MA2001 LINEAR ALGEBRA

MATRICES

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Definition of Matrix

- **Definition.** A **matrix** (plural **matrices**) is a rectangular array of numbers.
  - $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$
    - m is the number of **rows** in the matrix.
    - n is the number of **columns** in the matrix.
    - The **size** of the matrix is given by $m \times n$.
    - The (i, j)**-entry** is the entry in ith row & jth column.
  - In the given matrix, the (i, j)-entry is a_{ij}.
- **Remark.** Some books use $[\cdots]$ instead of $(\cdots)$.
  - Notations $|\cdots|$ and $\|\cdots\|$ are reserved to use later.

Examples

1. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & -1 \end{pmatrix}$ is a 3×2 matrix.
   - The $(1, 2)$-entry is 2 and the $(3, 1)$-entry is 0.
2. $\begin{pmatrix} \sqrt{2} & 3.1 & -2 \\ 3 & \frac{1}{2} & 0 \\ 0 & \pi & 0 \end{pmatrix}$ is a 3×3 matrix.
3. $\begin{pmatrix} 2 & 1 & 0 \end{pmatrix}$ is a 1×3 matrix.
4. $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is a 3×1 matrix.
5. (4) is a 1×1 matrix.
   - A 1×1 matrix is usually treated as a real number in computation. For instance, $(4) = 4$.

Notation of Matrices

- A matrix is usually denoted by capital letters $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$.
  - $m \times n$ matrix $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$.
    - a_{ij} is the (i, j)-entry of $\boldsymbol{A}$.
    - It is denoted by $\boxed{\boldsymbol{A} = (a_{ij})_{m\times n}}$.

    Sometimes, if the size of $\boldsymbol{A}$ is known (or not important)

    - simple notation: $\boxed{\boldsymbol{A} = (a_{ij})}$.

Examples

- Write down the following matrices explicitly.
  1. $\boldsymbol{A} = (a_{ij})_{2\times 3}$, where $a_{ij} = i + j$.

$$\begin{aligned} \boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} &= \begin{pmatrix} 1+1 & 1+2 & 1+3 \\ 2+1 & 2+2 & 2+3 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix} \end{aligned}$$

  2. $\boldsymbol{B} = (b_{ij})_{3\times 2}$, where $b_{ij} = \begin{cases} 1 & \text{if } i+j \text{ is even} \\ -1 & \text{if } i+j \text{ is odd} \end{cases}$

$$\boldsymbol{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Special Matrices

- A **row matrix** (**row vector**) is a matrix with only one row.
  - $\begin{pmatrix} 2 & 1 & 0 \end{pmatrix}$ is a row matrix (row vector).
- A **column matrix** (**column vector**) is a matrix with only one column.
  - $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is a column matrix (column vector).
- A **square matrix** is a matrix with the same number of rows and columns.
  - An $n \times n$ matrix is called a **square matrix** of **order** n.
  - $\begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix}$ is a square matrix of order 3.

Special Matrices

- Let $\boldsymbol{A} = (a_{ij})$ be a **square matrix** of order n.
  - The **diagonal** of $\boldsymbol{A}$ is the sequence of entries
    - $\boxed{a_{11}, a_{22}, \ldots, a_{nn}}$
  - $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$
  - $a_{ii}, i = 1, \ldots, n$, are the **diagonal entries**.
  - $a_{ij}, i \neq j$, are called the **non-diagonal entries**.
    - a_{ij} is $\begin{cases} \text{a diagonal entry} & \text{if } i = j, \\ \text{a non-diagonal entry} & \text{if } i \neq j. \end{cases}$

Special Matrices

- Let $\boldsymbol{A} = (a_{ij})$ be a **square matrix** of order n.
  - The **diagonal** of $\boldsymbol{A}$ is the sequence of entries
    - $\boxed{a_{11}, a_{22}, \ldots, a_{nn}}$
  - $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$.
- **Remark.** In some textbook,
  - The **diagonal** is also called the **principle diagonal** or **major diagonal**.
  - The **anti-diagonal** or **minor diagonal** refers to the diagonal from the right top to the left bottom.
    - $a_{1n}, a_{2,n-1}, \ldots, a_{n1}$.

Special Matrices

- A **square matrix** is called a **diagonal matrix** if all its non-diagonal entries are zero.
  - $\boldsymbol{A} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$.
  - $\boldsymbol{A} = (a_{ij})_{n\times n}$ is diagonal $\Leftrightarrow a_{ij} = 0$ for all $i \neq j$.

  Example.
  - $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is a diagonal matrix.
  - $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, a, b, c are (possibly zero) numbers.

Special Matrices

- A **diagonal matrix** is called a **scalar matrix** if all its diagonal entries are the same.
  - $\boldsymbol{A} = \begin{pmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{pmatrix}$, where c is a constant.
  - $\boldsymbol{A} = (a_{ij})_{n\times n}$ is scalar $\Leftrightarrow a_{ij} = \begin{cases} c & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$

  Example.
  - $\begin{pmatrix} 2001 & 0 \\ 0 & 2001 \end{pmatrix}$ is a scalar matrix.
  - $\begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}$, where c is a (possibly zero) number.

Special Matrices

- A **scalar matrix** is called an **identity matrix** if all its diagonal entries are 1.
  - Denote the identity matrix of order n (size $n \times n$) by $\boldsymbol{I}_n$.
    - If no confusion in order, write $\boldsymbol{I}$ instead of $\boldsymbol{I}_n$.
  - $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$.
  - $\boldsymbol{A} = (a_{ij})_{n\times n}$ is identity $\Leftrightarrow a_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$
- **Note**: There is exactly one identity matrix in order n.
  - $\boldsymbol{I}_1 = (1) = 1; \quad \boldsymbol{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Special Matrices

- A matrix with all entries equal to zero is a **zero matrix**.
  - Denote the zero matrix of size $m \times n$ by $\mathbf{0}_{m\times n}$.
    - If no confusion in size, write $\mathbf{0}$ instead of $\mathbf{0}_{m\times n}$.
  - $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$.
  - $\boldsymbol{A} = (a_{ij})_{m\times n}$ is zero $\Leftrightarrow a_{ij} = 0$ for all i, j.

  Note: There is exactly one zero matrix in size $m \times n$.
  - $\mathbf{0}_{1\times 1} = (0) = 0$; $\mathbf{0}_{1\times 3} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$.
  - $\mathbf{0}_{3\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$; $\mathbf{0}_{3\times 4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Special Matrices

- A **square matrix** is called **symmetric** if it is symmetric with respect to the diagonal.
  - $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{pmatrix}$.
  - $\boldsymbol{A} = (a_{ij})_{n\times n}$ is symmetric $\Leftrightarrow a_{ij} = a_{ji}$ for all i, j.
    - (There is no restriction to the diagonal entries.)
- **Examples.**
  - (2); $\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$; $\begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix}$

Special Matrices

- A **square matrix** is called **upper triangular** if all the entries **below** the diagonal are zero.
  - $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & 0 & 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$
  - $\boldsymbol{A} = (a_{ij})_{n\times n}$ is upper triangular $\Leftrightarrow a_{ij} = 0$ if $i > j$.
    - (There is no restriction to the diagonal entries.)
- **Examples.**
  - (2); $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$; $\begin{pmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{pmatrix}$

Special Matrices

- A **square matrix** is called **lower triangular** if all the entries **above** the diagonal are zero.
  - $\boldsymbol{A} = \begin{pmatrix} a_{11} & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & 0 & \cdots & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{pmatrix}.$
  - $\boldsymbol{A} = (a_{ij})_{n\times n}$ is lower triangular $\Leftrightarrow a_{ij} = 0$ if $i < j$.
    - (There is no restriction to the diagonal entries.)
- **Examples.**
  - (2); $\begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}$; $\begin{pmatrix} a & 0 & 0 & 0 \\ b & e & 0 & 0 \\ c & f & h & 0 \\ d & g & i & j \end{pmatrix}$

Special Matrices

- Both **upper triangular matrices** and **lower triangular matrices** are called **triangular matrices**.
  - (A matrix is both upper and lower triangular

    $\Leftrightarrow$ it is diagonal.)
- **More Examples.**
  - Square matrices:
    - $\begin{pmatrix} 4 \end{pmatrix} \begin{pmatrix} 0 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ -1 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 6 & 2 \\ 0 & 3 & 9 & -1 \\ 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 1 \end{pmatrix}$
  - Diagonal matrices:
    - $\begin{pmatrix} 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Special Matrices

- **More Examples.**
  - Scalar matrices:
    - $\begin{pmatrix} 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$
  - Identity matrices:
    - $\begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
  - Zero matrices:
    - $\begin{pmatrix} 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Special Matrices

- **More Examples.**
  - Symmetric matrices:
    - $\begin{pmatrix} 4 \end{pmatrix}$ $\begin{pmatrix} 0 & 4 \\ 4 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & 1 & 6 & -2 \\ 1 & 3 & 0 & -1 \\ 6 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix}$
  - Upper triangular matrices:
    - $\begin{pmatrix} 4 \end{pmatrix}$ $\begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}$
  - Lower triangular matrix:
    - $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix}$

Matrix Operations

Identical Matrices

- A matrix is completely determined by its size and entries.
- **Definition.** Two matrices are **equal** if
  - they have the same size (same number of rows, same number of columns), and
  - all the corresponding entries are the same.

  Let $\boldsymbol{A} = (a_{ij})_{m\times n}$ and $\boldsymbol{B} = (b_{ij})_{p\times q}$. Then
  - $\boldsymbol{A} = \boldsymbol{B} \Leftrightarrow \boxed{m = p \text{ \& } n = q \text{ \& } a_{ij} = b_{ij} \text{ for all } i, j}$
- **Examples.**
  - $\mathbf{0}_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; $\mathbf{0}_{2\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
    - $\mathbf{0}_{2\times 2}$ and $\mathbf{0}_{2\times 3}$ have different size $\Rightarrow \mathbf{0}_{2\times 2} \neq \mathbf{0}_{2\times 3}$.

Identical Matrices

- A matrix is completely determined by its size and entries.
- **Definition.** Two matrices are **equal** if
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  - all the corresponding entries are the same.

  Let $\boldsymbol{A} = (a_{ij})_{m\times n}$ and $\boldsymbol{B} = (b_{ij})_{p\times q}$. Then
  - $\boldsymbol{A} = \boldsymbol{B} \Leftrightarrow \boxed{m = p \;\&\; n = q \;\&\; a_{ij} = b_{ij} \text{ for all } i, j}$
- **Examples.**
  - $\boldsymbol{A} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$; $\boldsymbol{B} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$.
    - $\boldsymbol{A} = \boldsymbol{B} \Leftrightarrow \begin{cases} x = 1 \\ y = -1 \\ z = 2 \\ w = 4 \end{cases}$

Addition, Subtraction & Scalar Multiplication

- Let $\boldsymbol{A} = (a_{ij})_{m\times n}$ and $\boldsymbol{B} = (b_{ij})_{m\times n}$ be matrices.
  - **Addition**: $\boldsymbol{A} + \boldsymbol{B} = (a_{ij} + b_{ij})_{m\times n}$.
  - **Subtraction**: $\boldsymbol{A} - \boldsymbol{B} = (a_{ij} - b_{ij})_{m\times n}$.
- **Example.**
  - Let $\boldsymbol{A} = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \end{pmatrix}$.

$$\begin{aligned} \boldsymbol{A} + \boldsymbol{B} &= \begin{pmatrix} 2+1 & 3+2 & 4+3 \\ 4+(-1) & 5+(-1) & 6+(-1) \end{pmatrix} \\ &= \begin{pmatrix} 3 & 5 & 7 \\ 3 & 4 & 5 \end{pmatrix} \\ \boldsymbol{A} - \boldsymbol{B} &= \begin{pmatrix} 2-1 & 3-2 & 4-3 \\ 4-(-1) & 5-(-1) & 6-(-1) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 \\ 5 & 6 & 7 \end{pmatrix} \end{aligned}$$

Addition, Subtraction & Scalar Multiplication

- Let $\boldsymbol{A} = (a_{ij})_{m\times n}$ be a matrix, and c a constant.
  - **Scalar multiplication**: $c\boldsymbol{A} = (ca_{ij})_{m\times n}$.
- **Example.**
  - Let $\boldsymbol{A} = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}$. Then

$$4\boldsymbol{A} = \begin{pmatrix} 4\cdot 2 & 4\cdot 3 & 4\cdot 4 \\ 4\cdot 4 & 4\cdot 5 & 4\cdot 6 \end{pmatrix} = \begin{pmatrix} 8 & 12 & 16 \\ 16 & 20 & 24 \end{pmatrix}$$

- **Remarks.**
  - $(-1)\boldsymbol{A}$ is usually denoted by $-\boldsymbol{A}$.
  - It can be proved that $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\boldsymbol{B})$.
    - In the discussion we usually only consider addition and scalar multiplication.

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Properties

- **Theorem.** Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be matrices of the same size.
  - $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\boldsymbol{B})$.
  - Commutative Law for Matrix Addition:
    - $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$.
  - Associative Law for Matrix Addition:
    - $(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$.
  - Let $\mathbf{0}$ be the zero matrix of the same size as $\boldsymbol{A}$.
    - $\mathbf{0} + \boldsymbol{A} = \boldsymbol{A}$; $\boldsymbol{A} - \boldsymbol{A} = \mathbf{0}$; $0\boldsymbol{A} = \mathbf{0}$; $c\mathbf{0} = \mathbf{0}$.
  - Distributive Law for Scalar Multiplication over Addition:
    - $c(\boldsymbol{A} + \boldsymbol{B}) = c\boldsymbol{A} + c\boldsymbol{B}$.
    - $(c + d)\boldsymbol{A} = c\boldsymbol{A} + d\boldsymbol{A}$.
  - $c(d\boldsymbol{A}) = (cd)\boldsymbol{A}$, $1\boldsymbol{A} = \boldsymbol{A}$.

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Properties

- Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices of the same size.
  - Prove that $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\boldsymbol{B})$.

Proof. Suppose that $\boldsymbol{A} = (a_{ij})_{m \times n}$ and $\boldsymbol{B} = (b_{ij})_{m \times n}$.

1. Verify that LHS and RHS have the same size.
   - $\boldsymbol{A}$ is $m \times n$ & $\boldsymbol{B}$ is $m \times n \Rightarrow \boldsymbol{A} - \boldsymbol{B}$ is $m \times n$.
   - $\boldsymbol{B}$ is $m \times n \Rightarrow -\boldsymbol{B}$ is $m \times n$.
     - $\boldsymbol{A}, -\boldsymbol{B}$ are $m \times n \Rightarrow \boldsymbol{A} + (-\boldsymbol{B})$ is $m \times n$.
2. Verify: "(i,j)-entry of LHS" $=$ "(i,j)-entry of RHS".

$$\begin{aligned}(i,j)\text{-entry of } (\boldsymbol{A} - \boldsymbol{B}) &= a_{ij} - b_{ij} \\ &= a_{ij} + (-b_{ij}) \\ &= a_{ij} + (i,j)\text{-entry of } (-\boldsymbol{B}) \\ &= (i,j)\text{-entry of } [\boldsymbol{A} + (-\boldsymbol{B})]\end{aligned}$$

Properties

- Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be matrices of the same size.
  - Prove that $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$.

Proof. Let $\boldsymbol{A} = (a_{ij})_{m \times n}, \boldsymbol{B} = (b_{ij})_{m \times n}, \boldsymbol{C} = (c_{ij})_{m \times n}$.

