MA2001 LINEAR ALGEBRA

VECTOR SPACES

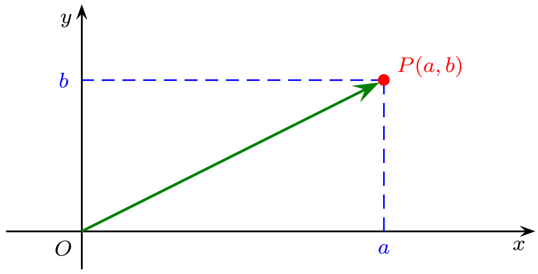
Goh Jun Le / Wang Fei

gohjunle@nus.edu.sg / matwf@nus.edu.sg

Department of Mathematics
Office: S17-06-25 / S17-06-16
Tel: 6601-1355 / 6516-2937

Vectors in xy-Plane

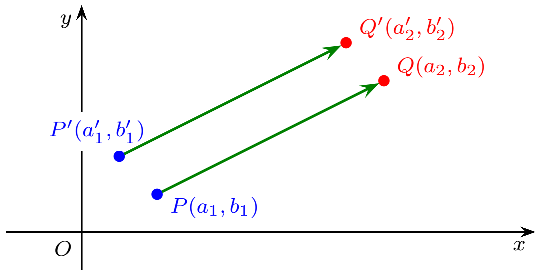
- Recall the xy-plane:



  - Every point P on the plane is represented by (a,b).
    - a is the x-coordinate and b is the y-coordinate.
- The arrow from the origin O to the point P is called a **vector**, denoted by $\overrightarrow{OP} = \boldsymbol{v} = (a,b)$.

Vectors in xy-Plane

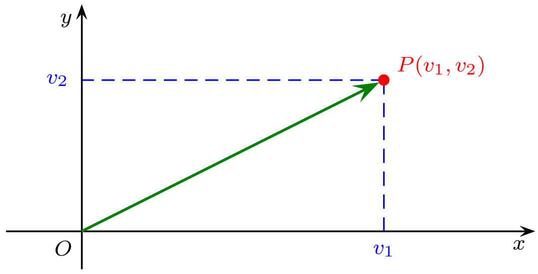
- A **vector** represents the change from the **initial point** to the **end point**.



If $\overrightarrow{PQ}$ is **parallel** shifted to $\overrightarrow{P'Q'}$, then
  - $\overrightarrow{PQ} = \overrightarrow{P'Q'}$,
    - that is, $(a_2 - a_1, b_2 - b_1) = (a_2' - a_1', b_2' - b_1')$,
    - that is, $a_2 - a_1 = a_2' - a_1'$ & $b_2 - b_1 = b_2' - b_1'$.

Length

- Let $\boldsymbol{v} = (v_1, v_2)$ be a vector in xy-plane.



  - Its **length** is $\boxed{\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2}}$.
  - If $\boldsymbol{v}$ is the vector from $P(a_1, b_1)$ to $Q(a_2, b_2)$.
    - $\boldsymbol{v} = (a_2 - a_1, b_2 - b_1)$.
    - $\|\boldsymbol{v}\| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$.

5 / 106

Scalar Multiplication

- **Scalar Multiplication.** Let $\boldsymbol{v} = (v_1, v_2)$ and $c \in \mathbb{R}$.
  - Then $\boxed{c\boldsymbol{v} = (cv_1, cv_2)}$

  Geometric interpretation:
  - $c\boldsymbol{v}$ is a vector parallel to $\boldsymbol{v}$ such that
    - its length is $|c|$ times the length of $\boldsymbol{v}$.

    1. If $c = 0$, then $c\boldsymbol{v} = 0\boldsymbol{v} = \mathbf{0}$ is the **zero vector**.
    2. If $c > 0$, then $c\boldsymbol{v}$ has the same direction as $\boldsymbol{v}$.
    3. If $c < 0$, then $c\boldsymbol{v}$ has the opposite direction of $\boldsymbol{v}$.

  In particular, $(-1)\boldsymbol{v}$ is the **negative** of $\boldsymbol{v}$, denoted by $-\boldsymbol{v}$.
- **Example.** Let $\boldsymbol{v} = (2, 1)$. Then
  - $0\boldsymbol{v} = (0, 0)$, $-\boldsymbol{v} = (-2, -1)$, $2\boldsymbol{v} = (4, 2)$.
  - $(-2)\boldsymbol{v} = (-4, -2) = -(2\boldsymbol{v}) = 2(-\boldsymbol{v})$.

6 / 106

Scalar Multiplication

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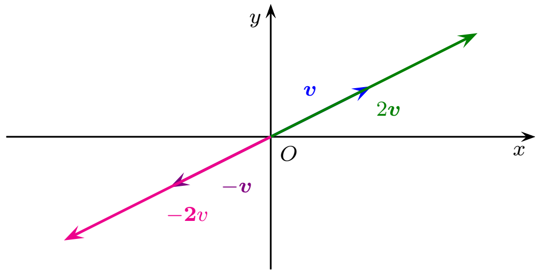
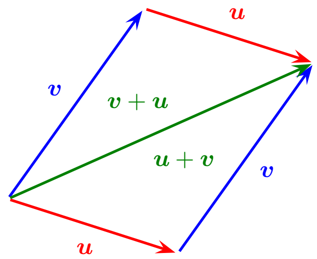


- **Properties & Exercises.**
  - $c(d\boldsymbol{v}) = (cd)\boldsymbol{v} = d(c\boldsymbol{v})$.
  - In particular, $-c\boldsymbol{v} = (-c)\boldsymbol{v} = c(-\boldsymbol{v})$, $-(-\boldsymbol{v}) = \boldsymbol{v}$.
  - $\|c\boldsymbol{v}\| = |c|\,\|\boldsymbol{v}\|$, $\boldsymbol{v} = \boldsymbol{0} \Leftrightarrow \|\boldsymbol{v}\| = 0$.

Addition and Subtraction

- **Addition.** Let $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$.
  - Then $\boxed{\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_2 + v_2)}$

  Geometric interpretation:
  - Parallel shift $\boldsymbol{v}$ so that its initial point is the same as the end of $\boldsymbol{u}$. Then $\boldsymbol{u} + \boldsymbol{v}$ is the vector from the initial point of $\boldsymbol{u}$ to the end point of $\boldsymbol{v}$.

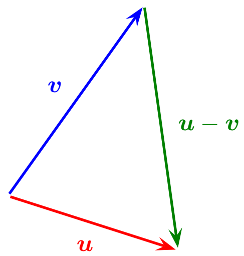


Addition and Subtraction

- **Subtraction.** Let $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$.
  - Then $\boxed{\boldsymbol{u} - \boldsymbol{v} = (u_1 - v_1, u_2 - v_2)}$

  Note that $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$.

  Geometric interpretation.
  - Parallel shift $\boldsymbol{v}$ so that $\boldsymbol{u}$ and $\boldsymbol{v}$ have the same initial point. Then $\boldsymbol{u} - \boldsymbol{v}$ is the vector from the end point of $\boldsymbol{v}$ to the end point of $\boldsymbol{u}$.

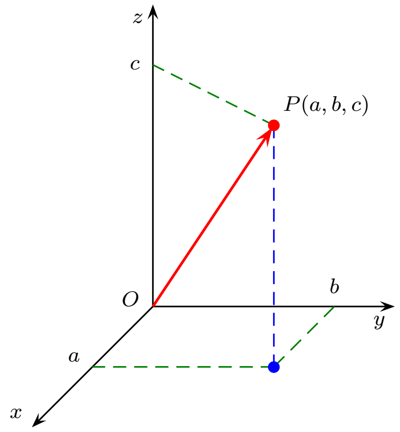


Addition and Subtraction

- **Example.** Let $\boldsymbol{u} = (2, 3)$ and $\boldsymbol{v} = (4, -5)$.
  - $\boldsymbol{u} + \boldsymbol{v} = (2 + 4, 3 + (-5)) = (6, -2)$.
  - $\boldsymbol{u} - \boldsymbol{v} = (2 - 4, 3 - (-5)) = (-2, 8)$.
- **Properties & Exercises.** Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in the xy-plane and $c, d \in \mathbb{R}$.
  - $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.
  - $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$.
  - $\boldsymbol{0} + \boldsymbol{v} = \boldsymbol{v}$, where $\boldsymbol{0}$ is the zero vector.
  - $\boldsymbol{v} + (-\boldsymbol{v}) = \boldsymbol{0}$.
  - $c(d\boldsymbol{v}) = (cd)\boldsymbol{v} = d(c\boldsymbol{v})$.
  - $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$.
  - $(c + d)\boldsymbol{v} = c\boldsymbol{v} + d\boldsymbol{v}$.
  - $1\boldsymbol{v} = \boldsymbol{v}$.

Vectors in xyz-Space

- Consider the xyz-space:



The **vector** $\boldsymbol{v} = \overrightarrow{OP}$ is the arrow from the origin O to P, denoted by $\boldsymbol{v} = (a, b, c)$.

Euclidean Spaces

- **Definition.** An **n-vector** or **ordered n-tuple** of real numbers is $\boldsymbol{v} = (v_1, v_2, \ldots, v_i, \ldots, v_n)$.
  - $v_i \in \mathbb{R}$ is the **ith component** or **ith coordinate** of $\boldsymbol{v}$.

  Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$.

  1. $\boldsymbol{u}$ and $\boldsymbol{v}$ are **equal** if $u_i = v_i$ for all $i = 1, \ldots, n$.
  2. The n-vector $\mathbf{0} = (0, 0, \ldots, 0)$ is the **zero vector**.
  3. Let $c \in \mathbb{R}$. The **scalar multiple** $c\boldsymbol{v}$ is
     - $c\boldsymbol{v} = (cv_1, cv_2, \ldots, cv_n)$.
  4. The **negative** of $\boldsymbol{v}$ is $(-1)\boldsymbol{v}$, denoted by $-\boldsymbol{v}$.
  5. The **addition** $\boldsymbol{u} + \boldsymbol{v}$ is
     - $\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$.
  6. The **subtraction** $\boldsymbol{u} - \boldsymbol{v}$ is
     - $\boldsymbol{u} - \boldsymbol{v} = (u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n)$.

Euclidean Spaces

- **Notation.** An n-vector $(v_1, v_2, \ldots, v_n)$ can be viewed as
  - a **row matrix** (**row vector**) $\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$,
  - a **column matrix** (**column vector**) $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$.
- **Properties.** Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be n-vectors and $c, d \in \mathbb{R}$.
  - $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.
  - $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$.
  - $\boldsymbol{v} + \boldsymbol{0} = \boldsymbol{v}$ and $\boldsymbol{v} + (-\boldsymbol{v}) = \boldsymbol{0}$.
  - $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$.
  - $(c + d)\boldsymbol{v} = c\boldsymbol{v} + d\boldsymbol{v}$.
  - $c(d\boldsymbol{v}) = (cd)\boldsymbol{v}$.
  - $1\boldsymbol{v} = \boldsymbol{v}$. (Verification is left as exercise.)

Euclidean Spaces

- The **Euclidean n-space** (or simply **n-space**) is the set of all n-vectors of real numbers.
  - $\mathbb{R}^n = \{(v_1, v_2, \ldots, v_n) \mid v_1, v_2, \ldots, v_n \in \mathbb{R}\}$.

  $\boldsymbol{v} \in \mathbb{R}^n$ if and only if $\boldsymbol{v}$ is of the form
  - $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ for real numbers $v_1, v_2, \ldots, v_n$.
- In particular,
  - If $n = 1$, then $\mathbb{R} = \mathbb{R}^1$ is the real line.
  - If $n = 2$, then $\mathbb{R}^2$ is the xy-plane.
  - If $n = 3$, then $\mathbb{R}^3$ is the xyz-space.
- Linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ in m equations and n variables.
  - $\boldsymbol{x}$ can be viewed as an n-vector, i.e., $\boldsymbol{x} \in \mathbb{R}^n$.

