Methods of Proof

Even : n is even $\Leftrightarrow \exists k \in \mathbb{Z}$ such that $n = 2k$

odd : n is odd $\Leftrightarrow \exists k \in \mathbb{Z}$ such that $n = 2k+1$

Prime : n is prime: $(n > 1) \wedge \forall r, s \in \mathbb{Z}^+,$
$\big(n = rs \rightarrow (r = 1 \wedge s = n) \vee (r = n \wedge s = 1)\big)$.

Composite : n is composite: $\exists r, s \in \mathbb{Z}^+ \ (n = rs \wedge (1 < r < n) \wedge (1 < s < n))$.

Proving Existential Statement by Constructive Proof

An existential statement:

$$\exists x \in D\ Q(x)$$

is true iff $Q(x)$ is true for <u>at least one</u> x in D.

To prove such statement, we may use constructive proofs of existence:

- Find an x in D that makes $Q(x)$ true; or
- Give a set of directions for finding such an x.

Disproving Universal Statement by Counterexample

Given a universal (conditional) statement:

$$\forall x \in D\ \big(P(x) \rightarrow Q(x)\big).$$

Showing this statement is false is equivalent to showing that its negation is true.

The negation of the above statement is an existential statement:

$$\exists x \in D\ \big(P(x) \wedge {\sim}Q(x)\big).$$

Proving Universal Statement by Exhaustion

Given a universal conditional statement:

$$\forall x \in D\ \big(P(x) \rightarrow Q(x)\big).$$

When D is finite or when only a finite number of elements satisfy $P(x)$, we may prove the statement by the method of exhaustion.

Proving Universal Statement by Generalizing from the Generic Particular

To show that every element of a set satisfies a certain property, suppose x is a *particular* but *arbitrarily chosen* element of the set, and show that x satisfies the property.

Rational Numbers

A real number r is rational if, and only if, it can be expressed as a quotient of two integers with a nonzero denominator.

A real number that is not rational is irrational.

r is rational $\Leftrightarrow \exists$ integers a and b such that $r = \frac{a}{b}$ and $b \neq 0$.

Theorem 4.2.1 : Every integer is a rational number

Theorem 4.2.2 : The sum of any two rational number is rational

Corollary 4.2.3 : The double of a rational number is rational

Theorem 4.3.1 : For all positive integers a and b, if $a | b$, then $a \le b$

Theorem 4.3.2 : The only divisors of 1 are 1 and -1

Theorem 4.3.3 : For all integers a, b and c, if $a | b$ and $b | c$ then $a | c$

Theorem 4.6.1 : There is no greatest integer

Indirect Proof : Proof by Contradiction

1. Suppose the statement to be proved, S, is false. That is, the negation of the statement, ${\sim}S$, is true.
2. Show that this supposition leads logically to a contradiction.
3. Conclude that the statement S is true.

Indirect Proof : Proof by Contraposition

Recall: Contrapositive of $p \rightarrow q$ is ${\sim}q \rightarrow {\sim}p$.

1. Statement to be proved: $\forall x \in D\ \big(P(x) \rightarrow Q(x)\big)$.
2. Rewrite the statement into its contrapositive form: $\forall x \in D\ \big({\sim}Q(x) \rightarrow {\sim}P(x)\big)$.
3. Prove the contrapositive statement by a direct proof.
   - 3.1 Suppose x is an (particular but arbitrarily chosen) element of D s.t. $Q(x)$ is false.
   - 3.2 Show that $P(x)$ is false.
4. Therefore, the original statement $\forall x \in D\ \big(P(x) \rightarrow Q(x)\big)$ is true.