CS2040S: Data Structures and Algorithms

Recitation 4

Goals:

- Motivate the design of an efficient search tree DS
- Specify the objectives of such a DS
- Conceptually understand B-trees and their various supporting operations
- Reason about the effectiveness of B-trees
- Appreciate the practical applications of B-trees

Problem 1. B-Trees

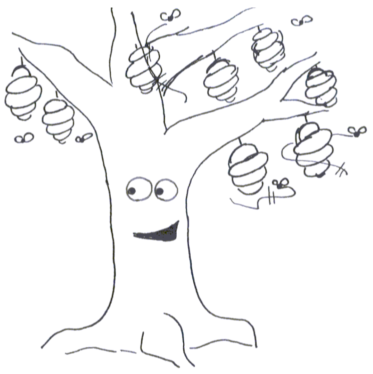


Image credit: Jeremy Fineman

In the first lecture you were asked whether the fastest way to search for data is to store them in an array, sort them and then perform binary search. The answer to that question is — surprisingly — no! You have seen Binary Search Trees (BSTs), where we design a DS that always maintains data in an (hierarchically) ordered manner. This will allow us to avoid the overhead of sorting before we search. However, you also learnt that unbalanced BSTs can be terribly inefficient for insertion, deletion and search operations (*why?*).

In this week's lecture, you were introduced to the idea of self-balancing BST such as an AVL-tree (other variants exist!). Today, we will learn about **B-trees**, which is another (very important!) self-balancing tree DS for maintaining ordered data. It is important to note that the 'B' in B-tree *does not* stand for "Binary" so don't get confused between B-Trees and BSTs. They are not the same!

1 (a, b)-trees

Before we talk about B-trees, we first introduce its family (generalized form) which are called (a, b)-trees. In an (a, b)-tree, a and b are parameters where $2 \leq a \leq (b+1)/2$. Respectively, a and b refer to the minimum and the maximum number of children an internal node in the tree (i.e. non-root, non-leaf) can have.

The main differences between binary trees and (a, b)-trees can be summarized in the following table.

Binary trees	(a, b)**-trees**
Each node *has at most* 2 children	Each node *can have more than* 2 children
Each node *stores exactly one* key	Each node *can store multiple* keys

1.1 Rules

Pro-Tip: Review these rules by concurrently referring to the $(2, 4)$-tree example in Figure 1.

Rule 1 ***"(a, b)-child Policy"***

Node type	**#Keys**		**#Children**	
	Min	Max	Min	Max
Root	1	$b-1$	2	b
Internal	$a-1$	$b-1$	a	b
Leaf	$a-1$	$b-1$	0	0

Table 1: For each type of node, we can either define this rule in terms of the number of children or equivalently in terms of the number of keys permitted.

With the exception of leaves, realize that the number of children is always one more than the number of keys (see Rule 2).

Rule 2 ***"Key ranges"***

A ***non-leaf node*** *(i.e. root or internal) must have one more child than its number of keys. This is to ensure that all value ranges due to its keys are covered in its subtrees. We shall henceforth refer to the permitted range of keys within a subtree to be its* ***key range****.*

Specifically, for a non-leaf node with k keys and $k+1$ children:

- *Its keys in sorted order are $v_1, v_2, \ldots, v_k$*
- *The subtrees due to its keys are $t_1, t_2, \ldots, t_{k+1}$*

Then:

- *First child t_1 has key range $\leq v_1$*
- *Final child t_{k+1} has key range $> v_k$*
- *All other children t_i where $i \in [2, k]$ has key range $(v_{i-1}, v_i]$*

Rule 3 ***"Leaf depth"***

*All **leaf nodes** must all be at the same depth (from root).*

Test yourself: Given these rules, is a BSTs just an (a, b)-tree where $a = 1$ and $b = 2$? Why or why not?

B-trees are simply $(B, 2B)$-trees. That is to say, they are a subcategory of (a, b)-trees such that $a = B$ and $b = 2B$. For instance, when $B = 2$, we have a $(2, 4)$-tree. (*FYI:* this is sometimes referred to as a (2,3,4)-tree). An example of a $(2, 4)$-tree is given below in Figure 1.

