CS2040S
Data Structures and Algorithms

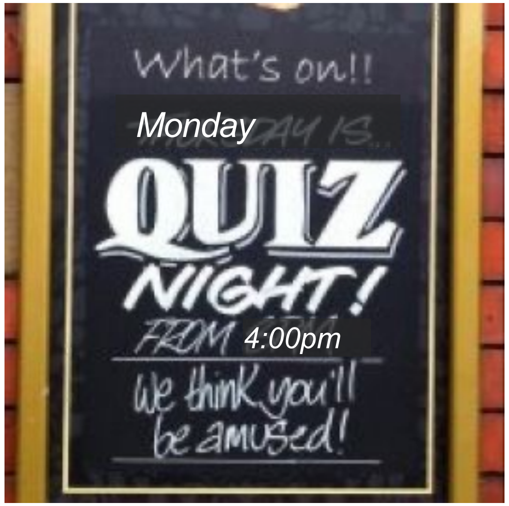
Union-Find

Today

Disjoint Set Data Structure

- Problem: Dynamic Connectivity
- Algorithm: Union-Find
- Applications

Midterm



Monday March 6: 4:00pm

- Location: MPSH 2A and MPSH 2B

Bring to quiz:

- One double-sided sheet of paper with any notes you like.
- Pens/pencils.
- You may not use anything else. (No calculators, no phones, etc.)

CS2040S Treasure Island
Hall of Fame (Fastest)

Zhang Jikun	4,206
Lim Zi Jia	4,215
Zhang Xiaorui	4,271
Lee Wei Ming	4,376
Gonzales Andres Rico III Marcilla	4,387
Andrew Yapp Wei Rong	4,475
Kieron Seven Lee Jun Wei	4,576
Luo Jiale	4,583
Maximilliano Utomo Quok	4,799
Teoh Tze Tzun	4,941

CS2040S Treasure Island
Hall of Fame (Cheapest)

Xu Shuyao	1,966,393
Zhang Jikun	1,994,837
P J Anthony	2,035,605
Andrew Yapp Wei Rong	2,125,446
Wang Ziwen	2,261,665
Luo Jiale	2,281,669
Fong Yi Yong Calvin	2,474,292
Doan Quoc Thinh	2,572,176
Lim Zi Jia	2,590,322
Zhang Xiaorui	2,605,167

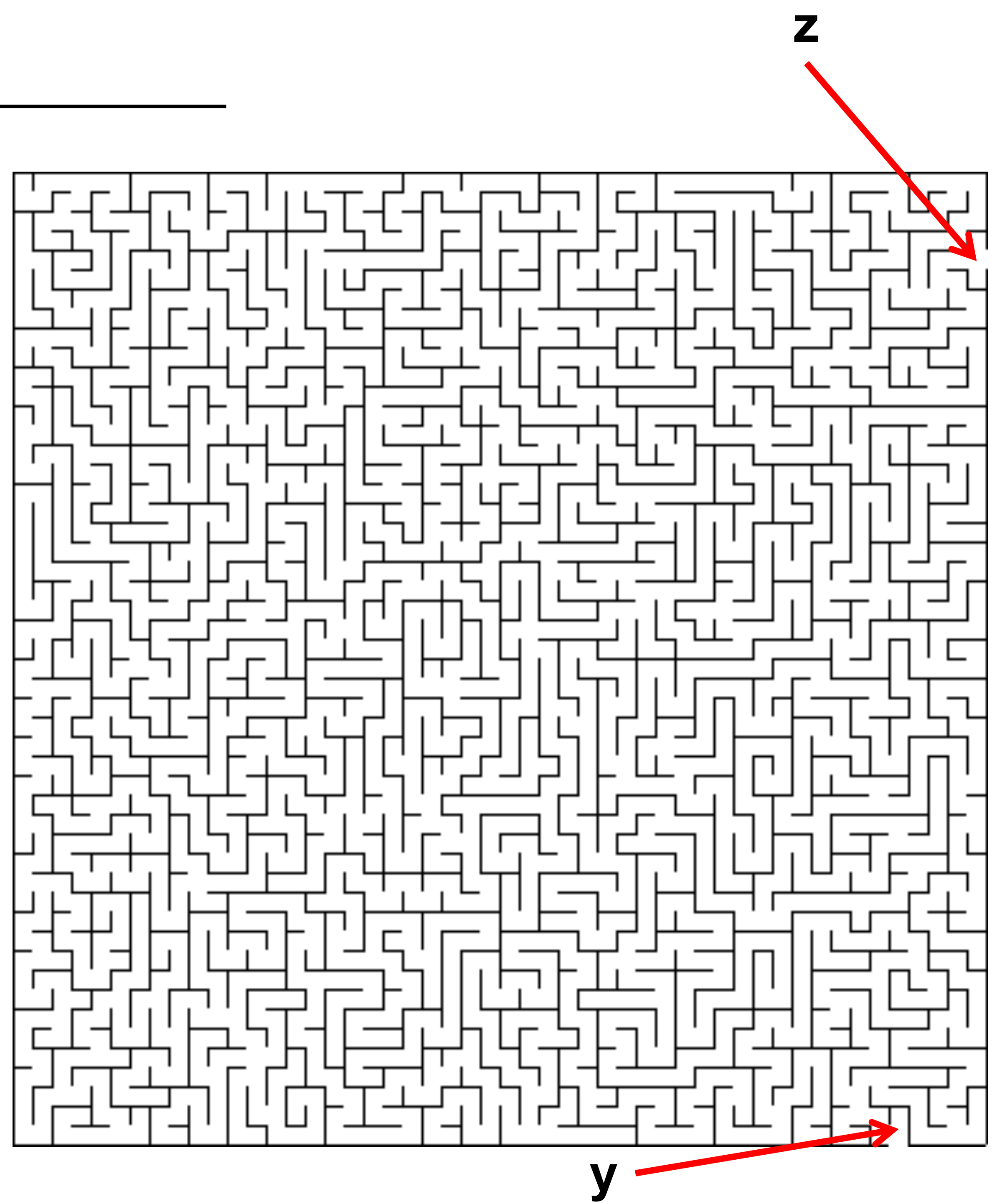
Today

Disjoint Set Data Structure

- Problem: Dynamic Connectivity
- Algorithm: Union-Find
- Applications

Mazes

Is there any route from y to z?



How do you pre-process?

Two steps:

1. Pre-process maze
2. Answer queries

isConnected(y,z) :

Returns true if there is a path from A to B, and false otherwise.

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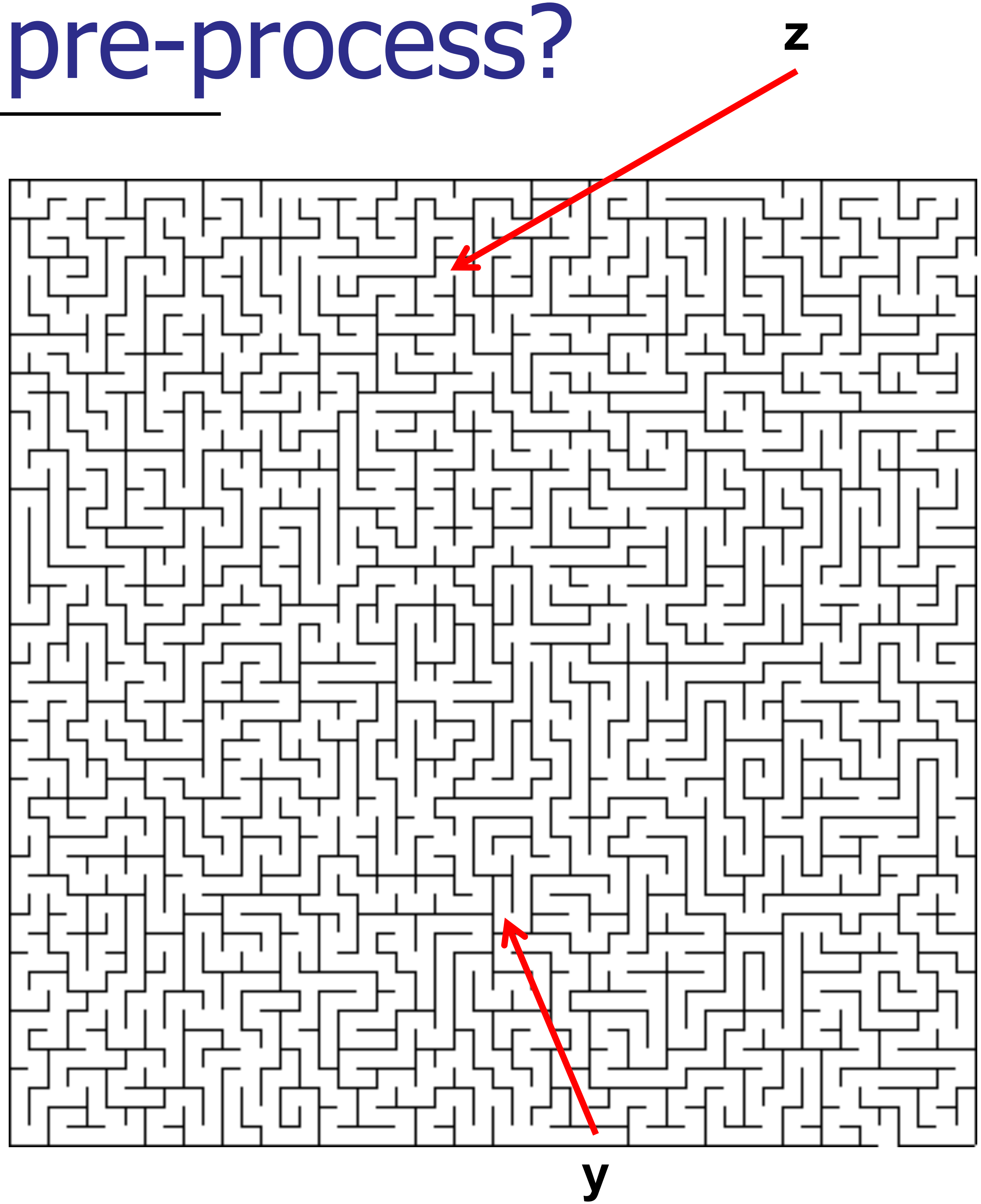
is open



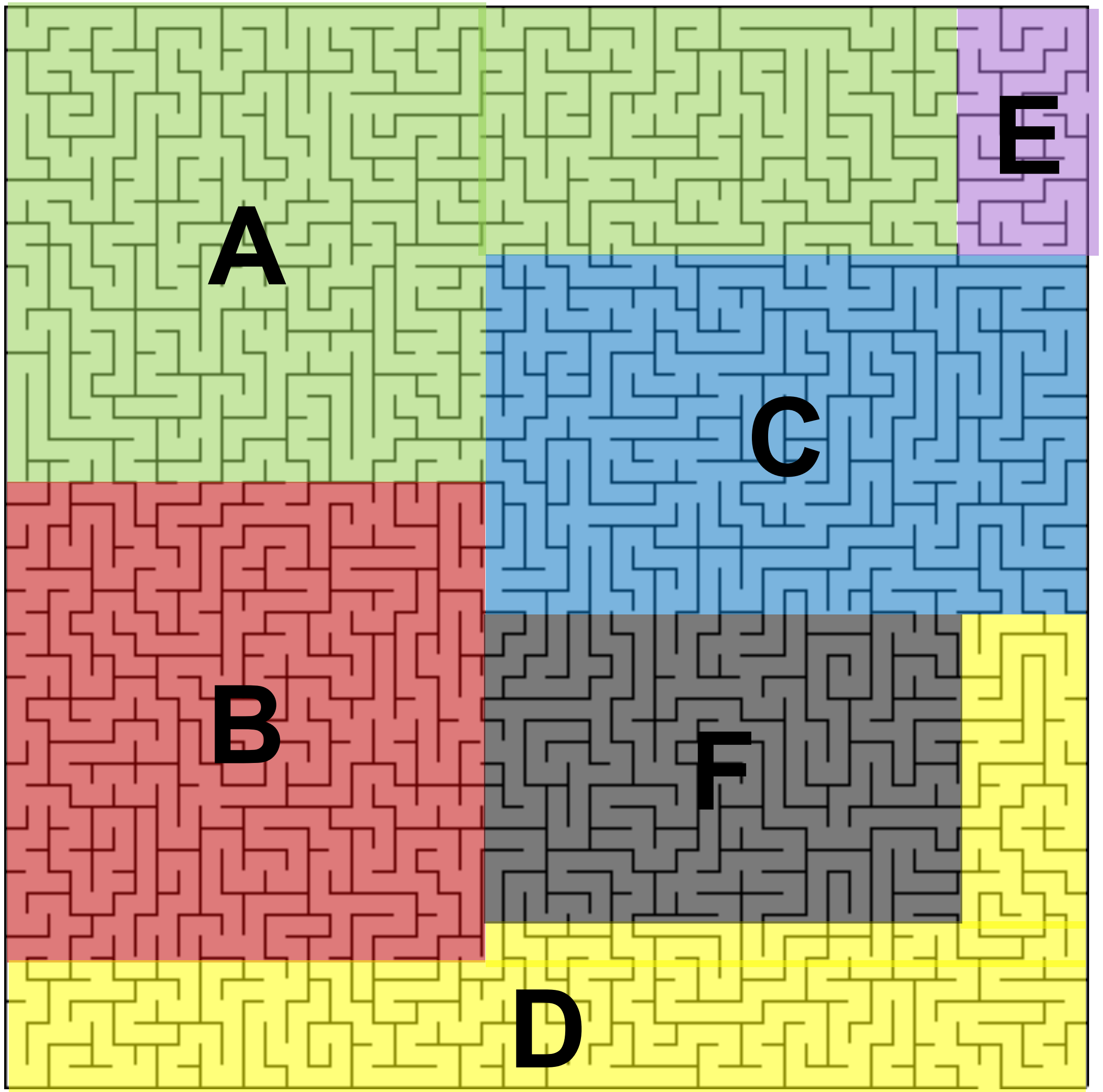
Mazes

Preprocess:

Identify connected locations. Label each location with its location id.

isConnected(y,z) :

Returns true if A and B have the same label.



Mazes

Preprocess:

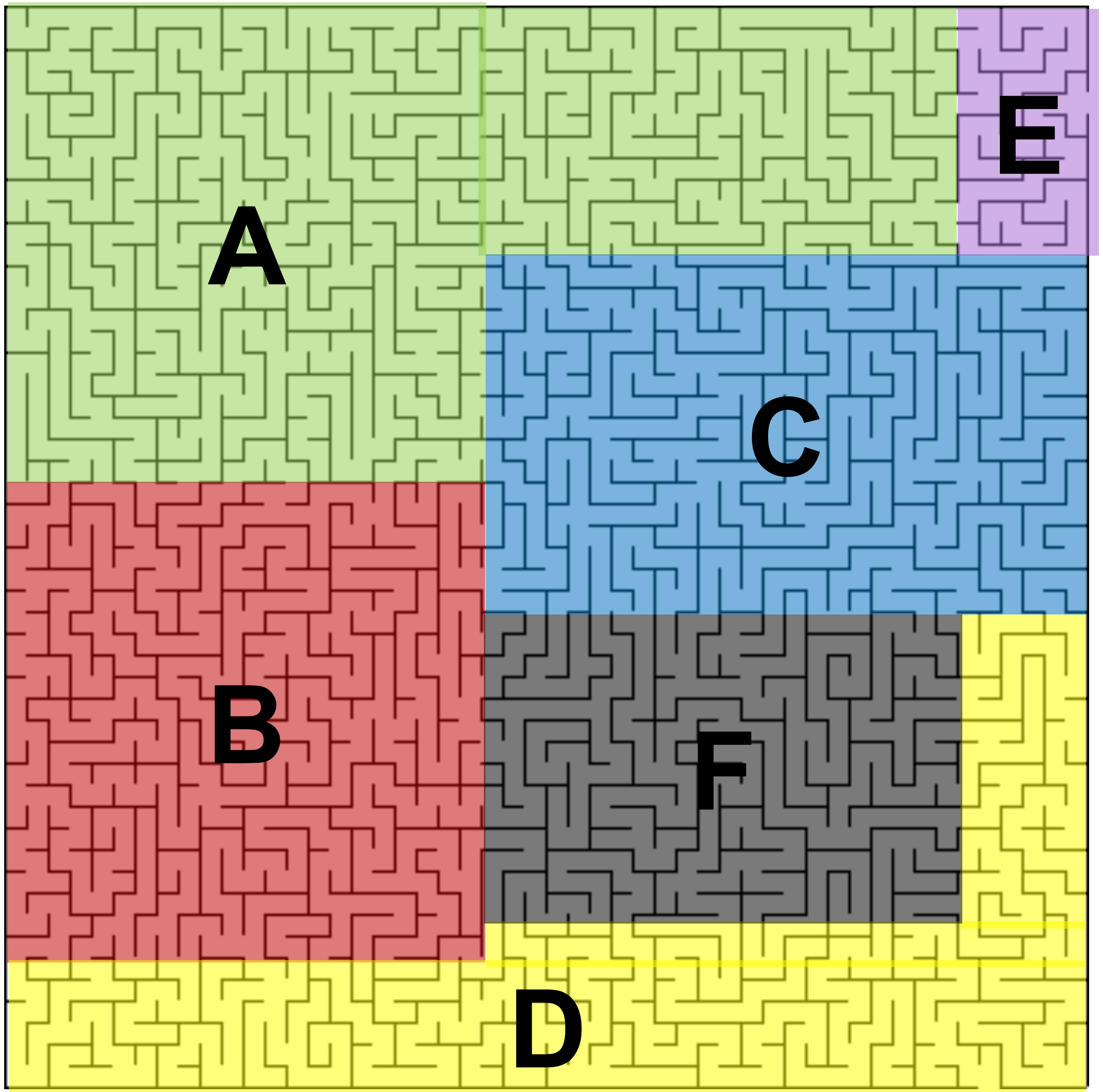
Prepare to answer queries.

destroyWall(x):

Remove walls from the maze using your superpowers.

isConnected(y, z):

Answer connectivity queries.



Mazes

Preprocess:

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destroyWall(x):

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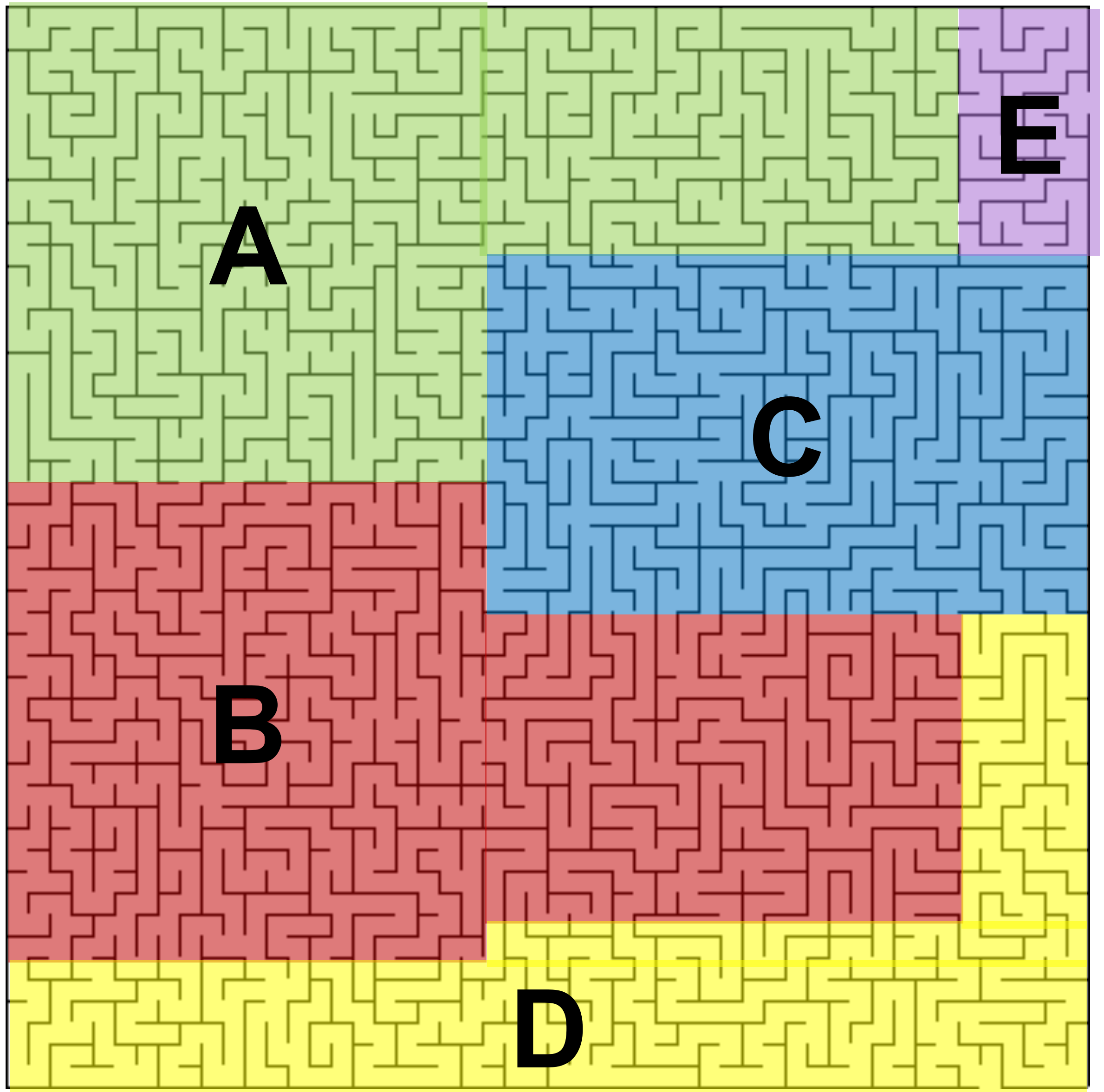
isConnected(y, z):

Answer connectivity queries.

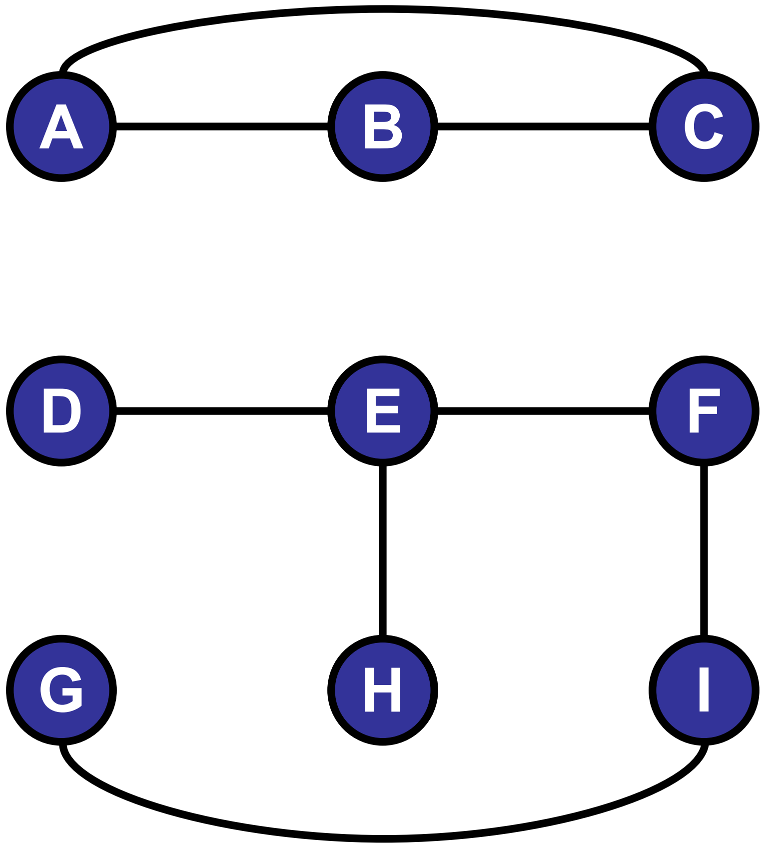


Dynamic Connectivity

Given a set of objects:

- Union: connect two objects
- Find: is there a path connecting the two objects?

union(E, F)
union(I, G)
union(D, E)
union(B, A)
find(G, D) = **false**
find(D, F) = **true**
union(B, C)
union(H, E)
union(A, C)
union(F, I)
find(G, D) = **true**



Dynamic Connectivity

Given a set of objects:

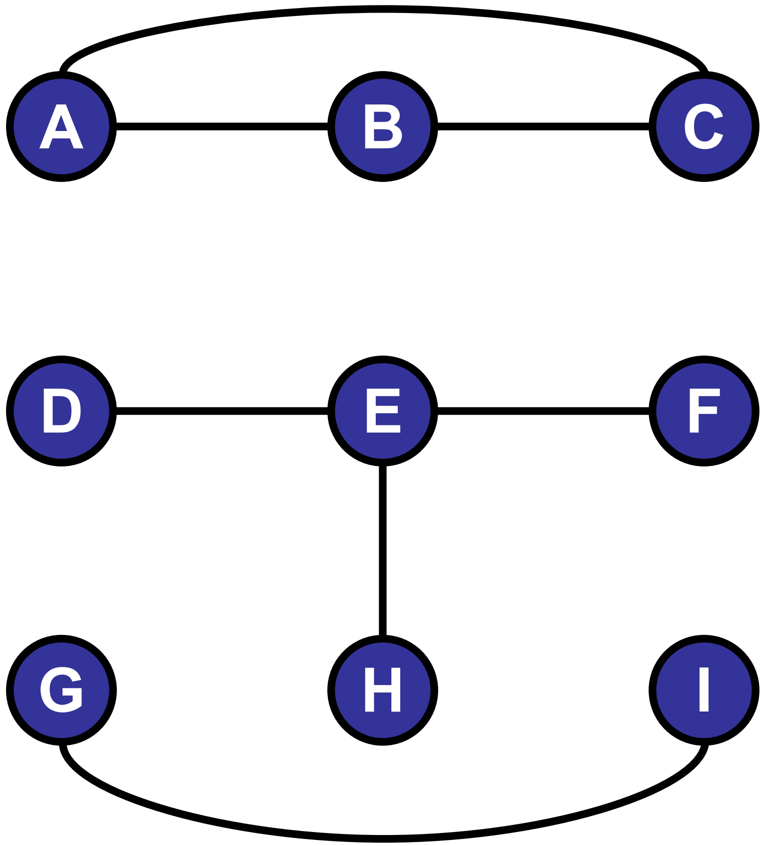
- Union: connect two objects
- Find: is there a path connecting the two objects?

Transitivity

- If **p** is connected to **q** and
  if **q** is connected to **r**,
  then **p** is connected to **r**.

Connected components:

- Maximal set of mutually
  connected objects.



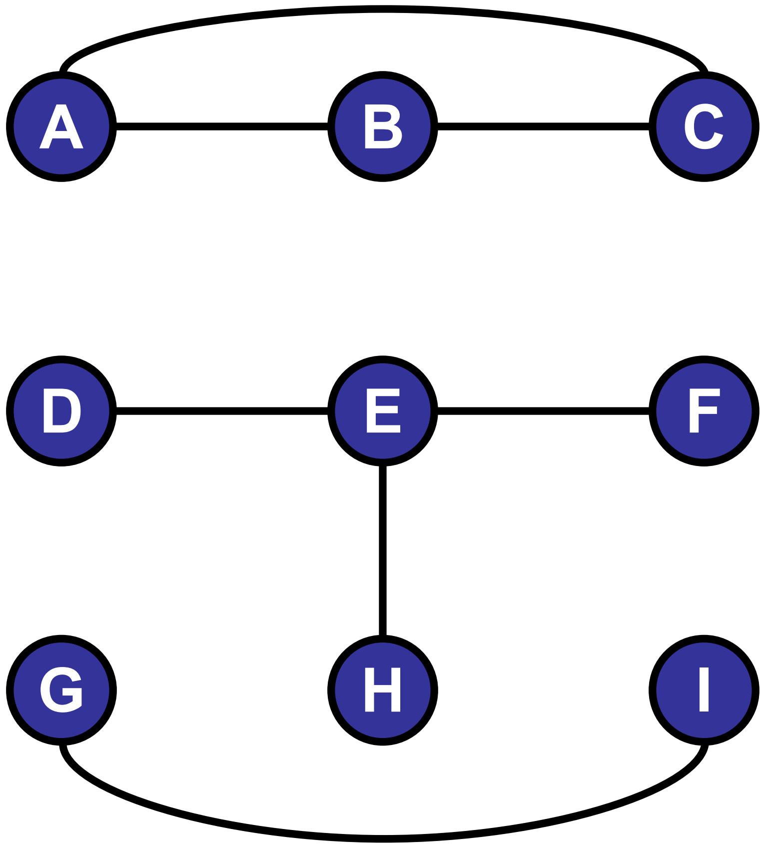
Dynamic Connectivity

Given a set of objects:

- Union: connect two objects
- Find: is there a path connecting the two objects?

Maintain sets of nodes:

{A, B, C}
{D, E, F, H}
{G, I}



Dynamic Connectivity

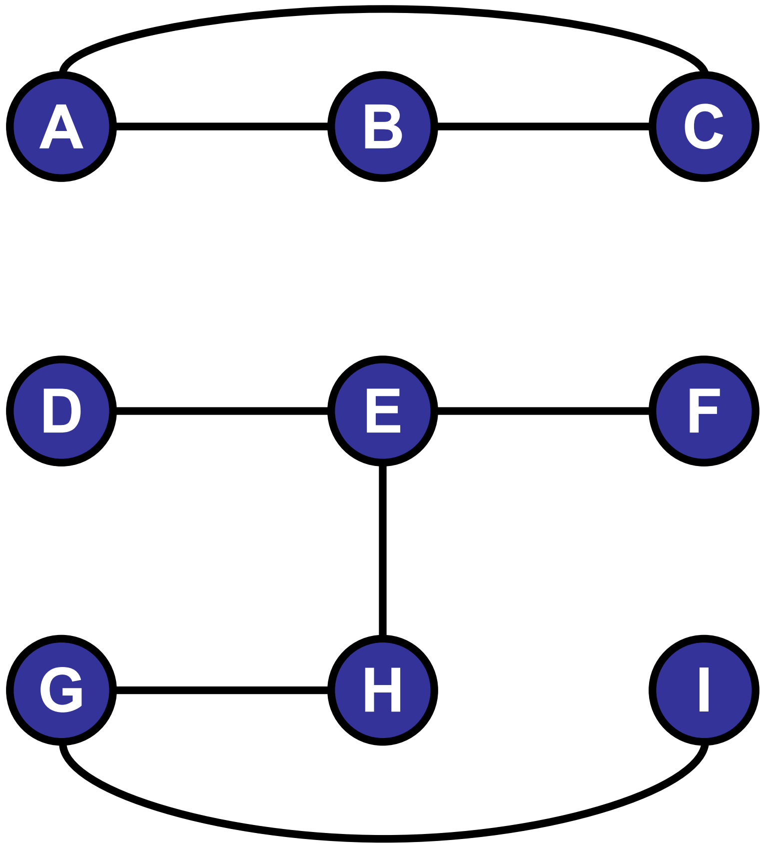
Given a set of objects:

- Union: connect two objects
- Find: is there a path connecting the two objects?