1. It is clear that LHS & RHS are both $m \times n$. (Why?)
2. Verify: "(i,j)-entry of LHS" $=$ "(i,j)-entry of RHS".

$$\begin{aligned}(i,j)\text{-entry of } \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) &= a_{ij} + [(i,j)\text{-entry of } \boldsymbol{B} + \boldsymbol{C}] \\ &= a_{ij} + (b_{ij} + c_{ij}) \\ &= (a_{ij} + b_{ij}) + c_{ij} \\ &= [(i,j)\text{-entry of } \boldsymbol{A} + \boldsymbol{B}] + c_{ij} \\ &= (i,j)\text{-entry of } (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}.\end{aligned}$$

$\therefore \quad \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}.$

- **Exercise**: Prove the remaining properties. (Question 2.18)

Matrix Multiplication

- Consider the following linear systems:
  - $\begin{cases} a_{11}y_1 + a_{12}y_2 = z_1 \\ a_{21}y_1 + a_{22}y_2 = z_2 \\ a_{31}y_1 + a_{32}y_2 = z_3 \end{cases} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$
  - $\begin{cases} b_{11}x_1 + b_{12}x_2 + b_{13}x_3 = y_1 \\ b_{21}x_1 + b_{22}x_2 + b_{23}x_3 = y_2 \end{cases} \quad \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$

$$\begin{cases} (a_{11}b_{11} + a_{12}b_{21})x_1 + (a_{11}b_{12} + a_{12}b_{22})x_2 + (a_{11}b_{13} + a_{12}b_{23})x_3 = z_1 \\ (a_{21}b_{11} + a_{22}b_{21})x_1 + (a_{21}b_{12} + a_{22}b_{22})x_2 + (a_{21}b_{13} + a_{22}b_{23})x_3 = z_2 \\ (a_{31}b_{11} + a_{32}b_{21})x_1 + (a_{31}b_{12} + a_{32}b_{22})x_2 + (a_{31}b_{13} + a_{32}b_{23})x_3 = z_3 \end{cases}$$

Can we use the coefficient matrices of the first two linear systems to obtain the coefficient matrix of their composite?

  - $\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{12}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \end{pmatrix}$

Matrix Multiplication

- Define matrix multiplication so that the **product** of
  - $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$ and $\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$ is given by
  - $\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{12}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \end{pmatrix}$

The $(3,2)$-entry of the product is $a_{31}b_{12} + a_{32}b_{22}$.

  - a_{31} and a_{32} are from the **3rd row** of the first matrix.
  - b_{12} and b_{22} are from the **2nd column** of the second.

In order to get the (i,j)-**entry** of the **product** matrix:

1. Find the ith **row** of the first matrix;
2. Find the jth **column** of the second matrix;
3. Multiply the corresponding entries.
4. Add the products together.

Matrix Multiplication

- **Definition.** Let $\boldsymbol{A} = (a_{ij})_{m\times p}$ and $\boldsymbol{B} = (b_{ij})_{p\times n}$.
  - $\boldsymbol{AB}$ is the $m \times n$ matrix such that its (i, j)-entry is
    - $\boxed{a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}}$

Note: No. of columns of $\boldsymbol{A}$ = the no. of rows of $\boldsymbol{B}$.

1. ith row of $\boldsymbol{A}$: $\begin{pmatrix} & \vdots & \\ a_{i1} \quad a_{i2} & \cdots & a_{ip} \\ & \vdots & \end{pmatrix}$
2. jth column of $\boldsymbol{B}$: $\begin{pmatrix} & b_{1j} & \\ & b_{2j} & \\ \cdots & \vdots & \cdots \\ & b_{pj} & \end{pmatrix}$
3. Multiply componentwise and add the products.

Examples

- $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix}$
  - $\begin{pmatrix} 1\cdot 1 + 2\cdot 2 + 3\cdot(-1) & 1\cdot 1 + 2\cdot 3 + 3\cdot(-2) \\ 4\cdot 1 + 5\cdot 2 + 6\cdot(-1) & 4\cdot 1 + 5\cdot 3 + 6\cdot(-2) \end{pmatrix}$
- $\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 7 & 9 \\ 14 & 19 & 24 \\ -9 & -12 & -15 \end{pmatrix}$
  - $\begin{pmatrix} 1\cdot 1 + 1\cdot 4 & 1\cdot 2 + 1\cdot 5 & 1\cdot 3 + 1\cdot 6 \\ 2\cdot 1 + 3\cdot 4 & 2\cdot 2 + 3\cdot 5 & 2\cdot 3 + 3\cdot 6 \\ (-1)\cdot 1 + (-2)\cdot 4 & (-1)\cdot 2 + (-2)\cdot 5 & (-1)\cdot 3 + (-2)\cdot 6 \end{pmatrix}$

- **Remark.** Matrix multiplication is **NOT commutative**.
  - $\boldsymbol{AB}$ is the **pre-multiplication** of $\boldsymbol{A}$ to $\boldsymbol{B}$ (to $\boldsymbol{B}$ by $\boldsymbol{A}$).
  - $\boldsymbol{BA}$ is the **post-multiplication** of $\boldsymbol{A}$ to $\boldsymbol{B}$ (to $\boldsymbol{B}$ by $\boldsymbol{A}$).

Properties

- **Theorem.**
  - Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ be $m \times p, p \times q, q \times n$ matrices, respectively.
    - Associative Law: $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$.
  - Let $\boldsymbol{A}$ be $m \times p$ matrix, $\boldsymbol{B}_1, \boldsymbol{B}_2$ be $p \times n$ matrices.
    - Distributive Law: $\boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2) = \boldsymbol{AB}_1 + \boldsymbol{AB}_2$.
  - Let $\boldsymbol{A}_1, \boldsymbol{A}_2$ be $m \times p$ matrices, $\boldsymbol{B}$ be $p \times n$ matrix.
    - Distributive Law: $(\boldsymbol{A}_1 + \boldsymbol{A}_2)\boldsymbol{B} = \boldsymbol{A}_1\boldsymbol{B} + \boldsymbol{A}_2\boldsymbol{B}$.
  - Let $\boldsymbol{A}$ be $m \times p$ and $\boldsymbol{B}$ be $p \times n$. For constant c,
    - $c(\boldsymbol{AB}) = (c\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}(c\boldsymbol{B})$.
  - Let $\boldsymbol{A}$ be an $m \times n$ matrix.
    - $\boldsymbol{A}\boldsymbol{0}_{n\times p} = \boldsymbol{0}_{m\times p}; \quad \boldsymbol{0}_{p\times m}\boldsymbol{A} = \boldsymbol{0}_{p\times n}$.
    - $\boldsymbol{A}\boldsymbol{I}_n = \boldsymbol{A}; \quad \boldsymbol{I}_m\boldsymbol{A} = \boldsymbol{A}$.

Properties

- Let $\boldsymbol{A}$ be an $m \times p$ matrix and $\boldsymbol{B}_1, \boldsymbol{B}_2$ be $p \times n$ matrices.
  - Prove that $\boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2) = \boldsymbol{AB}_1 + \boldsymbol{AB}_2$.

  Proof. Verify that LHS and RHS have the same size.

  - $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are $p \times n \Rightarrow \boldsymbol{B}_1 + \boldsymbol{B}_2$ is $p \times n$.

    $\Rightarrow \boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2)$ is $m \times n$.
  - $\boldsymbol{AB}_1$ is $m \times n$, and $\boldsymbol{AB}_2$ is $m \times n$

    $\Rightarrow \boldsymbol{AB}_1 + \boldsymbol{AB}_2$ is $m \times n$.

  Let $\boldsymbol{A} = (a_{ij})_{m\times p}$, $\boldsymbol{B}_1 = (b_{ij})_{p\times n}$ and $\boldsymbol{B}_2 = (b'_{ij})_{p\times n}$.

  - We shall verify that the following are equal:
    - "the (i, j)-entry of $\boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2)$" and
    - "the (i, j)-entry of $\boldsymbol{AB}_1 + \boldsymbol{AB}_2$",

    for all $i = 1, \ldots, m, j = 1, \ldots, n$.

Properties

- Let $\boldsymbol{A}$ be an $m \times p$ matrix and $\boldsymbol{B}_1, \boldsymbol{B}_2$ be $p \times n$ matrices.
  - Prove that $\boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2) = \boldsymbol{A}\boldsymbol{B}_1 + \boldsymbol{A}\boldsymbol{B}_2$.

  Proof. Let $\boldsymbol{A} = (a_{ij})$, $\boldsymbol{B}_1 = (b_{ij})$, $\boldsymbol{B}_2 = (b'_{ij})$.
  - $\boldsymbol{B}_1 + \boldsymbol{B}_2 = (b_{ij} + b'_{ij})$. Let $b''_{ij} = b_{ij} + b'_{ij}$.

$$
\begin{aligned}
&(i,j)\text{-entry of } \boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2) \\
&= a_{i1}b''_{1j} + a_{i2}b''_{2j} + \cdots + a_{ip}b''_{pj} \\
&= a_{i1}(b_{1j} + b'_{1j}) + a_{i2}(b_{2j} + b'_{2j}) + \cdots + a_{ip}(b_{pj} + b'_{pj}) \\
&= (a_{i1}b_{1j} + a_{i1}b'_{1j}) + (a_{i2}b_{2j} + a_{i2}b'_{2j}) + \cdots + (a_{ip}b_{pj} + a_{ip}b'_{pj}) \\
&= (a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}) + (a_{i1}b'_{1j} + a_{i2}b'_{2j} + \cdots + a_{ip}b'_{pj}) \\
&= (i,j)\text{-entry of } \boldsymbol{A}\boldsymbol{B}_1 + (i,j)\text{-entry of } \boldsymbol{A}\boldsymbol{B}_2 \\
&= (i,j)\text{-entry of } (\boldsymbol{A}\boldsymbol{B}_1 + \boldsymbol{A}\boldsymbol{B}_2).
\end{aligned}
$$

  $\therefore\ \boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2) = \boldsymbol{A}\boldsymbol{B}_1 + \boldsymbol{A}\boldsymbol{B}_2$.

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Powers of Square Matrices

- Let $\boldsymbol{A}$ be an $m \times n$ matrix.
  - $\boldsymbol{A}\boldsymbol{A}$ is well-defined $\Leftrightarrow m = n \Leftrightarrow \boldsymbol{A}$ is square.

  Definition. Let $\boldsymbol{A}$ be a **square** matrix of order n. For nonnegative integers k, the **powers** of $\boldsymbol{A}$ are defined as
  - $$\boxed{\boldsymbol{A}^k = \begin{cases} \boldsymbol{I}_n & \text{if } k = 0, \\ \underbrace{\boldsymbol{A}\boldsymbol{A}\cdots\boldsymbol{A}}_{k \text{ times}} & \text{if } k \geq 1. \end{cases}}$$

- **Example.** Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$. Then
  - $\boldsymbol{A}^2 = \boldsymbol{A}\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 4 & 11 \end{pmatrix}$.
  - $\boldsymbol{A}^3 = \boldsymbol{A}\boldsymbol{A}\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 11 & 30 \\ 15 & 41 \end{pmatrix}$.

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Properties

- Let $\boldsymbol{A}$ be a square matrix, and m, n nonnegative integers.
  - $\boldsymbol{A}^m \boldsymbol{A}^n = \boldsymbol{A}^{m+n}$, $(\boldsymbol{A}^m)^n = \boldsymbol{A}^{mn}$.
- Recall that matrix multiplication is NOT commutative.
  - In general, $(\boldsymbol{AB})^n \neq \boldsymbol{A}^n \boldsymbol{B}^n$ for $n = 2, 3, \ldots$.
  - For example, let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
    - $(\boldsymbol{AB})^2 = (\boldsymbol{AB})(\boldsymbol{AB}) = \boldsymbol{ABAB} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.
    - $\boldsymbol{A}^2 \boldsymbol{B}^2 = (\boldsymbol{AA})(\boldsymbol{BB}) = \boldsymbol{AABB} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.
- **Exercise.** Let $\boldsymbol{A}, \boldsymbol{B}$ be square matrices of the same size.
  - Suppose that $\boldsymbol{AB} = \boldsymbol{BA}$. Prove that
    - $(\boldsymbol{AB})^n = \boldsymbol{A}^n \boldsymbol{B}^n$ for all nonnegative integers n.

Matrix Representation

- Let $\boldsymbol{A} = (a_{ij})_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$.
  - Let $\boldsymbol{a}_i$ denote the ith row of $\boldsymbol{A}$, $i = 1, \ldots, m$.
    - $\boldsymbol{a}_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix}$
    - $\boldsymbol{a}_2 = \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}$
    - $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
    - $\boldsymbol{a}_m = \begin{pmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

    Then each $\boldsymbol{a}_i$ is a $1 \times n$ matrix (row vector).
    - $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_m \end{pmatrix}$.

Matrix Representation

- Let $\boldsymbol{A} = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$.
  - Let $\boldsymbol{b}_j$ denote the jth column of $\boldsymbol{A}$, $j = 1, \ldots, n$.
    - $\boldsymbol{b}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \boldsymbol{b}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, \boldsymbol{b}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$.

    Then each $\boldsymbol{b}_j$ is an $m \times 1$ matrix (column vector).
    - $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \end{pmatrix}$.

Matrix Representation

- Let $\boldsymbol{a} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.
  - Then $\boldsymbol{ab}$ is a 1×1 matrix, i.e., a real number.
    - $\boldsymbol{ab} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$.

  Note: $\boldsymbol{ba}$ is an $n \times n$ matrix.
  - (i, j)-entry is the ith entry of $\boldsymbol{b}$ times the jth entry of $\boldsymbol{a}$.
    - $\boldsymbol{ba} = \begin{pmatrix} b_1a_1 & b_1a_2 & \cdots & b_1a_n \\ b_2a_1 & b_2a_2 & \cdots & b_2a_n \\ \vdots & \vdots & \ddots & \vdots \\ b_na_1 & b_na_2 & \cdots & b_na_n \end{pmatrix}$.

Matrix Representation

- Suppose $\boldsymbol{A} = (a_{ij})_{m\times p}$.
  - Let $\boldsymbol{a}_i = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{ip} \end{pmatrix}$ be the ith row of $\boldsymbol{A}$.

  Suppose $\boldsymbol{B} = (b_{ij})_{p\times n}$.
  - Let $\boldsymbol{b}_j = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{pmatrix}$ be the jth column of $\boldsymbol{B}$.

$$\begin{aligned}\boldsymbol{a}_i\boldsymbol{b}_j &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} \\ &= (i,j)\text{-entry of } \boldsymbol{AB}.\end{aligned}$$

  - $\boldsymbol{AB} = \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 & \boldsymbol{a}_1\boldsymbol{b}_2 & \cdots & \boldsymbol{a}_1\boldsymbol{b}_n \\ \boldsymbol{a}_2\boldsymbol{b}_1 & \boldsymbol{a}_2\boldsymbol{b}_2 & \cdots & \boldsymbol{a}_2\boldsymbol{b}_n \\ \vdots & \vdots & & \vdots \\ \boldsymbol{a}_m\boldsymbol{b}_1 & \boldsymbol{a}_m\boldsymbol{b}_2 & \cdots & \boldsymbol{a}_m\boldsymbol{b}_n \end{pmatrix}$.

Matrix Representation

- The (i,j)-entry of $\boldsymbol{AB}$ is
  - The ith row of $\boldsymbol{A}$ times the jth column of $\boldsymbol{B}$.

$$\begin{aligned}\boldsymbol{a}_i\boldsymbol{B} &= \boldsymbol{a}_i\begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{a}_i\boldsymbol{b}_1 & \boldsymbol{a}_i\boldsymbol{b}_2 & \cdots & \boldsymbol{a}_i\boldsymbol{b}_n \end{pmatrix} \\ &= i\text{th row of } \boldsymbol{AB}\end{aligned}$$

$$\boldsymbol{AB} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_m \end{pmatrix}\boldsymbol{B} = \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{B} \\ \boldsymbol{a}_2\boldsymbol{B} \\ \vdots \\ \boldsymbol{a}_m\boldsymbol{B} \end{pmatrix}.$$

Matrix Representation

- The (i, j)-entry of $\boldsymbol{AB}$ is
  - The ith row of $\boldsymbol{A}$ times the jth column of $\boldsymbol{B}$.

$$\boldsymbol{A}\boldsymbol{b}_j = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_m \end{pmatrix} \boldsymbol{b}_j = \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{b}_j \\ \boldsymbol{a}_2\boldsymbol{b}_j \\ \vdots \\ \boldsymbol{a}_m\boldsymbol{b}_j \end{pmatrix} = j\text{th column of } \boldsymbol{AB}.$$

$$\begin{aligned} \boldsymbol{AB} &= \boldsymbol{A}\begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{A}\boldsymbol{b}_1 & \boldsymbol{A}\boldsymbol{b}_2 & \cdots & \boldsymbol{A}\boldsymbol{b}_n \end{pmatrix} \end{aligned}$$

- **Remark.** Matrices can be multiplied in blocks (provided that the sizes are matched).
  - Reference: Question 2.23.

