NATIONAL UNIVERSITY OF SINGAPORE

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MA2001 Linear Algebra I **Tutorial 10**

EXERCISE 6.9

For each of the following matrix $\boldsymbol{A}$,

(i) determine whether $\boldsymbol{A}$ is diagonalizable; and

(ii) if A is diagonalizable, find a matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$ and determine $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$.

Note that it requires eigenvalues and eigenvectors to determine the diagonalizability. We first review the definitions.

Let $\boldsymbol{A}$ be a square matrix of order n. If for some $\lambda \in \mathbb{R}$ and nonzero $\boldsymbol{v} \in \mathbb{R}^n$ such that $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$, then λ is called an *eigenvalue* of $\boldsymbol{A}$, and $\boldsymbol{v}$ an *eigenvector* associated to λ.

The *characteristic polynomial* $\det(\lambda\boldsymbol{I} - \boldsymbol{A})$ is a monic polynomial in variable λ of degree n. Its zeros (i.e., the roots to the *characteristic equation* $\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0$) are precisely all the eigenvalues of $\boldsymbol{A}$. That is,

$$\begin{aligned} \lambda \text{ is an eigenvalue of } \boldsymbol{A} &\Leftrightarrow \lambda\boldsymbol{I} - \boldsymbol{A} \text{ is a singular matrix} \\ &\Leftrightarrow \det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0. \end{aligned}$$

In particular,

(i) If $\boldsymbol{A}$ is a triangular matrix, then its eigenvalues are the diagonal entries.

(ii) If $\boldsymbol{A}$ is a square matrix of order 2, then the characteristic polynomial of $\boldsymbol{A}$ is

$$p_{\boldsymbol{A}}(\lambda) = \lambda^2 - \operatorname{tr}(\boldsymbol{A}) + \det(\boldsymbol{A}),$$

where $\operatorname{tr}(\boldsymbol{A})$ is the *trace* of $\boldsymbol{A}$, defined as the sum of diagonal entries of $\boldsymbol{A}$.

Let λ be an eigenvalue of $\boldsymbol{A}$. Then the eigenvectors of $\boldsymbol{A}$ associated to λ are precisely all the *nonzero* vectors in the nullspace of $\lambda\boldsymbol{I} - \boldsymbol{A}$. The nullspace of $\lambda\boldsymbol{I} - \boldsymbol{A}$ is the *eigenspace* of $\boldsymbol{A}$ associated to λ, denoted by E_λ or $E_{\boldsymbol{A},\lambda}$. Since $\lambda\boldsymbol{I} - \boldsymbol{A}$ is singular, we have $\dim E_\lambda \geq 1$.

A square matrix $\boldsymbol{A}$ is said to be *diagonalizable* if there exists an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$.

It is also prove that

(i) The eigenvectors associated to distinct eigenvalues are linearly independent.

(ii) A square matrix $\boldsymbol{A}$ of order n is diagonalizable $\Leftrightarrow$ $\boldsymbol{A}$ has n linearly independent eigenvectors.

(iii) If a square matrix $\boldsymbol{A}$ of order n has n distinct eigenvalues, then it is diagonalizable.

We have the **criterion for diagonalization**: Let $\boldsymbol{A}$ be a square matrix of order n.

(a) If $\boldsymbol{A}$ has a non-real eigenvalue, or equivalently, if $\det(\lambda\boldsymbol{I}-\boldsymbol{A})$ cannot be completely factorized as product of linear factors over $\mathbb{R}$, then $\boldsymbol{A}$ is not diagonalizable over $\mathbb{R}$.

(b) $\boldsymbol{A}$ has a real eigenvalue λ with *algebraic multiplicity* $a(\lambda)$ (it is the multiplicity of λ as a root of the characteristic equation) such that $\dim(E_\lambda) < a(\lambda)$, then $\boldsymbol{A}$ is not diagonalizable.

(c) Suppose that every eigenvalue λ of $\boldsymbol{A}$ is a real number and $\dim(E_\lambda) = a(\lambda)$.

  (i) For each eigenvalue λ of $\boldsymbol{A}$, solve the homogeneous linear system $(\lambda\boldsymbol{I}-\boldsymbol{A})\boldsymbol{x}=\boldsymbol{0}$ to get a basis for E_λ.

  (ii) Let the union of the bases for the eigenspaces be $\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n\}$. Let $\boldsymbol{P} = \begin{pmatrix}\boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n\end{pmatrix}$. Then $\boldsymbol{D} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ is a diagonal matrix such that its diagonal entries are the corresponding eigenvalues of the columns of $\boldsymbol{P}$.

After review, we are now ready to check the diagonalizability of the following matrices.

6.9(d). $\boldsymbol{A} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$.

Note that $\boldsymbol{A}$ is already a diagonal matrix. If we set $\boldsymbol{P}=\boldsymbol{I}$, then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}=\boldsymbol{A}$ is a diagonal matrix. Hence, we conclude that $\boldsymbol{A}$ is diagonalizable.

