CS2040S

Data Structures and Algorithms

On the importance of being balanced

Welcome!

Plan of the Day

Trees

- Terminology
- Traversals
- Operations

Balanced Trees

- Height-balanced binary search trees
- AVL trees
- Rotations

Announcements

Midterm : Monday March 6, 4pm (class time)

Location: MPSH 2A & 2B

Note: In person, pen and paper

Nota Bene: Please mark your calendar now.

Trees

On the importance of being balanced

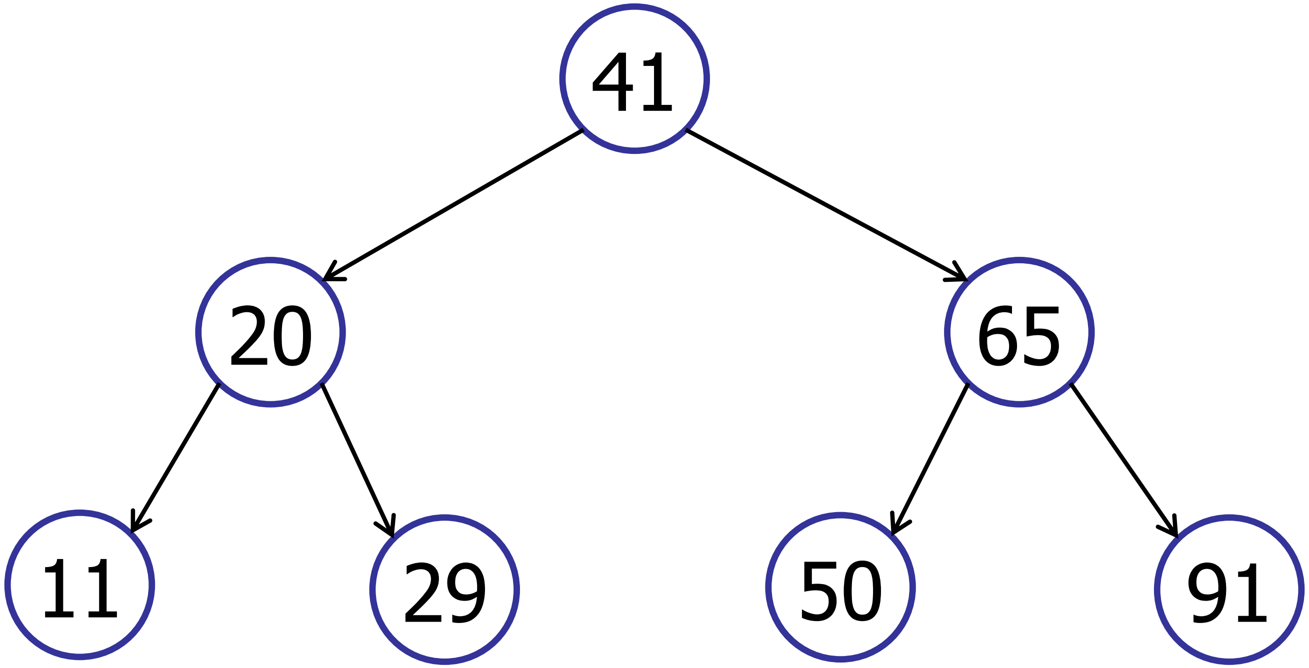
- Height-balanced binary search trees
- AVL trees
- Rotations

Dictionary Interface

A collection of (key, value) pairs:

interface	IDictionary	
void	insert(Key k, Value v)	*insert (k,v) into table*
Value	search(Key k)	*get value paired with k*
Key	successor(Key k)	*find next key > k*
Key	predecessor(Key k)	*find next key < k*
void	delete(Key k)	*remove key k (and value)*
boolean	contains(Key k)	*is there a value for k?*
int	size()	*number of (k,v) pairs*

Recap: Binary Search Trees



- Two children: v.left, v.right
- Key: v.key
- **BST Property**: all in left sub-tree < key < all in right sub-right

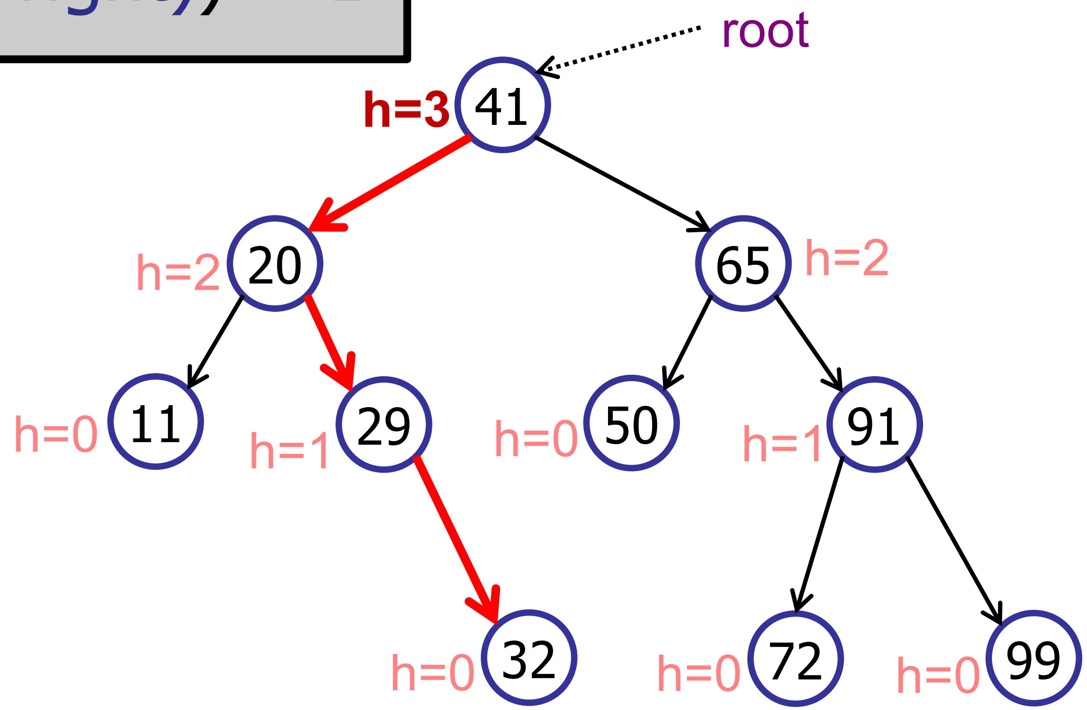
Binary Search Trees Heights

Height:

Number of edges on longest path from root to leaf.

$h(v) = 0$ (if v is a leaf)

$h(v) = \max(h(v.left), h(v.right)) + 1$



(For simplicity: $h(null) = -1$)

Binary Search Tree

Modifying Operations

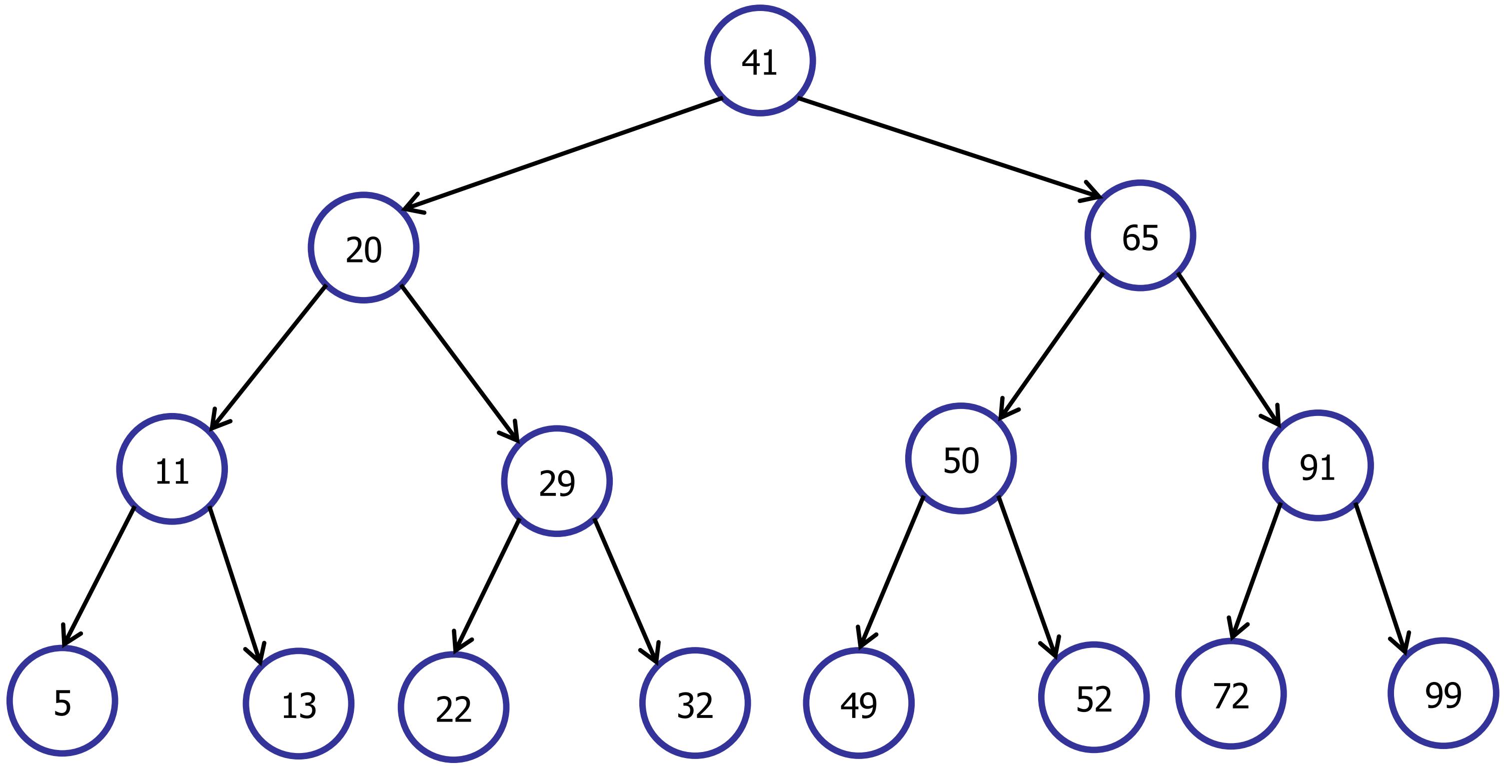
- insert
- delete

Query Operations:

- search
- predecessor, successor
- findMax, findMin
- in-order-traversal

The Importance of Being Balanced

Operations take O(h) time



Binary Search Tree

Modifying Operations

- insert: O(h)
- delete: O(h)

Query Operations:

- search: O(h)
- predecessor, successor: O(h)
- findMax, findMin: O(h)
- in-order-traversal: O(n)

What is the largest possible height h?

1. $\theta(1)$
2. $\theta(\log n)$
3. $\theta(\text{sqrt}(n))$
4. $\theta(n)$
5. $\theta(n^2)$

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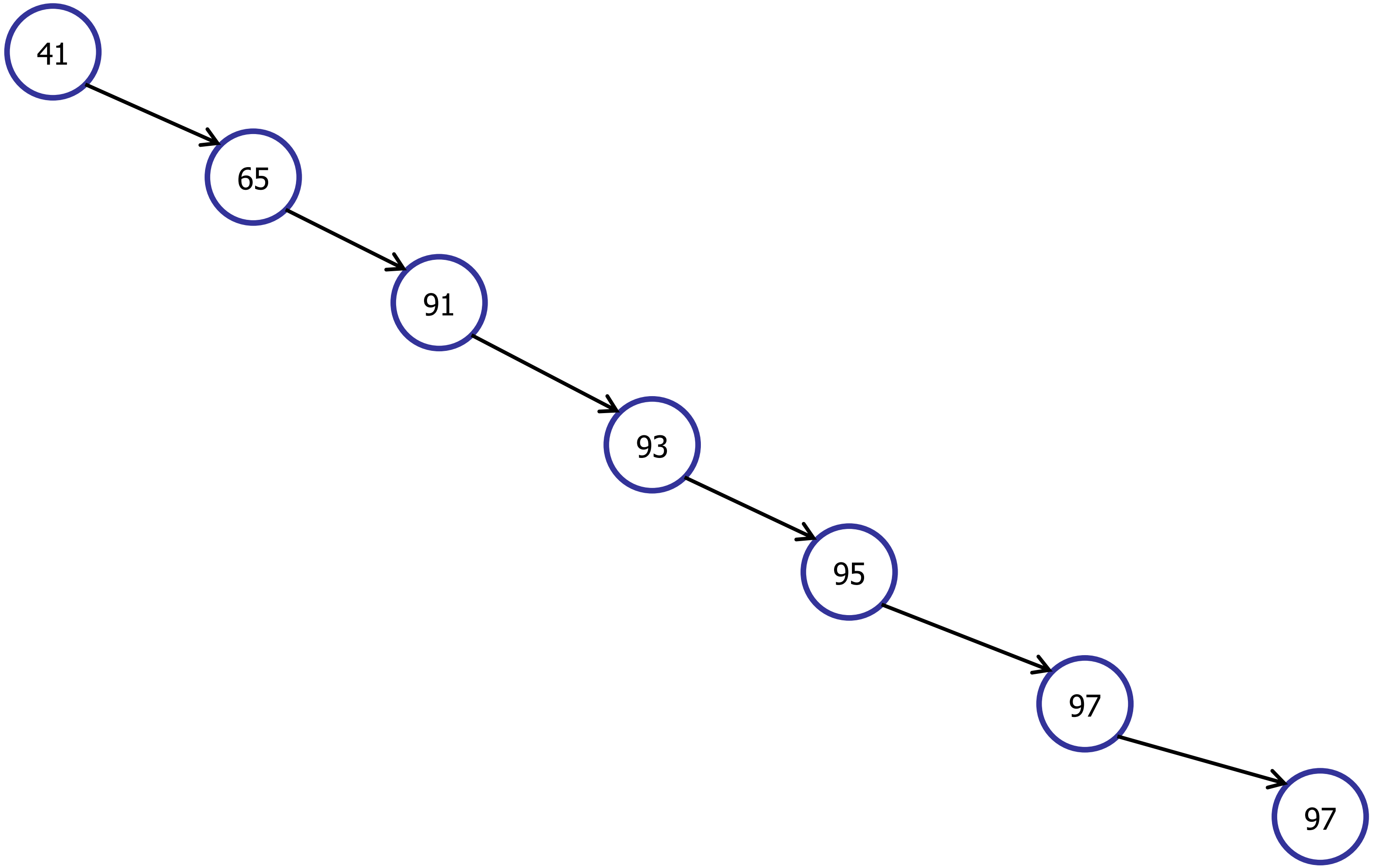
is open

The Importance of Being Balanced

Operations take O(h) time

$h \leq n$



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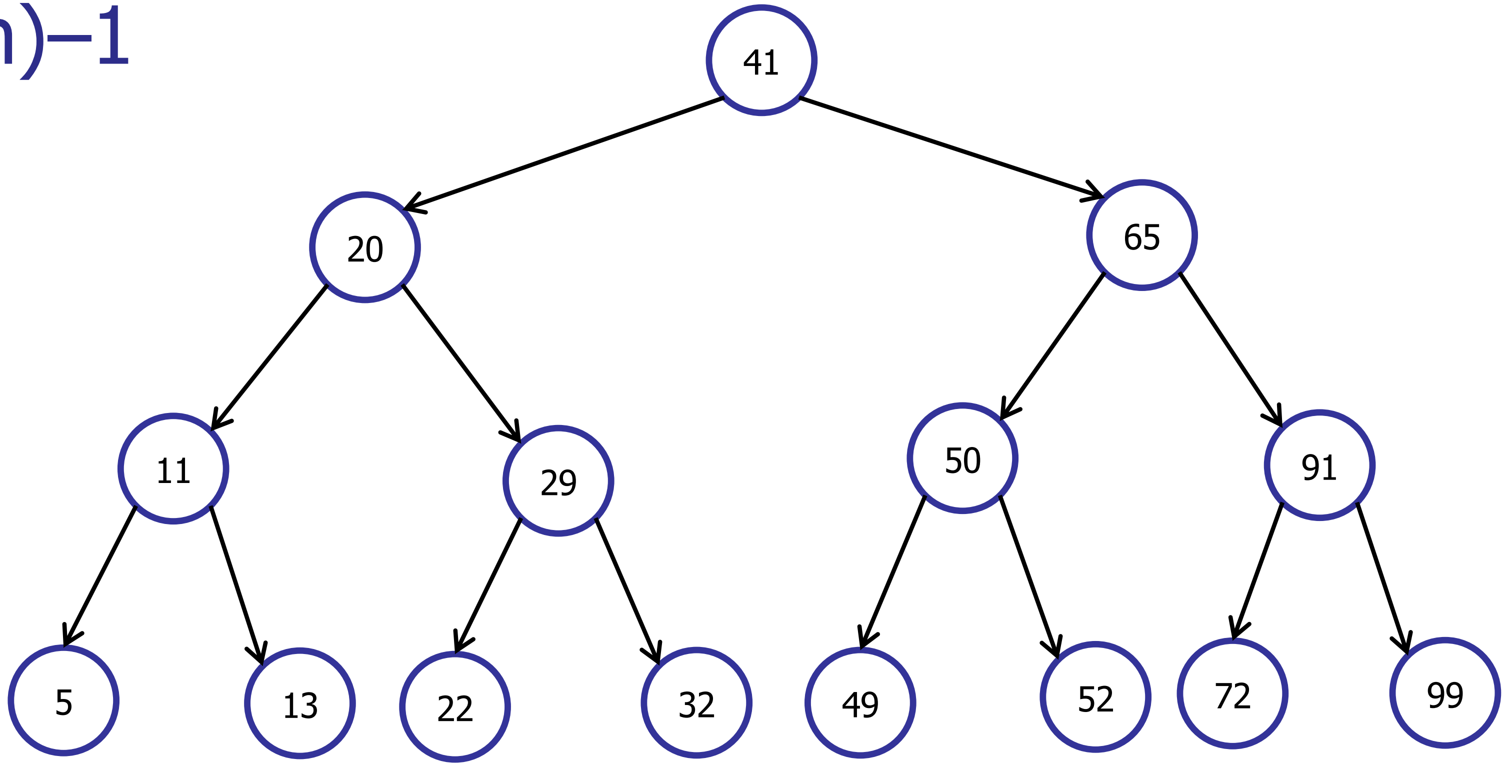
The Importance of Being Balanced