  Then the solution set of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is a subset of $\mathbb{R}^n$.

Implicit and Explicit Forms

- A linear system is given in the **implicit form**:
  - $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$
- **Example.** $\begin{cases} x + y + z = 0, \\ x - y + 2z = 1. \end{cases}$
  - An implicit form of the solution set is
    - $\{(x, y, z) \mid x + y + z = 0$ and $x - y + 2z = 1\}$.

    Geometrically, the solution set is the intersection of two non-parallel planes, which is a straight line in $\mathbb{R}^3$.

Implicit and Explicit Forms

- A linear system is given in the **implicit form**:
  - $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

  Its general solution is in the **explicit form**.
- **Example.** $\begin{cases} x + y + z = 0, \\ x - y + 2z = 1. \end{cases}$
  - $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 2 & 1 \end{array}\right) \xrightarrow{R_2+(-1)R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{array}\right)$.
    - $x = \frac{1}{2} - \frac{3}{2}t$, $y = -\frac{1}{2} + \frac{1}{2}t$, $z = t$, where $t \in \mathbb{R}$.
  - An explicit form of the solution set is
    - $\{(\frac{1}{2} - \frac{3}{2}t, -\frac{1}{2} + \frac{1}{2}t, t) \mid t \in \mathbb{R}\}$.

Lines in $\mathbb{R}^2$

- A **straight line** in $\mathbb{R}^2$ is of the form
  - $ax + by = c$, where a and b are not both zero.

  Implicit form: $\{(x, y) \mid ax + by = c\}$.

  Explicit form:
  - If $a \neq 0$, then $y = t$ and $x = \dfrac{c - bt}{a}$.
    - $\left\{\left(\dfrac{c - bt}{a}, t\right) \mid t \in \mathbb{R}\right\}$.
  - If $b \neq 0$, then $x = s$ and $y = \dfrac{c - as}{b}$.
    - $\left\{\left(s, \dfrac{c - as}{b}\right) \mid s \in \mathbb{R}\right\}$.

Lines in $\mathbb{R}^2$

- A **straight line** in $\mathbb{R}^2$ is determined by a point (x_0, y_0) on the line, and its direction vector $(a, b) \neq \mathbf{0}$.
  - A point on the line is of the form $(x_0, y_0) + t(a, b)$.

  Explicit form of the line:
  - $\{(x_0 + ta, y_0 + tb) \mid t \in \mathbb{R}\}$.

- **Example.** Suppose a line has an explicit form:
  - $\{(2 + 5t, 3 - 2t) \mid t \in \mathbb{R}\}$.

  Find an implicit form of the line.

  Solution. $x = 2 + 5t$ and $y = 3 - 2t$.
  - $t = \dfrac{x - 2}{5}$ and $t = \dfrac{3 - y}{2}$.
  - $\dfrac{x - 2}{5} = \dfrac{3 - y}{2} \Rightarrow \{(x, y) \mid 2x + 5y = 19\}$.

Planes in $\mathbb{R}^3$

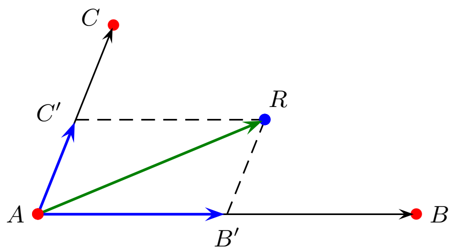
- A **plane** in $\mathbb{R}^3$ is of the form
  - $ax + by + cz = d$, where a, b, c are not all zero.

  Implicit form: $\{(x, y, z) \mid ax + by + cz = d\}$.

  Explicit form:
  - If $a \neq 0$, $\left\{ \left(\frac{d - bs - ct}{a}, s, t \right) \mid s, t \in \mathbb{R} \right\}$.
  - If $b \neq 0$, $\left\{ \left(s, \frac{d - as - ct}{b}, t \right) \mid s, t \in \mathbb{R} \right\}$.
  - If $c \neq 0$, $\left\{ \left(s, t, \frac{d - as - bt}{c} \right) \mid s, t \in \mathbb{R} \right\}$.

Planes in $\mathbb{R}^3$

- Three **non-collinear** points A, B, C determines a plane.



$$\begin{aligned} \boldsymbol{r} - \boldsymbol{a} = \overrightarrow{AR} &= \overrightarrow{AB'} + \overrightarrow{AC'} \\ &= s\overrightarrow{AB} + t\overrightarrow{AC} \\ &= s\boldsymbol{u} + t\boldsymbol{v}. \end{aligned}$$

  - $\boxed{\boldsymbol{r} = \boldsymbol{a} + s\boldsymbol{u} + t\boldsymbol{v}, \quad s, t \in \mathbb{R}}$

Planes in $\mathbb{R}^3$

- A **plane** in $\mathbb{R}^3$ can be explicitly represented as
  - $\{(x_0, y_0, z_0) + s(a_1, b_1, c_1) + t(a_2, b_2, c_2) \mid s, t \in \mathbb{R}\}$,

  (x_0, y_0, z_0) is a point on the plane, and (a_1, b_1, c_1) & (a_2, b_2, c_2) are non-parallel vectors parallel to the plane.
- **Example.** A plane is given by
  - $\{(1 + s - t, 2 + s - 2t, 4 - s - 3t) \mid s, t \in \mathbb{R}\}$.

  Let $x = 1 + s - t$, $y = 2 + s - 2t$, $z = 4 - s - 3t$.
  - $\left(\begin{array}{cc|c} 1 & -1 & x-1 \\ 1 & -2 & y-2 \\ -1 & -3 & z-4 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cc|c} 1 & -1 & x-1 \\ 0 & -1 & -x+y-1 \\ 0 & 0 & 5x-4y+z-1 \end{array}\right)$.

  The system is consistent. So $5x - 4y + z - 1 = 0$.
  - Implicit form:
    - $\{(x, y, z) \mid 5x - 4y + z = 1\}$.

Lines in $\mathbb{R}^3$

- A **straight line** in $\mathbb{R}^3$ is the intersection of two non-parallel planes. An implicit form is
  - $\{(x, y, z) \mid a_1x + b_1y + c_1z = d_1 \;\&\; a_2x + b_2y + c_2z = d_2\}$,

  a_i, b_i, c_i not all zero, and the planes are not parallel.
- **Example.** Suppose a line is the intersection of
  - $x + 2y + 3z = 4$ and $2x + 3y + 4z = 5$.

  Solve the system to have
  - $x = t - 2$, $y = -2t + 3$ and $z = t$.

  An explicit form of the line:
  - $\{(t - 2, -2t + 3, t) \mid t \in \mathbb{R}\}$.

  Note that $(t - 2, -2t + 3, t) = (-2, 3, 0) + t(1, -2, 1)$.

Lines in $\mathbb{R}^3$

- A **straight line** in $\mathbb{R}^3$ is determined by a point (x_0, y_0, z_0) on the line, and its direction vector $(a, b, c) \neq \mathbf{0}$.
  - A point on the line: $(x_0, y_0, z_0) + t(a, b, c)$.

  Explicit form: $\{(x_0 + ta, y_0 + tb, z_0 + tc) \mid t \in \mathbb{R}\}$.

  In order to have an implicit form, we need to find two non-parallel planes $ax + by + cz = d$ containing the line.
- **Example.** $\{(t - 2, -2t + 3, t + 1) \mid t \in \mathbb{R}\}$.
  - $x = t - 2$ and $y = -2t + 3$
    - $y = -2(2 + x) + 3 \Rightarrow 2x + y = -1$.
  - $x = t - 2$ and $z = t + 1$.
    - $x - z = -3$.

  An implicit form of the line is
  - $\{(x, y, z) \mid 2x + y = -1 \;\&\; x - z = -3\}$.

Linear Combinations and Linear Spans

Linear Combination

- Recall the operations on vectors in $\mathbb{R}^n$.
  - If $\boldsymbol{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ and $c \in \mathbb{R}$,
    - $c\boldsymbol{v} = (cv_1, \ldots, cv_n)$.
  - If $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} \in (v_1, \ldots, v_n) \in \mathbb{R}^n$,
    - $\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, \ldots, u_n + v_n)$.
- **Definition.** Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ be vectors in $\mathbb{R}^n$.
  - A **linear combination** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ has the form
    - $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$,

    where $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

  In particular, $\mathbf{0}$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$:
  - $\mathbf{0} = 0\boldsymbol{v}_1 + 0\boldsymbol{v}_2 + \cdots + 0\boldsymbol{v}_k$.

Examples

- Let $\boldsymbol{v}_1 = (2,1,3)$, $\boldsymbol{v}_2 = (1,-1,2)$ and $\boldsymbol{v}_3 = (3,0,5)$.
  - Is $\boldsymbol{v} = (3,3,4)$ a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$?

  Suppose that $\boldsymbol{v} = a\boldsymbol{v}_1 + b\boldsymbol{v}_2 + c\boldsymbol{v}_3$, i.e.,

$$\begin{aligned}(3,3,4) &= a(2,1,3) + b(1,-1,2) + c(3,0,5) \\ &= (2a + b + 3c, a - b, 3a + 2b + 5c).\end{aligned}$$

  Solve the linear system $\begin{cases} 2a + b + 3c = 3 \\ a - b = 3 \\ 3a + 2b + 5c = 4. \end{cases}$

  - $\left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -1 & 0 & 3 \\ 3 & 2 & 5 & 4 \end{array}\right) \xrightarrow[\text{elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$

  The system is consistent.

  - Therefore, $\boldsymbol{v}$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

26 / 106

Examples

- Let $\boldsymbol{v}_1 = (2,1,3)$, $\boldsymbol{v}_2 = (1,-1,2)$ and $\boldsymbol{v}_3 = (3,0,5)$.
  - Is $\boldsymbol{v} = (1,2,4)$ a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$?

  Suppose that $\boldsymbol{v} = a\boldsymbol{v}_1 + b\boldsymbol{v}_2 + c\boldsymbol{v}_3$, i.e.,

$$\begin{aligned}(1,2,4) &= a(2,1,3) + b(1,-1,2) + c(3,0,5) \\ &= (2a + b + 3c, a - b, 3a + 2b + 5c).\end{aligned}$$

  Solve the linear system $\begin{cases} 2a + b + 3c = 1 \\ a - b = 2 \\ 3a + 2b + 5c = 4. \end{cases}$

  - $\left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & -1 & 0 & 2 \\ 3 & 2 & 5 & 4 \end{array}\right) \xrightarrow[\text{elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 3 \end{array}\right)$

  The system is inconsistent.

  - Therefore, $\boldsymbol{v}$ is not a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

27 / 106

Linear Span

- **Definition.** Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a subset of $\mathbb{R}^n$.
  - The **set of all linear combinations** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$
    - $\{c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\}$

    is called the **linear span** of S (or $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$).
  - It is denoted by $\mathrm{span}(S)$ or $\mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$.

  $\boldsymbol{v}$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$

  $\Leftrightarrow \boldsymbol{v} \in \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$.
- **Example.** Let $S = \{(2,1,3), (1,-1,2), (3,0,5)\}$.
  - $(3,3,4) \in \mathrm{span}(S)$ but $(1,2,4) \notin \mathrm{span}(S)$.

  Example. Let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$.
  - $(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$.

  Therefore, $\mathrm{span}(S) = \mathbb{R}^3$.

Examples

- Let $S = \{(1,0,0,-1), (0,1,1,0)\}$ be a subset of $\mathbb{R}^4$.
  - Every vector in $\mathrm{span}(S)$ is of the form
    - $a(1,0,0,-1) + b(0,1,1,0) = (a,b,b,-a)$,

    where $a, b \in \mathbb{R}$.
  - $\mathrm{span}(S) = \{(a,b,b,-a) \mid a,b \in \mathbb{R}\}$.
- Let $V = \{(2a+b, a, 3b-a) \mid a,b \in \mathbb{R}\} \subseteq \mathbb{R}^3$.
  - Every vector in V is of the form
    - $(2a+b, a, 3b-a) = a(2,1,-1) + b(1,0,3)$,

    where $a, b \in \mathbb{R}$.
  - $V = \mathrm{span}\{(2,1,-1), (1,0,3)\}$.