In general, *any diagonal matrix is diagonalizable* (by the identity matrix $\boldsymbol{I}$).

6.9(b). $\boldsymbol{A} = \begin{pmatrix}1 & 1\\ -1 & 3\end{pmatrix}$.

(i) The characteristic polynomial of $\boldsymbol{A}$ is

$$\det(\lambda\boldsymbol{I}-\boldsymbol{A}) = \lambda^2 - \operatorname{tr}(\boldsymbol{A})\lambda + \det(\boldsymbol{A}) = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2.$$

Then $\boldsymbol{A}$ has a unique eigenvalue $\lambda = 2$.

(ii) Find the eigenspace of $\boldsymbol{A}$ associated to $\lambda=2$:

$$2\boldsymbol{I}-\boldsymbol{A} = \begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix} \xrightarrow{R_2-R_1} \begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}.$$

Then

$$(2\boldsymbol{I}-\boldsymbol{A})\boldsymbol{x}=\boldsymbol{0} \Leftrightarrow \boldsymbol{x} = t(1,1),\ t\in\mathbb{R}.$$

So $\dim(E_2)=1$ but the algebraic multiplicity of 2 is 2. It follows that $\boldsymbol{A}$ is not diagonalizable.

Alternatively, assume that $\boldsymbol{A}$ is diagonalized by $\boldsymbol{P}$. Then we must have

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D},$$

where $\boldsymbol{D}$ is the diagonal matrix whose diagonal entries are the eigenvalues of $\boldsymbol{A}$, i.e.,

$$\boldsymbol{D} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\boldsymbol{I}.$$

Then

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1} = \boldsymbol{P}(2\boldsymbol{I})\boldsymbol{P}^{-1} = 2(\boldsymbol{P}\boldsymbol{P}^{-1}) = 2\boldsymbol{I} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

which is clearly a contradiction.

6.9(h). $\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 1 \\ -2 & 2 & 1 \\ -2 & -1 & 0 \end{pmatrix}$. In Tutorial 9, we have calculated that

(i) $\boldsymbol{A}$ has eigenvalues $\lambda = -1$ and $\lambda = 1$.
(ii) E_{-1} has a basis $\{(-1,-1,1)\}$, and E_1 has a basis $\{(\frac{1}{2},1,0),(\frac{1}{2},0,1)\}$.

Let $\boldsymbol{P}$ be the matrix whose columns are the vectors in the bases, e.g., $\boldsymbol{P} = \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Then

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the diagonal matrix whose diagonal entries are the corresponding eigenvalues of $\boldsymbol{A}$.

6.9(j). $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. In Tutorial 9, we have calculated that

(i) $\boldsymbol{A}$ has eigenvalues $\lambda = -1$ and $\lambda = 1$.
(ii) E_{-1} has a basis $\{(-1,0,1,0),(0,-1,0,1)\}$ and E_1 has a basis $\{(1,0,1,0),(0,1,0,1)\}$.

Let $\boldsymbol{P}$ be the matrix whose columns are the vectors in the bases, e.g., $\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

Then

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

the diagonal matrix whose diagonal entries are the corresponding eigenvalues of $\boldsymbol{A}$.

6.9(f). $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 9 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(i) The characteristic polynomial of $\boldsymbol{A}$ is

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda & -1 & 0 \\ -9 & \lambda & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2) \begin{vmatrix} \lambda & -1 \\ -9 & \lambda \end{vmatrix} = (\lambda - 2)(\lambda^2 - 9) = (\lambda - 2)(\lambda - 3)(\lambda + 3).$$

Then the eigenvalues of $\boldsymbol{A}$ are $\lambda = 2$, $\lambda = 3$ and $\lambda = -3$.

Since $\boldsymbol{A}$ has 3 distinct eigenvalues, it is diagonalizable.

(ii) Find the basis for each eigenspace.

Let $\lambda = 2$. Then

$$2\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ -9 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 + \frac{9}{2}R_1} \begin{pmatrix} 2 & -1 & 0 \\ 0 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$(2\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \mathbf{0} \Leftrightarrow \boldsymbol{x} = t(0,0,1), \ t \in \mathbb{R}.$$

Hence, E_2 has a basis $\{(0,0,1)\}$.

Let $\lambda = 3$. Then

$$3\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} 3 & -1 & 0 \\ -9 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 + 3R_1} \begin{pmatrix} 3 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 3 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$(3\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \mathbf{0} \Leftrightarrow \boldsymbol{x} = t(\tfrac{1}{3},1,0), \ t \in \mathbb{R}.$$

Then E_3 has a basis $\{(\frac{1}{3},1,0)$.

(iii) Let $\lambda = -3$. Then

$$-3\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} -3 & -1 & 0 \\ -9 & -3 & 0 \\ 0 & 0 & -5 \end{pmatrix} \xrightarrow{R_2 - 3R_1} \begin{pmatrix} -3 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} -3 & -1 & 0 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$(-3\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \mathbf{0} \Leftrightarrow \boldsymbol{x} = t(-\tfrac{1}{3},1,0), \ t \in \mathbb{R}.$$

Then E_{-3} has a basis $\{(-\frac{1}{3},1,0)\}$.