Operations take O(h) time

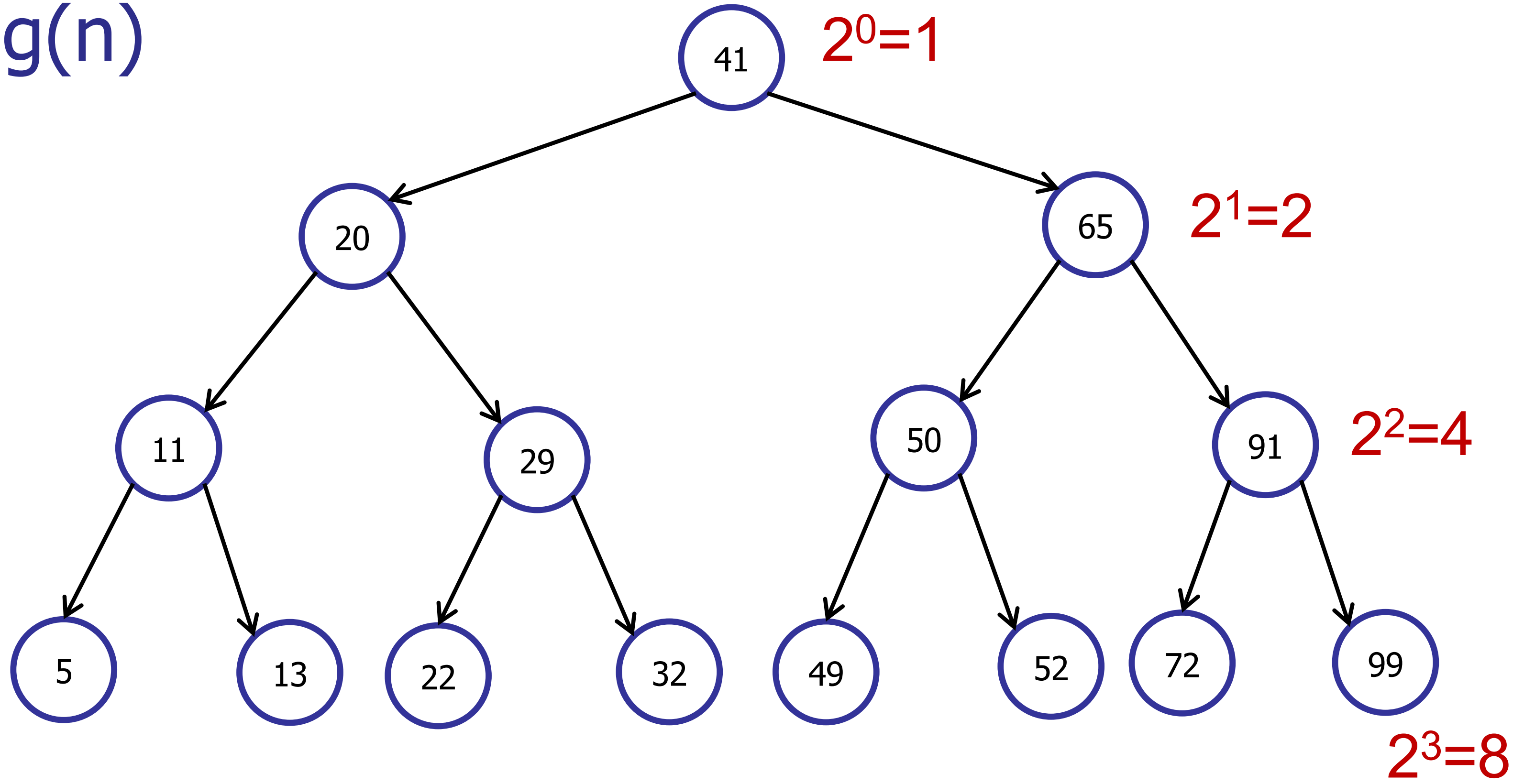
$h \geq \log(n) - 1$



The Importance of Being Balanced

Operations take O(h) time

$h+1 \geq \log(n)$



$n \leq 1 + 2 + 4 + \ldots + 2^h$

$\leq 2^0 + 2^1 + 2^2 + \ldots + 2^h < 2^{h+1}$

The Importance of Being Balanced

Operations take O(h) time

$\log(n) - 1 \leq h \leq n$

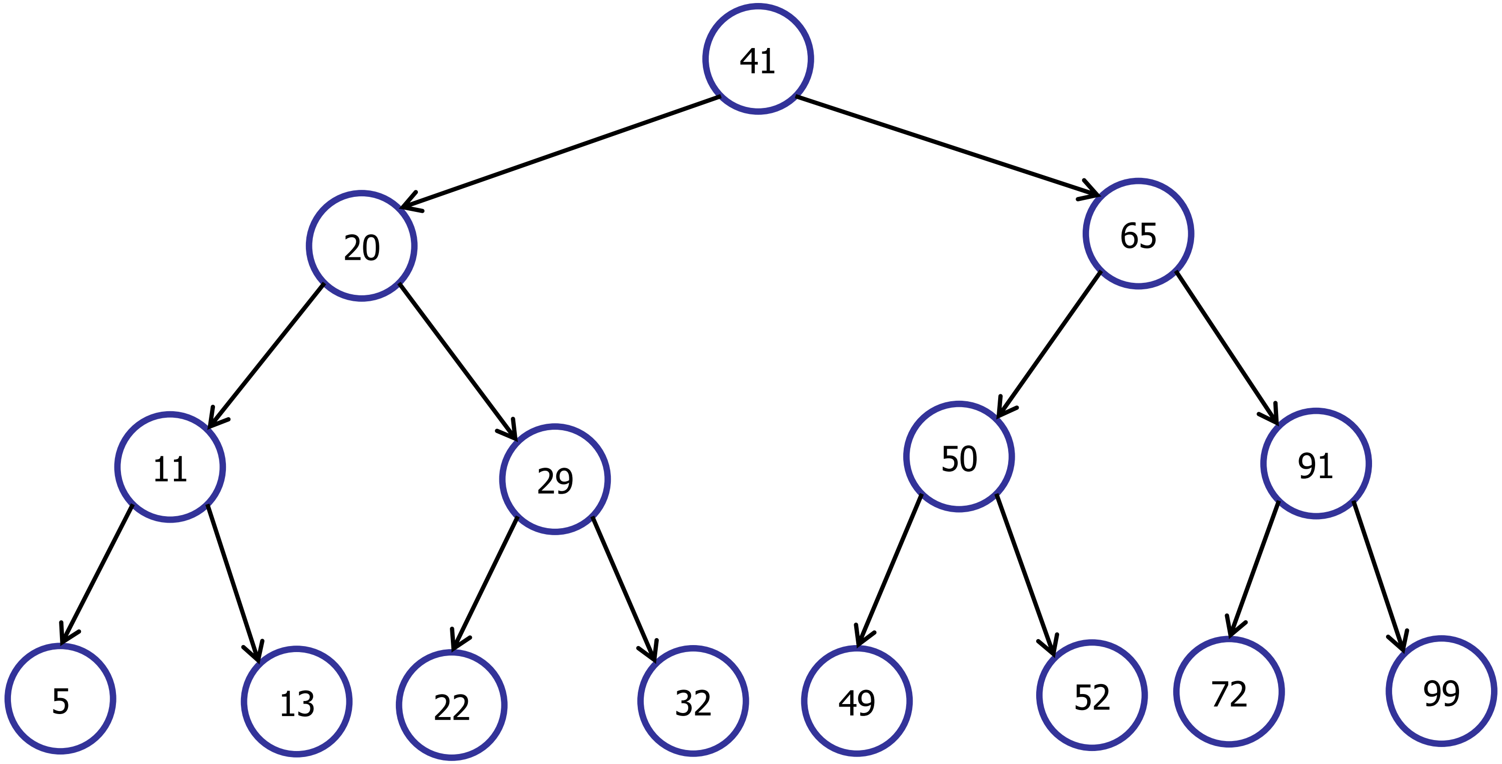
Key definition

A BST is <u>balanced</u> if $h = O(\log n)$

On a balanced BST: all operations run in $O(\log n)$ time.

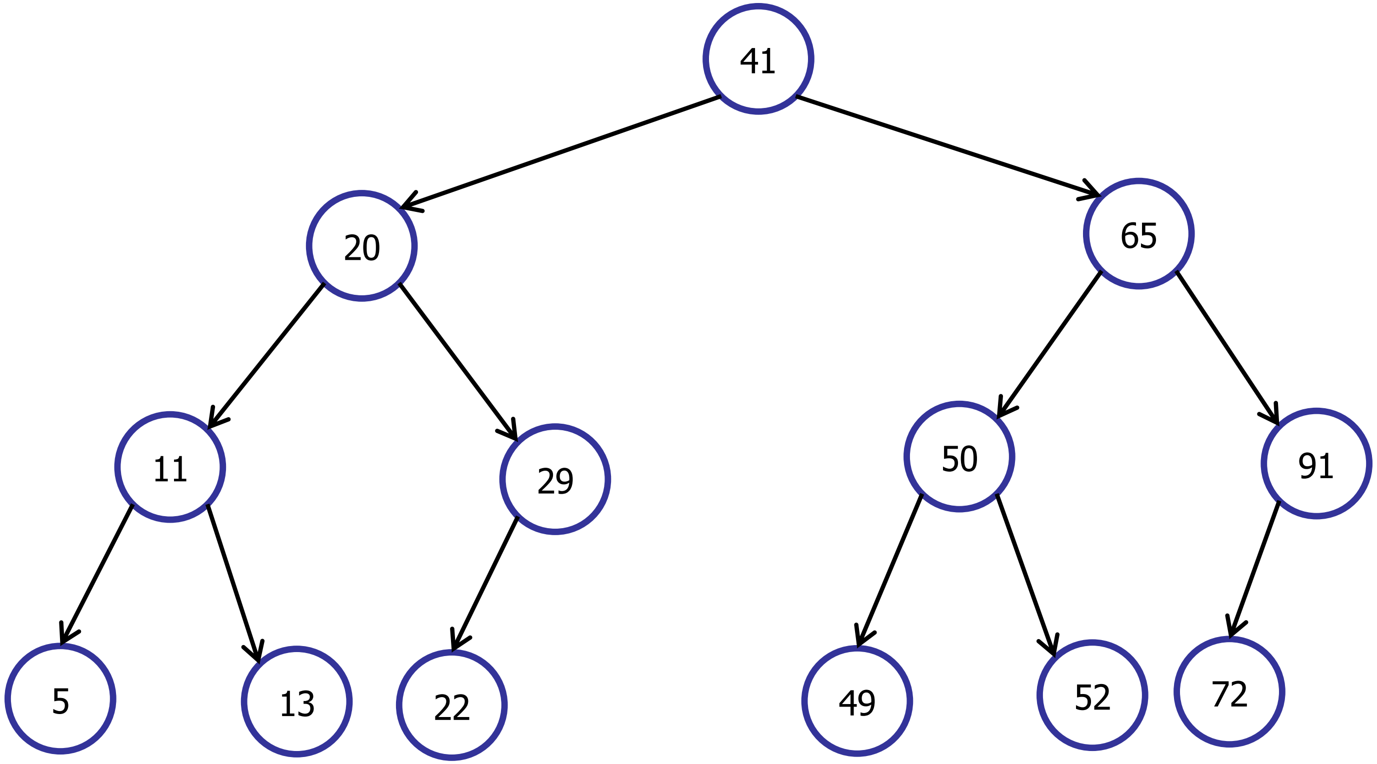
The Importance of Being Balanced

Perfectly balanced:



The Importance of Being Balanced

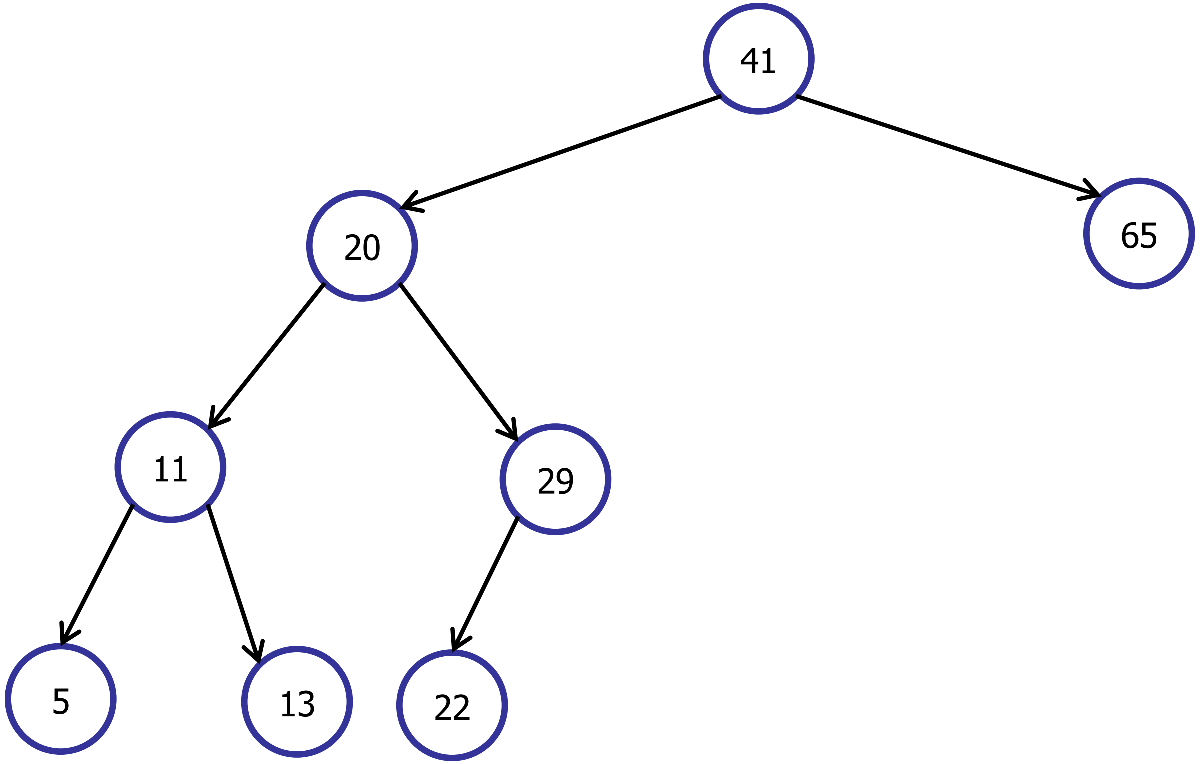
Almost perfectly balanced:



Every subtree has (almost) the same number of nodes.

The Importance of Being Balanced

Not perfectly balanced:



Left tree has 6, right tree has 1.

Balanced Search Trees

Many different flavors of balanced search trees

- AVL trees (Adelson-Velsii & Landis, 1962)
- B-trees / 2-3-4 trees (Bayer & McCreight, 1972)
- BB[α] trees (Nievergelt & Reingold 1973)
- Red-black trees (see CLRS 13)
- Splay trees (Sleator and Tarjan 1985)
- Treaps (Seidel and Aragon 1996)
- Skip Lists (Pugh 1989)
- Scapegoat Trees (Anderson 1989)

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The Importance of Being Balanced

How to get a balanced tree:

- Define a good property of a tree.
- Show that if the good property holds, then the tree is balanced.
- After every insert/delete, make sure the good property still holds. If not, fix it.

Invariant

AVL Trees [Adelson-Velskii & Landis 1962]

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Step 0: Augment

Step 1: Define Balance Condition

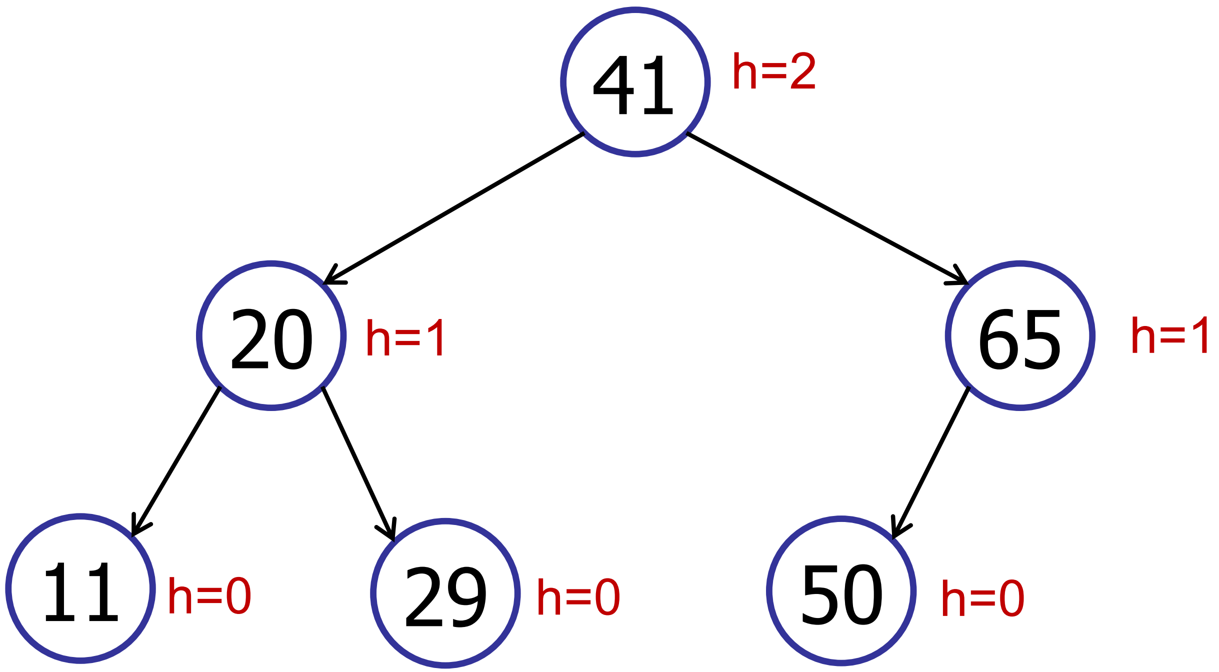
Step 2: Maintain Balance

AVL Trees [Adelson-Velskii & Landis 1962]

Step 0: Augment

- In every node v, store height:

v.height = h(v)



AVL Trees [Adelson-Velskii & Landis 1962]

Step 0: Augment

- In every node v, store height:

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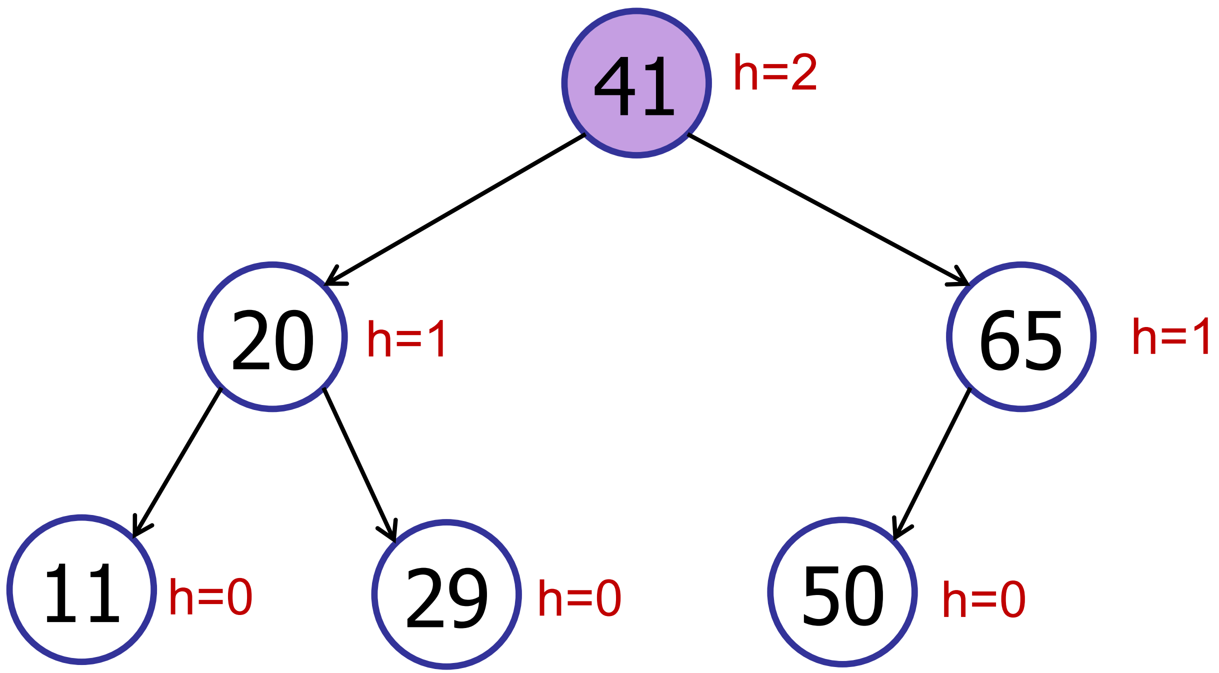
- On insert & delete update height:

```
insert(x)
        if (x < key)
                left.insert(x)
        else right.insert(x)
        height = max(left.height, right.height) + 1
```

