CS2040S – Data Structures and Algorithms

Lecture 17 – Connecting People - MST

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Outline

Minimum Spanning Tree (MST)

- Motivating Example & Some Definitions

Two Algorithms to solve MST (you have a choice!)

- Prim's (greedy algorithm with PriorityQueue)
  - PriorityQueue is discussed in Lecture 09
- Kruskal's (greedy algorithm, uses sorting and UFDS)
  - UFDS is discussed in Lecture 10

Review

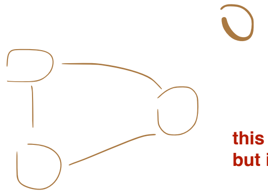
Definitions that we have learned before

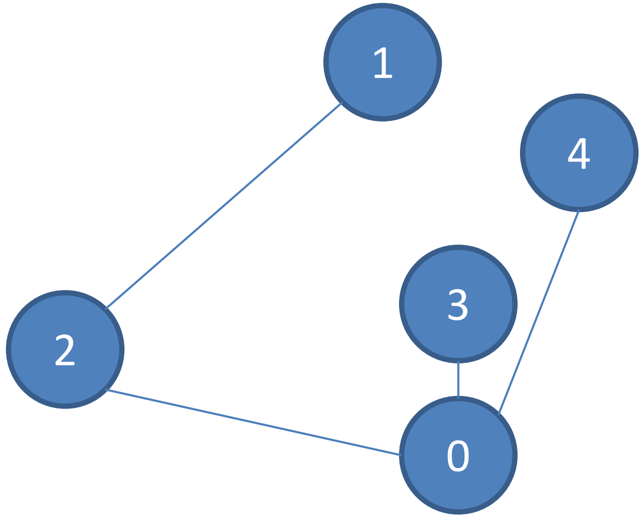
- **Tree T**
  - **T** is a **connected graph** that has **V** vertices and **V-1** edges
  - Important: One unique path between any two pair of vertices in **T**
- **Spanning** Tree **ST** of connected graph **G**
  - **ST** is a tree that spans (covers) every vertex in **G**
  - Recall the **BFS and DFS Spanning Tree**

Is This A Tree?

1. Yes, why ___________ it is connected graph with v vertices and v-1 edges
2. No, why ___________

keyword is connected because we cannot just say v vertices and v-1 edges

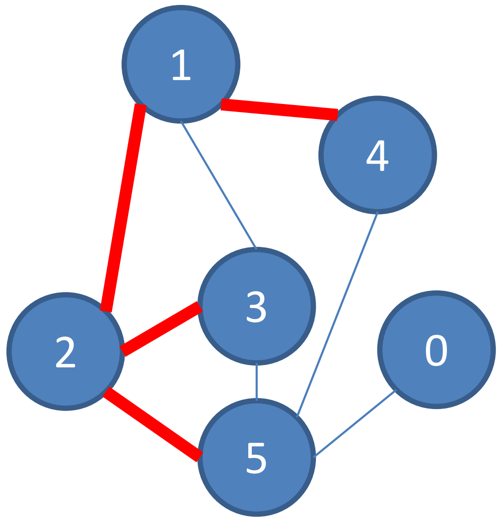




this also has v vertices and v-1 edges
but it is not a tree because it is NOT CONNECTED

Do the edges highlighted in red part form a spanning tree of the original graph?

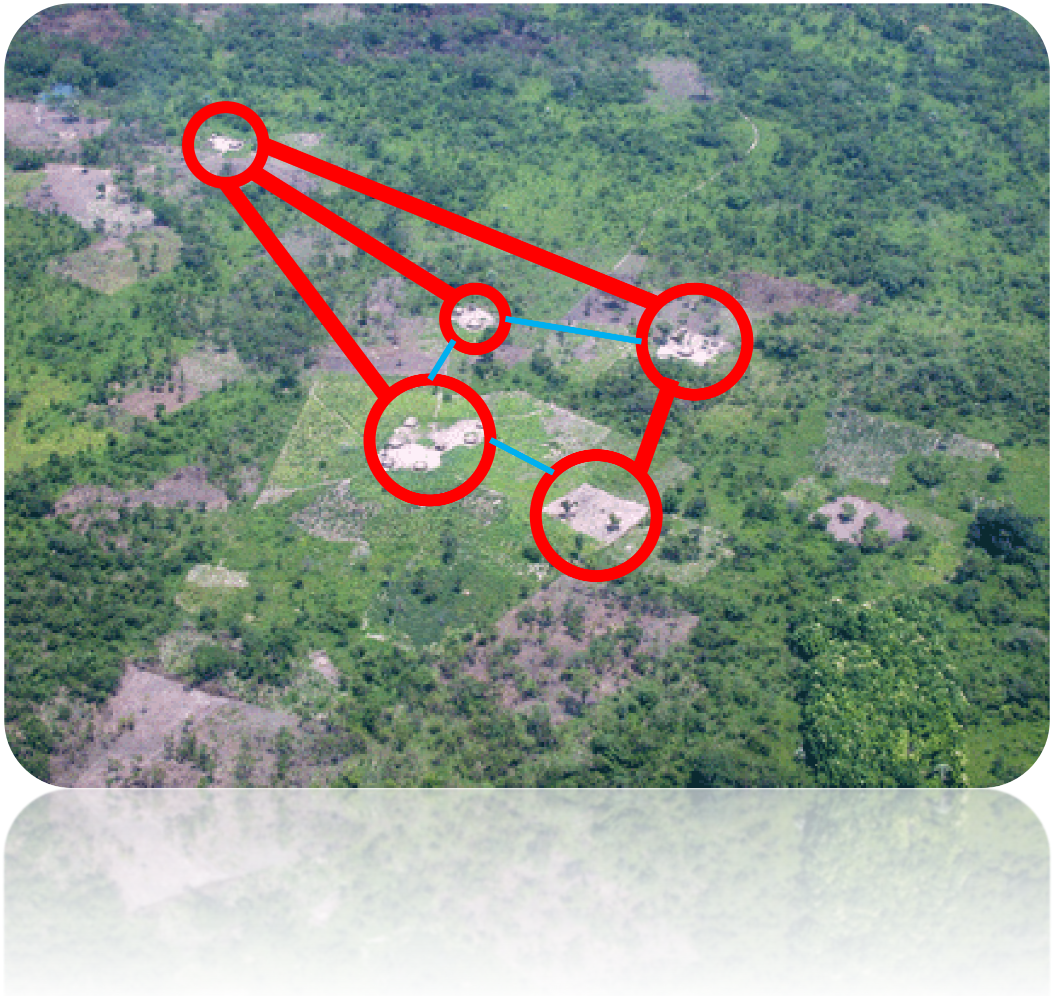
1. Yes, why __________
2. No, why vertex 0 is disconnected from the rest __________