Maintain sets of nodes:

{A, B, C}

{D, E, F, H, G, I}



Abstract Data Type

Disjoint Set (Union-Find)

```
public interface  DisjointSet<Key>
---------------------------------------------------------------------------
                  DisjointSet(int N)    constructor: N objects
         boolean  find(Key p, Key q)    are p and q in the
                                        same set?
            void  union(Key p, Key q)   replace sets containing
                                        p and q with their union
```

Roadmap

Disjoint Set Data Structure

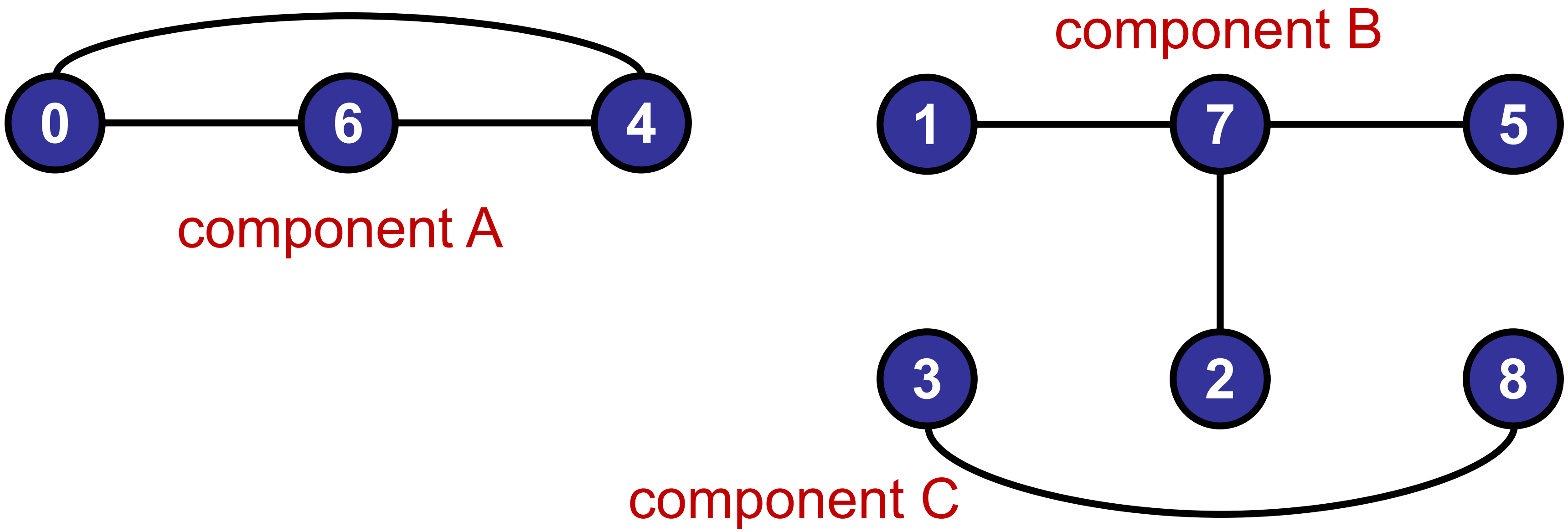
- Problem: Dynamic Connectivity
- Algorithm: Quick-Find
- Algorithm: Quick-Union
- Optimizations
- Applications

Quick Find

Data structure:

- Array: componentId
- Two objects are connected if they have the same component identifier.

object	0	1	2	3	4	5	6	7	8
component identifier	A	B	B	C	A	B	A	B	C



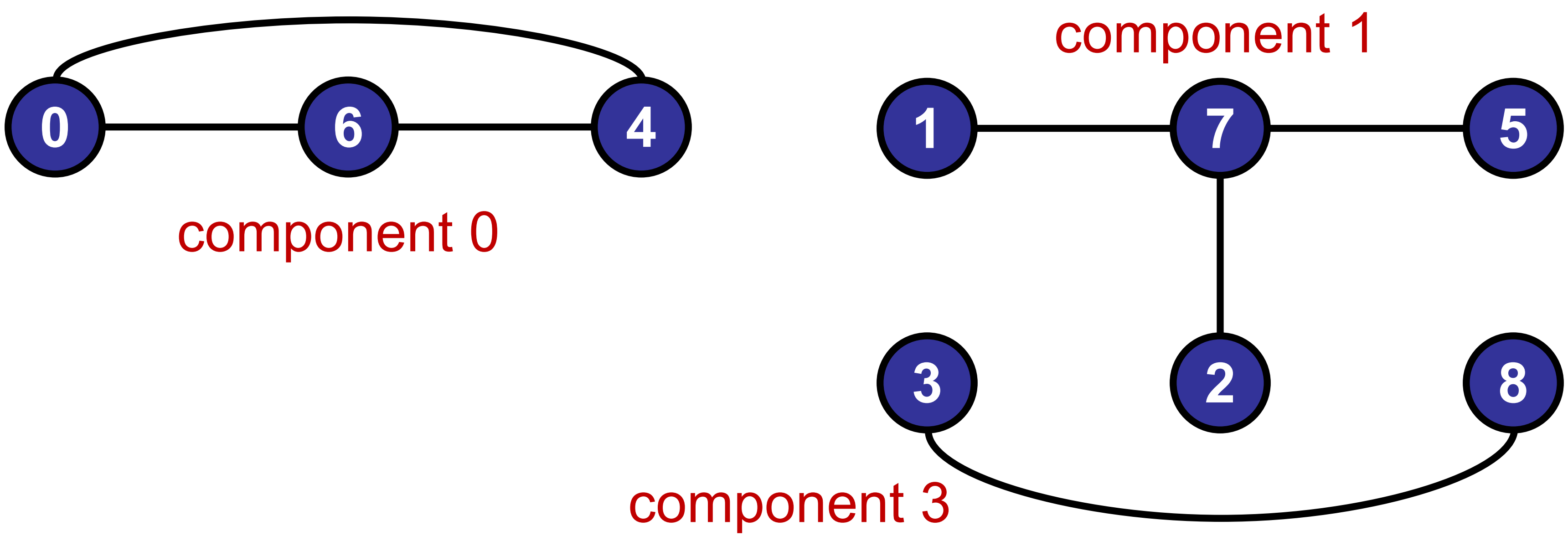
Quick Find

Assume objects are integers

Data structure:

- Integer array: int[] componentId
- Two objects are connected if they have the same component identifier.

object	0	1	2	3	4	5	6	7	8
component identifier	0	1	1	3	0	1	0	1	3

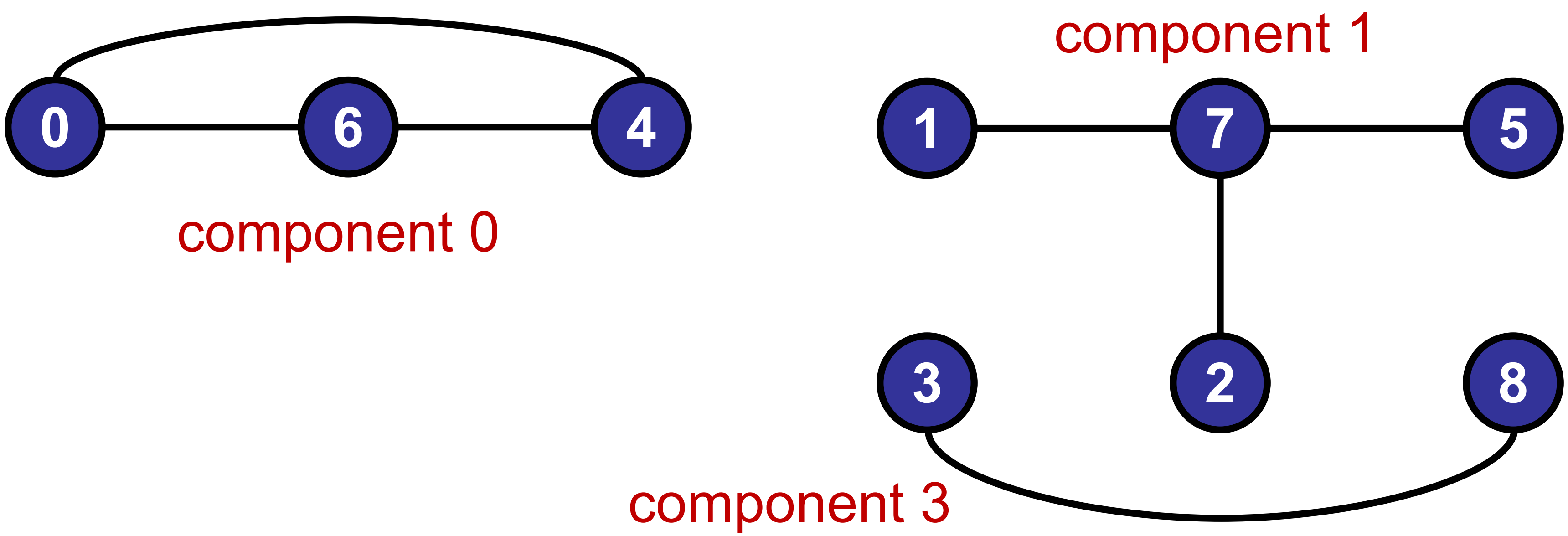


Quick Find

Data structure:

- Integer array: int[] componentId
- Two objects are connected if they have the same component identifier.

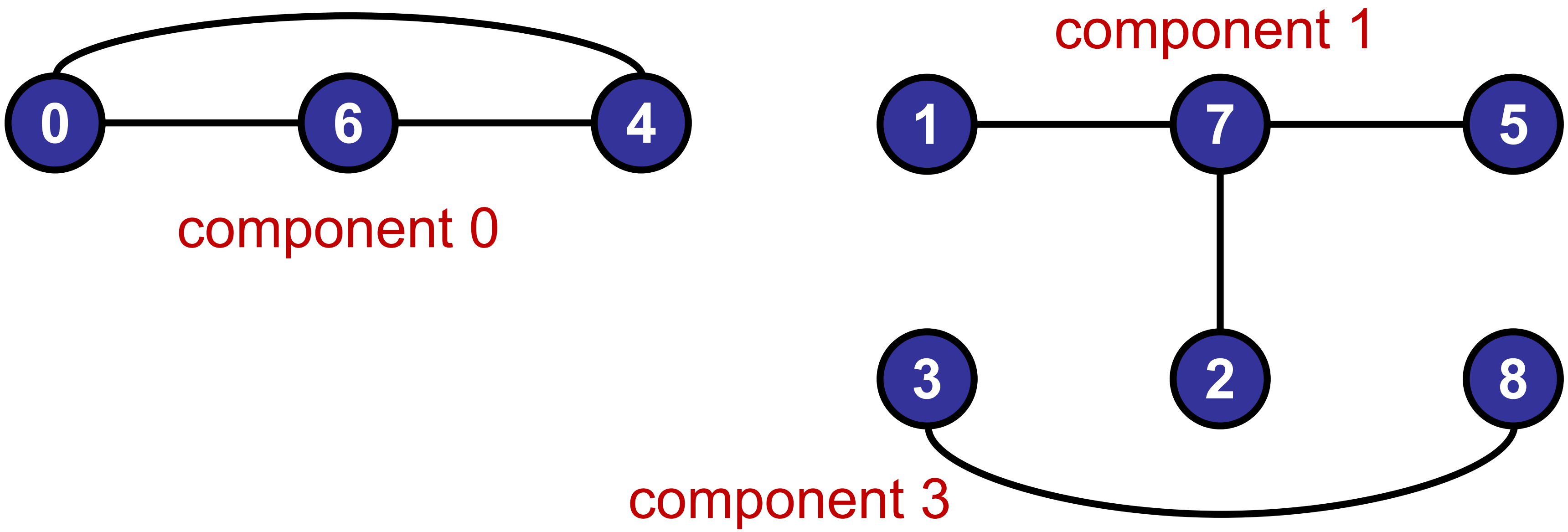
object	0	1	2	3	4	5	6	7	8
component identifier	0	1	1	3	0	1	0	1	3



Quick Find

```
find(int p, int q)
  return(componentId[p] == componentId[q]);
```

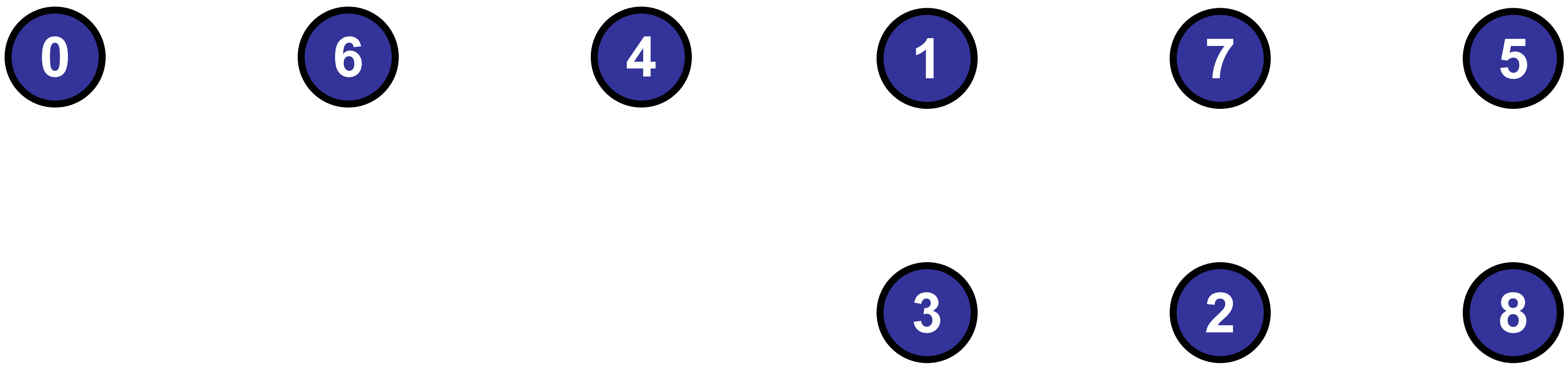
object	0	1	2	3	4	5	6	7	8
component identifier	0	1	1	3	0	1	0	1	3



Quick Find

Initial state of data structure:

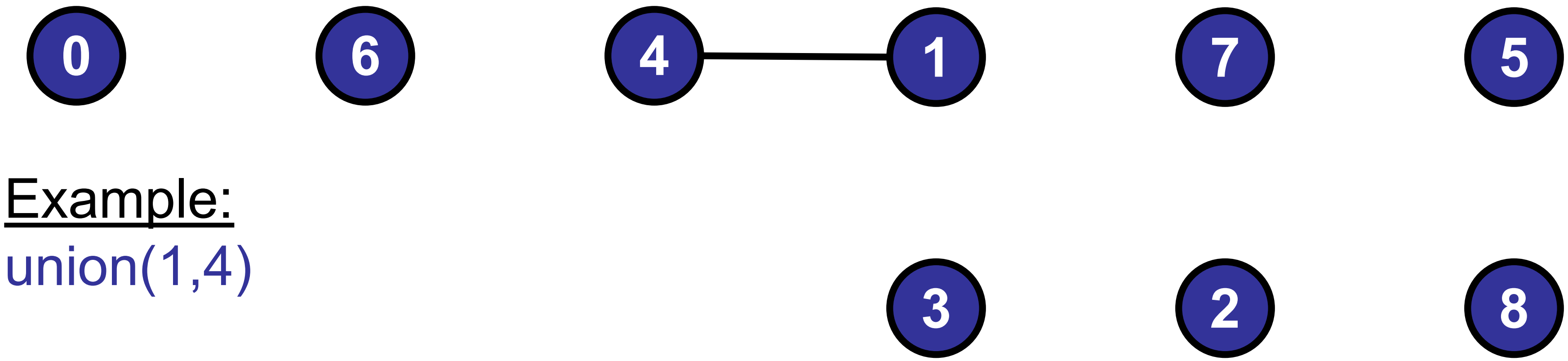
object	0	1	2	3	4	5	6	7	8
component identifier	0	1	2	3	4	5	6	7	8



Quick Find

```
union(int p, int q)
   updateComponent = componentId[q]
   for (int i=0; i<componentId.length; i++)
           if (componentId[i] == updateComponent)
               componentId[i] = componentId[p];
```

object	0	1	2	3	4	5	6	7	8
component identifier	0	1	2	3	1	5	6	7	8

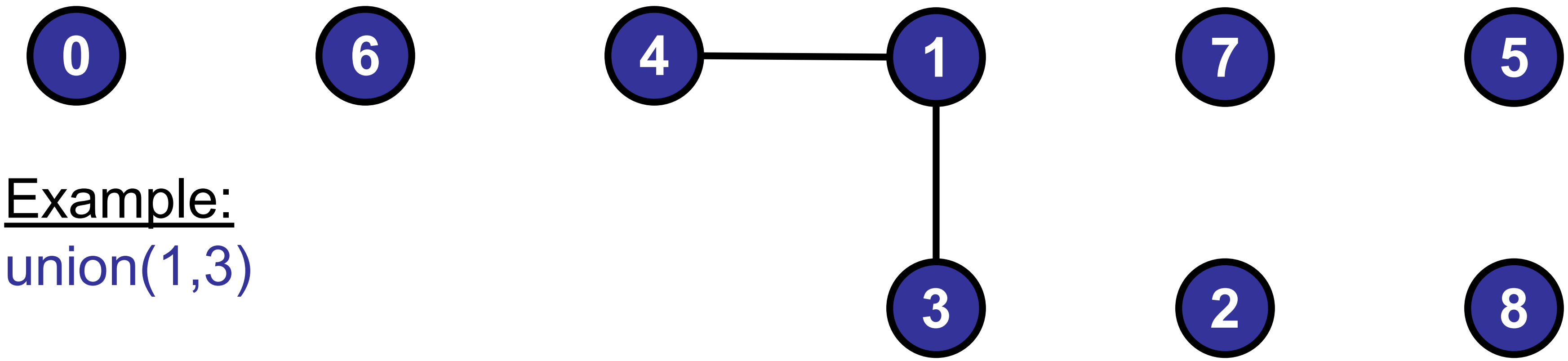


Example:
union(1,4)

Quick Find

```
union(int p, int q)
  updateComponent = componentId[q]
  for (int i=0; i<componentId.length; i++)
        if (componentId[i] == updateComponent)
           componentId[i] = componentId[p];
```

object	0	1	2	3	4	5	6	7	8
component identifier	0	1	2	1	1	5	6	7	8

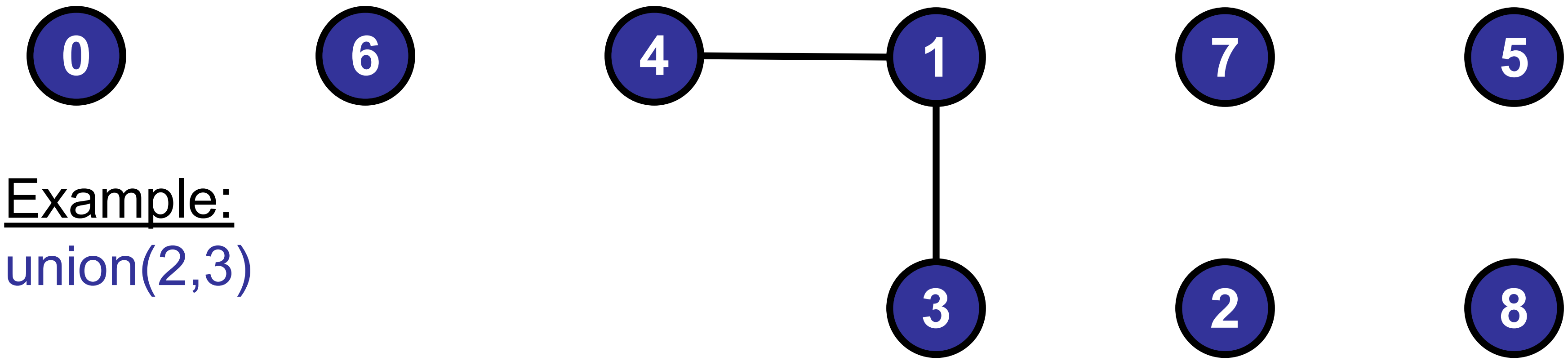


Example:
union(1,3)

Quick Find

```
union(int p, int q)
  updateComponent = componentId[q]
  for (int i=0; i<componentId.length; i++)
        if (componentId[i] == updateComponent)
           componentId[i] = componentId[p];
```

object	0	1	2	3	4	5	6	7	8
component identifier	0	1	2	1	1	5	6	7	8

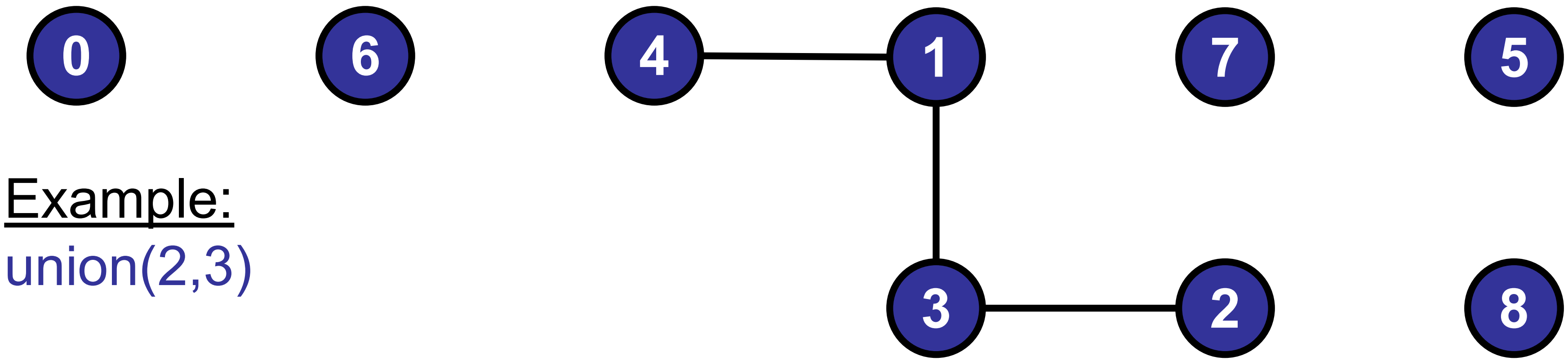


Example:
union(2,3)

Quick Find

```
union(int p, int q)
   updateComponent = componentId[q]
   for (int i=0; i<componentId.length; i++)
           if (componentId[i] == updateComponent)
               componentId[i] = componentId[p];
```

object	0	1	2	3	4	5	6	7	8
component identifier	0	2	2	2	2	5	6	7	8

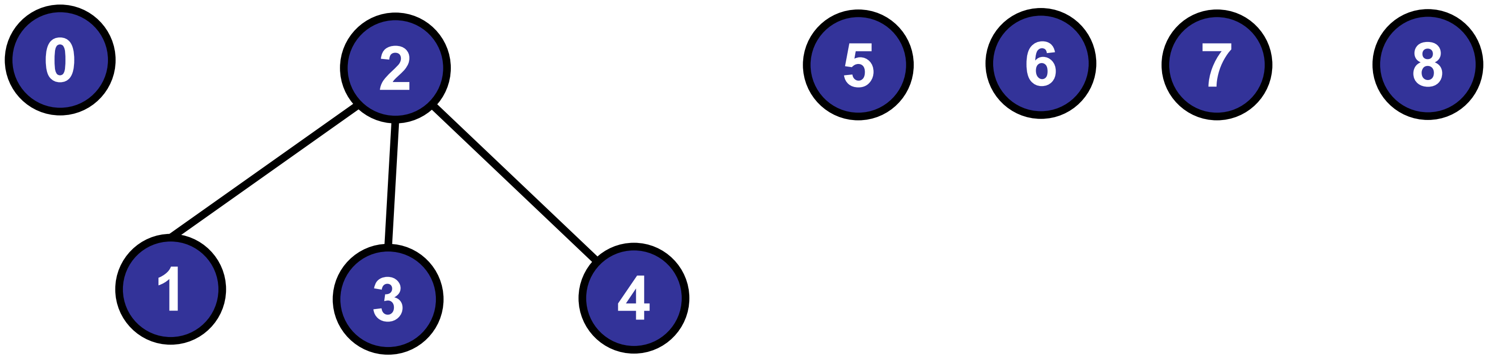


Example:
union(2,3)

Quick Find

Flat trees:

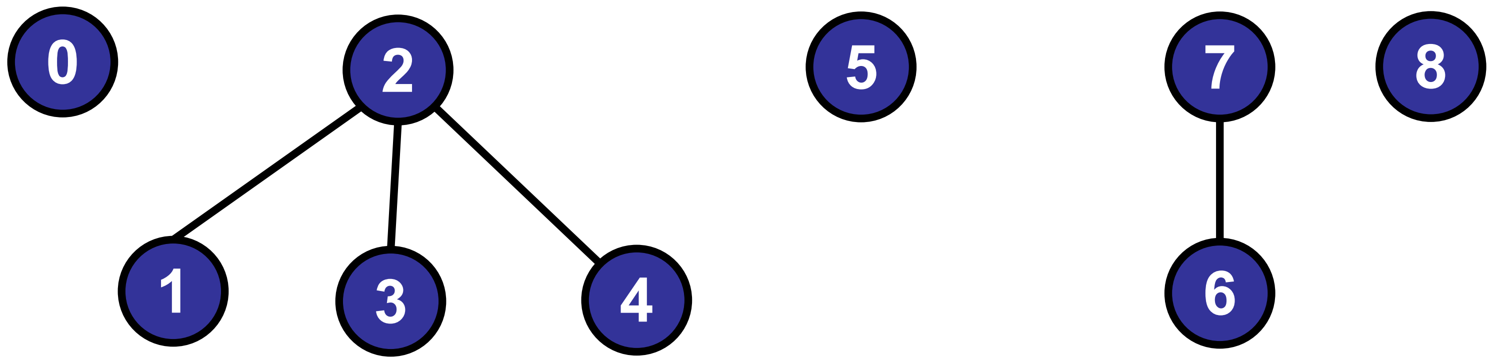
object	0	1	2	3	4	5	6	7	8
component identifier	0	2	2	2	2	5	6	7	8



Quick Find

Flat trees:

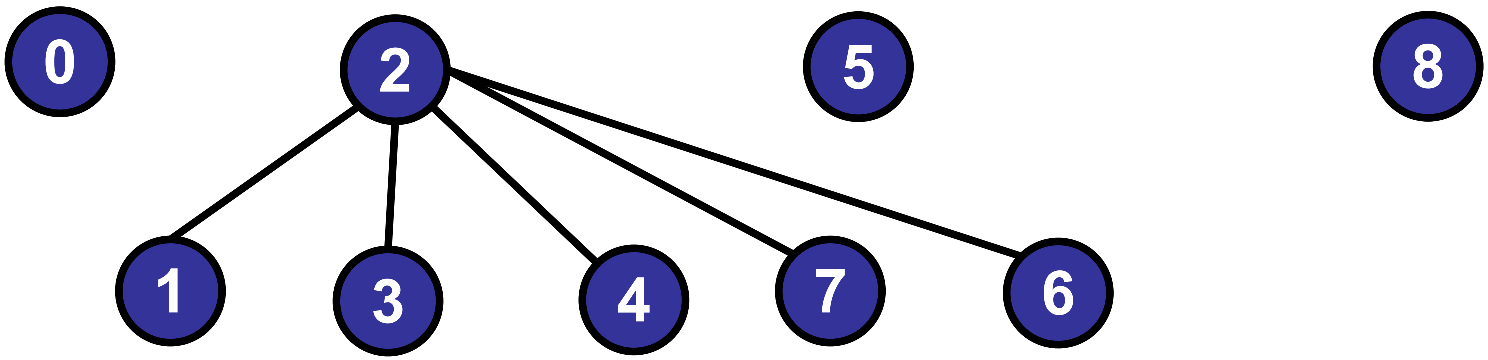
object	0	1	2	3	4	5	6	7	8
component identifier	0	2	2	2	2	5	7	7	8



Quick Find

`Flat trees:`

object	0	1	2	3	4	5	6	7	8
component identifier	0	2	2	2	2	5	2	2	8



Running time of (Find, Union):

1. O(1), O(1)
2. O(1), O(n)
3. O(n), O(1)
4. O(n), O(n)
5. O(log n), O(log n)
6. None of the above.

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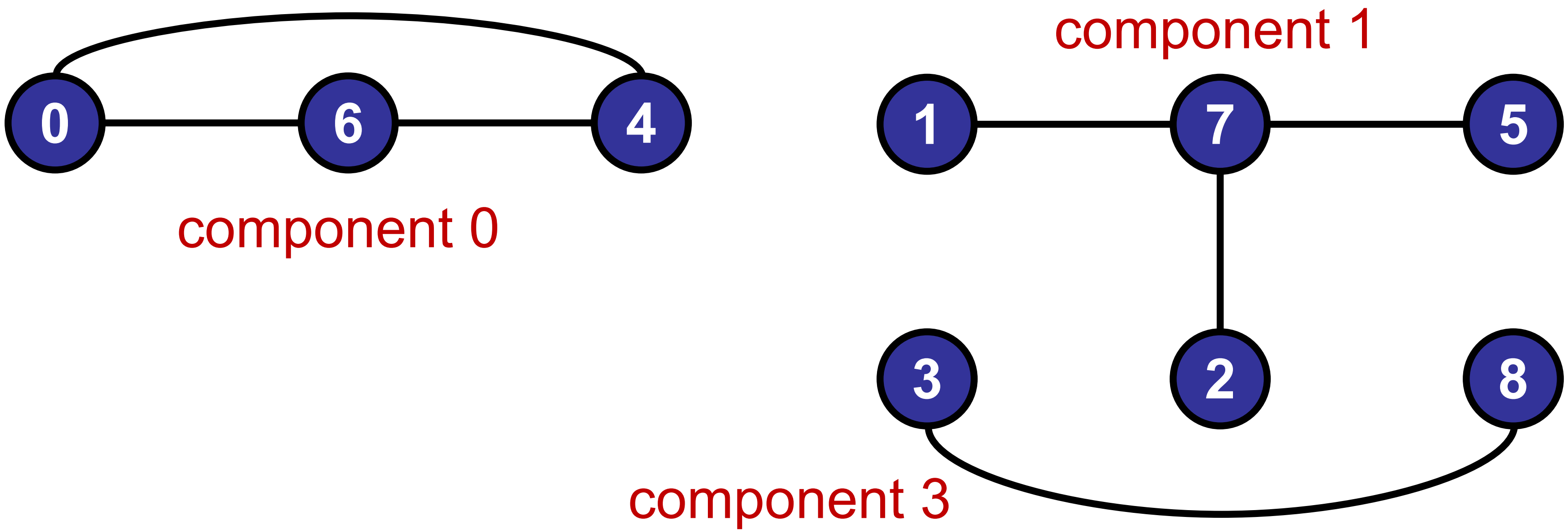
is open

