Solutions to Exam 2019-2020 Semester 1

1. (a) A light shines from the top of a lamp post 15 metres high. At time $t = 0$ a ball is projected vertically upwards from a point on the ground 10 metres away from the foot of the lamp post. It is known that the ball moves upwards a distance of $s = 10t - 4.9t^2$ metres above the ground in t seconds for $0 \le t \le 1.01$. If the speed of the shadow of the ball on the ground at $t = 1$ second is equal to u metre per second, Find the value of u. Give your answer correct to two decimal places.

   (b) Let a denote a positive constant. It is known that the graph of $y^2 = x^2a^3 - 3x^3a^2 + 3x^4a - x^5$ has a loop and that the area bounded by this loop is equal to $\frac{2019}{1521}$. Find the value of a. Give your answer correct to two decimal places.

   Answer. (a) 0.31, (b) $a = 1.65$.

   Solution. (a) Let the distance of the shadow of the ball from the foot of the lamp post be x metres at time t second. When $t = 0$, we have $x = 10$. By similar triangles, $\frac{s}{15} = \frac{x-10}{x}$. Thus $x = \frac{150}{15-s} = \frac{150}{15-10t+4.9t^2}$. Therefore, $\frac{dx}{dt} = \frac{150(10-9.8t)}{(15-10t+4.9t^2)^2}$. Consequently, $\left.\frac{dx}{dt}\right|_{t=1} = \frac{150(10-9.8)}{(15-10+4.9)^2} = \frac{150\times 0.2}{9.9^2} = 0.306 \approx 0.31$ metres per second. That is $u = 0.31$.

   (b) First $y^2 = x^2a^3 - 3x^3a^2 + 3x^4a - x^5 = x^2(a-x)^3$. Thus $y = 0 \Leftrightarrow x = 0$ or a. That means the loop is within the range for $x = 0$ to $x = a$. The upper and lower curves bounding the loop have equations given by $y = x(a-x)^{\frac{3}{2}}$ and $y = -x(a-x)^{\frac{3}{2}}$ respectively. Therefore, the area of the loop is $2\int_0^a x(a-x)^{\frac{3}{2}}\,dx = 2\int_0^a a(a-x)^{\frac{3}{2}} - (a-x)^{\frac{5}{2}}\,dx = 2\left[-\frac{2}{5}a(a-x)^{\frac{5}{2}} + \frac{2}{7}(a-x)^{\frac{7}{2}}\right]_0^a = 2[\frac{2}{5}aa^{\frac{5}{2}} - \frac{2}{7}a^{\frac{7}{2}}] = \frac{8}{35}a^{\frac{7}{2}}$. Hence $\frac{8}{35}a^{\frac{7}{2}} = \frac{2019}{1521} \Leftrightarrow a = 1.65$.

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2. (a) Find the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(3^n)(n!)^2(x-5)^{2n}}{(2n)!}$. Give your answer correct to two decimal places.

   (b) Let $f(x) = \displaystyle\int_0^x \frac{\ln(1+t^2)}{1+t}\,dt$. Find the **exact value** of $f^{(7)}(0)$.

   Answer. (a) 1.15, (b) 600.

   Solution. (a) We have $\displaystyle\lim_{n\to\infty}\left|\frac{\frac{(3^{n+1})((n+1)!)^2(x-5)^{2n+2}}{(2n+2)!}}{\frac{(3^n)(n!)^2(x-5)^{2n}}{(2n)!}}\right| = \lim_{n\to\infty}\frac{3(n+1)^2|x-5|^2}{(2n+2)(2n+1)} = \frac{3|x-5|^2}{4}$.

   By ratio test, the power series converges if $\frac{3|x-5|^2}{4} < 1 \Leftrightarrow |x-5| < \frac{2}{\sqrt{3}} = 1.15$, and diverges if $|x-5| > \frac{2}{\sqrt{3}}$. Thus the radius of convergence is 1.15.

(b) We have the Maclaurin series $\ln(1+x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$, for $|x| < 1$.

Thus $\ln(1+t^2) = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+2}}{n+1}$, for $|t| < 1$.

That is $\ln(1+t^2) = t^2 - \frac{t^4}{2} + \frac{t^6}{3} - \frac{t^8}{4} + \cdots$, for $|t| < 1$.

Then $\frac{\ln(1+t^2)}{1+t} = (1 - t + t^2 - t^3 + t^4 - \cdots)(t^2 - \frac{t^4}{2} + \frac{t^6}{3} - \frac{t^8}{4} + \cdots)$, for $|t| < 1$.

