CS2040S – Data Structures and Algorithms

Lecture 18 – Finding Shortest Way
from Here to There, Part I

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Outline

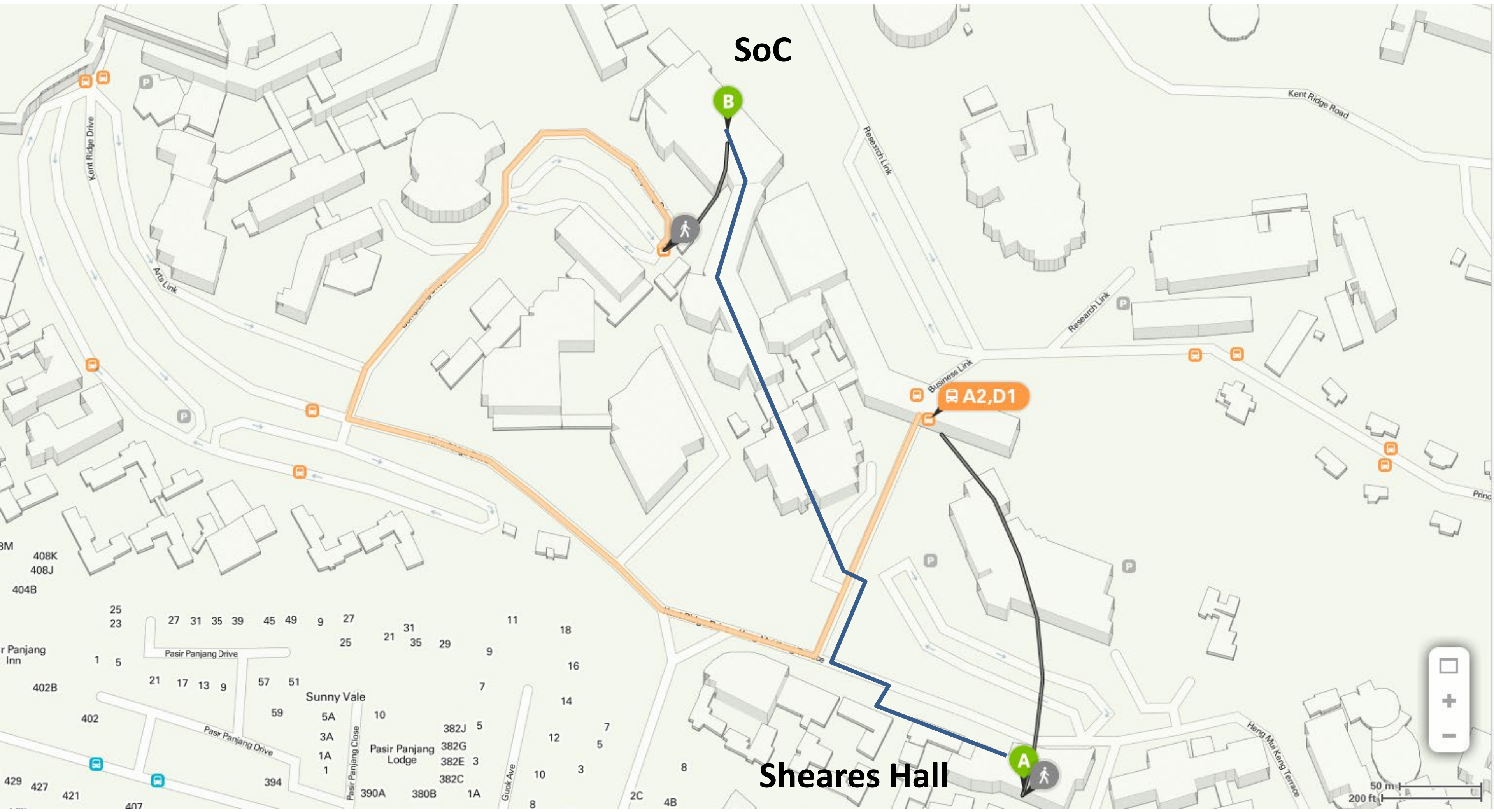
Single-Source Shortest Paths (SSSP) Problem

- Motivating example
- Some more definitions
- Discussion of negative weight edges and cycles

Algorithms to Solve SSSP Problem (CP4 Section 4.4)

- BFS algorithm (cannot be used for the general SSSP problem)
- Bellman Ford's algorithm
  - Precursor
  - Pseudo code, example animation, and later: Java implementation
  - Theorem, proof, and corollary about Bellman Ford's algorithm

Motivating Example



Review: Definitions that you know

- Vertex set **V** (e.g. street intersections, houses, etc)
- Edge set **E** (e.g. streets, roads, avenues, etc)
  - **Directed** (e.g. one way road, etc)
    - Note that we can use bi-directed edges for two way roads, etc.
  - **Weighted** (e.g. distance, time, toll, etc)
    - Weight function **w(a, b): E→R,** sets the weight of edge from **a** to **b**
- **Directed/Bi-directed Weighted** Graph: **G(V, E), w(a, b): E→R**