Example

- Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}$.
  - Let $\boldsymbol{a}_1 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$, $\boldsymbol{a}_2 = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$. Then

$$\begin{aligned} \boldsymbol{AB} &= \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \end{pmatrix} \boldsymbol{B} = \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{B} \\ \boldsymbol{a}_2\boldsymbol{B} \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix} \\ \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix} \\ \begin{pmatrix} 8 & 7 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix} \end{aligned}$$

Example

- Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}$.
  - Let $\boldsymbol{b}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\boldsymbol{b}_2 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$. Then

$$\begin{aligned}
\boldsymbol{AB} &= \boldsymbol{A}\begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{Ab}_1 & \boldsymbol{Ab}_2 \end{pmatrix} \\
&= \left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} 2 \\ 8 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 7 \end{pmatrix} \right) = \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix}
\end{aligned}$$

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Example

- Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}$.

$$\begin{aligned}
\boldsymbol{AB} &= \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 & \boldsymbol{a}_1\boldsymbol{b}_2 \\ \boldsymbol{a}_2\boldsymbol{b}_1 & \boldsymbol{a}_2\boldsymbol{b}_2 \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \\ \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} & \begin{pmatrix} 4 & 5 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix}
\end{aligned}$$

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Representation of Linear System

- Linear System of m equations in n variables $x_1, \dots, x_n$:
  - $\begin{cases} a_{11}x_1 + & a_{12}x_2 + \cdots + & a_{1n}x_n = b_1 \\ a_{21}x_1 + & a_{22}x_2 + \cdots + & a_{2n}x_n = b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + \cdots + & a_{mn}x_n = b_m \end{cases}$

$$\begin{aligned} b_i &= a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \\ &= \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \end{aligned}$$

Representation of Linear System

- Linear System of m equations in n variables $x_1, \dots, x_n$.
  - $\begin{cases} a_{11}x_1 + & a_{12}x_2 + \cdots + & a_{1n}x_n = b_1 \\ a_{21}x_1 + & a_{22}x_2 + \cdots + & a_{2n}x_n = b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + \cdots + & a_{mn}x_n = b_m \end{cases}$
  - $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, **coefficient matrix**.
    - $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, **variable matrix**.
    - $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$, **constant matrix**. Then $\boxed{\boldsymbol{Ax} = \boldsymbol{b}}$

Representation of Linear System

- Let $\boldsymbol{A} = (a_{ij})_{m\times n}$, $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$.
  - Then $\boldsymbol{Ax} = \boldsymbol{b}$ is the linear system of
    - m linear equations in n variables $x_1, \ldots, x_n$,
    - a_{ij} are the coefficients, and b_i are the constants.
  - Let $\boldsymbol{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$.
    - $x_1 = u_1, \ldots, x_n = u_n$ is a solution to the system

      $\Leftrightarrow \boldsymbol{Au} = \boldsymbol{b}$

      $\Leftrightarrow \boldsymbol{u}$ is a solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

Representation of Linear System

- **Problem.** Suppose that a linear system has more than one solutions. Then it has infinitely many solutions.

  Proof. Let the system be represented by $\boldsymbol{Ax} = \boldsymbol{b}$.
  - Suppose that it has two solutions $\boldsymbol{u}_1 \neq \boldsymbol{u}_2$.
    - $\boldsymbol{Au}_1 = \boldsymbol{b}$ and $\boldsymbol{Au}_2 = \boldsymbol{b}$.

    Consider $c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2$, where c_1, c_2 are constants.
    - $\boldsymbol{A}(c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2) = c_1\boldsymbol{Au}_1 + c_2\boldsymbol{Au}_2 = c_1\boldsymbol{b} + c_2\boldsymbol{b} = (c_1 + c_2)\boldsymbol{b}$.

    In particular,
    - If $c_1 + c_2 = 1$, then $\boldsymbol{A}(c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2) = 1\boldsymbol{b} = \boldsymbol{b}$.

    Therefore, $\boldsymbol{Ax} = \boldsymbol{b}$ has infinitely many solutions:
    - $\boldsymbol{x} = c_1\boldsymbol{u} + c_2\boldsymbol{u}_2$, where $c_1 + c_2 = 1$.

Representation of Linear System

- Consider $\begin{cases} 4x + 5y + 6z = 5 \\ x - y \quad\quad = 2 \\ \quad\quad y - z = 3. \end{cases}$

$$\begin{aligned} \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} &= \begin{pmatrix} 4 & 5 & 6 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} 4x + 5y + 6z \\ x - y \\ y - z \end{pmatrix} \\ &= \begin{pmatrix} 4x \\ x \\ 0 \end{pmatrix} + \begin{pmatrix} 5y \\ -y \\ y \end{pmatrix} + \begin{pmatrix} 6z \\ 0 \\ -z \end{pmatrix} \\ &= x \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} + z \begin{pmatrix} 6 \\ 0 \\ -1 \end{pmatrix} \end{aligned}$$

Representation of Linear System

- $\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$.

$$\begin{aligned} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} &= \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix} \\ &= x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{aligned}$$

  - Let $\boldsymbol{a}_j$ denote the jth column of $\boldsymbol{A}$. Then
    - $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} = x_1\boldsymbol{a}_1 + \cdots + x_n\boldsymbol{a}_n = \sum_{j=1}^{n} x_j\boldsymbol{a}_j$.

Transpose

- Let $\boldsymbol{A} = (a_{ij})_{m\times n}$ be a matrix.
  - The **transpose** of $\boldsymbol{A}$ is the $n \times m$ matrix $\boldsymbol{A}^{\mathrm{T}}$ (or $\boldsymbol{A}^{\mathrm{t}}$)
    - whose (i, j)-entry is a_{ji}.
- **Example.** Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$.
  - $\boldsymbol{A}^{\mathrm{T}} = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix}$ and $(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$.
- **Remarks.**
  - The ith row of $\boldsymbol{A}^{\mathrm{T}}$ is the ith column of $\boldsymbol{A}$.
  - The jth column of $\boldsymbol{A}^{\mathrm{T}}$ is the jth row of $\boldsymbol{A}$.

Properties

- **Theorem.** Let $\boldsymbol{A}$ be an $m \times n$ matrix.
  - $(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{A}$.
  - $\boldsymbol{A}$ is symmetric $\Leftrightarrow \boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}$.
  - Let c be a scalar. Then $(c\boldsymbol{A})^{\mathrm{T}} = c\boldsymbol{A}^{\mathrm{T}}$.
  - Let $\boldsymbol{B}$ be $m \times n$. Then $(\boldsymbol{A} + \boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}$.
  - Let $\boldsymbol{B}$ be $n \times p$. Then $(\boldsymbol{AB})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$.
- **Proof.** We only prove the last statement.
  - Left-hand side:
    - $\boldsymbol{AB}$ is $m \times p \Rightarrow (\boldsymbol{AB})^{\mathrm{T}}$ is $p \times m$.
  - Right-hand side:
    - $\boldsymbol{B}^{\mathrm{T}}$ is $p \times n$ & $\boldsymbol{A}^{\mathrm{T}}$ is $n \times m \Rightarrow \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$ is $p \times m$.

    So $(\boldsymbol{AB})^{\mathrm{T}}$ and $\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$ have the same size.

Properties

- **Proof.** (Continued) Let $\boldsymbol{A} = (a_{ij})_{m\times n}$ and $\boldsymbol{B} = (b_{ij})_{n\times p}$.

$$\begin{aligned}(i,j)\text{-entry of } (\boldsymbol{AB})^{\mathrm{T}} &= (j,i)\text{-entry of } \boldsymbol{AB}\\ &= a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jn}b_{ni}.\end{aligned}$$

  - Let $\boldsymbol{A}^{\mathrm{T}} = (a'_{ij})_{n\times m}$ and $\boldsymbol{B}^{\mathrm{T}} = (b'_{ij})_{p\times n}$.
    - $a'_{ij} = a_{ji}$ and $b'_{ij} = b_{ji}$.

$$\begin{aligned}(i,j)\text{-entry of } \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} &= b'_{i1}a'_{1j} + b'_{i2}a'_{2j} + \cdots + b'_{in}a'_{nj}\\ &= b_{1i}a_{j1} + b_{2i}a_{j2} + \cdots + b_{ni}a_{jn}\\ &= a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jn}b_{ni}.\end{aligned}$$

  $\therefore\ (\boldsymbol{AB})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$.

  Note: In general, $(\boldsymbol{AB})^{\mathrm{T}} \neq \boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}$.

Inverses of Square Matrices

Inverses of Numbers

- Let a and b be real numbers.
  - $a + x = b \Rightarrow x = b - a = b + (-a)$.

  The number $-a$ is the **additive inverse** of a.
  - $ax = b \Rightarrow x = b/a = a^{-1}b$, provided that $a \neq 0$.

  The number a^{-1} is the **multiplicative inverse** of a $(\neq 0)$.

- Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices of the same size.
  - $\boldsymbol{A} + \boldsymbol{X} = \boldsymbol{B} \Rightarrow \boldsymbol{X} = \boldsymbol{B} - \boldsymbol{A} = \boldsymbol{B} + (-\boldsymbol{A})$.

  So $-\boldsymbol{A}$ is the **additive inverse** of $\boldsymbol{A}$.

- Let $\boldsymbol{A}$ be an $m\times n$ matrix and $\boldsymbol{B}$ be an $m\times p$ matrix.
  - $\boldsymbol{AX} = \boldsymbol{B} \Rightarrow \boldsymbol{X} = \cdots$.

  It is expected to have a matrix $\boldsymbol{A}^{-1}$ so that $\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$.

Inverses of Square Matrices

- **Definition.** Let $\boldsymbol{A}$ be a **square matrix** of order n.
  - If there exists a square matrix $\boldsymbol{B}$ of order n so that
    - $\boxed{\boldsymbol{AB} = \boldsymbol{I}_n}$ and $\boxed{\boldsymbol{BA} = \boldsymbol{I}_n}$

    then $\boldsymbol{A}$ is called **invertible**, and $\boldsymbol{B}$ is an **inverse** of $\boldsymbol{A}$.
  - If $\boldsymbol{A}$ is not **invertible**, $\boldsymbol{A}$ is called **singular**.

  Note: Non-square matrix is neither invertible nor singular.

- **Example.** Suppose that $\boldsymbol{A}$ is invertible with inverse $\boldsymbol{B}$.

$$\begin{aligned} \boldsymbol{AX} = \boldsymbol{C} &\Rightarrow \boldsymbol{B}(\boldsymbol{AX}) = \boldsymbol{BC} \\ &\Rightarrow (\boldsymbol{BA})\boldsymbol{X} = \boldsymbol{BC} \\ &\Rightarrow \boldsymbol{IX} = \boldsymbol{BC} \\ &\Rightarrow \boldsymbol{X} = \boldsymbol{BC}. \end{aligned}$$

  ($\boldsymbol{A}$ and $\boldsymbol{C}$ must have the same number of rows.)

Examples

- Let $\boldsymbol{A} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$.
  - $\boldsymbol{AB} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I}$.
  - $\boldsymbol{BA} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I}$.

  Therefore, $\boldsymbol{A}$ is invertible and $\boldsymbol{B}$ is an inverse of $\boldsymbol{A}$.

- Solve the equation $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \boldsymbol{X} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$.
  - Pre-multiply the equation by $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$.

$$\begin{pmatrix} 12 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \boldsymbol{X} = \boldsymbol{IX} = \boldsymbol{X}.$$

Examples

- Prove that $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is invertible, and find its inverse.
    - Let $\boldsymbol{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$.
        - $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{AB} = \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix}$.

        Solve a linear system in a, b, c, d:
        - $\begin{cases} 1 = a+2c \\ 0 = b+2d \\ 0 = 3a+4c \\ 1 = 3b+4d \end{cases} \cdots \Rightarrow \cdots \begin{cases} a = -2 \\ b = 1 \\ c = 3/2 \\ d = -1/2. \end{cases}$

        Moreover, one must verify that $\boldsymbol{BA} = \cdots = \boldsymbol{I}$.
    - $\boldsymbol{A}$ is invertible with an inverse $\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$.

Examples

- Prove that $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is a singular matrix.
    - Assume that $\boldsymbol{A}$ is invertible (prove by contradiction).
        - Then $\boldsymbol{A}$ has an inverse $\boldsymbol{B}$: $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$.

        Let $\boldsymbol{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then
        - $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$.
            - $1 = a, 0 = b, 0 = a, 1 = b$, a contradiction!

        So such $\boldsymbol{B}$ does not exist.
    - Therefore, $\boldsymbol{A}$ is not invertible, i.e., it is singular.

    Remark: One also gets a contradiction by checking $\boldsymbol{BA} = \boldsymbol{I}$.

Properties

- **Theorem.** Let $\boldsymbol{A}$ be a square matrix.
  - If $\boldsymbol{A}$ is invertible, then its inverse is unique.
- **Proof.** Suppose that $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are both inverses of $\boldsymbol{A}$.
  - $\boldsymbol{A}\boldsymbol{B}_1 = \boldsymbol{B}_1\boldsymbol{A} = \boldsymbol{I}$ and $\boldsymbol{A}\boldsymbol{B}_2 = \boldsymbol{B}_2\boldsymbol{A} = \boldsymbol{I}$.

  We need to verify that $\boldsymbol{B}_1 = \boldsymbol{B}_2$:
  - $\boldsymbol{B}_1 = \boldsymbol{B}_1\boldsymbol{I} = \boldsymbol{B}_1(\boldsymbol{A}\boldsymbol{B}_2) = (\boldsymbol{B}_1\boldsymbol{A})\boldsymbol{B}_2 = \boldsymbol{I}\boldsymbol{B}_2 = \boldsymbol{B}_2$.
- **Notation.** The unique inverse of $\boldsymbol{A}$, if exists, is denoted by $\boldsymbol{A}^{-1}$.
  - $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$.
- Suppose that $\boldsymbol{A}$ is an invertible matrix. Then
  - If $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{B}$ ($\boldsymbol{A}$ and $\boldsymbol{B}$ have the same number of rows),
    - then $\boldsymbol{A}^{-1}\boldsymbol{B} = \boldsymbol{A}^{-1}(\boldsymbol{A}\boldsymbol{X}) = (\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{X} = \boldsymbol{I}\boldsymbol{X} = \boldsymbol{X}$.

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Properties

- **Cancellation Law.** Let $\boldsymbol{A}$ be an invertible matrix.
  - $\boldsymbol{A}\boldsymbol{B}_1 = \boldsymbol{A}\boldsymbol{B}_2 \Rightarrow \boldsymbol{B}_1 = \boldsymbol{B}_2$.
  - $\boldsymbol{C}_1\boldsymbol{A} = \boldsymbol{C}_2\boldsymbol{A} \Rightarrow \boldsymbol{C}_1 = \boldsymbol{C}_2$.
- **Proof.** Suppose that $\boldsymbol{A}\boldsymbol{B}_1 = \boldsymbol{A}\boldsymbol{B}_2$. Then
  - $\boldsymbol{B}_1$ is the solution to $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{A}\boldsymbol{B}_2$.
    - $\boldsymbol{B}_1 = \boldsymbol{A}^{-1}(\boldsymbol{A}\boldsymbol{B}_2) = (\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{B}_2 = \boldsymbol{I}\boldsymbol{B}_2 = \boldsymbol{B}_2$.

  The other statement is left as an exercise.
- **Remark.** The cancellation law fails if $\boldsymbol{A}$ is singular.
  - Recall that $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is singular.
    - $\boldsymbol{A}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ & $\boldsymbol{A}\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$

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Example

- Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find the condition when $\boldsymbol{A}$ is invertible.

  Let $\boldsymbol{B} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Suppose that $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$.

  ◦ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{AB} = \begin{pmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{pmatrix}$.

  Solve a linear system in x, y, z, w:

  ◦ $\begin{cases} ax \quad + bz \quad\quad = 1 \\ \quad ay \quad\quad + bw = 0 \\ cx \quad + dz \quad\quad = 0 \\ \quad cy \quad\quad + dw = 1 \end{cases} \Rightarrow \begin{cases} ax + bz = 1 \\ cx + dz = 0 \end{cases} \quad \begin{cases} ay + bw = 0 \\ cy + dw = 1 \end{cases}$

  - They are inconsistent $\Leftrightarrow a : c = b : d \Leftrightarrow ad = bc$.
  - They are consistent $\Leftrightarrow ad \neq bc$.

Example

- Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find the condition when $\boldsymbol{A}$ is invertible.

  ◦ If $ad = bc$, then $\boldsymbol{A}$ is singular. Suppose that $ad \neq bc$.