Examples

- Prove that $\mathrm{span}\{(1,0,1),(1,1,0),(0,1,1)\} = \mathbb{R}^3$.
  - It is clear: $\mathrm{span}\{(1,0,1),(1,1,0),(0,1,1)\} \subseteq \mathbb{R}^3$.

  Is $\mathbb{R}^3 \subseteq \mathrm{span}\{(1,0,1),(1,1,0),(0,1,1)\}$?
- Let $(x,y,z) \in \mathbb{R}^3$. We shall find $a,b,c \in \mathbb{R}$ such that
  - $(x,y,z) = a(1,0,1) + b(1,1,0) + c(0,1,1)$.
  - Equivalently, $\begin{cases} a + b \phantom{{}+c} = x \\ \phantom{a+{}} b + c = y \\ a \phantom{{}+b} + c = z \end{cases}$
    - $\left(\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 1 & 0 & 1 & z \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 0 & 0 & 2 & z-x+y \end{array}\right)$

    The system is always consistent for any $(x,y,z) \in \mathbb{R}^3$.

  Therefore, $\mathrm{span}\{(1,0,1),(1,1,0),(0,1,1)\} = \mathbb{R}^3$.

30 / 106

Examples

- Prove that $\mathrm{span}\{(1,1,1),(1,2,0),(2,1,3),(2,3,1)\} \neq \mathbb{R}^3$.
  - Clear: $\mathrm{span}\{(1,1,1),(1,2,0),(2,1,3),(2,3,1)\} \subseteq \mathbb{R}^3$.

  Is $\mathbb{R}^3 \nsubseteq \mathrm{span}\{(1,1,1),(1,2,0),(2,1,3),(2,3,1)\}$?
- Let $(x,y,z) \in \mathbb{R}^3$. Can we find $a,b,c,d \in \mathbb{R}$ such that
  - $(x,y,z) = a(1,1,1) + b(1,2,0) + c(2,1,3) + d(2,3,1)$?
  - Equivalently, $\begin{cases} a + b + 2c + 2d = x \\ a + 2b + c + 3d = y \\ a \phantom{{}+2b} + 3c + d = z \end{cases}$
    - $\left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & x \\ 1 & 2 & 1 & 3 & y \\ 1 & 0 & 3 & 1 & z \end{array}\right) \xrightarrow{\text{G.E.}} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & x \\ 0 & 1 & -1 & 1 & y-x \\ 0 & 0 & 0 & 0 & y+z-2x \end{array}\right)$

    The system is consistent $\Leftrightarrow y+z-2x = 0$.

  Therefore, $\mathrm{span}\{(1,1,1),(1,2,0),(2,1,3),(2,3,1)\} \neq \mathbb{R}^3$.
  - $\mathrm{span}\{(1,1,1),(1,2,0),(2,1,3),(2,3,1)\}$

    $= \{(x,y,z) \mid y+z-2x = 0\}$.

31 / 106

Criterion for $\mathrm{span}(S) = \mathbb{R}^n$

- Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$. Is $\mathrm{span}(S) = \mathbb{R}^n$?
  - For arbitrary $\boldsymbol{v} \in \mathbb{R}^n$, we shall check the consistency of
    - $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{v}$.

    View each $\boldsymbol{v}_j$ as a column vector $\boldsymbol{v}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$.

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \boldsymbol{v} = c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + c_k \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix}$$
$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}.$$

Criterion for $\mathrm{span}(S) = \mathbb{R}^n$

- Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$. Is $\mathrm{span}(S) = \mathbb{R}^n$?
  - For arbitrary $\boldsymbol{v} \in \mathbb{R}^n$, we shall check the consistency of
    - $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{v}$.

    View $\boldsymbol{v}_j$ as column vectors. Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_k \end{pmatrix}$.
    - The system can be written as $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{v}$.

  Let $\boldsymbol{R}$ be a row-echelon form of $\boldsymbol{A}$.
  - $(\boldsymbol{A} \mid \boldsymbol{v}) \xrightarrow[\text{Elimination}]{\text{Gaussian}} (\boldsymbol{R} \mid \boldsymbol{v}')$.
    - Since $\boldsymbol{v} \in \mathbb{R}^n$ is arbitrary, $\boldsymbol{v}' \in \mathbb{R}^n$ is also arbitrary.

$$\begin{aligned} \mathrm{span}(S) = \mathbb{R}^n &\Leftrightarrow \boldsymbol{A}\boldsymbol{x} = \boldsymbol{v} \text{ consistent for every } \boldsymbol{v} \in \mathbb{R}^n \\ &\Leftrightarrow \text{last column of } (\boldsymbol{R} \mid \boldsymbol{v}') \text{ non-pivot for any } \boldsymbol{v}' \in \mathbb{R}^n \\ &\Leftrightarrow \boldsymbol{R} \text{ has no zero row.} \end{aligned}$$

Criterion for $\mathrm{span}(S) = \mathbb{R}^n$

- Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$.
  1. View each $\boldsymbol{v}_j$ as a column vector.
  2. Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_k \end{pmatrix}$.
  3. Find a row-echelon form $\boldsymbol{R}$ of $\boldsymbol{A}$.
     - If $\boldsymbol{R}$ has a zero row, then $\mathrm{span}(S) \neq \mathbb{R}^n$.
     - If $\boldsymbol{R}$ has no zero row, then $\mathrm{span}(S) = \mathbb{R}^n$.
- **Example.**
  - $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

  $\therefore\ \mathrm{span}\{(1,0,1),(1,1,0),(0,1,1)\} = \mathbb{R}^3$.

Criterion for $\mathrm{span}(S) = \mathbb{R}^n$

- Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$.
  1. View each $\boldsymbol{v}_j$ as a column vector.
  2. Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_k \end{pmatrix}$.
  3. Find a row-echelon form $\boldsymbol{R}$ of $\boldsymbol{A}$.
     - If $\boldsymbol{R}$ has a zero row, then $\mathrm{span}(S) \neq \mathbb{R}^n$.
     - If $\boldsymbol{R}$ has no zero row, then $\mathrm{span}(S) = \mathbb{R}^n$.
- **Example.**
  - $\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 0 & 3 & 1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

  $\therefore\ \mathrm{span}\{(1,1,1),(1,2,0),(2,1,3),(2,3,1)\} \neq \mathbb{R}^3$.

Criterion for $\operatorname{span}(S) = \mathbb{R}^n$

- **Theorem.** Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a subset of $\mathbb{R}^n$.
  - If $k < n$, then $\operatorname{span}(S) \neq \mathbb{R}^n$.

  Proof. View each $\boldsymbol{v}_j$ as a column vector. Then
  - $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_k \end{pmatrix}$ is an $n \times k$ matrix.

  Let $\boldsymbol{R}$ be a row-echelon form of $\boldsymbol{A}$. Then $\boldsymbol{R}$ is $n \times k$.
  1. $\boldsymbol{R}$ has at most k pivot columns.
  2. $\boldsymbol{R}$ has at most k nonzero rows.
  3. $\boldsymbol{R}$ has at least $n - k > 0$ zero rows.

  Therefore, $\operatorname{span}(S) \neq \mathbb{R}^n$.

- **Examples.**
  - One vector cannot span $\mathbb{R}^2$.
  - Two vectors cannot span $\mathbb{R}^3$.

Properties of Linear Spans

- Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \subseteq \mathbb{R}^n$.
  - Then $\mathbf{0} = 0\boldsymbol{u}_1 + 0\boldsymbol{u}_2 + \cdots + 0\boldsymbol{u}_k \in \operatorname{span}(S)$.

  Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \in \operatorname{span}(S)$.
  - Each $\boldsymbol{v}_i$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$.
    - $\boldsymbol{v}_1 = a_{11}\boldsymbol{u}_1 + a_{12}\boldsymbol{u}_2 + \cdots + a_{1k}\boldsymbol{u}_k$.
    - $\boldsymbol{v}_2 = a_{21}\boldsymbol{u}_1 + a_{22}\boldsymbol{u}_2 + \cdots + a_{2k}\boldsymbol{u}_k$.

      $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$
    - $\boldsymbol{v}_r = a_{r1}\boldsymbol{u}_1 + a_{r2}\boldsymbol{u}_2 + \cdots + a_{rk}\boldsymbol{u}_k$.

    Then for any $c_1, c_2, \ldots, c_r \in \mathbb{R}$,

$$\begin{aligned} c_1\boldsymbol{v}_1 + \cdots + c_r\boldsymbol{v}_r &= c_1(a_{11}\boldsymbol{u}_1 + \cdots + a_{1k}\boldsymbol{u}_k) \\ &\quad + \cdots + c_r(a_{r1}\boldsymbol{u}_1 + \cdots + a_{rk}\boldsymbol{u}_k) \\ &= (c_1a_{11} + \cdots + c_ra_{r1})\boldsymbol{u}_1 \\ &\quad + \cdots + (c_1a_{1k} + \cdots + c_ra_{rk})\boldsymbol{u}_k. \end{aligned}$$

Properties of Linear Spans

- **Theorem.** Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a subset of $\mathbb{R}^n$.
  - $\mathbf{0} \in \mathrm{span}(S)$, where $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$.
  - Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \in \mathrm{span}(S)$, $c_1, c_2, \ldots, c_r \in \mathbb{R}$.
    - $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_r\boldsymbol{v}_r \in \mathrm{span}(S)$.
- **Remarks.** In particular,
  - Since $\mathbf{0} \in \mathrm{span}(S)$, $\mathrm{span}(S) \neq \varnothing$.
  - $\boldsymbol{v} \in \mathrm{span}(S)$ and $c \in \mathbb{R} \Rightarrow c\boldsymbol{v} \in \mathrm{span}(S)$.
    - $\mathrm{span}(S)$ is **closed** under scalar multiplication.
  - $\boldsymbol{u} \in \mathrm{span}(S)$ and $\boldsymbol{v} \in \mathrm{span}(S) \Rightarrow \boldsymbol{u} + \boldsymbol{v} \in \mathrm{span}(S)$.
    - $\mathrm{span}(S)$ is **closed** under addition.

38 / 106

Properties of Linear Spans

- **Theorem.** Given two subsets of $\mathbb{R}^n$:
  - $S_1 = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$, $S_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$.

  Then $\mathrm{span}(S_1) \subseteq \mathrm{span}(S_2)$

  $\Leftrightarrow$ Every $\boldsymbol{u}_i$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$.

  Proof. $\Rightarrow$: Suppose that $\mathrm{span}(S_1) \subseteq \mathrm{span}(S_2)$.
  - $\boldsymbol{u}_i = 0\boldsymbol{u}_1 + \cdots + 1\boldsymbol{u}_i + \cdots + 0\boldsymbol{u}_k \in \mathrm{span}(S_1)$.
    - Then $\boldsymbol{u}_i \in \mathrm{span}(S_2)$ by assumption.

    That is, $\boldsymbol{u}_i$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$.

  $\Leftarrow$: Suppose each $\boldsymbol{u}_i$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$.
  - Let $\boldsymbol{w} \in \mathrm{span}(S_1)$. There exist $c_1, c_2, \ldots, c_k \in \mathbb{R}$ s.t.
    - $\boldsymbol{w} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k \in \mathrm{span}(S_2)$.

    Therefore, $\mathrm{span}(S_1) \subseteq \mathrm{span}(S_2)$.

39 / 106

Properties of Linear Spans

- **Theorem.** Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_k \in \mathbb{R}^n$.
  - If $\boldsymbol{v}_k$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}$, then
    - $\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}\} = \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_k\}$.