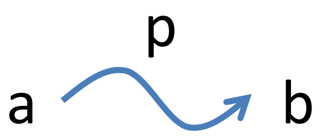
More Definitions (1)

- **Path** $p = \langle v_0, v_1, v_2, \ldots, v_k \rangle$
  - Where $(v_i, v_{i+1}) \in E, \forall_{0 \le i \le (k-1)}$
  - In SSSP, the path is usually a simple path (no repeated vertex), unless there is a negative cycle
- Shortcut notation: $v_0 \overset{p}{\leadsto} v_k$
  - Means that **p** is a path from v_0 to v_k
- **Path weight:** $PW(p) = \sum_{i=0}^{k-1} w(v_i, v_{i+1})$

More Definitions (2)

- **Shortest Path weight** from vertex **a** to **b**: **δ(a, b)**
  - δ is pronounced as 'delta'

$$\delta(a,b) = \begin{cases} \min(PW(p)) & \text{If there exists such path} \\ \infty & \text{If } \mathbf{b} \text{ is unreachable from } \mathbf{a} \end{cases}$$



- **<u>Single-Source</u> Shortest Paths** (SSSP) Problem:
  - Given **G(V, E)**, **w(a, b): E->R**, and a **<u>source vertex s</u>**
  - Find **δ(s, b)** from vertex **s** to each vertex **b**∈**V** (together with the corresponding shortest path)
    - i.e. From one source **to the rest**

given bidirected, directed weighted graph and source vertex

find shortest path weight from source to each vertex in the graph

More Definitions (3)

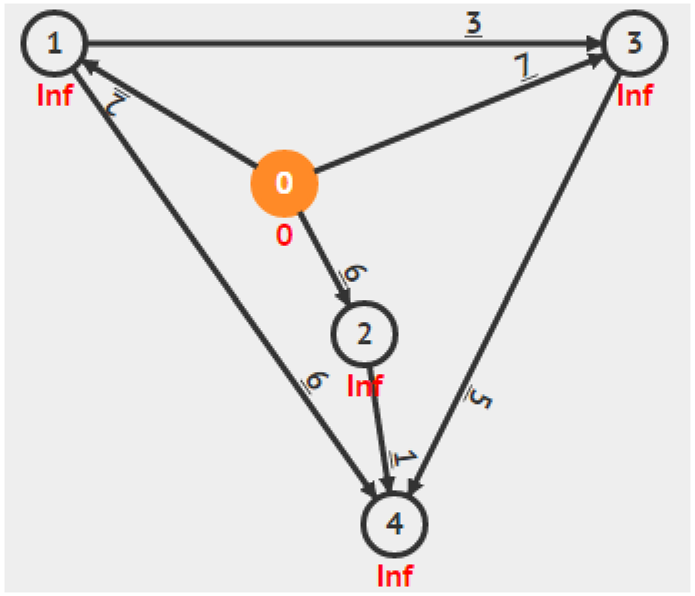
- **Additional Data Structures** to solve the SSSP Problem:
  - An array/Vector **D** of size **V** (**D** stands for 'distance')

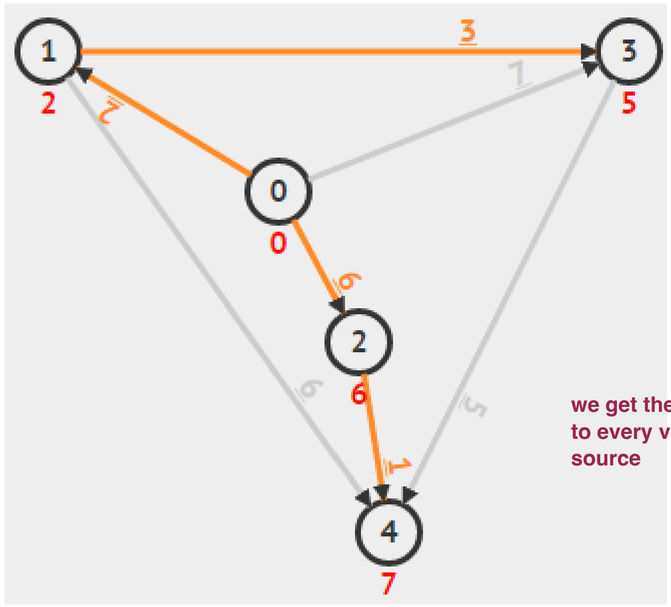
    D[v] stores shortest path cost of vertex v
    - Initially, **D[v] = 0** if **v** = **s**; otherwise **D[v] =** $\propto$ (a large number)
    - **D[v]** decreases as we find better paths

    shortest path estimate
    - **D[v] ≥ δ(s, v)** throughout the execution of SSSP algorithm
    - **D[v] = δ(s, v)** at the end of SSSP algorithm
  - An array/Vector **p** of size **V**
    - **p[v]** = the predecessor on best path from source **s** to **v**
    - **p[s]** = -1 (not defined)
    - Recall: The usage of this array/Vector **p** is already discussed in BFS/DFS Spanning Tree