Quick Find

```
find(int p, int q)
  return(componentId[p] == componentId[q]);
```

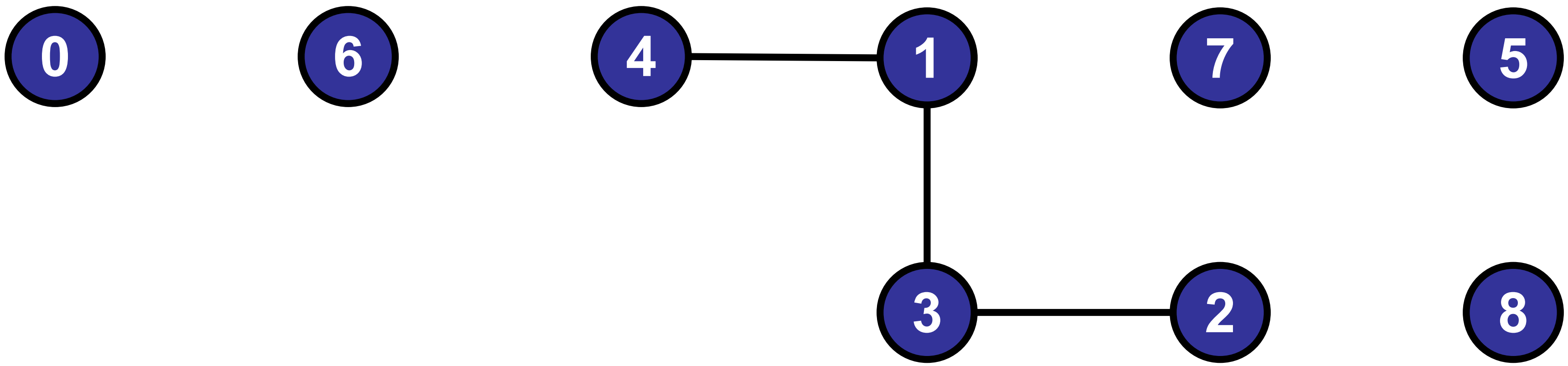
object	0	1	2	3	4	5	6	7	8
component identifier	0	1	1	3	0	1	0	1	3



Quick Find

```
union(int p, int q)
  updateComponent = componentId[q]
  for (int i=0; i<componentId.length; i++)
        if (componentId[i] == updateComponent)
           componentId[i] = componentId[p];
```

object	0	1	2	3	4	5	6	7	8
component identifier	0	2	2	2	2	5	6	7	8



Roadmap

Disjoint Set Data Structure

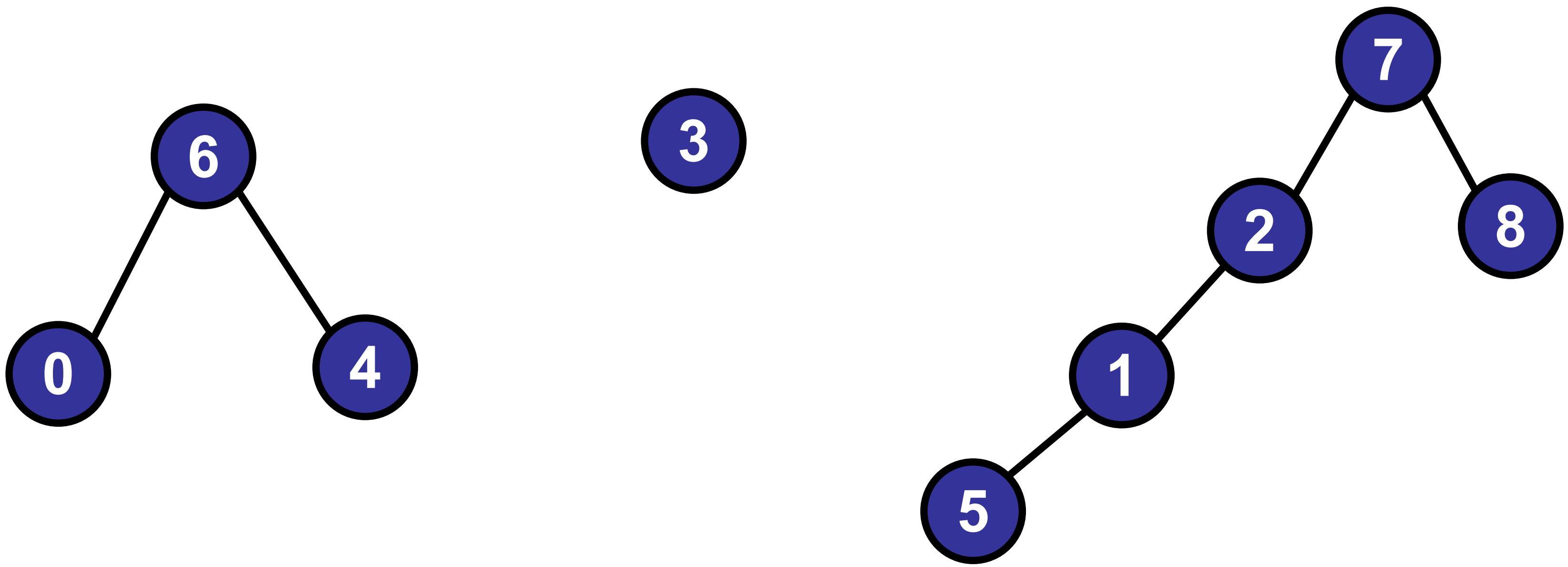
- Problem: Dynamic Connectivity
- Algorithm: Quick-Find
- Algorithm: Quick-Union
- Optimizations
- Applications

Quick Union

Data structure:

- Integer array: int[] parent
- Two objects are connected if they are part of the same tree.

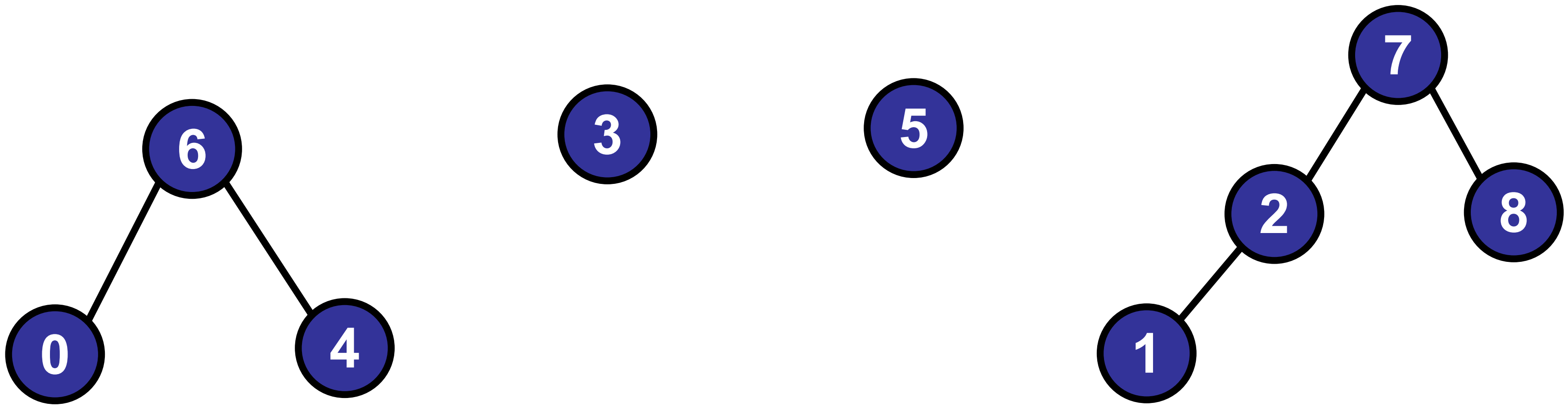
object	0	1	2	3	4	5	6	7	8
parent	6	2	7	3	6	1	6	7	7



Quick Union

```
find(int p, int q)
   while (parent[p] != p) p = parent[p];
   while (parent[q] != q) q = parent[q];
   return (p == q);
```

object	0	1	2	3	4	5	6	7	8
parent	6	2	7	3	6	1	6	7	7

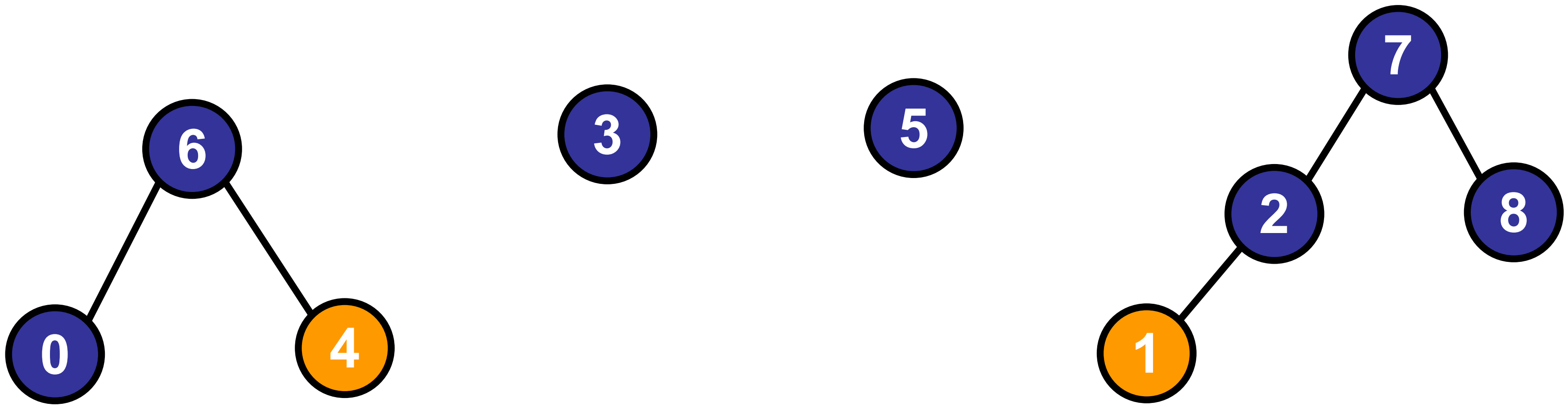


Quick Union

Example: find(4, 1)

4 → 6 → 6;

object	0	1	2	3	4	5	6	7	8
parent	6	2	7	3	6	1	6	7	7



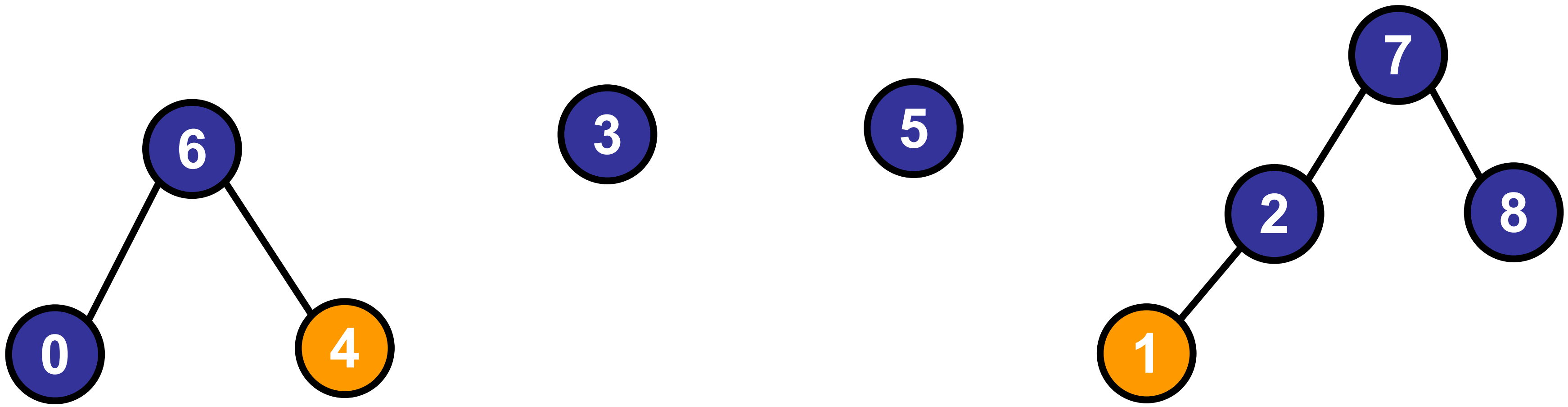
Quick Union

Example: `find(4, 1)`

4 → 6 → 6

1 → 2 → 7 → 7

object	0	1	2	3	4	5	6	7	8
parent	6	2	7	3	6	1	6	7	7

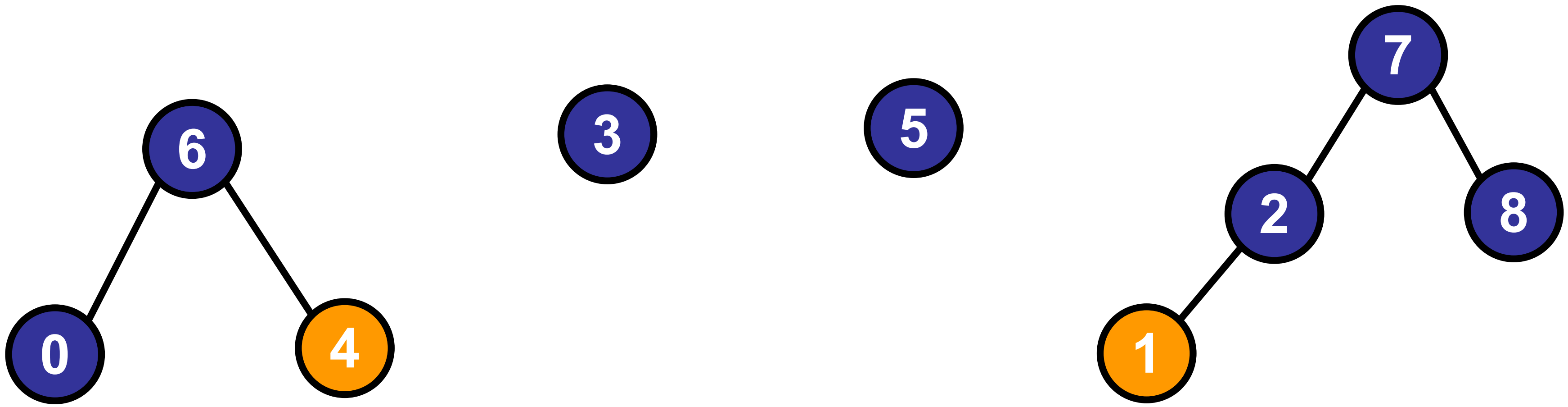


Quick Union

Example: `find(4, 1)`

```
4 → 6 → 6
1 → 2 → 7 → 7
return (6 == 7) → false
```

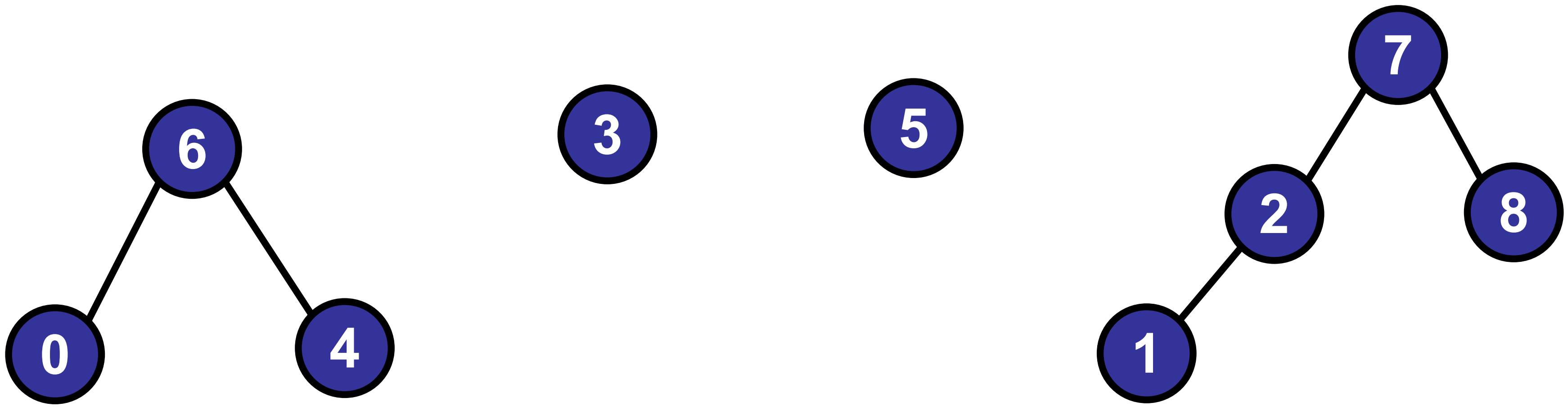
object	0	1	2	3	4	5	6	7	8
parent	6	2	7	3	6	1	6	7	7



Quick Union

```
find(int p, int q)
  while (parent[p] != p) p = parent[p];
  while (parent[q] != q) q =parent[q];
  return (p == q);
```

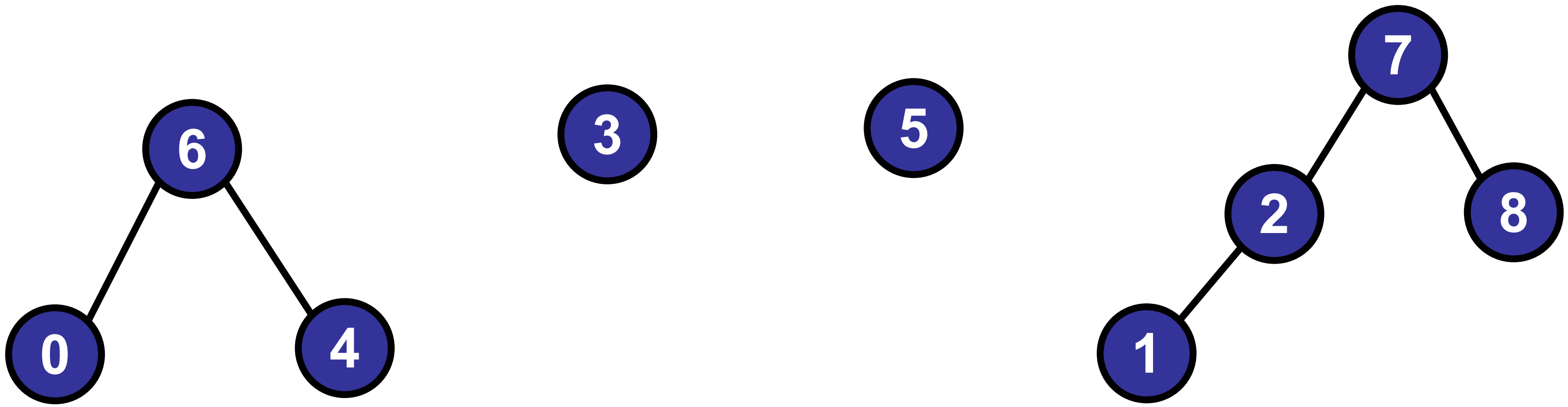
object	0	1	2	3	4	5	6	7	8
parent	6	2	7	3	6	1	6	7	7



Quick Union

```
union(int p, int q)
  while (parent[p] != p) p = parent[p];
  while (parent[q] != q) q = parent[q];
  parent[p] = q;
```

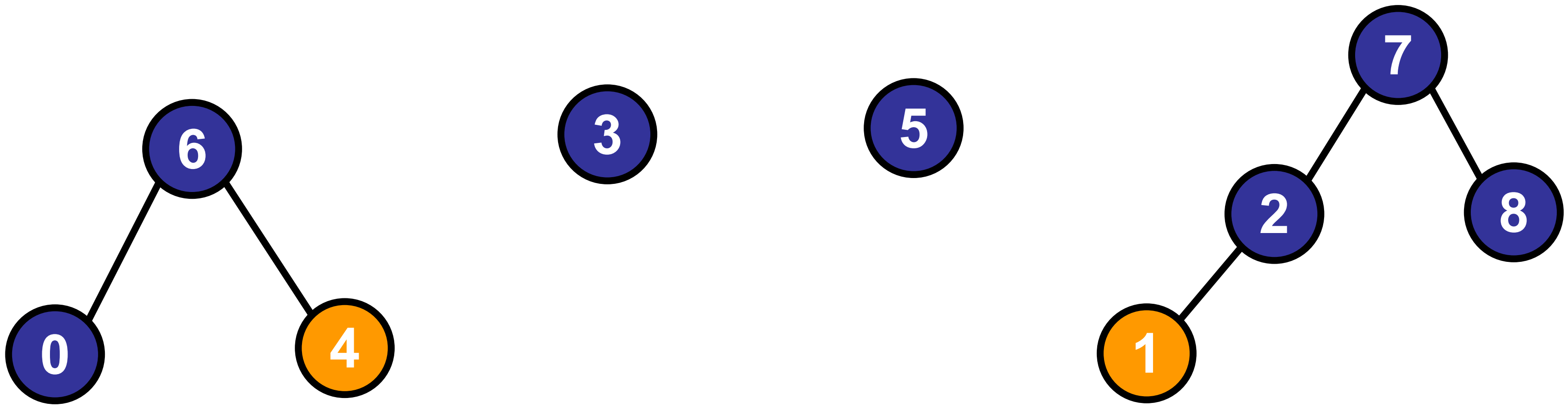
object	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
parent	**6**	**2**	**7**	**3**	**6**	**1**	**6**	**7**	**7**



Quick Union

`Example: union(1, 4)`

object	0	1	2	3	4	5	6	7	8
parent	6	2	7	3	6	1	6	7	7



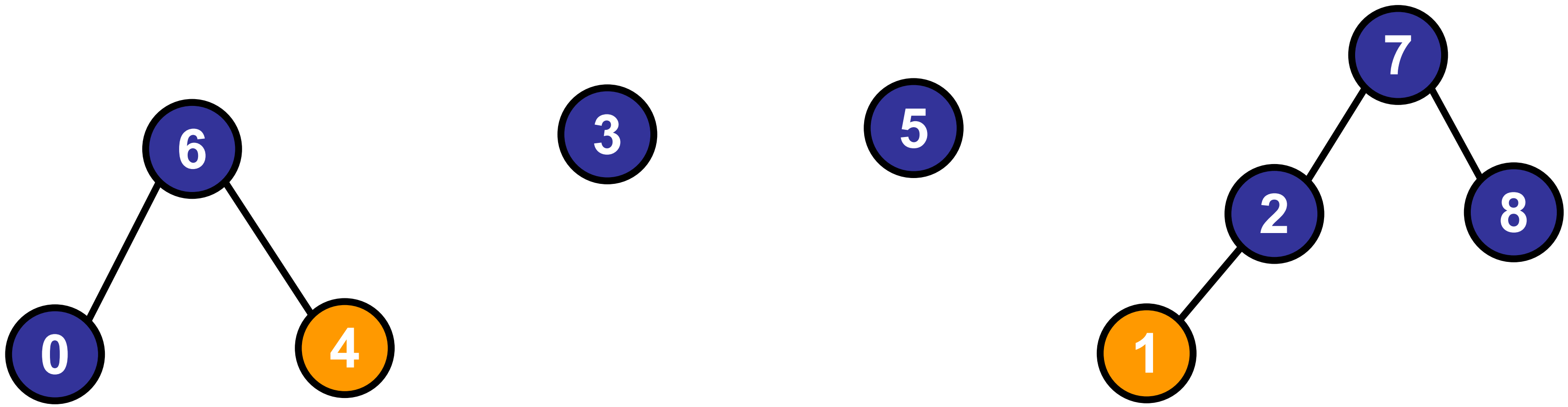
Quick Union

Example: `union(1, 4)`

4 → 6 → **6**

1 → 2 → 7 → **7**

object	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
parent	**6**	**2**	**7**	**3**	**6**	**1**	**6**	**7**	**7**

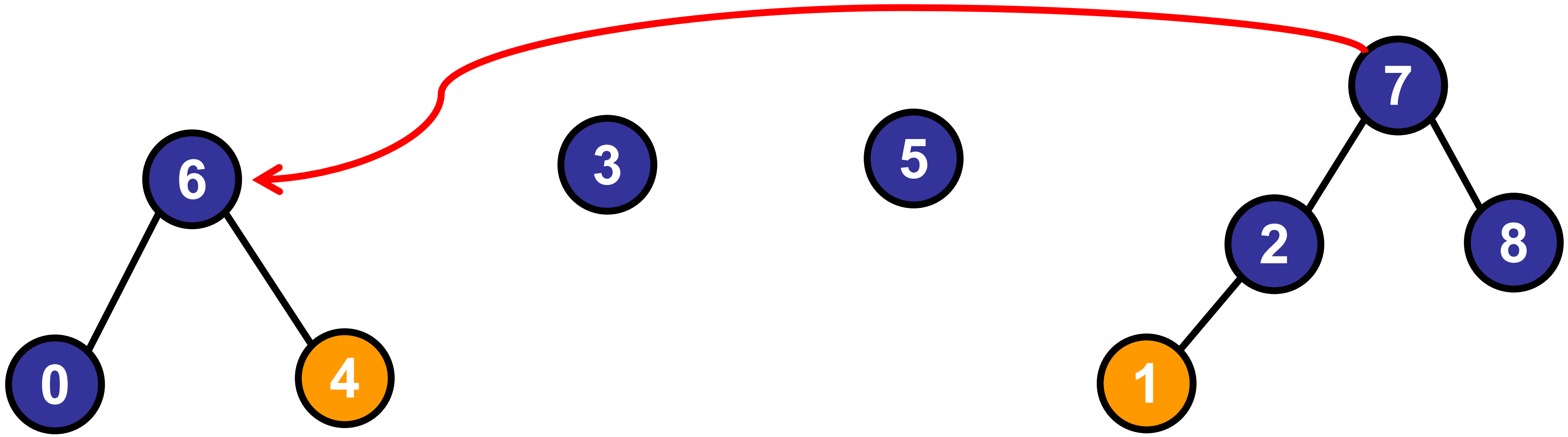


Quick Union

Example: union(1, 4)

```
4 → 6 → 6
1 → 2 → 7 → 7
parent[7] = 6;
```

object	0	1	2	3	4	5	6	7	8
parent	6	2	7	3	6	1	6	6	7

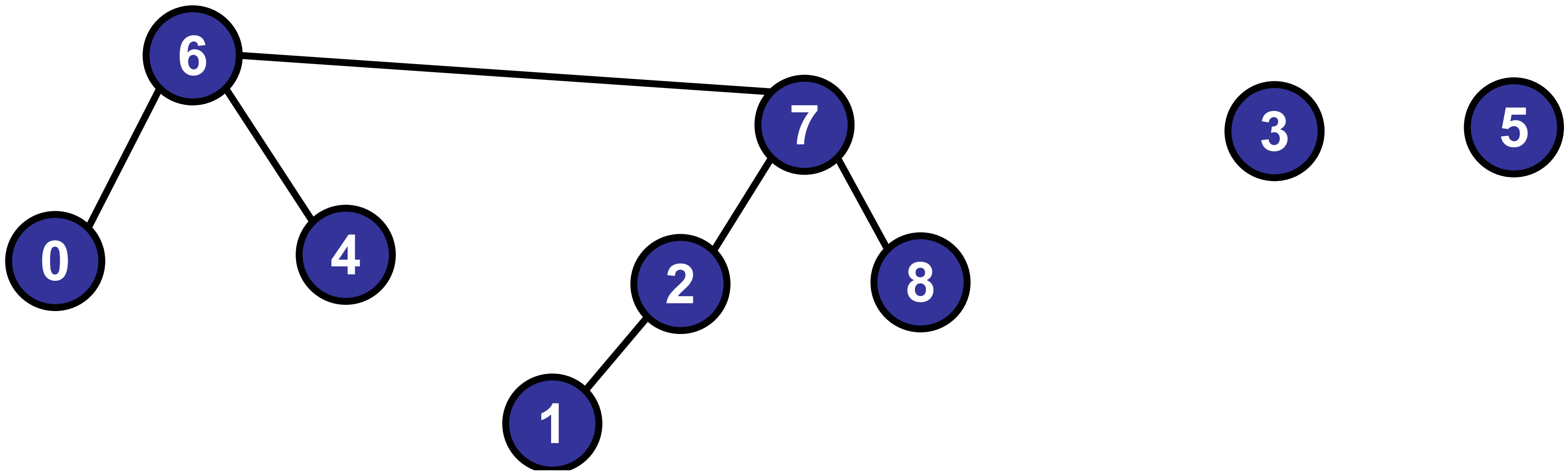


Quick Union

Example: union(1, 4)

```
4 → 6 → 6
1 → 2 → 7 → 7
parent[7] = 6;
```

object	0	1	2	3	4	5	6	7	8
parent	6	2	7	3	6	1	6	6	7



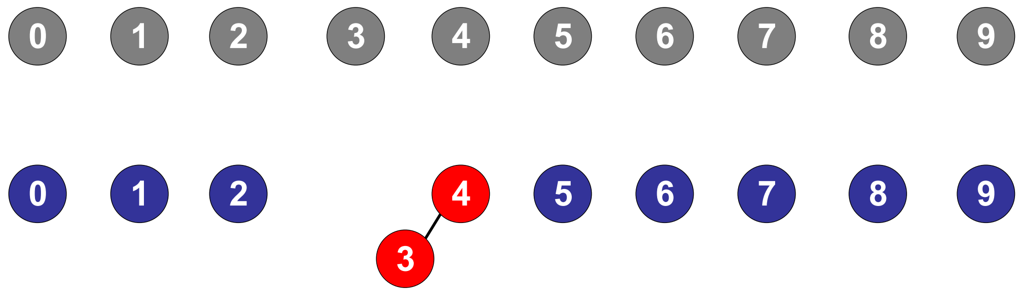
Example:

