CS1231S: Discrete Structures

Tutorial #8: Cardinality

(Week 10: 17 – 21 October 2022)

I. Discussion Questions

These are meant for you to discuss on Canvas. No answers will be provided.

D1 Is the set of perfect squares $\{0,1,4,9,16,\ldots\}$ countable? Prove or disprove it.

D2. Aiken spoke about a set being "uncountable and infinite". Dueet commented that Aiken must have meant "uncountably infinite." Comment on what Aiken and Dueet said.

D3. [AY2021/22 Semester 2 Exam Multiple-Response Question].
Which of the following sets are countable?

A. The set A of all points in the plane with rational coordinates.

B. The set B of all infinite sequences of integers.

C. The set C of all functions $f: \{0,1\} \to \mathbb{N}$.

D. The set D of all functions $f: \mathbb{N} \to \{0,1\}$.

E. The set E of all 2-element subsets of $\mathbb{N}$.

II. Tutorial Questions

1. In lecture example #3, we showed that $\mathbb{Z}$ is countable by defining a bijection $f: \mathbb{Z}^+ \to \mathbb{Z}$ as follows:

$$f(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even;} \\ -\dfrac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

The above is based on the definition $\aleph_0 = |\mathbb{Z}^+|$. Suppose we adopt the definition $\aleph_0 = |\mathbb{N}|$ instead, define a bijection $g: \mathbb{N} \to \mathbb{Z}$ using a single-line formula to show that $\mathbb{Z}$ is countable.

2. Let B be a countably infinite set and C a finite set. Show that $B \cup C$ is countable

(a) by using the sequence argument;

(b) by defining a bijection $g: \mathbb{N} \to B \cup C$.

3. Recall the definition of $\bigcup_{i=m}^{n} A_i$ in Tutorial 3.

   (a) Consider this claim:

   "Suppose $A_1, A_2, \cdots$ are finite sets. Then $\bigcup_{i=1}^{n} A_i$ is finite for any $n \geq 2$."

   The above statement is true. However, consider the following "proof":

   "We will prove by induction on n. Since A_1 and A_2 are finite, then $A_1 \cup A_2$ is finite, so the claim is true for $n = 2$. Now suppose the claim is true for $n = k$, so $\bigcup_{i=1}^{k} A_i$ is finite. Let $A_{k+1} = \emptyset$. Then $\bigcup_{i=1}^{k+1} A_i = (\bigcup_{i=1}^{k} A_i) \cup A_{k+1} = \bigcup_{i=1}^{k} A_i$ which is finite by the induction hypothesis, so the claim is true for $n = k + 1$. Therefore, the claim is true for all $n \geq 2$."

   What is wrong with this "proof"?

   (b) Prove the following is false: "Suppose $A_1, A_2, \cdots$ are finite sets. Then $\bigcup_{k=1}^{\infty} A_k$ is finite."
   [The point here is: induction takes you to any finite n, but not to infinity.]

4. Suppose $A_1, A_2, A_3, \cdots$ are countable sets.

   (a) Prove, by induction, that $\bigcup_{i=1}^{n} A_i$ is countable for any $n \in \mathbb{Z}^+$.

   (b) Does (a) prove that $\bigcup_{i=1}^{\infty} A_i$ is countable?

5. Let S_i be a countably infinite set for each $i \in \mathbb{Z}^+$. Prove that $\bigcup_{i \in \mathbb{Z}^+} S_i$ is countable.
   [Hint: Use this theorem covered in class: $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.]

6. Let B be a (not necessarily countable) infinite set and C be a finite set.
   Define a bijection $B \cup C \to B$.

7. Prove that a set B is infinite if and only if there is $A \subsetneq B$ such that $|A| = |B|$.

8. Prove that $\mathbb{C}$ (the set of complex numbers) is uncountable.

9. Let A be a countably infinite set. Prove that $\mathcal{P}(A)$ is uncountable.
   (Note: $\mathcal{P}(A)$ is the power set of A.)