NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 2, 2022/2023

MA2001 Linear Algebra **Homework Assignment 2**

1. For each of the following sets, decide if it is a subspace of $\mathbb{R}^n$. The value of n varies from question to question and should be clear from context.

(i) The line passing through the points $(1,2)$ and $(-2,-4)$.

Solution. The slope of the line is $((-4)-2)/((-2)-1) = 2$, and the line has equation

$$y = 2(x-1) + 2 = 2x,$$

which is homogeneous. Hence, the line is a subspace of $\mathbb{R}^2$.

(ii) $\{(x, y) \mid x + y = 1\}$.

Solution. Since $x+y=1$ is non-homogeneous, the solution set is not a subspace of $\mathbb{R}^2$.

(iii) $\{(x, y, z) \mid x + y = 1\}$.

Solution. Since $x+y=1$ is non-homogeneous, the solution set is not a subspace of $\mathbb{R}^3$.

(iv) $\{(x, y, z) \mid xyz = 0\}$.

Solution. Since $(1,1,0)$ and $(0,0,1)$ are both in the set, but $(1,1,0)+(0,0,1) = (1,1,1)$ is not, this set is not closed under addition, and thus it is not a subspace of $\mathbb{R}^3$.

(v) The solution set to the system $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Solution. Since $\boldsymbol{Ax} = \boldsymbol{b}$ is non-homogeneous, the solution set is not a subspace of $\mathbb{R}^2$.

(vi) $\{(w, x, y, z) \mid w + 2x - y + z = 0\}$.

Solution. Since $w+2x-y+z=0$ is homogeneous, the solution set is a subspace of $\mathbb{R}^4$.

(vii) The set of all $\boldsymbol{b} \in \mathbb{R}^2$ such that the system $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent, where $\boldsymbol{A}$ is a fixed 2×2 invertible matrix.

Solution. If $\boldsymbol{A}$ is invertible, then the system $\boldsymbol{Ax} = \boldsymbol{b}$ is always consistent. Hence, the set is $\mathbb{R}^2$, which is a subspace of $\mathbb{R}^2$.

2. Consider the following vectors in $\mathbb{R}^3$:

$$\boldsymbol{u}_1 = (1,-3,2), \qquad \boldsymbol{u}_2 = (0,2,-1),$$
$$\boldsymbol{v}_1 = (0,-3,-2), \quad \boldsymbol{v}_2 = (1,-1,1), \quad \boldsymbol{v}_3 = (2,0,1).$$

Solution. View each vector as a column vector.

$$\left(\begin{array}{ccc|c|c} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{array}\right) = \left(\begin{array}{ccc|c|c} 0 & 1 & 2 & 1 & 0 \\ -3 & -1 & 0 & -3 & 2 \\ -2 & 1 & 1 & 2 & -1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c|c} -3 & -1 & 0 & -3 & 2 \\ 0 & 1 & 2 & 1 & 0 \\ -2 & 1 & 1 & 2 & -1 \end{array}\right)$$
$$\xrightarrow{R_3 - \frac{2}{3}R_1} \left(\begin{array}{ccc|c|c} -3 & -1 & 0 & -3 & 2 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & \frac{5}{3} & 1 & 4 & -\frac{7}{3} \end{array}\right) \xrightarrow{R_3 - \frac{5}{3}R_2} \left(\begin{array}{ccc|c|c} -3 & -1 & 0 & -3 & 2 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & -\frac{7}{3} & \frac{7}{3} & -\frac{7}{3} \end{array}\right).$$

Since the columns corresponding to $\boldsymbol{u}_1, \boldsymbol{u}_2$ are non-pivot, both $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

Indeed, since all rows of the row-echelon form of $\begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{pmatrix}$ are nonzero, we conclude that

$$\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} = \mathbb{R}^3.$$

$$\left(\begin{array}{cc|c|c|c} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{array}\right) = \left(\begin{array}{cc|c|c|c} 1 & 0 & 0 & 1 & 2 \\ -3 & 2 & -3 & -1 & 0 \\ 2 & -1 & -2 & 1 & 1 \end{array}\right) \xrightarrow[R_3 - 2R_1]{R_2 + 3R_1} \left(\begin{array}{cc|c|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 2 & -3 & 2 & 6 \\ 0 & -1 & -2 & -1 & -3 \end{array}\right)$$
$$\xrightarrow{R_3 + \frac{1}{2}R_2} \left(\begin{array}{cc|c|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 2 & -3 & 2 & 6 \\ 0 & 0 & -\frac{7}{2} & 0 & 0 \end{array}\right)$$

Since the column corresponding to $\boldsymbol{v}_1$ is pivot, and those corresponding to $\boldsymbol{v}_2, \boldsymbol{v}_3$ are non-pivot, we have $\boldsymbol{v}_1 \notin \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $\boldsymbol{v}_2, \boldsymbol{v}_3 \notin \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.

In particular, since $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ contains only 2 vectors, it is not large enough to span $\mathbb{R}^3$.