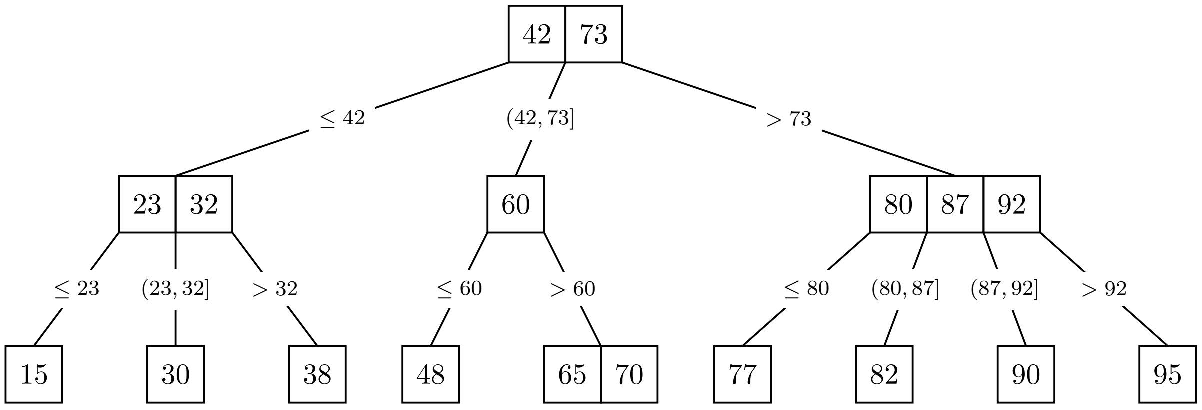


Figure 1: A $(2, 4)$-tree storing 18 keys. The range of keys contained within a subtree is indicated by the label on the edge from its parent.

Problem 1.a. Is an (a, b)-tree balanced? If it is, which rule(s) ensures that? If not, why?

Problem 1.b. What is the minimum and maximum height of an (a, b)-tree with n **keys**?

Problem 1.c. Write down the pseudocode for searching an (a, b)-tree. What data structure should we use for storing the keys and subtree links in a node?

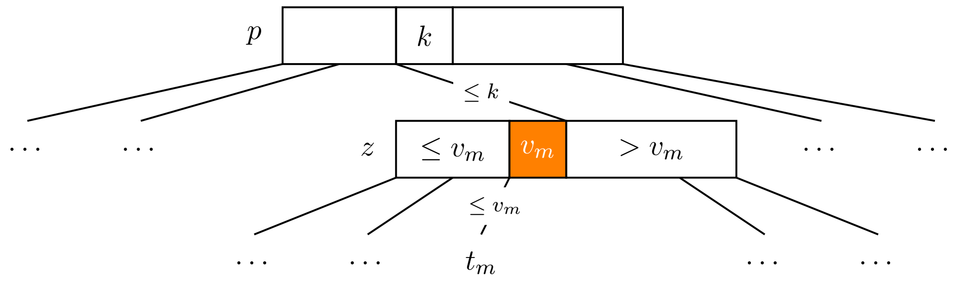
Problem 1.d. What is the cost of searching an (a, b)-tree with n **nodes**?

1.2 `split` operation

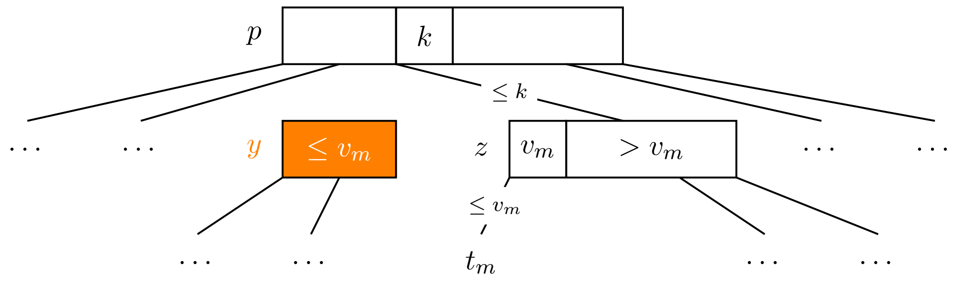
By definition, the three rules (Section 1.1) must hold before and after any operation that (a, b)-trees support. To ensure this, we need to take care of any potential violations that can occur across all operations. It should be obvious that `insertion` and `deletion` poses risks of violating the rules since they alter membership.

