Relations

Relation: Let A and B be Sets. A (binary) relation R from A to B is a Subset of $A \times B$

Given a ordered Pair (x, y) in $A \times B$, x is related to y by R or x is R-related to y, written xRy, iff $(x,y) \in R$

$x R y$ means $(x,y) \in R$
$x \not{R} y$ means $(x,y) \notin R$

Domain of R: $Dom(R)$ is the set $\{a \in A : aRb$ for some $b \in B\}$

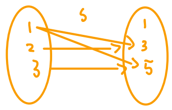
Codomain of R: $coDom(R)$ is the set B

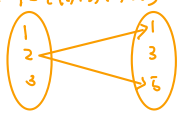
Range of R: $Range(R)$ is the set $\{b \in B : aRb$ for some $a \in A\}$

$A = \{1,2,3\}$, $B = \{2,4,9\}$ $Dom(R) = \{2,3\}$, $CoDom(R) = \{2,4,9\}$
$Range(R) = \{4,9\}$

Arrow Diagram: $A = \{1,2,3\}$ $B = \{1,3,5\}$

$(x,y) \in S \iff x < y$, $T = \{(2,1),(2,5)\}$





Inverse of a Relation

Let R be a relation from A to B.
R^{-1} from B to A is $R^{-1} = \{(y,x) \in B \times A : (x,y) \in R\}$

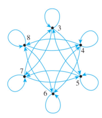
Let $A = \{2,3,4\}$ and $B = \{2,6,8\}$ and let R be the 'divides' relation from A to B $\forall (x,y) \in A \times B \ (xRy \iff x|y)$

$R^{-1}: \forall (y,x) \in B \times A \ (yR^{-1}x \iff y = kx)$

$R^{-1} = \{(2,2), (6,2), (8,2), (6,3), (8,4)\}$

Relation on a set A: Relation from A to A, Subset of $A \times A$
A^2, A^n

Directive graph: Represent A only once and draw an arrow from each Point of A to its related Point.
$A = \{3,4,5,6,7,8\}$ $xRy \iff 2|(x-y)$



Composition of Relations: Set A, B, C, Let $R \subseteq A \times B$
$S \subseteq B \times C$

Composition of R with S: $S \circ R$ is the relation from A to C such that:
$\forall x \in A, \forall z \in C \ (x\, S \circ R\, y \iff (\exists y \in B \ (xRy \wedge yRz)))$
$x \in A$ and $z \in C$ are $S \circ R$ related iff there is a path from x to z via some intermediate element $y \in B$ in the arrow diagram

Associative: $T \circ (S \circ R) = (T \circ S) \circ R = T \circ S \circ R$

Inverse of Composition: $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

N-ary Relations: Given n sets $A_1, A_2 \cdots, A_n$, n-ary relation R on $A_1 \times A_2 \cdots A_n$ is a subset of $A_1 \times A_2 \cdots A_n$
2-ary, 3-ary, 4-ary relation is called binary, ternary, quaternary relations

Let R be a relation on set A

R is **reflexive** iff $\forall x \in A \ (xRx)$
R is **symmetric** iff $\forall x, y \in A \ (xRy \Rightarrow yRx)$
R is **transitive** iff $\forall x, y, z \in A \ (xRy \wedge yRz \Rightarrow xRz)$

Transitive Closure of a Relation: The relation obtained by adding the least number of ordered pairs to ensure transitivity

Let A be a set and a relation on A. The transitive Closure of R is the relation R^t on A that satisfies:

1. R^t is transitive
2. $R \subseteq R^t$
3. If S is any other transitive relation that contains R, then $R^t \subseteq S$

Partition: $\mathcal{C}$ is a Partition of set A if the following hold:

(1) $\mathcal{C}$ is a set of which all elements are non-empty subsets of A
$\emptyset \neq S \subseteq A$ for all $S \in \mathcal{C}$

(2) Every element of A is in exactly one element of $\mathcal{C}$
$\forall x \in A \ \exists S \in \mathcal{C} \ (x \in S)$ and $\forall x \in A \ \exists S_1, S_2 \in \mathcal{C}$

Elements of a Partition are called **Component** of the Partition

$\forall x \in A \ \exists! S \in \mathcal{C} \ (x \in S)$

Relation Induced by a Partition

Given a Partition $\mathcal{C}$ of a set A, the relation R induced by the Partition is defined on A as follows: $\forall x, y \in A$,
$xRy \iff \exists$ a component S of $\mathcal{C}$ s.t $x, y \in S$

Let $A = \{0,1,2,3,4\}$
Partition of A: $\{\{0,3,4\},\{1\},\{2\}\}$.
$0R0, 0R3, 0R4, 3R0 \ldots$

Theorem 8.3.1: Let A be a set with a partition and let R be the relation by the partition
Then R is reflexive, symmetric and transitive

Equivalence Relation: Let A be a set and R a relation on A. R is an equivalence relation iff R is reflexive, symmetric and transitive

Equivalence Classes: For each $a \in A$, $[a]_\sim = \{x \in A : a \sim x\}$
$[a]_\sim$ ↳ equivalence class of a
$\sim$ ↳ equivalence relation on A

Lemma Rel.1 Equivalence classes: Let $\sim$ be an equivalence relation on a set A.
The following are equivalent for all $x, y \in A$

i) $x \sim y$ ii) $[x] = [y]$ iii) $[x] \cap [y] \neq \emptyset$

Theorem 8.3.4 The Partition: If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form a Partition of A; that is, the Union of the equivalence classes is all of A, and the intersection of any two distinct classes is empty.