As a conclusion:

(i) $\boldsymbol{u}_1 \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, $\boldsymbol{u}_2 \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$;

$\operatorname{span}\{\boldsymbol{u}_1\} \subseteq \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, $\operatorname{span}\{\boldsymbol{u}_2\} \subseteq \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, $\operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\} \subseteq \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

(ii) $\boldsymbol{v}_1 \notin \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, $\boldsymbol{v}_2 \in \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, $\boldsymbol{v}_3 \in \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$;

$\operatorname{span}\{\boldsymbol{v}_1\} \nsubseteq \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, $\operatorname{span}\{\boldsymbol{v}_2\} \subseteq \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, $\operatorname{span}\{\boldsymbol{v}_3\} \subseteq \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$;

$\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \nsubseteq \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_3\} \nsubseteq \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, $\operatorname{span}\{\boldsymbol{v}_2, \boldsymbol{v}_3\} \subseteq \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$;

$\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \nsubseteq \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.

(iii) $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \neq \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$. Here are two ways to see this. (1) The latter is $\mathbb{R}^3$ (as observed previously) but the former cannot be $\mathbb{R}^3$ as $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ has only two vectors. (2) Prove that $\boldsymbol{v}_3 \notin \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ by row-reducing $\left(\begin{array}{cc|c} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{array}\right)$ and noting that the third column is pivot.

3. Find the determinant of the following matrix:

$$\boldsymbol{A} = \begin{pmatrix} 2-x & 0 & 2 \\ 1 & 2-x & 0 \\ 3 & -2 & 2-x \end{pmatrix}.$$

Solution. Expand along the 1st row:

$$\begin{aligned} \det(\boldsymbol{A}) &= (2-x)\begin{vmatrix} 2-x & 0 \\ -2 & 2-x \end{vmatrix} + 2\begin{vmatrix} 1 & 2-x \\ 3 & -2 \end{vmatrix} \\ &= (2-x)\cdot(2-x)^2 + 2\cdot[1(-2)-3(2-x)] \\ &= -x^3+6x^2-6x-8. \end{aligned}$$

4. Let V be a subset of $\mathbb{R}^n$. Suppose that there are vectors $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ such that $\boldsymbol{v}_1 - 2\boldsymbol{v}_2 \notin V$. Prove that V is not a subspace of $\mathbb{R}^n$.

Proof. By definition, $\boldsymbol{v}_1 - 2\boldsymbol{v}_2 = v_1 + (-2)\boldsymbol{v}_2 \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Assume that V is a subspace of $\mathbb{R}^n$. Since $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$, we have $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \subseteq V$, and thus $\boldsymbol{v}_1 - 2\boldsymbol{v}_2 \in V$, a contradiction.

5. Let $\boldsymbol{u}_1 = (-3,0,2)$, $\boldsymbol{u}_2 = (1,4,0)$, $\boldsymbol{u}_3 = (1,2,-1)$, $\boldsymbol{u}_4 = (0,3,-2)$.

(i) Is $\operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\} = \mathbb{R}^3$?

(ii) Is $\operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \mathbb{R}^3$?

(iii) Is $\operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\} = \mathbb{R}^3$?

Solution. View each vector as a column vector.

$$\begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 & \boldsymbol{u}_4 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 & 0 \\ 0 & 4 & 2 & 3 \\ 2 & 0 & -1 & -2 \end{pmatrix} \xrightarrow{R_3+\frac{2}{3}R_1} \begin{pmatrix} -3 & 1 & 1 & 0 \\ 0 & 4 & 2 & 3 \\ 0 & \frac{2}{3} & -\frac{1}{3} & -2 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 & 0 \\ 0 & 4 & 2 & 3 \\ 0 & 0 & -\frac{2}{3} & -\frac{5}{2} \end{pmatrix}.$$

(i) All rows of row-echelon form of $\begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 & \boldsymbol{u}_4 \end{pmatrix}$ are nonzero, so

$$\operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\} = \mathbb{R}^3.$$

(ii) All rows of row-echelon form of $\begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{pmatrix}$ are nonzero, so

$$\operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \mathbb{R}^3.$$

(iii) The last row of row-echelon form of $\begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{pmatrix}$ is zero, so

$$\operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\} \neq \mathbb{R}^3.$$

Alternatively, since $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ has only 2 vectors, it is not large enough to span $\mathbb{R}^3$.

6. Suppose $\boldsymbol{A}$ is an $m \times n$ matrix with the property that for all $\boldsymbol{x} \in \mathbb{R}^n$, we have $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$. Prove that $\boldsymbol{A}$ is the zero matrix.

Proof. Let $\boldsymbol{A} = (a_{ij})_{m\times n}$. By assumption, for any $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$,

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0, \quad i = 1, \dots, m.$$

In particular, fix $i = 1, \dots, m$ and let $(x_1, x_2, \dots, x_n) = (a_{i1}, a_{i2}, \dots, a_{in})$, then

$$0 = a_{i1}a_{i1} + a_{i2}a_{i2} + \cdots + a_{in}a_{in} = a_{i1}^2 + a_{i2}^2 + \cdots + a_{in}^2,$$

which implies that $a_{i1} = a_{i2} = \cdots = a_{in} = 0$, i.e., the i^{th} row of $\boldsymbol{A}$ is $\boldsymbol{0}$. Since $i = 1, \dots, m$ is arbitrary, we conclude that $\boldsymbol{A} = \boldsymbol{0}$.