Example

we get the SSSP





we get the SPST from source to every vertex reachable from source

s = 0
Initially:
D[s] = D[0] = 0
D[v]= ∝ for the rest
Denoted as values in **red font/vertex**
p[s] = -1 (to say ‘no predecessor’)
p[v] = -1 for the rest
Denoted as **orange edges (none initially)**

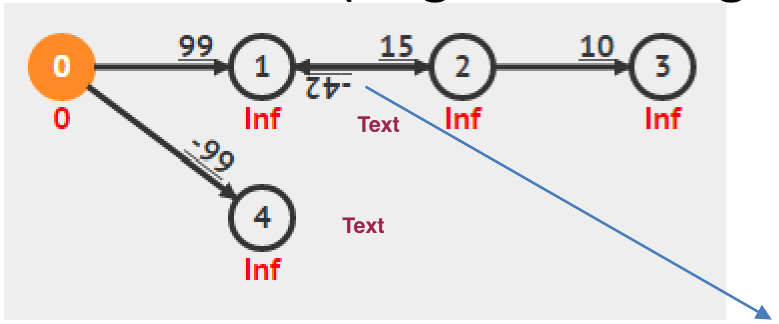
s = 0
At the end of algorithm:
D[s] = D[0] = 0 (unchanged)
D[v] = δ(s, v) for the rest
e.g. D[2] = 6, D[4] = 7
p[s] = -1 (source has no predecessor)
p[v] = the origin of **orange edges** for the rest
e.g. p[2] = 0, p[4] = 2

Negative Weight Edges and Cycles

negative weight cycle: cycle where if we add up all the weights in a cycle, the accumulative weight is negative

They exist in some applications

- Fictional application: Suppose you can travel back in time by passing through time tunnel (edges with negative weight)



Take this as a cycle

- Shortest paths from 0 to {1, 2, 3} are **undefined**

keeps cycling the path infinitely cause the cost becomes smaller and smaller

  – 1→2→1 is a negative cycle as it has negative total path (cycle) weight
  – One can take 0→$\underline{1→2→1→2→1}$→… indefinitely to get -∞
- Shortest path from 0 to 4 is ok, with δ(0, 4) = -99

SSSP Algorithms

This SSSP problem is a(nother) **well-known** CS problem

We will discuss three algorithms in this lecture:

1. O(**V+E**) BFS which fails on *general case* of SSSP problem but useful for a special case
   - Introducing the "initSSSP" and "Relax" operations
2. General SSSP algorithm (pre-cursor to Bellman Ford)
3. O(**VE**) Bellman Ford's SSSP algorithm
   - General idea of SSSP algorithm
   - Trick to ensure termination of the algorithm
   - Bonus: Detecting negative weight cycle

Initialization Step

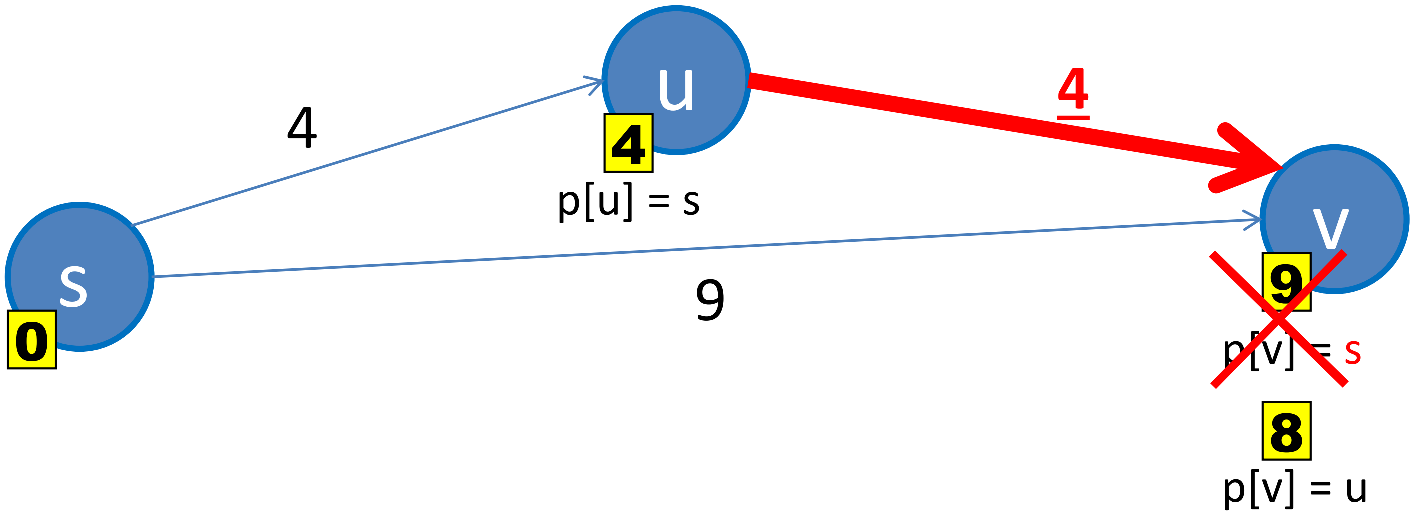
We will use this initialization step
for all our SSSP algorithms

```
initSSSP(s)
  for each v ∈ V // initialization phase
    set all paths in dist array to infinity
    D[v] ← 1000000000 // use 1B to represent INF
    p[v] ← -1 // use -1 to represent NULL
  D[s] ← 0 // this is what we know so far

  this is source -> 0 cost
```