Binary Search Trees

insert(27)



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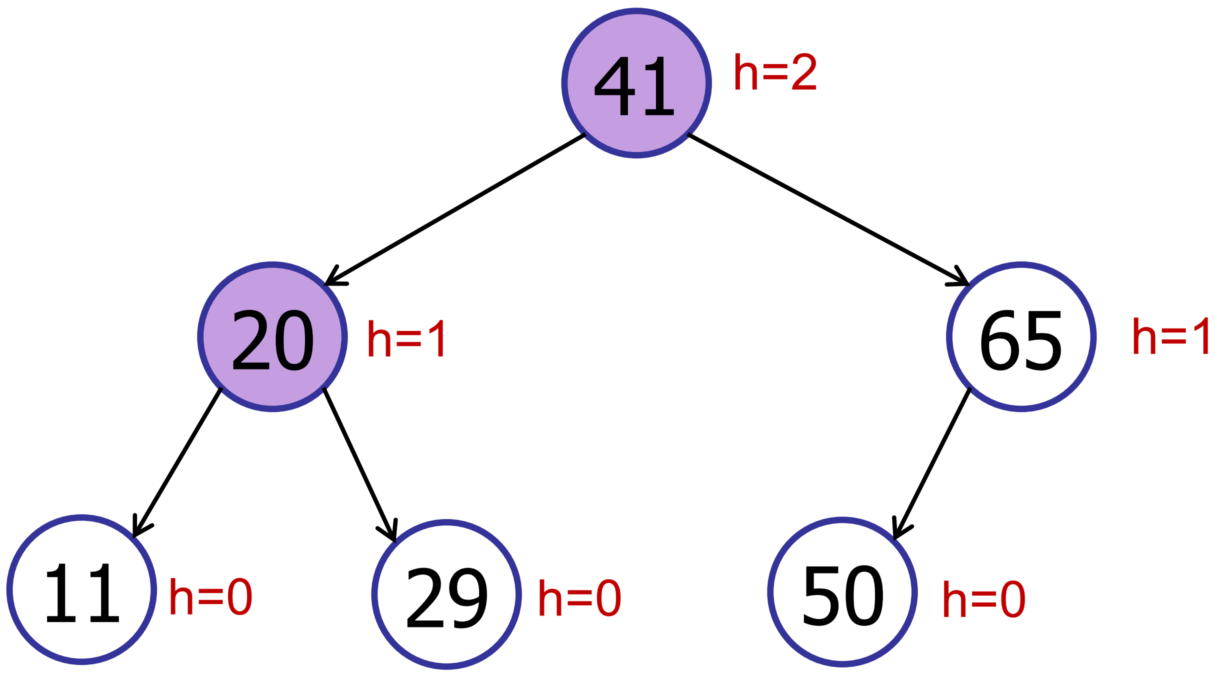

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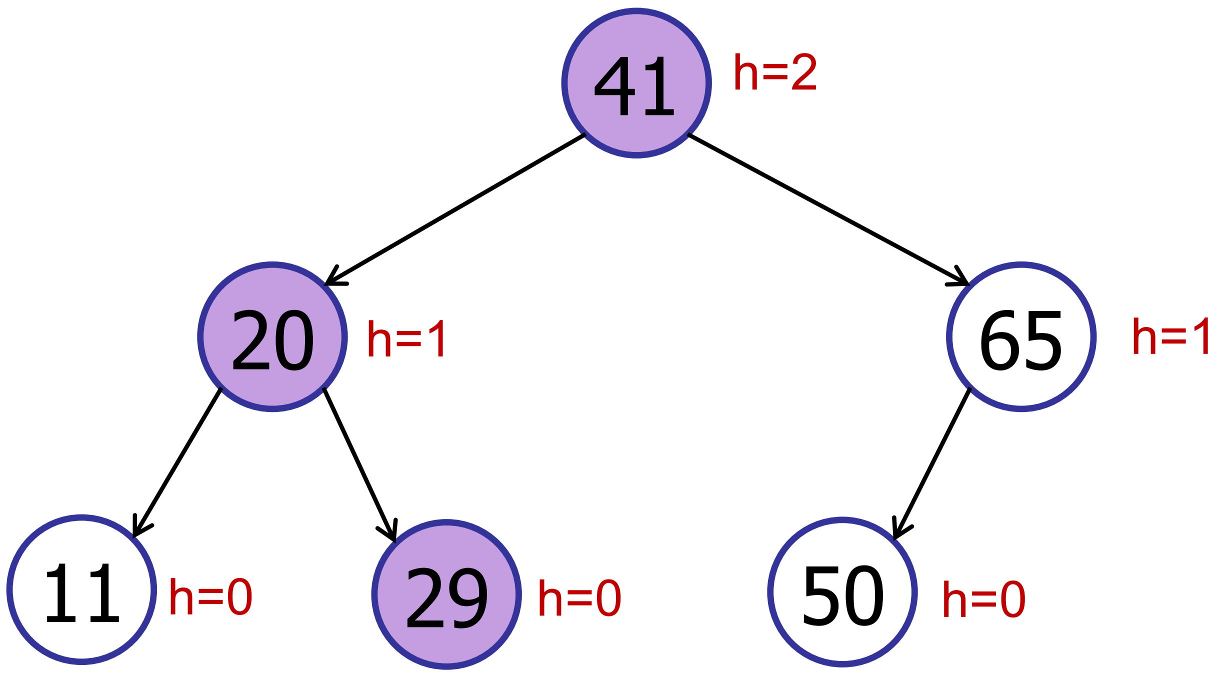

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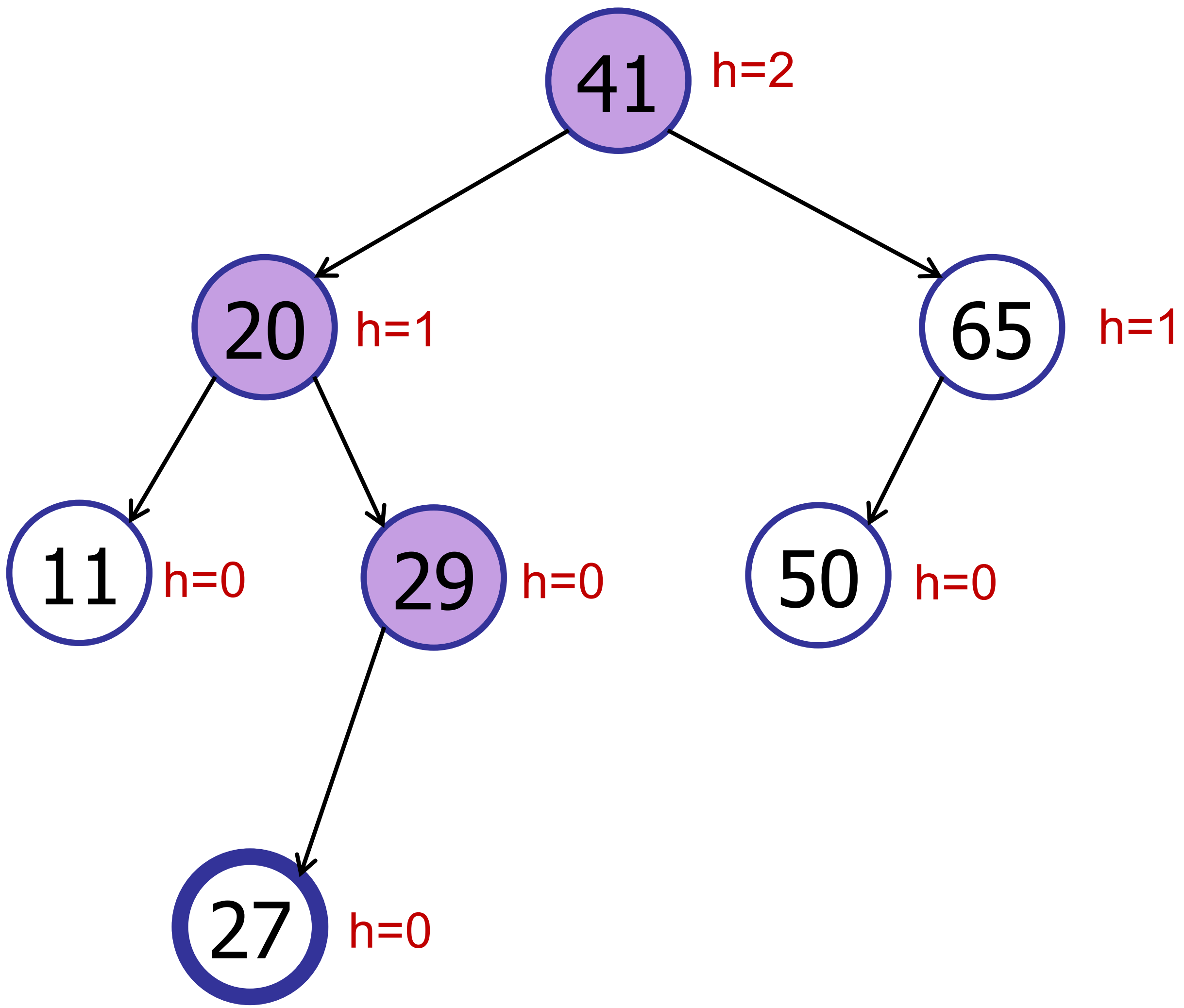

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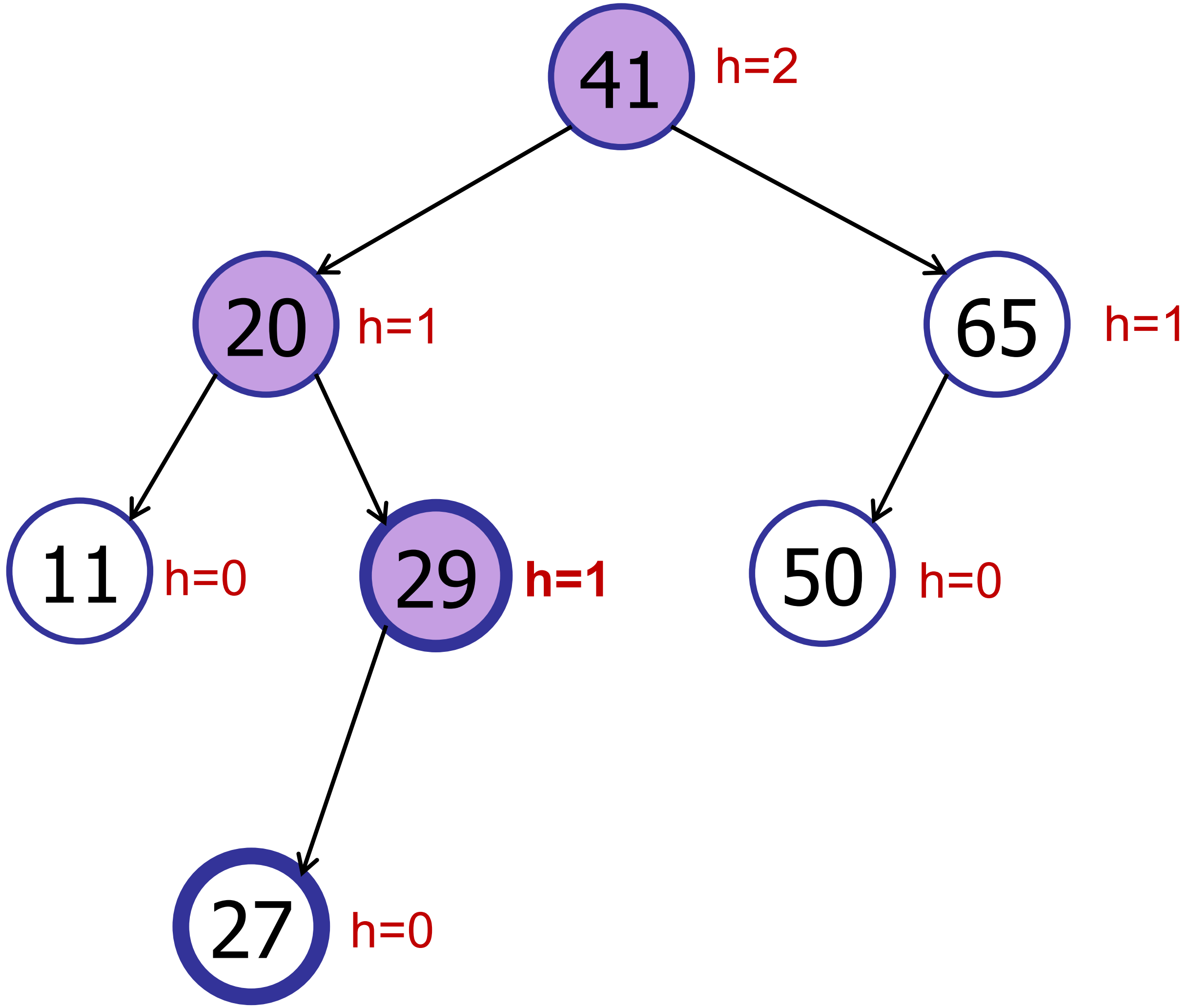

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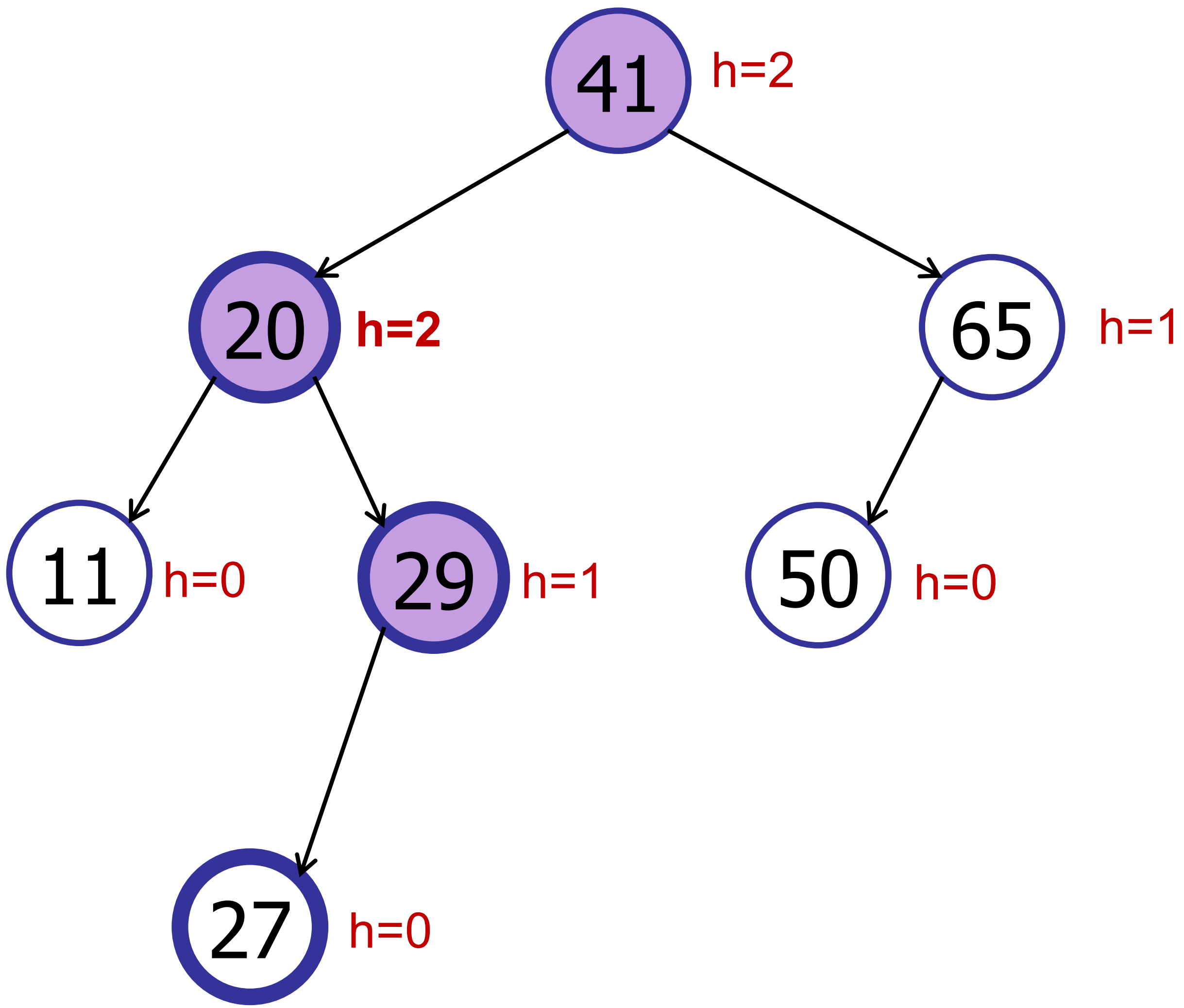