We will first look at `insertion`. This operation clearly risk nodes growing beyond permissible bounds. To address this, we will sometimes need to `split` large nodes into smaller ones. Firstly, keys in the violating node z are divided into two parts using the median index in its keylist (think partition step in QuickSort). We shall refer to the key at the median index as the *split key*. This ensures that each part contains half of z's keys along with their associated subtree links. The split key is then inserted into z's parent. Specifically, split operation on a non-root node z with b keys is outlined in the following algorithm:

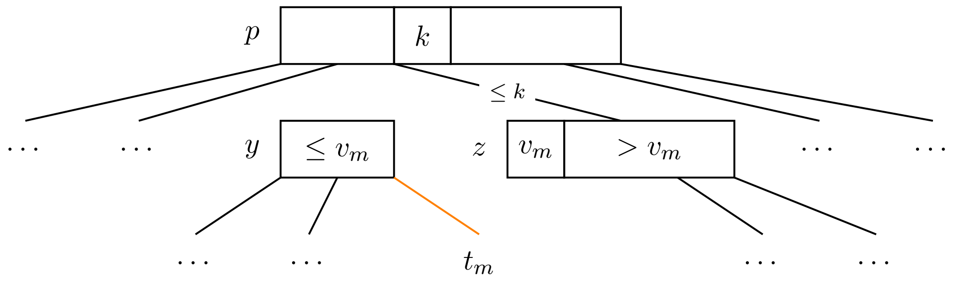
1. Find the median key v_m where the number of keys in z that are $\leq v_m$ (i.e. LHS) is $\lfloor (b-1)/2 \rfloor$ and the number of keys $> v_m$ (i.e. RHS) is $\lceil (b-1)/2 \rceil$.
   Since by definition $2 \leq a \leq (b+1)/2 \equiv 1 \leq a-1 \leq (b-1)/2$ for (a, b)-trees, each half have at least $a-1$ keys, satisfying Rule 1.



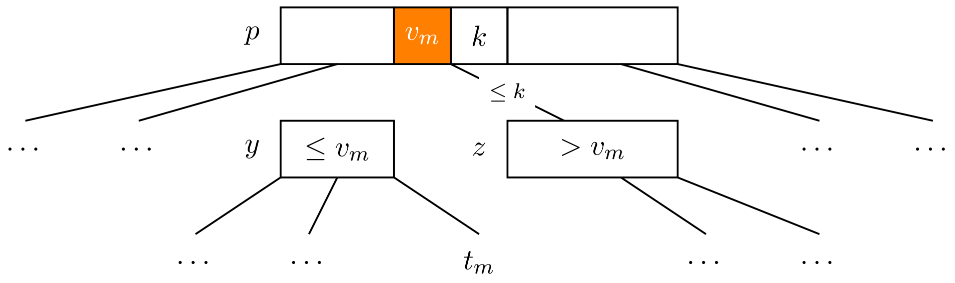
2. In z, separate all keys $\leq v_m$ as well as their associated subtree links to form a new adjacent node y.



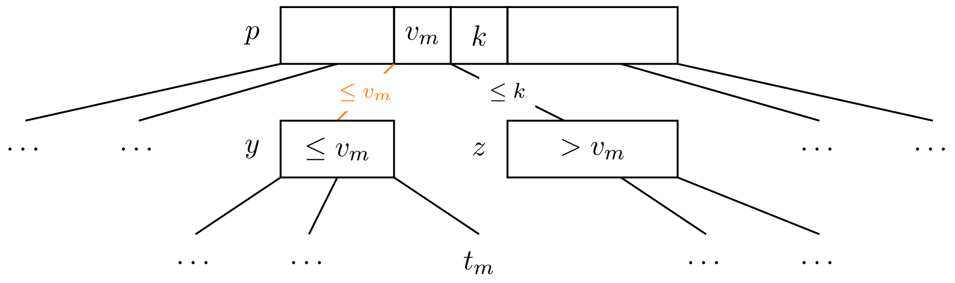
3. Since y will be missing a final child corresponding to key range $> y_{m-1}$, assign it to be t_m, the subtree associated with v_m. Remove the the association between t_m and v_m.



4. Move key v_m from z to p (z's parent). This newly inserted key should immediately precede the key in p which is associated to node z.



5. Add y as a child of p by associating it with newly inserted v_m in p. We can do this because all keys in y are guaranteed to be $\leq v_m$ by virtue of the split.



6. Return (v_m, y, z).

Test yourself: Why do we have to 'offer' a key to the parent? *Hint:* Recall Rule 1.

You should convince yourself that such a split operation on a **non-root** node z is safe (i.e. post-conditions satisfy all rules) so long as it satisfies the following preconditions:

1. z contains $\geq 2a$ keys
   Because after splitting and offering a key to the parent,
   - LHS has $\geq a - 1$ keys
   - RHS has $\geq a$ keys
2. z's parent contains $\leq b - 2$ keys

Test yourself: What happens if after the split operation, the parent node has $\geq b$ keys?