“Relaxation” Operation

given edge u -> v and the cost of the edge w(u,v)

```
relax(u, v, w(u,v))
  if D[v] > D[u]+w(u,v) // if SP can be shortened
    D[v] ← D[u]+w(u,v) // relax this edge
    p[v] ← u // remember/update the predecessor
    // if necessary, update some data structure
```



BFS for SSSP

When the graph is **unweighted/edges have same weight***, the SSSP can be viewed as a problem of finding the **least number of edges** traversed from source **s** to other vertices

* We can view every edge as having weight 1

The O(**V**+**E**) Breadth First Search (BFS) traversal algorithm precisely measures this (BFS Spanning Tree = Shortest Paths Spanning Tree)

Modified BFS

Do these three simple modifications:

1. Replace **`visited`** with **D** ☺ replace visited array with Dist array
2. At the start of BFS, set **`D[v] = INF`** (say, 1 Billion) for all **v** in **G**, except **`D[s] = 0`** ☺
3. Change this part (in the BFS loop) from:

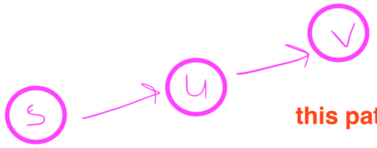
```
if visited[v] = 0 // if v is not visited before
  visited[v] = 1; // set v as reachable from u
```

into:

```
if D[v] = INF // if v is not visited before
  D[v] = D[u]+1; // v is 1 step away from u ☺
```

we alr have shortest path s to u + 1 more edge to get to v

+ 1 because each edge weight is 1



this path + 1 more edge

Modified BFS Pseudo Code (1)

```
for all v in V
  D[v] ← INF
  p[v] ← -1
Q ← {s} // start from s
D[s] ← 0

while Q is not empty
  u ← Q.dequeue()
  for all v adjacent to u // order of neighbor
    if D[v] = INF //  influences BFS
      D[v] ← D[u]+1 // visitation sequence
      p[v] ← u
      Q.enqueue(v)

// we can then use information stored in D/p
```