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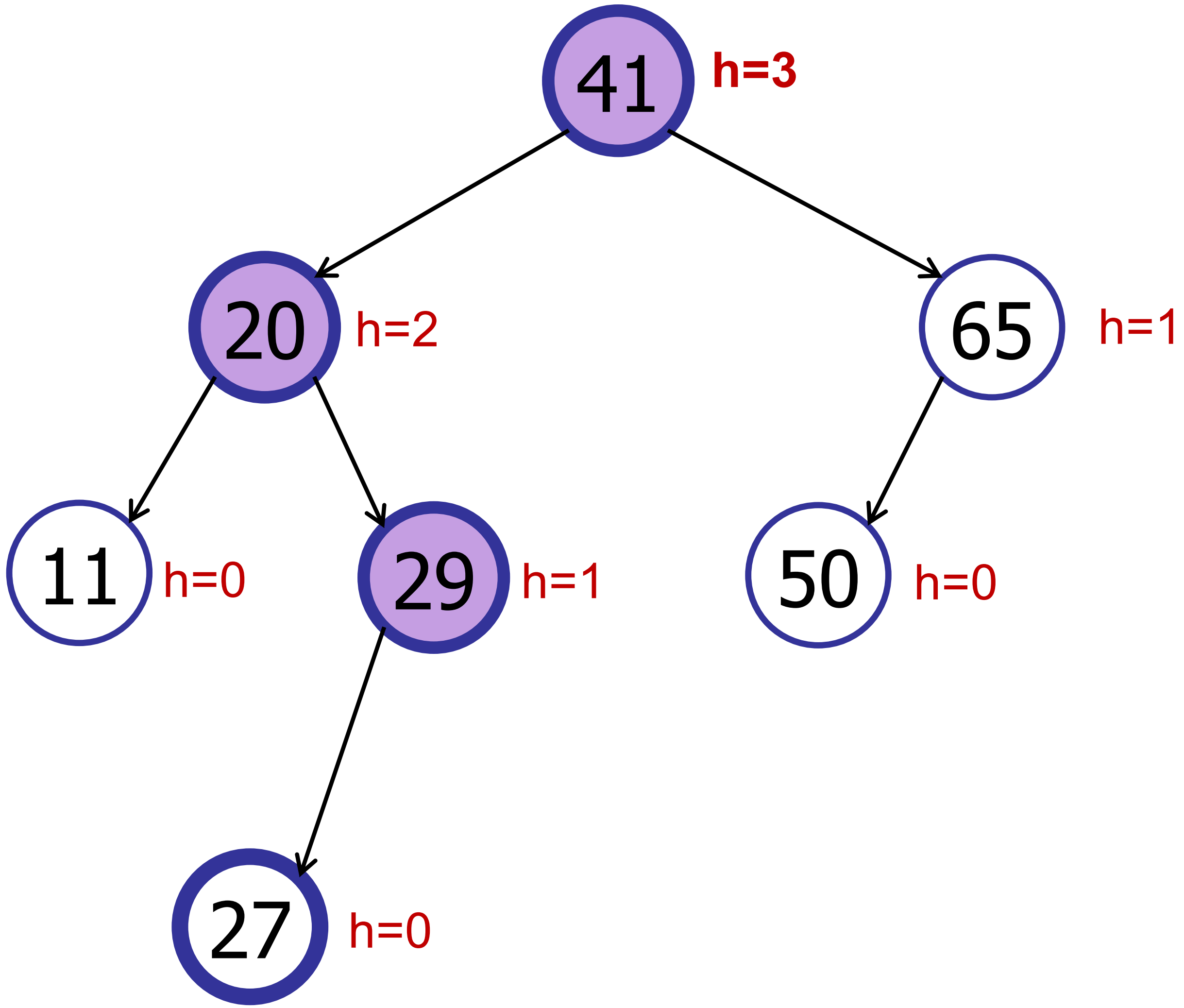

AVL Trees [Adelson-Velskii & Landis 1962]

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AVL Trees [Adelson-Velskii & Landis 1962]

Step 0: Augment

Step 1: Define Balance Condition

Step 2: Maintain Balance

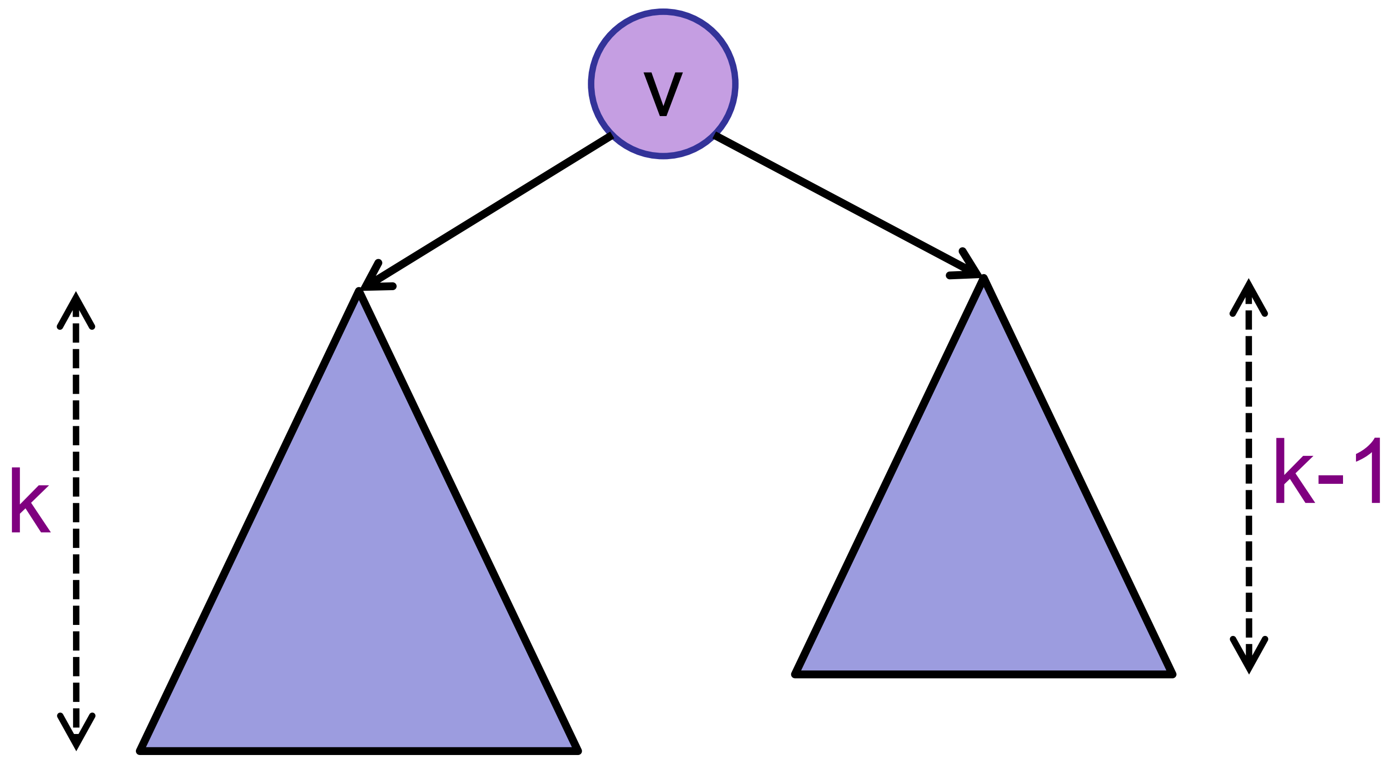
AVL Trees [Adelson-Velskii & Landis 1962]

Key definition

Step 1: Define Invariant

- A node v is **height-balanced** if:

$$|v.left.height - v.right.height| \leq 1$$



AVL Trees [Adelson-Velskii & Landis 1962]

Step 1: Define Invariant

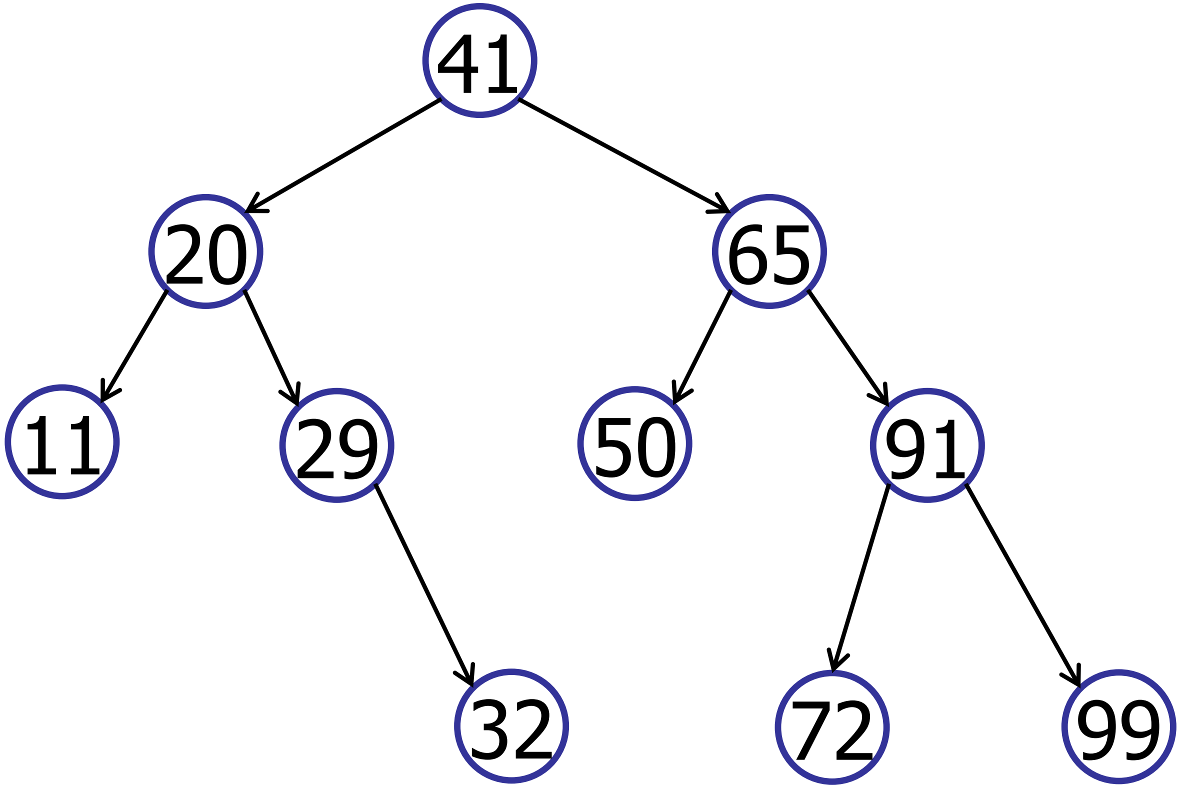
- A node v is height-balanced if:

$$|v.left.height - v.right.height| \leq 1$$

- A binary search tree is height balanced if **every** node in the tree is height-balanced.

Is this tree height-balanced?

1. Yes
2. No
3. I’m confused.



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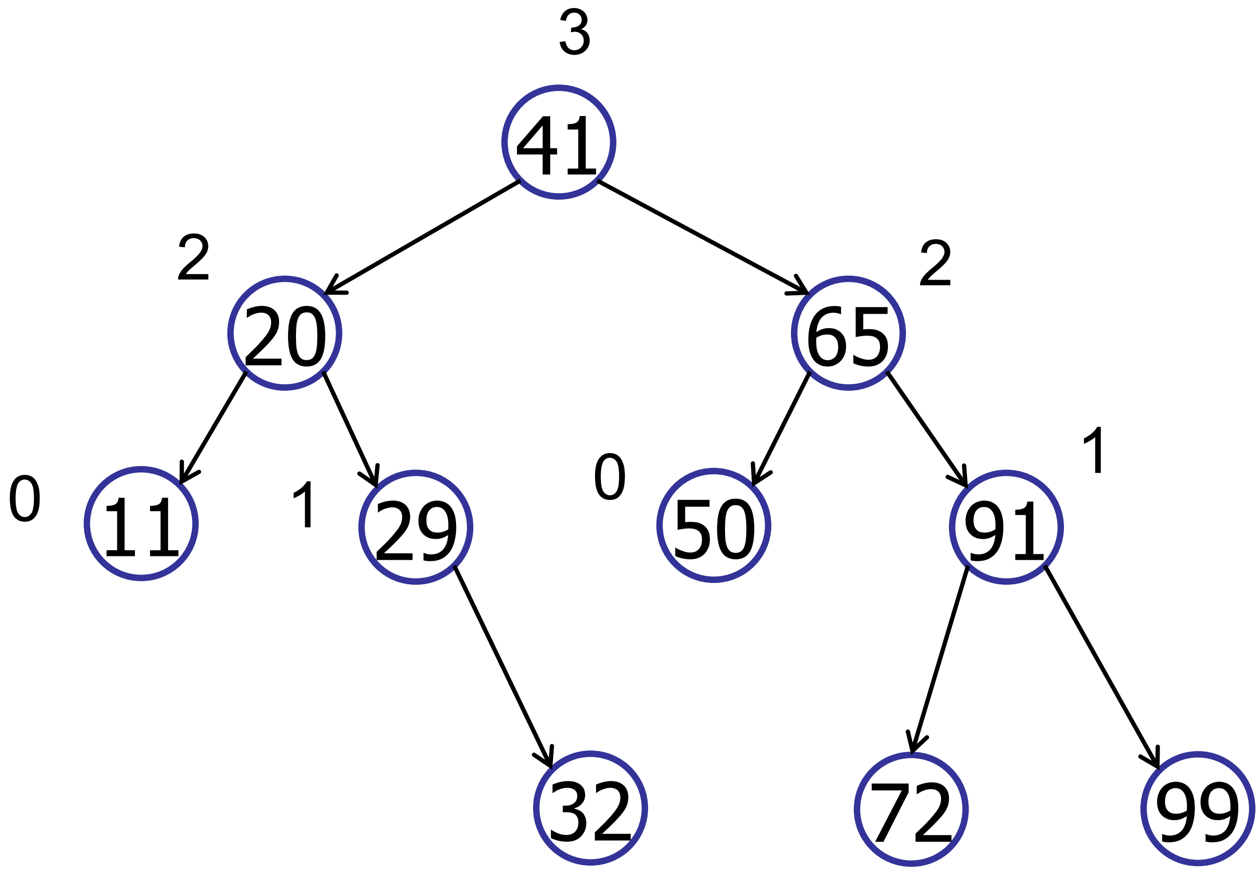

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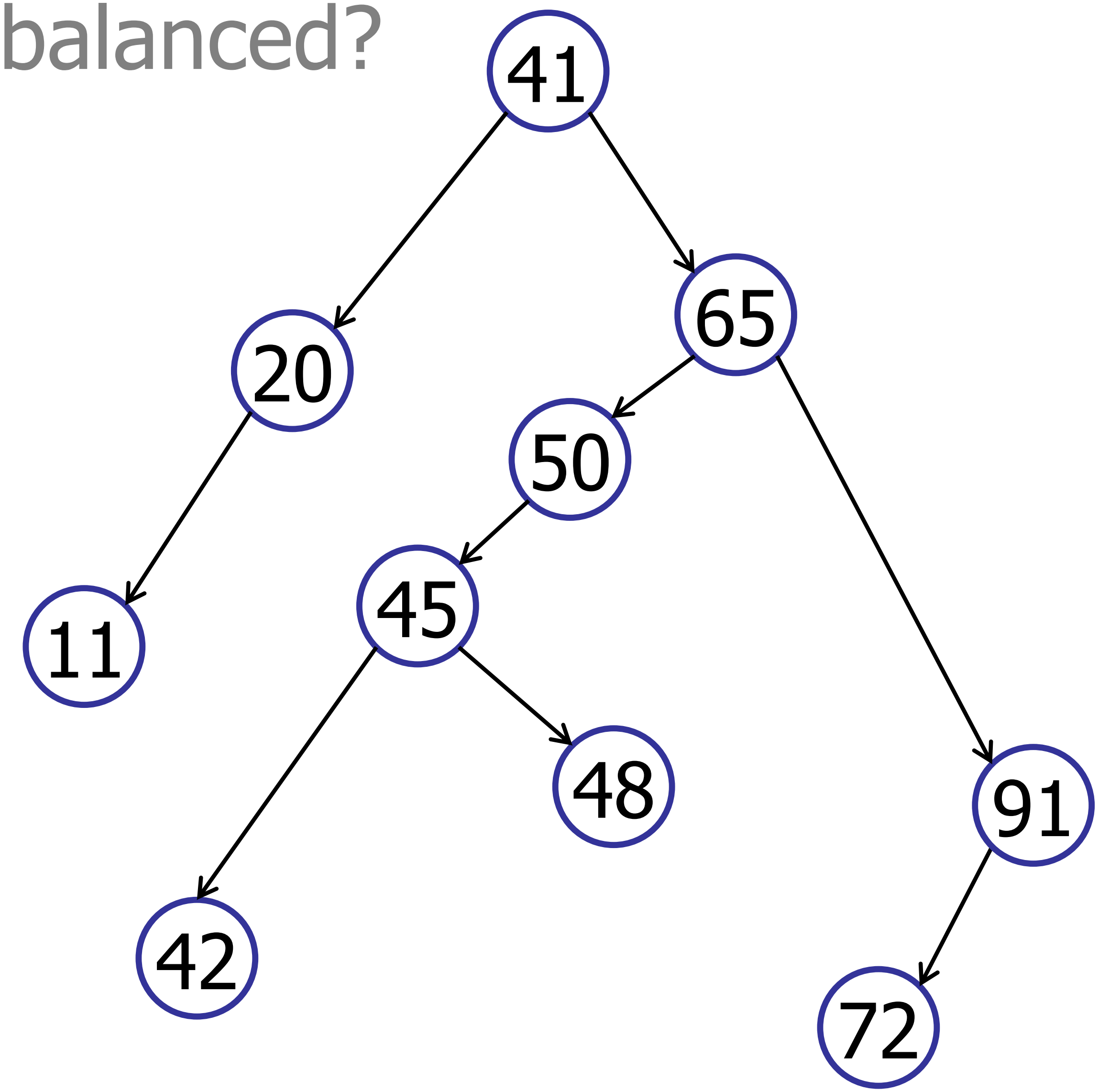

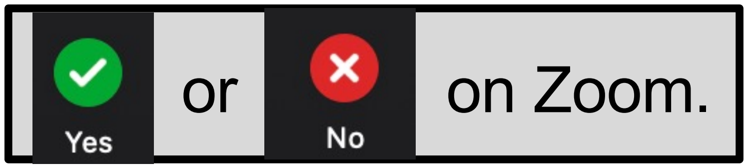

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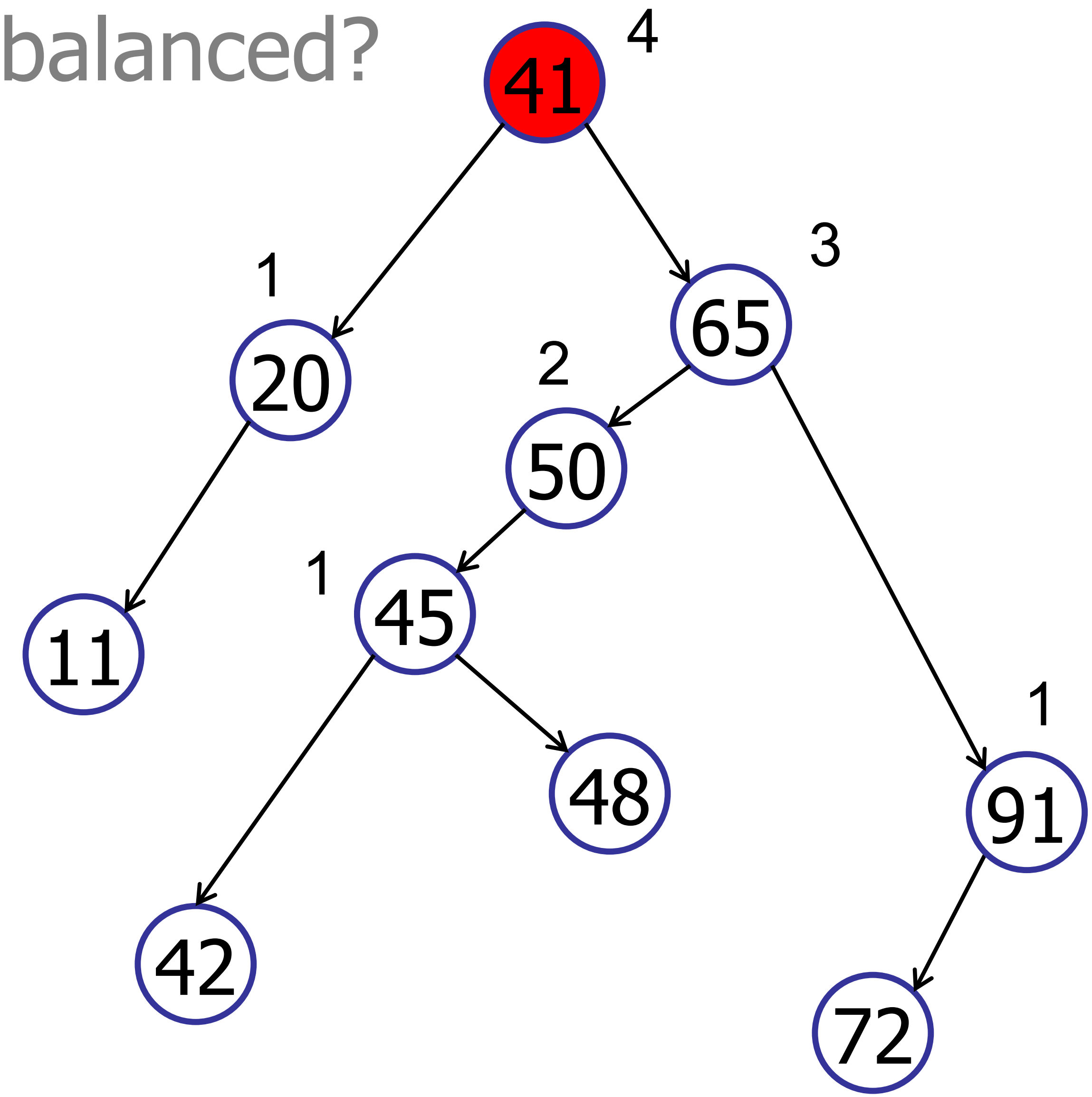

Height-Balanced Trees

Claim:

A height-balanced tree with n nodes has **at most** height $h < 2\log(n)$.