$A/\sim = \{[x]_\sim : x \in A\}$

Divisibility: Let $n, d \in \mathbb{Z}$. Then $d|n \iff n = dk$ for some $k \in \mathbb{Z}$

Congruence: Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. Then a is congruent to be modulo n iff $a - b = nk$ for some $k \in \mathbb{Z}$. In other words, $n | (a-b)$.
We write $a \equiv b \pmod n$

Congruence-mod n is an equivalence relation on $\mathbb{Z}$ for every $n \in \mathbb{Z}^+$. # of $[x]$ is n

Set of equivalence classes: Let A be a set and $\sim$ be an equivalence relation on A. Denote by $A/\sim$ the set of all equivalence classes with respect to $\sim$, i.e.,
$A/\sim = \{[x]_\sim : x \in A\}$.
↳ Read as "the quotient of A by $\sim$"

Theorem Rel.2 Equivalence classes from a partition: Let $\sim$ be an equivalence relation on a set A. Then $A/\sim$ is a Partition of A.

Summary

Definition: A **relation** on set A is a subset of A^2.

Definition: If R is a **relation** on a set A, then we write $x\,R\,y$ for $(x,y) \in R$.

Definition: A **partition** of a set A is a set $\mathcal{C}$ of non-empty subsets of A such that
$\forall x \in A\ \exists! S \in \mathcal{C}\ (x \in S)$.

Definition: A relation R on A is an **equivalence relation** if
- (reflexivity) $\forall x \in A\ (x\,R\,x)$;
- (symmetry) $\forall x, y \in A\ (x\,R\,y \Rightarrow y\,R\,x)$; and
- (transitivity) $\forall x, y, z \in A\ (x\,R\,y \wedge yRz \Rightarrow y\,R\,z)$.

Definition: Let $\sim$ be an equivalence relation on A. Then the set of equivalence classes is denoted by $A/\sim\, = \{[x]_\sim : x \in A\}$, where $[x]_\sim = \{y \in A : x \sim y\}$.

Proposition: The same-component relation w.r.t. a partition is an equivalence relation.

Theorem Rel.2: If $\sim$ is an equivalence relation on A, then $A/\sim$ is a partition of A.

Partial Order Relations

Antisymmetry: $\forall x,y \in A\ (xRy \wedge yRx \Rightarrow x=y)$
$\exists x,y \in A\ (xRy \wedge yRx \wedge x \neq y)$

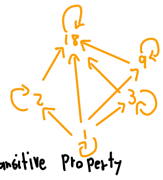
Partial Order: Let R be a relation on set A. Then R is a partial order relation iff R is reflexive, antisymmetric and transitive

Partially Ordered Set: A set A is called partially ordered set with respect to a partial order relation R on A, denoted by (A,R)

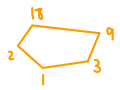
Theorem 4.3.3 Transitivity: For all integers a,b and c, if $a|b$ and $b|c$, then $a|c$

The symbol $\preccurlyeq$ is often used to refer to a general partial order, and the notation $x \preccurlyeq y$ is read "x is curly less than or equal to y".

Directed graph: $a|b \Leftrightarrow b = ka$ for some integer k



Hasse Diagrams: From a directed graph, eliminate
1. The loops at all the vertices
2. All arrows whose existence is implied by transitive property
3. The direction indicators on the arrows



Comparability: Suppose $\preccurlyeq$ is a partial order relation on a set. Elements a and b are comparable iff either $a \preccurlyeq b$ or $b \preccurlyeq a$.

Maximal: iff $\forall x \in A\ (c \preccurlyeq x \Rightarrow c = x)$

Minimal: iff $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$

Largest: iff $\forall x \in A\ (x \preccurlyeq c)$

Smallest: iff $\forall x \in A\ (c \preccurlyeq x)$

Total Order Relations: $\forall x,y \in A\ (xRy \vee yRx)$

Linearization of a partial order: Let $\preccurlyeq$ be a partial order on a set A. A linearization of $\preccurlyeq$ is a total order $\preccurlyeq^*$ on A such that

$$\forall x,y \in A\ (x \preccurlyeq y \Rightarrow x \preccurlyeq^* y)$$

Well-Ordered Set: Let $\preccurlyeq$ be a total order on a set A. A is well ordered iff every non-empty subset of A contains a smallest element
$\forall S \in P(A), S \neq \emptyset \Rightarrow (\exists x \in S\ \forall y \in S\ (x \preccurlyeq y))$

Asymmetric relation: $\forall x,y \in A\ (xRy \Rightarrow x \not R y)$