Department of Mathematics

National University of Singapore

(2022/23) Semester I MA1521 Calculus for Computing Tutorial 6

(1) For each of the following series, determine whether it converges or diverges.

$$\text{(a)} \sum_{n=1}^{\infty} \cos\frac{1}{n}, \qquad \text{(b)} \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

$$\text{(c)} \sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right), \quad \text{(d)} \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{1+n^2}}.$$

Ans. (a) divergent, (b) convergent, (c) convergent, (d) convergent.

(2) Find the radius of convergence of the following series.

$$\text{(a)} \sum_{n=1}^{\infty} (-1)^n \frac{(x+2)^n}{n}, \quad \text{(b)} \sum_{n=0}^{\infty} \frac{3^n x^n}{n!},$$

$$\text{(c)} \sum_{n=1}^{\infty} n^n x^n, \qquad \text{(d)} \sum_{n=1}^{\infty} \frac{(4x-5)^{2n+1}}{n^{3/2}}.$$

Ans. (a) 1, (b) ∞, (c) 0, (d) 1/4.

(3) If $\sum_{n=1}^{\infty} a_n x^n$ has a radius of convergence $R > 0$ and if $|b_n| \le |a_n|$ for all integer n, show that the radius of convergence of $\sum_{n=1}^{\infty} b_n x^n$ is larger than or equal to R.

(4) Suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$. Find the radius of convergence for each of the following series.

$$\text{(a)} \sum_{n=1}^{\infty} a_n 2^n x^n \qquad \text{(b)} \sum_{n=1}^{\infty} a_n (-1)^n x^{2n}.$$

Ans. (a) $\frac{1}{2}$, (b) 1.

(5) Find the Taylor series for the following functions:

(a) $\dfrac{x}{1-x}$ at $x = 0$,

(b) $\dfrac{1}{x^2}$ at $x = 1$,

(c) $\dfrac{x}{1+x}$ at $x = -2$.

Ans. (a) $\displaystyle\sum_{n=0}^{\infty} x^{n+1}$, (b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n$, (c) $2+\displaystyle\sum_{n=1}^{\infty}(x+2)^n$.

(6) Let

$$S = \sum_{n=0}^{\infty} \frac{1}{n!\,(n+2)}.$$

In this question, we will introduce two different ways to find the value of S, one by integration and the other by differentiation.

(i) Integrate the Taylor series of xe^x to show that $S = 1$.

(ii) Differentiate the Taylor series of $\frac{e^x-1}{x}$ to show that $S = 1$.

Further Exercises (Not to be discussed during tutorial)

1. Find the sum of the geometric series inside the interval of convergence

$$1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 - \cdots + (-\frac{x-3}{2})^n + \cdots .$$

Ans. $\dfrac{2}{x-1}$.

2. Let n be a positive integer. Prove that

$$\frac{1}{2}\int_0^1 t^{n-1}\,(1-t)^2\,dt = \frac{1}{n\,(n+1)\,(n+2)}.$$

Hence find the exact value of the infinite series

$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \frac{1}{5\cdot 6\cdot 7} + \frac{1}{7\cdot 8\cdot 9} + \cdots$$

Ans. $\ln 2 - \frac{1}{2}$.

3. In 1914, the Indian mathematician Srinivasa Ramanujan discovered the formula

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!(1103+26390n)}{(n!)^4(396)^{4n}}.$$

Approximate the series with only the $n = 0$ term and show that one can get 6 digits of π correct. Approximate the series using the $n = 0$ and $n = 1$ terms and show that one can get 14 digits of π correct. In general, each term of this remarkable series increases the accuracy by 8 digits. [$\pi = 3.14159265358979 \cdots$.]

Show that Ramanujan's series converges.

Let $z \in (-R, R)$. Since the radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$ is R, the series $\sum_{n=1}^{\infty} a_n z^n$ converges.

Consider the series $\sum_{n=1}^{\infty} b_n z^n$. We have $|b_n z^n| = |b_n||z^n| \le |a_n||z^n| = |a_n z^n|$ for all n. By comparison test, $\sum_{n=1}^{\infty} b_n z^n$ converges. This shows that radius of convergence of $\sum_{n=1}^{\infty} b_n x^n$ is at least R.

(a) $\lim_{n\to\infty} \left| \frac{a_{n+1}2^{n+1}}{a_n 2^n} \right| = 2\times \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 2$. Thus the radius of convergence of $\sum_{n=1}^{\infty} a_n 2^n x^n$ is $\frac{1}{2}$.

(b) First rewrite the power series as $\sum_{n=1}^{\infty} a_n(-1)^n x^{2n} = \sum_{n=1}^{\infty} a_n(-1)^n (x^2)^n$.

Then $\lim_{n\to\infty} \left| \frac{a_{n+1}(-1)^{n+1}(x^2)^{n+1}}{a_n(-1)^n(x^2)^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x|^2 = |x|^2$. It follows by ratio test that the given series converges for $|x|^2 < 1$ and diverges for $|x|^2 > 1$. Or equivalently, the series converges for $|x| < 1$ and diverges for $|x| > 1$. Therefore, the radius of convergence of the given power series is 1.