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A height-balanced tree with n nodes has **at most** height $h < 2\log(n)$.

$\Leftrightarrow h/2 < \log(n)$

$\Leftrightarrow 2^{h/2} < 2^{\log(n)}$

$\Leftrightarrow 2^{h/2} < n$

Show: a height-balanced tree with height h has **at least** $n > 2^{h/2}$ nodes

Height-Balanced Trees

Strategy:

Assume tree is height-balanced with height h.

Show that it has **at least** $n > 2^{h/2}$ nodes.

➔ Since every height-balanced tree with height h has **at least** $n > 2^{h/2}$ nodes, we know that every height-balanced tree with n nodes has height $h < 2\log(n)$.

If height were bigger, it would need to have more nodes!

Height-Balanced Trees

Proof:

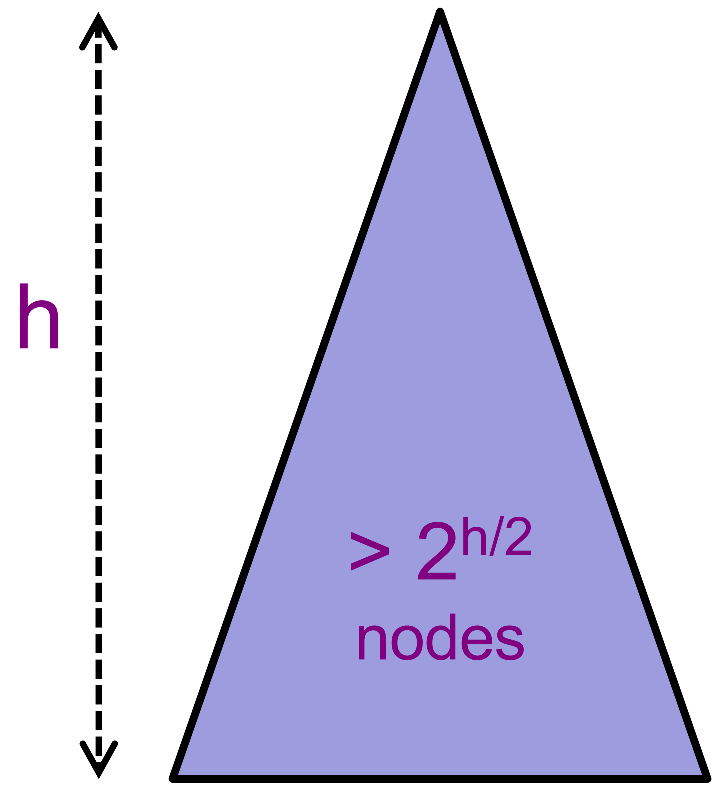
Let n_h be the minimum number of nodes in a height-balanced tree of height h.

Show:

$n_h > 2^{h/2}$

Note:

$n_h > n_{h-1}$

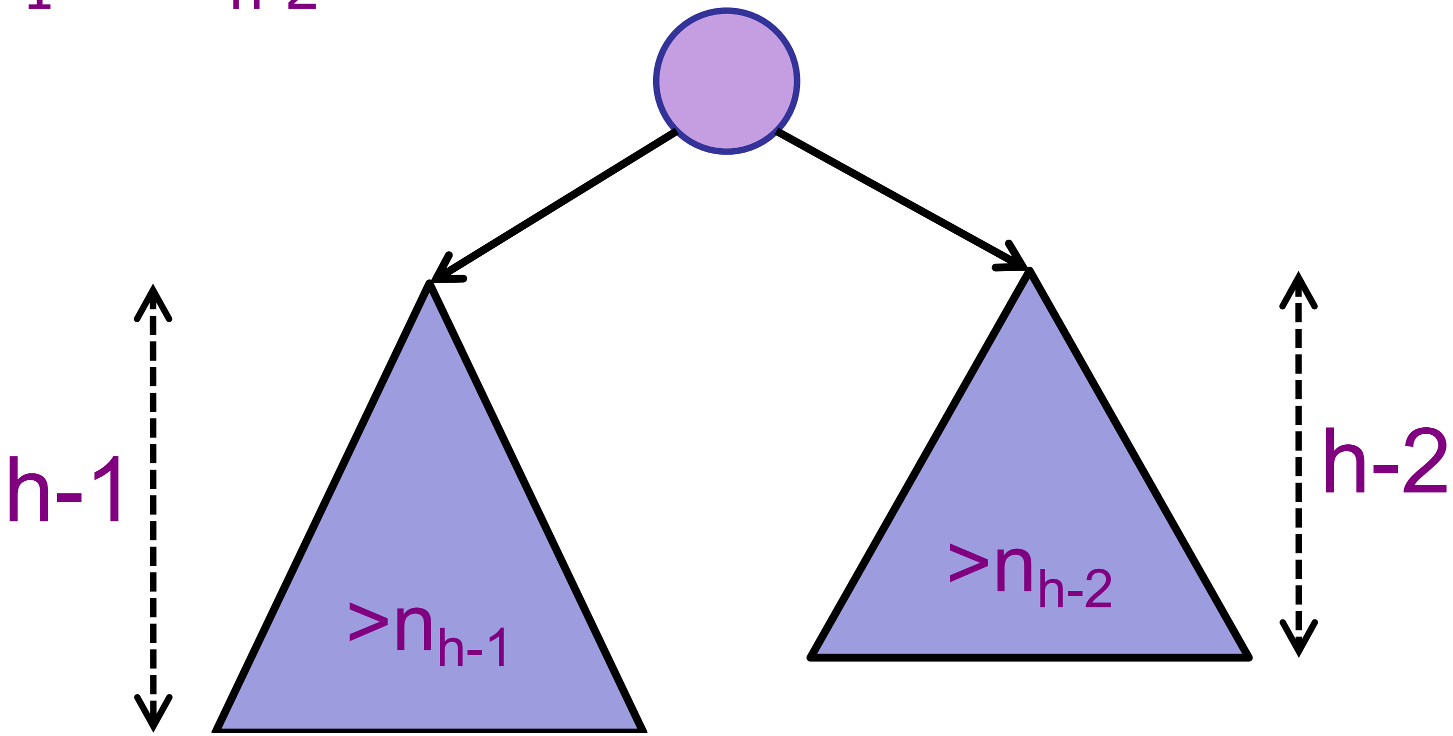


Height-Balanced Trees

Proof:

Let n_h be the minimum number of nodes in a height-balanced tree of height h.

$n_h \geq 1 + n_{h-1} + n_{h-2}$



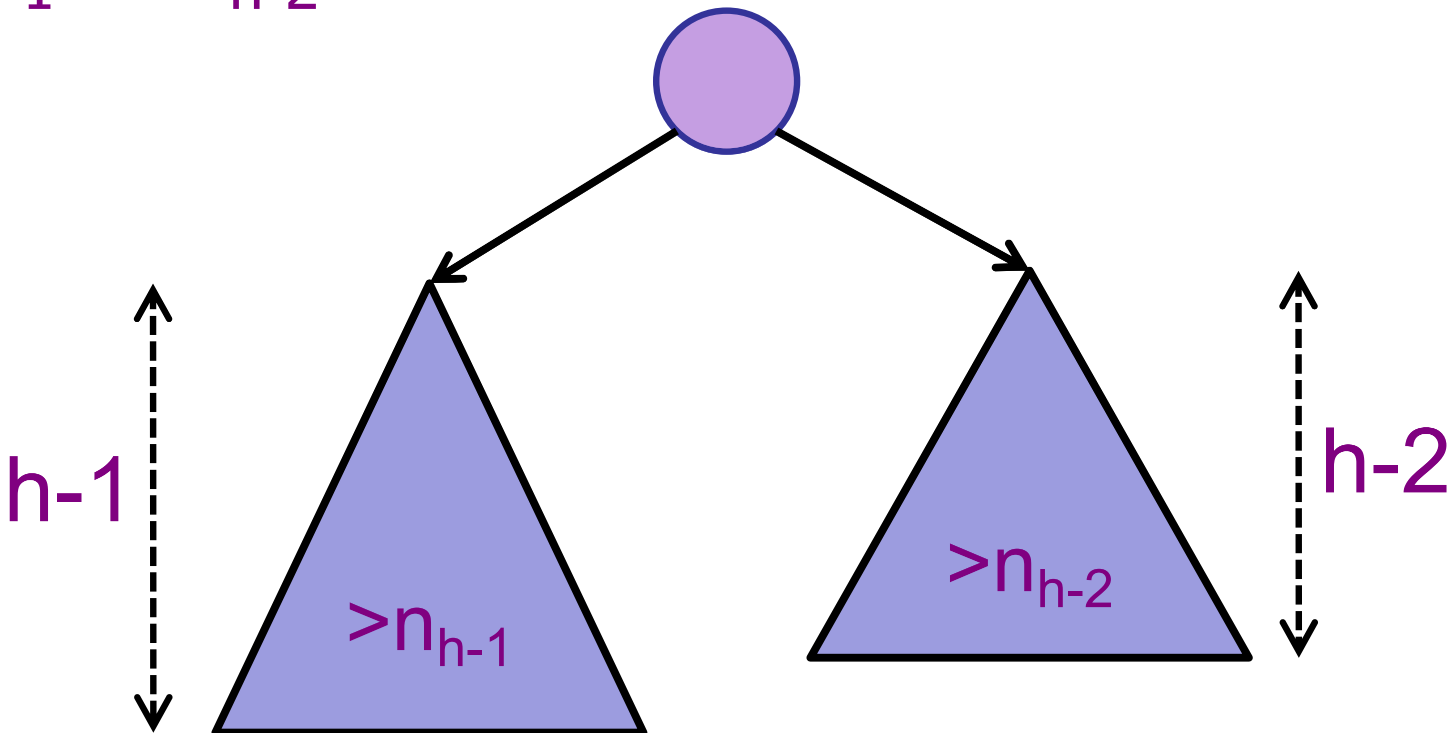
Height-Balanced Trees

Proof:

Let n_h be the minimum number of nodes in a height-balanced tree of height h.

$$n_h \geq 1 + n_{h-1} + n_{h-2}$$

$$\geq 2n_{h-2}$$



Height-Balanced Trees

Proof:

Let n_h be the minimum number of nodes in a height-balanced tree of height h.

$$n_h \geq 1 + n_{h-1} + n_{h-2}$$

$$\geq 2n_{h-2}$$

$$\geq 4n_{h-4}$$

$$\geq 8n_{h-6}$$

$$\geq \ldots$$

How many times?

Base case:
$n_0 = 1$

Height-Balanced Trees

Proof:

Let n_h be the minimum number of nodes in a height-balanced tree of height h.

$$n_h \geq 1 + n_{h-1} + n_{h-2}$$

$$\geq 2^1 n_{h-2}$$

$$\geq 2^2 n_{h-4}$$

$$\geq 2^3 n_{h-6}$$

$$\geq \ldots \geq 2^k n_0$$

What is k?

Base case:
$n_0 = 1$

Height-Balanced Trees

Proof:

Let n_h be the minimum number of nodes in a height-balanced tree of height h.

$$n_h \geq 1 + n_{h-1} + n_{h-2}$$

$$\geq 2n_{h-2}$$

$$\geq 2^{h/2}\, n_0$$

$$\geq 2^{h/2}$$

Base case:
$n_0 = 1$

Height-Balanced Trees

Claim:

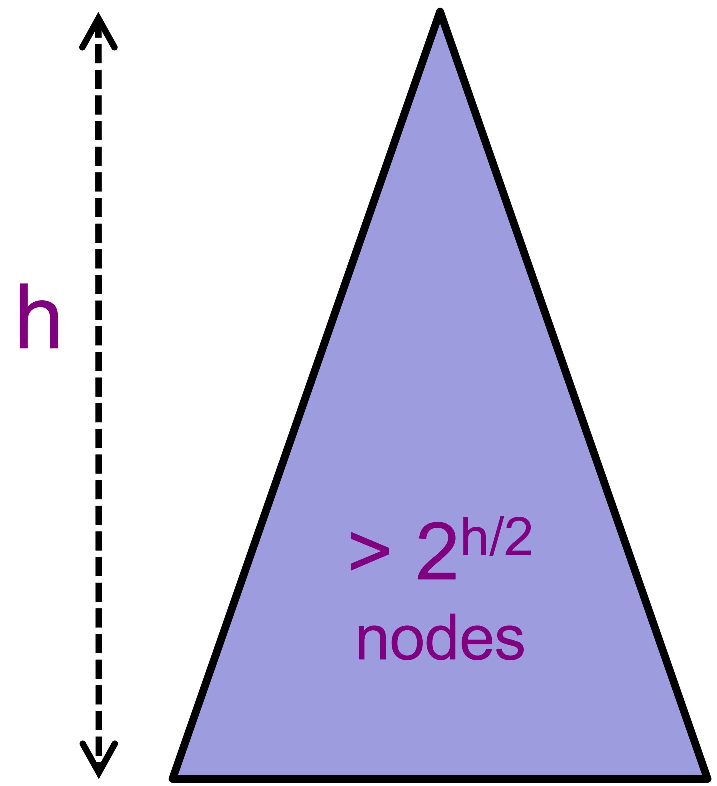
A height-balanced tree with n nodes has height $h < 2\log(n)$.

Show:

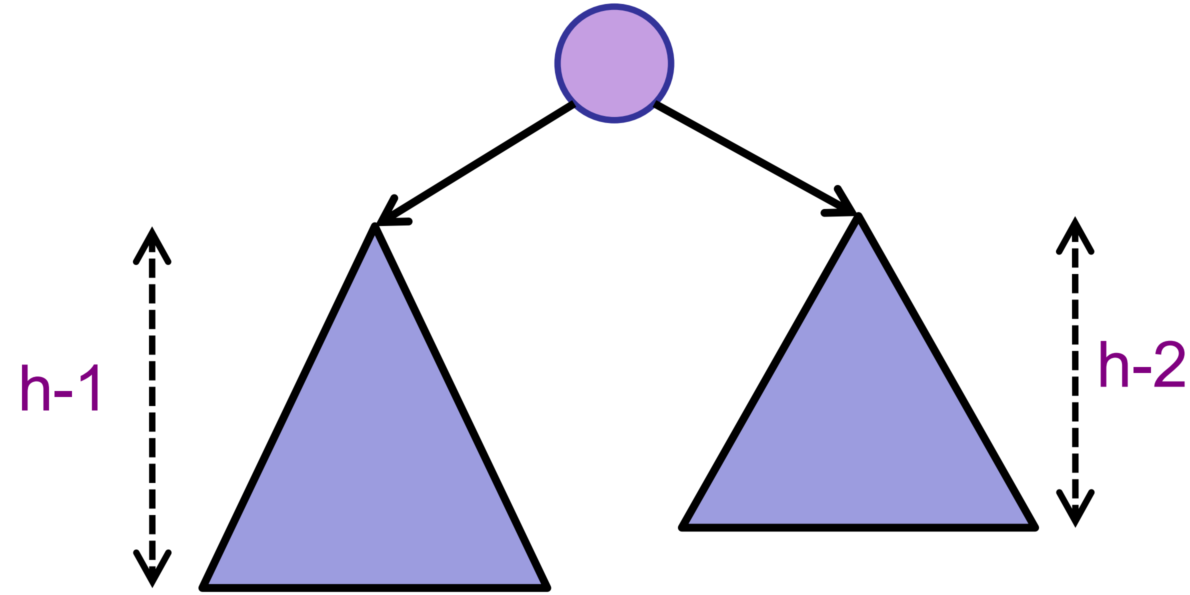
$$n > 2^{h/2}$$

$$\Leftrightarrow$$

$$h < 2\log(n)$$



Height-Balanced Trees



Show (induction):

F_n = n^{th} Fibonacci number

$n_h = F_{h+2} - 1 \cong \phi^{h+1}/\sqrt{5} - 1$ (rounded to nearest int)

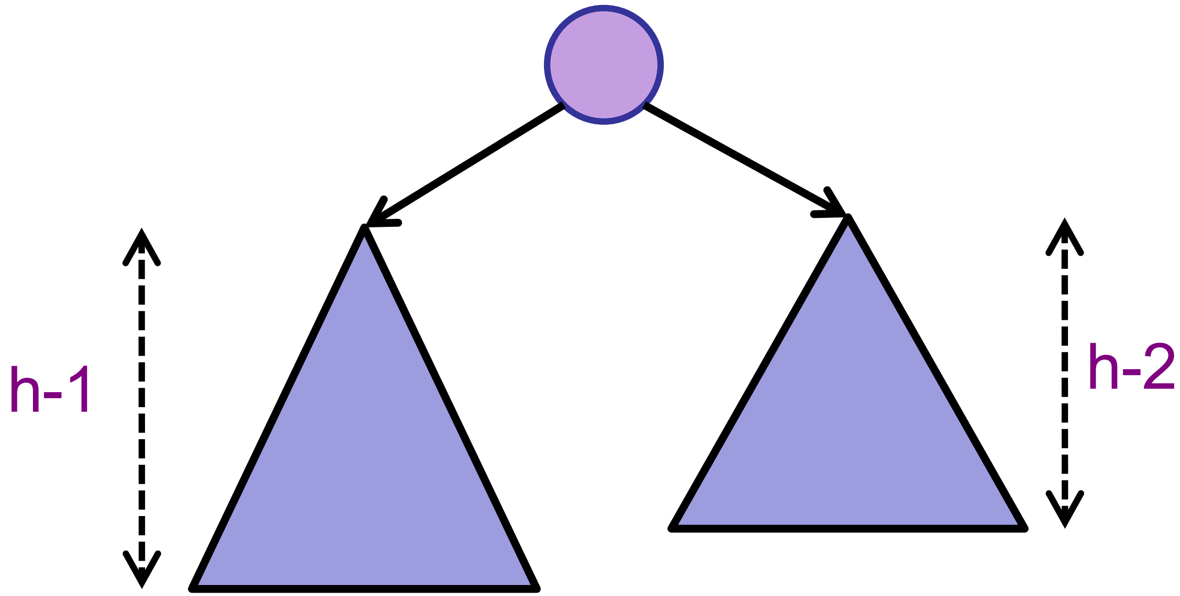
$h \cong \log(n) / \log(\phi)$ $\phi \cong 1.618$

$h \cong 1.44 \log(n)$

Height-Balanced Trees

Claim:

A height-balanced tree is balanced, i.e., has height $h = O(\log n)$.