(iv) Let $\boldsymbol{P}$ be the matrix whose columns are vectors in the bases of the eigenspaces, e.g., $\boldsymbol{P} = \begin{pmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Then

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix},$$

the diagonal matrix whose diagonal entries are the corresponding eigenvalues of $\boldsymbol{A}$.

EXERCISE 6.16

A square matrix $\boldsymbol{A} = (a_{ij})_{n\times n}$ is called a *stochastic matrix* if $a_{ij} \geq 0$ for all $i, j = 1, \ldots, n$ and the sum of each column

$$\sum_{i=1}^{n} a_{ij} = a_{1j} + a_{2j} + \cdots + a_{nj} = 1, \qquad j = 1, \ldots, n.$$

(a) Show that 1 is an eigenvalue of $\boldsymbol{A}$, and if λ is an eigenvalue of $\boldsymbol{A}$, then $|\lambda| \leq 1$.

(b) Show that $\boldsymbol{B} = \begin{pmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.95 & 0.05 \\ 0 & 0.05 & 0.95 \end{pmatrix}$ is stochastic, and diagonalize it.

6.16(a). We write down all the conditions: $a_{ij} \geq 0$, and the sum of each column is 1. So

$$\begin{aligned} a_{11} + a_{21} + \cdots + a_{n1} &= 1 \\ a_{12} + a_{22} + \cdots + a_{n2} &= 1 \\ \vdots \qquad\quad \vdots \qquad & \\ a_{1n} + a_{2n} + \cdots + a_{nn} &= 1. \end{aligned}$$

We should be able to express these equations in matrix form

$$\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Note that the coefficient matrix is $\boldsymbol{A}^{\mathrm{T}}$. If we write $\boldsymbol{u} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$. Then the given information becomes $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u} = \boldsymbol{u}$, or equivalently,

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u} = 1\boldsymbol{u}.$$

This means that

(i) $\lambda = 1$ is an eigenvalue of $\boldsymbol{A}$, and

(ii) $\boldsymbol{u} \neq \mathbf{0}$ is an eigenvector of $\boldsymbol{A}$ associated to $\lambda = 1$.

However, we are expected to show that 1 is an eigenvalue of $\boldsymbol{A}$ instead of $\boldsymbol{A}^{\mathrm{T}}$. Can we find the relation between the eigenvalues of $\boldsymbol{A}$ and eigenvalues of $\boldsymbol{A}^{\mathrm{T}}$?

We have seen that the eigenvalues of a square matrix are precisely all zeros to its characteristic polynomial. Then can we find the relation between the characteristic polynomials of $\boldsymbol{A}$ and $\boldsymbol{A}^{\mathrm{T}}$?

The characteristic polynomial of $\boldsymbol{A}$ is the determinant of the matrix $\lambda\boldsymbol{I} - \boldsymbol{A}$, while the characteristic polynomial of $\boldsymbol{A}^{\mathrm{T}}$ is the determinant of the matrix $\lambda\boldsymbol{I} - \boldsymbol{A}^{\mathrm{T}}$.

We realize that $\lambda\boldsymbol{I} - \boldsymbol{A}$ and $\lambda\boldsymbol{I} - \boldsymbol{A}^{\mathrm{T}}$ are the transpose of each other:

$$(\lambda\boldsymbol{I} - \boldsymbol{A})^{\mathrm{T}} = \lambda\boldsymbol{I}^{\mathrm{T}} - \boldsymbol{A}^{\mathrm{T}} = \lambda\boldsymbol{I} - \boldsymbol{A}^{\mathrm{T}}.$$

In particular, they have the same determinant. Hence, for any square matrix $\boldsymbol{A}$,

(i) $\boldsymbol{A}$ and $\boldsymbol{A}^{\mathrm{T}}$ have the same characteristic polynomial;
(ii) $\boldsymbol{A}$ and $\boldsymbol{A}^{\mathrm{T}}$ have the same eigenvalues (including algebraic multiplicities).

Since we have shown that 1 is an eigenvalue of $\boldsymbol{A}^{\mathrm{T}}$, it is also an eigenvalue of $\boldsymbol{A}$.

Next, we shall prove that if λ is an eigenvalue of $\boldsymbol{A}$, then $|\lambda| \leq 1$.

Since $\boldsymbol{A}$ have $\boldsymbol{A}^{\mathrm{T}}$ have the same eigenvalues, it is equivalent to showing that if λ is an eigenvalue of $\boldsymbol{A}^{\mathrm{T}}$, then $|\lambda| \leq 1$.