  - $\begin{cases} ax + bz = 1 \\ cx + dz = 0 \end{cases} \Rightarrow x = \dfrac{d}{ad-bc}, z = \dfrac{-c}{ad-bc}$.
  - $\begin{cases} ay + bw = 0 \\ cy + dw = 1 \end{cases} \Rightarrow y = \dfrac{-b}{ad-bc}, w = \dfrac{a}{ad-bc}$.

  Let $\boldsymbol{B} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \dfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

  - One verifies that $\boldsymbol{AB} = \boldsymbol{I}$ and $\boldsymbol{BA} = \boldsymbol{I}$.

  Conclusion: $\boldsymbol{A}$ is invertible $\Leftrightarrow ad - bc \neq 0$.

  ◦ If $\boldsymbol{A}$ is invertible, then $\boldsymbol{A}^{-1} = \dfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Properties

- **Theorem.** Let $\boldsymbol{A}, \boldsymbol{B}$ be invertible matrices of same size.
  - Let $c \neq 0$. $c\boldsymbol{A}$ is invertible, and $(c\boldsymbol{A})^{-1} = \frac{1}{c}\boldsymbol{A}^{-1}$.
  - $\boldsymbol{A}^{\mathrm{T}}$ is invertible, and $(\boldsymbol{A}^{\mathrm{T}})^{-1} = (\boldsymbol{A}^{-1})^{\mathrm{T}}$.
  - $\boldsymbol{A}^{-1}$ is invertible, and $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$.
  - $\boldsymbol{AB}$ is invertible, and $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.
- **Proof.** To prove that $\boldsymbol{A}$ is invertible with $\boldsymbol{A}^{-1} = \boldsymbol{M}$,
  - Verify that $\boldsymbol{AM} = \boldsymbol{MA} = \boldsymbol{I}$.

  To prove that $\boldsymbol{A}^{\mathrm{T}}$ is invertible with inverse $(\boldsymbol{A}^{-1})^{\mathrm{T}}$.
  - We shall verify that $\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{A}^{-1})^{\mathrm{T}} = (\boldsymbol{A}^{-1})^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{I}$.
    - $\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{A}^{-1})^{\mathrm{T}} = (\boldsymbol{A}^{-1}\boldsymbol{A})^{\mathrm{T}} = \boldsymbol{I}^{\mathrm{T}} = \boldsymbol{I}$.
    - $(\boldsymbol{A}^{-1})^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} = (\boldsymbol{A}\boldsymbol{A}^{-1})^{\mathrm{T}} = \boldsymbol{I}^{\mathrm{T}} = \boldsymbol{I}$.

  Other properties are left as exercises.

Properties

- Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_k$ be invertible matrices of same size.
  - $(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_k)^{-1} = \boldsymbol{A}_k^{-1}\cdots\boldsymbol{A}_2^{-1}\boldsymbol{A}_1^{-1}$.

  In particular, $(\underbrace{\boldsymbol{A}\boldsymbol{A}\cdots\boldsymbol{A}}_{k \text{ times}})^{-1} = \underbrace{\boldsymbol{A}^{-1}\cdots\boldsymbol{A}^{-1}\boldsymbol{A}^{-1}}_{k \text{ times}}$.
  - $(\boldsymbol{A}^k)^{-1} = (\boldsymbol{A}^{-1})^k$.
- **Definition.** Let $\boldsymbol{A}$ be an invertible matrix.
  - For any positive integer k, $\boxed{\boldsymbol{A}^{-k} = (\boldsymbol{A}^{-1})^k}$
- **Exercise.** Let $\boldsymbol{A}$ be an invertible matrix.
  - For any integers m and n,
    - $\boldsymbol{A}^{m+n} = \boldsymbol{A}^m\boldsymbol{A}^n$ and $(\boldsymbol{A}^m)^n = \boldsymbol{A}^{mn}$.
- **Note.** If $\boldsymbol{A}$ is singular, then $\boldsymbol{A}^{-1}$ is undefined.

Elementary Operations

- Recall the **elementary row operations** of matrices.
  - Multiply a row by a nonzero constant.
  - Interchange two rows.
  - Add a constant multiple of a row to another row.
- What is the resulting matrix by applying an elementary row operation to the identity matrix $\boldsymbol{I}$?
  - $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{cR_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
  - cR_i, where $c \neq 0$:
    - Replace the ith diagonal entry by c:
      - $a_{ii} = 1 \mapsto c$.

Elementary Operations

- Recall the **elementary row operations** of matrices.
  - Multiply a row by a nonzero constant.
  - Interchange two rows.
  - Add a constant multiple of a row to another row.
- What is the resulting matrix by applying an elementary row operation to the identity matrix $\boldsymbol{I}$?
  - $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
  - $R_i \leftrightarrow R_j$, where $i \neq j$:
    - The ith and jth diagonal entries become 0.
    - The (i, j) and (j, i)-entries become 1.
      - $a_{ii} = a_{jj} = 1 \mapsto 0$, $a_{ij} = a_{ji} = 0 \mapsto 1$.

Elementary Operations

- Recall the **elementary row operations** of matrices.
  - Multiply a row by a nonzero constant.
  - Interchange two rows.
  - Add a constant multiple of a row to another row.
- What is the resulting matrix by applying an elementary row operation to the identity matrix $\boldsymbol{I}$?
  - $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2+cR_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
  - $R_i + cR_j$, where $i \neq j$:
    - The (i, j)-entry becomes c.
      - $a_{ij} = 0 \mapsto c$.

Elementary Matrices

- **Definition.** A square matrix is called an **elementary matrix** if it can be obtained from the identity matrix by performing a single elementary row operation.
  - cR_i, where $c \neq 0$: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
  - $R_i \leftrightarrow R_j$, where $i \neq j$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
  - $R_i + cR_j$, where $i \neq j$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Connection to Matrix Multiplication

- Let $\boldsymbol{E}$ be an elementary matrix.
  - Suppose that it is obtained from $\boldsymbol{I}$ by
    - Multiplying the ith row by a nonzero number c.

  Let $\boldsymbol{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$.

  - $\boldsymbol{EA} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ca_{31} & ca_{32} & ca_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$.

  Pre-multiplication by $\boldsymbol{E}$ to $\boldsymbol{A}$

  $\Leftrightarrow$ Multiplying the ith row of $\boldsymbol{A}$ by number c.

Connection to Matrix Multiplication

- Let $\boldsymbol{E} = (e_{ij})_{m\times m}$ be obtained from $\boldsymbol{I}_m$ by multiplying kth row by c.
  - $e_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \neq k, \\ c & \text{if } i = j = k. \end{cases}$

  Let $\boldsymbol{A} = (a_{ij})_{m\times n}$. Then $\boldsymbol{EA}$ is well-defined.

  - Let $i \neq k$. Then (i, j)-entry of $\boldsymbol{EA}$ is

$$\begin{aligned} & e_{i1}a_{1j} + \cdots + e_{ii}a_{ij} + \cdots + e_{im}a_{mj} \\ = {} & 0 + \cdots + 1 \cdot a_{ij} + \cdots + 0 = a_{ij}. \end{aligned}$$

  - Let $i = k$. Then (i, j)-entry of $\boldsymbol{EA}$ is

$$\begin{aligned} & e_{i1}a_{1j} + \cdots + e_{ii}a_{ij} + \cdots + e_{im}a_{mj} \\ = {} & 0 + \cdots + c \cdot a_{ij} + \cdots + 0 = ca_{ij}. \end{aligned}$$

  Therefore, $\boldsymbol{EA}$ is the matrix obtained from $\boldsymbol{A}$ by multiplying the kth row by c.

Connection to Matrix Multiplication

- Let $\boldsymbol{E}$ be an elementary matrix.
  - Suppose that it is obtained from $\boldsymbol{I}$ by
    - Interchanging the ith row and the jth row.

Let $\boldsymbol{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$

- $\boldsymbol{EA} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{41} & a_{42} & a_{43} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

Pre-multiplication by $\boldsymbol{E}$ to $\boldsymbol{A}$

$\Leftrightarrow$ Interchanging the ith row and jth row of $\boldsymbol{A}$.

Connection to Matrix Multiplication

- Let $\boldsymbol{E} = (e_{ij})$ be obtained from $\boldsymbol{I}_m$ by interchanging ℓth and kth rows $(\ell < k)$.
  - $e_{ij} = \begin{cases} 1 & \text{if } i = j \neq \ell, k \text{ or } (i,j) = (\ell, k),\ (k, \ell), \\ 0 & \text{otherwise}. \end{cases}$

  Let $\boldsymbol{A} = (a_{ij})_{m \times n}$. Then $\boldsymbol{EA}$ is well-defined.
  - Let $i \neq \ell, k$. Then (i, j)-entry of $\boldsymbol{EA}$ is

$$\begin{aligned} & e_{i1}a_{1j} + \cdots + e_{ii}a_{ij} + \cdots + e_{im}a_{mj} \\ = {} & 0 + \cdots + 1 \cdot a_{ij} + \cdots + 0 = a_{ij}. \end{aligned}$$

  - Let $i = \ell$ (the case $i = k$ is similar). Then (i, j)-entry of $\boldsymbol{EA}$

$$\begin{aligned} & e_{i1}a_{1j} + \cdots + e_{ii}a_{ij} + \cdots + e_{ik}a_{kj} + \cdots + e_{im}a_{mj} \\ = {} & 0 + \cdots + 0 \cdot a_{ij} + \cdots + 1 \cdot a_{kj} + \cdots + 0 = a_{kj}. \end{aligned}$$

  Therefore, $\boldsymbol{EA}$ is the matrix obtained from $\boldsymbol{A}$ by interchanging ℓth and kth rows.

Connection to Matrix Multiplication

- Let $\boldsymbol{E}$ be an elementary matrix.
  - Suppose that it is obtained from $\boldsymbol{I}$ by
    - Adding c times of the jth row to the ith row.

Let $\boldsymbol{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$

- $\boldsymbol{EA} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+ca_{41} & a_{22}+ca_{42} & a_{23}+ca_{43} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{33} \end{pmatrix}$

Pre-multiplication by $\boldsymbol{E}$ to $\boldsymbol{A}$

$\Leftrightarrow$ Adding c times the jth row to the ith row of $\boldsymbol{A}$.

Connection to Matrix Multiplication

- Let $\boldsymbol{E} = (e_{ij})$ be obtained from $\boldsymbol{I}_m$ by adding c times of ℓth row to kth row.
  - $e_{ij} = \begin{cases} 1 & \text{if } i = j, \\ c & \text{if } (i,j) = (k,\ell), \\ 0 & \text{otherwise.} \end{cases}$

Let $\boldsymbol{A} = (a_{ij})_{m\times n}$. Then $\boldsymbol{EA}$ is well-defined.

- Let $i \neq k$. Then (i,j)-entry of $\boldsymbol{EA}$ is

$$\begin{aligned} & e_{i1}a_{1j} + \cdots + e_{ii}a_{ij} + \cdots + e_{im}a_{mj} \\ &= 0 + \cdots + 1 \cdot a_{ij} + \cdots + 0 = a_{ij}. \end{aligned}$$

- Let $i = k$. Then (i,j)-entry of $\boldsymbol{EA}$ is (w.l.o.g., assume $k < \ell$)

$$\begin{aligned} & e_{i1}a_{1j} + \cdots + e_{ii}a_{ij} + \cdots + e_{i\ell}a_{\ell j} + \cdots + e_{im}a_{mj} \\ &= 0 + \cdots + 1 \cdot a_{ij} + \cdots + c \cdot a_{\ell j} + \cdots + 0 = a_{ij} + ca_{\ell j}. \end{aligned}$$

Therefore, $\boldsymbol{EA}$ is obtained from $\boldsymbol{A}$ by adding c times of ℓth row to the kth row.

Connection to Multiplication

- **Theorem.**
  - Let $\boldsymbol{E}$ be the elementary matrix obtained
    - by performing an elementary row operation to $\boldsymbol{I}_m$.

    Then for any $m \times n$ matrix $\boldsymbol{A}$, $\boldsymbol{EA}$ can be obtained
    - by performing same elementary row operation to $\boldsymbol{A}$.
- Let $\boldsymbol{A}$ be an $m \times n$ matrix.
  - $\boldsymbol{I}_m \xrightarrow{cR_i} \boldsymbol{E} \Rightarrow \boldsymbol{A} \xrightarrow{cR_i} \boldsymbol{EA}$.
  - $\boldsymbol{I}_m \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E} \Rightarrow \boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{EA}$.
  - $\boldsymbol{I}_m \xrightarrow{R_i + cR_j} \boldsymbol{E} \Rightarrow \boldsymbol{A} \xrightarrow{R_i + cR_j} \boldsymbol{EA}$.

Example

- Let $\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix}$.
  - $\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & -3 \\ 0 & 4 & 2 \end{pmatrix} = \boldsymbol{A}_1$.
    - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \boldsymbol{E}_1$.

$$\begin{aligned} \boldsymbol{E}_1\boldsymbol{A} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & -3 \\ 0 & 4 & 2 \end{pmatrix} = \boldsymbol{A}_1. \end{aligned}$$

Example

- Let $\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix}$.
  - $\boldsymbol{A}_1 = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & -3 \\ 0 & 4 & 2 \end{pmatrix} \xrightarrow{R_2+2R_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 4 & 2 \end{pmatrix} = \boldsymbol{A}_2$.
    - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2+2R_1} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{E}_2$.

$$\begin{aligned} \boldsymbol{E}_2\boldsymbol{A}_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & -3 \\ 0 & 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 4 & 2 \end{pmatrix} = \boldsymbol{A}_2. \end{aligned}$$

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Example

- Let $\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix}$.
  - $\boldsymbol{A}_2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 4 & 2 \end{pmatrix} \xrightarrow{R_3+(-4)R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} = \boldsymbol{A}_3$.
    - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3+(-4)R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} = \boldsymbol{E}_3$.

$$\begin{aligned} \boldsymbol{E}_3\boldsymbol{A}_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} = \boldsymbol{A}_3. \end{aligned}$$

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Example

- Let $\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix}$.
  - $\boldsymbol{A}_3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{A}_4$.
    - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} = \boldsymbol{E}_4$.

$$\begin{aligned} \boldsymbol{E}_4\boldsymbol{A}_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{A}_4. \end{aligned}$$

Example

- Let $\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix}$.
  - $\boldsymbol{A}_4 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1+(-2)R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{A}_5$.
    - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1+(-2)R_3} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{E}_5$.

$$\begin{aligned} \boldsymbol{E}_5\boldsymbol{A}_4 &= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{A}_5. \end{aligned}$$

Example

- Let $\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix}$.
  - $\boldsymbol{A}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2+(-1)R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{A}_6 = \boldsymbol{I}$.
    - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2+(-1)R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{E}_6$.

$$\begin{aligned} \boldsymbol{E}_6\boldsymbol{A}_5 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{A}_6 = \boldsymbol{I}. \end{aligned}$$

Invertibility

- **Theorem.** Every elementary matrix is invertible.
  - The inverse of an elementary matrix is elementary.

Proof. There are three types of elementary matrices.

- Suppose that $\boldsymbol{I} \xrightarrow{cR_i} \boldsymbol{E}$, where $c \neq 0$.
  - Then $\boldsymbol{E} \xrightarrow{\frac{1}{c}R_i} \boldsymbol{I}$.

  Let $\boldsymbol{D}$ denote the elementary matrix $\boldsymbol{I} \xrightarrow{\frac{1}{c}R_i} \boldsymbol{D}$.
  - $\boldsymbol{I} \xrightarrow[\boldsymbol{E}]{cR_i} \boldsymbol{E} \xrightarrow[\boldsymbol{D}]{\frac{1}{c}R_i} \boldsymbol{I}$. So $\boldsymbol{DE} = \boldsymbol{I}$.
  - $\boldsymbol{I} \xrightarrow[\boldsymbol{D}]{\frac{1}{c}R_i} \boldsymbol{D} \xrightarrow[\boldsymbol{E}]{cR_i} \boldsymbol{I}$. So $\boldsymbol{ED} = \boldsymbol{I}$.

  It follows that $\boldsymbol{E}$ is invertible and $\boldsymbol{E}^{-1} = \boldsymbol{D}$.

The other two cases are similar and left as exercises.

