-1)^2 y+\frac{4}{3}y^3 \right]_{y=0}^{y=2} dx \\
&= \int_0^3 [2+2(x-1)^2+\frac{32}{3}]\,dx = \left[\frac{38}{3}x+\frac{2}{3}(x-1)^3 \right]_0^3 = 44.
\end{aligned}
$$

3. $$\iint\limits_R xy^2\,dy\,dx = \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} xy^2\,dy\,dx$$

$$= \int_0^a \left[\frac{1}{3}xy^3 \right]_{y=a-x}^{y=\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{1}{3}x(a^2-x^2)^{3/2} - \frac{1}{3}x(a-x)^3\,dx = \frac{a^5}{20}.$$