In discussion of an eigenvalue, we must associate it with an eigenvector, and vice versa. Let λ be an eigenvalue of $\boldsymbol{A}^{\mathrm{T}}$ and $\boldsymbol{v} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \neq \mathbf{0}$ an associated eigenvector. Then

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{v} = \lambda\boldsymbol{v} \Leftrightarrow \begin{cases} a_{11}c_1 + a_{21}c_2 + \cdots + a_{n1}a_n = \lambda c_1 \\ a_{12}c_1 + a_{22}c_2 + \cdots + a_{n2}c_n = \lambda c_1 \\ \quad\vdots \qquad\qquad \vdots \\ a_{1n}c_1 + a_{2n}c_2 + \cdots + a_{nn}c_n = \lambda c_n \end{cases}$$

We analyze, in particular, the first equation:

$$a_{11}c_1 + a_{21}c_2 + \cdots + a_{n1}c_n = \lambda c_1.$$

In order to get an upper bound for $|\lambda|$, we first take absolute values:

$$|a_{11}c_1 + a_{21}c_2 + \cdots + a_{n1}c_n| = |\lambda||c_1|.$$

An upper bound is clearly obtained by *triangle inequality*:

$$|\lambda||c_1| \leq |a_{11}||c_1| + |a_{21}||c_2| + \cdots + |a_{n1}||c_n| = a_{11}|c_1| + a_{21}|c_2| + \cdots + a_{n1}|c_n|.$$

Of course we must use the condition

$$a_{11} + a_{21} + \cdots + a_{n1} = 1,$$

in which the coefficients for $a_{11}, a_{21}, \ldots, a_{21}$ are the same, while $|c_1|, |c_2|, \ldots, |c_n|$ may be distinct.

If $|c_1| \leq M, |c_2| \leq M, \ldots, |c_n| \leq M$, then we may continue to get

$$|\lambda|\,|c_i| \leq a_{11}M + a_{21}M + \cdots + a_{n1}M = (a_{11} + a_{21} + \cdots + a_{n1})M = M.$$

What is the *best* choice for M? In the sense of *best*, we hope to have the *least* M so that

$$|c_1| \leq M, |c_2| \leq M, \ldots, |c_n| \leq M.$$

So the *best* choice for M is

$$M = \max\{|c_1|, |c_2|, \ldots, |c_n|\},$$

the maximum value among $|c_1|, |c_2|, \ldots, |c_n|$.

By definition, there is an index $i = 1, \ldots, n$ such that $|c_i| = M$. Consider the i^{th} equation:

$$\lambda c_i = a_{1i}c_1 + a_{2i}c_2 + \cdots + a_{ni}c_n.$$

Then

$$\begin{aligned} |\lambda|M = |\lambda|\,|c_i| &= |a_{1i}c_1 + a_{2i}c_2 + \cdots + a_{ni}c_n| \\ &\leq a_{1i}|c_1| + a_{2i}|c_2| + \cdots + a_{ni}|c_n| \\ &\leq a_{1i}M + a_{2i}M + \cdots + a_{ni}M \\ &= (a_{1i} + a_{2i} + \cdots + a_{ni})M = M. \end{aligned}$$

We should be able to arrive the desired result $|\lambda| \leq 1$ by dividing both sides by M. We can always do so as long as $M > 0$.

By definition, M is the largest value among all $|c_1|, |c_2|, \ldots, |c_n|$. If $M = 0$, then

$$|c_1| = |c_2| = \cdots = |c_n| = 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0,$$

and this implies that $\boldsymbol{v} = \mathbf{0}$.

We shall emphasize again that *an eigenvector is must be a nonzero vector*. So this is impossible. Therefore, $M > 0$. Then dividing M both sides to $|\lambda|M \leq M$ yields $|\lambda| \leq 1$.

6.16(b). The verification that $\boldsymbol{B} = \begin{pmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.95 & 0.05 \\ 0 & 0.05 & 0.95 \end{pmatrix}$ is a stochastic matrix is straightforward: (i) every entry is ≥ 0, and (ii) the sum of each column is 1:

- The 1^{st} column: $0.95 + 0.05 + 0 = 1$.
- The 2^{nd} column: $0 + 0.95 + 0.05 = 1$.
- The 3^{rd} column: $0 + 0.05 + 0.95 = 1$.

We now diagonalize $\boldsymbol{B}$ using the same method as **Exercise 6.9**.

(i) Find the characteristic polynomial of $\boldsymbol{B}$:

$$\det(\lambda\boldsymbol{I}-\boldsymbol{B}) = \begin{vmatrix} \lambda-0.95 & 0 & 0 \\ -0.05 & \lambda-0.95 & -0.05 \\ 0 & -0.05 & \lambda-0.95 \end{vmatrix} = (\lambda-0.95)\begin{vmatrix} \lambda-0.95 & -0.05 \\ -0.05 & \lambda-0.95 \end{vmatrix}$$
$$= (\lambda-0.95)[(\lambda-0.95)(\lambda-0.95)-(-0.05)(-0.05)] = (\lambda-0.95)(\lambda-1)(\lambda-0.9).$$

Then $\boldsymbol{B}$ has eigenvalues $\lambda=1$, $\lambda=0.95$ and $\lambda=0.9$.

Since $\boldsymbol{B}$ has 3 distinct eigenvalues, it is diagonalizable.

(ii) Find the bases for eigenspaces.