The algorithm outlined before applies to splitting a non-root node z, but what happens when z is actually the root node? What's different now is that there is no parent to offer the split key v_m to! Hence following the same procedure, now we simply create a new node, offer it the split key v_m and then promote it to become the new root node. Recall from Rule 1 that we have made it legal to have a root with one key — the purpose of that is to simply support this root-splitting operation!

Specifically, the root-splitting operation can be outlined as follows:

Unchanged steps	1. Pick median key v_m as split key
	2. Split z into LHS and RHS using v_m
	3. Create a new node y
	4. Move LHS split from z to y
New steps	5. Create a new empty node r
	6. Insert v_m into r
	7. Promote r to new root node
	8. Assign y and z to be the left and right child of r respectively
	9. Assign previous subtree t_m associated with v_m to be the final child of y

Problem 1.e. Come up with an example each for splitting an internal node, and for splitting a root node.

1.3 Insertion

Alike BSTs, new items are also inserted only in the leaves of an (a, b)-tree. With the `split` operation defined, it is now easy to outline the full insertion procedure of an item x into the tree:

```
w ← BTree.root
while true do
    if w contains b − 1 keys then
        v_m, y, z ← Split(w, v_m)
        if x ≤ v_m then
            w ← y
        else
            w ← z
    if w is a leaf then
        break
    else
        w ← GetSubtree(w, x)                    // Subtree with keyrange containing x
InsertKey(w, x)
```

Notice in the procedure outlined above, we preemptively split any nodes we encounter along the way which are at maximum key capacity (i.e. $b-1$). This is known as a *proactive* approach. With such an approach, we are guaranteed that the parent of the node to be split will not have too many keys after the addition of the split key. As opposed to the proactive approach, the *passive* approach is a lazier strategy whereby the new item is first inserted and then we recursively check upwards for violation, potentially having to perform splitting all the way up to the root node.

Problem 1.f. Prove that:

- If node w is split, then it satisfies the preconditions of the split routine
- Before x is inserted into w (at the last step), there are $< b$ keys in w (so key x can be added)
- All three rules of the (a, b)-tree are satisfied after an insertion

Problem 1.g. What is the cost of inserting into an (a, b)-tree with n nodes?

1.4 `merge` and `share` operations

Deletion in an (a, b)-tree risks having nodes shrink too small. In order to support deletion in an (a, b)-tree without violating rules afterwards, we need to introduce two different operations on nodes, each for handling a separate scenario after removing a key from the tree. The operations are `merge` and `share`.

Suppose z is the offending node and y is its smallest sibling (with least number of keys). Then immediately after a key is deleted from z, the corresponding operation to use are summarised in the following table:

Scenario	**Operation**	**Algorithm**
y and z have $< b - 1$ keys together	`merge(`y`, `z`)`	1. In parent, delete key v (the key separating the siblings) 2. Add v to the keylist of y 3. In y, merge in z's keylist and treelist 4. Remove z
y and z have $\geq b - 1$ keys together	`share(`y`, `z`)`	1. `merge(`y`, `z`)` 2. Split newly combined node using the regular `split` operation

1.5 Deletion

Prior to deletion, we need to first ascertain which type of node is the key contained in. Deleting a key off a root or internal node immediately raises the issue of handling its orphaned child below, which can get troublesome. You have learnt from AVL trees that to delete a node with two children, we can swap its key with that of its predecessor/successor and then delete off the predecessor/successor node instead. We can employ a similar strategy for (a, b)-trees because realize that predecessor/successor keys of a root/internal node will always be contained within leaf nodes (why?).

In other words, if the key to delete does not belong to a leaf node in the tree, then we can first swap it out with its predecessor (or successor) key before removing it from the tree. Doing so is convenient because it circumvents the issue of having to deal with orphaned nodes.

Now we are ready to specify the deletion routine as follows:

```
$w \leftarrow$ Search($BTree$, $x$)
if $w$ is not leaf then
    $p_w, p_x \leftarrow$ GetPredecessor($w$, $x$) // $p_w$: predecessor node, $p_x$: predecessor key
    SwapKeys($w$, $x$, $p_w$, $p_x$)
    $w \leftarrow p_w$
DeleteKey($w$, $x$)
while true do
    if $w$ contains $< a - 1$ keys then                // If rule 1 violated, merge/share
        $z \leftarrow$ GetSmallestSibling($w$)
        if $w$ and $z$ together contain $< b - 1$ keys then then
            $w \leftarrow$ Merge($w$, $z$)                 // Set $w$ to be the newly merged node
        else
            $s_L, s_R \leftarrow$ Share($w$, $z$)
            $w \leftarrow s_L$                       // Arbitrarily set $w$ to be the left node
    else
        break                                  // End routine once no more violations
    if $w$ is not root then
        $w \leftarrow$ GetParent($w$)                                // Recurse upwards
    else
        break                                    // End routine once handled root
```