Motivating Example

Government Project

- Want to link rural villages with roads
- The cost to build a road depends on the terrain, etc
- You only have limited budget
- How are you going to build the roads?



Definitions (1)

- Vertex set **V** (e.g. street intersections, houses, etc)
- Edge set **E** (e.g. streets, roads, avenues, etc)
  - Generally undirected (e.g. bidirectional road, etc)
  - Weighted (e.g. distance, time, toll, etc)
- Weight function **w(a, b): E→R**
  - Sets the weight of edge from **a** to **b**
- **Weighted** Graph **G**: **G(V, E), w(a, b): E→R**
- **Connected** undirected graph **G**
  - There is a path from any vertex **a** to any other vertex **b** in **G**
- The graph **G** we're concerned with is **connected undirected** and **weighted** when dealing with **MST**

More Definitions (2)

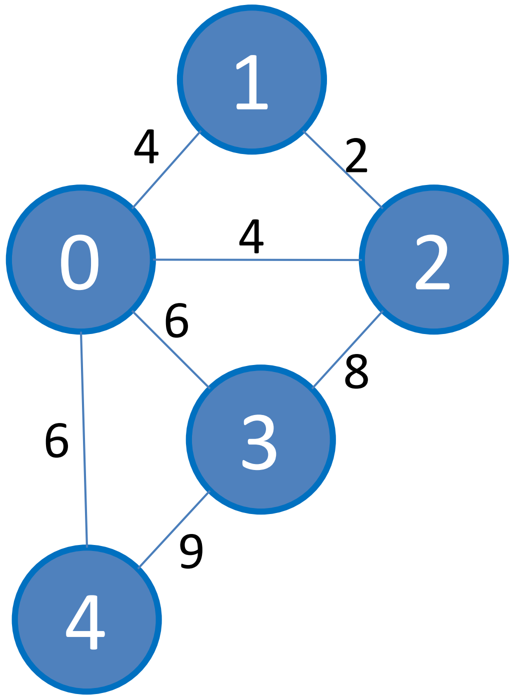
- Spanning Tree **ST** of connected undirected weighted graph **G**
  - Let **w(ST)**, weight of **ST**, denotes the total weight of edges in **ST**

$$w(ST) = \sum_{(a,b)\in ST} w(a,b)$$

- **Minimum Spanning Tree (MST)** of connected undirected weighted graph **G**
  - **MST** of **G** is an **ST** of **G** with the minimum possible **w(ST)**

More Definitions (3)

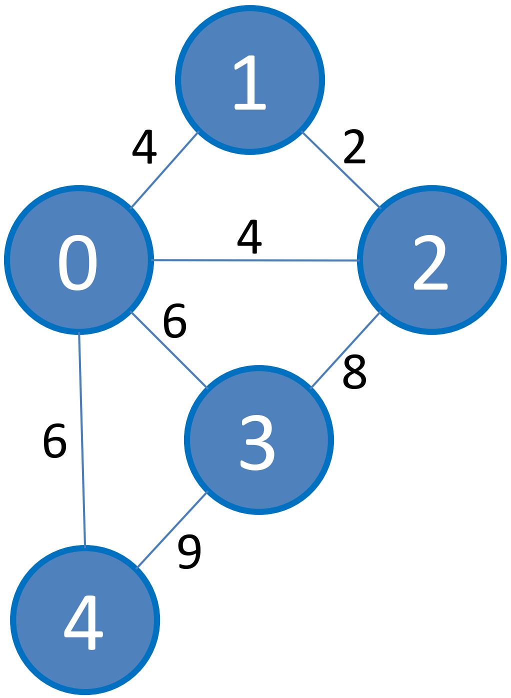
- **The (standard) MST Problem**
  - Input: A connected undirected weighted graph **G(V, E)**
  - Select some edges of **G** such that the graph forms a spanning tree, but with minimum total weight
  - Output: Minimum Spanning Tree **(MST)** of **G**



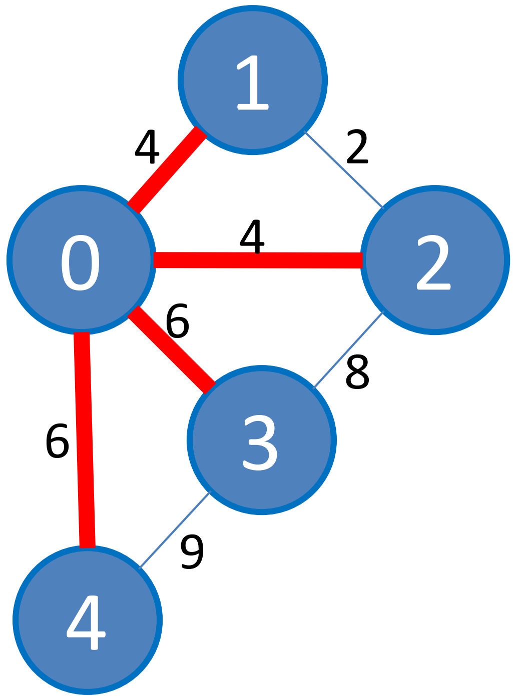
Example

we only get multiple spanning trees if there are cycles
cause we have alternative ways of getting to the same vertex