Let $\lambda=1$.

$$1\boldsymbol{I}-\boldsymbol{B} = \begin{pmatrix} 0.05 & 0 & 0 \\ -0.05 & 0.05 & -0.05 \\ 0 & -0.05 & 0.05 \end{pmatrix} \xrightarrow{R_2+R_1} \begin{pmatrix} 0.05 & 0 & 0 \\ 0 & 0.05 & -0.05 \\ 0 & -0.05 & 0.05 \end{pmatrix} \xrightarrow{R_3+R_2} \begin{pmatrix} 0.05 & 0 & 0 \\ 0 & 0.05 & -0.05 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$(1\boldsymbol{I}-\boldsymbol{B})\boldsymbol{x}=\boldsymbol{0} \Leftrightarrow \boldsymbol{x}=t(0,1,1),\ t\in\mathbb{R}.$$

So E_1 has a basis $\{(0,1,1)\}$.

Let $\lambda=0.95$.

$$0.95\boldsymbol{I}-\boldsymbol{B} = \begin{pmatrix} 0 & 0 & 0 \\ -0.05 & 0 & -0.05 \\ 0 & -0.05 & 0 \end{pmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{pmatrix} -0.05 & 0 & -0.05 \\ 0 & 0 & 0 \\ 0 & -0.05 & 0 \end{pmatrix} \xrightarrow{R_2\leftrightarrow R_3} \begin{pmatrix} -0.05 & 0 & -0.05 \\ 0 & -0.05 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$(0.95\boldsymbol{I}-\boldsymbol{B})\boldsymbol{x}=\boldsymbol{0} \Leftrightarrow \boldsymbol{x}=t(-1,0,1),\ t\in\mathbb{R}.$$

So $E_{0.95}$ has a basis $\{(-1,0,1)\}$.

Let $\lambda=0.9$.

$$0.9\boldsymbol{I}-\boldsymbol{B} = \begin{pmatrix} -0.05 & 0 & 0 \\ -0.05 & -0.05 & -0.05 \\ 0 & -0.05 & -0.05 \end{pmatrix} \xrightarrow{R_2-R_1} \begin{pmatrix} -0.05 & 0 & 0 \\ 0 & -0.05 & -0.05 \\ 0 & -0.05 & -0.05 \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} -0.05 & 0 & 0 \\ 0 & -0.05 & -0.05 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$(0.9\boldsymbol{I}-\boldsymbol{B})\boldsymbol{x}=\boldsymbol{0} \Leftrightarrow \boldsymbol{x}=t(0,-1,1),\ t\in\mathbb{R}.$$

So $E_{0.9}$ has a basis $\{(0,-1,1)\}$.

(iii) Let $\boldsymbol{P}$ be the matrix whose columns are the vectors in the bases of eigenspaces, e.g., $\boldsymbol{P} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$. Then

$$\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.95 & 0 \\ 0 & 0 & 0.9 \end{pmatrix},$$

the diagonal matrix whose diagonal entries are the corresponding eigenvalues of $\boldsymbol{B}$.

EXERCISE 6.20

Find a general formula for a_n such that

(a) $a_n = 3a_{n-1} - 2a_{n-2}$ with $a_0 = 0$ and $a_1 = 1$.
(b) $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 1$ and $a_1 = 0$.

This is an application of diagonalization.

Suppose that a square matrix $\boldsymbol{A}$. Then we can find an invertible matrix $\boldsymbol{A}$ and a diagonal matrix $\boldsymbol{D}$ such that

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}.$$

Note that the powers of $\boldsymbol{D}$ can be evaluated easily:

$$\boldsymbol{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \Rightarrow \boldsymbol{D}^k = \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{pmatrix}.$$

We can thus easily find the powers of $\boldsymbol{A}$:

$$\boldsymbol{A}^k = \boldsymbol{P}\boldsymbol{D}^k\boldsymbol{P}^{-1}.$$

Suppose that the values of a_0 and a_1 are given, and

$$a_n = \alpha a_{n-1} + \beta a_{n-2}, \qquad n \geq 2.$$

Then

$$\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} a_n \\ \alpha a_n + \beta a_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}.$$

Let $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ \beta & \alpha \end{pmatrix}$. Then $\boldsymbol{A}\boldsymbol{x}_{n-1} = \boldsymbol{x}_n$.

Note that the characteristic polynomial of $\boldsymbol{A}$ is

$$\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda & -1 \\ -\beta & \lambda - \alpha \end{vmatrix} = \lambda(\lambda - \alpha) - (-1)(-\beta) = \lambda^2 - \alpha\lambda - \beta.$$

Suppose that $\boldsymbol{A}$ can be diagonalized as $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$. Then

$$\boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0 = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}\boldsymbol{x}_0.$$

6.20(a). $a_n = 3a_{n-1} - 2a_{n-2}$ with $a_0 = 0$ and $a_1 = 1$.