AVL Trees [Adelson-Velskii & Landis 1962]

Step 0: Augment

Step 1: Define Balance Condition ← invariant

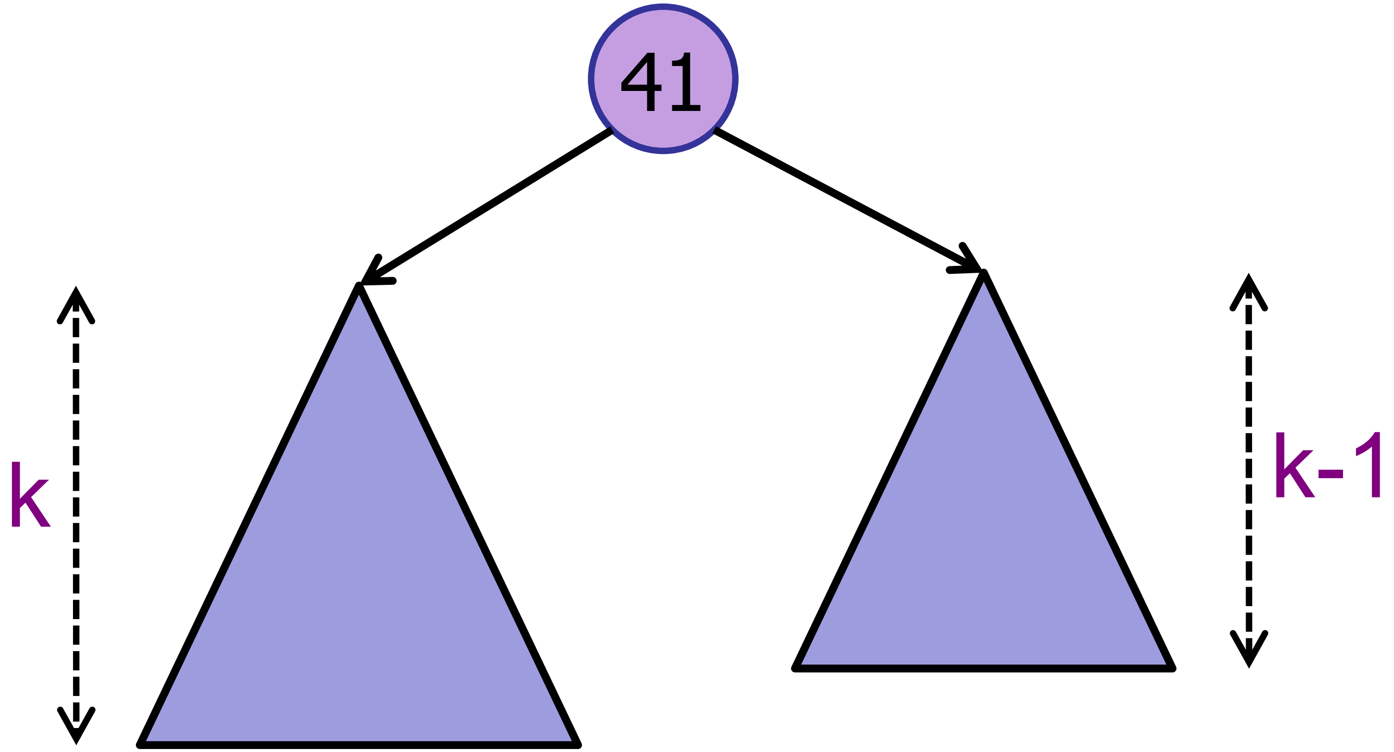
Step 2: Maintain Balance

It’s good that we don’t have to

Balance perfectly

AVL Trees [Adelson-Velskii & Landis 1962]

Step 2: Show how to maintain height-balance



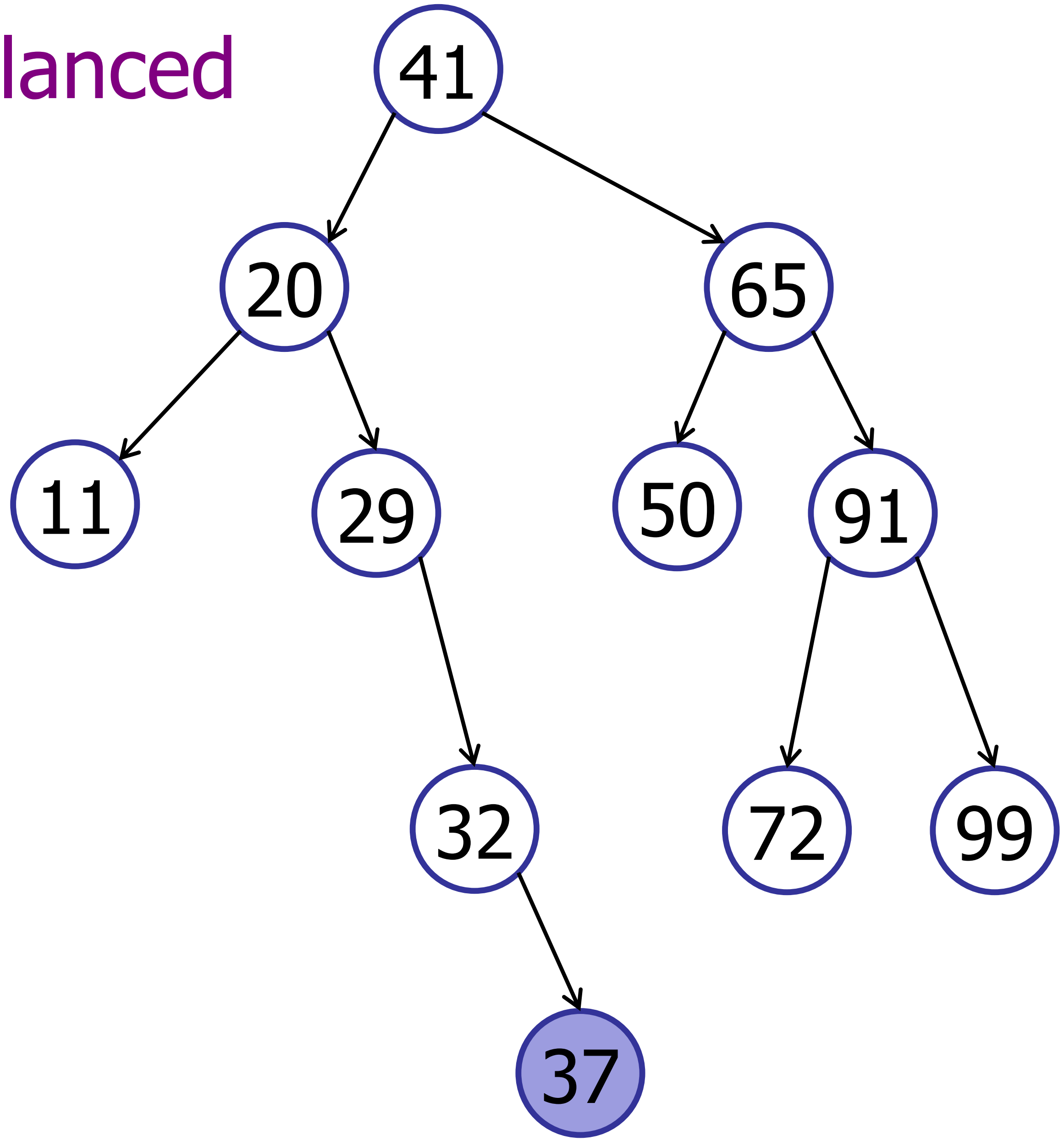
Inserting in an AVL Tree

Before insertion, balanced

insert(37)

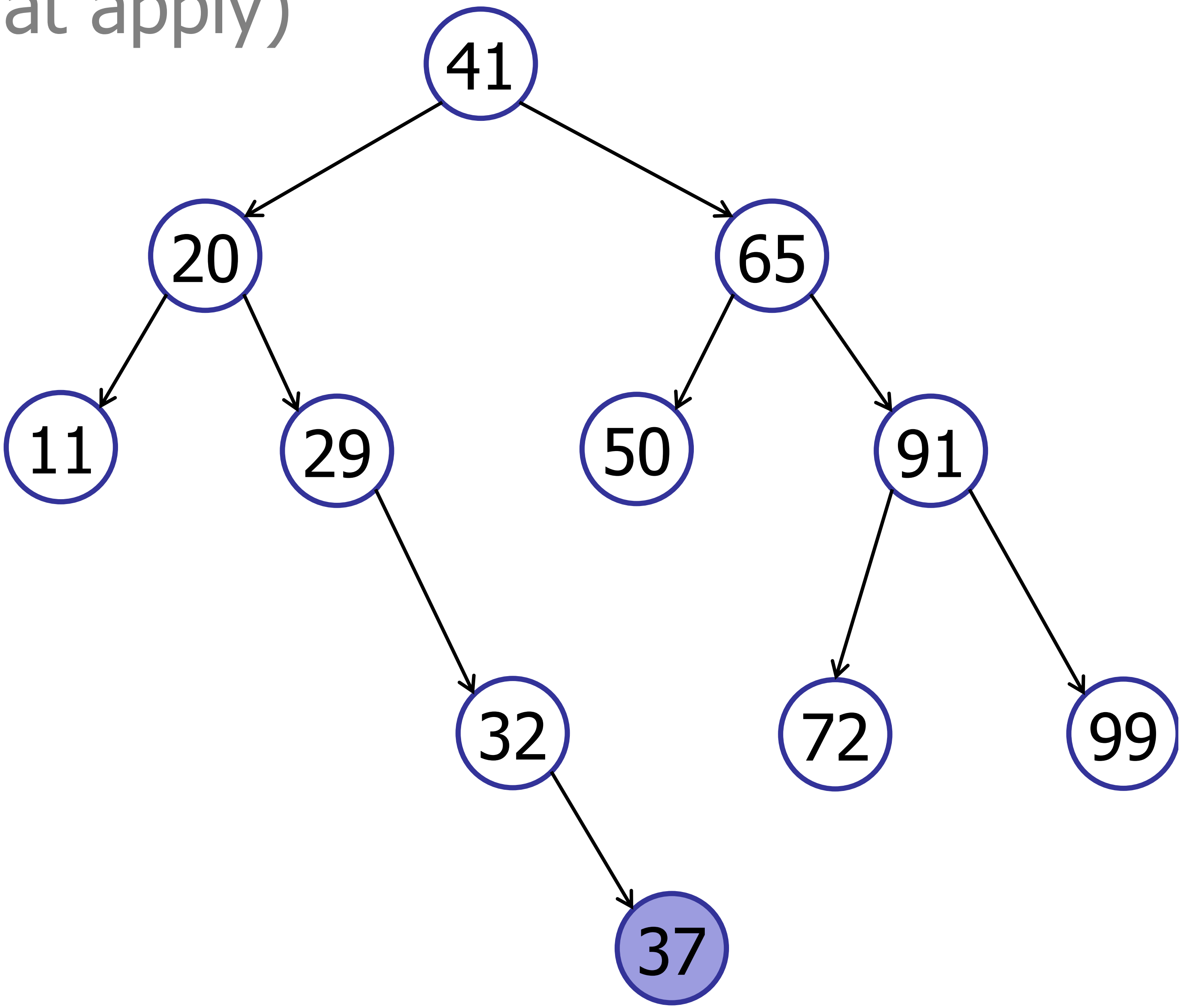
No longer balanced after insertion!

Need to rebalance!



Which nodes need rebalancing?
(click all that apply)

1. 41
2. 20
3. 11
4. 29
5. 32
6. 37
7. 65



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is open

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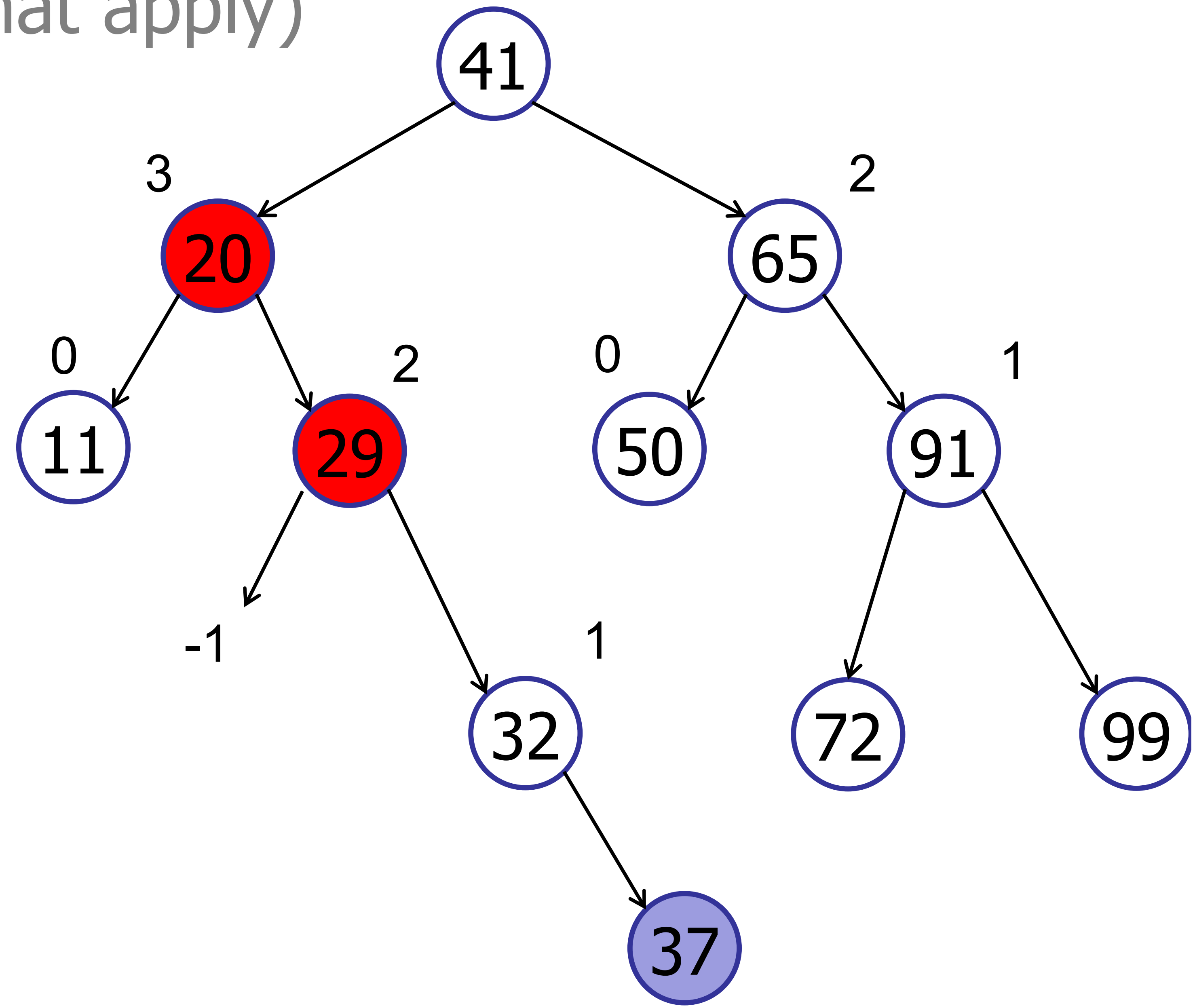

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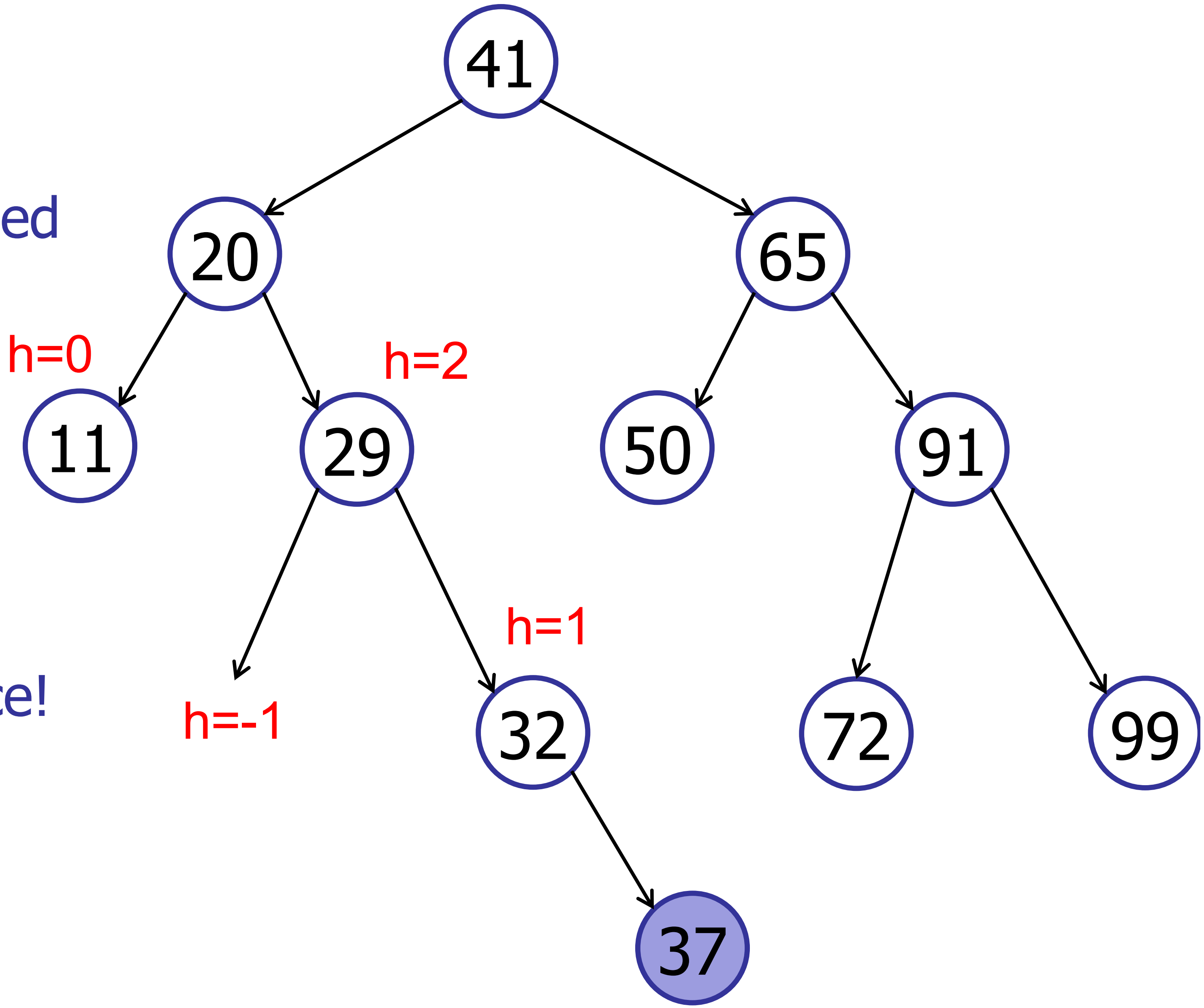