Unlike our insertion algorithm, the deletion procedure above employs a *passive* strategy whereby it first deletes the target key, carry out `merge/share` on the node (if necessary) and then recursively check upwards for violation, potentially having to perform `merge/share` all the way up to the root. The *proactive* approach to deletion is left for you as an exercise.

Problem 1.h. Come up with an example each for merge and share operations, as well as a key deletion. Prove that for a merge or share, the necessary preconditions are satisfied before the operation, and that the three rules of an (a, b)-tree are satisfied afterwards. Thereafter, prove that after a delete operation, the three rules of an (a, b)-tree are satisfied. What is the cost of deleting from an (a, b)-tree?

Problem 2. External Memory (or why we care about B-Trees)[1]

In general, large amounts of data are stored on disk. And searching big data is where efficiency is most important and (a, b)-trees shine. In general, data is stored on disk in blocks, e.g., $B_1, B_2, \ldots, B_m$. Each block stores a chunk of memory of size B. You can think of each block as an array of size B.

Note that when you want to access a memory location in some block B_j, the entire block is read into memory. You can assume that your memory can hold M blocks at a time. Transfering a block from disk to memory is expensive: it requires spinning up the harddrive platter, moving the arm to the right track (using, e.g., the elevator algorithm!), and reading the disk while the platter spins. By contrast, reading the block once it is stored in memory is almost instantaneous (by comparison), e.g., many, many orders of magnitude faster. (See Table 2)

To estimate the cost of searching data on disk, let us only count the number of blocks we have to move from disk to memory. (We can ignore the cost of accessing a block once it is in memory. If we run out of space in memory (e.g., we need more than M blocks), then one of the blocks we do not need should be kicked out. For today, let's assume that never happens.

Memory unit	**Size**	**Block size**	**Access time (CPU clock cycles)**
L1 cache	64KB	64B	4
L2 cache	256KB	64B	10
L3 cache	up to 40MB	64B	40–74
Main memory	128GB	16KB	200-350
(Solid-State) Drive	Arbitrarily big	16KB	20,000
(Magnetic-Disk) Drive	Arbitrarily big	16KB	20,000,000

Table 2: Some typical numbers (from the Haswell architecture data sheet)

CPU caches are housed within the CPU which means they are blazing fast memory units. This is why they sit at the top of the memory hierarchy. CPU caches are organized in levels, with L1 being the fastest and L3 being the slowest. To keep our discussions simple, we will be ignoring them today.

Notice that accessing the disk, especially if it is a magnetic disk (as is typical for 100TB disks), is 100,000 times slower than accessing memory. Just as a numerical example, if 90% of your data is found in the L1 cache, 8% of your data is found in the L2 cache, and 2% of your data is in main memory, that means that you will be spending 57% of your time waiting on main memory. If you have to get any data from disk, you will be spending 99% of your time waiting for the disk!

We will be focusing on data stored on disk, and counting the number of block transfers between disk and main memory.

[1]This is an optional problem. It will not be covered during the recitation

Problem 2.a. Assume your data is stored on disk. Your data is a sorted array of size n, and spans many blocks. What is the number of block transfers needed (i.e. cost) of doing a *linear search* for an item? Leave your answer in terms of n and B.

Problem 2.b. Assume your data is stored on disk. Your data is a sorted array of size n, and spans many blocks. What is the number of block transfers needed (i.e. cost) of doing a *binary search* for an item? Leave your answer in terms of n and B.

Now, imagine you are storing your data in a B-tree. (Notice that you might choose $a = B/2$ and $b = B$, or $a = B/4$ and $b = B/2$, etc., depending on how you want to optimize.)

Notice that each node in your B-tree can now be stored in some constant number of blocks, for example, one block stores the key list, one block stores the subtree links, and one block stores the other information (e.g., the parent pointer, pointers to the two other blocks, and any other auxiliary information).

Problem 2.c. What is the cost of searching a keylist in a B-tree node? What is the cost of splitting a B-tree node? What is the cost of merging or sharing B-tree nodes?

Problem 2.d. What is the cost of searching a B-tree? What is the cost of inserting or deleting in a B-tree?