P	0	1	2	3	4	5	6	7	8	9

0 1 2 3 4 5 6 7 8 9

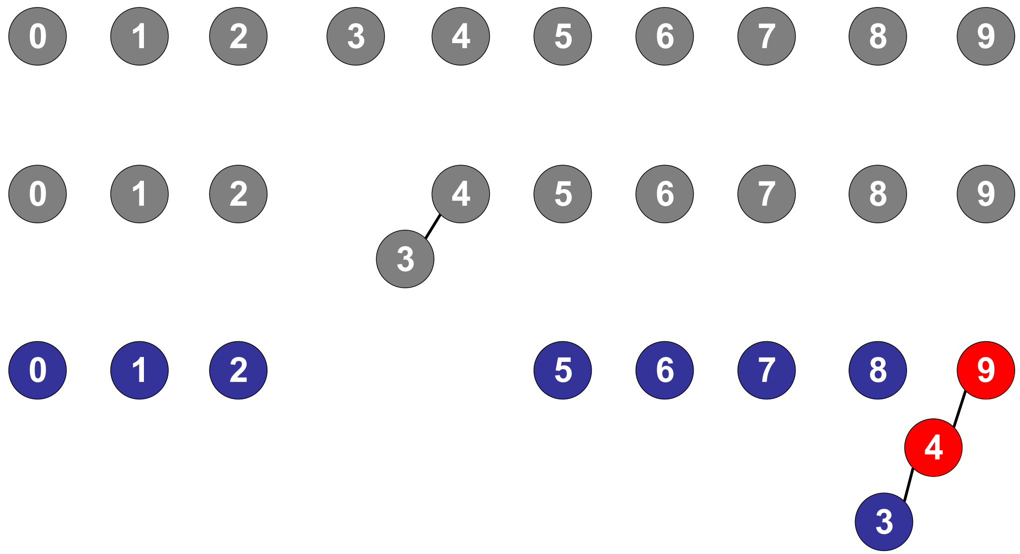
Example:

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9



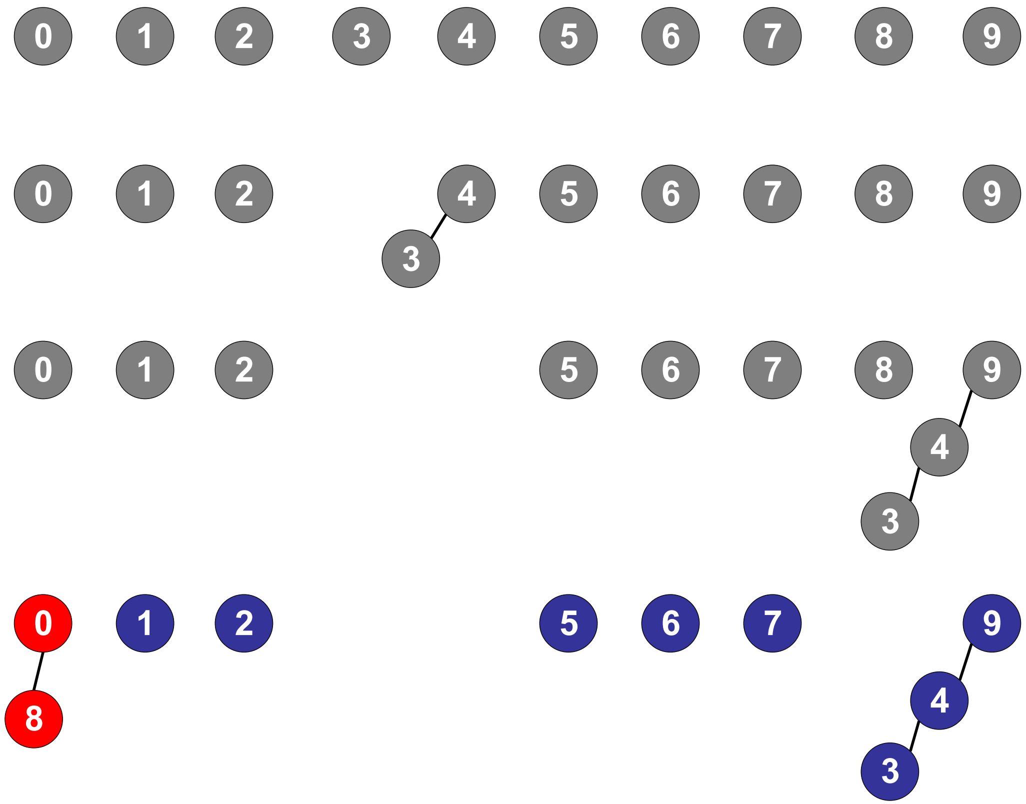
Example:

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	9	5	6	7	8	9



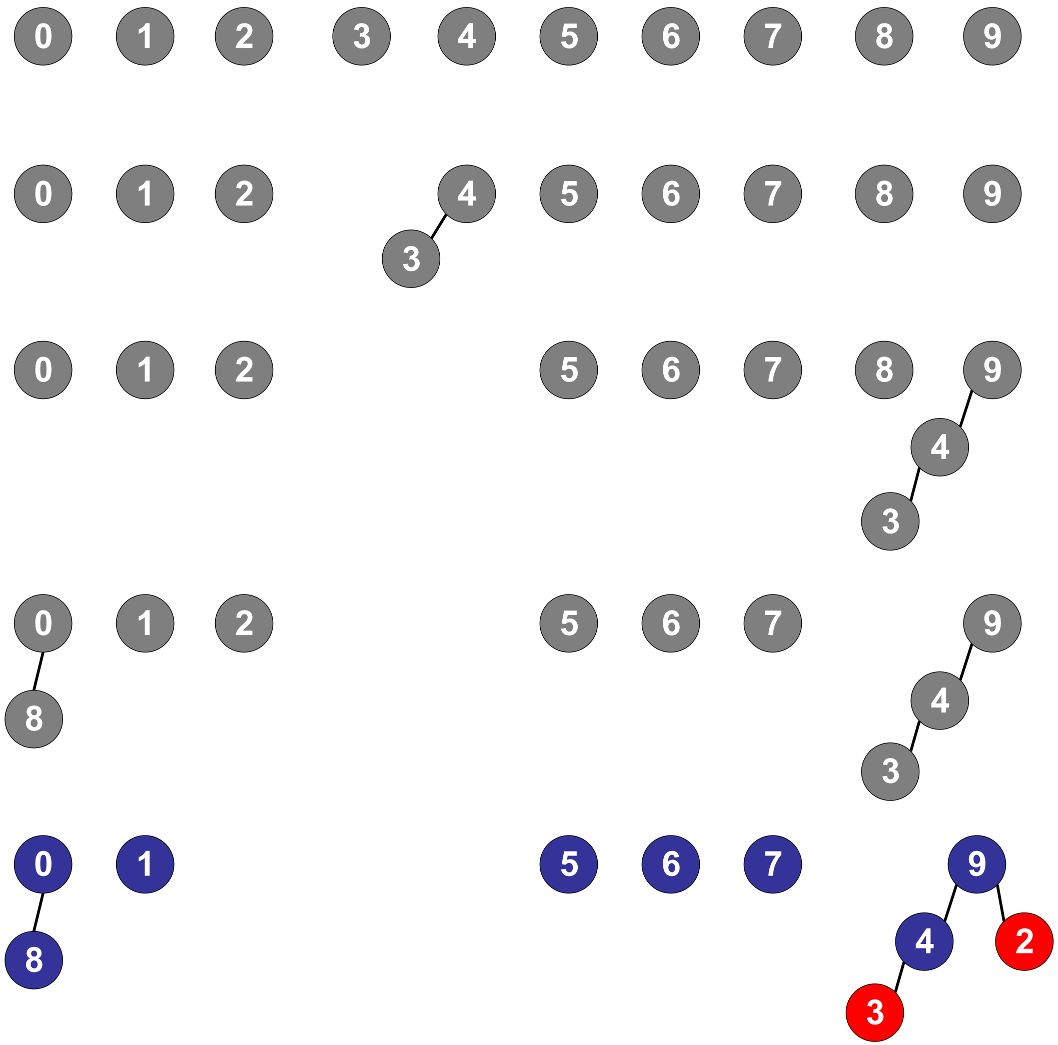
Example:

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	9	5	6	7	8	9
8-0	0	1	2	4	9	5	6	7	0	9



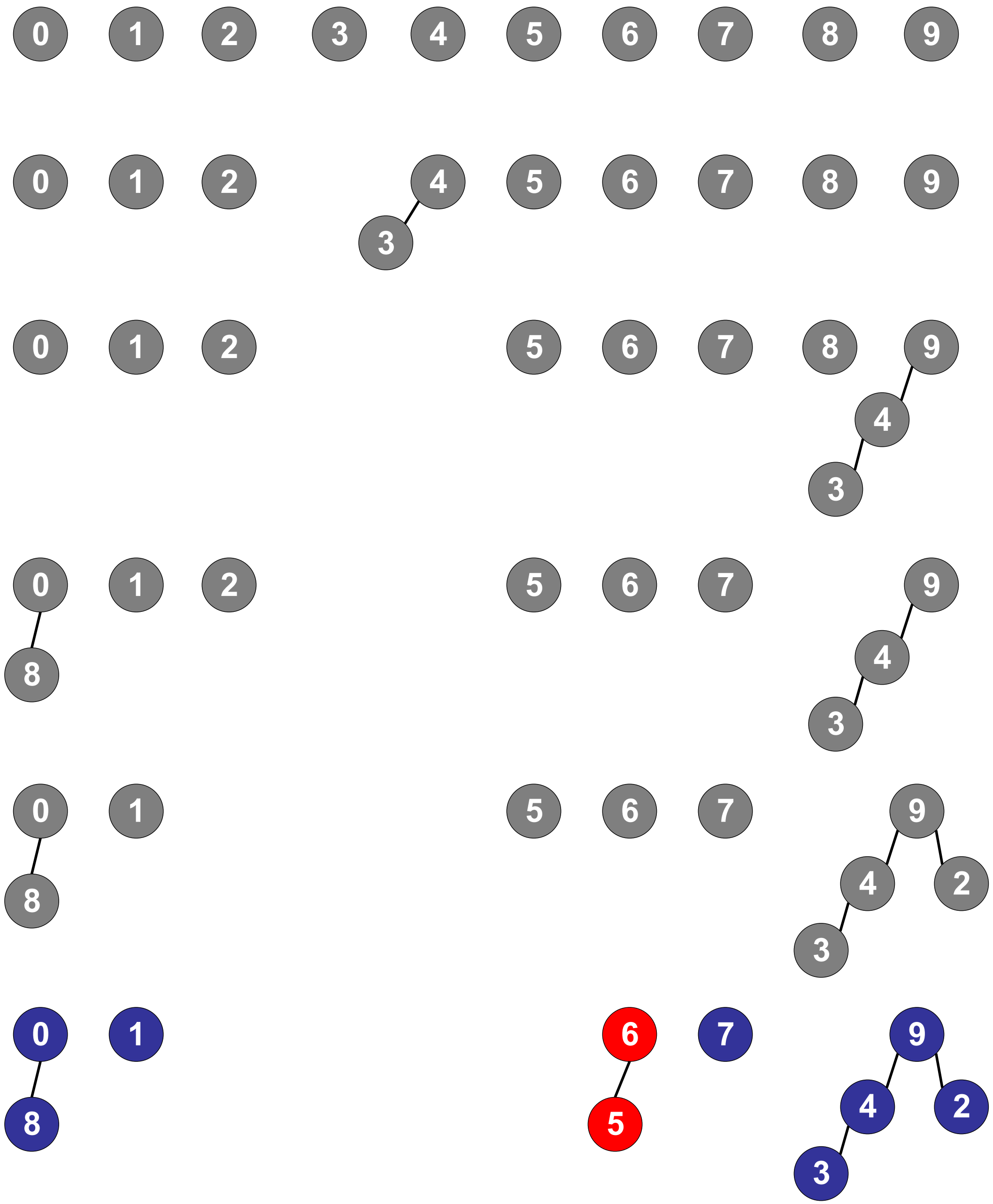
Example:

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	9	5	6	7	8	9
8-0	0	1	2	4	9	5	6	7	0	9
2-3	0	1	9	4	9	5	6	7	0	9



Example:

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	9	5	6	7	8	9
8-0	0	1	2	4	9	5	6	7	0	9
2-3	0	1	9	4	9	5	6	7	0	9
5-6	0	1	9	4	9	6	6	7	0	9

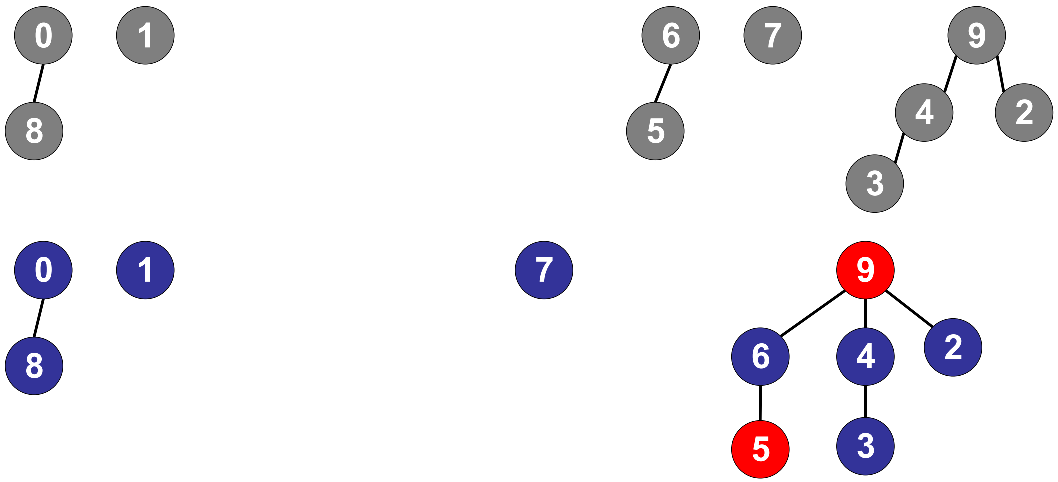


Example:

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	9	5	6	7	8	9
8-0	0	1	2	4	9	5	6	7	0	9
2-3	0	1	9	4	9	5	6	7	0	9
5-6	0	1	9	4	9	6	6	7	0	9

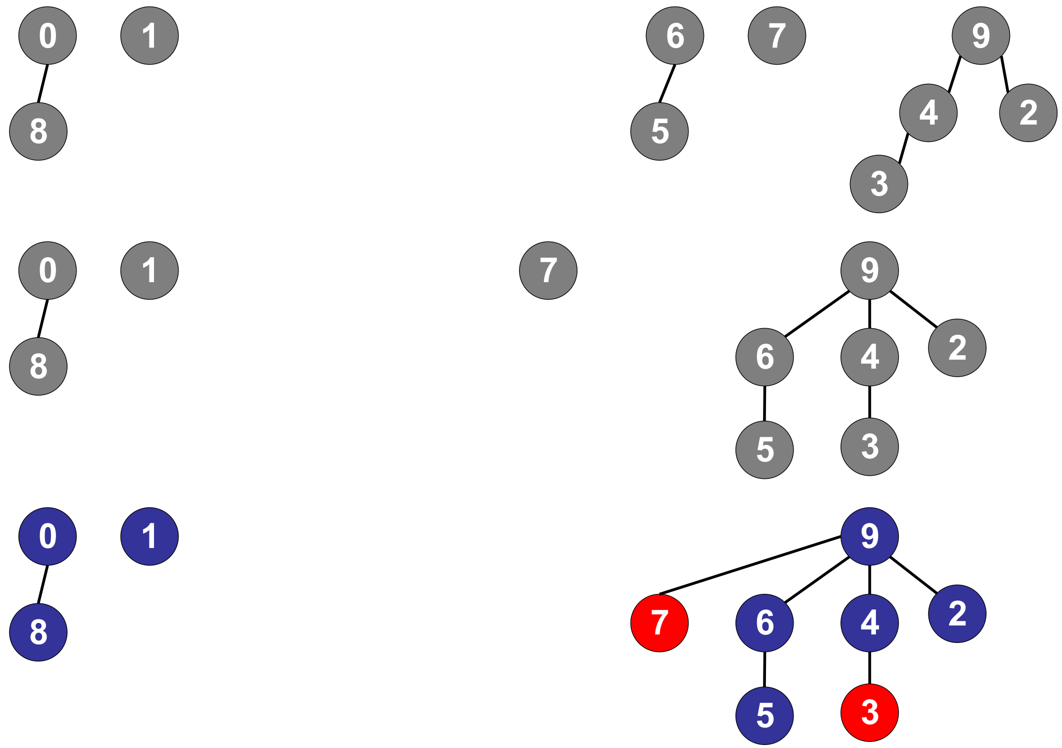
Example:

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	9	5	6	7	8	9
8-0	0	1	2	4	9	5	6	7	0	9
2-3	0	1	9	4	9	5	6	7	0	9
5-6	0	1	9	4	9	6	6	7	0	9
5-9	0	1	9	4	9	6	9	7	0	9



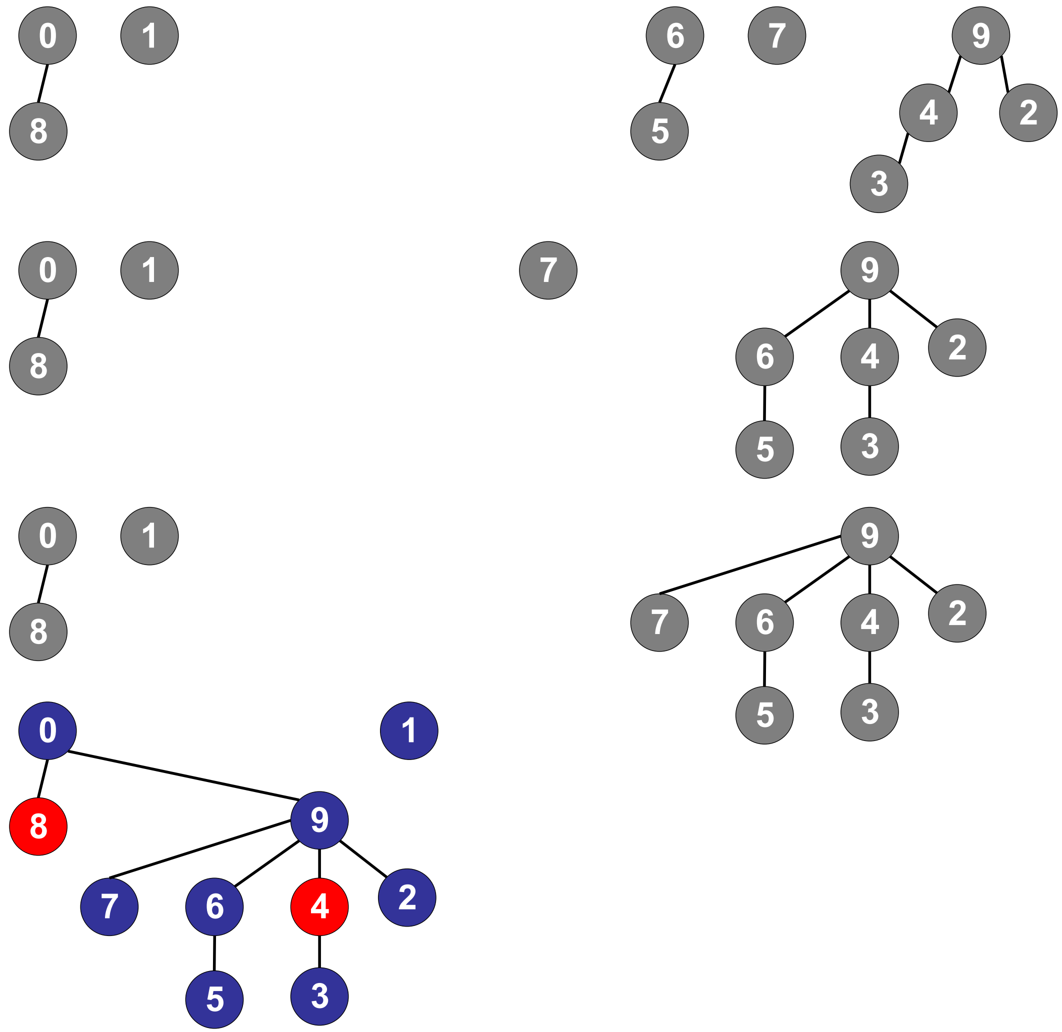
Example:

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	9	5	6	7	8	9
8-0	0	1	2	4	9	5	6	7	0	9
2-3	0	1	9	4	9	5	6	7	0	9
5-6	0	1	9	4	9	6	6	7	0	9
5-9	0	1	9	4	9	6	9	7	0	9
7-3	0	1	9	4	9	6	9	9	0	9



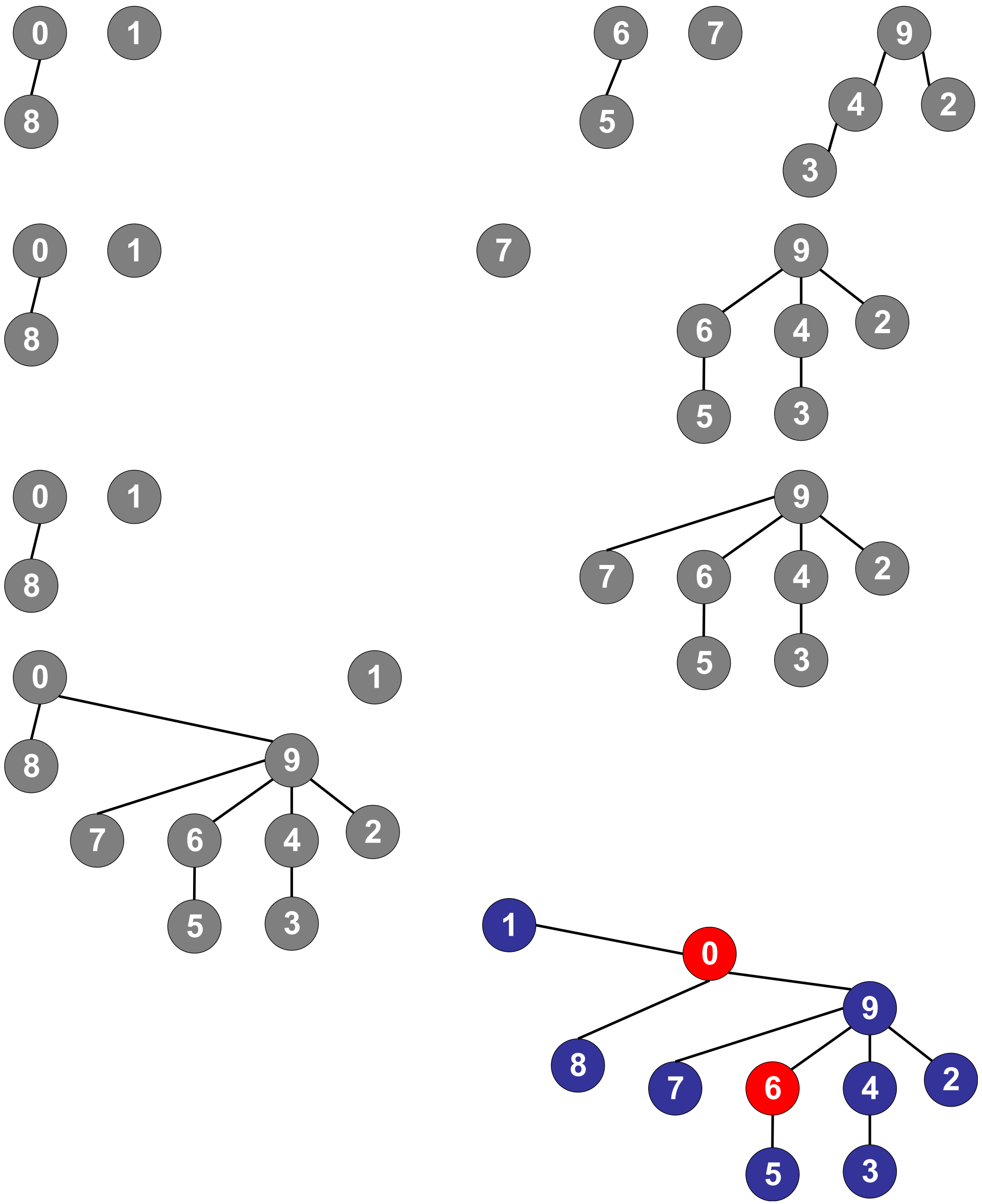
Example:

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	9	5	6	7	8	9
8-0	0	1	2	4	9	5	6	7	0	9
2-3	0	1	9	4	9	5	6	7	0	9
5-6	0	1	9	4	9	6	6	7	0	9
5-9	0	1	9	4	9	6	9	7	0	9
7-3	0	1	9	4	9	6	9	9	0	9
4-8	0	1	9	4	9	6	9	9	0	0



Example:

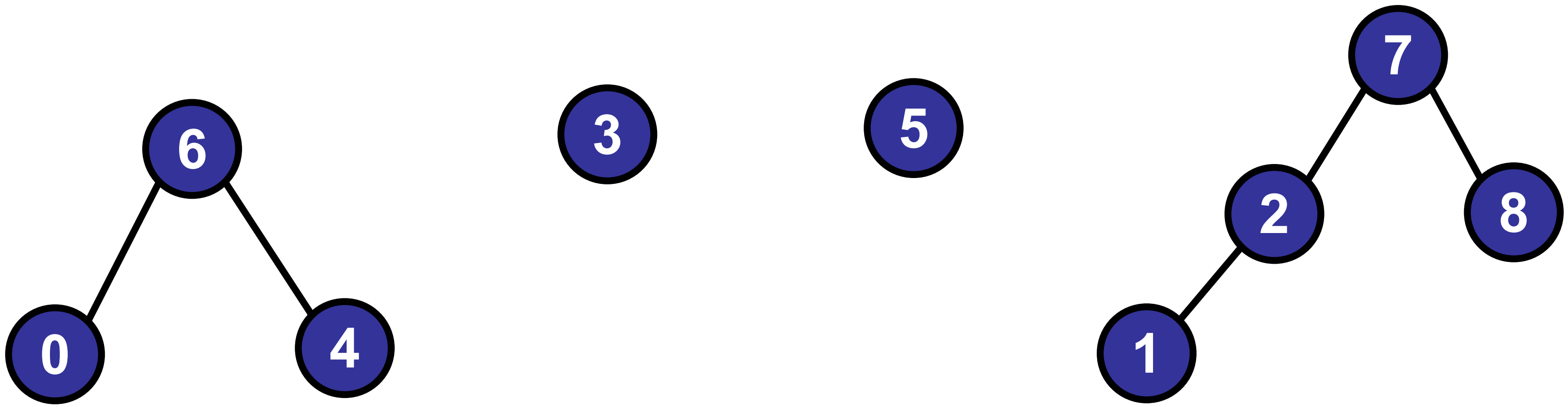
P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	9	5	6	7	8	9
8-0	0	1	2	4	9	5	6	7	0	9
2-3	0	1	9	4	9	5	6	7	0	9
5-6	0	1	9	4	9	6	6	7	0	9
5-9	0	1	9	4	9	6	9	7	0	9
7-3	0	1	9	4	9	6	9	9	0	9
4-8	0	1	9	4	9	6	9	9	0	0
6-1	1	1	9	4	9	6	9	9	0	0



Quick Union

```
union(int p, int q)
   while (parent[p] != p) p = parent[p];
   while (parent[q] != q) q = parent[q];
   parent[p] = q;
```

object	0	1	2	3	4	5	6	7	8
parent	6	2	7	3	6	1	6	7	7



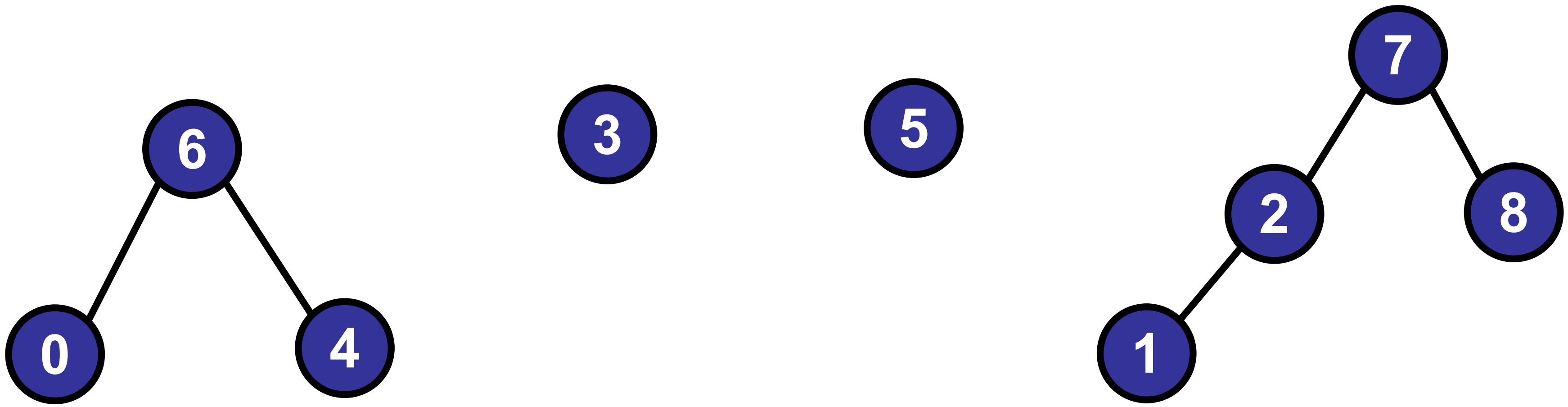
Running time of (Find, Union):

1. O(1), O(1)
2. O(1), O(n)
3. O(n), O(1)
4. O(n), O(n)
5. O(log n), O(log n)
6. None of the above.

Quick Union

```
find(int p, int q)
   while (parent[p] != p) p = parent[p];
   while (parent[q] != q) q = parent[q];
   return (p == q);
```

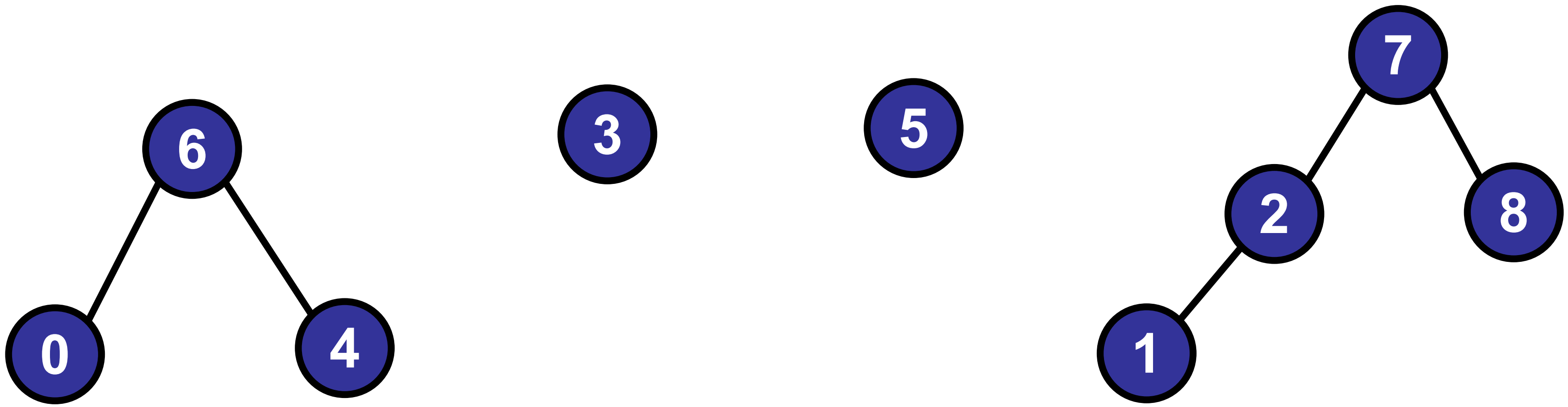
object	0	1	2	3	4	5	6	7	8
parent	6	2	7	3	6	1	6	7	7