Let $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$ and $\boldsymbol{x}_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$. Then $\boldsymbol{A}\boldsymbol{x}_{n-1} = \boldsymbol{x}_n$ with $\boldsymbol{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(i) The characteristic polynomial of $\boldsymbol{A}$ is

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda - 3 \end{vmatrix} = \lambda^3 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2).$$

Then $\boldsymbol{A}$ has eigenvalues $\lambda = 1$ and $\lambda = 2$.

(ii) Find the bases for the eigenspaces.

Let $\lambda = 1$. Then

$$1\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Then $E_1 = \{(t,t) \mid t \in \mathbb{R}\} = \operatorname{span}\{(1,1)\}$.

Let $\lambda = 2$. Then

$$2\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}.$$

Then $E_2 = \{(\frac{1}{2}t, t) \mid t \in \mathbb{R}\} = \operatorname{span}\{(\frac{1}{2}, 1)\}$.

(iii) Let $\boldsymbol{P} = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. So

$$\boldsymbol{x}_n = \boldsymbol{A}^n\boldsymbol{x}_0 = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}\boldsymbol{x}_0 = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & 2^n \end{pmatrix} \frac{1}{1 \cdot 1 - 1 \cdot \frac{1}{2}} \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2^n - 1 \\ 2^{n+1} - 1 \end{pmatrix}.$$

In particular, the first entry gives the general formula of the sequence:

$$a_n = 2^n - 1.$$

6.20(b). $a_n = a_{n-1} + 2a_{n-1}$ with $a_0 = 1$ and $a_1 = 0$.

Let $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ and $\boldsymbol{x}_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$. Then $\boldsymbol{A}\boldsymbol{x}_{n-1} = \boldsymbol{x}_n$ and $\boldsymbol{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(i) Find the characteristic polynomial of $\boldsymbol{A}$:

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda & -1 \\ -2 & \lambda - 1 \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2).$$

Then $\boldsymbol{A}$ has eigenvalues $\lambda = -1$ and $\lambda = 2$.

(ii) Find the bases for the eigenspaces of $\boldsymbol{A}$.

Let $\lambda = -1$. Then

$$-1\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Then $E_{-1} = \{(-t, t) \mid t \in \mathbb{R}\} = \operatorname{span}\{(-1, 1)\}$.

Let $\lambda = 2$. Then

$$2\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix} \xrightarrow{R_2 + R_1} \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}.$$

Then $E_2 = \{(\frac{1}{2}t, t) \mid t \in \mathbb{R}\} = \operatorname{span}\{(\frac{1}{2}, 1)\}$.

(iii) Let $\boldsymbol{P} = \begin{pmatrix} -1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$. Then

$$\boldsymbol{x}_n = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}\boldsymbol{x}_0 = \begin{pmatrix} -1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & 2^n \end{pmatrix} \frac{1}{(-1)\cdot 1 - 1 \cdot \frac{1}{2}} \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}(-1)^n + \frac{1}{3}2^n \\ -\frac{2}{3}(-1)^n + \frac{2}{3}2^n \end{pmatrix}.$$

In particular, the first entry gives a general formula of the sequence

$$a_n = \tfrac{2}{3}(-1)^n + \tfrac{1}{3}2^n.$$

Exercise 6.24

For each of the following, find a matrix $\boldsymbol{P}$ that orthogonally diagonalizes $\boldsymbol{A}$ and determine $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P}$.

Suppose $\boldsymbol{A}$ is diagonalizable. Then there exist an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$ is a diagonal matrix.

If the order of $\boldsymbol{A}$ is large, then it is complicated to find $\boldsymbol{P}^{-1}$ (apply Gauss-Jordan elimination to $(\boldsymbol{P} \mid \boldsymbol{I}) \dashrightarrow (\boldsymbol{I} \mid \boldsymbol{P}^{-1})$). On the other hand, if $\boldsymbol{P}$ is an *orthogonal matrix*, then its inverse is the same as the transpose $\boldsymbol{P}^{-1} = \boldsymbol{P}^{\mathrm{T}}$, which can be easily obtained.

Let $\boldsymbol{A}$ be a square matrix. If there exists an orthogonal matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$ is a diagonal matrix, then $\boldsymbol{A}$ is said to be *orthogonally diagonalizable.*

Surprisingly, the orthogonally diagonalizable matrices are precisely all the symmetric matrices:

A square matrix $\boldsymbol{A}$ is orthogonally diagonalizable $\Leftrightarrow$ $\boldsymbol{A}$ is a symmetric matrix.

Moreover, suppose that $\boldsymbol{A}$ is a symmetric (orthogonally diagonalizable) matrix. Then

(i) All eigenvalues of $\boldsymbol{A}$ are real numbers.
(ii) Eigenvectors associated to different eigenvalues of $\boldsymbol{A}$ are orthogonal.

We have the following algorithm for orthogonal diagonalize a symmetric matrix $\boldsymbol{A}$:

(i) Solve the characteristic equation $\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0$ to find the eigenvalues of $\boldsymbol{A}$.

(ii) For each eigenspace E_λ, find an orthonormal basis S_λ.

(iii) Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ be the union of these orthonormal bases.