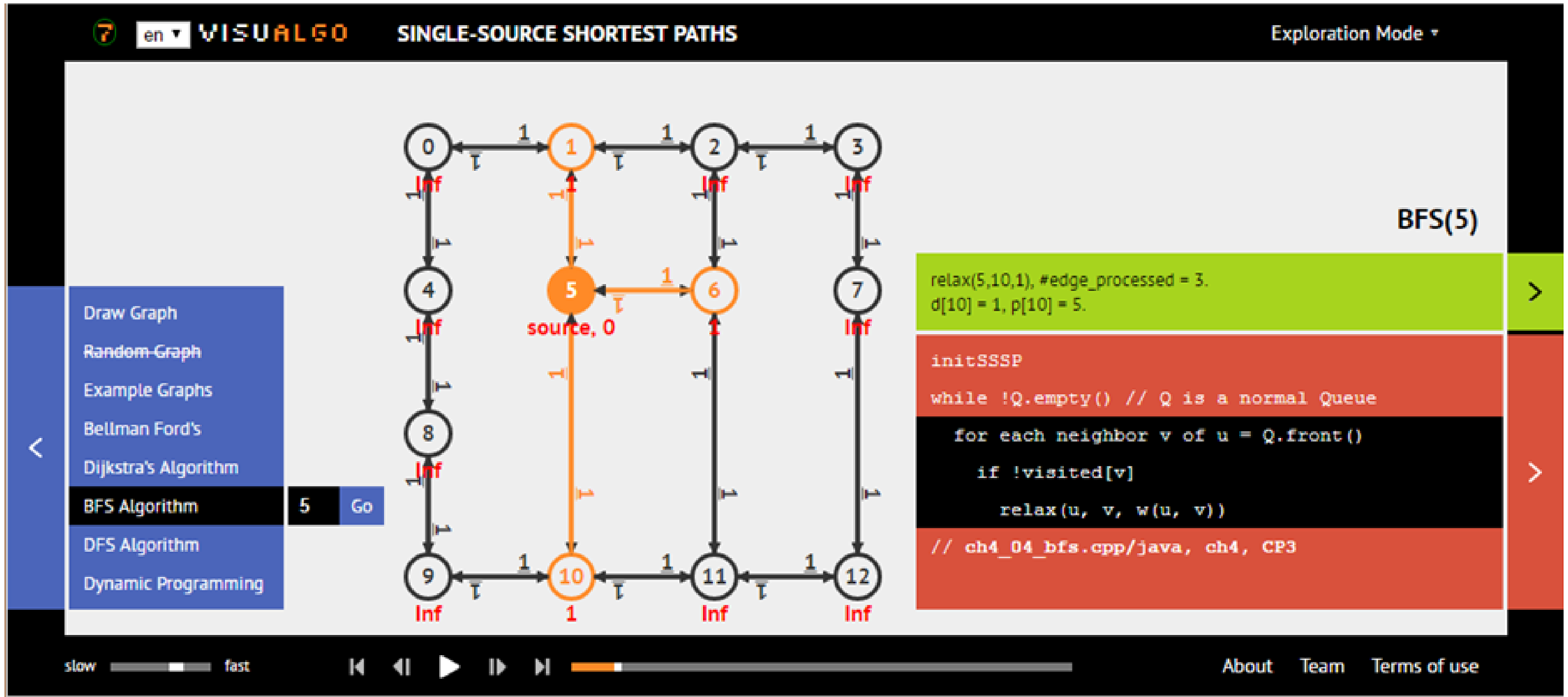
Initialization phase

Main loop

SSSP: BFS on Unweighted Graph

Ask VisuAlgo to perform BFS *from various sources* on the sample Graph (CP4 4.2)

In the screen shot below, we show the start of BFS from source vertex 5 (the same example as in Lecture 16)



relax(5,10,1), #edge_processed = 3.
d[10] = 1, p[10] = 5.

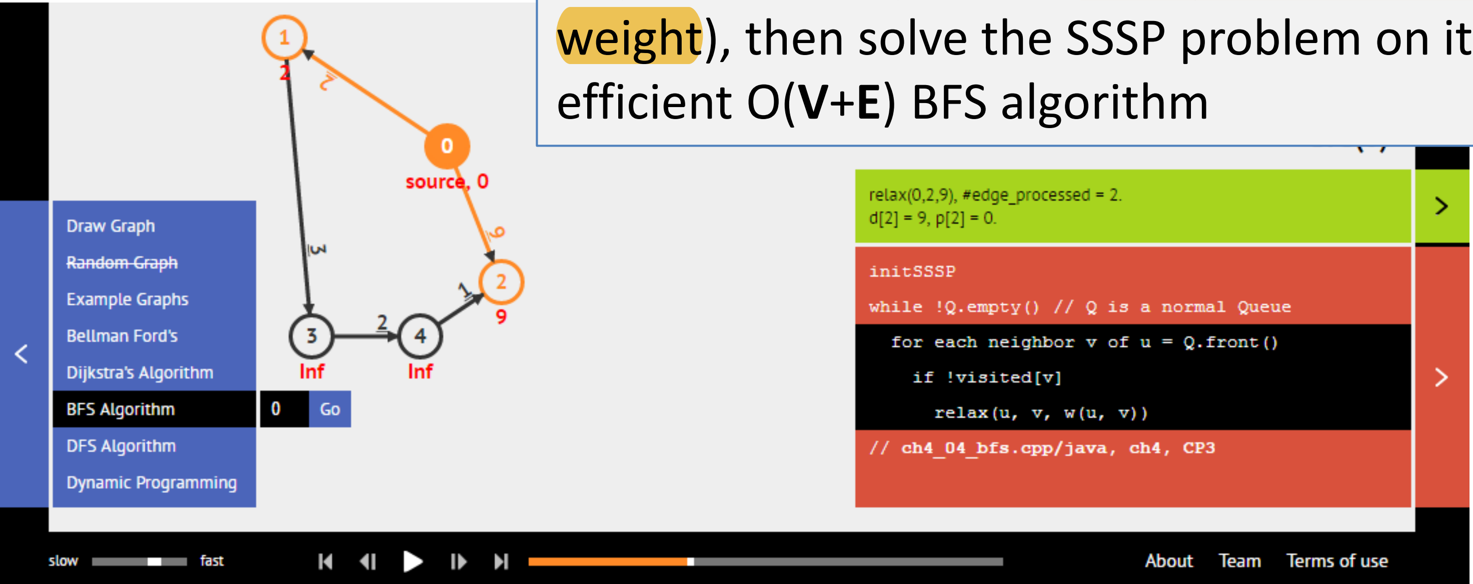
```
initSSSP
while !Q.empty() // Q is a normal Queue
  for each neighbor v of u = Q.front()
    if !visited[v]
      relax(u, v, w(u, v))
// ch4_04_bfs.cpp/java, ch4, CP3
```

But BFS will not work on general cases

The shortest path from 0 to 2 is not path 0→2 with weight 9, but a "detour" path 0→1→3→4→2 with weight 2+3+2+1= 8

- BFS cannot detect this and will only report path 0→2 (wrong answer)
- You can draw this graph @ VisuAlgo and try it for yourself

