CS1231S: Discrete Structures

Tutorial #3: Sets

Answers

Note that the sets here are finite sets, unless otherwise stated.

1. Let $\mathcal{P}(A)$ denotes the power set of A. Find the following:

   a. $\mathcal{P}(\{a,b,c\})$; ***Answer:*** $\mathcal{P}(\{a,b,c\}) = \big\{\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{a,b\},\ \{a,c\},\ \{b,c\},\ \{a.b,c\}\big\}$.

   b. $\mathcal{P}\Big(\mathcal{P}\big(\mathcal{P}(\emptyset)\big)\Big)$. ***Answer:*** $\mathcal{P}\Big(\mathcal{P}\big(\mathcal{P}(\emptyset)\big)\Big) = \Big\{\emptyset,\ \{\emptyset\},\ \{\{\emptyset\}\},\ \{\emptyset,\{\emptyset\}\}\Big\}$.

2. Let $A = \{5,6,7,\ldots,12\}$. Find the following:

   a. $\{n \in A : n \text{ is even}\}$; ***Answer:*** $\{6,8,10,12\}$

   b. $\{n \in A : n = m^2 \text{ for some } m \in \mathbb{Z}\}$; ***Answer:*** $\{9\}$

   c. $\{-5,-4,-3,\ldots,5\} \setminus \{1,2,3,\ldots,10\}$; ***Answer:*** $\{-5,-4,-3,-2,-1,0\}$

   d. $\overline{\{5,7,9\} \cup \{9,11\}}$, where A is considered the universal set;

      Answer: $\overline{\{5,7,9\} \cup \{9,11\}} = \overline{5,7,9,11} = \{6,8,10,12\}$ where A is the universal set.

   e. $\{(x,y) \in \{1,3,5\} \times \{2,4\} : x + y \geq 6\}$. ***Answer:*** $\{(3,4),\ (5,2),\ (5,4)\}$

3. (Past year's exam question.)
   Denote by $|n|$ the absolute value of the integer n, i.e.,

$$|n| = \begin{cases} n, & \text{if } n \geq 0; \\ -n, & \text{if } n < 0. \end{cases}$$

   Given the set $S = \{-9,-6,1,3,5,8\}$, for each of the following statements, state whether it is true or false, with explanation.

   a. $\exists z \in S\ \forall x,y \in S\ z > |x-y|$.

   b. $\exists z \in S\ \forall x,y \in S\ z < |x-y|$.

   Answers:

   a. False.
      It suffices to show that its negation, $\forall z \in S\ \exists x,y \in S\ z \leq |x-y|$ is true.
      Take any $z \in S$. Let $x = 8$ and $y = -9$. Then $x,y \in S$ and $|x-y| = |8-(-9)| = 17$ which is larger than every element in S.

   b. True.
      Let $z = -6 \in S$ (or the other negative value in S, i.e. -9). Then $|x-y| \geq 0 > -1$ for all $x,y \in S$.

4. Let $A = \{2n + 1 : n \in \mathbb{Z}\}$ and $B = \{2n - 5 : n \in \mathbb{Z}\}$. Is $A = B$? Prove that your answer is correct. What does this tell us about how odd numbers may be defined?

Answer:

Yes, $A = B$. Proof as shown below. (Recall that: $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$.)

1. ($\subseteq$)
   1.1. Let $a \in A$.
   1.2. Use the definition of A to find an integer n such that $a = 2n + 1$.
   1.3. Then $a = 2n + 1 = 2(n + 3) - 5$.
   1.4. $n + 3 \in \mathbb{Z}$ (by closure of integers under $+$).
   1.5. Therefore, $a \in B$ (by the definition of B).
2. ($\supseteq$)
   2.1. Let $b \in B$.
   2.2. Use the definition of B to find an integer n such that $b = 2n - 5$.
   2.3. Then $b = 2n - 5 = 2(n - 3) + 1$.
   2.4. $n - 3 \in \mathbb{Z}$ (by closure of integers under $-$).
   2.5. Therefore, $b \in A$ (by the definition of A).
3. Therefore, $A = B$ (by the definition of set equality).

We use this definition of odd integers: An integer n is odd if and only if $n = 2k + 1$ for some integer k. The above tells us that we may define the set of odd numbers in many other ways.