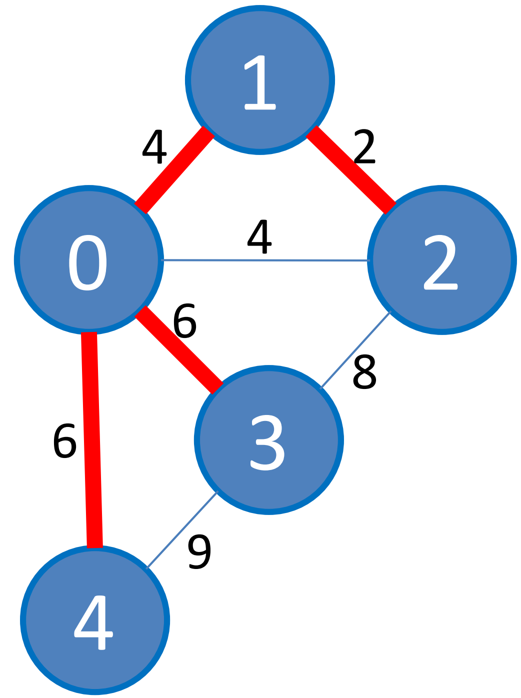
The Original Graph



A Spanning Tree
Cost: 4+4+6+6 = 20

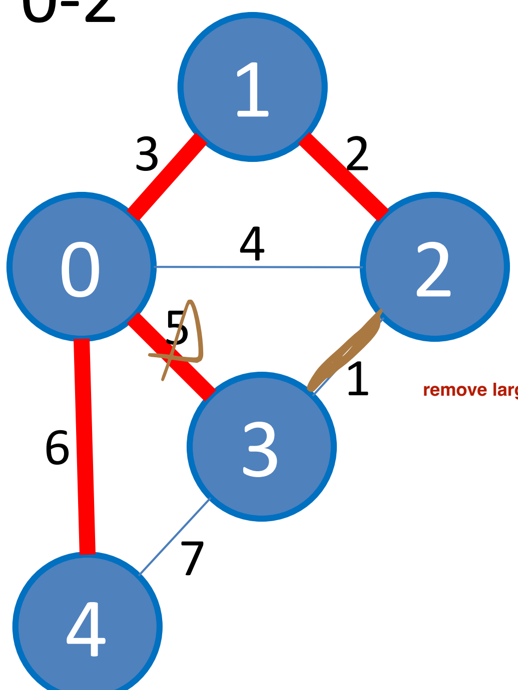


An MST
Cost: 4+6+6+2 = 18



Do the edges highlighted in red part form an MST of the original graph?

1. No, we must replace edge 0-3 with edge 2-3
2. No, we must replace edge 1-2 with 0-2
3. Yes



remove largest edge in the cycle which is 5

Brute force/Complete Search Solution?

- Consider all cycles in the graph and break them!
  - For each cycle remove the largest edge
  - If 1 or more edges in a cycle has already been removed previously move on to the next cycle

- Cycle property: For any cycle C in graph G(V,E), if weight of an edge e is larger than every other edge in C, e cannot be included in the MST of G(V,E) basically for each cycle remove the largest edge

- How to get all cycles in the graph?
  - Not so easy except for some special graphs … (Can you think of a way to do this?)
  - Can have up to O(V!) different cycles!
  - Listing down one by one is slow !

MST Algorithms

MST is a well-known Computer Science problem

Several efficient (polynomial) algorithms:

- Jarnik's/Prim's greedy algorithm
  - Uses PriorityQueue Data Structure taught in Lecture 09
- Kruskal's greedy algorithm
  - Uses Union-Find Data Structure taught in Lecture 10
- Boruvka's greedy algorithm (not discussed here)
- And a few more advanced variants/special cases…

Do you still remember Prim's/Kruskal's algorithms from CS1231/S?

1. Yes and I also know how to ***implement*** them
2. Yes, but I have not try implementing them yet
3. I forgot that particular CS1231/S material…
   but I know it exists
4. Eh?? These two algorithms were covered before in CS1231/S??
5. I didn't take CS1231/S ☹

Prim's Algorithm

Very simple pseudo code

pick a source vertex

```
1. T ← {s}, a starting vertex s (usually vertex 0)
```

PQ stores a pair(w(u,v) the weight of the edge uv and the other end of the edge V)

```
2. enqueue edges connected to s (only the other ending
   vertex and edge weight) into a priority queue PQ
   that orders elements based on increasing weight
```