  Proof. It follows from the definition of linear span that
  - $\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}\} \subseteq \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_k\}$.

  Since $\boldsymbol{v}_k$ is a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}$,
  - $\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_k\} \subseteq \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}\}$.

  $\therefore \quad \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}\} = \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_k\}$.
- **Example.** Let $\boldsymbol{v}_1 = (1, 1, 0, 2)$, $\boldsymbol{v}_2 = (1, 0, 0, 1)$.
  - Let $\boldsymbol{v}_3 = \boldsymbol{v}_1 - \boldsymbol{v}_2 = (0, 1, 0, 1)$.

  Then $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

Properties of Linear Spans

- Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a subset of $\mathbb{R}^n$.

$$\begin{aligned} \boldsymbol{v} \in \operatorname{span}(S) &\Leftrightarrow \boldsymbol{v} = c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k \text{ for some } c_i \in \mathbb{R} \\ &\Leftrightarrow \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \boldsymbol{v}. \end{aligned}$$

1. View each $\boldsymbol{v}_j$ as a column vector.
2. Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k \end{pmatrix}$.
3. Check if the linear system $\boldsymbol{Ax} = \boldsymbol{v}$ is consistent.
   - If $\boldsymbol{Ax} = \boldsymbol{v}$ is consistent, then $\boldsymbol{v} \in \operatorname{span}(S)$.
   - If $\boldsymbol{Ax} = \boldsymbol{v}$ is inconsistent, then $\boldsymbol{v} \notin \operatorname{span}(S)$.

Examples

- Let $\boldsymbol{u}_1 = (1,0,1)$, $\boldsymbol{u}_2 = (1,1,2)$, $\boldsymbol{u}_3 = (-1,2,1)$, and
  - $\boldsymbol{v}_1 = (1,2,3)$, $\boldsymbol{v}_2 = (2,-1,1)$.

  Prove that $\mathrm{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

  Solution. Step 1: $\mathrm{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.
  - Show that $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \in \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

$$\left(\begin{array}{cc|c|c|c} 1 & 2 & 1 & 1 & -1 \\ 2 & -1 & 0 & 1 & 2 \\ 3 & 1 & 1 & 2 & 1 \end{array}\right) \xrightarrow{\text{G.E.}} \left(\begin{array}{cc|c|c|c} 1 & 2 & 1 & 1 & -1 \\ 0 & -5 & -2 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

  - The systems $\begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{pmatrix} \boldsymbol{x} = \boldsymbol{u}_j$ are all consistent.
    - Then $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \in \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

    Therefore, $\mathrm{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

42 / 106

Examples

- Let $\boldsymbol{u}_1 = (1,0,1)$, $\boldsymbol{u}_2 = (1,1,2)$, $\boldsymbol{u}_3 = (-1,2,1)$, and
  - $\boldsymbol{v}_1 = (1,2,3)$, $\boldsymbol{v}_2 = (2,-1,1)$.

  Prove that $\mathrm{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

  Solution. Step 2: $\mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \subseteq \mathrm{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$.
  - Show that $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathrm{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$.

$$\left(\begin{array}{ccc|c|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 1 & 2 & 1 & 3 & 1 \end{array}\right) \xrightarrow{\text{G.E.}} \left(\begin{array}{ccc|c|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

  - The systems $\begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{pmatrix} \boldsymbol{x} = \boldsymbol{v}_j$ are all consistent.
    - Then $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathrm{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$.

    Therefore, $\mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \subseteq \mathrm{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$.

  We can conclude that $\mathrm{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

43 / 106

Examples

- Let $\boldsymbol{u}_1 = (1, 0, 0, 1)$, $\boldsymbol{u}_2 = (0, 1, -1, 2)$, $\boldsymbol{u}_3 = (2, 1, -1, 4)$.
  $\boldsymbol{v}_1 = (1, 1, 1, 1)$, $\boldsymbol{v}_2 = (-1, 1, -1, 1)$, $\boldsymbol{v}_3 = (-1, 1, 1, -1)$.
  - $\left(\begin{array}{ccc|c|c|c} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{array} \right)$
    $$\xrightarrow{\text{G.E.}} \left(\begin{array}{ccc|c|c|c} 1 & -1 & -1 & 1 & 0 & 2 \\ 0 & 2 & 2 & -1 & 1 & -1 \\ 0 & 0 & 2 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$
  - The systems $\begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{pmatrix} \boldsymbol{x} = \boldsymbol{u}_j$ are consistent.
    - Then $\mathrm{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.
  - $\left(\begin{array}{ccc|c|c|c} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 & \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{array} \right)$
    $$\xrightarrow{\text{G.E.}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$
  - $\begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{pmatrix} \boldsymbol{x} = \boldsymbol{v}_j$ are inconsistent for $j = 1, 3$.
    - Then $\mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \nsubseteq \mathrm{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$.

Subspaces

Subspaces

- **Definition.** Let V be a subset of $\mathbb{R}^n$. Then V is called a **subspace** of $\mathbb{R}^n$ if there exist $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ s.t.
  - $V = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$.

  More precisely,
  - V is the **subspace spanned** by $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$;
  - $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ **spans** the subspace V.
- **Remark.**
  - Let $\mathbf{0} \in \mathbb{R}^n$ be the zero vector. Then
    - $\{\mathbf{0}\} = \mathrm{span}\{\mathbf{0}\}$ is the **zero space**.
  - Let $\boldsymbol{e}_i$ denote the n-vector whose ith coordinate is 1 and elsewhere 0, e.g., $\boldsymbol{e}_2 = (0, 1, 0, \ldots, 0)$.
    - Then for every $\boldsymbol{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$
      - $\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + \cdots + v_n\boldsymbol{e}_n$.

    $\mathbb{R}^n = \mathrm{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is a subspace of $\mathbb{R}^n$.

Examples

- In order to show that a subset V of $\mathbb{R}^n$ is a subspace:
  - Find $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$.
  - Show that every $\boldsymbol{v} \in V$ is of the form
    - $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$, $c_1, c_2, \ldots, c_k \in \mathbb{R}$.
- Let $V_1 = \{(a+4b, a) \mid a, b \in \mathbb{R}\}$.
  - $(a+4b, a) = a(1,1) + b(4,0)$ for all $a, b \in \mathbb{R}$.

  Then $V_1 = \mathrm{span}\{(1,1),(4,0)\}$ is a subspace of $\mathbb{R}^2$.
- Let $V_2 = \{(x,y,z) \mid x+y-z=0\}$.
  - $x+y-z=0$ can be explicitly solved:
    - $(x,y,z) = (-s+t, s, t)$, where $s, t \in \mathbb{R}$.
    - $(-s+t, s, t) = s(-1,1,0) + t(1,0,1)$.

  $V_2 = \mathrm{span}\{(-1,1,0),(1,0,1)\}$ is a subspace of $\mathbb{R}^3$.

Examples

- Recall that a subspace V is of the form $\mathrm{span}(S)$. Then
  - $\mathbf{0} \in V$,
  - $c \in \mathbb{R}$ & $\boldsymbol{v} \in V \Rightarrow c\boldsymbol{v} \in V$,
  - $\boldsymbol{u} \in V$ & $\boldsymbol{v} \in V \Rightarrow \boldsymbol{u}+\boldsymbol{v} \in V$.

  If any of the above fails, then V is not a subspace (of $\mathbb{R}^n$).
- Let $V_3 = \{(1,a) \mid a \in \mathbb{R}\}$.
  - $\mathbf{0} = (0,0) \notin V_3$. So V is not a subspace of $\mathbb{R}^2$.
- Let $V_4 = \{(x,y,z) \mid x^2 \leq y^2 \leq z^2\}$.
  - $(1,2,3) \in V_4$ because $1^2 \leq 2^2 \leq 3^2$.
  - $(1,2,-3) \in V_4$ because $1^2 \leq 2^2 \leq (-3)^2$.
    - $(1,2,3) + (1,2,-3) = (2,4,0) \notin V_4$.

  Therefore, V_4 is not a subspace of $\mathbb{R}^3$.

Subspaces of $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$

- Let $\boldsymbol{v}$ be a nonzero vector in $\mathbb{R}^n$, $n = 1, 2, 3$.
  - $V = \mathrm{span}\{\boldsymbol{v}\} = \{c\boldsymbol{v} \mid c \in \mathbb{R}\}$.

  This is a line through the origin.

  1. $n = 1$: $\boldsymbol{v} = v \in \mathbb{R}$, and V is the whole $\mathbb{R}^1 = \mathbb{R}$.
  2. $n = 2$: $\boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^2$.
     - $V = \{(x, y) \mid v_2x - v_1y = 0\}$.
  3. $n = 3$: $\boldsymbol{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$.
     - $V = \{(cv_1, cv_2, cv_3) \mid c \in \mathbb{R}\}$.
     - If $v_1 \neq 0$, then V is the intersection of planes
       - $v_2x - v_1y = 0$ and $v_3x - v_1z = 0$.

       $V = \{(x, y, z) \mid v_2x - v_1y = 0 \;\&\; v_3x - v_1z = 0\}$

Subspaces of $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$

- Let $\boldsymbol{u}, \boldsymbol{v}$ be nonzero vectors in $\mathbb{R}^n$, $n = 2, 3$.
  - $V = \mathrm{span}\{\boldsymbol{u}, \boldsymbol{v}\} = \{s\boldsymbol{u} + t\boldsymbol{v} \mid s, t \in \mathbb{R}\}$.

  If $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel, then $V = \mathrm{span}\{\boldsymbol{u}\} = \mathrm{span}\{\boldsymbol{v}\}$.

  Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel. Then

  - $V = \mathrm{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ is a plane containing the origin.

  1. $n = 2$: Then V the the whole $\mathbb{R}^2$.
  2. $n = 3$: Let $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$.
     - $V = \{s(u_1, u_2, u_3) + t(v_1, v_2, v_3) \mid s, t \in \mathbb{R}\}$.

     We can find an implicit form of V:

     - $V = \{(x, y, z) \mid ax + by + cz = 0\}$,

     where a, b, c are not all zero.

Subspaces of $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$

- Subspaces of $\mathbb{R}^1$:
  - $\{0\}$,
  - $\mathbb{R}$.
- Subspaces of $\mathbb{R}^2$:
  - $\{\mathbf{0}\} = \{(0,0)\}$,
  - A straight line passing through the origin $(0,0)$,
  - $\mathbb{R}^2$.
- Subspaces of $\mathbb{R}^3$:
  - $\{\mathbf{0}\} = \{(0,0,0)\}$,
  - A straight line passing through the origin $(0,0,0)$,
  - A plane containing the origin $(0,0,0)$,
  - $\mathbb{R}^3$.

  A subspace of $\mathbb{R}^i$, $i = 1, 2, 3$, is always the solution set of a homogeneous linear system.

Solution Space

- **Theorem.** The **solution set** of a **homogeneous** linear system of n variables is a **subspace** of $\mathbb{R}^n$.

  Proof. Recall that a homogeneous system is consistent.

  - If the system has only the trivial solution,
    - then the solution set $\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.
  - Suppose that the system has infinitely many solutions.
    - Use Gauss-Jordan elimination to find RREF. By setting the variables corresponding to non-pivot columns as arbitrary parameters $t_1, \ldots, t_k$, solve the variables corresponding to pivot columns.
      - $x_1 = r_{11}t_1 + r_{12}t_2 + \cdots + r_{1k}t_k$.
      - $x_2 = r_{21}t_1 + r_{22}t_2 + \cdots + r_{2k}t_k$.
      - $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$
      - $x_n = r_{n1}t_1 + r_{n2}t_2 + \cdots + r_{nk}t_k$.

Solution Space

- **Theorem.** The **solution set** of a **homogeneous** linear system of n variables is a **subspace** of $\mathbb{R}^n$.

  Proof. Recall that a homogeneous system is consistent.