(iv) Then $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{pmatrix}$ is an orthogonal matrix, which diagonalizes $\boldsymbol{A}$. The diagonal entries of $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P}$ are the corresponding eigenvalues of $\boldsymbol{A}$.

6.24(e). $\boldsymbol{A} = \begin{pmatrix} 0 & -2 & 1 \\ -2 & 3 & -2 \\ 1 & -2 & 0 \end{pmatrix}$.

(i) Find the characteristic polynomial

$$\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda & 2 & -1 \\ 2 & \lambda-3 & 2 \\ -1 & 2 & \lambda \end{vmatrix} = [\lambda^2(\lambda-3) + (-4) + (-4)] - [4\lambda + 4\lambda + (\lambda-3)]$$
$$= \lambda^3 - 3\lambda^2 - 9\lambda - 5$$
$$= (\lambda+1)^2(\lambda-5).$$

Then $\boldsymbol{A}$ has eigenvalues $\lambda = -1$ and $\lambda = 5$.

(ii) Find the orthonormal bases for the eigenspaces of $\boldsymbol{A}$.

Let $\lambda = -1$. Then

$$-1\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 + 2R_1} \begin{pmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$E_{-1} = \{(2s - t, s, t) \mid s, t \in \mathbb{R}\} = \operatorname{span}\{(2,1,0), (-1,0,1)\}.$$

Apply Gram-Schmidt process to get an orthonormal basis:

$$\boldsymbol{v}_1 = (2,1,0),$$
$$\boldsymbol{v}_2 = (-1,0,1) - \frac{(-1,0,1)\cdot(2,1,0)}{(2,1,0)\cdot(2,1,0)}(2,1,0) = \left(-\tfrac{1}{5}, \tfrac{2}{5}, 1\right).$$

Normalize:

$$\boldsymbol{w}_1 = \boldsymbol{v}_1 / \|\boldsymbol{v}_1\| = \left(\tfrac{2}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}, 0\right),$$
$$\boldsymbol{w}_2 = \boldsymbol{v}_2 / \|\boldsymbol{v}_2\| = \left(-\tfrac{1}{\sqrt{30}}, \tfrac{2}{\sqrt{30}}, \tfrac{5}{\sqrt{30}}\right).$$

Let $\lambda = 5$. Then

$$5\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix} \xrightarrow[R_3 + \frac{1}{5}R_1]{R_2 - \frac{2}{5}R_1} \begin{pmatrix} 5 & 2 & -1 \\ 0 & \frac{6}{5} & \frac{12}{5} \\ 0 & \frac{12}{5} & \frac{24}{5} \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 5 & 2 & -1 \\ 0 & \frac{6}{5} & \frac{12}{5} \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$E_5 = \{(t, -2t, t) \mid t \in \mathbb{R}\} = \operatorname{span}\{(1,-2,1)\}.$$

Normalize

$$\boldsymbol{w}_3 = (1,-2,1)/\|(1,-2,1)\| = \left(\tfrac{1}{\sqrt{6}}, -\tfrac{2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}\right).$$

(iii) Let $\boldsymbol{P} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & -\frac{2}{\sqrt{6}} \\ 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{pmatrix}$. Then $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{w}_3 \end{pmatrix}$ is an orthogonal matrix such that

$$\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Note that the diagonal entries of $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P}$ are the corresponding eigenvalues of $\boldsymbol{A}$.

6.24(g). $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

In a similar way to **6.9(j)**, we can calculate that

(i) $\boldsymbol{A}$ has eigenvalues $\lambda = -1$ and $\lambda = 1$.

(ii) E_{-1} has a basis $\{(-1,1,0,0), (0,0,-1,1)\}$.

(iii) E_1 has a basis $\{(1,1,0,0), (0,0,1,1)\}$.

Note that the bases that we obtained for E_{-1} and E_1 are already orthogonal. We can simply normalize them to obtain orthonormal bases:

$$\begin{aligned} \boldsymbol{w}_1 &= (-1,1,0,0)/\|(-1,1,0,0)\| = \left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0, 0\right), \\ \boldsymbol{w}_2 &= (0,0,-1,1)/\|(0,0,-1,1)\| = \left(0, 0, -\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \\ \boldsymbol{w}_3 &= (1,1,0,0)/\|(1,1,0,0)\| = \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0, 0\right), \\ \boldsymbol{w}_4 &= (0,0,1,1)/\|(0,0,1,1)\| = \left(0, 0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right). \end{aligned}$$

Let $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{w}_3 & \boldsymbol{w}_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then $\boldsymbol{P}$ is an orthogonal matrix such that

$$\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Again, the diagonal entries of $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P}$ are the corresponding eigenvalues of $\boldsymbol{A}$.

EXERCISE 6.29

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ and $\boldsymbol{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

(a) Show that $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$.

(b) If $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ and $\boldsymbol{v} \neq 0$. Show that $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$.

(c) Suppose $\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & a_1 & a_2 & a_3 \\ \frac{1}{2} & b_1 & b_2 & b_3 \\ \frac{1}{2} & c_1 & c_2 & c_3 \\ \frac{1}{2} & d_1 & d_2 & d_3 \end{pmatrix}$ is an orthogonal matrix. Find $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P}$.