min heap

so we keep choosing the smallest weight edge

if the VERTEX hasnt been included in the spanning tree yet

add it to the spanning tree

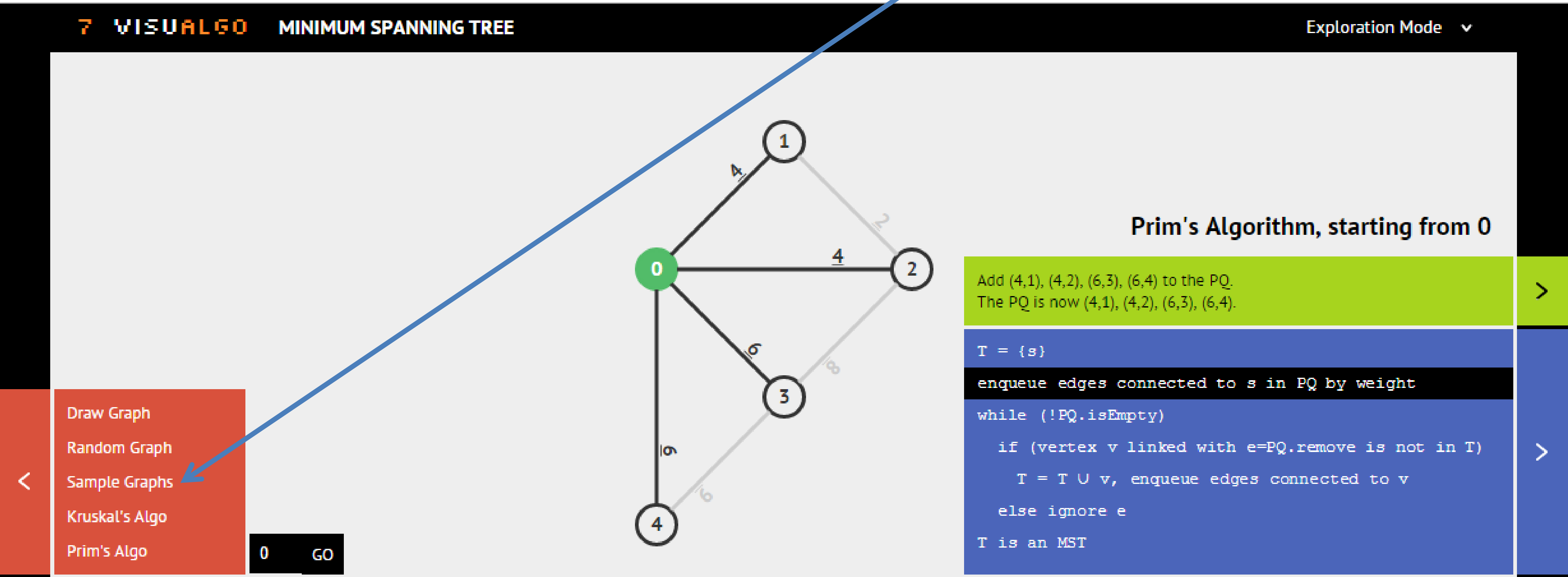
```
3. while there are unprocessed edges left in PQ
     take out the front most edge e
     if vertex v linked with this edge e is not taken yet
       T ← T ∪ v (including this edge e)
       enqueue each edge adjacent to v into the PQ if it
       is not already in T

4.T is an MST
```

taken array prevents forming of cycles

MST Algorithm: Prim's

Ask VisuAlgo to perform Prim's *from various sources* on the sample Graph (CP3 4.10), *then try other graphs*

In the screen shot below, we show the start of **Prim(0)**



Easy Java Implementation

You need to use two known Data Structures
to be able to implement Prim's algorithm:

1. A priority queue **PQ** (we can use Java PriorityQueue), and
2. A boolean array **taken** (to decide if a vertex has been taken or not)

With these DSes, we can run Prim's in O(**E** log **V**) using Adjacency list

Max size of PQ is O(E)

- We only process each edge once (enqueue and dequeue it), O(**E**)
  - Each time, we enqueue/dequeue from a PQ in O(log **E**) enq/deq takes log E time
  - As **E** = O($\mathbf{V}^2$), we have O(log **E**) = O(log $\mathbf{V}^2$) = O(2 log **V**) = O(log **V**)
    E can range from O(V) for a tree to O(V^2) for a complete graph
  - Total time O(**E**)*O(**logV**) = O(**E**log**V**)

Let's have a quick look at PrimDemo.java

Why Does Prim's Work? (1)

First, we have to realize that **Prim's algorithm** is a **greedy algorithm**

This is because **at each step**, it always try to select the next valid edge e with **minimal weight** (greedy!)

Greedy algorithm is usually simple to implement

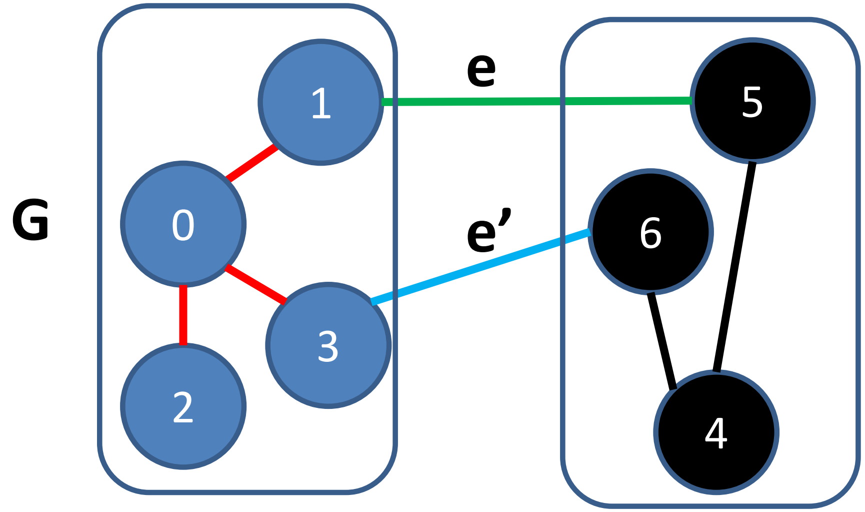
- However, it usually requires "proof of correctness"
- Here, we will just see a quick proof

Cut Property of a Connected Graph G

- **Cut of a connected graph**: any partition of vertices of G into 2 disjoint subset (vertices in one set is not in the other). An example is shown below.

edges that link two disjoint sets

- **Cut Set**: The set of edges that cross a cut (e and e' in the example)
- **Cut Property of a connected graph**: For any cut of the graph, if the weight of an edge **e** in the cut-set is strictly smaller than the weights of other edges of the Cut Set, then **e** belongs to all MSTs of the graph.