Invertibility

- Let $\boldsymbol{E}$ be an elementary matrix.
  - $\boldsymbol{I} \xrightarrow{cR_i} \boldsymbol{E} \Rightarrow \boldsymbol{I} \xrightarrow{\frac{1}{c}R_i} \boldsymbol{E}^{-1}$.
  - $\boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E} \Rightarrow \boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}^{-1}$. (So $\boldsymbol{E} = \boldsymbol{E}^{-1}$)
  - $\boldsymbol{I} \xrightarrow{R_i + cR_j} \boldsymbol{E} \Rightarrow \boldsymbol{I} \xrightarrow{R_i + (-c)R_j} \boldsymbol{E}^{-1}$.
- Suppose that matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent.
  - $\boldsymbol{A} = \boldsymbol{A}_0 \xrightarrow{\text{ero}1} \boldsymbol{A}_1 \xrightarrow{\text{ero}2} \boldsymbol{A}_2 \cdots \to \boldsymbol{A}_{k-1} \xrightarrow{\text{ero}k} \boldsymbol{A}_k = \boldsymbol{B}$.

  Let $\boldsymbol{E}_i$ be the elementary matrix corresponding to the ith elementary row operation: $\boldsymbol{I} \xrightarrow{\text{ero}i} \boldsymbol{E}_i$.
  - $\boldsymbol{A} = \boldsymbol{A}_0 \xrightarrow[\boldsymbol{E}_1]{\text{ero}1} \boldsymbol{A}_1 \xrightarrow[\boldsymbol{E}_2]{\text{ero}2} \boldsymbol{A}_2 \cdots \to \boldsymbol{A}_{k-1} \xrightarrow[\boldsymbol{E}_k]{\text{ero}k} \boldsymbol{A}_k = \boldsymbol{B}$.

  Then $\boldsymbol{B} = \boldsymbol{E}_k \boldsymbol{E}_{k-1} \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A}$.

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Invertibility

- **Theorem.** Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent

  $\Leftrightarrow$ there exist elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$

  such that $\boldsymbol{B} = \boldsymbol{E}_k \boldsymbol{E}_{k-1} \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A}$.
- **Remarks.** Suppose for elementary matrices $\boldsymbol{E}_i$,
  - $\boldsymbol{B} = \boldsymbol{E}_k \boldsymbol{E}_{k-1} \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A}$.
    - $\boldsymbol{A} \xrightarrow{\boldsymbol{E}_1} \bullet \xrightarrow{\boldsymbol{E}_2} \bullet \to \cdots \to \bullet \xrightarrow{\boldsymbol{E}_{k-1}} \bullet \xrightarrow{\boldsymbol{E}_k} \boldsymbol{B}$.
    - $\boldsymbol{A} \xleftarrow{\boldsymbol{E}_1^{-1}} \bullet \xleftarrow{\boldsymbol{E}_2^{-1}} \bullet \leftarrow \cdots \leftarrow \bullet \xleftarrow{\boldsymbol{E}_{k-1}^{-1}} \bullet \xleftarrow{\boldsymbol{E}_k^{-1}} \boldsymbol{B}$.

  $\therefore$ $\boldsymbol{A} = \boldsymbol{E}_1^{-1} \boldsymbol{E}_2^{-1} \cdots \boldsymbol{E}_{k-1}^{-1} \boldsymbol{E}_k^{-1} \boldsymbol{B}$.

  We can now prove the theorem stated in Chapter 1:
- **Theorem.** Suppose that the augmented matrices of two linear systems are row equivalent.
  - Then the two systems have the same solution set.

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Invertibility

- **Proof.** Let the two linear systems be
  - $\boldsymbol{Ax} = \boldsymbol{b}$ and $\boldsymbol{Cx} = \boldsymbol{d}$.

  Then the associated augmented matrices are
  - $\left(\boldsymbol{A} \mid \boldsymbol{b}\right)$ and $\left(\boldsymbol{C} \mid \boldsymbol{d}\right)$.

  There exist elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$ so that
  - $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \left(\boldsymbol{A} \mid \boldsymbol{b}\right) = \left(\boldsymbol{C} \mid \boldsymbol{d}\right)$.
    - $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{C}$ and $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{b} = \boldsymbol{d}$.

  Let $\boldsymbol{u}$ be a solution to $\boldsymbol{Ax} = \boldsymbol{b}$, i.e., $\boldsymbol{Au} = \boldsymbol{b}$.
  - $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{Au} = \boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{b} \Rightarrow \boldsymbol{Cu} = \boldsymbol{d}$.

  So $\boldsymbol{u}$ is also a solution to $\boldsymbol{Cx} = \boldsymbol{d}$.

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Invertibility

- **Proof.** Let the two linear systems be
  - $\boldsymbol{Ax} = \boldsymbol{b}$ and $\boldsymbol{Cx} = \boldsymbol{d}$.

  Then the associated augmented matrices are
  - $\left(\boldsymbol{A} \mid \boldsymbol{b}\right)$ and $\left(\boldsymbol{C} \mid \boldsymbol{d}\right)$.

  There exist elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$ so that
  - $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \left(\boldsymbol{A} \mid \boldsymbol{b}\right) = \left(\boldsymbol{C} \mid \boldsymbol{d}\right)$.
    - $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{C}$ and $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{b} = \boldsymbol{d}$.
    - $\boldsymbol{A} = \boldsymbol{E}_1^{-1} \cdots \boldsymbol{E}_k^{-1} \boldsymbol{C}$ and $\boldsymbol{b} = \boldsymbol{E}_1^{-1} \cdots \boldsymbol{E}_k^{-1} \boldsymbol{d}$.

  Let $\boldsymbol{v}$ be a solution to $\boldsymbol{Cx} = \boldsymbol{d}$, i.e., $\boldsymbol{Cv} = \boldsymbol{d}$.
  - $\boldsymbol{E}_1^{-1} \cdots \boldsymbol{E}_k^{-1} \boldsymbol{Cv} = \boldsymbol{E}_1^{-1} \cdots \boldsymbol{E}_k^{-1} \boldsymbol{d} \Rightarrow \boldsymbol{Av} = \boldsymbol{b}$.

  So $\boldsymbol{v}$ is also a solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

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Main Theorem for Invertible Matrices

- **Theorem.** Let $\boldsymbol{A}$ be a square matrix. Then the followings are equivalent:
  1. $\boldsymbol{A}$ is an invertible matrix.
  2. Linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution.
  3. Linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
  4. The reduced row-echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}$.
  5. $\boldsymbol{A}$ is the product of elementary matrices.

Main Theorem for Invertible Matrices

- **Proof.** We prove 1 $\Rightarrow$ 2 $\Rightarrow$ 3 $\Rightarrow$ 4 $\Rightarrow$ 5 $\Rightarrow$ 1:

1 $\Rightarrow$ 2: Suppose $\boldsymbol{A}$ is invertible.

- $\boldsymbol{Ax} = \boldsymbol{b} \Rightarrow \boldsymbol{x} = \boldsymbol{Ix} = \boldsymbol{A}^{-1}\boldsymbol{Ax} = \boldsymbol{A}^{-1}\boldsymbol{b}$.

2 $\Rightarrow$ 3: Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution $\boldsymbol{u}$.

- If $\boldsymbol{Ax} = \boldsymbol{0}$ has a solution $\boldsymbol{v}$, then $\boldsymbol{Av} = \boldsymbol{0}$.
  - $\boldsymbol{A}(\boldsymbol{u} - \boldsymbol{v}) = \boldsymbol{Au} - \boldsymbol{Av} = \boldsymbol{b} - \boldsymbol{0} = \boldsymbol{b}$.
  - $\boldsymbol{u} - \boldsymbol{v}$ is also a solution to $\boldsymbol{Ax} = \boldsymbol{b}$.
- By uniqueness, $\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{v}$; so $\boldsymbol{v} = \boldsymbol{0}$.

3 $\Rightarrow$ 4: Suppose $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.

- Let $\boldsymbol{R}$ be the reduced row-echelon form of $\boldsymbol{A}$.
  - Except the last column, all other columns of $(\boldsymbol{R} \mid \boldsymbol{0})$ are pivot columns.
- Note that $\boldsymbol{R}$ is a square matrix. So $\boldsymbol{R} = \boldsymbol{I}$.

Main Theorem for Invertible Matrices

- **Proof.** We prove 1 ⇒ 2 ⇒ 3 ⇒ 4 ⇒ 5 ⇒ 1:

4 ⇒ 5: Suppose that the reduced row-echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}$.

- $\boldsymbol{I}$ can be obtained from $\boldsymbol{A}$ by elementary row operations.
  - $\boldsymbol{A} \xrightarrow{\text{ero}1} \bullet \cdots \to \cdots \bullet \xrightarrow{\text{ero}k} \boldsymbol{I}$.
- Let $\boldsymbol{E}_i$ be the elementary matrix corresponding to the ith elementary row operation.
  - $\boldsymbol{I} = \boldsymbol{E}_k \boldsymbol{E}_{k-1} \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A}$.
- Then $\boldsymbol{A} = \boldsymbol{E}_1^{-1} \boldsymbol{E}_2^{-1} \cdots \boldsymbol{E}_{k-1}^{-1} \boldsymbol{E}_k^{-1}$.
  - (Recall: each $\boldsymbol{E}_i^{-1}$ is also an elementary matrix.)

5 ⇒ 1: Suppose that $\boldsymbol{A} = \boldsymbol{E}_1 \boldsymbol{E}_2 \cdots \boldsymbol{E}_k$,

- where $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_k$ are elementary matrices.

Each $\boldsymbol{E}_i$ is invertible ⇒ $\boldsymbol{A}$ is invertible.

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Find Inverse

- Let $\boldsymbol{A}$ be an invertible matrix of order n. Its reduced row-echelon form is $\boldsymbol{I}_n$.
  - There exist elementary matrices $\boldsymbol{E}_i$ such that
    - $\boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I}_n$.

    Then $\boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 = \boldsymbol{A}^{-1}$.

  Consider the $n \times 2n$ matrix $(\boldsymbol{A} \mid \boldsymbol{I})$.
  - Apply the ele. row oper. corresponding to $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_k$:

$$
\begin{aligned}
(\boldsymbol{A} \mid \boldsymbol{I}_n) &\xrightarrow{\boldsymbol{E}_1} (\boldsymbol{E}_1 \boldsymbol{A} \mid \boldsymbol{E}_1) \\
&\xrightarrow{\boldsymbol{E}_2} (\boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} \mid \boldsymbol{E}_2 \boldsymbol{E}_1) \\
&\to \cdots \to \cdots \\
&\xrightarrow{\boldsymbol{E}_k} (\boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} \mid \boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1) \\
&= (\boldsymbol{I}_n \mid \boldsymbol{A}^{-1}).
\end{aligned}
$$

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Find Inverse

- **Theorem.** Let $\boldsymbol{A}$ be an invertible matrix.
  - The reduced row-echelon form of $(\boldsymbol{A} \mid \boldsymbol{I})$ is $\boxed{(\boldsymbol{I} \mid \boldsymbol{A}^{-1})}$
- **Example.** Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$. Find $\boldsymbol{A}^{-1}$.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3+(-1)R_1]{R_2+(-2)R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3+2R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{(-1)R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

Find Inverse

- **Theorem.** Let $\boldsymbol{A}$ be an invertible matrix.
  - The reduced row-echelon form of $(\boldsymbol{A} \mid \boldsymbol{I})$ is $\boxed{(\boldsymbol{I} \mid \boldsymbol{A}^{-1})}$
- **Example.** Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$. Find $\boldsymbol{A}^{-1}$.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \xrightarrow[R_2+3R_3]{R_1+(-3)R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\xrightarrow{R_1+(-2)R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

Therefore, $\boldsymbol{A}^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$.

Find Inverse

- A square matrix is **invertible**

$\Leftrightarrow$ Its reduced row-echelon form is $\boldsymbol{I}$
$\Leftrightarrow$ All the columns in its row-echelon form are pivot.
$\Leftrightarrow$ All the rows in its row-echelon form are nonzero.

A square matrix is **singular**

$\Leftrightarrow$ Its reduced row-echelon form is not $\boldsymbol{I}$
$\Leftrightarrow$ Some columns in its row-echelon form are non-pivot.
$\Leftrightarrow$ Some rows in its row-echelon form are zero.

Example. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 4 \\ 3 & 6 & 3 \end{pmatrix}$. Then $\boldsymbol{A}$ is singular.

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 4 \\ 3 & 6 & 3 \end{pmatrix} \xrightarrow[R_3+(-3)R_1]{R_2+(-2)R_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 3 \end{pmatrix} \xrightarrow{R_3+(-\frac{3}{4})R_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Find Inverse

- **Theorem.** Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of the same size. If $\boldsymbol{AB} = \boldsymbol{I}$, then
  - $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible, and $\boldsymbol{A}^{-1} = \boldsymbol{B}$, $\boldsymbol{B}^{-1} = \boldsymbol{A}$.

Proof. Consider the linear system $\boldsymbol{Bx} = \boldsymbol{0}$.

- $\boldsymbol{Bx} = \boldsymbol{0} \Rightarrow \boldsymbol{ABx} = \boldsymbol{A0} \Rightarrow \boldsymbol{x} = \boldsymbol{0}$.
- $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution $\Rightarrow \boldsymbol{B}$ is invertible.

$\boldsymbol{B}^{-1}$ exists such that $\boldsymbol{BB}^{-1} = \boldsymbol{B}^{-1}\boldsymbol{B} = \boldsymbol{I}$.

- $\boldsymbol{AB} = \boldsymbol{I} \Rightarrow \boldsymbol{ABB}^{-1} = \boldsymbol{IB}^{-1} \Rightarrow \boldsymbol{A} = \boldsymbol{B}^{-1}$.

Therefore, $\boldsymbol{A}$ is invertible, and $\boldsymbol{A}^{-1} = (\boldsymbol{B}^{-1})^{-1} = \boldsymbol{B}$.

- **Corollary and Exercise.** Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_k$ be square matrices of the same size.
  - $\boldsymbol{A}_1\boldsymbol{A}_2 \cdots \boldsymbol{A}_k$ is invertible $\Leftrightarrow$ all $\boldsymbol{A}_i$ are invertible.
  - $\boldsymbol{A}_1\boldsymbol{A}_2 \cdots \boldsymbol{A}_k$ is singular $\Leftrightarrow$ some $\boldsymbol{A}_i$ are singular.

Find Inverse

- Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\boldsymbol{B} = \dfrac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $ad \neq bc$.
  - One verifies that $\boldsymbol{AB} = \boldsymbol{I}$.
  - Using the theorem, $\boldsymbol{A}^{-1} = \boldsymbol{B}$.
    - The verification that $\boldsymbol{BA} = \boldsymbol{I}$ is not necessary.
- Let $\boldsymbol{A}$ be a square matrix such that $\boldsymbol{A}^2 - 3\boldsymbol{A} - 4\boldsymbol{I} = \boldsymbol{0}$.
  - Prove that $\boldsymbol{A}$ is invertible, and find $\boldsymbol{A}^{-1}$.

  Wrong Proof. $\boldsymbol{0} = (\boldsymbol{A} - 4\boldsymbol{I})(\boldsymbol{A} + \boldsymbol{I})$.

  - $\boldsymbol{A} - 4\boldsymbol{I} = \boldsymbol{0}$ or $\boldsymbol{A} + \boldsymbol{I} = \boldsymbol{0} \Rightarrow \boldsymbol{A} = 4\boldsymbol{I}$ or $\boldsymbol{A} = -\boldsymbol{I}$.

  Proof. $4\boldsymbol{I} = \boldsymbol{A}^2 - 3\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{A} - 3\boldsymbol{I})$.

  - $\boldsymbol{I} = \frac{1}{4}\boldsymbol{A}(\boldsymbol{A} - 3\boldsymbol{I}) = \boldsymbol{A}\left[\frac{1}{4}(\boldsymbol{A} - 3\boldsymbol{I})\right]$.
  - So $\boldsymbol{A}$ is invertible with $\boldsymbol{A}^{-1} = \frac{1}{4}(\boldsymbol{A} - 3\boldsymbol{I})$.

Column Operations

- Recall that the pre-multiplication of an elementary matrix $\Leftrightarrow$ corresponding elementary row operation.

  Question. What is the effect of **post-multiplication** of an elementary matrix?

  - Answer: Corresponding elementary column operation.
- **Elementary column operations**:
  - kC_i: multiply ith column by a nonzero constant k.
  - $C_i \leftrightarrow C_j$: interchange ith and jth columns.
  - $C_i + kC_j$: add k times jth column to ith column.