  - If the system has only the trivial solution,
    - then the solution set $\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.
  - Suppose that the system has infinitely many solutions.
    - $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = t_1 \begin{pmatrix} r_{11} \\ r_{21} \\ \vdots \\ r_{n1} \end{pmatrix} + t_2 \begin{pmatrix} r_{12} \\ r_{22} \\ \vdots \\ r_{n2} \end{pmatrix} + \cdots + t_k \begin{pmatrix} r_{1k} \\ r_{2k} \\ \vdots \\ r_{nk} \end{pmatrix}.$

    The solution set is spanned by

    - $(r_{11}, r_{21}, \ldots, r_{n1}), \ldots, (r_{1k}, r_{2k}, \ldots, r_{nk})$.

    So the solution set is a subspace of $\mathbb{R}^n$.

Examples

- The **solution set** of a **homogeneous** linear system is called the **solution space** of the system.

  We will see later that a subspace of $\mathbb{R}^n$ is always the solution space of a homogeneous linear system.

- $\begin{cases} x - 2y + 3z = 0 \\ 2x - 4y + 6z = 0 \\ 3x - 6y + 9z = 0 \end{cases}.$
  - $\left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 2 & -4 & 6 & 0 \\ 3 & -6 & 9 & 0 \end{array}\right) \xrightarrow[R_3+(-3)R_1]{R_2+(-2)R_1} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$
  - $(x, y, z) = (2s - 3t, s, t) = s(2, 1, 0) + t(-3, 0, 1)$.

  The solution space is $\operatorname{span}\{(2, 1, 0), (-3, 0, 1)\}$.

Examples

- $\begin{cases} x - 2y + 3z = 0 \\ -3x + 7y - 8z = 0 \\ -2x + 4y - 6z = 0 \end{cases}$.
  - $\left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -3 & 7 & -8 & 0 \\ -2 & 4 & -6 & 0 \end{array}\right) \xrightarrow{\text{G.-J.E.}} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$
  - $(x, y, z) = (-5t, -t, t) = t(-5, -1, 1)$.

  The solution space is $\mathrm{span}\{(-5, -1, 1)\}$.
- $\begin{cases} x - 2y + 3z = 0 \\ -3x + 7y - 8z = 0 \\ 4x + y + 2z = 0 \end{cases}$.
  - $\left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -3 & 7 & -8 & 0 \\ 4 & 1 & 2 & 0 \end{array}\right) \xrightarrow{\text{G.-J.E.}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$
  - $(x, y, z) = (0, 0, 0)$. The solution space is $\{(0, 0, 0)\}$.

Linear Independence

Linear Independence

- In $\mathbb{R}^3$, a plane containing the origin can be spanned by two non-parallel vectors: $V = \mathrm{span}\{\boldsymbol{u}, \boldsymbol{v}\}$.
  - If a plane is spanned by more than two vectors, then
    - some vectors in the spanning set is redundant.
- Suppose that $V = \mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$.
  - Recall that if $\boldsymbol{v}_k$ is a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}$,
    - $\mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_k\} = \mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}\}$.

    Continuing this procedure, we can remove the redundant vectors in the spanning set to obtain
    - $V = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$,

    so that any $\boldsymbol{v}_i$ is NOT a linear combination of the other vectors.

Linear Independence

- **Definition.** Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ be a subset of $\mathbb{R}^n$.
  - The equation $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}$

  has a **trivial solution** $c_1 = c_2 = \cdots = c_k = 0$.

  1. If the equation has a **non-trivial solution**, then
     - S is a **linearly dependent set**,
     - $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ are **linearly dependent**.

     There exist $c_1, c_2, \ldots, c_k \in \mathbb{R}$ not all zero such that

     - $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}$.
  2. If the equation has **only the trivial solution**, then
     - S is a **linearly independent set**,
     - $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ are **linearly independent**.

     $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0} \Rightarrow c_1 = \cdots = c_k = 0$.

Examples

- Let $S = \{(1, -2, 3), (5, 6, -1), (3, 2, 1)\}$.
  - $c_1(1, -2, 3) + c_2(5, 6, -1) + c_3(3, 2, 1) = (0, 0, 0)$.
    - $c_1 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 5 \\ 6 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.
    - $\begin{pmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.
    - $\begin{pmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{pmatrix} \xrightarrow{\text{G.E.}} \begin{pmatrix} 1 & 5 & 3 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{pmatrix}$.

    The 3rd column is non-pivot.

    - The system has infinitely many solutions.

  Therefore, S is a linearly dependent set.

Examples

- Let $S = \{(1,0,0,1), (0,2,1,0), (1,-1,1,1)\}$.
  - $c_1(1,0,0,1) + c_2(0,2,1,0) + c_3(1,-1,1,1) = \mathbf{0}$.
    - $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.
    - $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow[\text{elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix}$.

    All the columns are pivot.
    - The system has only the trivial solution.

  Therefore, S is a linearly independent set.

Properties

- Let S_1, S_2 be finite subsets of $\mathbb{R}^n$ such that $S_1 \subseteq S_2$.
  - S_1 linearly dependent $\Rightarrow$ S_2 linearly dependent.
  - S_2 linearly independent $\Rightarrow$ S_1 linearly independent.
- $c\mathbf{0} = \mathbf{0}$ has infinitely many solutions $c \in \mathbb{R}$.
  - $\{\mathbf{0}\}$ is linearly dependent.
  - If $\mathbf{0} \in S$ ($\subseteq \mathbb{R}^n$) then S is linearly dependent.
- Let $\boldsymbol{v} \in \mathbb{R}^n$. Then $c\boldsymbol{v} = \mathbf{0} \Leftrightarrow c = 0$ or $\boldsymbol{v} = \mathbf{0}$.
  - $\{\boldsymbol{v}\}$ is linearly independent $\Leftrightarrow \boldsymbol{v} \neq \mathbf{0}$.
- Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$. Then

$$\begin{aligned} \{\boldsymbol{u}, \boldsymbol{v}\} \text{ is linearly dependent} \Leftrightarrow\ & \boldsymbol{u} = a\boldsymbol{v} \text{ for some } a \in \mathbb{R} \\ & \text{or } \boldsymbol{v} = a\boldsymbol{u} \text{ for some } a \in \mathbb{R} \end{aligned}$$

Properties

- **Theorem.** Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$, $k \geq 2$.
  - S is linearly dependent

    $\Leftrightarrow$ there exists $\boldsymbol{v}_i$ such that it is a linear combination of other vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_k$.
- **Proof.** $\Rightarrow$: Suppose S is linearly dependent.
  - There exist $c_1, c_2, \ldots, c_k \in \mathbb{R}$ not all zero such that
    - $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \mathbf{0}$.

    Suppose that $c_i \neq 0$. Then
    - $\boldsymbol{v}_i = -\frac{c_1}{c_i}\boldsymbol{v}_1 - \cdots - \frac{c_{i-1}}{c_i}\boldsymbol{v}_{i-1} - \frac{c_{i+1}}{c_i}\boldsymbol{v}_{i+1} - \cdots - \frac{c_n}{c_i}\boldsymbol{v}_n$.

    $\Leftarrow$: Suppose $\boldsymbol{v}_i$ is a linear combination of other vectors. Then there exist $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_k \in \mathbb{R}$ such that
    - $\boldsymbol{v}_i = c_1\boldsymbol{v}_1 + \cdots + c_{i-1}\boldsymbol{v}_{i-1} + c_{i+1}\boldsymbol{v}_{i+1} + \cdots + c_k\boldsymbol{v}_k$.

    $c_1\boldsymbol{v}_1 + \cdots + c_{i-1}\boldsymbol{v}_{i-1} + (-1)\boldsymbol{v}_i + c_{i+1}\boldsymbol{v}_{i+1} + \cdots + c_k\boldsymbol{v}_k = \mathbf{0}$

    $\therefore$ $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_i, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_k\}$ is linearly dept.

Properties

- **Theorem.** Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$, $k \geq 2$.
  - S is linearly dependent

    $\Leftrightarrow$ there exists $\boldsymbol{v}_i$ such that it is a linear combination of other vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_k$.
  - S is linearly independent

    $\Leftrightarrow$ no vector in S can be written as a linear combination of other vectors.
- **Remarks.** Suppose $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is linearly dependent. Let $V = \mathrm{span}(S)$.
  - Some $\boldsymbol{v}_i \in S$ is a linear combination of other vectors.
  - Remove $\boldsymbol{v}_i$ from S and repeat the procedure until we obtain a linearly independent set S'.
  - Then $\mathrm{span}(S') = V$ and S' has no “redundant vector” to span V.

Examples

- Let $S_1 = \{(1,0), (0,4), (2,4)\}$.
  - Note that $(2,4) = 2(1,0) + 1(0,4)$.

  Then S_1 is linearly dependent. Moreover,
  - $\mathrm{span}(S_1) = \mathrm{span}\{(1,0), (0,4)\}$.
- Let $S_2 = \{(-1,0,0), (0,3,0), (0,0,7)\}$.
  - $(-1,0,0)$ is the only vector whose 1st component $\neq 0$
  - $(0,3,0)$ is the only vector whose 2nd component $\neq 0$.
  - $(0,0,7)$ is the only vector whose 3rd component $\neq 0$.

  Any vector is NOT a linear combination of the other two vectors.

  $\therefore$ S_2 is linearly independent.

Properties

- **Theorem.** Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$.
  - If $k > n$, then S is linearly dependent.

  Proof. Consider $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \mathbf{0}$.
  - View each $\boldsymbol{v}_j$ as a column vector.
  - Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_k \end{pmatrix}$.
    - Determine if $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution.
  - $\boldsymbol{A}$ is of size $n \times k$; so is the RREF $\boldsymbol{R}$.
    - $\boldsymbol{R}$ has at most n nonzero rows.
    - $\boldsymbol{R}$ has at most n pivot columns.
    - $\boldsymbol{R}$ has at least $k - n > 0$ non-pivot columns.
  - Then $\boldsymbol{Ax} = \mathbf{0}$ has non-trivial solutions.

  $\therefore$ S is linearly dependent.

Properties

- **Theorem.** Suppose $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$ is linearly independent.
  - If $\boldsymbol{v}_{k+1} \in \mathbb{R}^n$ is not in $\mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$.
    - then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}\}$ is linearly independent.

  Proof. Suppose $c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k + c_{k+1}\boldsymbol{v}_{k+1} = \mathbf{0}$.
  - If $c_{k+1} \neq 0$, then
    - $\boldsymbol{v}_{k+1} = -\dfrac{c_1}{c_{k+1}}\boldsymbol{v}_1 - \cdots - \dfrac{c_k}{c_{k+1}}\boldsymbol{v}_k$,
    - $\boldsymbol{v}_{k+1} \in \mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$, contradiction!
  - So $c_{k+1} = 0$. This implies
    - $c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k = \mathbf{0}$.
    - $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is linearly independent.
      $\Rightarrow c_1 = c_2 = \cdots = c_k = 0$.

  Therefore, $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}\}$ is linearly independent.

Bases

Motivation

- Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$ be linearly independent.
  1. Suppose $\mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \neq \mathbb{R}^n$.
  2. Pick $\boldsymbol{v}_{k+1} \in \mathbb{R}^n$ but $\boldsymbol{v}_{k+1} \notin \mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$.
  3. Then $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}\}$ is linearly independent.
  4. Repeat this procedure until
     - $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \ldots, \boldsymbol{v}_m\}$ is linearly independent &
     - $\mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \ldots, \boldsymbol{v}_m\} = \mathbb{R}^n$.
- If $m > n$, then $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ is linearly dependent.
  If $m < n$, then $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ cannot span $\mathbb{R}^n$.
  - We must have $n = m$.

  $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is linearly independent, and spans $\mathbb{R}^n$.

Vector Spaces

- **Definition.** A set V is called a **vector space** if
  - V is a subspace of $\mathbb{R}^n$ for some positive integer n.