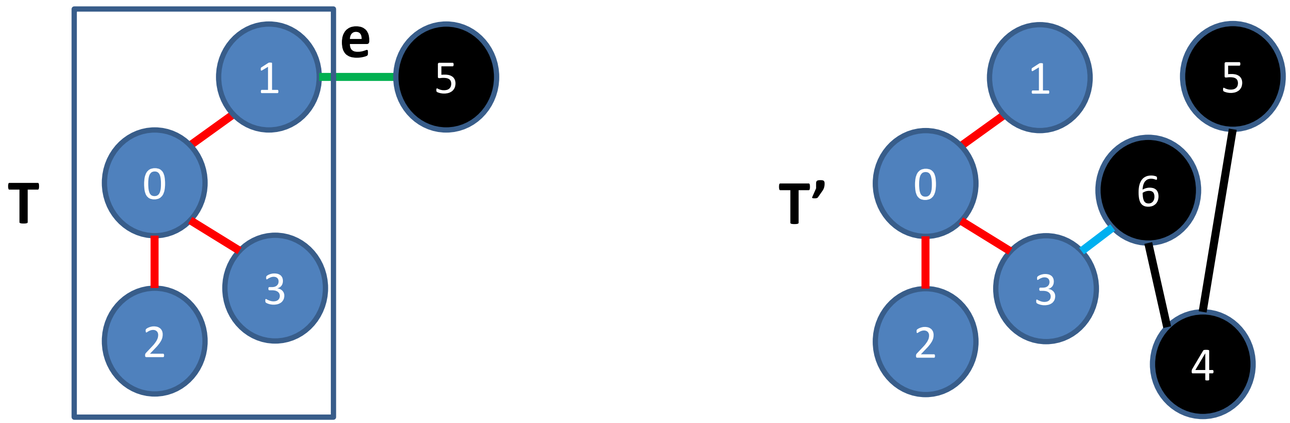
use the smallest edge in the cut set to link two disjoint sets that are already msts

Why Does Prim's Work? (2)

with visual explanation

Proof by contradiction:

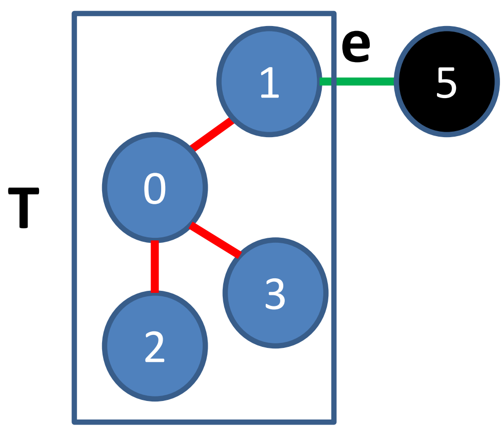
- Assume that edge **e** is the first edge at iteration k chosen by Prim's which is not in any valid MST.
- Let **T** be the tree generated by Prim's before adding **e**.
- Now **T** must be a subtree of some valid MST **T'**

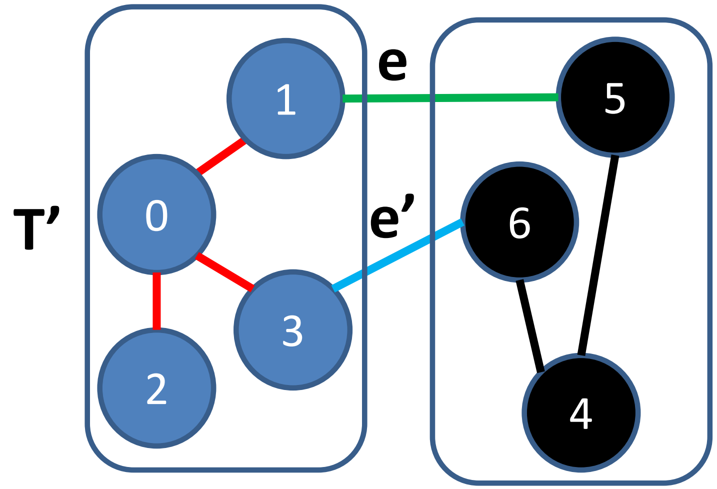


Why Does Prim's Work? (3)

with visual explanation

- Adding edge **e** to **T'** will now create a cycle.
- Since e has 1 endpoint in **T** (the valid endpoint) and one endpoint outside **T**, trace around this cycle in **T'** until we get to some edge **e'** that goes back to **T**
- Vertices of **T** (blue) and vertices outside **T** (black) forms 2 disjoints subsets of **T'**. This is a cut of **T'**, where {**e**,**e'**} is the cut set

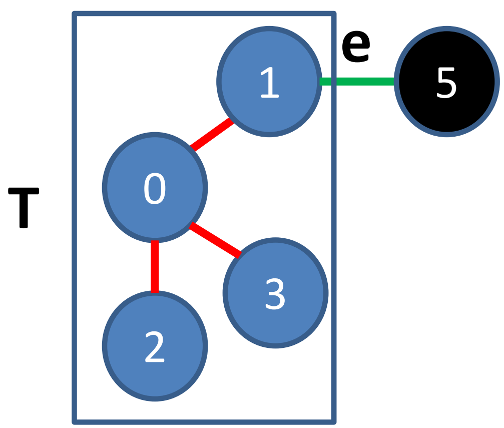


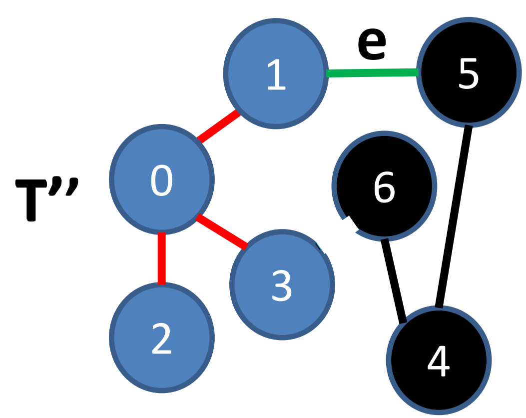


Why Does Prim's Work? (4)

with visual explanation

- By Prim's algorithm **e** and **e'** must be candidate edges at iteration k, but **e** was chosen meaning w(**e**) ≤ w(**e'**) by the **cut property**
- Now replacing **e'** with **e** in **T'** must give us tree **T''** covering all vertices of the graph s.t w(**T''**) ≤ w(**T'**)
- Contradiction that **e** is first edge chosen wrongly





*Prim's variant for Dense Graphs (1)

if sparse graph, E = O(v), O(vlogv)