Rule of Thumb:
If you know for sure that your graph is unweighted (all edges have weight 1 or all edges have the same constant weight), then solve the SSSP problem on it using the more efficient O(**V**+**E**) BFS algorithm



relax(0,2,9), #edge_processed = 2.
d[2] = 9, p[2] = 0.

```
initSSSP
while !Q.empty() // Q is a normal Queue
  for each neighbor v of u = Q.front()
    if !visited[v]
      relax(u, v, w(u, v))
// ch4_04_bfs.cpp/java, ch4, CP3
```

Reference: CP4 Section 4.4

visualgo.net/sssp

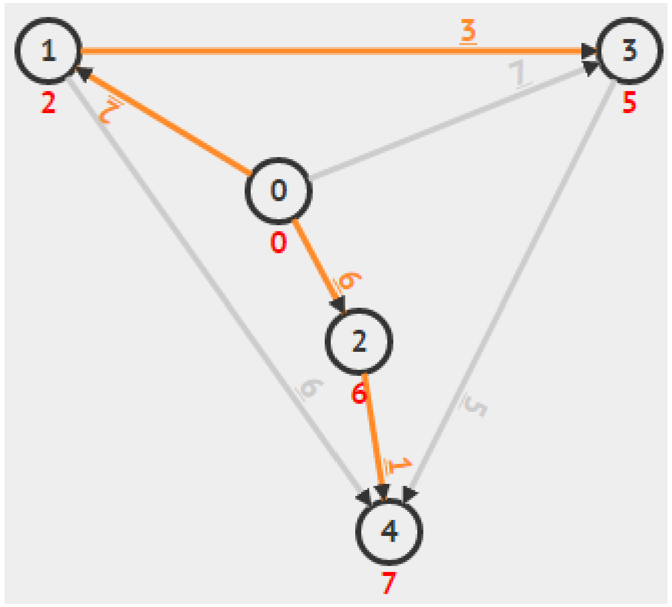
BELLMAN FORD'S SSSP ALGORITHM

Precursor to Bellman Ford

How do we determine when an algorithm has solved the SSSP?

- when for all edges (u,v), D[v] <= D[u] + w(u,v) (i.e no edge can be relaxed further)

Validate this condition on the example in slide 8



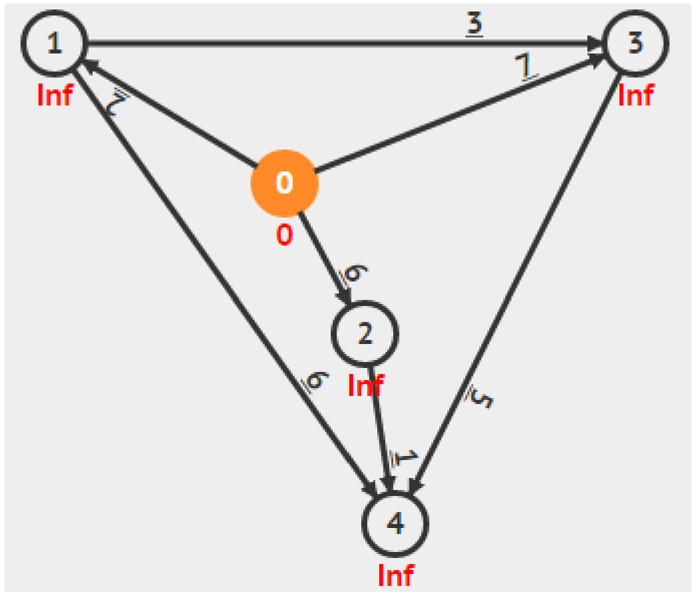
Very simple algorithm to solve SSSP

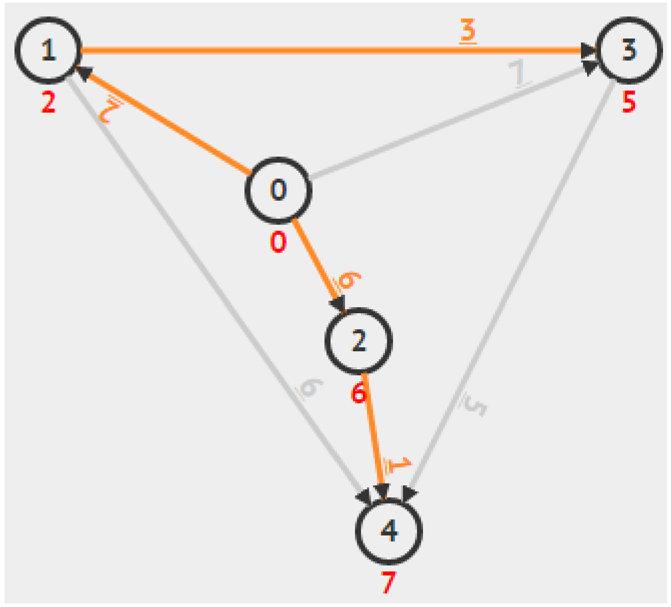
```
initSSSP(s) // as defined in previous two slides

repeat // main loop
  select edge(u, v) ∈ E in arbitrary manner
  relax(u, v, w(u, v)) // as defined in previous slide
until all edges have D[v] <= D[u] + w(u, v)
```

keep looping until we cannot relax the edges anymore

Let’s Play a Simple Game

(Demo on Whiteboard – cannot be done on VisuAlgo)





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Algorithm Analysis

If given a graph without negative weight cycle, when will this simple SSSP algorithm terminate?