Quick Union

```
union(int p, int q)
   while (parent[p] != p) p = parent[p];
   while (parent[q] != q) q = parent[q];
   parent[p] = q;
```

object	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
parent	**6**	**2**	**7**	**3**	**6**	**1**	**6**	**7**	**7**



Union-Find Summary

Quick-find is slow:

- Union is expensive
- Tree is flat

Quick-union is slow:

- Trees too tall (i.e., unbalanced)
- Union *and* find are expensive.

	find	union
quick-find	O(1)	O(n)
quick-union	O(n)	O(n)

Roadmap

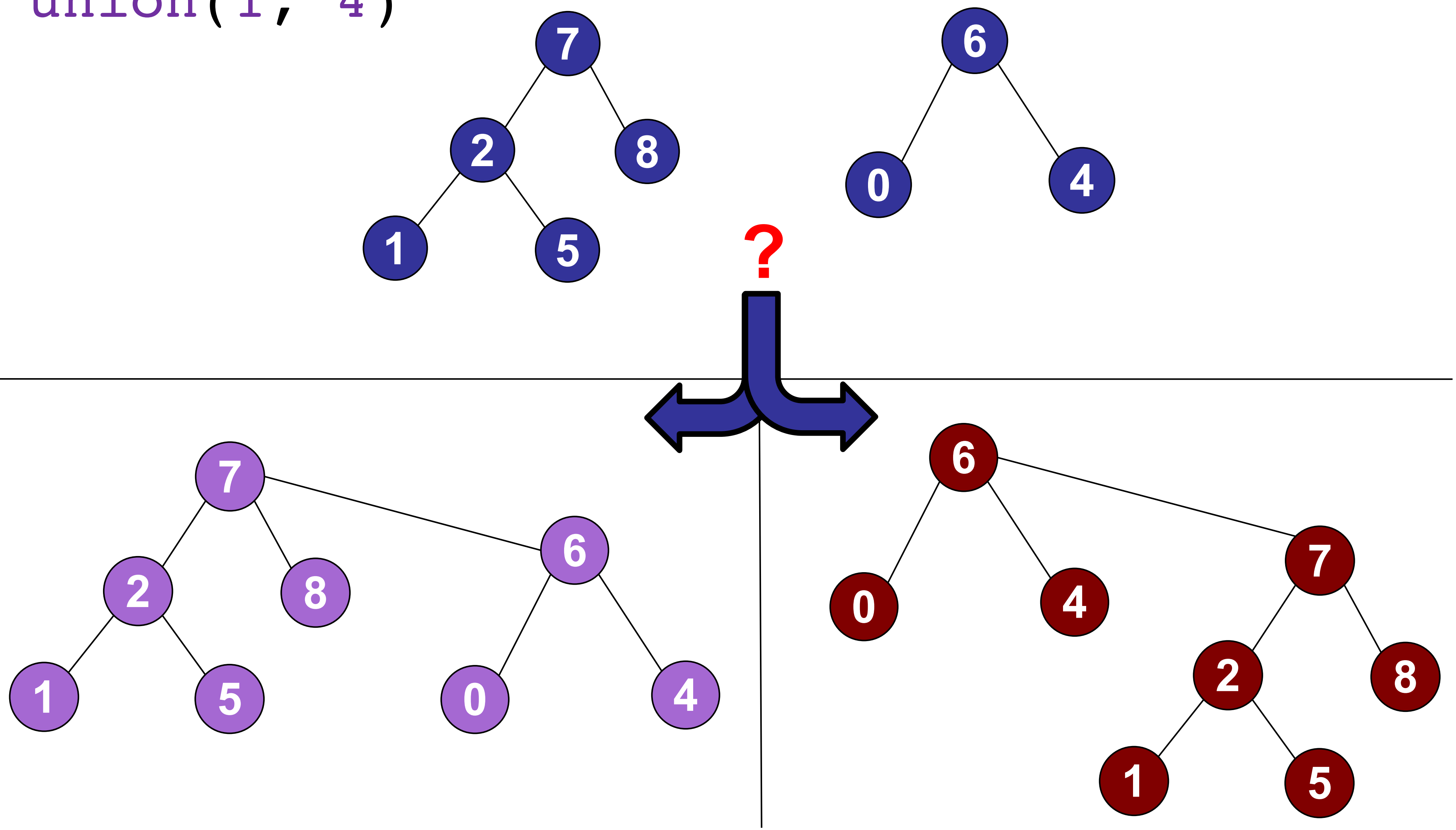
Disjoint Set Data Structure

- Problem: Dynamic Connectivity
- Algorithm: Quick-Find
- Algorithm: Quick-Union
- Optimizations
- Applications

Weighted Union

Question: which tree should you make the root?

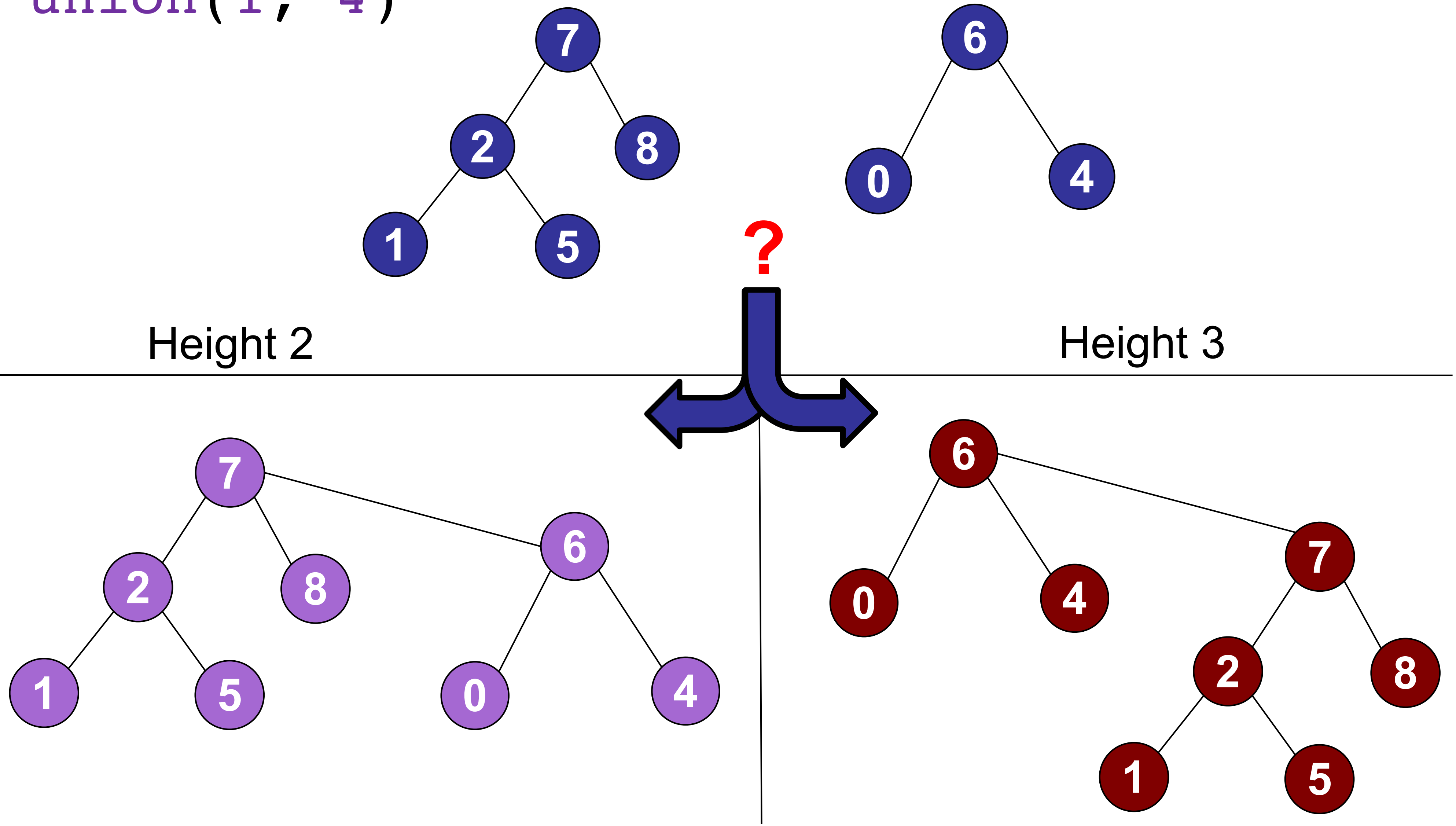
`union(1, 4)`



Weighted Union

Question: which tree should you make the root?

`union(1, 4)`



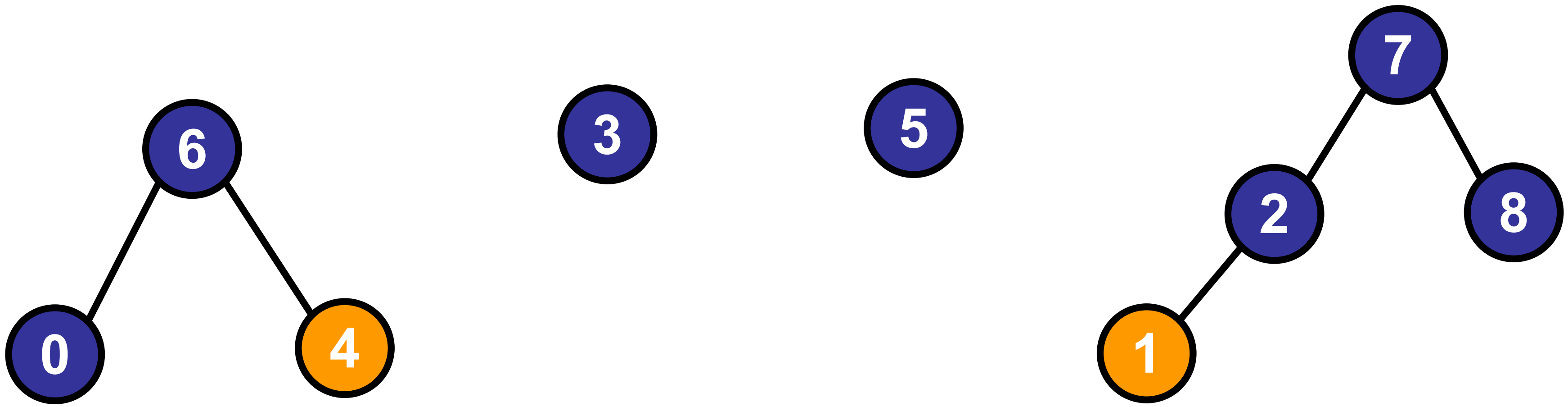
Weighted Union

```
union(int p, int q)
   while (parent[p] !=p) p = parent[p];
   while (parent[q] !=q) q = parent[q];
   if (size[p] > size[q] {
           parent[q] = p;    // Link q to p
           size[p] = size[p] + size[q];
   }
   else {
           parent[p] = q; // Link p to q
           size[q] = size[p] + size[q];
   }
```

Weighted Union

```
union(1, 4)
```

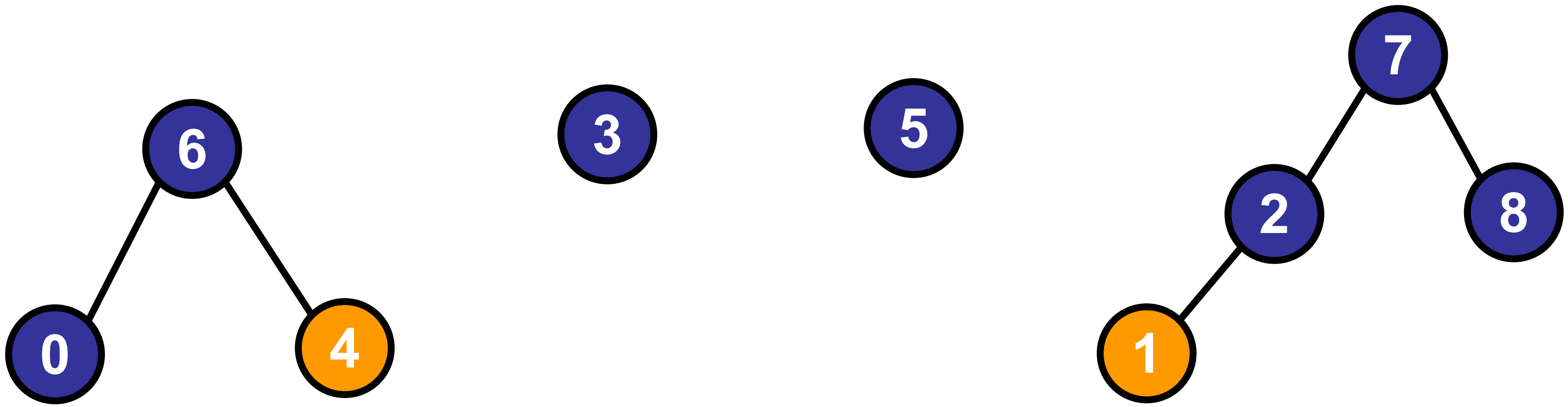
object	0	1	2	3	4	5	6	7	8
size	1	1	2	1	1	1	3	4	1
parent	6	2	7	3	6	1	6	7	7



Weighted Union

`union(1, 4)`

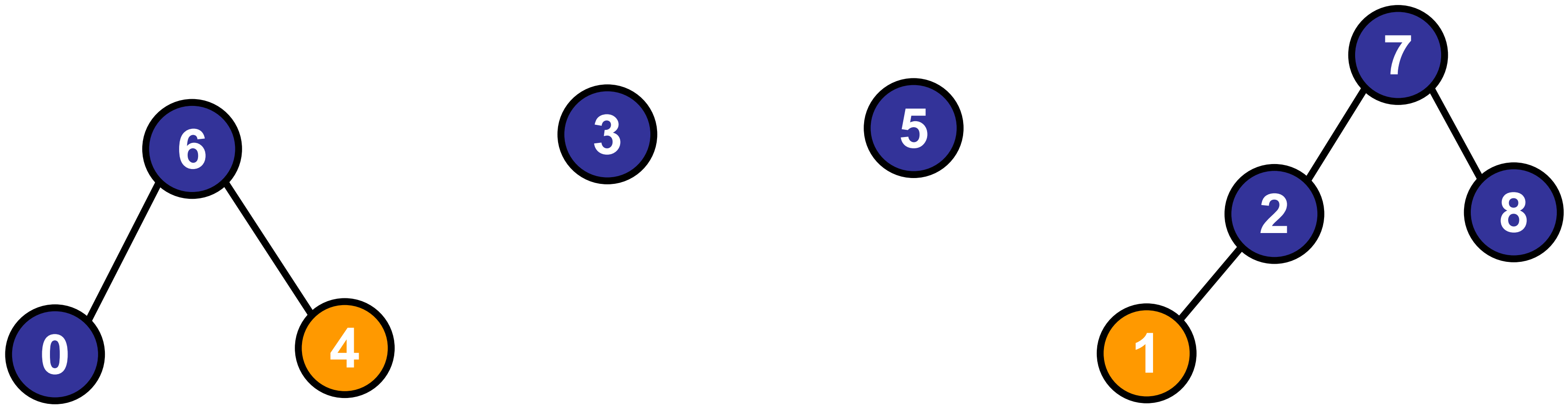
object	0	1	2	3	4	5	6	7	8
size	1	1	2	1	1	1	3	4	1
parent	6	2	7	3	6	1	6	7	7



Weighted Union

`union(1, 4)`

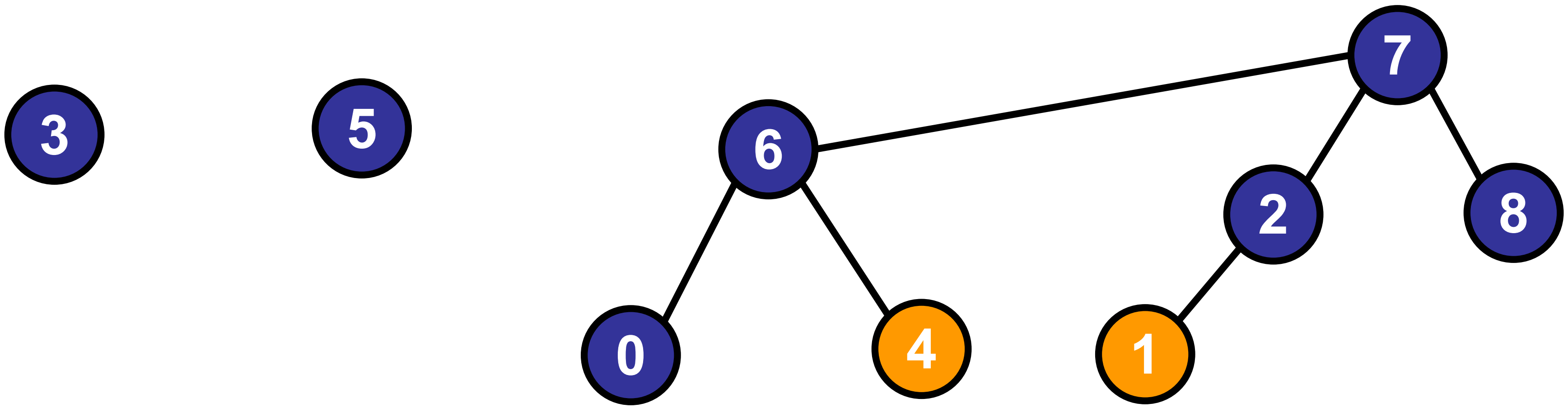
object	0	1	2	3	4	5	6	7	8
size	1	1	2	1	1	1	3	4	1
parent	6	2	7	3	6	1	6	7	7



Weighted Union

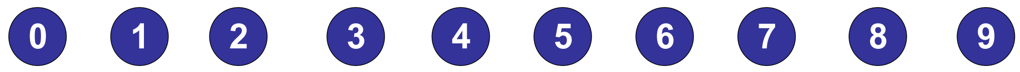
`union(1, 4)`

object	0	1	2	3	4	5	6	7	8
size	1	1	2	1	1	1	3	7	1
parent	6	2	7	3	6	1	6	7	7



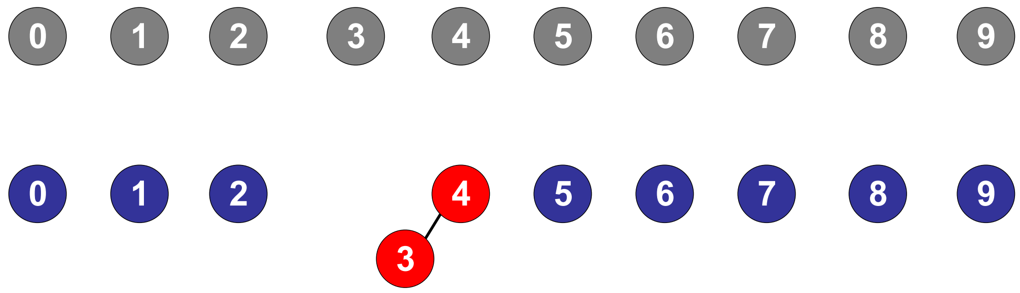
Example: Weighted Union

P	0	1	2	3	4	5	6	7	8	9



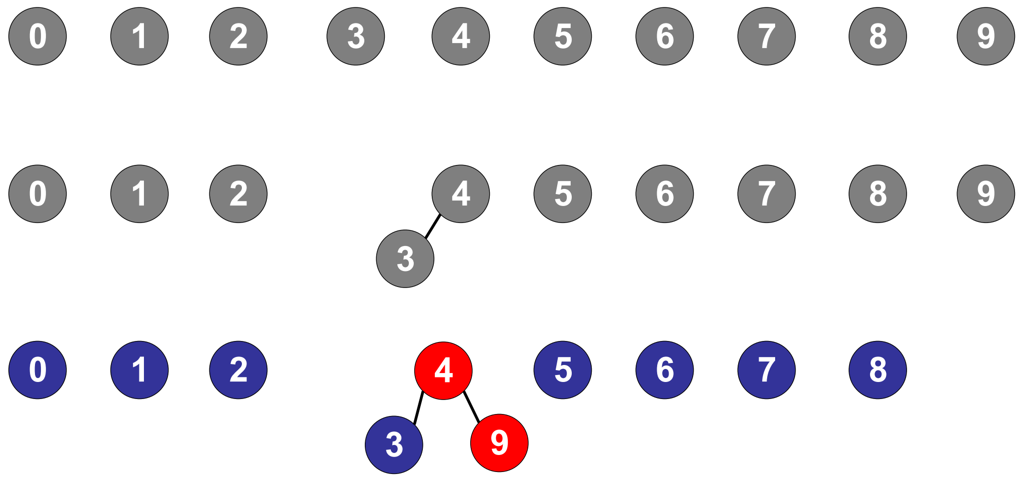
Example: Weighted Union

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9



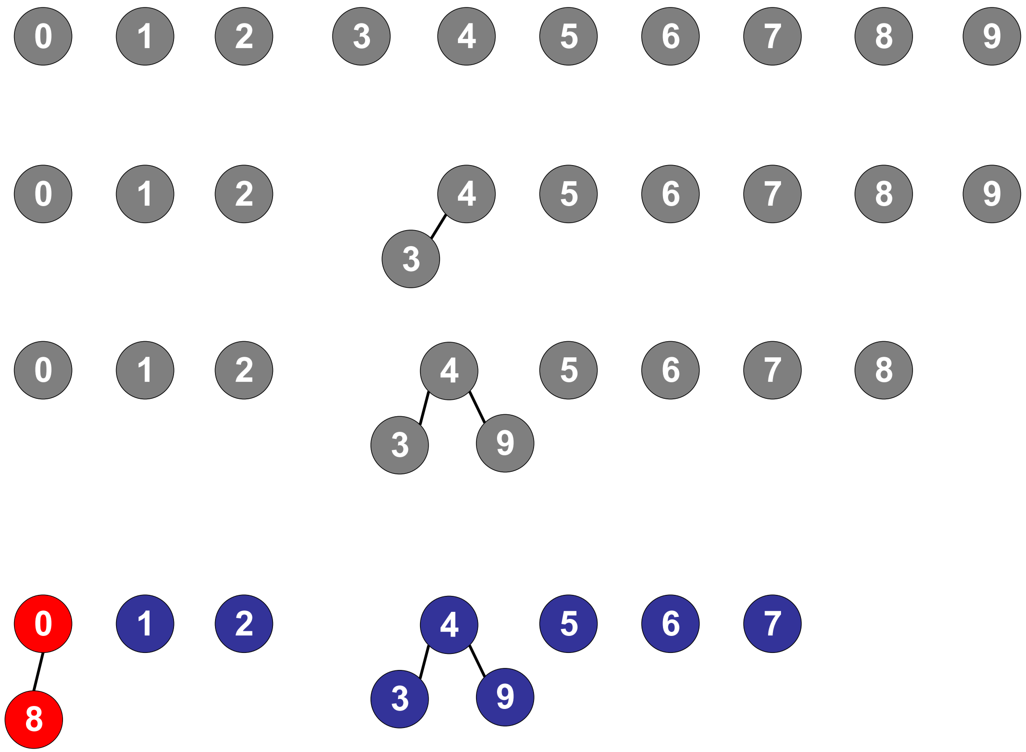
Example: Weighted Union

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	4	5	6	7	8	4



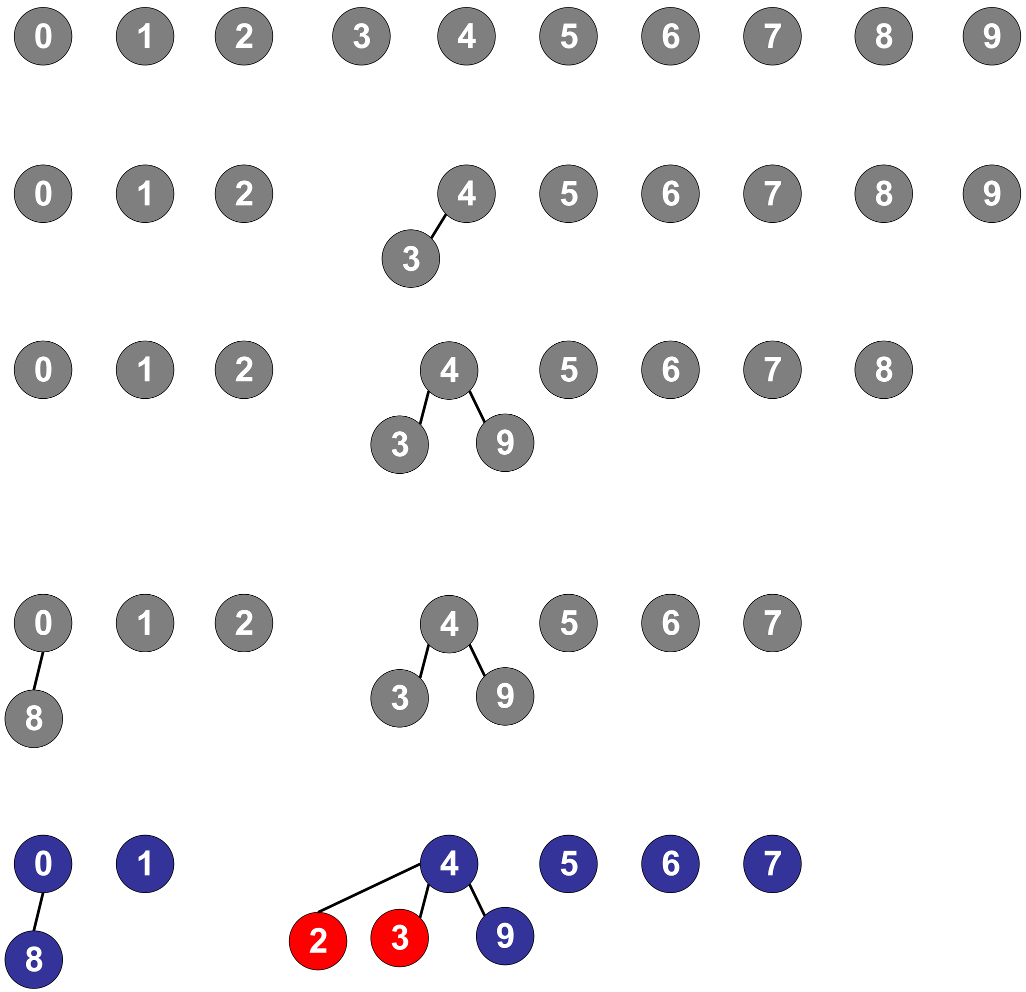
Example: Weighted Union

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	4	5	6	7	8	4
8-0	0	1	2	4	4	5	6	7	0	4



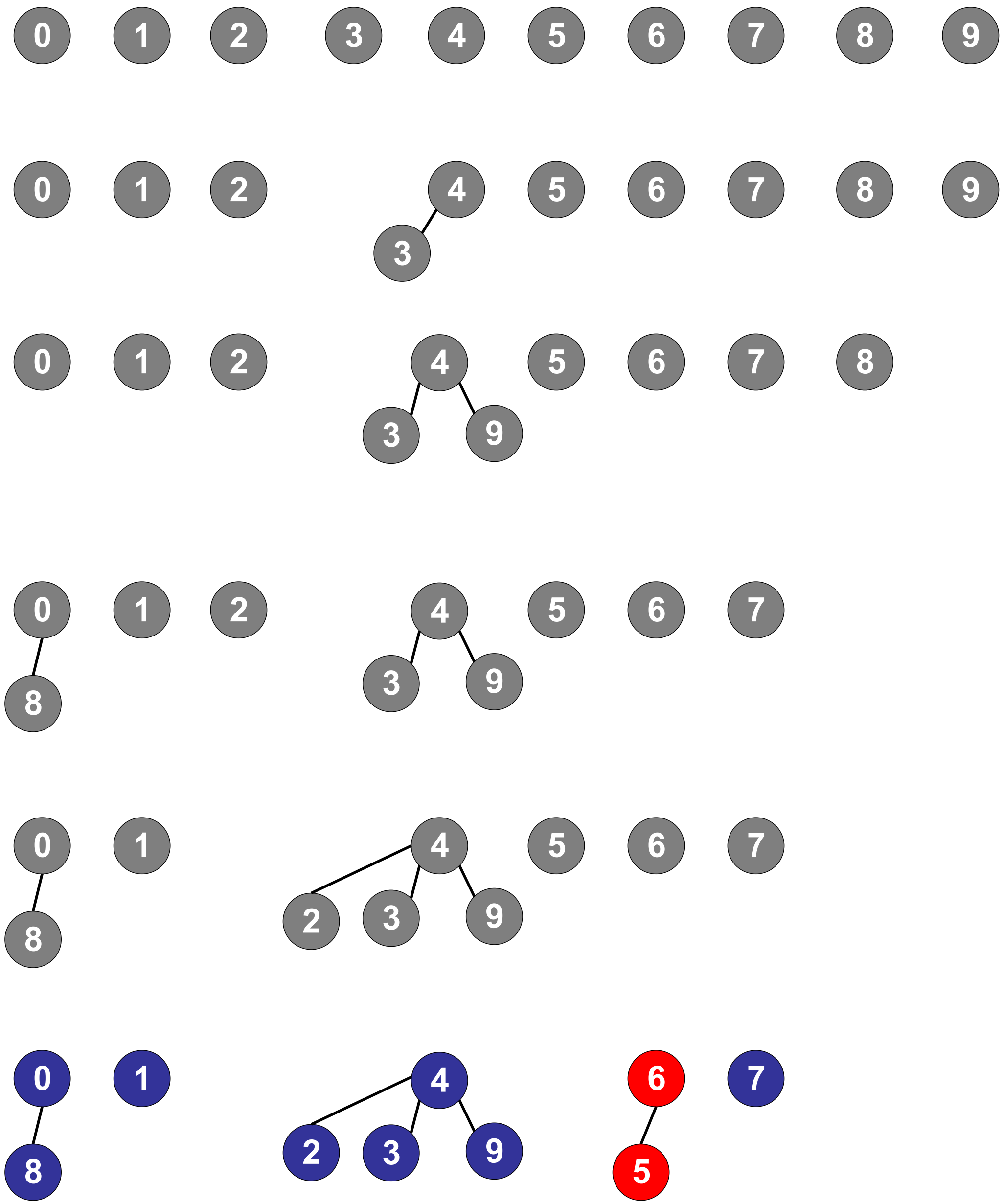
Example: Weighted Union

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	4	5	6	7	8	4
8-0	0	1	2	4	4	5	6	7	0	4
2-3	0	1	4	4	4	5	6	7	0	4