- For dense graphs where E = $O(V^2)$, time complexity of Prim's is O(ElogV) = **$O(V^2 \log V)$**
- We can improve this time complexity …. ironically by replacing the PQ with a simple array **A** of size V
  - For each **A**[v], store a pair info <w(u,v),u> of the smallest weighted edge (u,v) to v among all edges to v that has been explored as Prim's is executed
  - *Why only track the smallest weighted explored edge to a vertex v?* because we want MST

**this variant is not in Visualgo*

Prim's variant for Dense Graphs (2)

Pseudo code is as follows

make edge weight inf

```
1.  Initialize A[v] = <+inf,v> for all v
2.  A[s] = <0,s>
3.  While not all vertices are in T
```

find edge with min weight

```
      Scan A to get v where A[v].first is minimum in A
      T 🡐 T ∪ v and the adjoining edge
```

set edge weight back to inf because we dont want to deq it again

```
      A[v] = <+inf, A[v].second> // will not choose it again
      for all u adjacent to v // all neighbors of v
        if (u is not in T && A[u].first > w(v,u))
          A[u] = <w(v,u),v>
```

if we find a smaller edge, make it the new edge to traverse

```
4. T is an MST
```

Prim's variant for Dense Graphs (3)

Time complexity of Prim's variant for dense graphs

```
1.  Initialize A[v] = <+inf,v> for all v
2.  A[s] = <0,s>
3.  While not all vertices are in T // O(V) iterations
      scan A to get v where A[v].first is minimum in A // O(V)
      T 🡐 T ∪ v and the adjoining edge
       A[v] = <+inf, A[v].second>
      for all u adjacent to v // O(V) due to O(V) neighbors
        if (u is not in T && A[u].first > w(v,u))
          A[u] = <w(v,u),v>

4. T is an MST
```

always takes O(v^2) time regardless if its sparse or dense graph

Total Time Complexity = $\mathbf{O(V^2)}$, better than $O(V^2 \log V)$ for standard Prim's algorithm

Coming up next: Kruskal’s algorithm

Kruskal's Algorithm

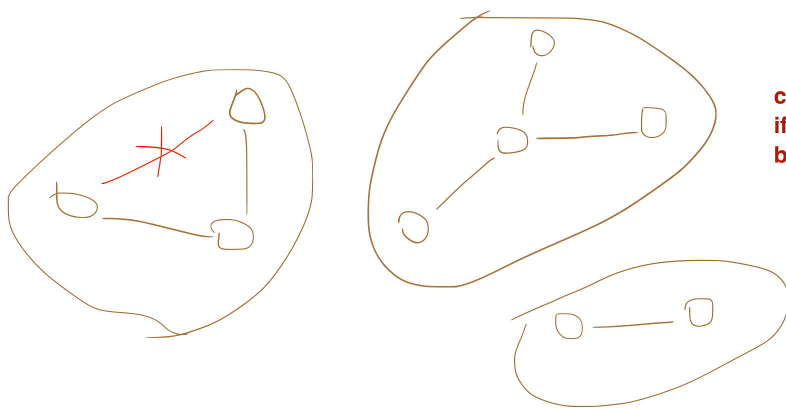
Very simple pseudo code

best graph data structure to store the graph is edge list because it gives us all the edges and we can sort it in increasing order

```
sort the set of E edges by increasing weight
T ← {}
while there are unprocessed edges left
  pick an unprocessed edge e with min cost
  if adding e to T does not form a cycle
    add e to T
T is an MST
```

simply go down the edge list since it is already sorted

kruskal's works by forming a forest of trees by picking out the overall smallest edges so it forms a cycle when the two vertices it links are in the same component/subtree



can represent each subtree as a disjoint set
if they are in the same set, dont add the edge
but if not, we can add the edge to the disjoint set

Kruskal's Implementation (1)

```
sort the set of E edges by increasing weight // O(?)
T ← {}
while there are unprocessed edges left // O(E)
  pick an unprocessed edge e with min cost // O(?)
  if adding e to T does not form a cycle // O(?)
    add e to the T // O(1)
T is an MST
```

ElogE
merge sort

O(E)

while loop
O(E alpha(E))

O(alpha(E))

use UFDS find set operation

Total time complexity ElogE

To sort the edges:

- We use **EdgeList** to store graph information
- Then use "any" sorting algorithm that we have seen before

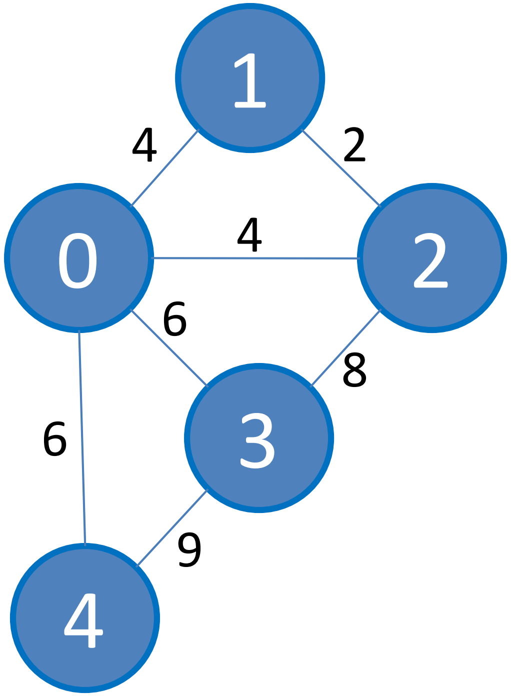
To test for cycles:

- We use **Union-Find Disjoint Sets**

Sorting Edges in Edge List

Adjacency Matrix/List that we have learned previously are *not suitable* for edge-sorting task!