  Let $\boldsymbol{E}$ be the matrix obtained from $\boldsymbol{I}$ by a single elementary column operation.

  - Then $\boldsymbol{E}$ is an **elementary matrix**.
    - (i.e., $\boldsymbol{E}$ can be obtained from $\boldsymbol{I}$ by a single elementary row operation.)

Column Operations

- If $\boldsymbol{E}$ is obtained from $\boldsymbol{I}_n$ by a single elementary column operation, then $\boldsymbol{E}$ is an elementary matrix.
  - $\boldsymbol{I} \xrightarrow{kC_i} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{kR_i} \boldsymbol{E}$.
  - $\boldsymbol{I} \xrightarrow{C_i \leftrightarrow C_j} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}$.
  - $\boldsymbol{I} \xrightarrow{C_i + kC_j} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{R_j + kR_i} \boldsymbol{E}$.
- Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then $\boldsymbol{A}^{\mathrm{T}}$ is $n \times m$.
  - Suppose that $\boldsymbol{I}_n \xrightarrow{kC_i} \boldsymbol{E}$. Note that $\boldsymbol{E} = \boldsymbol{E}^{\mathrm{T}}$.
    - $\boldsymbol{I}_n \xrightarrow{kR_i} \boldsymbol{E}^{\mathrm{T}} \Rightarrow \boldsymbol{A}^{\mathrm{T}} \xrightarrow{kR_i} \boldsymbol{E}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} = (\boldsymbol{A}\boldsymbol{E})^{\mathrm{T}}$.

    Then $\boldsymbol{A} \xrightarrow{kC_i} \boldsymbol{A}\boldsymbol{E}$.

Column Operations

- If $\boldsymbol{E}$ is obtained from $\boldsymbol{I}_n$ by a single elementary column operation, then $\boldsymbol{E}$ is an elementary matrix.
  - $\boldsymbol{I} \xrightarrow{kC_i} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{kR_i} \boldsymbol{E}$.
  - $\boldsymbol{I} \xrightarrow{C_i \leftrightarrow C_j} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}$.
  - $\boldsymbol{I} \xrightarrow{C_i + kC_j} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{R_j + kR_i} \boldsymbol{E}$.
- Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then $\boldsymbol{A}^{\mathrm{T}}$ is $n \times m$.
  - Suppose that $\boldsymbol{I}_n \xrightarrow{C_i \leftrightarrow C_j} \boldsymbol{E}$. Note that $\boldsymbol{E} = \boldsymbol{E}^{\mathrm{T}}$.
    - $\boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}^{\mathrm{T}} \Rightarrow \boldsymbol{A}^{\mathrm{T}} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} = (\boldsymbol{A}\boldsymbol{E})^{\mathrm{T}}$.

    Then $\boldsymbol{A} \xrightarrow{C_i \leftrightarrow C_j} \boldsymbol{A}\boldsymbol{E}$.

Column Operations

- If $\boldsymbol{E}$ is obtained from $\boldsymbol{I}_n$ by a single elementary column operation, then $\boldsymbol{E}$ is an elementary matrix.
  - $\boldsymbol{I} \xrightarrow{kC_i} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{kR_i} \boldsymbol{E}$.
  - $\boldsymbol{I} \xrightarrow{C_i \leftrightarrow C_j} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}$.
  - $\boldsymbol{I} \xrightarrow{C_i + kC_j} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{R_j + kR_i} \boldsymbol{E}$.
- Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then $\boldsymbol{A}^{\mathrm{T}}$ is $n \times m$.
  - Suppose that $\boldsymbol{I} \xrightarrow{C_i + kC_j} \boldsymbol{E}$.
    - Then $\boldsymbol{I} \xrightarrow{R_i + kR_j} \boldsymbol{E}^{\mathrm{T}}$.
    - $\boldsymbol{A}^{\mathrm{T}} \xrightarrow{R_i + kR_j} \boldsymbol{E}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} = (\boldsymbol{A}\boldsymbol{E})^{\mathrm{T}}$.

  Then $\boldsymbol{A} \xrightarrow{C_i + kC_j} \boldsymbol{A}\boldsymbol{E}$.

Column Operations

- If $\boldsymbol{E}$ is obtained from $\boldsymbol{I}_n$ by a single elementary column operation, then $\boldsymbol{E}$ is an elementary matrix.
  - $\boldsymbol{I} \xrightarrow{kC_i} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{kR_i} \boldsymbol{E}$.
  - $\boldsymbol{I} \xrightarrow{C_i \leftrightarrow C_j} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}$.
  - $\boldsymbol{I} \xrightarrow{C_i + kC_j} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{R_j + kR_i} \boldsymbol{E}$.
- Let $\boldsymbol{A}$ be an $m \times n$ matrix, and $\boldsymbol{E}$ an $n \times n$ elementary matrix. Then
  - The post-multiplication of $\boldsymbol{E}$ to $\boldsymbol{A}$

  $\Leftrightarrow$ Corresponding elementary column operation to $\boldsymbol{A}$.
    - $\boldsymbol{I} \xrightarrow{kC_i} \boldsymbol{E} \Rightarrow \boldsymbol{A} \xrightarrow{kC_i} \boldsymbol{A}\boldsymbol{E}$.
    - $\boldsymbol{I} \xrightarrow{C_i \leftrightarrow C_j} \boldsymbol{E} \Rightarrow \boldsymbol{A} \xrightarrow{C_i \leftrightarrow C_j} \boldsymbol{A}\boldsymbol{E}$.
    - $\boldsymbol{I} \xrightarrow{C_i + kC_j} \boldsymbol{E} \Rightarrow \boldsymbol{A} \xrightarrow{C_i + kC_j} \boldsymbol{A}\boldsymbol{E}$.

Examples

- Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix}$.
    - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{2C_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \boldsymbol{E}_1$.
    - $\boldsymbol{A}\boldsymbol{E}_1 = \cdots = \begin{pmatrix} 1 & 0 & 4 & 3 \\ 2 & -1 & 6 & 6 \\ 1 & 4 & 8 & 0 \end{pmatrix}$
        - $\boldsymbol{A} \xrightarrow{2C_3} \boldsymbol{A}\boldsymbol{E}_1$.

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Examples

- Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix}$.
    - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_2 \leftrightarrow C_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \boldsymbol{E}_2$.
    - $\boldsymbol{A}\boldsymbol{E}_2 = \cdots = \begin{pmatrix} 1 & 3 & 2 & 0 \\ 2 & 6 & 3 & -1 \\ 1 & 0 & 4 & 4 \end{pmatrix}$
        - $\boldsymbol{A} \xrightarrow{C_2 \leftrightarrow C_4} \boldsymbol{A}\boldsymbol{E}_2$.

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Examples

- Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix}$.
  - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_1+2C_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \boldsymbol{E}_3$.
  - $\boldsymbol{A}\boldsymbol{E}_3 = \cdots = \begin{pmatrix} 5 & 0 & 2 & 3 \\ 8 & -1 & 3 & 6 \\ 9 & 4 & 4 & 0 \end{pmatrix}$.
    - $\boldsymbol{A} \xrightarrow{C_1+2C_3} \boldsymbol{A}\boldsymbol{E}_3$.

Determinant

Determinant of 2×2 Matrix

- Recall that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible $\Leftrightarrow ad - bc \neq 0$.
- **Definition.** Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
  - The **determinant** of $\boldsymbol{A}$ is $\boxed{\det(\boldsymbol{A}) = |\boldsymbol{A}| = ad - bc}$

  Therefore, $\boldsymbol{A}$ is invertible $\Leftrightarrow \det(\boldsymbol{A}) \neq 0$.
- **Definition.** If $\boldsymbol{A} = (a)$, it is natural to set $\det(\boldsymbol{A}) = a$.
- **Properties & Exercises.** Let $\boldsymbol{A}, \boldsymbol{B}$ be 2×2 matrices.
  - $\det(\boldsymbol{I}_2) = 1$.
  - $\boldsymbol{A} \xrightarrow{cR_i} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = c\det(\boldsymbol{A})$.
  - $\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = -\det(\boldsymbol{A})$.
  - $\boldsymbol{A} \xrightarrow{R_i + cR_j} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = \det(\boldsymbol{A})$.

Determinant of 2×2 Matrix

- Consider linear system $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.
  - Suppose $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is invertible.
    - $\boldsymbol{A}^{-1} = \dfrac{1}{\det(\boldsymbol{A})} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \boldsymbol{A}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \cdots = \frac{1}{\det(\boldsymbol{A})} \begin{pmatrix} a_{22}b_1 - a_{12}b_2 \\ -a_{21}b_1 + a_{11}b_2 \end{pmatrix}$$

- $x_1 = \dfrac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$ and $x_2 = \dfrac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$.

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Determinant of 2×2 Matrix

- One can verify that

$$\begin{aligned} \begin{vmatrix} a_{11} + a'_{11} & a_{12} + a'_{12} \\ a_{21} & a_{22} \end{vmatrix} &= (a_{11} + a'_{11})a_{22} - (a_{12} + a'_{12})a_{21} \\ &= (a_{11}a_{22} - a_{12}a_{21}) + (a'_{11}a_{22} - a'_{12}a_{21}) \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a'_{11} & a'_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{aligned}$$

  In particular,

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= \begin{vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{vmatrix} + a_{12} \begin{vmatrix} 0 & 1 \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11} \cdot a_{22} - a_{12} \cdot a_{21} \\ &= a_{11} \cdot \det(a_{22}) - a_{12} \cdot \det(a_{21}). \end{aligned}$$

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Determinant of 3×3 Matrix

- Let $\boldsymbol{A}$ be a square matrix. It is expected that
  - $\det(\boldsymbol{I}) = 1$.
  - $\boldsymbol{A}$ is invertible $\Leftrightarrow \det(\boldsymbol{A}) \neq 0$.
    - Equivalently, $\boldsymbol{A}$ is singular $\Leftrightarrow \det(\boldsymbol{A}) = 0$.
  - $\boldsymbol{A} \xrightarrow{cR_i} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = c\det(\boldsymbol{A})$.
  - $\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = -\det(\boldsymbol{A})$.
  - $\boldsymbol{A} \xrightarrow{R_i + cR_j} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = \det(\boldsymbol{A})$.
- Suppose $\boldsymbol{A}$ is invertible. Then there exist elementary row operations
  - $\boldsymbol{A} \xrightarrow{\text{ero1}} \boldsymbol{A}_1 \xrightarrow{\text{ero2}} \boldsymbol{A}_2 \to \cdots \to \boldsymbol{A}_{k-1} \xrightarrow{\text{ero}k} \boldsymbol{A}_k = \boldsymbol{I}$.

  Then $\det(\boldsymbol{A})$ can be evaluated backwards.
- **Example.** $\boldsymbol{A} \xrightarrow{R_1 \leftrightarrow R_3} \bullet \xrightarrow{3R_2} \bullet \xrightarrow{R_2 + 2R_4} \boldsymbol{I} \Rightarrow \det(\boldsymbol{A}) = -\frac{1}{3}$.

Determinant of 3×3 Matrix

- It is also expected that
  - $\det\begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix} + \det\begin{pmatrix} R_1' \\ R_2 \\ \vdots \\ R_n \end{pmatrix} = \det\begin{pmatrix} R_1 + R_1' \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$
- Consider 3×3 matrix $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$
  - It is expected to have $\det(\boldsymbol{A})$:
    - $\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

Determinant of 3×3 Matrix

- If $a_{11} = 0$, the determinant is supposed to be 0. Suppose $a_{11} \neq 0$.
$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow{\frac{1}{a_{11}} R_1} \begin{pmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\xrightarrow[R_3 + (-a_{31}) R_1]{R_2 + (-a_{21}) R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

We use the same elementary row operations:

  - $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \to$ RREF and $\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \to$ RREF.

Determinant of 3×3 Matrix

- If $a_{12} = 0$, the determinant is supposed to be 0. Suppose $a_{12} \neq 0$.
$\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{12} \begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{23} \\ 0 & a_{31} & a_{33} \end{vmatrix} = -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$$\begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow{\frac{1}{a_{12}} R_1} \begin{pmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\xrightarrow[R_3 + (-a_{32}) R_1]{R_2 + (-a_{22}) R_1} \begin{pmatrix} 0 & 1 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{23} \\ 0 & a_{31} & a_{33} \end{pmatrix}$$

We use the same elementary row operations:

  - $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{23} \\ 0 & a_{31} & a_{33} \end{pmatrix} \to$ RREF and $\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} \to$ RREF.

Determinant of 3×3 Matrix

- If $a_{13} = 0$, the determinant is supposed to be 0. Suppose $a_{13} \neq 0$.
$$\begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{13} \begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{22} \\ 0 & a_{31} & a_{32} \end{vmatrix} = a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix}$$
$$\begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow{\frac{1}{a_{13}}R_1} \begin{pmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$
$$\xrightarrow[R_3+(-a_{33})R_1]{R_2+(-a_{23})R_1} \begin{pmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_3} \xrightarrow{C_2 \leftrightarrow C_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{22} \\ 0 & a_{31} & a_{32} \end{pmatrix}$$
We use the same elementary row operations:
  - $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{22} \\ 0 & a_{31} & a_{32} \end{pmatrix}$ $\rightarrow$ RREF and $\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$ $\rightarrow$ RREF.

Determinant of 3×3 Matrix

- **Definition.** Let $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.
  - $\boxed{\det(\boldsymbol{A}) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}$
- **Notation.** Let $\boldsymbol{A} = (a_{ij})_{n \times n}$.
  - Let $\boldsymbol{M}_{ij}$ be the **submatrix** obtained from $\boldsymbol{A}$ by deleting the ith row and jth column.
  - If $\boldsymbol{A} = (a_{ij})_{3\times 3}$, then $\det(\boldsymbol{A})$ is given by
    - $a_{11} \det(\boldsymbol{M}_{11}) - a_{12} \det(\boldsymbol{M}_{12}) + a_{13} \det(\boldsymbol{M}_{13})$
  - Let $A_{ij} = (-1)^{i+j} \det(\boldsymbol{M}_{ij})$, the (i, j)**-cofactor** of $\boldsymbol{A}$.
  - If $\boldsymbol{A} = (a_{ij})_{3\times 3}$, then
    - $\boxed{\det(\boldsymbol{A}) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}}$.

Example

- Let $\boldsymbol{B} = (b_{ij})_{3\times 3} = \begin{pmatrix} -3 & -2 & 4 \\ 4 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$.
  - $(1,1)$-cofactor: $(-1)^{1+1}\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 3\cdot 4 - 1\cdot 2 = 10.$
  - $(1,2)$-cofactor:
    - $(-1)^{1+2}\begin{vmatrix} 4 & 1 \\ 0 & 4 \end{vmatrix} = -(4\cdot 4 - 1\cdot 0) = -16.$
  - $(1,3)$-cofactor: $(-1)^{1+3}\begin{vmatrix} 4 & 3 \\ 0 & 2 \end{vmatrix} = 4\cdot 2 - 3\cdot 0 = 8.$

$$\begin{aligned} \det(\boldsymbol{B}) &= (-3)\cdot 10 + (-2)\cdot(-16) + 4\cdot 8 \\ &= 34. \end{aligned}$$

An Alternative Formula

- Let $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

$$\begin{aligned} \det(\boldsymbol{A}) &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ &\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) \\ &\quad -(a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31}). \end{aligned}$$

  - The positive terms come from the
    - 3 (broken) diagonals from the top left to bottom right.

    The negative terms come from the
    - 3 (broken) diagonals from the top right to bottom left.