  If W and V are vector spaces such that $W \subseteq V$,
  - then W is a **subspace** of V.
- **Examples.**
  - Let $U = \operatorname{span}\{(1,1,1)\}$, $V = \operatorname{span}\{(1,1,-1)\}$ and
    $W = \operatorname{span}\{(1,0,0),(0,1,1)\}$.

  Then U, V, W are vector spaces (subspace of $\mathbb{R}^3$).
  - $(1,1,1) = (1,0,0) + (0,1,1) \in W$.
    - Then $U \subseteq W$; so U is a subspace of W.
  - $(1,1,-1) \notin \operatorname{span}\{(1,0,0),(0,1,1)\}$.
    - Then $V \not\subseteq W$; so V is NOT a subspace of W.

Bases

- **Definition.** Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ be a subset of a vector space V. Then S is called a **basis** (plural **bases**) for V if
  - S is **linearly independent**, and $\operatorname{span}(S) = V$.
- **Example.** Show that $S = \{(1,2,1),(2,9,0),(3,3,4)\}$ is a basis for $\mathbb{R}^3$.
  1. Prove that S is linearly independent.
     - Let $c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4) = \mathbf{0}$.
     - $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & -3 \\ 0 & 0 & -\frac{1}{5} \end{pmatrix}$.
       - All the three columns are pivot.
       - The system has only the trivial solution.

     Therefore, S is linearly independent.

Bases

- **Definition.** Let $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ be a subset of a vector space V. Then S is called a **basis** (plural **bases**) for V if
  - S is **linearly independent**, and $\mathrm{span}(S) = V$.
- **Example.** Show that $S = \{(1,2,1),(2,9,0),(3,3,4)\}$ is a basis for $\mathbb{R}^3$.
  2. Prove that $\mathrm{span}(S) = \mathbb{R}^3$.
     - $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & -3 \\ 0 & 0 & -\frac{1}{5} \end{pmatrix}.$
       - A row-echelon form has no zero row.

       Therefore, $\mathrm{span}(S) = \mathbb{R}^3$.

  We can conclude that S is a basis for $\mathbb{R}^3$.

Examples

- Let $V = \mathrm{span}\{(1,1,1,1),(1,-1,-1,1),(1,0,0,1)\}$.
  - $S = \{(1,1,1,1),(1,-1,-1,1)\}$. Is S a basis for V?
    1. $\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix} \xrightarrow{\text{Guassian elimination}} \begin{pmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$
       - All the two columns are pivot.
       - $c_1(1,1,1,1) + c_2(1,-1,-1,1) = \mathbf{0}$

         has only the trivial solution.

       So S is linearly independent.
    2. $(1,0,0,1) = \frac{1}{2}(1,1,1,1) + \frac{1}{2}(1,-1,-1,1).$
       - So $(1,0,0,1) \in \mathrm{span}(S)$.

       So $\mathrm{span}(S) = V$.

  Therefore, S is a basis for V.

Examples

- Let $S = \{(1,1,1,1), (0,0,1,2), (-1,0,0,1)\}$.
  - Let $|S|$ be the number of vectors in S. Then $|S| = 3$.
  - So $\operatorname{span}(S) \neq \mathbb{R}^4$; thus S is NOT a basis for $\mathbb{R}^4$.
- Let $V = \operatorname{span}(S)$, $S = \{(1,1,1), (0,0,1), (1,1,0)\}$.
  - $(1,1,1) = (0,0,1) + (1,1,0)$.
  - So S is linearly dependent; thus S is not a basis for V.
- **Remarks.**
  - A basis for a vector space V contains
    - smallest possible number of vectors that spans V,
    - largest possible number of vectors that is linearly independent.
  - For convenience, $\varnothing$ is said to be the **basis** for $\{\mathbf{0}\}$.
  - Other than $\{\mathbf{0}\}$, any vector space has infinitely many different bases.

Properties

- **Theorem.** Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a subset of a vector space V. Then the following are equivalent:
  - S is a basis for V.
  - Every vector $\boldsymbol{v} \in V$ can be **uniquely** expressed as
    - $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$, $c_i \in \mathbb{R}$.
- **Proof.** $\Rightarrow$: Suppose S is a basis. Then $\operatorname{span}(S) = V$.
  - For every $\boldsymbol{v} \in V$, there exist $c_1, c_2, \ldots, c_k \in \mathbb{R}$ s.t.
    - $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$.
  - Suppose $\boldsymbol{v} = d_1\boldsymbol{v}_1 + d_2\boldsymbol{v}_2 + \cdots + d_k\boldsymbol{v}_k$, $d_i \in \mathbb{R}$.
    - $\mathbf{0} = (c_1 - d_1)\boldsymbol{v}_1 + \cdots + (c_k - d_k)\boldsymbol{v}_k$.

    Since S is linearly independent,
    - $c_1 - d_1 = c_2 - d_2 = \cdots = c_k - d_k = 0$;
    - that is, $c_1 = d_1, c_2 = d_2, \ldots, c_k = d_k$.

Properties

- **Theorem.** Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a subset of a vector space V. Then the following are equivalent:
  - S is a basis for V.
  - Every vector $\boldsymbol{v} \in V$ can be **uniquely** expressed as
    - $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$, $c_i \in \mathbb{R}$.
- **Proof.** $\Leftarrow$: Suppose that every vector $\boldsymbol{v} \in V$ can be uniquely expressed as
  - $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$, $c_i \in \mathbb{R}$.

  Then by definition $\mathrm{span}(S) = V$.

  Let $\mathbf{0} \in V$. Suppose $\mathbf{0} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$.
  - Note that $\mathbf{0} = 0\boldsymbol{v}_1 + 0\boldsymbol{v}_2 + \cdots + 0\boldsymbol{v}_k$.

  By the uniqueness, $c_1 = 0, c_2 = 0, \ldots, c_k = 0$.
  - So S is linearly independent.

Coordinate Vector

- **Definition.** Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a basis for a vector space V.
  - For every $\boldsymbol{v} \in V$, there exist unique $c_1, \ldots, c_k \in \mathbb{R}$ s.t.
    - $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$.

    $c_1, c_2, \ldots, c_k$ are the **coordinates** of $\boldsymbol{v}$ **relative** to S.
    - $(c_1, c_2, \ldots, c_k)$ is the **coordinate vector** of $\boldsymbol{v}$ **relative** to the basis S, denoted by $(\boldsymbol{v})_S$.
- **Remark.** The order of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ is fixed.
  - Let $S_1 = \{(1,1), (-1,1)\}$ be a basis for $\mathbb{R}^2$. (Check!)
    - Let $\boldsymbol{v} = 2(1,1) + 3(-1,1) = (-1,5)$.
      - Then $(\boldsymbol{v})_{S_1} = (2,3)$.

    Let $S_2 = \{(-1,1), (1,1)\}$. Then $(\boldsymbol{v})_{S_2} = (3,2)$.

Examples

- Let $S = \{(1,2,1),(2,9,0),(3,3,4)\}$.
  - One can check that S is a basis for $\mathbb{R}^3$. (Exercise!)

  Let $\boldsymbol{v} = (5,-1,9)$. Solve
  - $\boldsymbol{v} = a(1,2,1) + b(2,9,0) + c(3,3,4)$.
    - $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 2 & 9 & 3 & -1 \\ 1 & 0 & 4 & 9 \end{array}\right) \xrightarrow{\text{G.-J.E.}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$.

    $(\boldsymbol{v})_S = (a,b,c) = (1,-1,2)$.

  Suppose that $(\boldsymbol{w})_S = (-1,3,2)$.

$$\begin{aligned} \boldsymbol{w} &= (-1)(1,2,1) + 3(2,9,0) + 2(3,3,4) \\ &= (11,31,7). \end{aligned}$$

77 / 106

Examples

- Let $\boldsymbol{v} = (2,3) \in \mathbb{R}^2$.
  - Let $S_1 = \{(1,0),(0,1)\}$ be a basis for $\mathbb{R}^2$.
    - $\boldsymbol{v} = 2(1,0) + 3(0,1)$; so $(\boldsymbol{v})_{S_1} = (2,3)$.
  - Let $S_2 = \{(1,-1),(1,1)\}$ be a basis for $\mathbb{R}^2$.
    - $\left(\begin{array}{cc|c} 1 & 1 & 2 \\ -1 & 1 & 3 \end{array}\right) \xrightarrow{\text{G.-J.E.}} \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \end{array}\right)$.
    - $(\boldsymbol{v})_{S_2} = (-\frac{1}{2}, \frac{5}{2})$.
  - Let $S_3 = \{(1,0),(1,1)\}$ be a basis for $\mathbb{R}^2$.
    - $\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 3 \end{array}\right) \xrightarrow{R_1+(-1)R_2} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3 \end{array}\right)$.
    - $(\boldsymbol{v})_{S_3} = (-1,3)$.

78 / 106

Standard Basis

- **Definition.** Let $E = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ be a subset of $\mathbb{R}^n$,
  - $\boldsymbol{e}_1 = (1, 0, \ldots, 0),\ \boldsymbol{e}_2 = (0, 1, \ldots, 0), \ldots, \boldsymbol{e}_n = (0, 0, \ldots, 1)$.

  1. Let $\boldsymbol{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$. Then
     - $\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + \cdots + v_n\boldsymbol{e}_n$.

     So $\mathrm{span}(E) = \mathbb{R}^n$.
  2. Suppose that $c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n = \boldsymbol{0}$. Then
     - $(c_1, c_2, \ldots, c_n) = (0, 0, \ldots, 0)$.

     So E is linearly independent.

  $\boldsymbol{E}$ is called the **standard basis** for $\mathbb{R}^n$.
  - For any $\boldsymbol{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$,
    - $(\boldsymbol{v})_E = (v_1, v_2, \ldots, v_n) = \boldsymbol{v}$.

Properties

- **Theorem.** Let S be a basis for a vector space V.
  - $(\boldsymbol{v})_S = \boldsymbol{0} \Leftrightarrow \boldsymbol{v} = \boldsymbol{0}$.
  - For any $c \in \mathbb{R}$ and $\boldsymbol{v} \in \mathbb{R}$, $(c\boldsymbol{v})_S = c(\boldsymbol{v})_S$.
  - For any $\boldsymbol{u}, \boldsymbol{v} \in V$, $(\boldsymbol{u} + \boldsymbol{v})_S = (\boldsymbol{u})_S + (\boldsymbol{v})_S$.

  Proof. Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$.
  - $\Rightarrow$: Suppose $(\boldsymbol{v})_S = (\overbrace{0, 0, \ldots, 0}^{k}) = \boldsymbol{0} \in \mathbb{R}^k$.
    - $\boldsymbol{v} = 0\boldsymbol{v}_1 + \cdots + 0\boldsymbol{v}_k = \boldsymbol{0} \in V$.

    $\Leftarrow$: Let $\boldsymbol{v} = \boldsymbol{0} \in V$. Then $\boldsymbol{v} = 0\boldsymbol{v}_1 + \cdots + 0\boldsymbol{v}_k$.
    - $(\boldsymbol{v})_S = (\overbrace{0, \ldots, 0}^{k}) = \boldsymbol{0} \in \mathbb{R}^k$.

Properties

- **Theorem.** Let S be a basis for a vector space V
  - $(\boldsymbol{v})_S = \mathbf{0} \Leftrightarrow \boldsymbol{v} = \mathbf{0}$.
  - For any $c \in \mathbb{R}$ and $\boldsymbol{v} \in \mathbb{R}$, $(c\boldsymbol{v})_S = c(\boldsymbol{v})_S$.
  - For any $\boldsymbol{u}, \boldsymbol{v} \in V$, $(\boldsymbol{u} + \boldsymbol{v})_S = (\boldsymbol{u})_S + (\boldsymbol{v})_S$.

  Proof. Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$.
  - Let $(\boldsymbol{v})_S = (c_1, \ldots, c_k)$. Then
    - $\boldsymbol{v} = c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k$.
    - $c\boldsymbol{v} = cc_1\boldsymbol{v}_1 + \cdots + cc_k\boldsymbol{v}_k$.