To sort **EdgeList**, we use ***one liner* Java Collections.sort …**



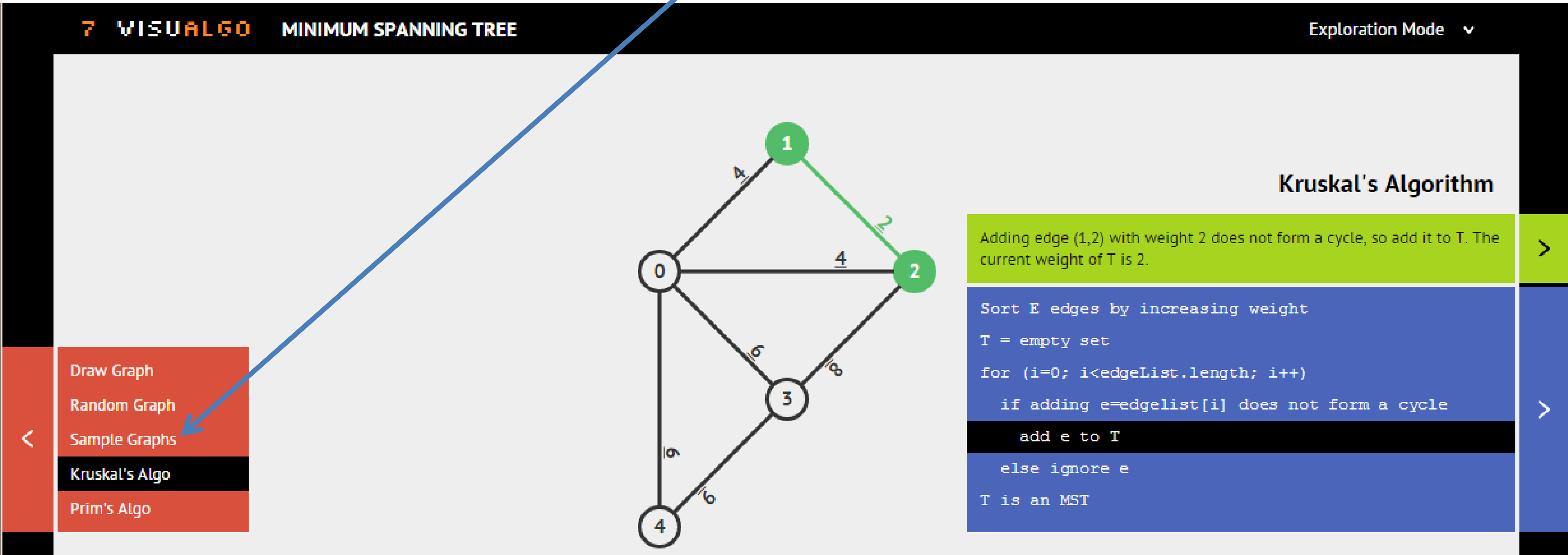
i	w	u	v
0	2	1	2
1	4	0	1
2	4	0	2
3	6	0	3
4	6	0	4
5	8	2	3
6	9	3	4

MST Algorithm: Kruskal’s

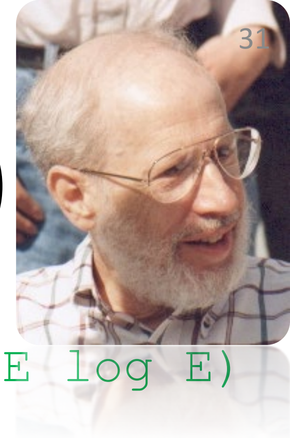
Ask VisuAlgo to perform Kruskal’s on the sample Graph (CP3 4.10), *then try other graphs*

In the screen shot below, we show the start of **Kruskal**

(there is no parameter for this algorithm)



Kruskal's Implementation (2)



```
sort the set of E edges by increasing weight // O(E log E)
T ← {}
while there are unprocessed edges left // O(E)
  pick an unprocessed edge e with min cost // O(1)
  if adding e to T does not form a cycle // O(α(V)) = O(1)
    add e to the T // O(1)
T is an MST
```

To sort the edges, we need $O(\mathbf{E} \log \mathbf{E})$

To test for cycles, we need $O(\alpha(\mathbf{V}))$ – small, assume constant $O(\mathbf{1})$

In overall

- Kruskal's runs in $O(\mathbf{E} \log \mathbf{E} + \cancel{\mathbf{E}\,\alpha(\mathbf{V})})$ // E log E dominates!
- As $\mathbf{E} = O(\mathbf{V}^2)$, thus Kruskal's runs in $O(\mathbf{E} \log \mathbf{V}^2) = O(\mathbf{E} \log \mathbf{V})$

Let's have a quick look at KruskalDemo.java

Why Does Kruskal’s Work? (1)

Kruskal’s algorithm is also a **greedy algorithm**

Because **at each step**, it always try to select the next unprocessed edge **e** with **minimal weight** (greedy!)

Simple proof on how this greedy strategy works

- Almost the same as that for Prim’s

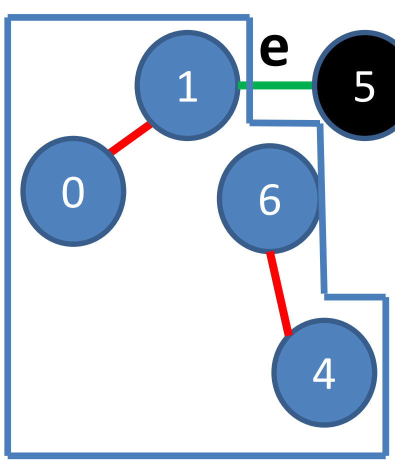
Why Does Kruskal's Work? (2)

with visual explanation

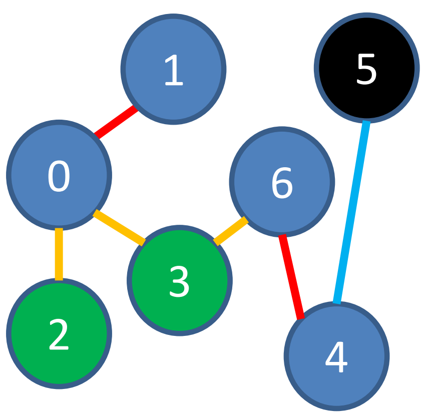
Proof by contradiction:

- Assume that edge **e** is the first edge at iteration k chosen by Kruskal's which is not in any valid MST.
- Let **F** be the forest generated by Kruskal's before adding **e**.
- Now **F** must be a part of some valid MST **T'**

F



T'

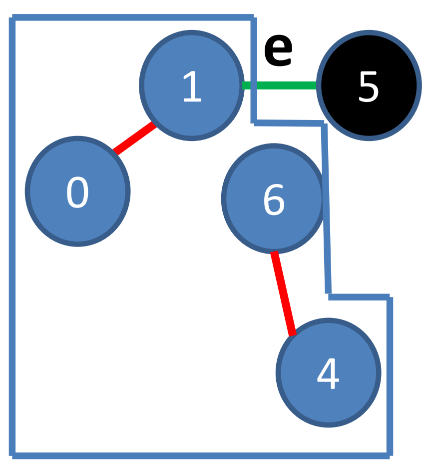


Why Does Kruskal's Work? (3)

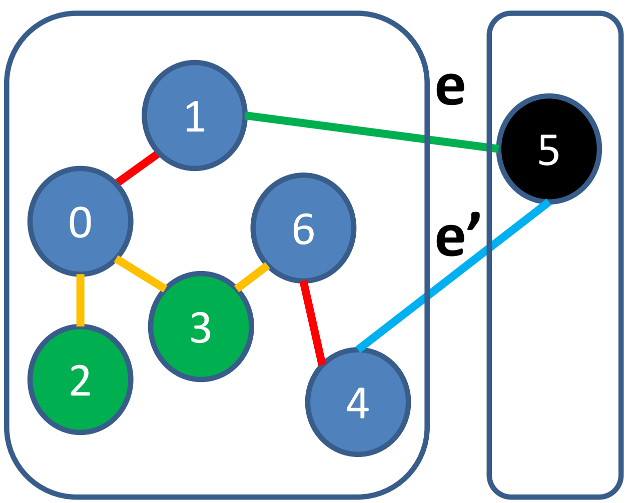
with visual explanation

- Putting **e** into **T'** will create a cycle.
- Tracing the cycle, let **V'** be the set of vertices encountered in the cycle that is outside of **F** {only 5 in the example}, where **e'** is the edge leading back into **F**
- We can create a cut of **T'** where the 2 partitions are {all vertices except **V'**} ({0,1,2,3,4,6} in the example) and **V'** ({5} in the example), and the cut set is {**e**,**e'**} in the example

F



T'

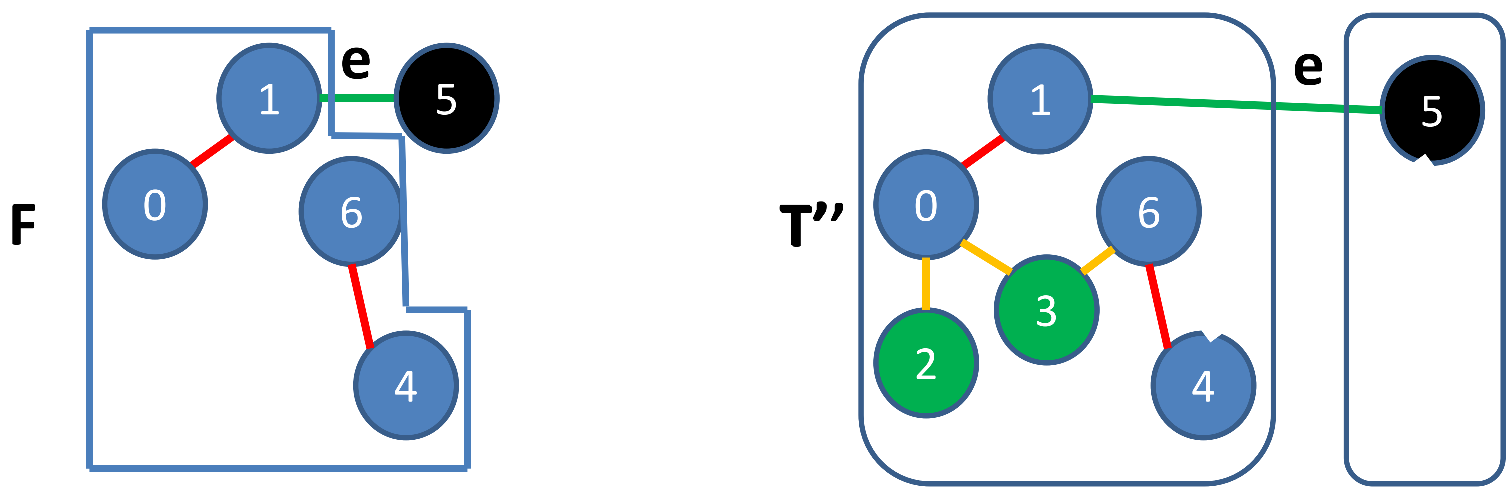


Why Does Kruskal's Work? (4)

with visual explanation

- At iteration k, both **e** and **e'** are candidate (they are not chosen and do not form a cycle if chosen).
- Since **e** was chosen, w(**e**) ≤ w(**e'**) by the **cut property**
- Now replacing **e'** with **e** in **T'** must give us tree **T''** covering all vertices of the graph s.t w(**T''**) ≤ w(**T'**)
- Contradiction that **e** is first edge chosen wrongly

meaning every edge chosen by kruskal's must be part of the mst



If given an MST problem, I will…

1. Use/code Kruskal's algorithm
2. Use/code Prim's algorithm if graph very dense, use prims variant
3. No preference…

Summary

Introduce the MST problem (covered briefly in CS1231/S)

Discuss 2 algorithms

- Prim's algorithm (uses PriorityQueue ADT) & a variant for dense graphs where $E=O(V^2)$ (uses an array instead of PQ)
- Kruskal's algorithm (uses Edge List and UFDS)
- Can view the above 2 algorithms as making use of the Cut Property as opposed to the Cycle Property to construct the MST of any given connected weighted graph

You *may* learn MST/Prim's/Kruskal's again in CS3230