Example

- Let $\boldsymbol{B} = (b_{ij})_{3\times 3} = \begin{pmatrix} -3 & -2 & 4 \\ 4 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$.

$$\begin{aligned} \det(\boldsymbol{B}) &= [(-3)\cdot 3\cdot 4 + (-2)\cdot 1\cdot 0 + 4\cdot 4\cdot 2] \\ &\quad - [(-3)\cdot 1\cdot 2 + (-2)\cdot 4\cdot 4 + 4\cdot 3\cdot 0] \\ &= (-36+0+32) - (-6-32+0) = 34. \end{aligned}$$

- Find the determinant of $\boldsymbol{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$$\begin{aligned} \det(\boldsymbol{I}_3) &= (1\cdot 1\cdot 1 + 0\cdot 0\cdot 0 + 0\cdot 0\cdot 0) \\ &\quad - (1\cdot 0\cdot 0 + 0\cdot 0\cdot 1 + 0\cdot 1\cdot 0) \\ &= 1 - 0 = 1. \end{aligned}$$

Elementary Row Operation

- Let $\boldsymbol{A} = (a_{ij})_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.
  - $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow{cR_2} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ ca_{21} & ca_{21} & ca_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \boldsymbol{B}$.

$$\begin{aligned} \det(\boldsymbol{B}) &= (a_{11}ca_{22}a_{33} + a_{12}ca_{23}a_{31} + a_{13}ca_{21}a_{31}) \\ &\quad - (a_{11}ca_{23}a_{32} + a_{12}ca_{21}a_{33} + a_{13}ca_{22}a_{31}) \\ &= c\cdot[(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{31}) \\ &\quad - (a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31})] \\ &= c\det(\boldsymbol{A}). \end{aligned}$$

  In particular, $\boldsymbol{I} \xrightarrow{cR_i} \boldsymbol{E} \Rightarrow \det(\boldsymbol{E}) = c\cdot\det(\boldsymbol{I}) = c$.
  - $\boldsymbol{A} \xrightarrow{cR_i} \boldsymbol{EA} \Rightarrow \det(\boldsymbol{EA}) = c\det(\boldsymbol{A}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

Elementary Row Operation

- Let $\boldsymbol{A} = (a_{ij})_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.
  - $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{pmatrix} = \boldsymbol{B}$.

$$\begin{aligned} \det(\boldsymbol{B}) &= (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{11}) \\ &\quad - (a_{31}a_{23}a_{12} + a_{32}a_{21}a_{13} + a_{33}a_{22}a_{11}) \\ &= (-1)\cdot[(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) \\ &\quad - (a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31})] \\ &= -\det(\boldsymbol{A}). \end{aligned}$$

  In particular, $\boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E} \Rightarrow \det(\boldsymbol{E}) = -\det(\boldsymbol{I}) = -1$.
  - $\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{EA} \Rightarrow \det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

Elementary Row Operation

- Let $\boldsymbol{A} = (a_{ij})_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.
  - $\boldsymbol{A} \xrightarrow{R_2 + cR_1} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + ca_{11} & a_{22} + ca_{12} & a_{23} + ca_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \boldsymbol{B}$.

$$\begin{aligned} \det(\boldsymbol{B}) &= [a_{11}(a_{22} + ca_{12})a_{33} + a_{12}(a_{23} + ca_{13})a_{31} \\ &\qquad + a_{13}(a_{21} + ca_{11})a_{32}] \\ &\quad - [a_{11}(a_{23} + ca_{13})a_{32} + a_{12}(a_{21} + ca_{11})a_{33} \\ &\qquad + a_{13}(a_{22} + ca_{12})a_{31}] \\ &\overset{\text{Exercise}}{=\cdots\cdots\cdots=} \det(\boldsymbol{A}). \end{aligned}$$

  In particular, $\boldsymbol{I} \xrightarrow{R_i + cR_j} \boldsymbol{E} \Rightarrow \det(\boldsymbol{E}) = \det(\boldsymbol{I}) = 1$.
  - $\boldsymbol{A} \xrightarrow{R_i + cR_j} \boldsymbol{EA} \Rightarrow \det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

Elementary Row Operation

- Let $\boldsymbol{A}$ be a square matrix of order 3.
  - For any elementary matrix $\boldsymbol{E}$ of order 3,
    - $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

  This property can be used to find $\det(\boldsymbol{A})$.
- Let $\boldsymbol{R}$ be the reduced row-echelon form of $\boldsymbol{A}$.
  - Then $\boldsymbol{R} = \boldsymbol{E}_k\boldsymbol{E}_{k-1}\cdots\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A}$, $\boldsymbol{E}_i$ elementary.

$$\begin{aligned}\det(\boldsymbol{R}) &= \det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1}\cdots\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A})\\ &= \cdots\cdots\cdots\cdots\cdots\cdots\\ &= \det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1})\cdots\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_1\boldsymbol{A})\\ &= \det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1})\cdots\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_1)\det(\boldsymbol{A}).\end{aligned}$$

  If $\boldsymbol{A}$ is **invertible**, $\boldsymbol{R} = \boldsymbol{I}$, and $\det(\boldsymbol{R}) = 1$.
  - $\det(\boldsymbol{A}) = [\det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1})\cdots\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_1)]^{-1}$

Elementary Row Operation

- Let $\boldsymbol{A}$ be a square matrix of order 3.
  - For any elementary matrix $\boldsymbol{E}$ of order 3,
    - $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

  This property can be used to find $\det(\boldsymbol{A})$.
- Let $\boldsymbol{R}$ be the reduced row-echelon form of $\boldsymbol{A}$.
  - Then $\boldsymbol{R} = \boldsymbol{E}_k\boldsymbol{E}_{k-1}\cdots\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A}$, $\boldsymbol{E}_i$ elementary.

$$\det(\boldsymbol{R}) = \det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1})\cdots\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_1)\det(\boldsymbol{A}).$$

  If $\boldsymbol{A}$ is **singular**, then the last row of $\boldsymbol{R}$ is zero.
  - $\boldsymbol{R} \xrightarrow{2R_3} \boldsymbol{R} \Rightarrow 2\det(\boldsymbol{R}) = \det(\boldsymbol{R}) \Rightarrow \det(\boldsymbol{R}) = 0.$

  Note that $\det(\boldsymbol{E}) \neq 0$ for any elementary matrix $\boldsymbol{E}$.
  - We must have $\det(\boldsymbol{A}) = 0$.

Example

- Let $\boldsymbol{A} = \begin{pmatrix} -3 & -2 & 4 \\ 4 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$.

$$\boldsymbol{A} \xrightarrow[\boldsymbol{E}_1]{R_2+\frac{4}{3}R_1} \bullet \xrightarrow[\boldsymbol{E}_2]{R_3+(-6)R_2} \bullet \xrightarrow[\boldsymbol{E}_3]{-\frac{1}{3}R_1} \bullet \xrightarrow[\boldsymbol{E}_4]{3R_2} \bullet$$
$$\xrightarrow[\boldsymbol{E}_5]{-\frac{1}{34}R_3} \bullet \xrightarrow[\boldsymbol{E}_6]{R_1+\frac{4}{3}R_3} \bullet \xrightarrow[\boldsymbol{E}_7]{R_2+(-19)R_3} \bullet \xrightarrow[\boldsymbol{E}_8]{R_1+(-\frac{2}{3}R_2)} \boldsymbol{I}.$$

  - $\det(\boldsymbol{E}_i) = 1$ if $i = 1, 2, 6, 7, 8$.
  - $\det(\boldsymbol{E}_3) = -\frac{1}{3}$, $\det(\boldsymbol{E}_4) = 3$, $\det(\boldsymbol{E}_5) = -\frac{1}{34}$.

  $\det(\boldsymbol{A}) = [\det(\boldsymbol{E}_1)\cdots\det(\boldsymbol{E}_8)]^{-1} = (\frac{1}{34})^{-1} = 34.$

- We will show that in order to find $\det(\boldsymbol{A})$ it suffices to use the Gaussian elimination to get a row-echelon form of $\boldsymbol{A}$.

Determinant

- **Definition.** Let $\boldsymbol{A} = (a_{ij})_{n\times n}$. Its **determinant** is:
  - If $n = 1$, define $\det(\boldsymbol{A}) = a_{11}$.
  - If $n > 1$, let A_{ij} be its (i, j)-cofactor, define
    - $\det(\boldsymbol{A}) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$.
- **Example.** Let $\boldsymbol{A} = (a_{ij})_{4\times 4} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$.

$$\det(\boldsymbol{A}) = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$
$$+ a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}.$$

Determinant

- **Example.** Find $\det(\boldsymbol{C})$ if $\boldsymbol{C} = \begin{pmatrix} 0 & -1 & 2 & 0 \\ 2 & -3 & 3 & -2 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$.

$$\begin{aligned}\det(\boldsymbol{C}) &= 0 \cdot \begin{vmatrix} -3 & 3 & -2 \\ 2 & 4 & 0 \\ 0 & 2 & -1 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 2 & 3 & -2 \\ 0 & 4 & 0 \\ 0 & 2 & -1 \end{vmatrix} \\ &+ 2 \cdot \begin{vmatrix} 2 & -3 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & -3 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{vmatrix} \\ &= 0 \cdot 10 - (-1) \cdot (-8) + 2 \cdot (-4) - 0 \cdot 8 = -16.\end{aligned}$$

- **Warning**: The "diagonal expansion" of $\det(\boldsymbol{A})$ for 2×2 or 3×3 matrices is no longer valid if the order ≥ 4.

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Properties

- **Theorem.** $\det(\boldsymbol{I}) = 1$. For any square matrices,
  - If $\boldsymbol{A} \xrightarrow{cR_i} \boldsymbol{B}$, then $\det(\boldsymbol{B}) = c\det(\boldsymbol{A})$.
    - In particular, If $\boldsymbol{I} \xrightarrow{cR_i} \boldsymbol{E}$, then $\det(\boldsymbol{E}) = c$.
  - If $\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B}$, then $\det(\boldsymbol{B}) = -\det(\boldsymbol{A})$.
    - In particular, if $\boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}$, then $\det(\boldsymbol{E}) = -1$.
  - If $\boldsymbol{A} \xrightarrow{R_i + cR_j} \boldsymbol{B}$, then $\det(\boldsymbol{B}) = \det(\boldsymbol{A})$.
    - In particular, if $\boldsymbol{I} \xrightarrow{R_i + cR_j} \boldsymbol{E}$, then $\det(\boldsymbol{E}) = 1$.
- **Theorem.** Let $\boldsymbol{A}$ be a square matrix.
  - For any elementary matrix $\boldsymbol{E}$ of the same order,
    - $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

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Properties

- **Theorem.** Suppose a square matrix $\boldsymbol{A}$ has a **zero row**.
  - Then $\det(\boldsymbol{A}) = 0$.

  Proof. Suppose the ith row of square matrix $\boldsymbol{A}$ is $\mathbf{0}$.
  - $\boldsymbol{A} \xrightarrow{2R_i} \boldsymbol{A} \Rightarrow \det(\boldsymbol{A}) = 2\det(\boldsymbol{A}) \Rightarrow \det(\boldsymbol{A}) = 0.$
- Suppose square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are **row equivalent**.
  - There exist elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$ s.t.
    - $\boldsymbol{B} = \boldsymbol{E}_k \cdots \boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A}$.

    $\det(\boldsymbol{B}) = \det(\boldsymbol{E}_k) \cdots \det(\boldsymbol{E}_2)\det(\boldsymbol{E}_1)\det(\boldsymbol{A}).$

  Note that $\det(\boldsymbol{E}) \neq 0$ for every elementary matrix $\boldsymbol{E}$.
  - $\det(\boldsymbol{A}) = 0 \Leftrightarrow \det(\boldsymbol{B}) = 0$.
  - Equivalently, $\det(\boldsymbol{A}) \neq 0 \Leftrightarrow \det(\boldsymbol{B}) \neq 0$.

Properties

- **Theorem.** Suppose a square matrix $\boldsymbol{A}$ has a **zero row**.
  - Then $\det(\boldsymbol{A}) = 0$.

  Proof. Suppose the ith row of square matrix $\boldsymbol{A}$ is $\mathbf{0}$.
  - $\boldsymbol{A} \xrightarrow{2R_i} \boldsymbol{A} \Rightarrow \det(\boldsymbol{A}) = 2\det(\boldsymbol{A}) \Rightarrow \det(\boldsymbol{A}) = 0.$
- **Theorem.** $\det(\boldsymbol{A}) = 0 \Leftrightarrow \boldsymbol{A}$ is **singular**.
  - Equivalently, $\det(\boldsymbol{A}) \neq 0 \Leftrightarrow \boldsymbol{A}$ is **invertible**.

  Proof. Suppose $\boldsymbol{A}$ is invertible. Then
  - $\boldsymbol{A}$ is row equivalent to $\boldsymbol{I}$.
    - $\det(\boldsymbol{I}) = 1 \neq 0 \Rightarrow \det(\boldsymbol{A}) \neq 0.$

    Suppose $\boldsymbol{A}$ is singular. Then the RREF of $\boldsymbol{A}$ is not $\boldsymbol{I}$.
    - The RREF of $\boldsymbol{A}$ has a zero row $\Rightarrow \det(\boldsymbol{A}) = 0.$

Properties

- **Theorem.** Let $\boldsymbol{A}, \boldsymbol{B}$ be square matrices of the same size.
  - Then $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$.

Proof. Suppose that $\boldsymbol{A}$ is invertible. Then

- $\boldsymbol{A} = \boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_k$ for elementary matrices $\boldsymbol{E}_i$.

$$\begin{aligned}\det(\boldsymbol{AB}) &= \det(\boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_k\boldsymbol{B})\\ &= \det(\boldsymbol{E}_1)\det(\boldsymbol{E}_2)\cdots\det(\boldsymbol{E}_k)\det(\boldsymbol{B})\\ &= \det(\boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_k)\det(\boldsymbol{B})\\ &= \det(\boldsymbol{A})\det(\boldsymbol{B}).\end{aligned}$$

Suppose that $\boldsymbol{A}$ is singular. Then $\boldsymbol{AB}$ is singular.

- $\det(\boldsymbol{A}) = 0$ and $\det(\boldsymbol{AB}) = 0$.

Then $\det(\boldsymbol{AB}) = 0 = \det(\boldsymbol{A})\det(\boldsymbol{B})$.

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Properties

- **Theorem.** For any square $\boldsymbol{A}$, $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^{\mathrm{T}})$.

Proof. Suppose $\boldsymbol{A}$ is singular.

- Then $\boldsymbol{A}^{\mathrm{T}}$ is also singular, because
  - $\boldsymbol{A}^{\mathrm{T}}$ is invertible $\Rightarrow \boldsymbol{A} = (\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}}$ is invertible.

  For this case, $\det(\boldsymbol{A}) = 0 = \det(\boldsymbol{A}^{\mathrm{T}})$.

Suppose $\boldsymbol{A}$ is invertible. Then

- $\boldsymbol{A} = \boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_k$ for elementary matrices $\boldsymbol{E}_i$.

Note that $\det(\boldsymbol{E}) = \det(\boldsymbol{E}^{\mathrm{T}})$ for elementary matrix $\boldsymbol{E}$.

$$\begin{aligned}\det(\boldsymbol{A}^{\mathrm{T}}) &= \det(\boldsymbol{E}_k^{\mathrm{T}}\cdots\boldsymbol{E}_2^{\mathrm{T}}\boldsymbol{E}_1^{\mathrm{T}})\\ &= \det(\boldsymbol{E}_k^{\mathrm{T}})\cdots\det(\boldsymbol{E}_2^{\mathrm{T}})\det(\boldsymbol{E}_1^{\mathrm{T}})\\ &= \det(\boldsymbol{E}_k)\cdots\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_1)\\ &= \det(\boldsymbol{E}_1)\det(\boldsymbol{E}_2)\cdots\det(\boldsymbol{E}_k)\\ &= \det(\boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_k) = \det(\boldsymbol{A}).\end{aligned}$$

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Properties

- **Theorem.** Suppose $\boldsymbol{A} = (a_{ij})_{n\times n}$ is **upper triangular**.
  - Then $\det(\boldsymbol{A}) = a_{11}a_{22}\cdots a_{nn}$.
- **Example.**
  - $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 10 & 11 & 12 \\ 0 & 0 & 0 & 13 & 14 \\ 0 & 0 & 0 & 0 & 15 \end{vmatrix} = 1\cdot 6\cdot 10\cdot 13\cdot 15 = 11700.$
- **Remarks.**
  - The result is also true for **lower triangular matrices**.
  - Note that a row-echelon form of a square matrix is always upper triangular.
    - To find the determinant using elementary row operation, it suffices to use Gaussian elimination to get a row-echelon form.

Properties

- **Theorem.** Suppose $\boldsymbol{A} = (a_{ij})_{n\times n}$ is **upper triangular**.
  - Then $\det(\boldsymbol{A}) = a_{11}a_{22}\cdots a_{nn}$.

Idea of Proof. Suppose that $a_{ii} \neq 0$ for all $i = 1, \ldots, n$.