    $(c\boldsymbol{v})_S = (cc_1, \ldots, cc_k) = c(c_1, \ldots, c_k) = c(\boldsymbol{v})_S$.
  - Let $(\boldsymbol{u})_S = (c_1, \ldots, c_k)$, $(\boldsymbol{v})_S = (d_1, \ldots, d_k)$.
    - $\boldsymbol{u} = c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k$, $\boldsymbol{v} = d_1\boldsymbol{v}_1 + \cdots + d_k\boldsymbol{v}_k$.
    - $\boldsymbol{u} + \boldsymbol{v} = (c_1 + d_1)\boldsymbol{v}_1 + \cdots + (c_k + d_k)\boldsymbol{v}_k$.

    $(\boldsymbol{u} + \boldsymbol{v})_S = (c_1 + d_1, \ldots, c_k + d_k) = (\boldsymbol{u})_S + (\boldsymbol{v})_S$.

81 / 106

Properties

- **Theorem.** Let S be a basis for a vector space V.
  - For any $\boldsymbol{u}, \boldsymbol{v} \in V$, $\boldsymbol{u} = \boldsymbol{v} \Leftrightarrow (\boldsymbol{u})_S = (\boldsymbol{v})_S$.
  - For any $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \in V$ and $c_1, c_2, \ldots, c_r \in \mathbb{R}$,
    - $(c_1\boldsymbol{v}_1 + \cdots + c_r\boldsymbol{v}_r)_S = c_1(\boldsymbol{v}_1)_S + \cdots + c_r(\boldsymbol{v}_r)_S$.

  Proof. Left as exercises.
- **Theorem.** Let S be a basis for a vector space V.
  - Suppose $|S| = k$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \in V$.
    1. $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ are linearly independent

       $\Leftrightarrow (\boldsymbol{v}_1)_S, \ldots, (\boldsymbol{v}_r)_S$ are linearly independent.
    2. $\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} = V$

       $\Leftrightarrow \operatorname{span}\{(\boldsymbol{v}_1)_S, \ldots, (\boldsymbol{v}_r)_S\} = \mathbb{R}^k$.

82 / 106

Properties

- **Proof.** 1. $\Rightarrow$: Suppose $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ are linearly independent.
  - Consider equation $c_1(\boldsymbol{v}_1)_S + \cdots + c_r(\boldsymbol{v}_r)_S = \boldsymbol{0} \in \mathbb{R}^k$.
    - $(c_1\boldsymbol{v}_1 + \cdots + c_r\boldsymbol{v}_r)_S = (\boldsymbol{0})_S$, where $\boldsymbol{0} \in V$.

    Then $c_1\boldsymbol{v}_1 + \cdots + c_r\boldsymbol{v}_r = \boldsymbol{0}$.
    - $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ linearly independent $\Rightarrow c_1 = \cdots = c_r = 0$.

    Therefore, $(\boldsymbol{v}_1)_S, \ldots, (\boldsymbol{v}_r)_S$ are linearly independent.

  $\Leftarrow$: Suppose $(\boldsymbol{v}_1)_S, \ldots, (\boldsymbol{v}_r)_S$ are linearly independent.
  - Consider equation $c_1\boldsymbol{v}_1 + \cdots + c_r\boldsymbol{v}_r = \boldsymbol{0} \in V$.
    - $(c_1\boldsymbol{v}_1 + \cdots + c_r\boldsymbol{v}_r)_S = (\boldsymbol{0})_S$.

    Then $c_1(\boldsymbol{v}_1)_S + \cdots + c_r(\boldsymbol{v}_r)_S = \boldsymbol{0} \in \mathbb{R}^k$.
    - $(\boldsymbol{v}_1)_S, \ldots, (\boldsymbol{v}_r)_S$ are linearly independent

      $\Rightarrow c_1 = \cdots = c_r = 0$.

    Therefore, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ are linearly independent.

Properties

- **Proof.** 2. $\Rightarrow$: Suppose $\mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} = V$.
  - Let $\boldsymbol{w} = (c_1, \ldots, c_k) \in \mathbb{R}^k$. If $S = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$,
    - then $\boldsymbol{v} = c_1\boldsymbol{u}_1 + \cdots + c_k\boldsymbol{u}_k \in V$, $(\boldsymbol{v})_S = \boldsymbol{w}$.

    Since $\mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} = V$, there exist $d_i \in \mathbb{R}$ s.t.
    - $\boldsymbol{v} = d_1\boldsymbol{v}_1 + \cdots + d_r\boldsymbol{v}_r$.

    Then $\boldsymbol{w} = (\boldsymbol{v})_S = d_1(\boldsymbol{v}_1)_S + \cdots + d_r(\boldsymbol{v}_r)_S$.

    Therefore, $\mathrm{span}\{(\boldsymbol{v}_1)_S, \ldots, (\boldsymbol{v}_r)_S\} = \mathbb{R}^k$.
- 2. $\Leftarrow$: Suppose $\mathrm{span}\{(\boldsymbol{v}_1)_S, \ldots, (\boldsymbol{v}_r)_S\} = \mathbb{R}^k$.
  - Let $\boldsymbol{v} \in V$. Then $(\boldsymbol{v})_S \in \mathbb{R}^k$. There exist $c_i \in \mathbb{R}^k$ s.t.
    - $(\boldsymbol{v})_S = c_1(\boldsymbol{v}_1)_S + \cdots + c_r(\boldsymbol{v}_r)_S$.
    - $(\boldsymbol{v})_S = (c_1\boldsymbol{v}_1 + \cdots + c_r\boldsymbol{v}_r)_S$.

    Then $\boldsymbol{v} = c_1\boldsymbol{v}_1 + \cdots + c_r\boldsymbol{v}_r$.

    Therefore, $\mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} = V$.

Criterion for Bases

- Let $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ be a subset of $\mathbb{R}^n$.
  - If $k > n$, then S is linearly dependent.
  - If $k < n$, then $\operatorname{span}(S) \neq \mathbb{R}^n$.

  If S is a basis, then $k = n$.
- **Theorem.** Let V be a vector space having a basis with k vectors.
  - Any subset of V of $> k$ vectors is linearly dependent.
  - Any subset of V of $< k$ vectors cannot span V.
- **Corollary.** All bases of a vector space have same size.
  - To be more precise, if S_1 and S_2 are two bases of a vector space V,
    - then $|S_1| = |S_2|$.

Criterion for Bases

- **Proof.** Let S be a basis of V with $|S| = k$.
  - Let $T = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_r\}$ be a subset of V.
    - Then $\{(\boldsymbol{v}_1)_S, \dots, (\boldsymbol{v}_r)_S\}$ is a subset of $\mathbb{R}^k$.

    1. Suppose $r > k$.
       - $\{(\boldsymbol{v}_1)_S, \dots, (\boldsymbol{v}_r)_S\}$ is linearly dependent in $\mathbb{R}^k$.

       Then $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_r\}$ is linearly dependent in V.
    2. Suppose $r < k$.
       - $\operatorname{span}\{(\boldsymbol{v}_1)_S, \dots, (\boldsymbol{v}_r)_S\} \neq \mathbb{R}^k$.

       Then $\operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_r\} \neq V$.

Dimension

- **Definition.** Let V be a vector space and S a basis for V.
  - The **dimension** of V is $\dim(V) = |S|$.
- **Examples.**
  - $\varnothing$ is a (the) basis for $\{\mathbf{0}\}$.
    - Then $\dim(\{\mathbf{0}\}) = |\varnothing| = 0$.
  - $\mathbb{R}^n$ has the standard basis $E = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$.
    - Then $\dim(\mathbb{R}^n) = n$.
  - In $\mathbb{R}^2$ and $\mathbb{R}^3$, every straight line through the origin is of the form $\operatorname{span}\{\boldsymbol{v}\}$ with $\boldsymbol{v} \neq \mathbf{0}$.
    - The dimension of such a straight line is 1.
  - In $\mathbb{R}^3$, every plane containing the origin is of the form $\operatorname{span}\{\boldsymbol{u}, \boldsymbol{v}\}$, where $\boldsymbol{u}, \boldsymbol{v}$ are linearly independent.
    - The dimension of such a plane is 2.

Dimension of Solution Space

- Let $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ be a homogeneous linear system.
  - Recall that the solution set is a vector space V.

  Let $\boldsymbol{R}$ be a row-echelon form of $\boldsymbol{A}$.

$$\begin{aligned} & \text{no. of non-pivot colns of } \boldsymbol{R} \\ = {} & \text{no. of arbitrary parameters in soln} \\ = {} & \text{the dimension of } V. \end{aligned}$$

- **Example.** $x + y + z = 0$.
  - $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$
    - The 2nd and 3rd columns are non-pivot.
    - The dimension of the solution space is 2.

Example

- $$\begin{cases} 2v + 2w - \ \ x \qquad + z = 0 \\ -v - \ \ w + 2x - 3y + z = 0 \\ \qquad\qquad x + \ \ y + z = 0 \\ v + \ \ w - 2x \qquad - z = 0 \end{cases}$$

    ◦ $$\begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$$

    $$\xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

    - The 2nd and 5th columns are non-pivot.
    - The solution space has dimension 2.

Properties

- **Theorem.** Let S be a subset of a vector space V. The following are equivalent:
    1. S is a basis for V.
    2. S is linearly independent, and $|S| = \dim(V)$.
    3. S spans V, and $|S| = \dim(V)$.
- To check whether a subset S is a basis for a vector space V, simply check any **two** of the following three conditions:
    - ◦ S is linearly independent,
    - ◦ S spans V,
    - ◦ $|S| = \dim(V)$.
- **Example.** Let $S = \{(2, 0, -1), (4, 0, 7), (-1, 1, 4)\}$.
    - ◦ One can check that S is linearly independent.

    Since $|S| = 3$, S is a basis for $\mathbb{R}^3$.

Properties

- **Proof.** "1 $\Rightarrow$ 2" and "1 $\Rightarrow$ 3" are clear.

  "2 $\Rightarrow$ 1": Suppose S is linearly indept. & $|S| = \dim(V)$.

  ○ It suffices to show that $\mathrm{span}(S) = V$.

  Assume $\mathrm{span}(S) \neq V$. Pick $\boldsymbol{v} \in V$ but $\boldsymbol{v} \notin \mathrm{span}(S)$.

  ○ Then $S \cup \{\boldsymbol{v}\}$ is linearly independent.

  But $|S \cup \{\boldsymbol{v}\}| = \dim(V) + 1 > \dim(V)$.

  ○ Then $S \cup \{\boldsymbol{v}\}$ is linearly dependent.

  Therefore, we must have $\mathrm{span}(S) = V$.

  ○ Since S is linearly independent, S is a basis for V.

Properties

- **Proof.** "1 $\Rightarrow$ 2" and "1 $\Rightarrow$ 3" are clear.

  "3 $\Rightarrow$ 1": Suppose S spans V & $|S| = \dim(V)$.

  ○ It suffices to show that S is linearly independent.

  Assume that S is linearly dependent.

  ○ Then there exists $\boldsymbol{v} \in S$ such that $\boldsymbol{v}$ is a linear combination of other vectors in S.

  ○ Hence, $\mathrm{span}(S - \{\boldsymbol{v}\}) = \mathrm{span}(S) = V$.

  On the other hand, $|S - \{\boldsymbol{v}\}| = \dim(V) - 1 < \dim(V)$.

  ○ Then $\mathrm{span}(S - \{\boldsymbol{v}\}) \neq V$.

  Therefore, S must be linearly independent.

  ○ Since S spans V, S is a basis for V.

Properties

- **Theorem.** Let U be a subspace of a vector space V.
  - $U = V \Leftrightarrow \dim(U) = \dim(V)$.

  Proof. $\Rightarrow$: Clear!