6.29(a). We emphasize again that *eigenvalue and eigenvector come together as a pair.*

In order to show that (a nonzero vector) $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$, we shall show that $\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{u}$ for some eigenvalue λ.

The verification is straightforward:

$$\boldsymbol{A}\boldsymbol{u} = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 4 \end{pmatrix} = 4\boldsymbol{u}.$$

Since $\boldsymbol{u} \neq \boldsymbol{0}$, we conclude that $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$ associated to the eigenvalue 4.

6.29(b). In order to show that $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$, we need to show that $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$ for a constant $\lambda \in \mathbb{R}$.

It is given that $\boldsymbol{u} \cdot \boldsymbol{v}$, which is convenient to convert to the matrix form $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = 0$. How can we link this condition to $\boldsymbol{A}\boldsymbol{v}$?

The matrix multiplication can be decomposed in blocks — the first matrix can be decomposed into rows and the second can be decomposed into columns. In particular,

$$\boldsymbol{A}\boldsymbol{v} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\boldsymbol{v} = \begin{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}\boldsymbol{v} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}\boldsymbol{v} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}\boldsymbol{v} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}\boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} \\ \boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} \\ \boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} \\ \boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \boldsymbol{0}.$$

Since $\boldsymbol{v} \neq \boldsymbol{0}$, the only way to express $\boldsymbol{0} = \lambda\boldsymbol{v}$ is setting $\lambda = 0$:

$$\boldsymbol{A}\boldsymbol{v} = \boldsymbol{0} = 0\boldsymbol{v}.$$

Therefore, $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$ associated to the eigenvalue 0.

Note that the eigenspace E_0 is the nullspace of $\boldsymbol{A}$. So $\dim(E_0) = \text{nullity}(\boldsymbol{A}) = 3$.

Before we proceed to **6.29(c)**, we summarize our findings:

(i) $\boldsymbol{A}$ is symmetric; so it is orthogonally diagonalizable. The vectors associated to distinct eigenvalues are orthogonal.

(ii) $\boldsymbol{A}$ has eigenvalues 0 and 4.

- E_4 contains an eigenvector $\boldsymbol{u}$; so $E_4 \supseteq \text{span}\{\boldsymbol{u}\}$ and $\dim(E_4) \geq 1$.
- E_0 is the nullspace of $\boldsymbol{A}$, and it is also the orthogonal complement of E_4: $(E_4)^\perp = E_0$. Moreover, $\dim(E_0) = 3$.

Of course we can compute directly to see all eigenvalues of $\boldsymbol{A}$. Can we conclude immediately from the above for its eigenvalues and eigenvectors?

Given a diagonalizable matrix of order n, the sum of the dimensions of the eigenspaces must be n. So for this matrix $\boldsymbol{A}$,

$$4 \geq \dim(E_0) + \dim(E_4) \geq 3 + 1 = 4.$$

There is no room for eigenvalues other than 0 and 4. Moreover, $\dim(E_4) = 1$, i.e., $E_4 = \text{span}\{\boldsymbol{u}\}$.

6.29(c). It is given that $\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & a_1 & a_2 & a_3 \\ \frac{1}{2} & b_1 & b_2 & b_3 \\ \frac{1}{2} & c_1 & c_2 & c_3 \\ \frac{1}{2} & d_1 & d_2 & d_3 \end{pmatrix}$ is an orthogonal matrix. Then its columns must be normalized. Write

$$\boldsymbol{w} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad \boldsymbol{w}_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix}, \quad \boldsymbol{w}_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}, \quad \boldsymbol{w}_1 = \begin{pmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{pmatrix}.$$

Then $\{\boldsymbol{w}, \boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ is an orthonormal basis for $\mathbb{R}^4$.

Note that $\boldsymbol{w}$ normalized from $\boldsymbol{u}$:

$$\boldsymbol{u}/\|\boldsymbol{u}\| = (1,1,1,1)/\|(1,1,1,1)\| = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) = \boldsymbol{w}.$$

Then $\{\boldsymbol{w}\}$ is an orthonormal basis for $\text{span}\{\boldsymbol{u}\} = E_4$.

Recall that E_0 is the set of all vectors orthogonal to $E_4 = \text{span}\{\boldsymbol{w}\}$. We now have an orthonormal set of vectors $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ that is orthogonal to $E_4 = \text{span}\{\boldsymbol{w}\}$. Then

$$\text{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\} \subseteq E_0.$$

Since $\dim(E_0) = 3$ and $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ is linearly independent (because it is an orthonormal set), we conclude that $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ is an orthonormal basis for E_0.

Let $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{w} & \boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{w}_3 \end{pmatrix}$. Then the columns of $\boldsymbol{P}$ are linearly independent eigenvectors of $\boldsymbol{P}$. So

$$\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the diagonal matrix whose diagonal entries are the corresponding eigenvalues of $\boldsymbol{A}$.