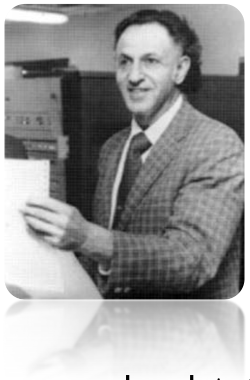
A: Depends on your luck…

A: Can be very slow…

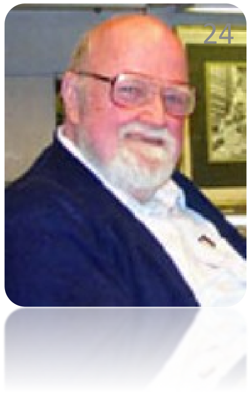
The main problem is in this line:

```
select edge(u, v) ∈ E in arbitrary manner
```

Next, we will study **Bellman Ford's** algorithm that do these relaxations in a *better order*!



Bellman Ford's Algorithm



```
initSSSP(s)

// Simple Bellman Ford's algorithm runs in O(VE)
for i = 1 to |V|-1 // O(V) here
  for each edge(u, v) ∈ E // O(E) here
    relax(u, v, w(u,v)) // O(1) here

// At the end of Bellman Ford's algorithm,
// D[v] = δ(s, v) if no negative weight cycle exist

// Q: Why "relaxing all edges V-1 times" works?
```

V - 1 (each edge in edge set) passes of relaxation of edges

inner for loop goes through each edge in edge set

if using edge list: O(VE)

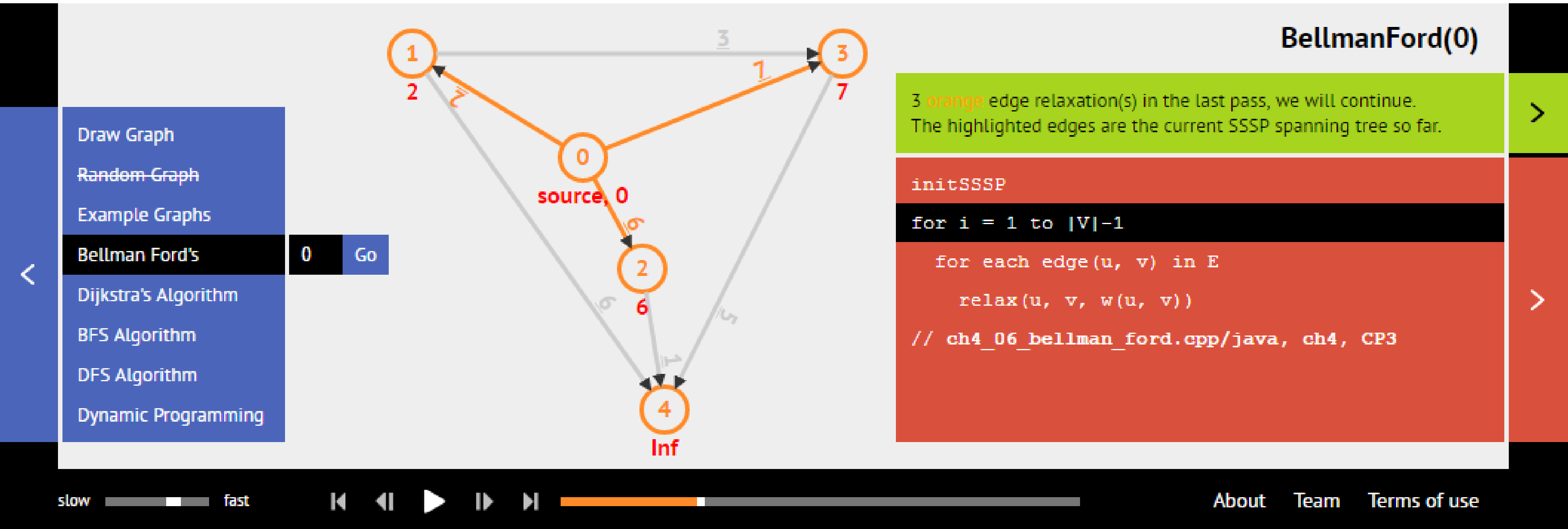
could be O(V^2) smaller connected graph or even O(V^3) for dense graph

for each iteration of outer for loop, we go through each edge and try to relax it once, the order of which we relax the edges doesnt matter

SSSP: Bellman Ford's

Ask VisuAlgo to perform Bellman Ford's algorithm *from various sources* on the sample Graph (CP4 4.14)

The screen shot below is *the first pass* of all **E** edges of **BellmanFord(0)**

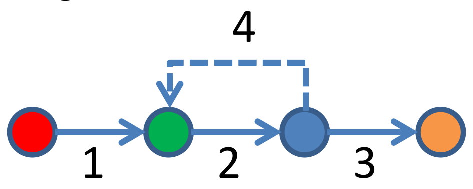


BellmanFord(0)