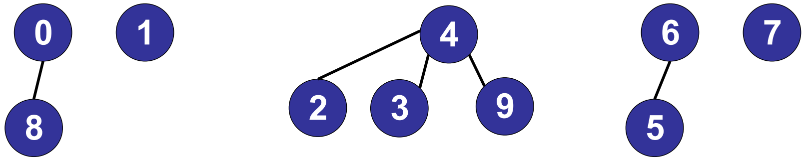
Example: Weighted Union

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	4	5	6	7	8	4
8-0	0	1	2	4	4	5	6	7	0	4
2-3	0	1	4	4	4	5	6	7	0	4
5-6	0	1	4	4	4	6	6	7	0	4



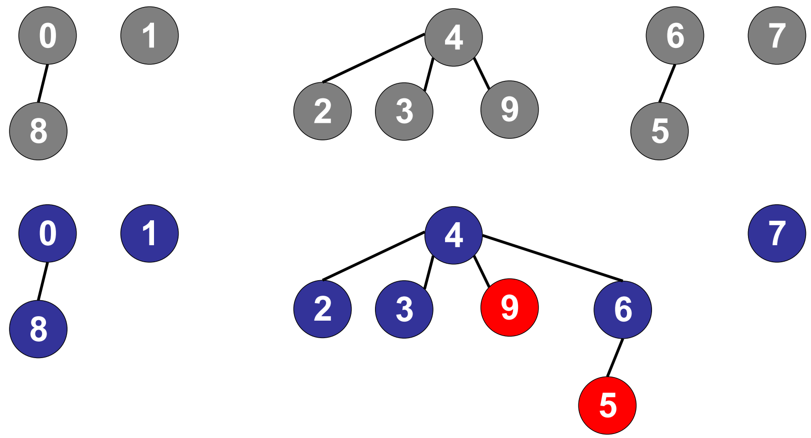
Example: Weighted Union

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	4	5	6	7	8	4
8-0	0	1	2	4	4	5	6	7	0	4
2-3	0	1	4	4	4	5	6	7	0	4
5-6	0	1	4	4	4	6	6	7	0	4



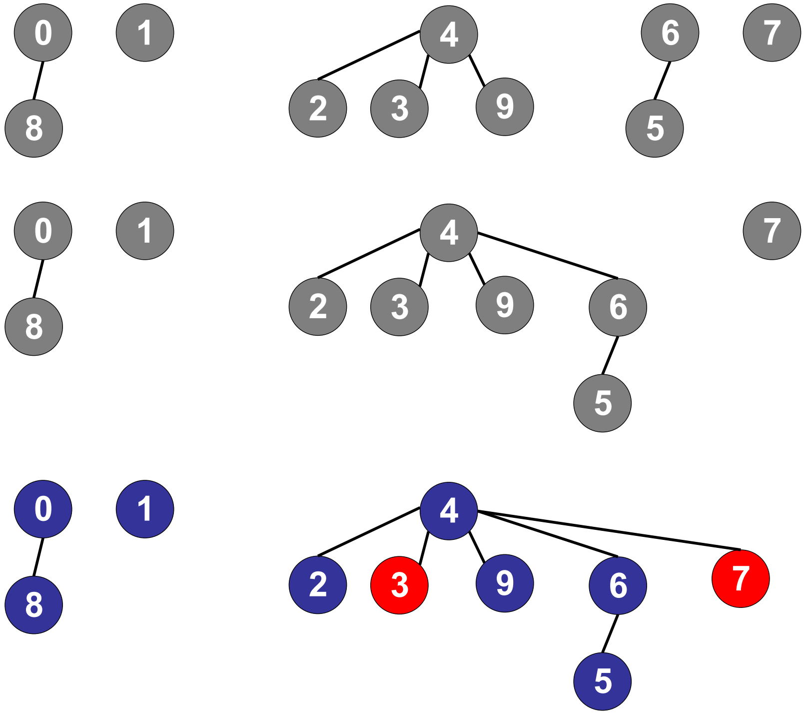
Example: Weighted Union

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	4	5	6	7	8	4
8-0	0	1	2	4	4	5	6	7	0	4
2-3	0	1	4	4	4	5	6	7	0	4
5-6	0	1	4	4	4	6	6	7	0	4
5-9	0	1	4	4	4	6	4	7	0	4



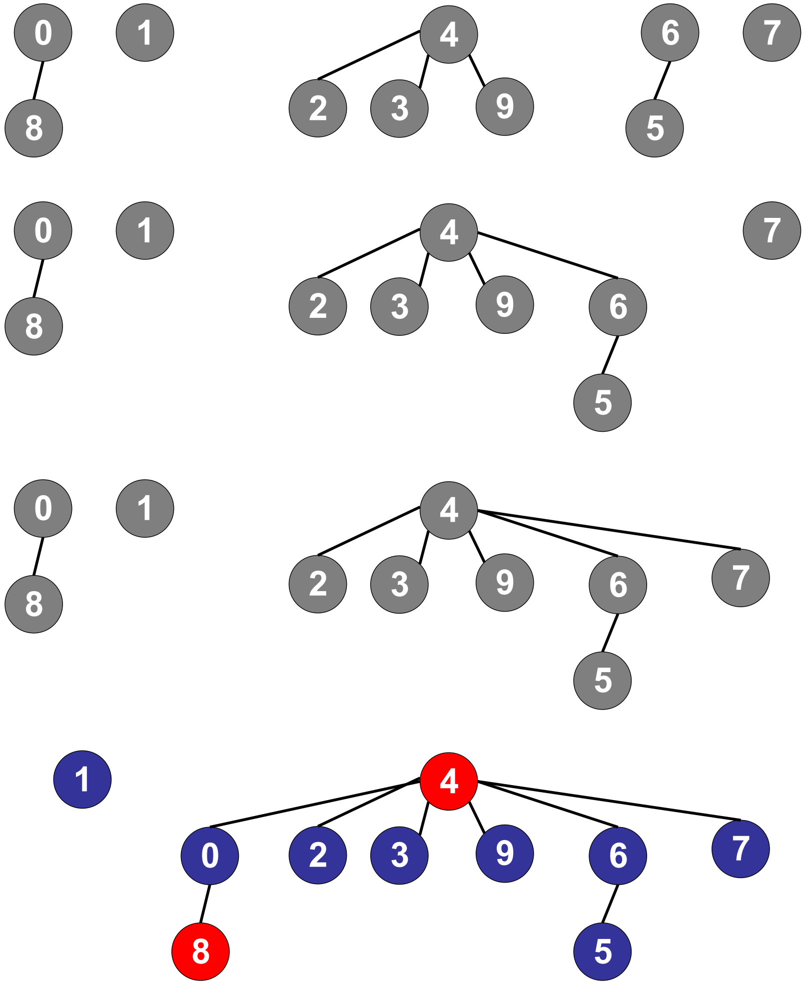
Example: Weighted Union

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	4	5	6	7	8	4
8-0	0	1	2	4	4	5	6	7	0	4
2-3	0	1	4	4	4	5	6	7	0	4
5-6	0	1	4	4	4	6	6	7	0	4
5-9	0	1	4	4	4	6	4	7	0	4
7-3	0	1	4	4	4	6	4	4	0	4



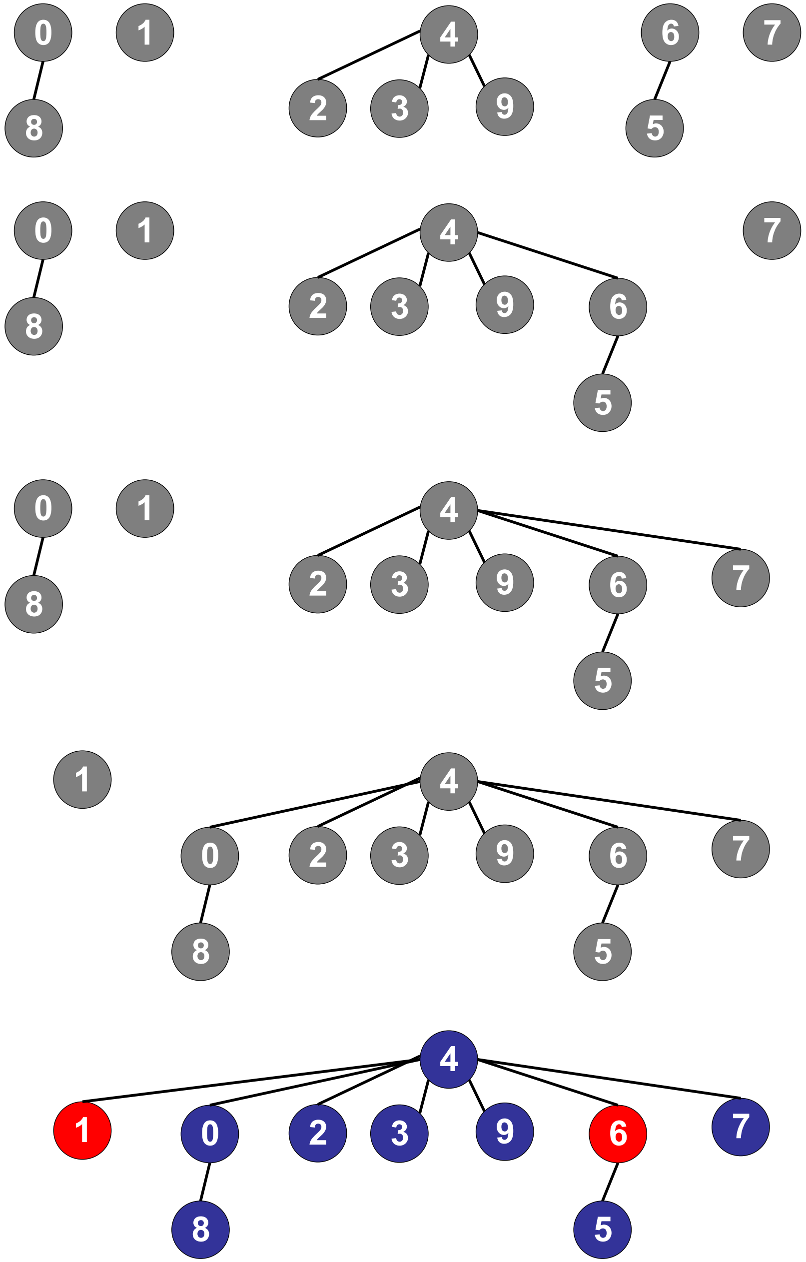
Example: Weighted Union

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	4	5	6	7	8	4
8-0	0	1	2	4	4	5	6	7	0	4
2-3	0	1	4	4	4	5	6	7	0	4
5-6	0	1	4	4	4	6	6	7	0	4
5-9	0	1	4	4	4	6	4	7	0	4
7-3	0	1	4	4	4	6	4	4	0	4
4-8	4	1	4	4	4	6	4	4	0	4



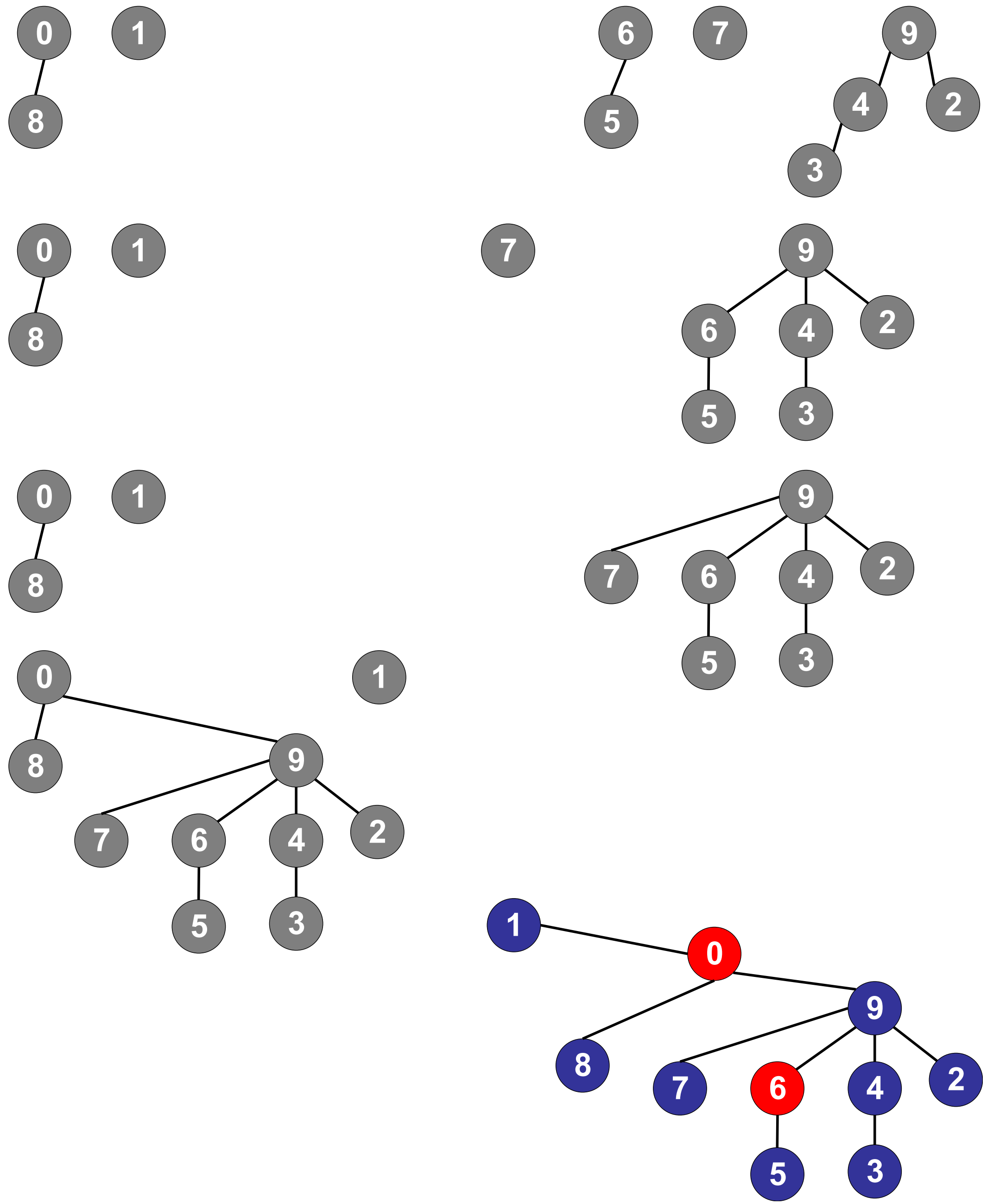
Example: Weighted Union

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	4	5	6	7	8	4
8-0	0	1	2	4	4	5	6	7	0	4
2-3	0	1	4	4	4	5	6	7	0	4
5-6	0	1	4	4	4	6	6	7	0	4
5-9	0	1	4	4	4	6	4	7	0	4
7-3	0	1	4	4	4	6	4	4	0	4
4-8	4	1	4	4	4	6	4	4	0	4
6-1	4	4	4	4	4	6	4	4	0	4



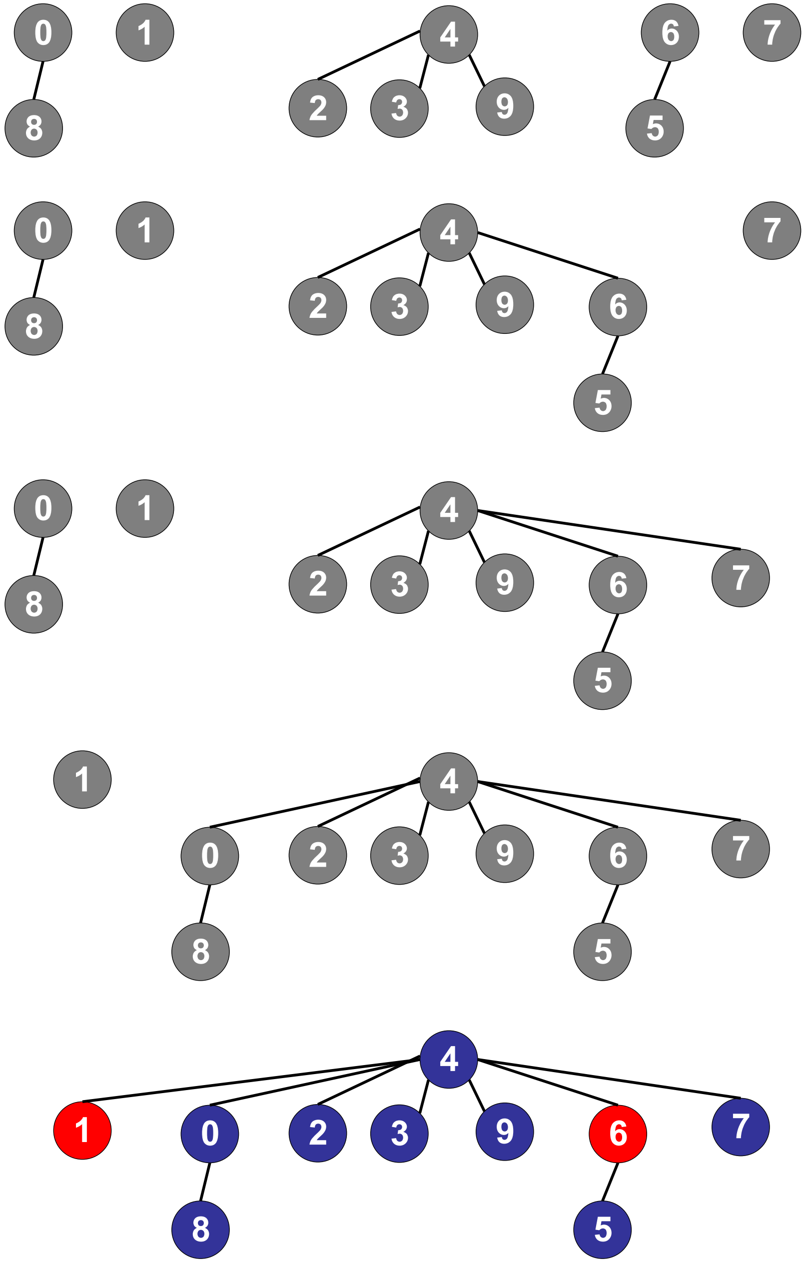
Example: (Unweighted) Quick Union

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	9	5	6	7	8	9
8-0	0	1	2	4	9	5	6	7	0	9
2-3	0	1	9	4	9	5	6	7	0	9
5-6	0	1	9	4	9	6	6	7	0	9
5-9	0	1	9	4	9	6	9	7	0	9
7-3	0	1	9	4	9	6	9	9	0	9
4-8	0	1	9	4	9	6	9	9	0	0
6-1	1	1	9	4	9	6	9	9	0	0



Example: Weighted Union

P	0	1	2	3	4	5	6	7	8	9
3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	4	5	6	7	8	4
8-0	0	1	2	4	4	5	6	7	0	4
2-3	0	1	4	4	4	5	6	7	0	4
5-6	0	1	4	4	4	6	6	7	0	4
5-9	0	1	4	4	4	6	4	7	0	4
7-3	0	1	4	4	4	6	4	4	0	4
4-8	4	1	4	4	4	6	4	4	0	4
6-1	4	4	4	4	4	6	4	4	0	4



Maximum depth of tree?

1. $O(1)$
2. $O(\log n)$
3. $O(n)$
4. $O(n \log n)$
5. $O(n^2)$
6. None of the above.

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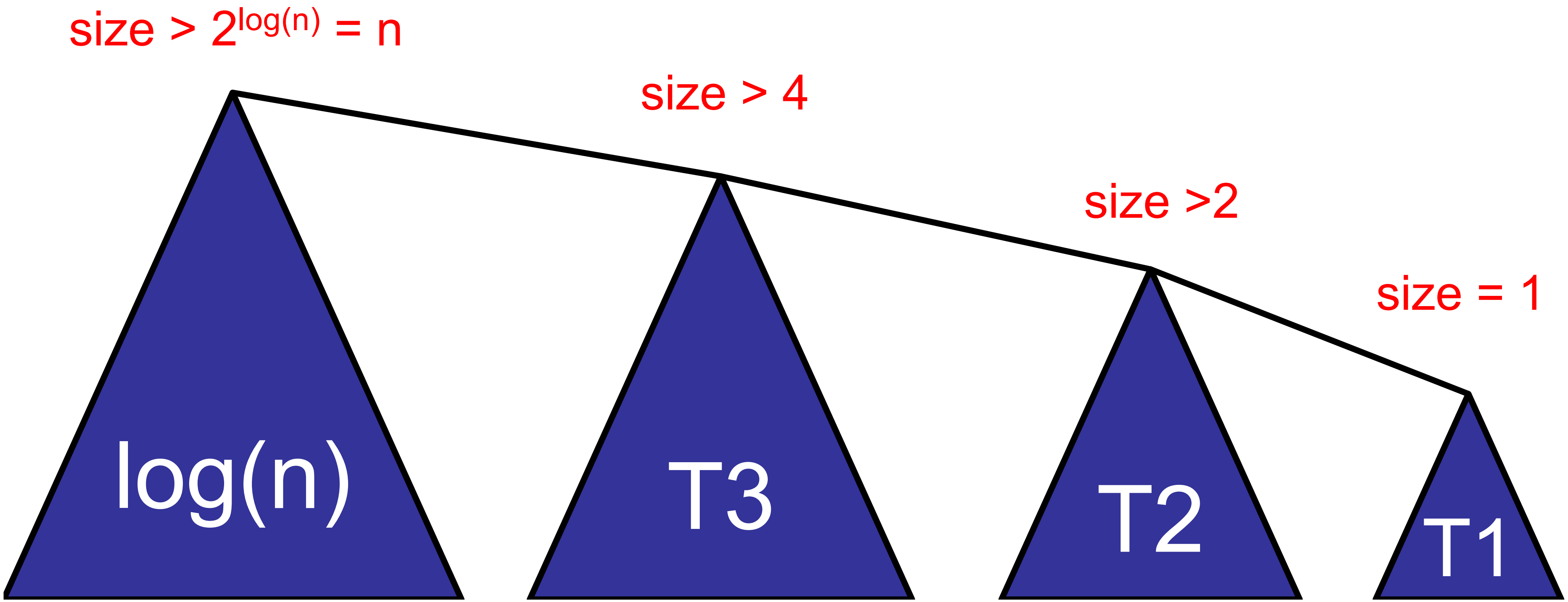
is open

Weighted Union

Key idea:

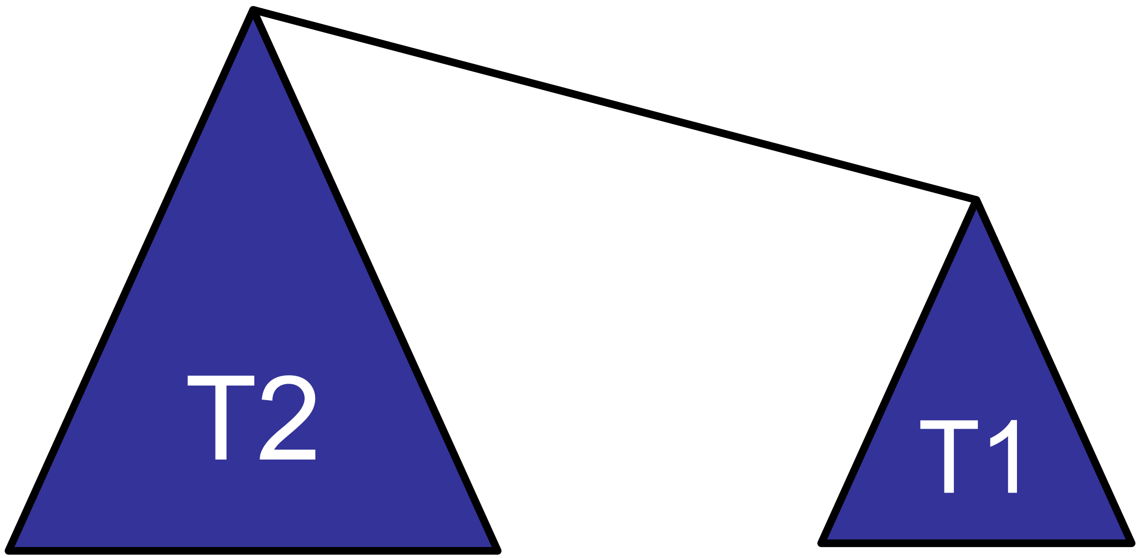
height only increases when total size doubles



Weighted Union

Analysis:

- Base case: tree of height 0 contains 1 object.

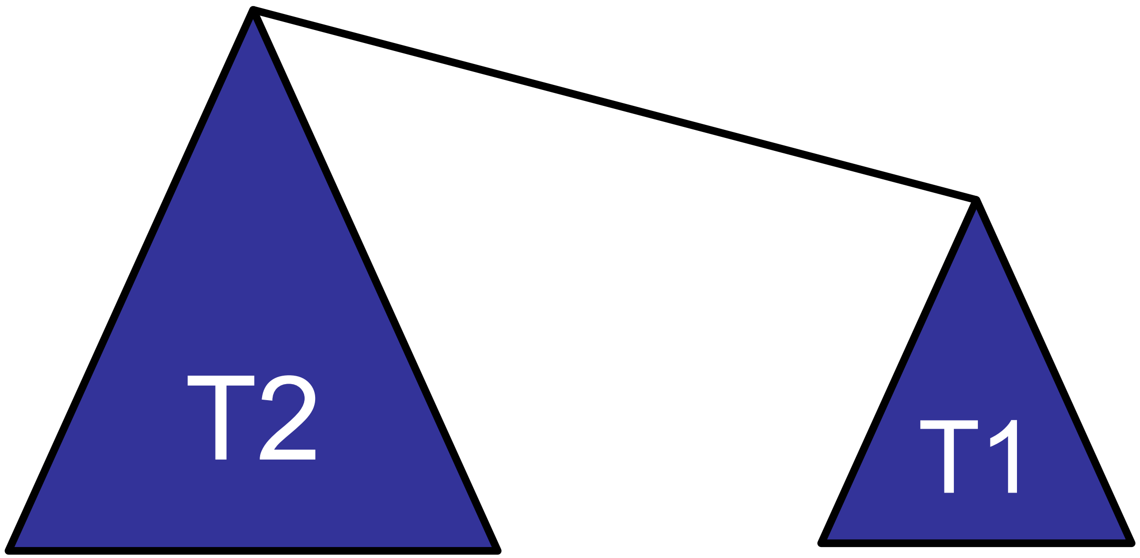


Weighted Union

Claim:

A tree of height k has size at least 2^k.

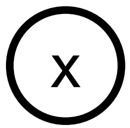
➔ height of tree of size n is at most log(n)



Weighted Union

Analysis:

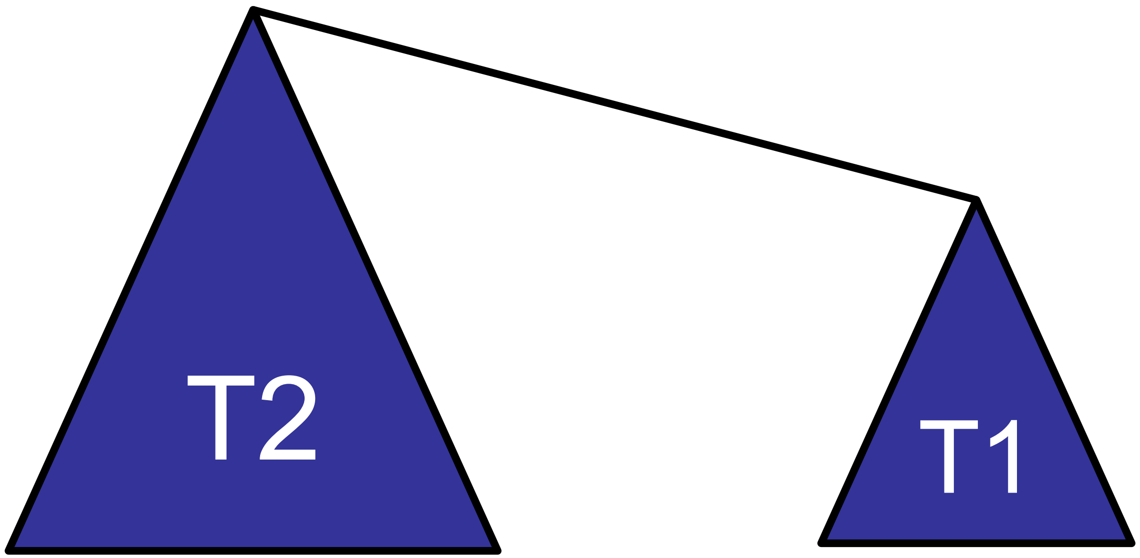
- Base case: tree of height 0 contains 1 object.



Weighted Union

Induction:

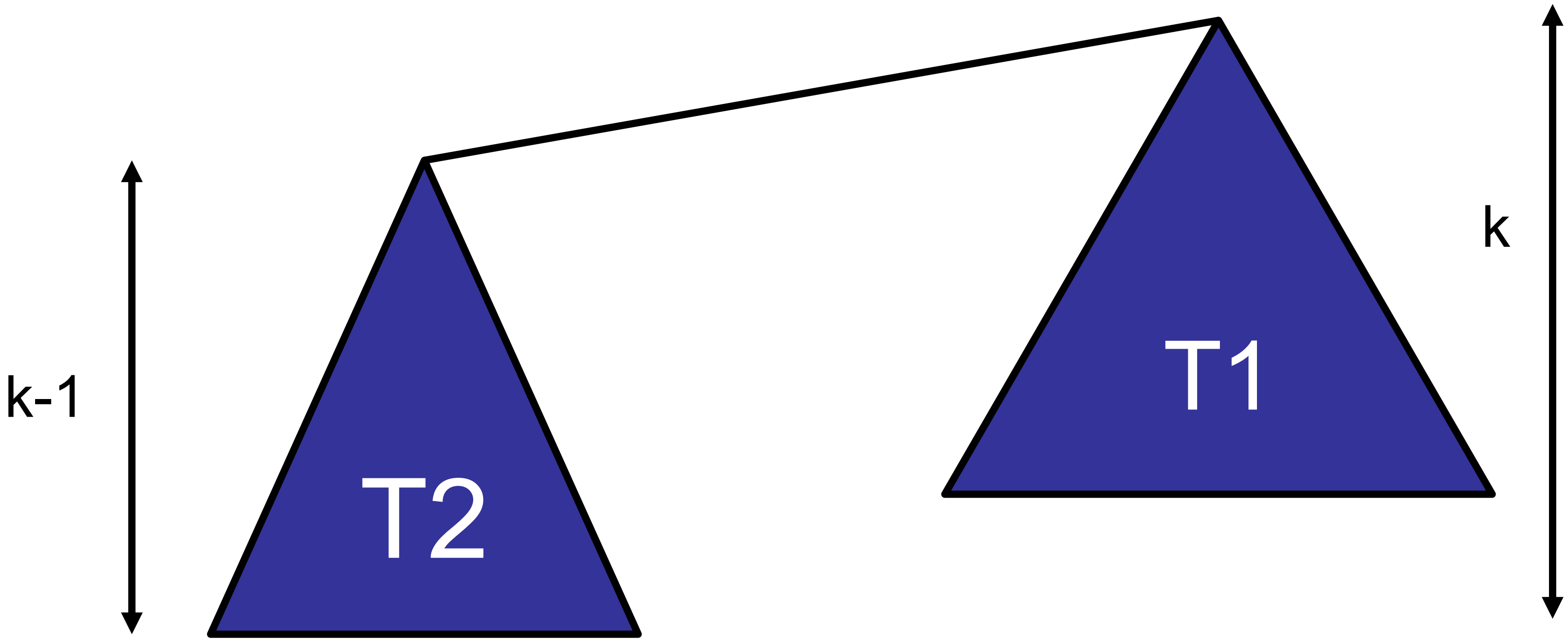
- **Assume:** A tree of height k-1 has size at least 2^{k-1}.
- **Show:** A tree of height k has size at least 2^k.