- Illustration:

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & * & * & * \\ 0 & a_{22} & * & * \\ 0 & 0 & a_{33} & * \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \xrightarrow{\text{multiply } a_{ii}^{-1} \text{ to } R_i} \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} = \boldsymbol{B}$$

$$\xrightarrow{\text{convert to RREF}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \boldsymbol{I}.$$

Therefore, $\det(\boldsymbol{A}) = a_{11}\cdots a_{nn}\det(\boldsymbol{I}) = a_{11}\cdots a_{nn}$.

Properties

- **Theorem.** Suppose $\boldsymbol{A} = (a_{ij})_{n\times n}$ is **upper triangular**.
  - Then $\det(\boldsymbol{A}) = a_{11}a_{22}\cdots a_{nn}$.

Idea of Proof. Suppose that $a_{ii} = 0$ for some i.

- Illustration: Assume that $a_{22} = 0$ but $a_{33} \neq 0$, $a_{44} \neq 0$.

$$\begin{pmatrix} * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & \otimes & * \\ 0 & 0 & 0 & \otimes \end{pmatrix} \xrightarrow[\text{to the 3rd and 4th rows}]{\text{row operations}} \begin{pmatrix} * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[\text{to the 2nd row}]{\text{add multiples of 3rd and 4th rows}} \begin{pmatrix} * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\boldsymbol{A}$ is row equivalent to a singular matrix; $\det(\boldsymbol{A}) = 0 = a_{11}\cdots a_{nn}$.

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Cofactor Expansion

- **Theorem.** Let $\boldsymbol{A}$ be a square matrix of order n.
  - Let A_{ij} denote the (i, j)-cofactor of $\boldsymbol{A}$.

Then for any i and j,

- $\det(\boldsymbol{A}) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$.
- $\det(\boldsymbol{A}) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$.

Idea of Proof. Suppose $\boldsymbol{A}$ is 4×4. Let $\boldsymbol{B} = (b_{ij})$ be obtained from $\boldsymbol{A}$ by

- Moving the 4th row to the top. Then $b_{1j} = a_{4j}$.

- $\boldsymbol{A} = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} \xrightarrow{R_3\leftrightarrow R_4, R_2\leftrightarrow R_3, R_2\leftrightarrow R_1} \begin{pmatrix} R_4 \\ R_1 \\ R_2 \\ R_3 \end{pmatrix} = \boldsymbol{B}$.

So $\det(\boldsymbol{B}) = (-1)^{4-1}\det(\boldsymbol{A})$.

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Cofactor Expansion

- **Idea of Proof.** Suppose $\boldsymbol{A}$ is 4×4. Let $\boldsymbol{B} = (b_{ij})$ be obtained from $\boldsymbol{A}$ by
  - Moving the 4th row to the top. Then $b_{1j} = a_{4j}$.
    - $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \dashrightarrow \begin{pmatrix} a_{41} & a_{42} & a_{43} & a_{44} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} = \boldsymbol{B}$.

    So $\det(\boldsymbol{B}) = (-1)^{4-1} \det(\boldsymbol{A})$.

  The submatrix obtained from $\boldsymbol{B}$ by deleting its 1st row and jth column is the same as that obtained from $\boldsymbol{A}$ by deleting its 4th row and jth column, say $\boldsymbol{M}_{ij}$.
  - Let B_{ij} be the (i,j)-cofactor of $\boldsymbol{B}$.
    - $B_{1j} = (-1)^{1+j} \det(\boldsymbol{M}_{ij})$, $A_{4j} = (-1)^{4+j} \det(\boldsymbol{M}_{ij})$.

    So $B_{1j} = (-1)^{4-1} A_{4j}$, $j = 1, \ldots, n$.

Cofactor Expansion

- **Proof.** Fix i. Let $\boldsymbol{B} = (b_{ij})$ be obtained from $\boldsymbol{A}$ by
  - Moving the ith row to the top.

  Then $b_{1j} = a_{ij}$, $B_{1j} = (-1)^{i-1} A_{ij}$ for any j.

$$\begin{aligned} \det(\boldsymbol{A}) &= (-1)^{i-1} \det(\boldsymbol{B}) \\ &= (-1)^{i-1} \cdot (b_{11} B_{11} + b_{12} B_{12} + \cdots + b_{1n} B_{1n}) \\ &= (-1)^{i-1} \cdot [a_{i1} (-1)^{i-1} A_{i1} + a_{i2} (-1)^{i-1} A_{i2} \\ &\qquad + \cdots + a_{in} (-1)^{i-1} A_{in}] \\ &= (-1)^{2i-2} (a_{i1} A_{i1} + a_{i2} A_{i2} + \cdots + a_{in} A_{in}) \\ &= a_{i1} A_{i1} + a_{i2} A_{i2} + \cdots + a_{in} A_{in}. \end{aligned}$$

  The is called the **cofactor expansion along the ith row**.
- The proof for the **cofactor expansion along the jth column** is left as exercises. (*Hint*: Consider $\boldsymbol{A}^{\mathrm{T}}$.)

Cofactor Expansion

- In evaluating the determinant using cofactor expansion,
  - expand along the row or column with the most zeros.
- **Example.** $\boldsymbol{A} = \begin{pmatrix} 0 & -1 & 2 & 0 \\ 2 & -3 & 3 & -2 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$.

$$\begin{aligned} \det(\boldsymbol{A}) &= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} + a_{41}A_{41} \\ &= 2 \cdot (-1)^{2+1} \begin{vmatrix} -1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 2 & -1 \end{vmatrix} \\ &= -2 \cdot (-1) \cdot (-1)^{3+3} \begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} \\ &= -16. \end{aligned}$$

Finding Determinant

- Find $\det(\boldsymbol{A})$ if $\boldsymbol{A}$ is a square matrix of order n.
  - If $\boldsymbol{A}$ has a zero row/column, then $\det(\boldsymbol{A}) = 0$.
  - If $\boldsymbol{A}$ is triangular, $\det(\boldsymbol{A}) = a_{11} \cdots a_{nn}$.
  - Suppose that $\boldsymbol{A}$ is not triangular.
    - If $n = 2$, use formula $\det(\boldsymbol{A}) = a_{11}a_{22} - a_{12}a_{21}$.
    - If a row/coln has many 0, use cofactor expansion.
    - Otherwise, use ele. row operations to get REF:
      - $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.
- Note the following formulas (exercises for the last two):
  - $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^{\mathrm{T}})$.
  - $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$.
  - $\det(c\boldsymbol{A}) = c^n \det(\boldsymbol{A})$, where $\boldsymbol{A}$ is $n \times n$.
  - $\det(\boldsymbol{A}^{-1}) = \det(\boldsymbol{A})^{-1}$ if $\boldsymbol{A}$ is invertible.

Examples

- Find $\det(\boldsymbol{A})$, where $\boldsymbol{A} = \begin{pmatrix} 3 & -1 & 1 & 1 \\ 3 & -1 & 2 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$.

$$\begin{pmatrix} 3 & -1 & 1 & 1 \\ 3 & -1 & 2 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix} \xrightarrow[\boldsymbol{E}_1]{R_2+(-1)R_1} \begin{pmatrix} 3 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$$

$$\xrightarrow[\boldsymbol{E}_2]{R_2 \leftrightarrow R_3} \begin{pmatrix} 3 & -1 & 1 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix} \xrightarrow[\boldsymbol{E}_3]{R_4+(-2)R_3} \begin{pmatrix} 3 & -1 & 1 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \boldsymbol{B}.$$

  - $\det(\boldsymbol{B}) = 3 \cdot 2 \cdot 1 \cdot (-1) = -6.$
    - $\det(\boldsymbol{A}) = [\det(\boldsymbol{E}_1)\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_3)]^{-1}\det(\boldsymbol{B}) = 6.$

Examples

- Find $\det(\boldsymbol{A})$, where $\boldsymbol{A} = \begin{pmatrix} 3 & -1 & 1 & 1 \\ 3 & -1 & 2 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$.
  - Expand along the 4th row: $\det(\boldsymbol{A}) = (-1)(-6) = 6$.
    - $2 \cdot (-1)^{4+3} \begin{vmatrix} 3 & -1 & 1 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{vmatrix} + (-1) \cdot (-1)^{4+4} \begin{vmatrix} 3 & -1 & 1 \\ 3 & -1 & 2 \\ 0 & 2 & 4 \end{vmatrix}$
    - $\begin{vmatrix} 3 & -1 & 1 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{vmatrix} \overset{R_2+(-1)R_1}{=\!=\!=\!=} \begin{vmatrix} 3 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix} = 0$
    - $\begin{vmatrix} 3 & -1 & 1 \\ 3 & -1 & 2 \\ 0 & 2 & 4 \end{vmatrix} = 3 \cdot \begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} - 3 \begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix} = -6.$

Examples

- Let $\boldsymbol{A} = \begin{pmatrix} -3 & -2 & 4 \\ 4 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.
  - It is given that $\det(\boldsymbol{A}) = 34$ and $\det(\boldsymbol{B}) = -1$.

1. $\det(\boldsymbol{A}^{\mathrm{T}}) = \det(\boldsymbol{A}) = 34$.
2. $\det(2\boldsymbol{A}) = 2^3 \det(\boldsymbol{A}) = 8 \cdot 34 = 272$.
3. $\det(\boldsymbol{A}^{-1}) = [\det(\boldsymbol{A})]^{-1} = \frac{1}{34}$.
4. $\det(\boldsymbol{AB}) = \det(\boldsymbol{A}) \det(\boldsymbol{B}) = 34 \cdot (-1) = -34$.
5. $\det(\boldsymbol{BA}) = \det(\boldsymbol{B}) \det(\boldsymbol{A}) = (-1) \cdot 34 = -34$.

Adjoint Matrix

- **Definition.** Let $\boldsymbol{A}$ be a square matrix of order n. The **(classical) adjoint** (or **adjugate**, or **adjunct**) of $\boldsymbol{A}$ is
  - $\boxed{\mathbf{adj}(\boldsymbol{A}) = (A_{ji})_{n \times n}}$

  where A_{ij} is the (i, j)-cofactor of $\boldsymbol{A}$.

- **Example.** Let $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Then
  - $A_{11} = a_{22}$, $A_{12} = -a_{21}$, $A_{21} = -a_{12}$, $A_{22} = a_{11}$.
  - $\mathbf{adj}(\boldsymbol{A}) = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.

  Recall that if $\boldsymbol{A}$ is invertible, then
  - $\boldsymbol{A}^{-1} = \dfrac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.

  It is conjectured that $\boldsymbol{A}^{-1} = [\det(\boldsymbol{A})]^{-1}\, \mathbf{adj}(\boldsymbol{A})$.

Adjoint Matrix

- **Theorem.** Let $\boldsymbol{A}$ be a square matrix. Then
  - $\boldsymbol{A}[\mathbf{adj}(\boldsymbol{A})] = \det(\boldsymbol{A})\boldsymbol{I}$.

  Proof. Let $\boldsymbol{A}[\mathbf{adj}(\boldsymbol{A})] = (c_{ij})$. Then
  - $c_{ij} = a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn}$.

  Let $i = j$. Then $c_{ii} = a_{i1}A_{i1} + \cdots + a_{in}A_{in} = \det(\boldsymbol{A})$.

  Suppose that $i \neq j$. Let $\boldsymbol{B}$ be the matrix obtained from $\boldsymbol{A}$ by replacing jth row by the ith row.
  - $\boldsymbol{B} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = -\det(\boldsymbol{B}) \Rightarrow \det(\boldsymbol{B}) = 0.$

$$\begin{aligned} c_{ij} &= a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} \\ &= b_{j1}B_{j1} + b_{j2}B_{j2} + \cdots + b_{jn}B_{jn} \\ &= \det(\boldsymbol{B}) = 0. \end{aligned}$$

  Therefore, $\boldsymbol{A}[\mathbf{adj}(\boldsymbol{A})] = \det(\boldsymbol{A})\boldsymbol{I}$.

Adjoint Matrix

- **Theorem.** Let $\boldsymbol{A}$ be a square matrix. Then
  - $\boldsymbol{A}[\mathbf{adj}(\boldsymbol{A})] = \det(\boldsymbol{A})\boldsymbol{I}$.
  - $[\mathbf{adj}(\boldsymbol{A})]\boldsymbol{A} = \det(\boldsymbol{A})\boldsymbol{I}$. (Exercise!)

  If $\boldsymbol{A}$ is invertible, then $\boldsymbol{A}^{-1} = \dfrac{1}{\det(\boldsymbol{A})}\,\mathbf{adj}(\boldsymbol{A})$.
- **Exercises.** Let $\boldsymbol{A}, \boldsymbol{B}$ be invertible matrices of order n.
  - Find $[\mathbf{adj}(\boldsymbol{A})]^{-1}$ and $\mathbf{adj}(\boldsymbol{A}^{-1})$.
  - Find $\det(\mathbf{adj}(\boldsymbol{A}))$ and $\mathbf{adj}(\mathbf{adj}(\boldsymbol{A}))$.
  - Prove that $\mathbf{adj}(\boldsymbol{AB}) = \mathbf{adj}(\boldsymbol{B})\,\mathbf{adj}(\boldsymbol{A})$.
- **Challenging Problems.** Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are not necessarily invertible.
  - Find $\det(\mathbf{adj}(\boldsymbol{A}))$ and $\mathbf{adj}(\mathbf{adj}(\boldsymbol{A}))$.
  - Is it true that $\mathbf{adj}(\boldsymbol{AB}) = \mathbf{adj}(\boldsymbol{B})\,\mathbf{adj}(\boldsymbol{A})$?

Example

- Let $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$. $\det(\boldsymbol{A}) = (-1)\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2$.

$$\mathbf{adj}(\boldsymbol{A}) = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -3 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\mathbf{adj}(\boldsymbol{A}) = \begin{pmatrix} \frac{3}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Cramer's Rule

- Let $\boldsymbol{A} = (a_{ij})_{n\times n}$ be an invertible matrix.
  - The linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution
    - $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$.

  Recall that $\boldsymbol{A}^{-1} = \dfrac{1}{\det(\boldsymbol{A})}[\mathbf{adj}(\boldsymbol{A})]$. Let $\boldsymbol{b} = (b_i)_{n\times 1}$.

  - $x_j = \dfrac{1}{\det(\boldsymbol{A})}(A_{1j}b_1 + A_{2j}b_2 + \cdots + A_{nj}b_n)$.

  Fix j, and let $\boldsymbol{A}_j$ be the matrix obtained by replacing the jth column of $\boldsymbol{A}$ by $\boldsymbol{b}$. Then

  - b_i is the (i,j)-entry of $\boldsymbol{A}_j$.
  - A_{ij} is the (i,j)-cofactor of $\boldsymbol{A}_j$.

  Therefore, $x_j = \dfrac{\det(\boldsymbol{A}_j)}{\det(\boldsymbol{A})}$, $j = 1, \ldots, n$.

Cramer's Rule

- **Cramer's Rule.** Let $\boldsymbol{A}$ be an invertible matrix of order n.
  - For every column matrix $\boldsymbol{b}$ of size $n \times 1$, the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution
    - $\boldsymbol{x} = \dfrac{1}{\det(\boldsymbol{A})} \begin{pmatrix} \det(\boldsymbol{A}_1) \\ \vdots \\ \det(\boldsymbol{A}_n) \end{pmatrix}$,

    $\boldsymbol{A}_j$ is obtained from $\boldsymbol{A}$ by replacing its jth coln by $\boldsymbol{b}$.
- **Example.** Let $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.
  - Suppose that $\boldsymbol{A}$ is invertible. $\boldsymbol{Ax} = \boldsymbol{b}$ implies
    - $\boldsymbol{x} = \dfrac{1}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \begin{pmatrix} \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \\ \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} \end{pmatrix}$

Example

- $\begin{pmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$. $\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix} = 60.$
  - $x = \dfrac{\begin{vmatrix} 0 & 1 & 3 \\ 4 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}}{60} = \dfrac{132}{60} = 2.2$
  - $y = \dfrac{\begin{vmatrix} 1 & 0 & 3 \\ 2 & 4 & 2 \\ 3 & 3 & 0 \end{vmatrix}}{60} = \dfrac{-24}{60} = -0.4$
  - $z = \dfrac{\begin{vmatrix} 1 & 1 & 0 \\ 2 & -2 & 4 \\ 3 & 9 & 3 \end{vmatrix}}{60} = \dfrac{-36}{60} = -0.6$