  $\Leftarrow$: Suppose $\dim(U) = \dim(V)$.
  - Let S be a basis for U. Then
    - S is linearly independent (in U, and thus) in V.
    - $|S| = \dim(U) = \dim(V)$.

    Then S is also a basis for V.
  - Therefore, $V = \mathrm{span}(S) = U$.
- **Corollary.** Let U be a subspace of a vector space V.
  - $U \neq V \Leftrightarrow \dim(U) < \dim(V)$.

94 / 106

Properties

- **Theorem.** Let $\boldsymbol{A}$ be a square matrix of order n. Then the following are equivalent:
  1. $\boldsymbol{A}$ is invertible.
  2. $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution.
  3. $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
  4. The reduced row-echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}_n$.
  5. $\boldsymbol{A}$ is a product of elementary matrices.
  6. $\det(\boldsymbol{A}) \neq 0$.
  7. The rows of $\boldsymbol{A}$ form a basis for $\mathbb{R}^n$.
  8. The columns of $\boldsymbol{A}$ form a basis for $\mathbb{R}^n$.
- We have proved the equivalence of 1 to 6.
  - It remains to show that "1 $\Leftrightarrow$ 7" & "1 $\Leftrightarrow$ 8".

95 / 106

Properties

- **Proof.** "1 ⇔ 8": Let $\boldsymbol{a}_j$ be the jth column of $\boldsymbol{A}$.
  - $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{pmatrix}$.

$$\begin{aligned} & \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\} \text{ is a basis for } \mathbb{R}^n \\ \Leftrightarrow\ & \operatorname{span}\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\} = \mathbb{R}^n \\ \Leftrightarrow\ & \text{a row-echelon form of } \boldsymbol{A} \text{ has no zero row} \\ \Leftrightarrow\ & \boldsymbol{A} \text{ is invertible.} \end{aligned}$$

"1 ⇔ 7":

$$\begin{aligned} & \text{rows of } \boldsymbol{A} \text{ form a basis for } \mathbb{R}^n \\ \Leftrightarrow\ & \text{columns of } \boldsymbol{A}^{\mathrm{T}} \text{ form a basis for } \mathbb{R}^n \\ \Leftrightarrow\ & \boldsymbol{A}^{\mathrm{T}} \text{ is invertible} \\ \Leftrightarrow\ & \boldsymbol{A} \text{ is invertible.} \end{aligned}$$

Examples

- Let $\boldsymbol{v}_1 = (1, 1, 1)$, $\boldsymbol{v}_2 = (-1, 1, 2)$ and $\boldsymbol{v}_3 = (1, 0, 1)$.
  - Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$. Then $\det(\boldsymbol{A}) = 3$.
    - $\boldsymbol{A}$ is invertible.
  - So $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis for $\mathbb{R}^3$.
- $\boldsymbol{v}_1 = (1, 1, 1, 1)$, $\boldsymbol{v}_2 = (1, -1, 1, -1)$, $\boldsymbol{v}_3 = (0, 1, -1, 0)$,

  $\boldsymbol{v}_4 = (2, 1, 1, 0)$.
  - $\begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \xrightarrow[\text{elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.
  - So $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is NOT a basis for $\mathbb{R}^4$.

Coordinate Vector

- Let $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ be a basis for a vector space V.
  - Then every vector $\boldsymbol{v} \in V$ can be uniquely expressed as a linear combination of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$:
    - $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$, where $c_i \in \mathbb{R}$.

    Then $(c_1, c_2, \dots, c_k) = (\boldsymbol{v})_S$ is the **coordinate vector** of $\boldsymbol{v}$ relative to the basis S.

  View each $\boldsymbol{v}_i$ as a column vector. Then
  - $\begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \boldsymbol{v}.$

Coordinate Vector

- Let $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ be a basis for a vector space V.
  - Then every vector $\boldsymbol{v} \in V$ can be uniquely expressed as a linear combination of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$:
    - $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$, where $c_i \in \mathbb{R}$.

    The **column vector** $[\boldsymbol{v}]_S = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$ is also called
    - the **coordinate vector** of $\boldsymbol{v}$ relative to S.
- View $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ as column vectors.
  - Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_k \end{pmatrix}$. Then
    - $\boldsymbol{A}[\boldsymbol{v}]_S = \boldsymbol{v}$ for every $\boldsymbol{v} \in V$.

Transition Matrix

- Let S and T be bases for a vector space V.
  - $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$.

  Let $\boldsymbol{w} \in V$. What is the relation between $[\boldsymbol{w}]_S$ and $[\boldsymbol{w}]_T$?
- Suppose all $\boldsymbol{u}_j, \boldsymbol{v}_j$ and $\boldsymbol{w}$ are viewed as column vectors.
  - Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_k \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k \end{pmatrix}$.

  Let $\boldsymbol{w} \in V$. Then

$$\begin{aligned} \boldsymbol{w} &= \boldsymbol{A}[\boldsymbol{w}]_S = \begin{pmatrix} \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_k \end{pmatrix} [\boldsymbol{w}]_S \\ &= \begin{pmatrix} \boldsymbol{B}[\boldsymbol{u}_1]_T & \cdots & \boldsymbol{B}[\boldsymbol{u}_k]_T \end{pmatrix} [\boldsymbol{w}]_S \\ &= \boldsymbol{B} \begin{pmatrix} [\boldsymbol{u}_1]_T & \cdots & [\boldsymbol{u}_k]_T \end{pmatrix} [\boldsymbol{w}]_S \end{aligned}$$

  So $\begin{pmatrix} [\boldsymbol{u}_1]_T & \cdots & [\boldsymbol{u}_k]_T \end{pmatrix} [\boldsymbol{w}]_S$ is the coordinate vector of $\boldsymbol{w}$ relative to the basis T; that is,
  - $\begin{pmatrix} [\boldsymbol{u}_1]_T & \cdots & [\boldsymbol{u}_k]_T \end{pmatrix} [\boldsymbol{w}]_S = [\boldsymbol{w}]_T$.

Transition Matrix

- **Definition.** Let V be a vector space, and
  - $S = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ and T be bases for V.

  $\begin{pmatrix} [\boldsymbol{u}_1]_T & \cdots & [\boldsymbol{u}_k]_T \end{pmatrix}$ is the **transition matrix** from S to T.
  - Denote it by $\boldsymbol{P}$. Then $\boldsymbol{P}[\boldsymbol{w}]_S = [\boldsymbol{w}]_T$ for all $\boldsymbol{w} \in V$.
- **Example.** Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.
  - $\boldsymbol{u}_1 = (1, 0, -1)$, $\boldsymbol{u}_2 = (0, -1, 0)$, $\boldsymbol{u}_3 = (1, 0, 2)$.
  - $\boldsymbol{v}_1 = (1, 1, 1)$, $\boldsymbol{v}_2 = (1, 1, 0)$, $\boldsymbol{v}_3 = (-1, 0, 0)$.

  View all vectors as column vectors.
  - $\left(\begin{array}{ccc|c|c|c} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{array} \right)$

$$\xrightarrow{\text{G.-J.E.}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array} \right) = \begin{pmatrix} \boldsymbol{I} \mid \boldsymbol{P} \end{pmatrix}.$$

Transition Matrix

- **Definition.** Let V be a vector space, and
  - $S = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ and T be bases for V.

  $\begin{pmatrix}[\boldsymbol{u}_1]_T & \cdots & [\boldsymbol{u}_k]_T\end{pmatrix}$ is the **transition matrix** from S to T.
  - Denote it by $\boldsymbol{P}$. Then $\boldsymbol{P}[\boldsymbol{w}]_S = [\boldsymbol{w}]_T$ for all $\boldsymbol{w} \in V$.
- **Example.** Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.
  - $\boldsymbol{u}_1 = (1, 0, -1)$, $\boldsymbol{u}_2 = (0, -1, 0)$, $\boldsymbol{u}_3 = (1, 0, 2)$.
  - $\boldsymbol{v}_1 = (1, 1, 1)$, $\boldsymbol{v}_2 = (1, 1, 0)$, $\boldsymbol{v}_3 = (-1, 0, 0)$.

  Transition matrix from S to T is $\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix}$.
  - Suppose $(\boldsymbol{w})_S = (2, -1, 2)$.
    - $[\boldsymbol{w}]_T = \boldsymbol{P}[\boldsymbol{w}]_S = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$. $(\boldsymbol{w})_T = (2, -1, -3)$.

Properties

- **Theorem.** Let S and T be bases for a vector space V.
  - Let $\boldsymbol{P}$ be the transition matrix from S to T. Then
    - $\boldsymbol{P}$ is an invertible matrix.
    - $\boldsymbol{P}^{-1}$ is the transition matrix from T to S.
- **Proof.** Let $\boldsymbol{Q}$ be the transition matrix from T to S.
  - It suffices to show that $\boldsymbol{QP} = \boldsymbol{I}$.

  Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$. Then
  - $[\boldsymbol{v}_1]_S = \boldsymbol{e}_1, [\boldsymbol{v}_2]_S = \boldsymbol{e}_2, \ldots, [\boldsymbol{v}_k]_S = \boldsymbol{e}_k$.

$$\begin{aligned}
\boldsymbol{QP} = \boldsymbol{QPI} = \boldsymbol{QP}\begin{pmatrix}\boldsymbol{e}_1 & \cdots & \boldsymbol{e}_k\end{pmatrix} &= \begin{pmatrix}\boldsymbol{QP}\boldsymbol{e}_1 & \cdots \boldsymbol{QP}\boldsymbol{e}_k\end{pmatrix} \\
&= \begin{pmatrix}\boldsymbol{QP}[\boldsymbol{v}_1]_S & \cdots & \boldsymbol{QP}[\boldsymbol{v}_k]_S\end{pmatrix} \\
&= \begin{pmatrix}\boldsymbol{Q}[\boldsymbol{v}_1]_T & \cdots & \boldsymbol{Q}[\boldsymbol{v}_k]_T\end{pmatrix} \\
&= \begin{pmatrix}[\boldsymbol{v}_1]_S & \cdots & [\boldsymbol{v}_k]_S\end{pmatrix} \\
&= \begin{pmatrix}\boldsymbol{e}_1 & \cdots & \boldsymbol{e}_k\end{pmatrix} = \boldsymbol{I}.
\end{aligned}$$

Examples

- Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, $\boldsymbol{u}_1 = (1,1)$, $\boldsymbol{u}_2 = (1,-1)$.

  $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, $\boldsymbol{v}_1 = (1,0)$, $\boldsymbol{v}_2 = (1,1)$.

  Note that both S and T are bases for $\mathbb{R}^2$.

  - $\left(\begin{array}{cc|c|c} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{array}\right) \xrightarrow{R_1+(-1)R_2} \left(\begin{array}{cc|c|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right)$
    - Transition matrix from S to T: $\boldsymbol{P} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$.
  - $\left(\begin{array}{cc|c|c} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{array}\right) \xrightarrow{\text{G.-J.E}} \left(\begin{array}{cc|c|c} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right)$
    - Transition matrix from T to S: $\boldsymbol{Q} = \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$.
  - One checks easily that $\boldsymbol{PQ} = \boldsymbol{QP} = \boldsymbol{I}$.

Examples

- Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.
  - $S = \{(1,0,-1), (0,-1,0), (1,0,2)\}$;
  - $T = \{(1,1,1), (1,1,0), (-1,0,0)\}$.

  We have computed the transition matrix from S to T:

  - $\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix}$.

  Then the transition matrix from T to S is

  - $\boldsymbol{P}^{-1} = \cdots = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -1 & -1 & 0 \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$.

  For any $\boldsymbol{w} \in \mathbb{R}^3 = \mathrm{span}(S) = \mathrm{span}(T)$,

  - $\boldsymbol{P}[\boldsymbol{w}]_S = [\boldsymbol{w}]_T$ and $\boldsymbol{P}^{-1}[\boldsymbol{w}]_T = [\boldsymbol{w}]_S$.