5. Using definitions of set operations (also called the **element method**), prove that for all sets A, B, C,

$$A \cap (B \setminus C) = (A \cap B) \setminus C.$$

Answer:

1.	$A \cap (B \setminus C) = \{x : x \in A \wedge x \in (B \setminus C)\}$	by the definition of $\cap$
2.	$= \{x : x \in A \wedge (x \in B \wedge x \notin C)\}$	by the definition of $\setminus$
3.	$= \{x : (x \in A \wedge x \in B) \wedge x \notin C\}$	by the associativity of $\wedge$
4.	$= \{x : (x \in A \cap B) \wedge x \notin C\}$	by the definition of $\cap$
5.	$= (A \cap B) \setminus C$	by the definition of $\setminus$

6. (Past year's midterm test question.)
Using **set identities** (Theorem 6.2.2), prove that for all sets A, B and C,

$$A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C).$$

Answer:

$A \setminus (B \setminus C)$

1. $= A \setminus (B \cap \bar{C})$ by the Set Difference Law
2. $= A \cap \overline{(B \cap \bar{C})}$ by the Set Difference Law
3. $= A \cap (\bar{B} \cup \bar{\bar{C}})$ by De Morgan's Law
4. $= A \cap (\bar{B} \cup C)$ by the Double Complement Law
5. $= (A \cap \bar{B}) \cup (A \cap C)$ by the Distributive Law
6. $= (A \setminus B) \cup (A \cap C)$ by the Set Difference Law

7. For sets A and B, define $A \oplus B = (A \setminus B) \cup (B \setminus A)$.
   a. Let $A = \{1,4,9,16\}$ and $B = \{2,4,6,8,10,12,14,16\}$. Find $A \oplus B$.
   b. Using set identities (Theorem 6.2.2), prove that for all sets A and B,

$$A \oplus B = (A \cup B) \setminus (A \cap B).$$

Answers:

a. $A \setminus B = \{1,9\}$; $B \setminus A = \{2,6,8,10,12,14\}$. Therefore, $A \oplus B = \{1,2,6,8,9,10,12,14\}$.

b.

1.	$A \oplus B$	
2.	$= (A \setminus B) \cup (B \setminus A)$	by the definition of $\oplus$
3.	$= \big((A \cap \bar{B}) \cup (B \cap \bar{A})\big)$	by the Set Difference Law
4.	$= \big((A \cap \bar{B}) \cup B\big) \cap \big((A \cap \bar{B}) \cup \bar{A}\big)$	by the Distributive Law
5.	$= \big(B \cup (A \cap \bar{B})\big) \cap \big(\bar{A} \cup (A \cap \bar{B})\big)$	by the Commutative Law
6.	$= \big((B \cup A) \cap (B \cup \bar{B})\big) \cap \big((\bar{A} \cup A) \cap (\bar{A} \cup \bar{B})\big)$	by the Distributive Law
7.	$= \big((B \cup A) \cap U\big) \cap \big(U \cap (\bar{A} \cup \bar{B})\big)$	by the Complement Law
8.	$= \big((A \cup B) \cap U\big) \cap \big((\bar{A} \cup \bar{B}) \cap U\big)$	by the Commutative Law
9.	$= (A \cup B) \cap (\bar{A} \cup \bar{B})$	by the Identity Law
10.	$= (A \cup B) \cap \overline{(A \cap B)}$	by De Morgan's Law
11.	$= (A \cup B) \setminus (A \cap B)$	by the Set Difference Law

8. Let A and B be set. Show that $A \subseteq B$ if and only if $A \cup B = B$.

Answer:

1. (⇒)
    1.1. Suppose $A \subseteq B$.
    (To show $A \cup B = B$, we need to show $A \cup B \subseteq B$ and $B \subseteq A \cup B$.)
    1.2. (To show $A \cup B \subseteq B$)
        1.2.1. Let $z \in A \cup B$.
        1.2.2. Then $z \in A$ or $z \in B$ (by the definition of ∪).
        1.2.3. Case 1: Suppose $z \in A$, then $z \in B$ as $A \subseteq B$ from line 1.1.
        1.2.4. Case 2: Suppose $z \in B$, then $z \in B$.
        1.2.5. In either case, we have $z \in B$.
    1.3. (To show $A \cup B \supseteq B$)
        1.3.1. Let $z \in B$.
        1.3.2. Then $z \in A$ or $z \in B$ (by generalization).
        1.3.3. So $z \in A \cup B$ (by the definition of ∪).
    1.4. Therefore, $A \cup B = B$ (by the definition of set equality).

2. (⇐)
    2.1. Suppose $A \cup B = B$.
    2.2. Let $z \in A$.
        2.2.1. Then $z \in A$ or $z \in B$ (by generalization).
        2.2.2. So $z \in A \cup B$ (by the definition of ∪).
        2.2.3. So $z \in B$ since $A \cup B = B$ (from line 2.1).
    2.3. Therefore, $A \subseteq B$.

3. Therefore, $A \subseteq B$ if and only if $A \cup B = B$ (from 1 and 2).

9. Hogwarts School of Witchcraft and Wizardry where Harry Potter attended was divided into 4 houses: Gryffindor, Hufflepuff, Ravenclaw and Slytherin.

Let $HSWW$ be the set of students in the Hogwarts School of Witchcraft and Wizardry, and G, H, R and S be the sets of students in the 4 houses.