Alternative proof: By assumption, the system $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b} = \boldsymbol{0}$.

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ be the columns of $\boldsymbol{A}$. This means that $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\} = \{\boldsymbol{0}\}$. Consequently, $\boldsymbol{v}_1 = \boldsymbol{v}_2 = \cdots = \boldsymbol{v}_n = \boldsymbol{0}$, i.e., $\boldsymbol{A} = \boldsymbol{0}$.

7. Define $V = \{(t, 2t, 3t-1) \mid t \in \mathbb{R}\}$.

(i) Express V in implicit form, i.e., come up with a linear system whose solution set is V.

(ii) Is there a homogeneous linear system whose solution set is V? Justify your answer.

Solution. (i) Let $(x, y, z) = (t, 2t, 3t-1)$, i.e., $x = t$, $y = 2t$ and $z = 3t - 1$. Then

$$t = x = y/2 = (z+1)/3.$$

Then V is the solution set of the linear system

$$x = y/2 \quad \text{and} \quad x = (z+1)/3,$$

or

$$V = \{(x, y, z) \mid 2x - y = 0 \text{ and } 3x - z = 1\}.$$

(ii) Let $(x, y, z) = (0, 0, 0)$. Then $3x - z = 0 \neq 1$. So $(0, 0, 0) \notin V$. Consequently, V is not a subspace of $\mathbb{R}^3$, and thus it is not the solution set of any homogeneous linear system.

8. Consider the linear system $\boldsymbol{Ax} = \boldsymbol{0}$, where $\boldsymbol{A} = \begin{pmatrix} 2 & 3 & 1 & -1 & 2 \\ -2 & 0 & -2 & 1 & -1 \\ 2 & 2 & -2 & 2 & 0 \end{pmatrix}$.

(i) Prove that the solution set of the given homogeneous linear system is a subspace by expressing it as a span of certain vectors (that you have to specify).

(ii) Use the previous part to write down a general solution for the linear system $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{b} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$.

Solution. (i) Apply Gauss-Jordan elimination to $\boldsymbol{A}$:

$$\boldsymbol{A} \xrightarrow[R_3-R_1]{R_2+R_1} \begin{pmatrix} 2 & 3 & 1 & -1 & 2 \\ 0 & 3 & -1 & 0 & 1 \\ 0 & -1 & -3 & 3 & -2 \end{pmatrix} \xrightarrow{R_3+\frac{1}{3}R_2} \begin{pmatrix} 2 & 3 & 1 & -1 & 2 \\ 0 & 3 & -1 & 0 & 1 \\ 0 & 0 & -\frac{10}{3} & 3 & -\frac{5}{3} \end{pmatrix}$$

$$\xrightarrow[-\frac{3}{10}R_3]{\substack{\frac{1}{2}R_1 \\ \frac{1}{3}R_2}} \begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{9}{10} & \frac{1}{2} \end{pmatrix} \xrightarrow[R_2+\frac{1}{3}R_3]{R_1-\frac{1}{2}R_3} \begin{pmatrix} 1 & \frac{3}{2} & 0 & -\frac{1}{20} & \frac{3}{4} \\ 0 & 1 & 0 & -\frac{3}{10} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{9}{10} & \frac{1}{2} \end{pmatrix} \xrightarrow{R_1-\frac{3}{2}R_2} \begin{pmatrix} 1 & 0 & 0 & \frac{2}{5} & 0 \\ 0 & 1 & 0 & -\frac{3}{10} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{9}{10} & \frac{1}{2} \end{pmatrix}.$$

Let $x_4 = s$ and $x_5 = t$ be arbitrary parameters. Then

$$x_1 = -\frac{2}{5}s, \quad x_2 = \frac{3}{10}s - \frac{1}{2}t, \quad x_3 = \frac{9}{10}s - \frac{1}{2}t.$$

So

$$(x_1, x_2, x_3, x_4, x_5) = s\left(-\frac{2}{5}, \frac{3}{10}, \frac{9}{10}, 1, 0\right) + t\left(0, -\frac{1}{2}, -\frac{1}{2}, 0, 1\right).$$

Hence, the solution set of $\boldsymbol{A}$ is

$$\operatorname{span}\left\{(-\tfrac{2}{5}, \tfrac{3}{10}, \tfrac{9}{10}, 1, 0),\ (0, -\tfrac{1}{2}, -\tfrac{1}{2}, 0, 1)\right\}.$$

(ii) Note that the 3rd column of $\boldsymbol{A}$ is $\boldsymbol{b}$. So $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution $\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

So $\boldsymbol{Ax} = \boldsymbol{b}$ has general solution

$$\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -\frac{2}{5} \\ \frac{3}{10} \\ \frac{9}{10} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}, \quad s, t \in \mathbb{R}.$$