Thus $\frac{\ln(1+t^2)}{1+t} = (t^2 - \frac{t^4}{2} + \frac{t^6}{3} + \cdots) + (-t^3 + \frac{t^5}{2} + \cdots) + (t^4 - \frac{t^6}{2} + \cdots) + (-t^5 + \cdots) + (t^6 + \cdots) + \cdots = t^2 - t^3 + \frac{t^4}{2} - \frac{t^5}{2} + \frac{5t^6}{6} + \cdots$, for $|t| < 1$.

Therefore, $f(x) = \int_0^x \frac{\ln(1+t^2)}{1+t}\, dt = \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{10} - \frac{x^6}{12} + \frac{5x^7}{42} + \cdots$, for $|x| < 1$.

Thus $f^{(7)}(0) = \frac{7! \times 5}{42} = 600$.

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3. (a) Find the directional derivative of the function $f(x,y,z) = xy^2 e^{\frac{2}{3}z}$ at the point $(1,2,3)$ in the direction of the vector which joins $(1,2,3)$ to $(3,2,1)$. Give your answer correct to two decimal places.

   (b) Find the exact value of the double integral $\iint_R (1+x)\, dA$, where R is the finite region in the third quadrant bounded above by the curve $x^2 + y = 0$ and bounded below by the curve $x + y^2 = 0$. Give your answer in the form of a fraction $\frac{m}{n}$ where m and n are positive integers without any common factors.

   Answer. (a) 6.97, (b) $\frac{11}{60}$.

   Solution. (a) First we have $\nabla f = \langle f_x, f_y, f_z \rangle = \langle y^2 e^{\frac{2}{3}z}, 2xye^{\frac{2}{3}z}, \frac{2}{3}xy^2 e^{\frac{2}{3}z} \rangle$. Thus $\nabla f(1,2,3) = \langle 4e^2, 4e^2, \frac{8}{3}e^2 \rangle$. The vector which joins $(1,2,3)$ to $(3,2,1)$ is $\langle 3-1, 2-2, 1-3 \rangle = \langle 2, 0, -2 \rangle$, whose length is $\sqrt{8}$. Thus the unit vector in the direction of the vector which joins $(1,2,3)$ to $(3,2,1)$ is $\frac{1}{\sqrt{8}}\langle 2,0,-2 \rangle = \langle \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \rangle$.

   Therefore, the required directional derivative is $\langle 4e^2, 4e^2, \frac{8}{3}e^2 \rangle \cdot \langle \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \rangle = \frac{4e^2}{3\sqrt{2}} = 6.97$.

   (b) The two curves intersect at $(0,0)$ and $(-1,-1)$.

   Thus $\iint_R (1+x)\, dA = \int_{-1}^{0} \int_{-\sqrt{-x}}^{-x^2} (1+x)\, dy dx = \int_{-1}^{0} [(1+x)y]_{-\sqrt{-x}}^{-x^2}\, dx$
   $= \int_{-1}^{0} (1+x)(-x^2 + \sqrt{-x})\, dx = \int_{-1}^{0} -x^2 + \sqrt{-x} - x^3 + x\sqrt{-x}\, dx$
   $= \int_{-1}^{0} -x^2 + (-x)^{\frac{1}{2}} - x^3 - (-x)^{\frac{3}{2}}\, dx = \left[-\frac{1}{3}x^3 - \frac{2}{3}(-x)^{\frac{3}{2}} - \frac{1}{4}x^4 + \frac{2}{5}(-x)^{\frac{5}{2}}\right]_{-1}^{0}$
   $= -(\frac{1}{3} - \frac{2}{3} - \frac{1}{4} + \frac{2}{5}) = \frac{11}{60}$.

   Alternatively, $\iint_R (1+x)\, dA = \int_{-1}^{0} \int_{-\sqrt{-y}}^{-y^2} (1+x)\, dx dy = \int_{-1}^{0} \left[x + \frac{x^2}{2}\right]_{-\sqrt{-y}}^{-y^2} dy = \int_{-1}^{0} -y^2 + \frac{y^4}{2} + \sqrt{-y} + \frac{y}{2}\, dy = \left[-\frac{y^3}{3} + \frac{y^5}{10} - \frac{2}{3}(-y)^{\frac{3}{2}} + \frac{y^2}{4}\right]_{-1}^{0} = -(\frac{1}{3} - \frac{1}{10} - \frac{2}{3} + \frac{1}{4}) = \frac{11}{60}$. ■

4. (a) Let k denote a positive constant. Let R denote the plane circular disc region centred at the origin with radius k on the xy-plane. Let D denote the solid region under the surface of the function $z = e^{-\frac{x^2+y^2}{k^2}}$ and over the region R. If the volume of D equals 101, find the value of k. Give your answer correct to two decimal places.

