NATIONAL UNIVERSITY OF SINGAPORE

MA2001 — LINEAR ALGEBRA

2021 – 2022   SEMESTER II

5 March 2022,   8:30 am – 9:30 am

INSTRUCTIONS TO CANDIDATES

1. Use A4 size paper and pen (blue or black ink) to write your answers. You **CANNOT** use electronic devices (e.g., iPad) to write answers.
2. Write down your student number clearly on the top left of every page of the answers.
3. Write on one side of the paper only. Start a new question in a new page. Write the question number and page number on the top right corner of each page (e.g., Q1P1, Q1P2, . . . , Q4P1, Q4P2).
4. This test paper contains **FOUR (4)** questions. Answer **ALL** questions.
5. The total mark for this paper is **FIFTY (50)**.
6. This is an **OPEN BOOK** test. You may use any hardcopy or softcopy in your computer or in LumiNUS. But you cannot use any software for computation and cannot use Internet except LumiNUS.
7. You may use non-graphing calculators. However, you should lay out systematically the various steps in the calculations.
8. Join the Zoom conference and turn on the video setting at all time during the test. Adjust your camera such that your face, upper body (including your hands), as well as your computer screen are captured on Zoom.
9. You may go for a short toilet break (not more than 5 minutes) during the test.
10. At the end of the test,
    (i) Scan or take pictures of your work (make sure the images can be read clearly).
    (ii) Merge all your images into one PDF file in correct order.
    (iii) Name the PDF file by "Student Number" (e.g., A1234567X.pdf).
    (iv) Upload your PDF into the LumiNUS Folder "Test – Submission".
    (v) Review your submission to ensure that it is successful.
    (vi) The Test Submission folder will close on 5 March 2022, 9:45 am. After the folder is closed, test answers that are not submitted will not be accepted.

Question 1 [15 marks]

Consider the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 2 & 0 & -2 & 4 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}.$$

(a) Find the determinant of $\boldsymbol{A}$.

(b) Find the inverse of $\boldsymbol{A}$.

Question 2 [10 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and define

$$V = \left\{ (a,b,c,d) \in \mathbb{R}^4 \mid \boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{A} \text{ where } \boldsymbol{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}.$$

Find a finite subset S of $\mathbb{R}^4$ such that $V = \mathrm{span}(S)$.

Question 3 [10 marks]

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, where

$$\begin{aligned} \boldsymbol{u}_1 &= (1,1,1,1), \quad \boldsymbol{u}_2 = (3,2,-1,2), \quad \boldsymbol{u}_3 = (0,4,1,2), \\ \boldsymbol{v}_1 &= (4,7,1,5), \quad \boldsymbol{v}_2 = (5,0,0,2), \quad \boldsymbol{v}_3 = (4,-1,-1,1). \end{aligned}$$

Determine whether each of the vectors in S belongs to $\mathrm{span}(T)$. If so, express it as a linear combination of vectors in T.

Question 4 [15 marks]

Determine whether each of the following statements is true or false. You need to **justify** your answers.

(a) Let $\boldsymbol{A}, \boldsymbol{B}$ be $m \times n$ and $n \times m$ matrices respectively. If $m > n$, then $\boldsymbol{AB}$ is singular.

(b) Let $\boldsymbol{A}, \boldsymbol{B}$ be $m \times n$ and $n \times m$ matrices respectively. If $m < n$, then $\boldsymbol{AB}$ is singular.

(c) Let $\boldsymbol{A}$ be a square matrix. If $\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{A} = \boldsymbol{0}$, then $\boldsymbol{A}$ is singular.

(d) Let $\boldsymbol{A}$ be a square matrix. If $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{I}$ and $\det(\boldsymbol{A}) < 0$, then $\boldsymbol{A} + \boldsymbol{I}$ is singular.

(e) Let $\boldsymbol{A}$ be a square matrix. If $\mathbf{adj}(\mathbf{adj}(\mathbf{adj}(\boldsymbol{A}))) = \boldsymbol{0}$, then $\boldsymbol{A}$ is singular.

End of Paper