CS1231S: Discrete Structures

Tutorial #5: Relations & Partial Orders

(Week 7: 26 – 30 September 2022)

I. Discussion Questions

These are meant for you to discuss on Canvass. No answers will be provided.

D1. Let R be a binary relation on a non-empty set A. If $R = \emptyset$, then is R reflexive? Symmetric? Transitive?

D2. Suppose a binary relation R on a non-empty set A is reflexive, transitive, symmetric and antisymmetric. What can you conclude about R? Explain.

D3. Asymmetry is defined in question 6 as follows: A binary relation R on a set A is **asymmetric** iff

$$\forall x, y \in A\ (x\ R\ y \Rightarrow y \not R\ x).$$

Are there any binary relations that are both symmetric and asymmetric?

II. Tutorial Questions

1. Let S be the set of all strings over the alphabet $\mathcal{A} = \{s, u\}$, i.e. an element of S is a sequence of characters, each of which is either s or u. Examples of elements of S are: ε (the empty string), $s, u, sus, usssuu$, and $susussssu$.

   Define a relation R on S by the following: $\forall a, b \in S\ \big(aRb\ \Leftrightarrow len(a) \le len(b)\big)$
   where $len(x)$ denotes the length of x, i.e. the number of characters in x.

   Is R antisymmetric? Prove or disprove it.

2. Consider the "divides" relation on each of the following sets of integers. For each of them, draw a Hasse diagram, and find all minimal, maximal, smallest and largest elements.

   (a) $A = \{1, 2, 4, 5, 10, 15, 20\}$.

   (b) $B = \{2, 3, 4, 6, 8, 9, 12, 18\}$.

3. Let $\mathcal{P}(A)$ denote the power set of set A. Prove that the binary relation $\subseteq$ on $\mathcal{P}(A)$ is a partial order.

4. Let $B = \{0,1\}$ and define the binary relation R on $B \times B$ as follows:

$$\forall (a, b), (c, d) \in B \times B\ \big((a, b)\ R\ (c, d) \Leftrightarrow (a \le c) \wedge (b \le d)\big).$$

   (a) Prove that R is a partial order.

   (b) Draw the Hasse diagram for R.

   (c) Find the maximal, largest, minimal and smallest elements.

   (d) Is $(B \times B, R)$ well-ordered?

5. Let R be a binary relation on a non-empty set A. Let $x, y \in A$. Define a relation S on A by

$$x\ S\ y \Leftrightarrow x = y \lor x\ R\ y \text{ for all } x, y \in A.$$

Show that:

(a) S is reflexive;

(b) $R \subseteq S$; and

(c) if S' is another reflexive relation on A and $R \subseteq S'$, then $S \subseteq S'$.

What is this relation S called? (Hint: Refer to Transitive Closure in Lecture 6).

6. Let R be a binary relation on a set A.
We have defined antisymmetry in class: R is **antisymmetric** iff

$$\forall x, y \in A\ (x\ R\ y \land y\ R\ x \Rightarrow x = y).$$

We define asymmetry here. R is **asymmetric** iff

$$\forall x, y \in A\ (x\ R\ y \Rightarrow y \not R\ x).$$

(a) Find a binary relation on A that is both asymmetric and antisymmetric.

(b) Find a binary relation on A that is not asymmetric but antisymmetric.

(c) Find a binary relation on A that is asymmetric but not antisymmetric.

(d) Find a binary relation on A that is neither asymmetric nor antisymmetric.

7. Consider a set A and a total order $\preccurlyeq$ on A. Show that all minimal elements are smallest.

8. **Definitions.** Consider a partial order $\preccurlyeq$ on a set A and let $a, b \in A$.

   - We say a, b are **comparable** iff $a \preccurlyeq b$ or $b \preccurlyeq a$.
   - We say a, b are **compatible** iff there exists $c \in A$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

In question 3, you are given the "divides" relation on $A = \{1, 2, 4, 5, 10, 15, 20\}$. List out the pairs of distinct elements in A that are (a) comparable; (b) compatible. Use the notation $\{x, y\}$ to represent the pair of elements x and y.

9. Let $A = \{a, b, c, d\}$. Consider the following partial order on A:

$$R = \{(a, a),\ (a, b),\ (a, c),\ (a, d),\ (b, b),\ (b, c),\ (c, c),\ (d, d)\}.$$

(a) Draw a Hasse diagram of R..

(b) Draw Hasse diagrams of all the linearizations of R.

10. For each of the following statements, state whether it is true or false and justify your answer.

(a) In all partially ordered sets, any two comparable elements are compatible.

(b) In all partially ordered sets, any two compatible elements are comparable.