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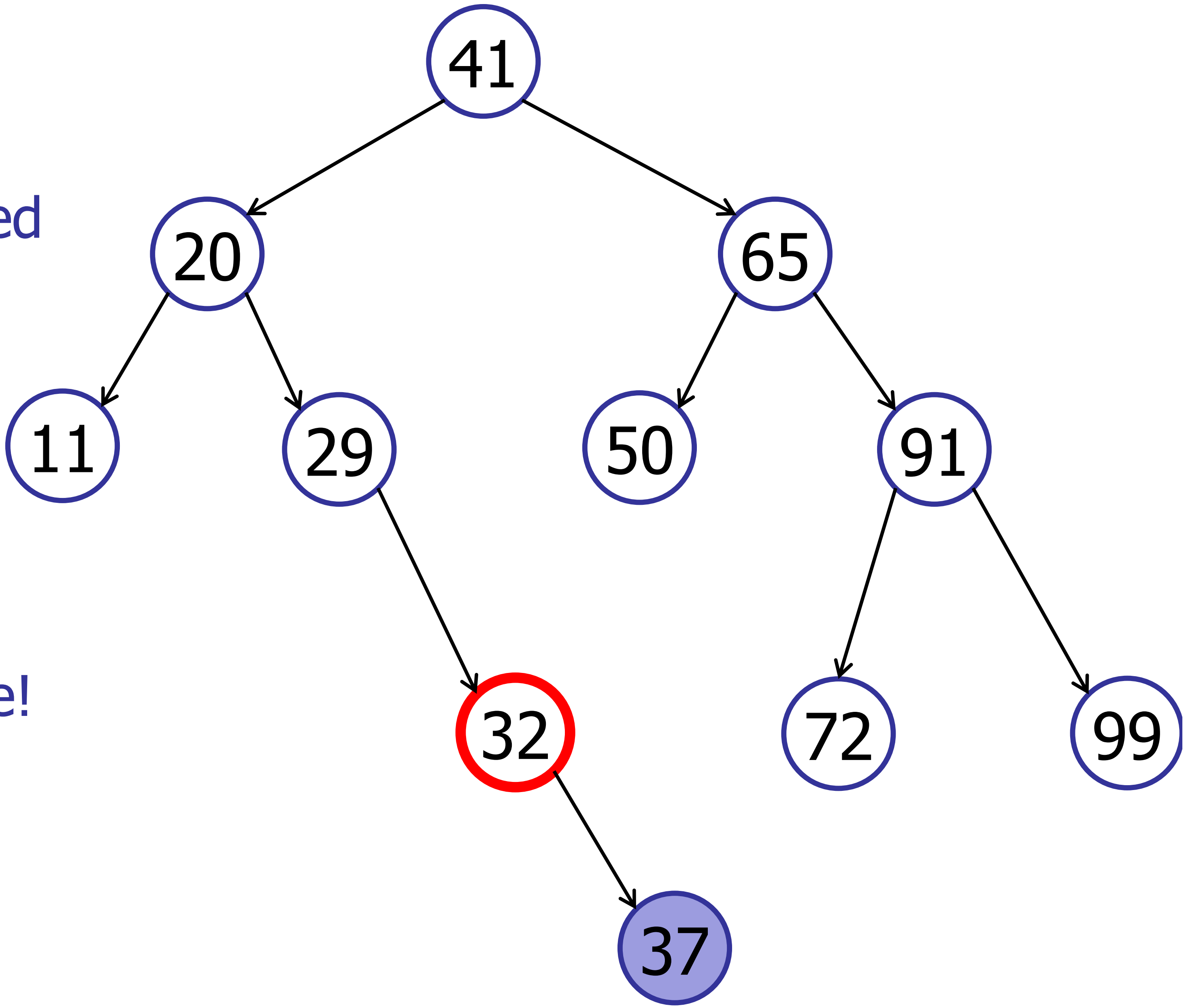

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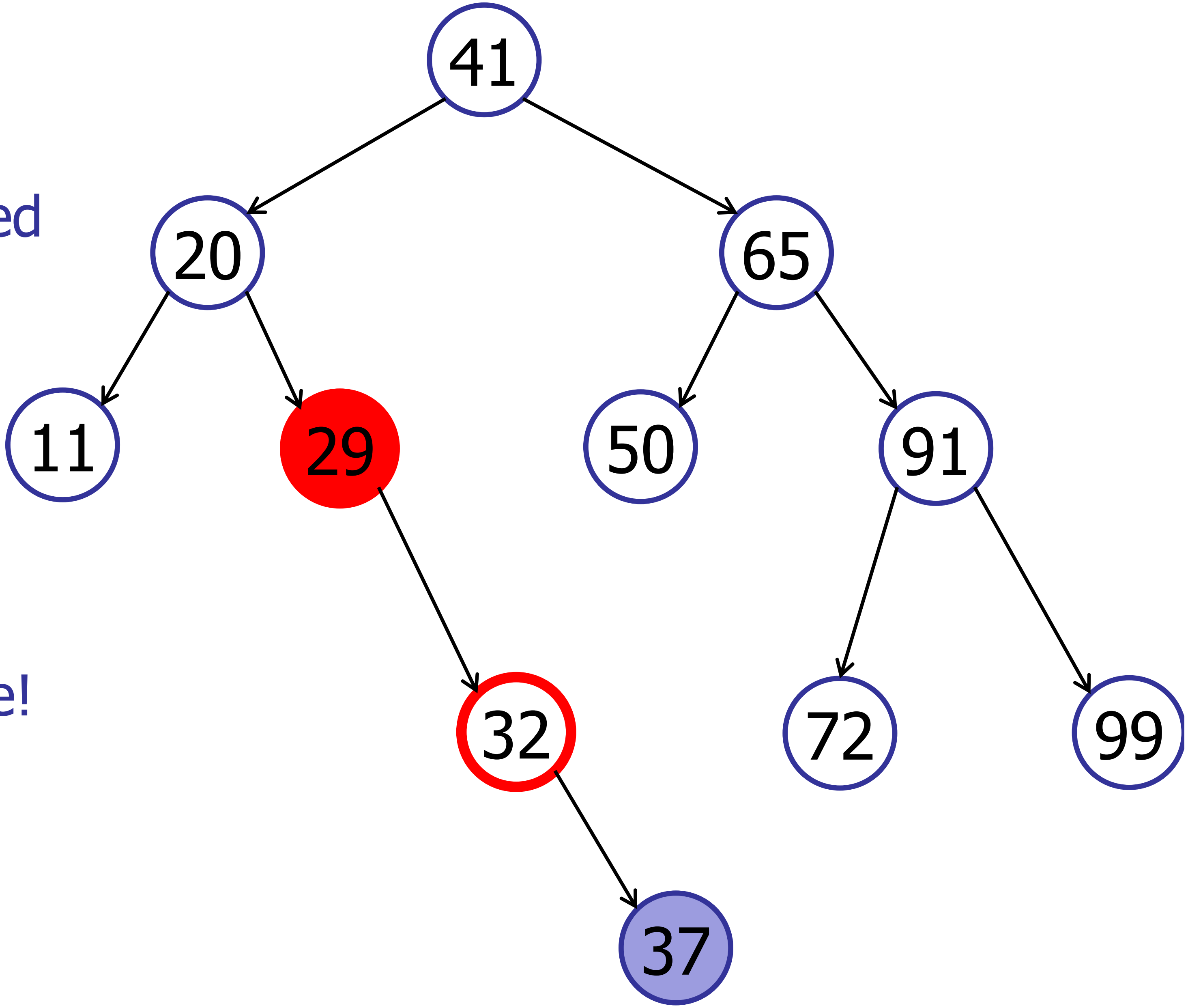

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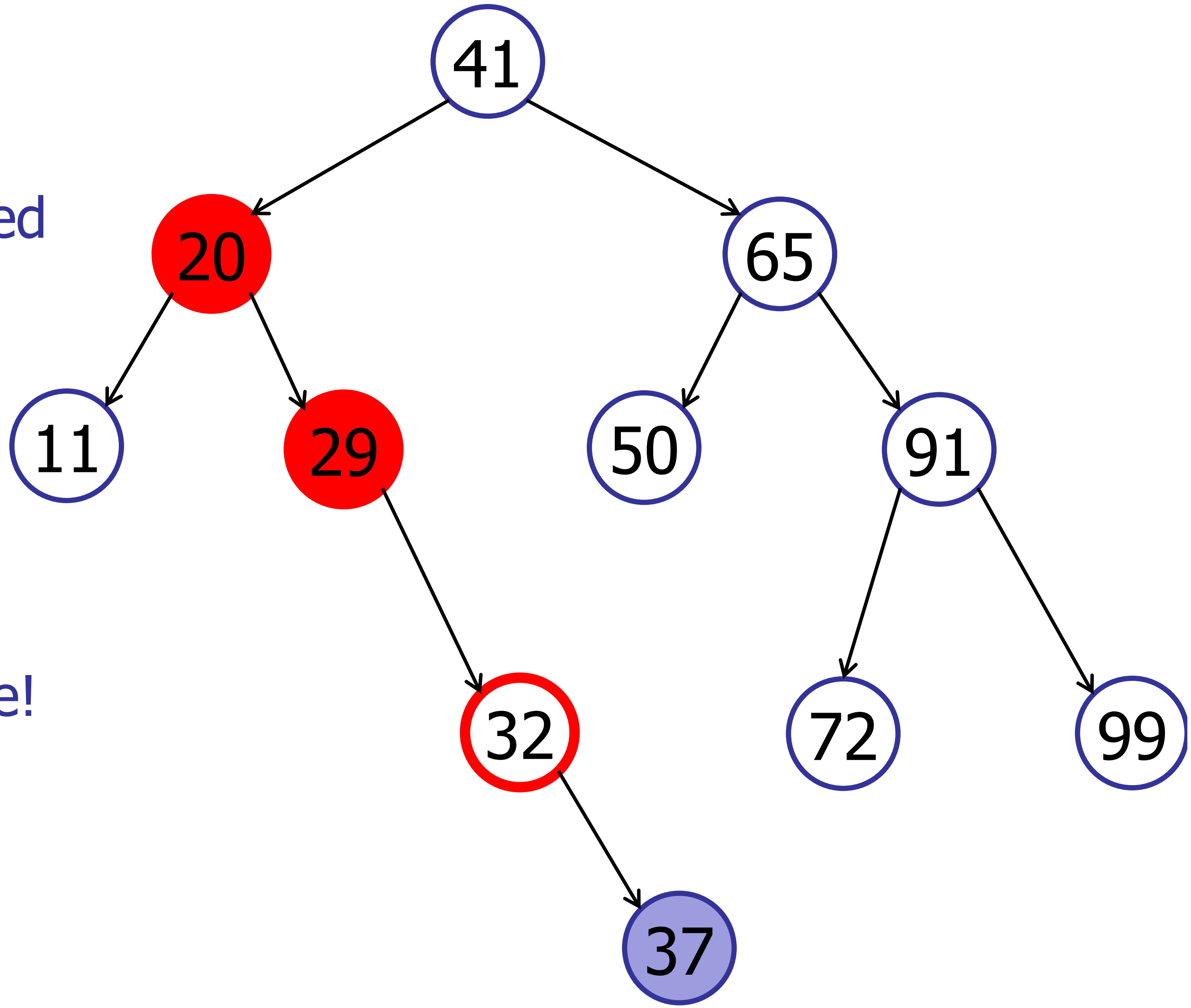

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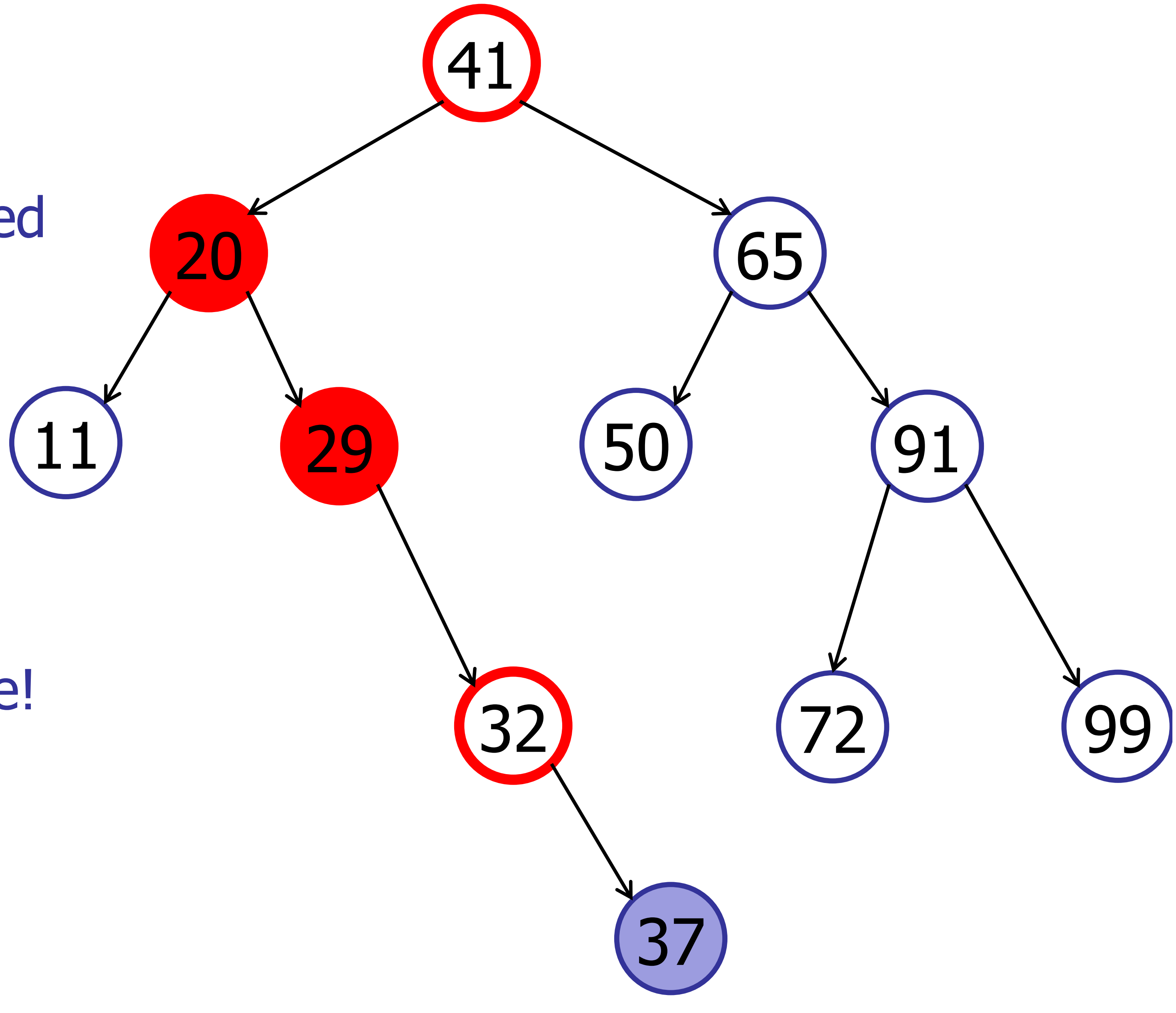
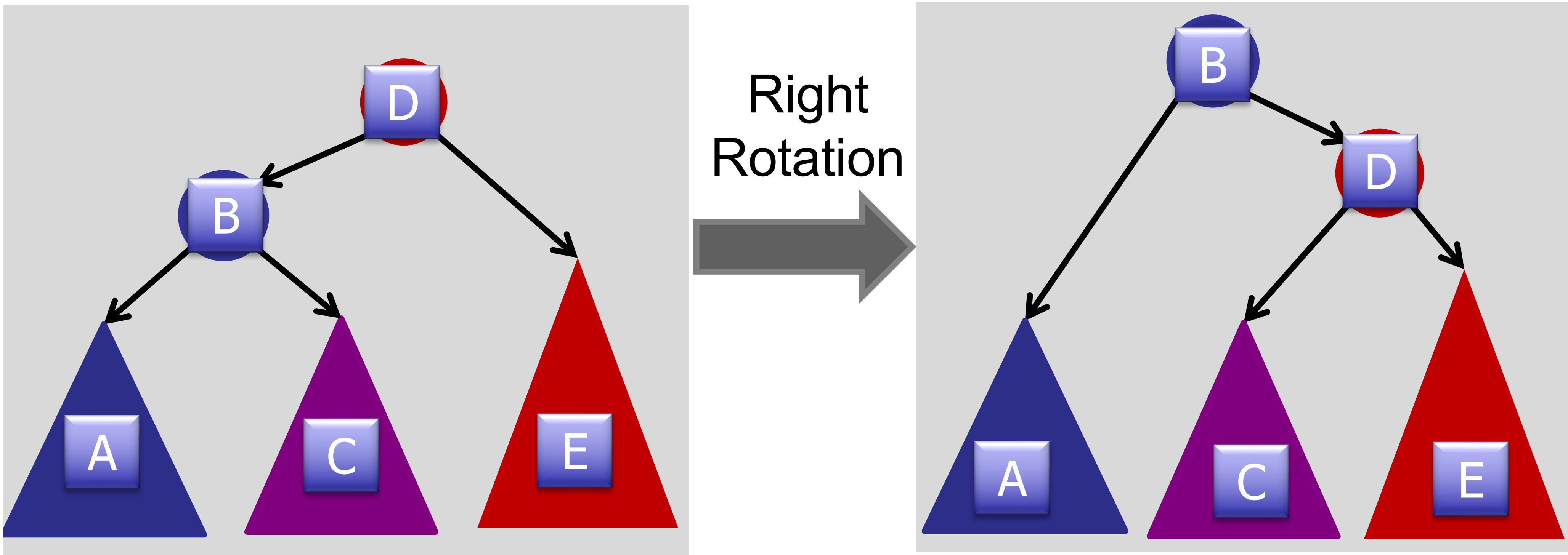


Trick to rebalance the tree

Tree rotation!