Weighted Union

How do you get a tree of height k?

Make tree of height (k-1) the child of another tree.

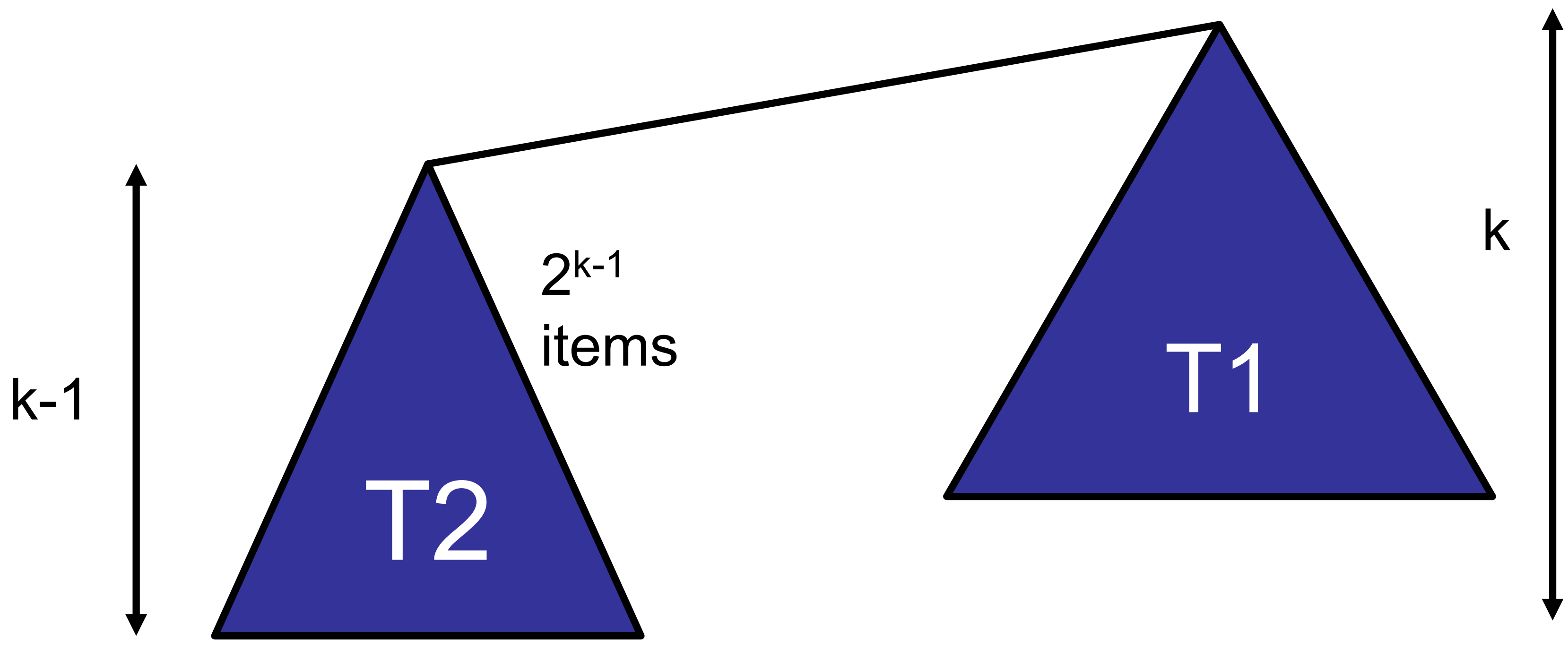


Weighted Union

How do you get a tree of height k?

Make tree of height (k-1) the child of another tree.

Tree T2 has size at least 2^{k-1} by induction.

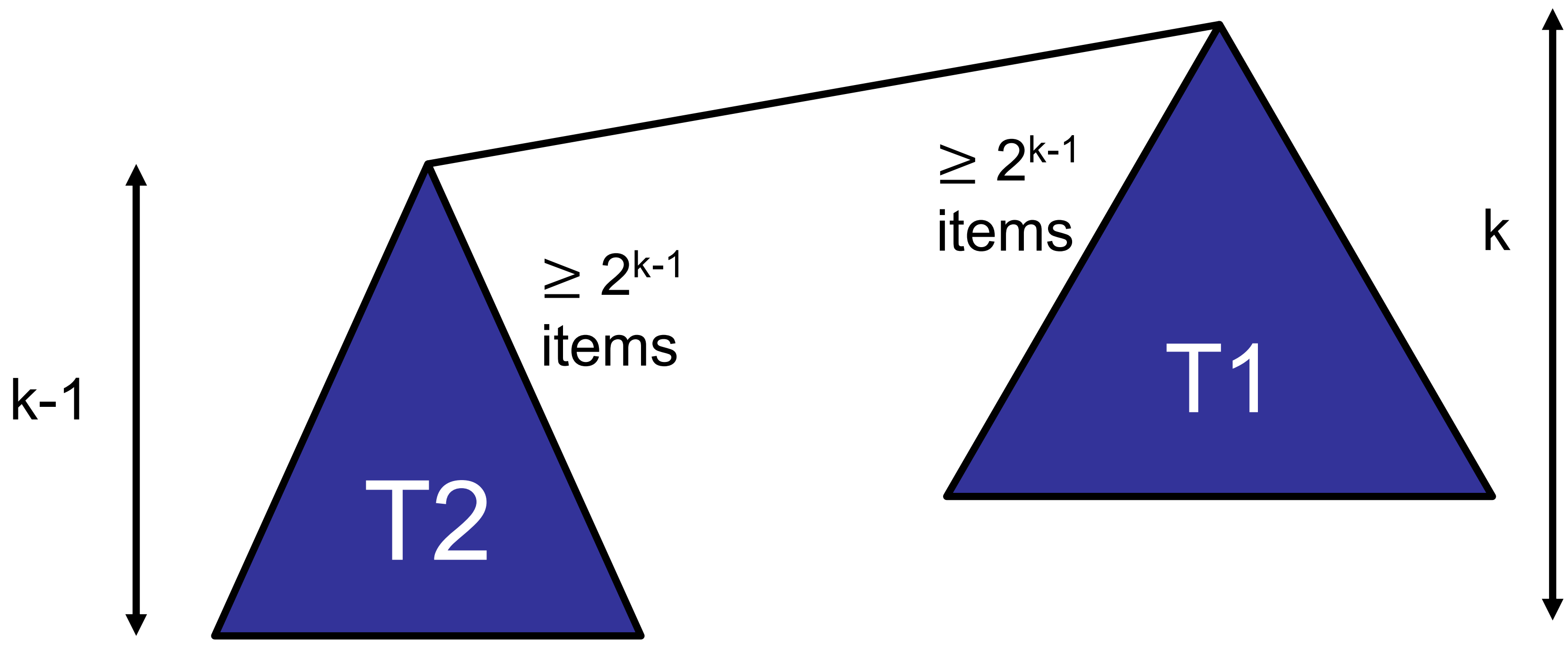


Weighted Union

How do you get a tree of height k?

Tree T2 has size at least 2^{k-1} by induction.

➔ $\text{size}[T1] \geq \text{size}[T2] \geq 2^{k-1}$ by union-by-weight-rule



Weighted Union

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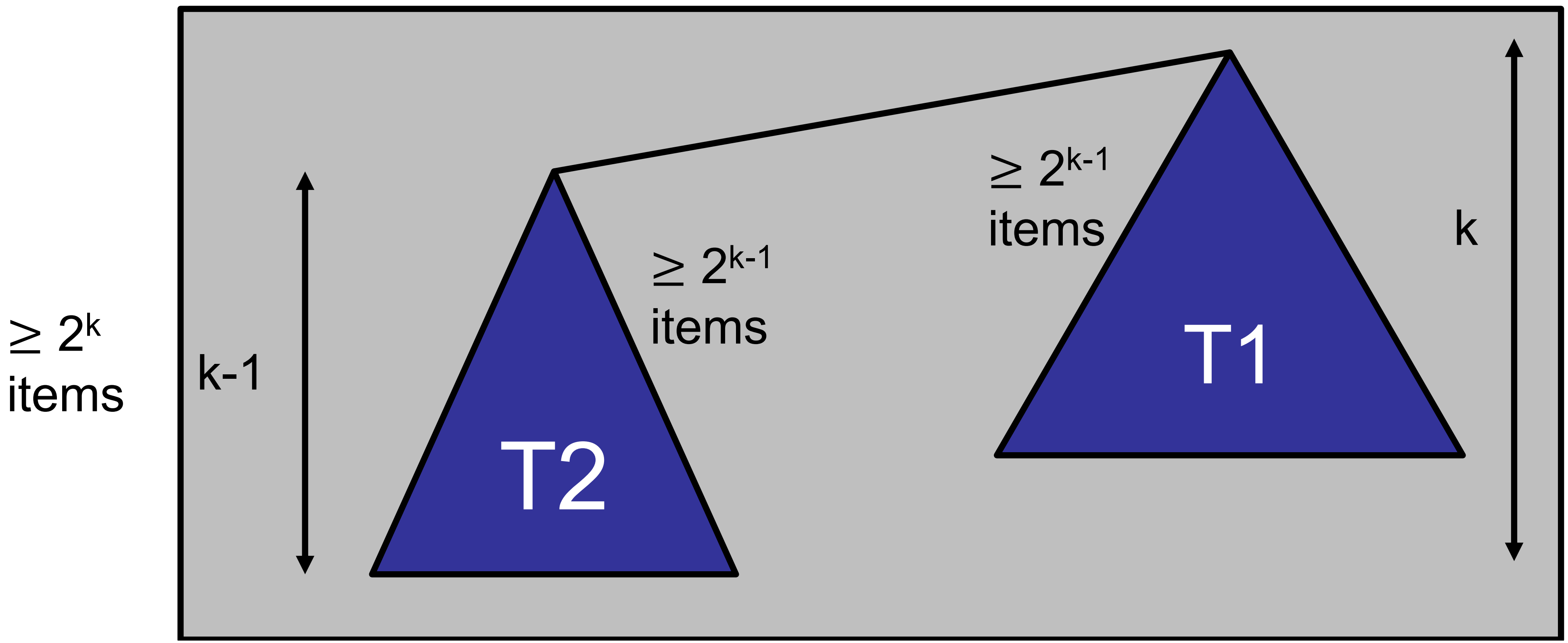
➔ $\text{size}[T1] \geq \text{size}[T2] \geq 2^{k-1}$ by union-by-weight-rule

➔ $\text{size}[T1 + T2] \geq 2^{k-1} + 2^{k-1} \geq 2^k$

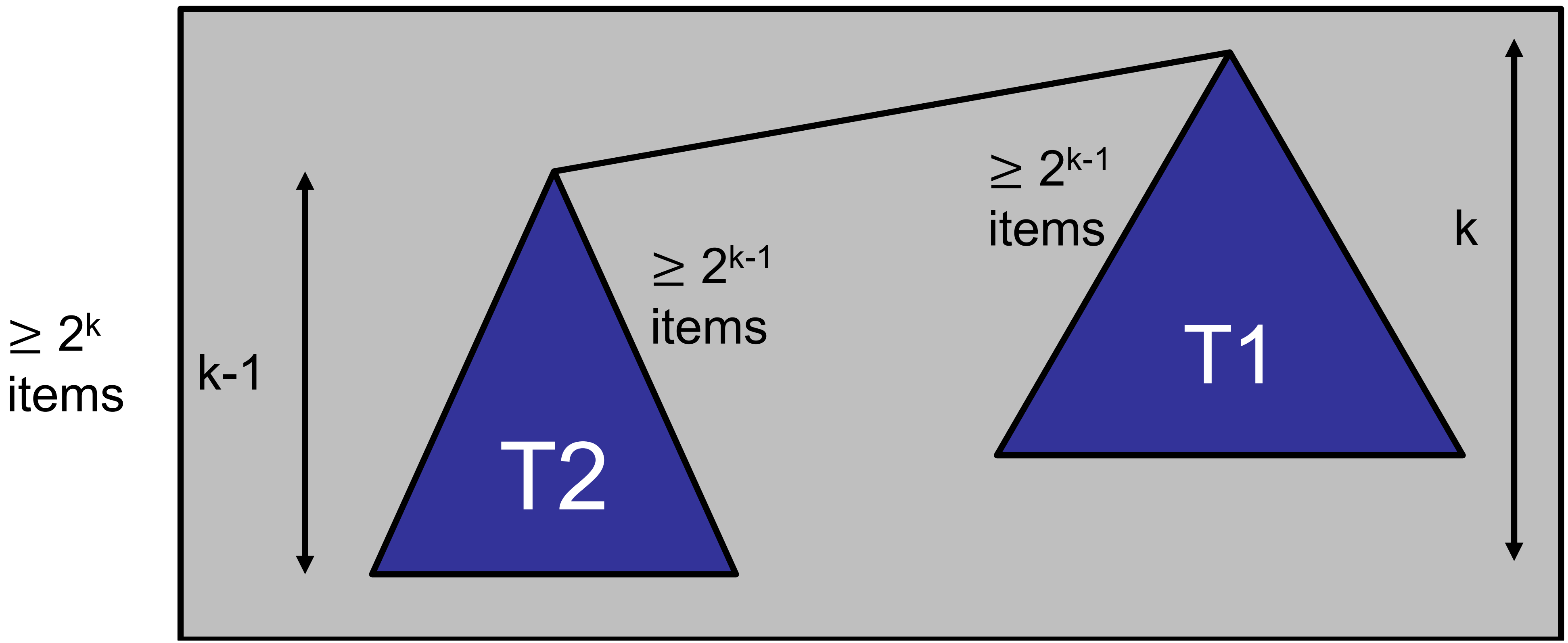


Weighted Union

Claim:

A tree of height k has size at least 2^k.

➔ height of tree of size n is at most log(n)



Running time of (Find, Union):

1. O(1), O(1)
2. O(1), O(n)
3. O(n), O(1)
4. O(n), O(n)
5. O(log n), O(log n)
6. None of the above.

ARCHIPELAGO

is open

Weighted Union

```
union(int p, int q) {
  while (parent[p] !=p) p = parent[p];
  while (parent[q] !=q) q = parent[q];
  if (size[p] > size[q] {
          parent[q] = p;    // Link q to p
          size[p] = size[p] + size[q];
  }
  else {
          parent[p] = q; // Link p to q
          size[q] = size[p] + size[q];
  }
}
```

Union-Find Summary

Quick-find and Quick-union are slow:

- Union and/or find is expensive
- Quick-union: tree is too deep

Weighted-union is faster:

- Trees too balanced: O(log n)
- Union *and* find are O(log n)

	find	**union**
quick-find	O(1)	O(n)
quick-union	O(n)	O(n)
weighted-union	O(log n)	O(log n)

Union-Find Summary

Notes:

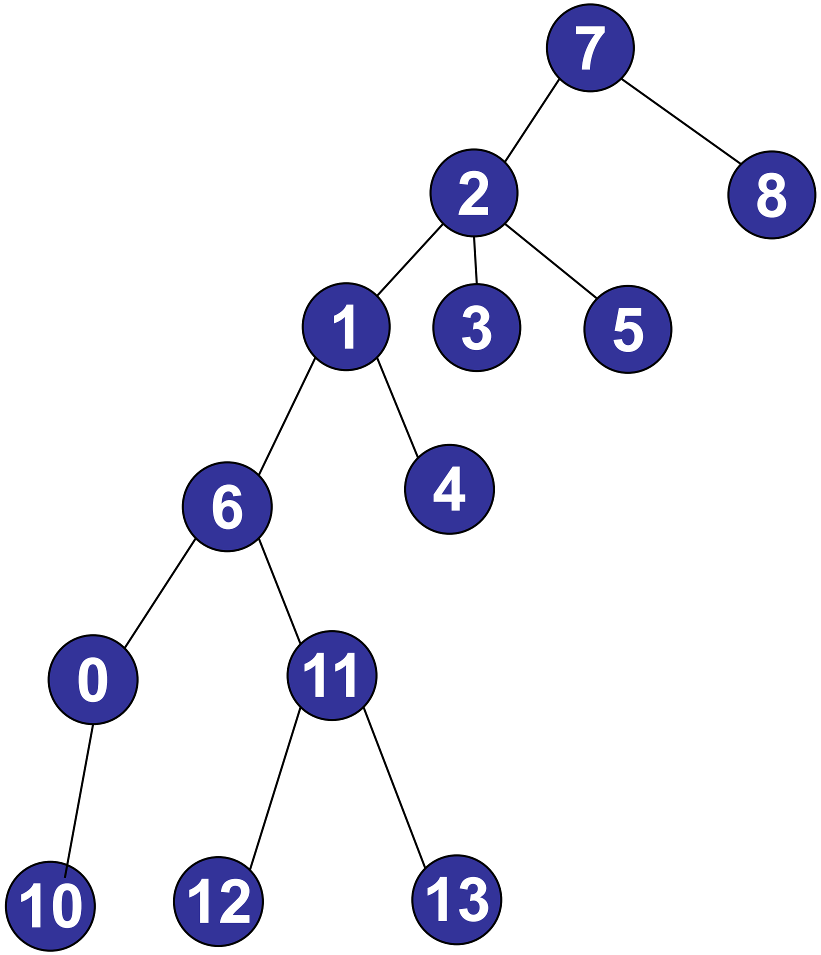
- Some prefer union-by-rank (where rank = log(size))
- Some prefer union-by-height (same idea)

Important property:

- weight/rank/size/height of subtree does not change except at root (so only update root on union).
- weight/rank/size/height only increases when tree size doubles.

Path Compression

After finding the root: set the parent of each traversed node to the root.



Path Compression

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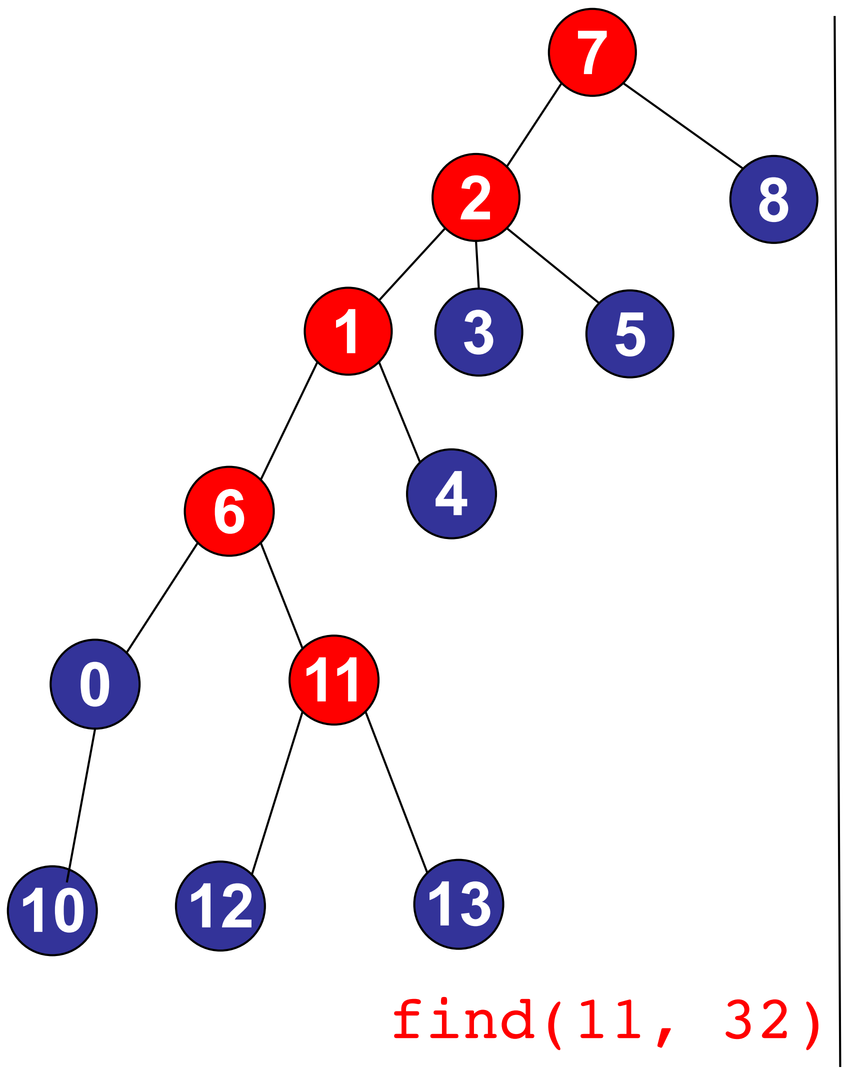

Path Compression

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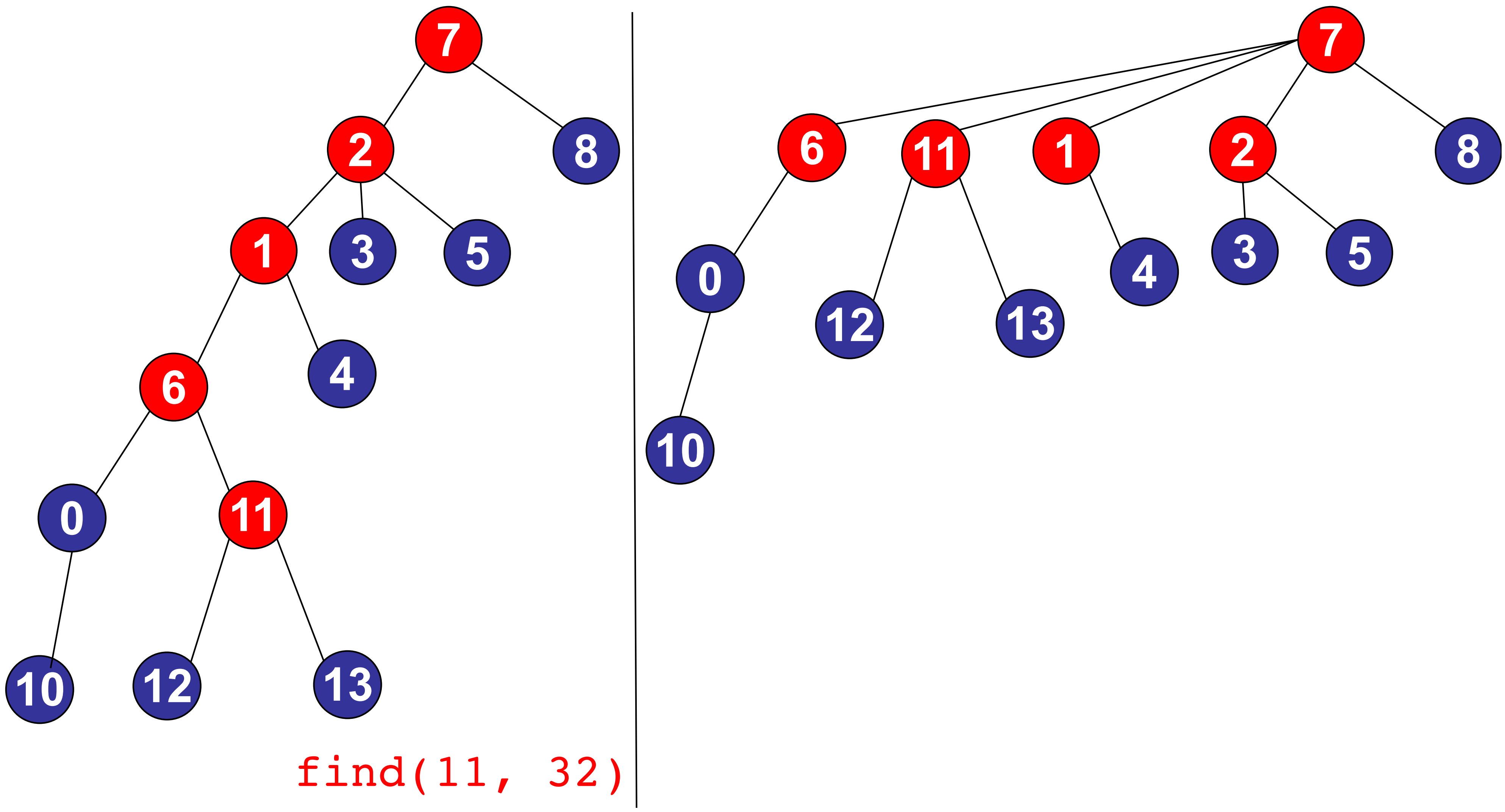

Path Compression

```
findRoot(int p) {
  root = p;
  while (parent[root] != root) root = parent[root];
  return root;
}
```

Path Compression

```
findRoot(int p) {
  root = p;
  while (parent[root] != root) root = parent[root];
  while (parent[p] != p) {
        temp = parent[p];
        parent[p] = root;
        p = temp;
  }
  return root;
}
```

Alternative Path Compression

```
findRoot(int p) {
  root = p;
  while (parent[root] != root) {
        parent[root] = parent[parent[root]];
        root = parent[root];
  }
  return root;
}
```

Make every other node in the path point to its grandparent!

- Simple
- Works as well!

Weight Union with Path Compression

Theorem: [Tarjan 1975]

Starting from empty, any sequence of m union/find operations on n objects takes: $O(n + m\alpha(m, n))$time.

Weight Union with Path Compression

Theorem: [Tarjan 1975]

Starting from empty, any sequence of m union/find operations on n objects takes: $O(n + m\alpha(m, n))$ time.

Inverse Ackermann function: always ≤ 5 in this universe.

n	α(n, n)
4	0
8	1
32	2
8,192	3
2^{65533}	4

Weight Union with Path Compression

Theorem: [Tarjan 1975]

Starting from empty, any sequence of m union/find operations on n objects takes: $\mathrm{O}(n + m\alpha(m, n))$time.

Proof:

Weight Union with Path Compression

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Proof:

- Very difficult.
- Algorithm: very simple to implement.

Weight Union with Path Compression

Theorem: [Tarjan 1975]

Starting from empty, any sequence of m union/find operations on n objects takes: $O(n + m\alpha(m, n))$ time.

Proof:

- Very difficult.
- Algorithm: very simple to implement.

Can we do better? No!

- Proof: impossible to achieve linear time.

Union-Find Summary

Weighted-union is faster:

- Trees are flat: O(log n)
- Union *and* find are O(log n)

Weighted Union + Path Compression is very fast:

- Trees very flat.
- On average, almost linear performance per operation.

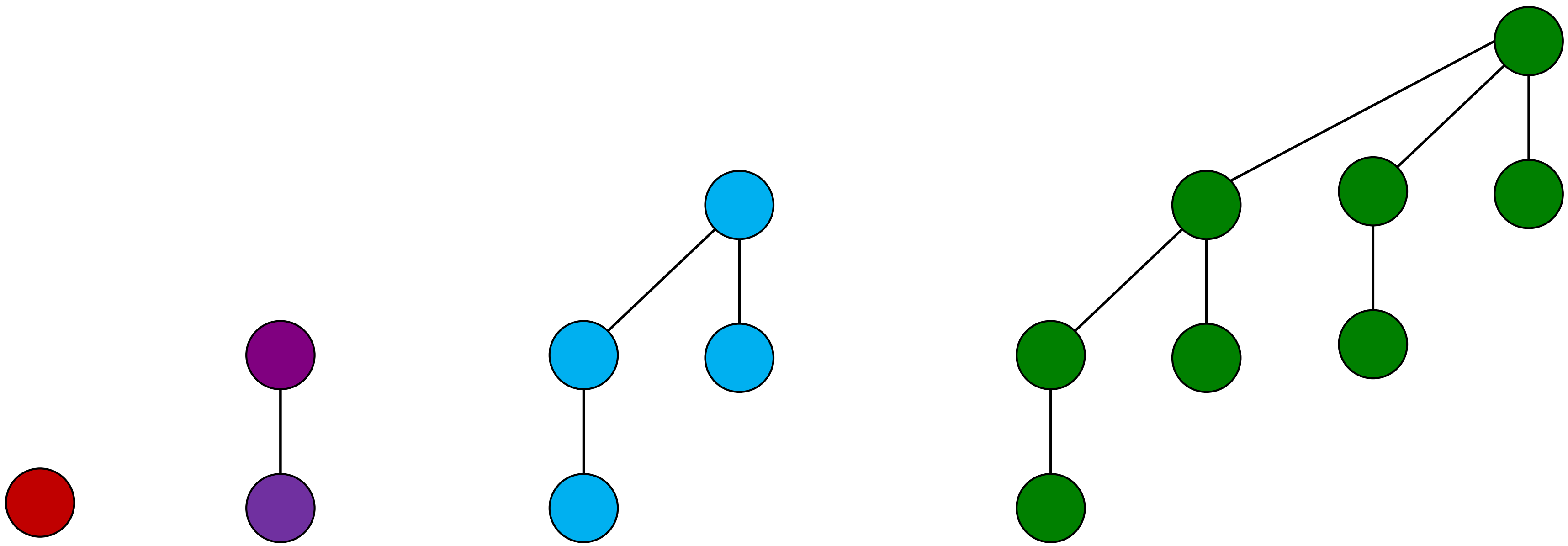
	find	union
quick-find	O(1)	O(n)
quick-union	O(n)	O(n)
weighted-union	O(log n)	O(log n)
weighted-union with path-compression	$\alpha(m, n)$	$\alpha(m, n)$

Union-Find Summary

Path Compression **without** weighted union?