What are the necessary conditions for $\{G, H, R, S\}$ to be a partition of $HSWW$? Explain in English and the write logical statements.

Answers:

The necessary conditions are every student is in exactly one of the four houses, and every house has at least one student.

$G \cap H = G \cap R = G \cap S = H \cap R = H \cap S = R \cap S = \emptyset$.
(That is, the houses are mutually disjoint sets.)

$G \cup H \cup R \cup S = HSWW$. (That is, every Hogwarts student is in one of the houses.)

$G \neq \emptyset \wedge H \neq \emptyset \wedge R \neq \emptyset \wedge S \neq \emptyset$. (That is, every house has at least one student.)

For questions 10 to 12, for sets $A_m, A_{m+1}, \dots, A_n$, we define the following:

$$\bigcup_{i=m}^{n} A_i = A_m \cup A_{m+1} \cup \dots \cup A_n$$

and

$$\bigcap_{i=m}^{n} A_i = A_m \cap A_{m+1} \cap \dots \cap A_n$$

10. Let $A_i = \{x \in \mathbb{Z} : x \geq i\}$ for all integers i. Write down $\bigcup_{i=2}^{5} A_i$ and $\bigcap_{i=2}^{5} A_i$ in **roster notation**.

    Answers:

    $$\bigcup_{i=2}^{5} A_i = \{2,3,4,5,\dots\}$$

    $$\bigcap_{i=2}^{5} A_i = \{5,6,7,8,\dots\}$$

11. Let $V_i = \left\{x \in \mathbb{R} : -\frac{1}{i} \leq x \leq \frac{1}{i}\right\} = \left[-\frac{1}{i}, \frac{1}{i}\right]$ for all positive integers i.

    a. What is $\bigcup_{i=1}^{4} V_i$? **Answer:** $\bigcup_{i=1}^{4} V_i = [-1,1]$.

    b. What is $\bigcap_{i=1}^{4} V_i$? **Answer:** $\bigcap_{i=1}^{4} V_i = \left[-\frac{1}{4}, \frac{1}{4}\right]$.

    c. What is $\bigcup_{i=1}^{n} V_i$, where n is a positive integer? **Answer:** $\bigcup_{i=1}^{n} V_i = [-1,1]$.

    d. What is $\bigcap_{i=1}^{n} V_i$, where n is a positive integer? **Answer:** $\bigcap_{i=1}^{n} V_i = \left[-\frac{1}{n}, \frac{1}{n}\right]$.

    e. Are $V_1, V_2, V_3, \dots$ mutually disjoint?

       Answer: $\boldsymbol{V_1}, V_2, V_3, \dots$ are not mutually disjoint. They have the common element 0.

       $V_1 = [-1,1];\ V_2 = \left[-\frac{1}{2}, \frac{1}{2}\right];\ V_3 = \left[-\frac{1}{3}, \frac{1}{3}\right];\ V_4 = \left[-\frac{1}{4}, \frac{1}{4}\right]; \cdots; V_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$

12. Let $B_1, B_2, B_3, \ldots, B_k$ and $C_1, C_2, C_3, \ldots, C_l$ be sets such that

$$\bigcup_{i=1}^{k} B_i \subseteq \bigcap_{j=1}^{l} C_j$$

Show that $B_i \subseteq C_j$ for any $i \in \{1,2, \ldots, k\}$ and any $j \in \{1,2, \ldots, l\}$.

Answer:

1. Let $r \in \{1,2, \ldots, k\}$ and $s \in \{1,2, \ldots, l\}$.
2. Take any $z \in B_r$.
   2.1. Then $z \in B_1 \vee z \in B_2 \vee \ldots \vee z \in B_k$ as $r \in \{1,2, \ldots, k\}$.
   2.2. So, $z \in B_1 \cup B_2 \cup \ldots \cup B_k = \bigcup_{i=1}^{k} B_i$ (by the definition of $\cup$).
   2.3. Hence, $z \in \bigcap_{j=1}^{l} C_j = C_1 \cap C_2 \cap \ldots \cap C_l$ (as we are given $\bigcup_{i=1}^{k} B_i \subseteq \bigcap_{j=1}^{l} C_j$).
   2.4. Thus $z \in C_1 \wedge z \in C_2 \wedge \ldots \wedge z \in C_l$ (by the definition of $\cap$).
   2.5. In particular, $z \in C_s$ as $s \in \{1,2, \ldots, l\}$.
3. Therefore, $B_i \subseteq C_j$ for any $i \in \{1,2, \ldots, k\}$ and any $j \in \{1,2, \ldots, l\}$.

Note: I asked students to draw a Venn diagram for $\bigcup_{i=1}^{k} B_i \subseteq \bigcap_{j=1}^{l} C_j$ (try it yourself!) and the above proof is derived from the diagram. There are other ways to write the proof, for instance, using Theorem 6.2.1.