3 orange edge relaxation(s) in the last pass, we will continue.
The highlighted edges are the current SSSP spanning tree so far.

```
initSSSP
for i = 1 to |V|-1
  for each edge(u, v) in E
    relax(u, v, w(u, v))
// ch4_06_bellman_ford.cpp/java, ch4, CP3
```

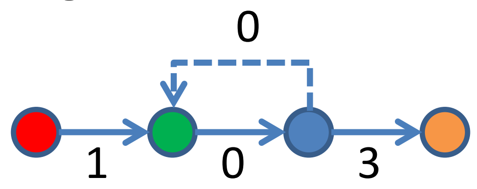
Theorem 1 : If **G = (V, E)** contains no negative weight cycle, then the shortest path **p** from **s** to **v** is a **simple path**



Let's do a **Proof by Contradiction!**

1. Suppose the shortest path **p** is not a simple path
2. Then **p** contains one (or more) cycle(s)
3. Suppose there is a cycle **c** in **p** with positive weight
4. If we remove **c** from **p**,
   then we have a shorter 'shortest path' than **p**
5. This contradicts the fact that **p** is a shortest path

Theorem 1 : If **G = (V, E)** contains no negative weight cycle, then the shortest path **p** from **s** to **v** is a **simple path**



6. Even if **c** is a cycle with zero total weight (it is possible!), we can still remove **c** from **p** without increasing the shortest path weight of **p**
7. So, **p** is a simple path (from point 5) or can always be made into a simple path (from point 6)

because we cannot have any repeated edges (if not there would be a cycle) so the longest shortest path in terms of number of edges is V -1

In other words, path **p** has at most **|V|-1** edges from the source **s** to the "furthest possible" vertex **v** in **G** (in terms of number of edges in the shortest path)

Theorem 2 : If **G = (V, E)** contains no negative weight cycle, then after Bellman Ford's terminates $\mathbf{D[v] = \delta(s, v), \forall v \in V}$

shortest path cost for all vertices v in V

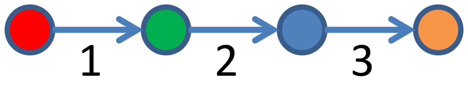
Let's do a **Proof by Induction**!

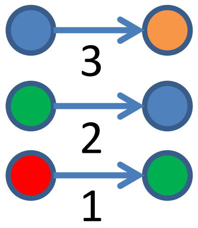
1. Define $\mathbf{v_i}$ to be any vertex that has shortest path **p** requiring *i* number of edges from s
2. Initially $\mathbf{D[v_0] = \delta(s, v_0) = 0}$, as $\mathbf{v_0}$ is just **s**
3. After **1** pass through **E**, we have $\mathbf{D[v_1] = \delta(s, v_1)}$
4. After **2** passes through **E**, we have $\mathbf{D[v_2] = \delta(s, v_2)}$, ...
5. After **k** passes through **E**, we have $\mathbf{D[v_k] = \delta(s, v_k)}$

Theorem 2 : If **G = (V, E)** contains no negative weight cycle, then after Bellman Ford's terminates **D[v] = δ(s, v), ∀v ∈ V**

6. When there is no negative weight cycle, the shortest path **p** will be simple (see the previous proof)
7. Thus, after **|V|-1** iterations, the "furthest" vertex $\mathbf{v_{|V|-1}}$ from **s** has $\mathbf{D[v_{|V|-1}] = \delta(s, v_{|V|-1})}$
   - Even if edges in **E** are processed in the *worst possible order*

at worst case, we have to do v-1 relaxations because of worst ordering





"Side Effect" of Bellman Ford's

Corollary: If a value **D[v]** *fails to converge* after **|V|-1** passes, then there exists a negative-weight cycle reachable from **s**

Additional check after running Bellman Ford's:

```
for each edge(u, v) ∈ E
  if (D[u] != INF && D[v] > D[u]+w(u, v))
    report negative weight cycle exists in G
```

if after running relaxation loop V - 1 times, just report negative weight cycle by doing extra check if we can still relax the edges, if can still relax(distance reduces) it means there is -ve weight cycle

Java Implementation

See BellmanFordDemo.java

- Implemented using **AdjacencyList** ☺
  - **AdjacencyList** or **EdgeList** can be used to have an O(**VE**) Bellman Ford's

Show performance on:

- Small graph without negative weight cycle → OK, in O(**VE**)
- Small graph with negative weight cycle → terminate in O(**VE**)
  - Plus we can report that negative weight cycle exists
- Small graph; some negative edges; no negative cycle → OK

Summary

Introducing the SSSP problem

Revisiting BFS algorithm for unweighted SSSP problem

- But it fails on general case

Introducing Bellman Ford's algorithm

V - 1 relaxations on all the edges -> edge set relaxed v-1 times so time complexity is O(VE)

- This one solves SSSP for general weighted graph in O(**VE**)
- Can also be used to detect the presence of -ve weight cycle