(b) You started an experiment with 100 mg of a radioactive substance X which has a half life of 30 minutes. After 0.82 hour, you had m mg of X left. Find the value of m. Give your answer correct to the nearest integer.

Answer. (a) 7.13, (b) 30.

Solution. (a) Using polar coordinates, the volume is $\iint_R e^{-\frac{x^2+y^2}{k^2}}\, dA = \int_0^{2\pi}\int_0^k e^{-\frac{r^2}{k^2}}\, r dr d\theta$ $= 2\pi\left[-\frac{k^2}{2}e^{-\frac{r^2}{k^2}}\right]_0^k = \pi k^2(1-e^{-1})$. Thus $\pi k^2(1-e^{-1}) = 101 \Leftrightarrow k = \sqrt{\frac{101}{\pi(1-e^{-1})}} = 7.13$.

(b) Let y in mg be the amount of the substance X at time t in minutes. We have $y = 100e^{-\frac{\ln 2}{T}t}$, where T is the half-life. That is $y = 100e^{-\frac{\ln 2}{30}t}$. Therefore, $m = 100e^{-\frac{\ln 2}{30}0.82\times 60} = 100e^{-1.64\ln 2} = 100(2)^{-1.64} = 32.08 \approx 32$ to the nearest integer.

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5. (a) Let r denote a positive constant. At time $t = 0$ a tank contains 100 grams of salt dissolved in 100 litres of water. Assume that water containing 3 grams of salt per litre is entering the tank at a rate of r litre per minute and that the well stirred solution is draining from the tank at the same rate. It is known that at time $t = 45$ minutes, there are 200 grams of salt in the tank. Find the value of r. Give your answer correct to two decimal places.

(b) Let $y(x)$ be the solution of the differential equation

$$x\frac{dy}{dx} + 2y = \frac{\cos x}{x}, \text{ with } x > 0 \text{ and } y(\pi) = 1.$$

Find the value of $y(\frac{\pi}{6})$. Give your answer correct to two decimal places.

Answer. (a) 1.54, (b) 37.82.

Solution. (a) First note that the volume of the solution remains constant which is 100 litres. Let Q be the amount of salt in grams at time t in minutes. The concentration of salt in the solution is $Q/100$ gram per litre. Suppose at time $t + dt$, the amount of salt is $Q + dQ$. Then

$$dQ = \text{salt input} - \text{salt output} = 3 \times r \times dt - r \times \frac{Q}{100} \times dt.$$

Thus

$$\frac{dQ}{dt} = 3r - \frac{rQ}{100}.$$

In the standard form, we have $\frac{dQ}{dt} + \frac{rQ}{100} = 3r$. Multiplying throughout by the integrating factor $e^{\frac{rt}{100}}$, we obtain $\frac{d}{dt}(e^{\frac{rt}{100}}Q) = 3re^{\frac{rt}{100}}$. Integrating,

$e^{\frac{rt}{100}}Q = 300e^{\frac{rt}{100}} + C$. Thus $Q = 300 + Ce^{-\frac{rt}{100}}$. At $t = 0, Q = 100$. Thus $100 = 300 + C \Leftrightarrow C = -200$. Therefore, $Q = 300 - 200e^{-\frac{rt}{100}}$. At $t = 45, Q = 200$. Thus $200 = 300 - 200e^{-\frac{45r}{100}} \Leftrightarrow r = \frac{100\ln 2}{45} = 1.54$. We may rewrite the function Q in the form: $Q = 300 - 200 \times 2^{-\frac{t}{45}}$.

(b) Rewrite the DE in the standard form: $\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2}$. An integrating factor is $e^{\int \frac{2}{x}\,dx} = x^2$. Thus multiplying the above equation by x^2, we have $(x^2y)' = x^2y' + 2xy = \cos x$. Integrating, $x^2y = \sin x + C$. Using $y(\pi) = 1$, we have $C = \pi^2$. Therefore, $y = \frac{\pi^2 + \sin x}{x^2}$. Thus $y(\frac{\pi}{6}) = \frac{\pi^2 + \sin\frac{\pi}{6}}{(\frac{\pi}{6})^2} = \frac{\pi^2 + \frac{1}{2}}{(\frac{\pi}{6})^2} = 36 + \frac{18}{\pi^2} = 37.82$.

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