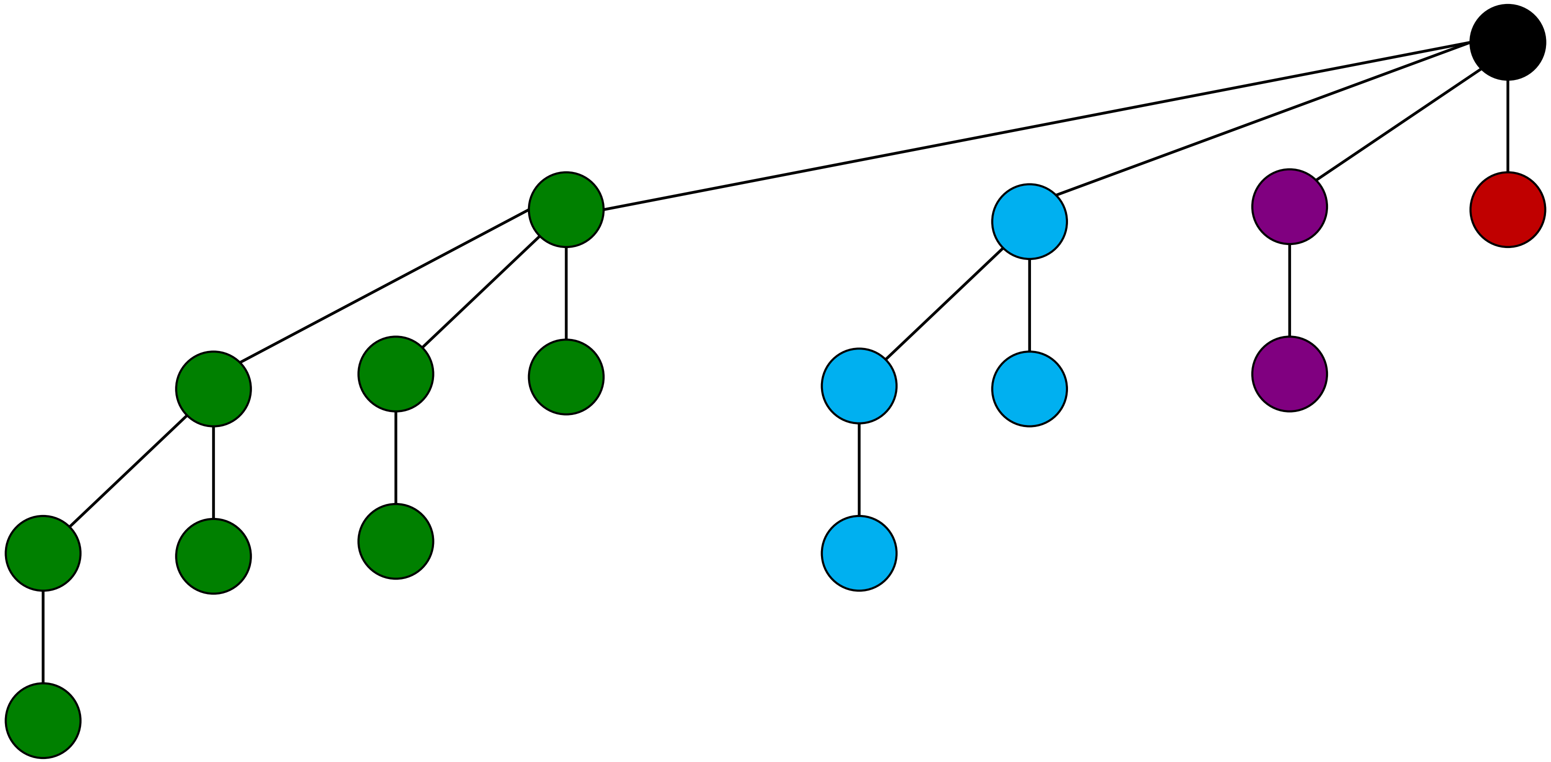
	find	**union**
quick-find	O(1)	O(n)
quick-union	O(n)	O(n)
weighted-union	O(log n)	O(log n)
path compression	O(log n)	O(log n)
weighted-union with path-compression	α(m, n)	α(m, n)

Binomial Trees:



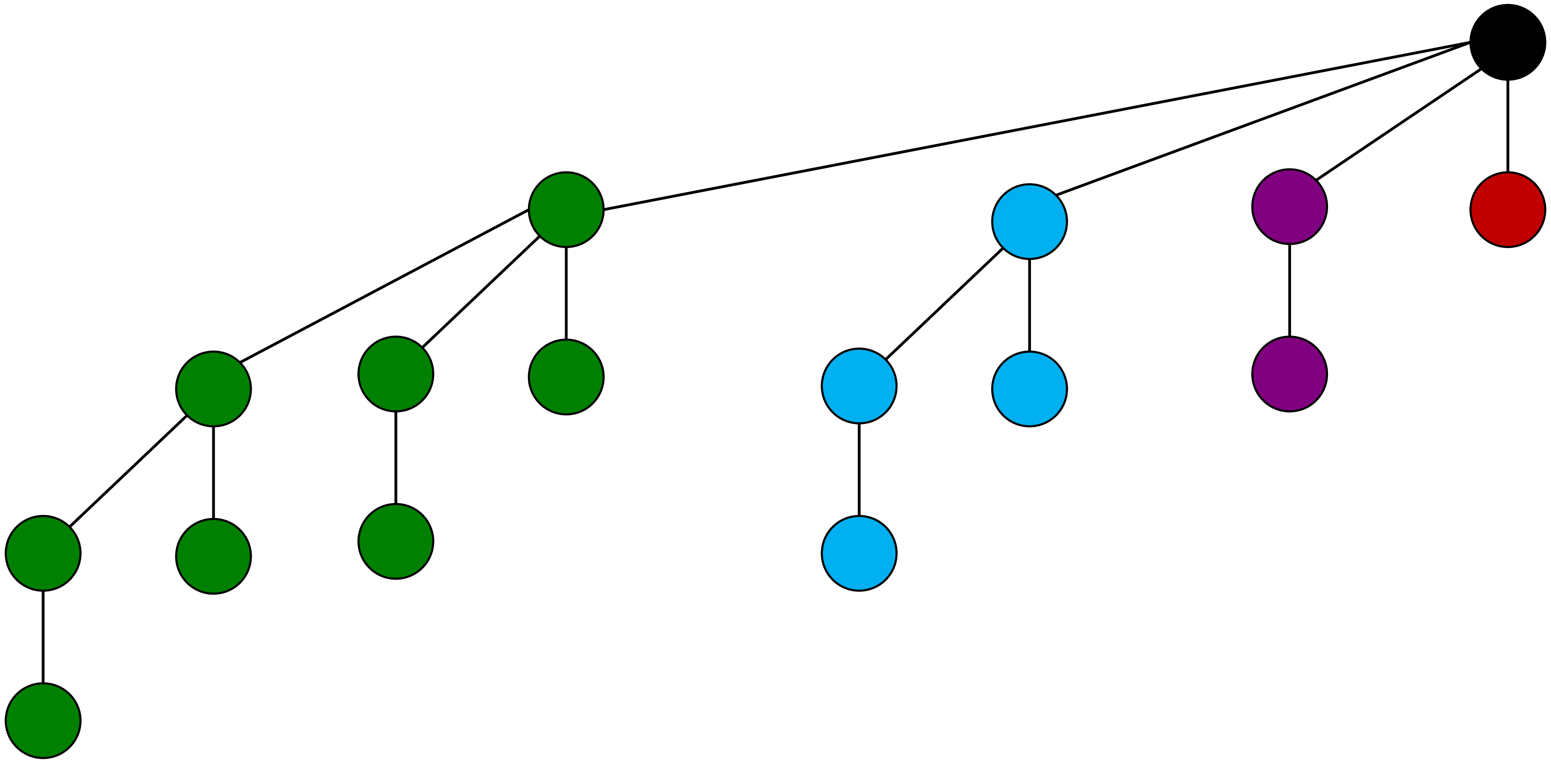
B0 B1 B2 B3

Binomial Trees:



B4 = (root + B0 + B1 + B2 + B3) = (B3 + B3)

Binomial Trees:



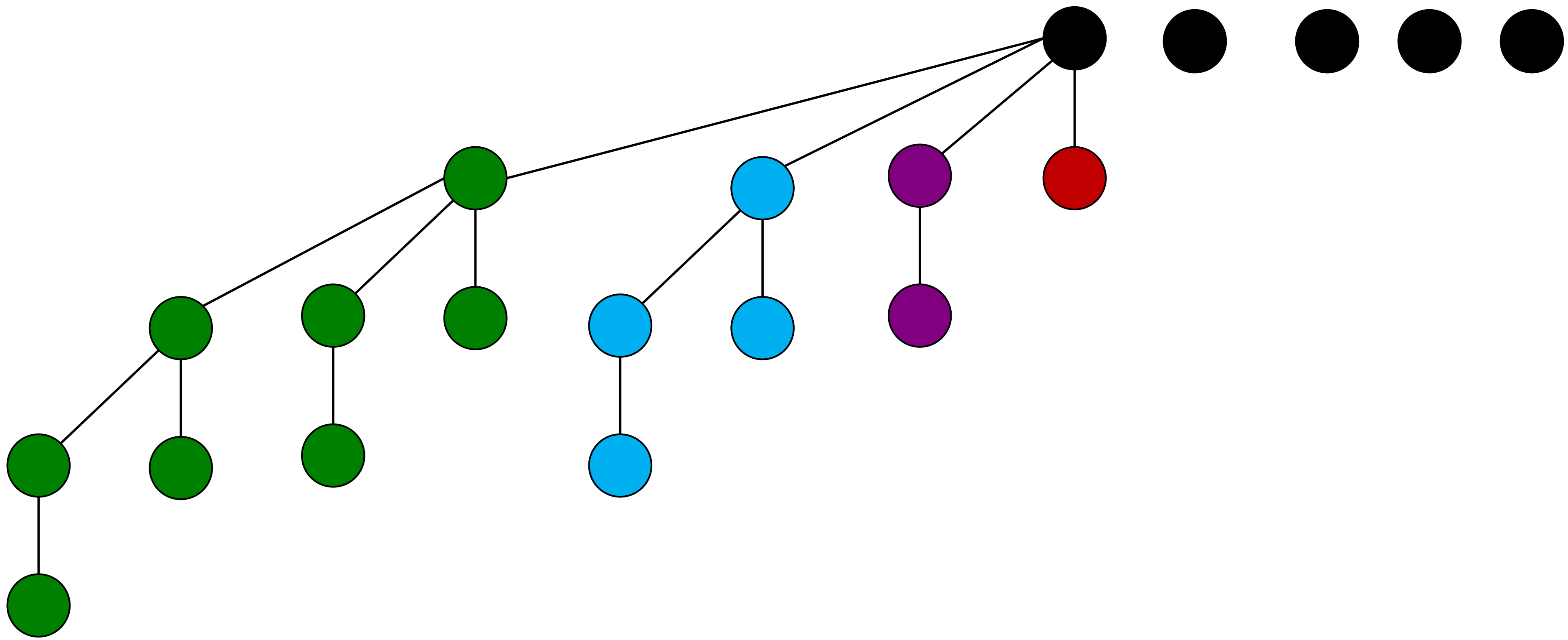
size(Bk) = $\Theta(2^k)$

height(Bk) = k-1

Union Find Example:

Step 1: Build Binomial tree using union operations.

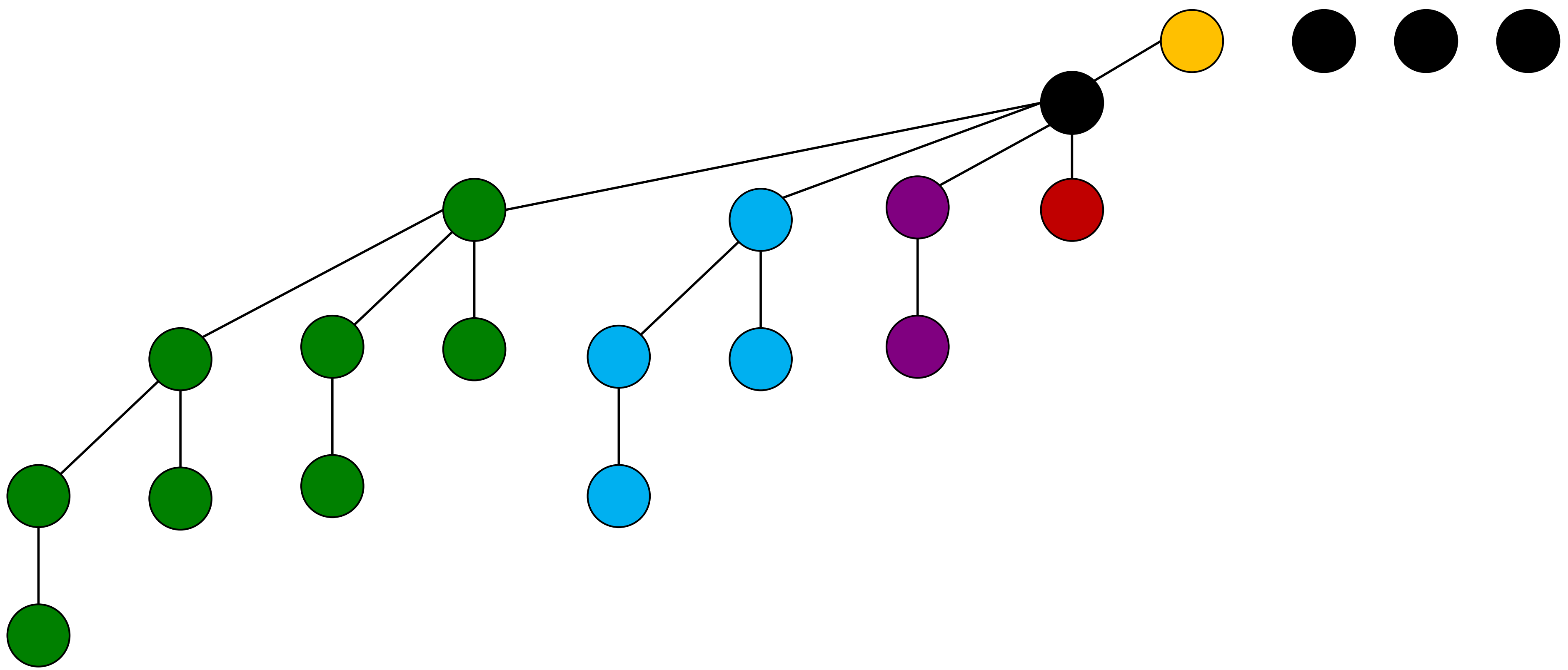
- Leave some extra objects free.



Union Find Example:

Step 1: Build Binomial tree using union operations.

Step 2: Union: create new root [O(1)]

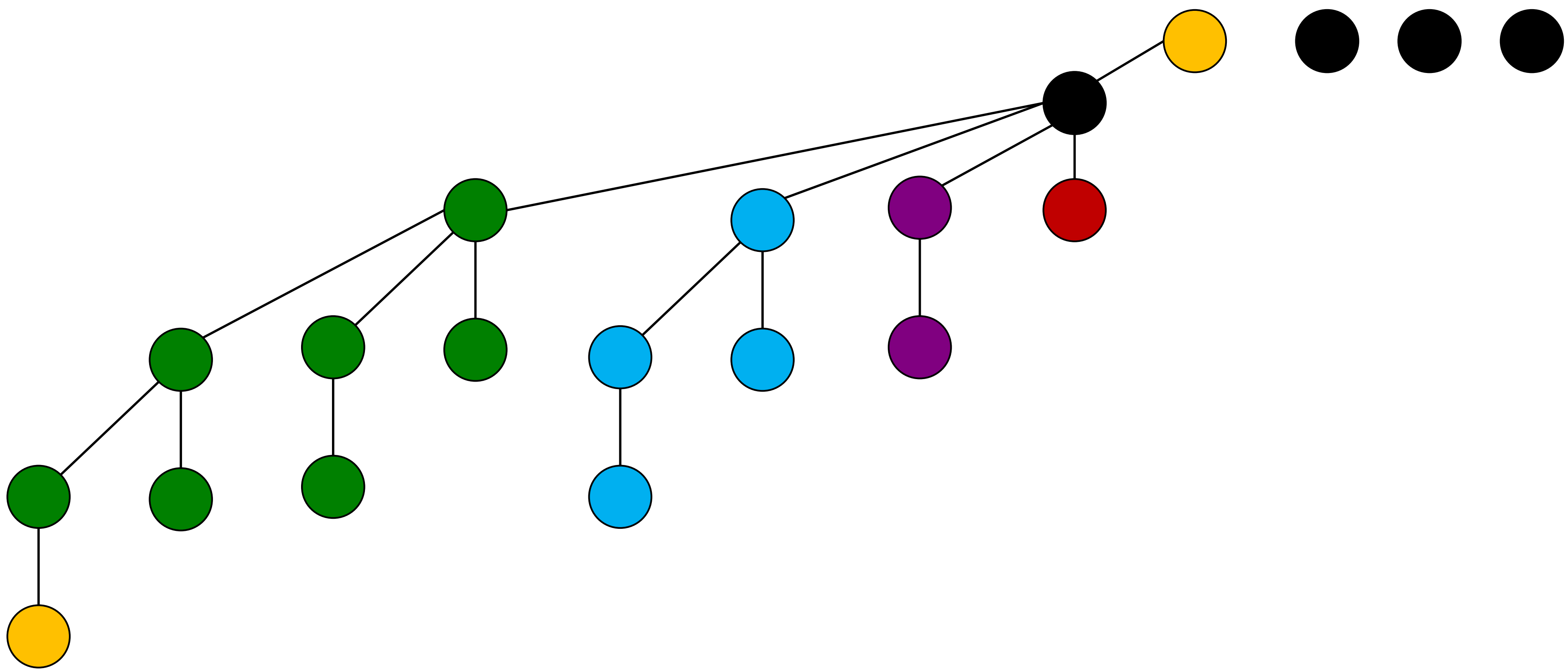


Union Find Example:

Step 1: Build Binomial tree using union operations.

Step 2: Union: create new root [O(1)]

Step 3: Find deepest leaf [O(log n)]



Union Find Example:

Step 1: Build Binomial tree using union operations.

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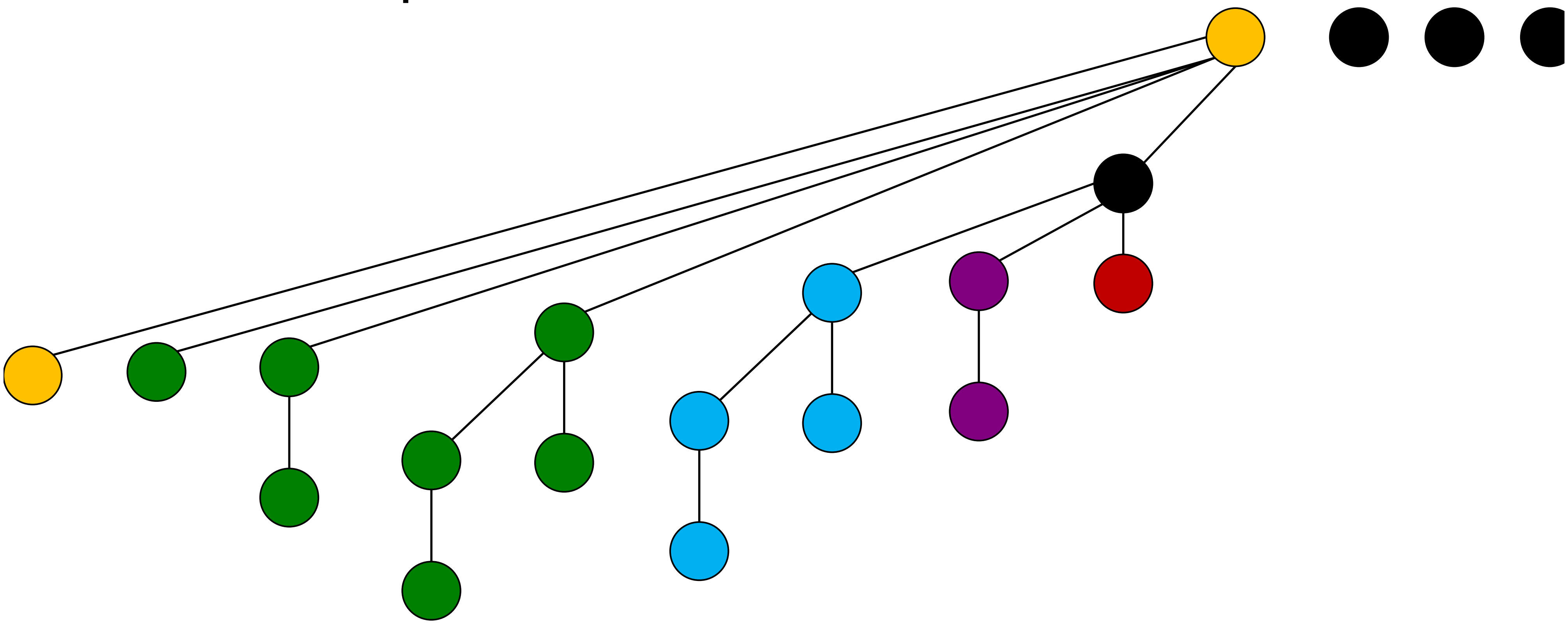
- Path compression…



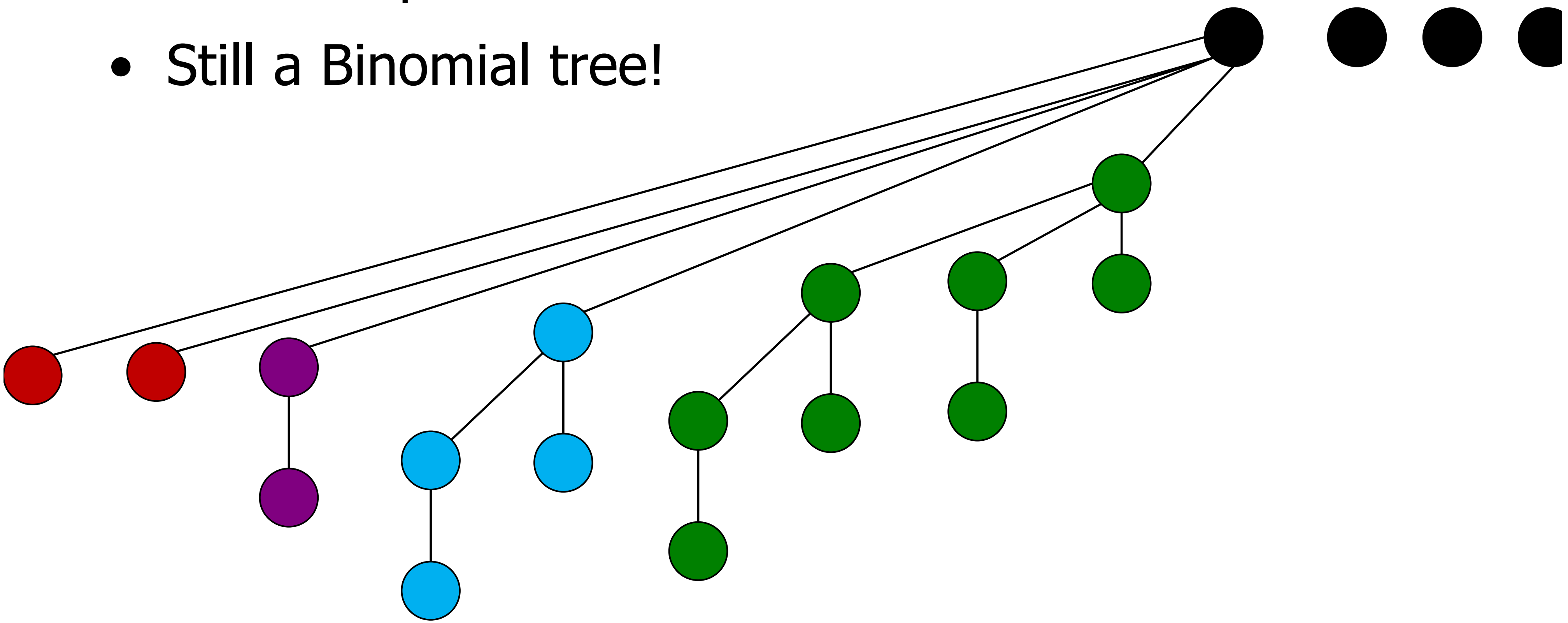
Union Find Example:

Step 1: Build Binomial tree using union operations.

Step 2: Union: create new root [O(1)]

Step 3: Find deepest leaf [O(log n)]

- Path compression…
- Still a Binomial tree!



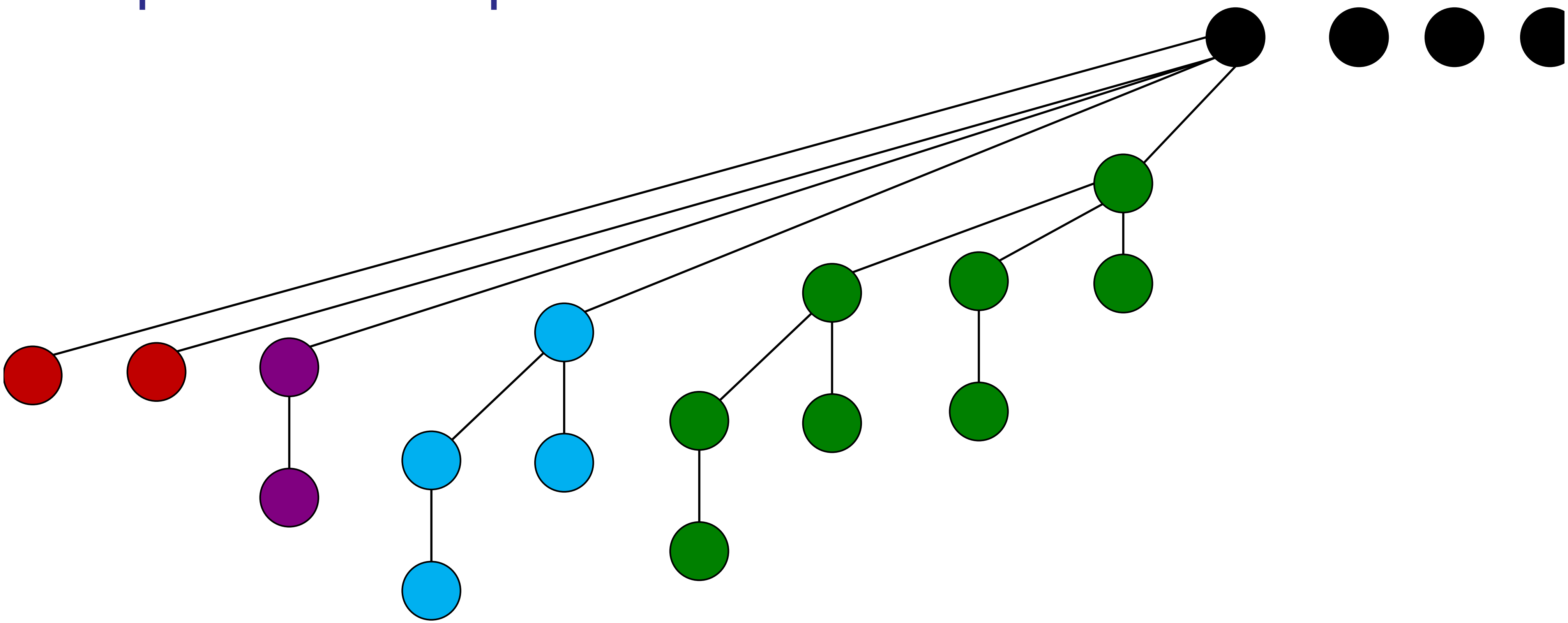
Union Find Example:

Step 1: Build Binomial tree using union operations.

Step 2: Union: create new root [O(1)]

Step 3: Find deepest leaf [O(log n)]

Step 4: Goto step 2.



Union-Find Summary

Path Compression **without** weighted union?

	find	**union**
quick-find	O(1)	O(n)
quick-union	O(n)	O(n)
weighted-union	O(log n)	O(log n)
path compression	O(log n)	O(log n)
weighted-union with path-compression	$\alpha(m, n)$	$\alpha(m, n)$

Union-Find Summary

What about Union-Split-Find?

Insert and delete edges.

Dynamic graph connectivity in polylogarithmic worst case time

Bruce M. Kapron * Valerie King * Ben Mountjoy *

Abstract

The dynamic graph connectivity problem is the following: given a graph on a fixed set of n nodes which is undergoing a sequence of edge insertions and deletions, answer queries of the form $q(a, b)$: "Is there a path between nodes a and b?" While data structures for this problem with polylogarithmic *amortized* time per operation have been known since the mid-1990's, these data structures have $\Theta(n)$ worst case time. In fact, no previously known solution has worst case time per operation which is $o(\sqrt{n})$.

We present a solution with worst case times $O(\log^4 n)$ per edge insertion, $O(\log^5 n)$ per edge deletion, and $O(\log n/\log\log n)$ per query. The answer to each query is correct if the answer is "yes" and is correct with high probability if the answer is "no". The data structure is based on a simple novel idea which can be used to quickly identify an edge in a cutset.

Our technique can be used to simplify and significantly

Though the problem of improving the worst case update time from $O(\sqrt{n})$ has been posed in the literature many times, there has been no improvement since 1985. In the words of Pătraşcu and Thorup, it is "perhaps the most fundamental challenge in dynamic graph algorithms today" [11].

Nearly every dynamic connectivity data structure maintains a spanning forest F. Dealing with edge insertions is relatively easy. The challenge is to find a replacement edge when a tree edge is deleted, splitting a tree into two subtrees. A replacement edge is an edge reconnecting the two subtrees, or, in other words, in the cutset of the cut $(T, V\backslash T)$ where T is one of the subtrees. An edge with both endpoints in the same subtree we call *internal* to the tree.

Applications

Many applications:

- Mazes
  - Are two locations connected?
- Games:
  - Can you get from one state to another?

Applications

Many applications:

- Networks
  - Are two locations connected?
- Least-common-ancestor:
  - Which node in a tree network is the closest ancestor?

Applications

Many applications:

- Programming languages
  - Hinley-Milner polymorphic type inference
  - Equivalence of finite state automata
  - Image processing in Matlab

- Physics:
  - Hoshen-Kopelman algorithm
  - Percolation
  - Conductance / insulation

Roadmap

Disjoint Set Data Structure

- Problem: Dynamic Connectivity
- Algorithm: Quick-Find
- Algorithm: Quick-Union
- Optimizations
- Applications

Post mid-term…

Arnab Bhattacharyya

Interests:
Algorithms
Sampling Algorithms
Streaming Algorithms
Learning

Coming topics:

- Hashing and Hash Tables
- Graphs
- Graph exploration
- Shortest paths in graphs
- Minimum spanning trees

CS2040S
Data Structures and Algorithms

Disjoint Sets and Union-Find