Tree Rotations

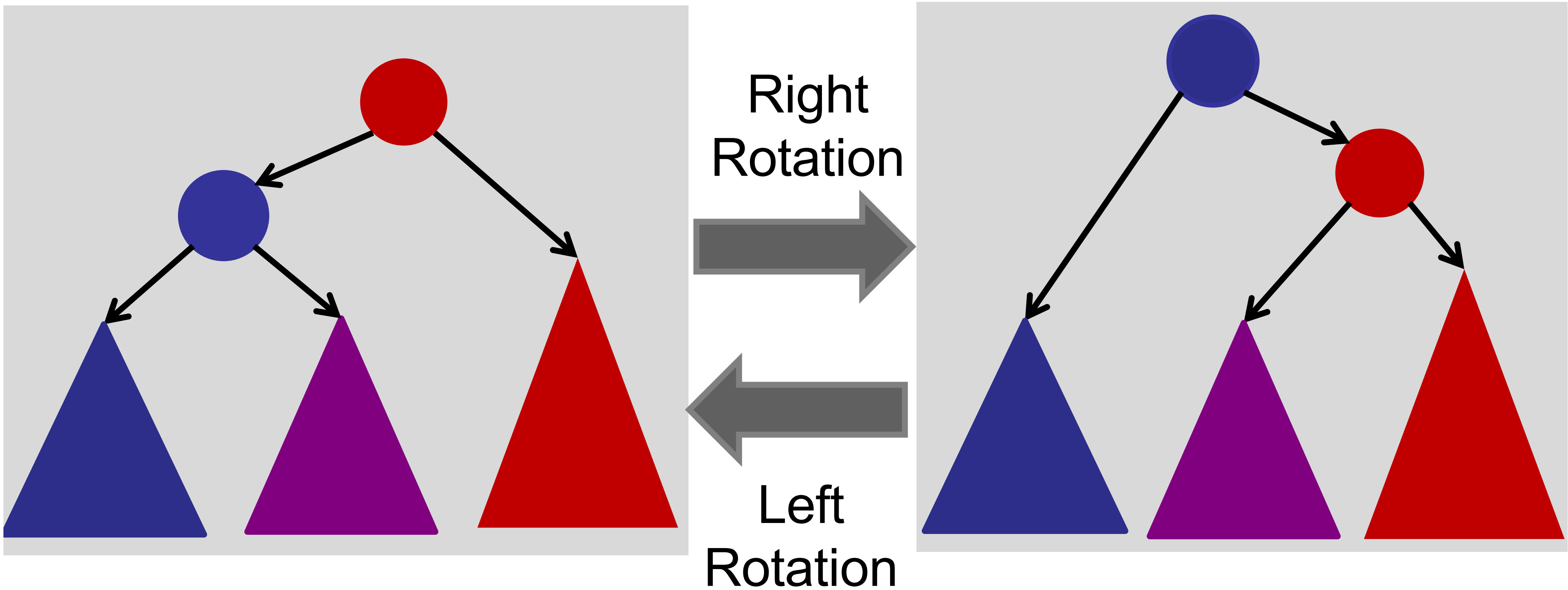


A < B < C < D < E

Rotations maintain ordering of keys.

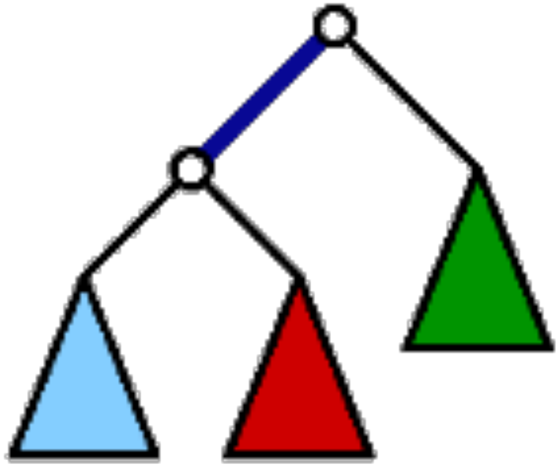
⇒ Maintains BST property.

Tree Rotations

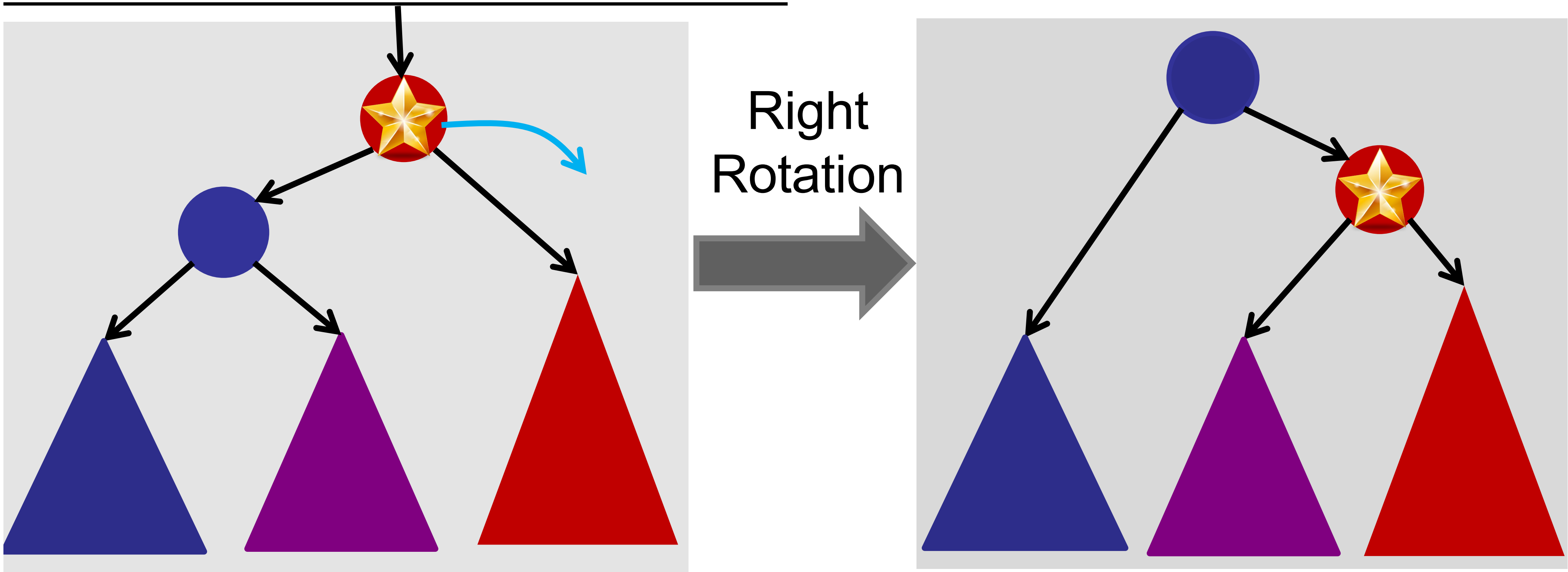


Wait….

What is a left rotation and what is a right rotation!?

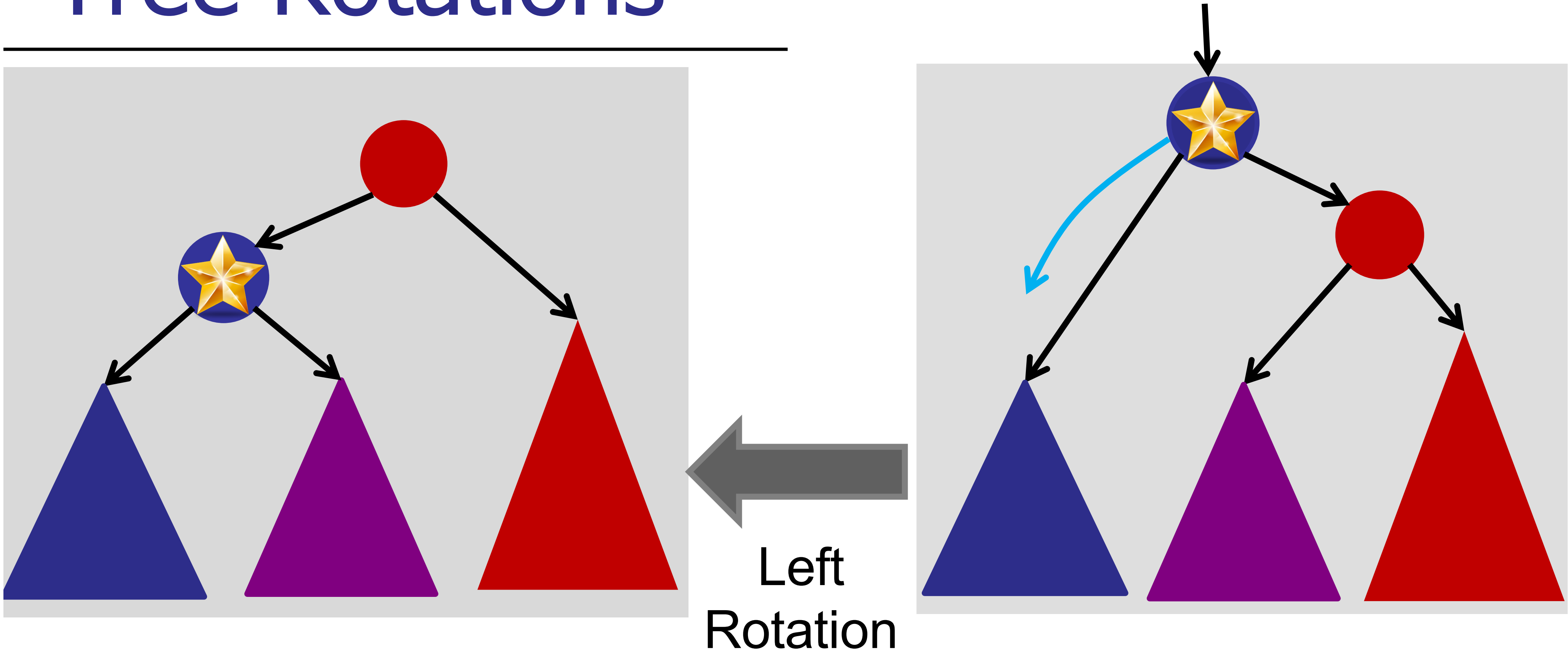


The way to remember it



The root of the subtree moves right

Tree Rotations

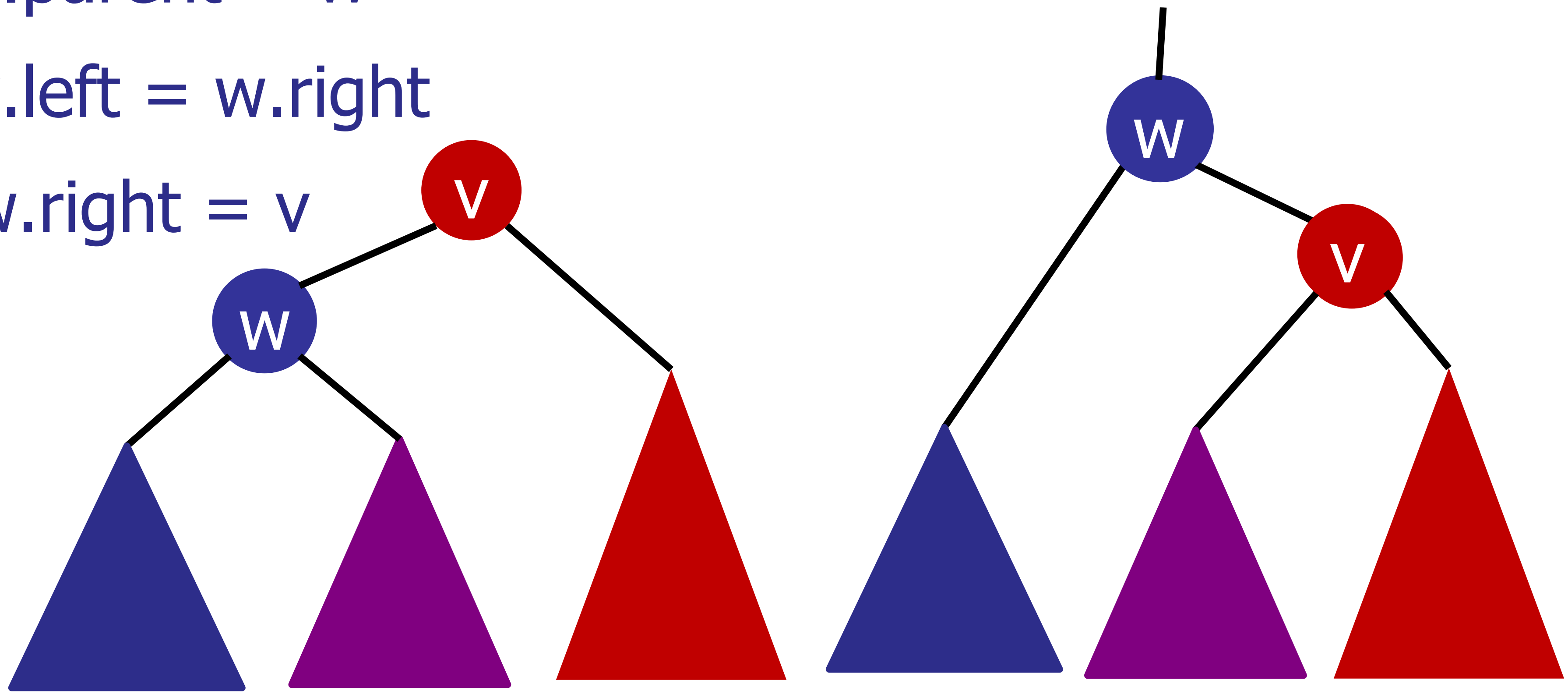


The root of the subtree moves left

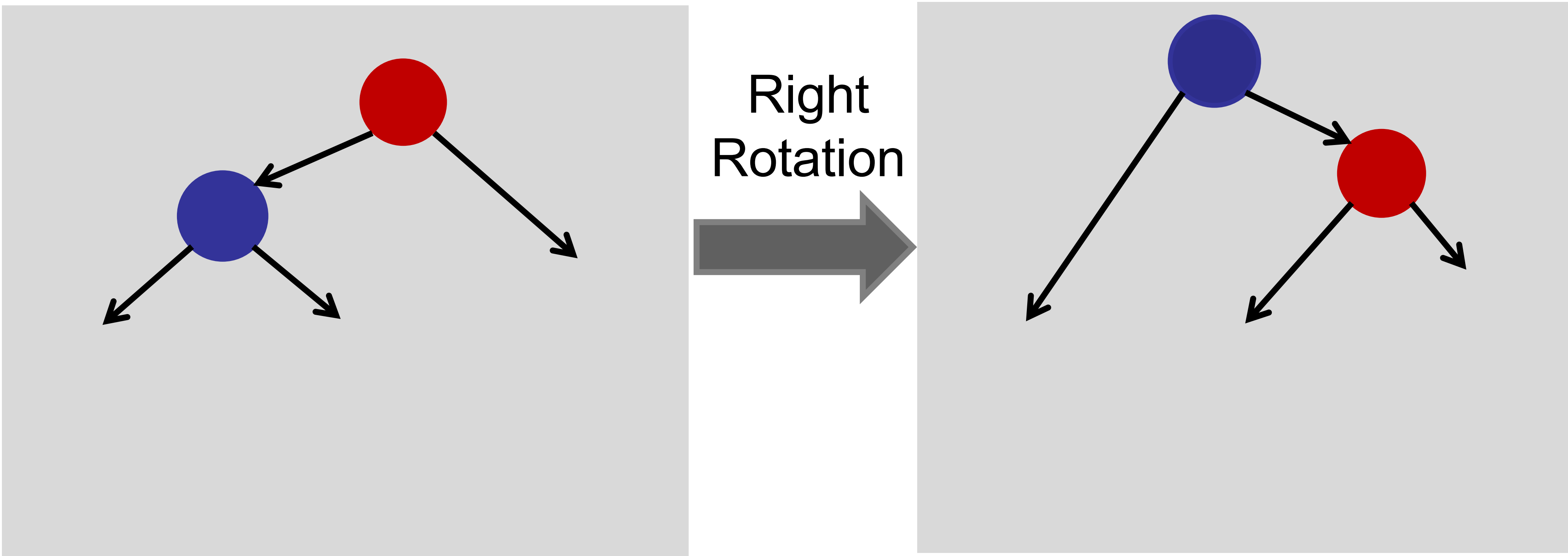
Rotations

right-rotate(v)   // assume v has left != null

    w = v.left
    w.parent = v.parent
    v.parent = w
    v.left = w.right
    w.right = v



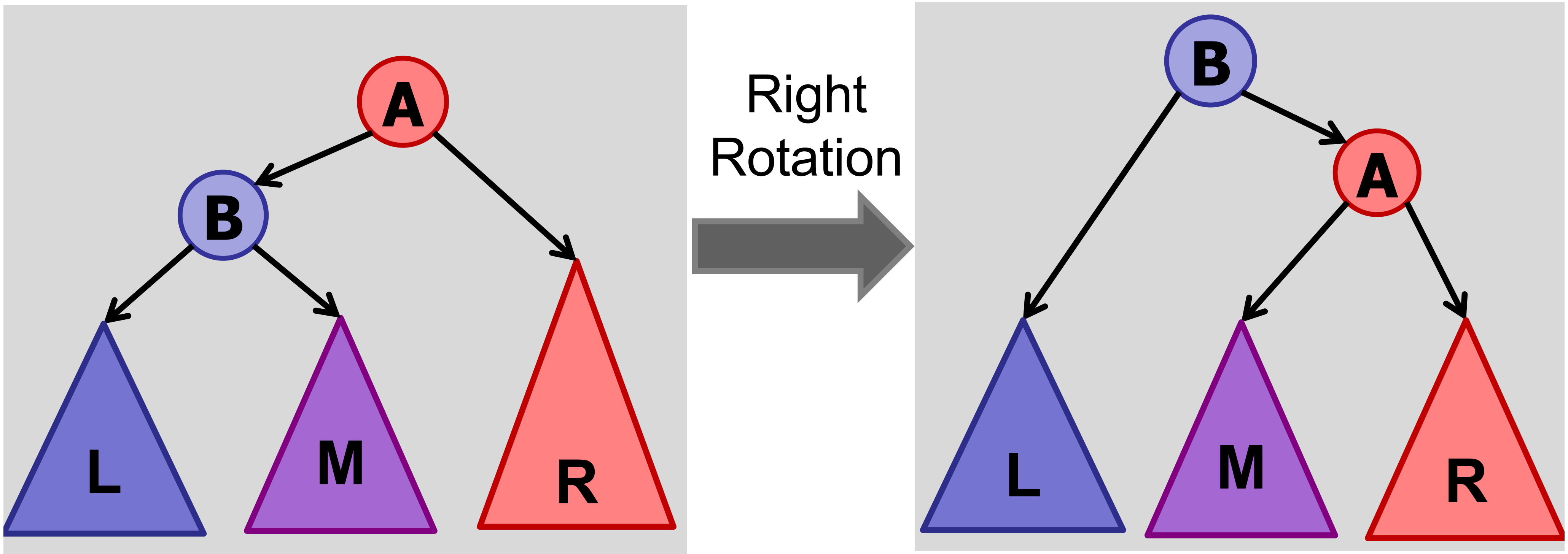
Tree Rotations



rotate-right requires a left child

rotate-left requires a right child

Tree Rotations



After insert:

Use tree rotations to restore balance.

Height is out-of-balance by 1