Solutions to Tutorial 8

MA1521 Calculus for Computing

1. $V = I \times R \implies I = \frac{V}{R}$.

   (i) $\dfrac{\partial I}{\partial V} = \dfrac{1}{R}$.

   If $R = 15$, then $\dfrac{\partial I}{\partial V} = \dfrac{1}{15} \approx 0.0667$ A/V.

   (ii) $\dfrac{\partial I}{\partial R} = -\dfrac{V}{R^2}$. If $V = 120$ and $R = 20$, then $\dfrac{\partial I}{\partial R} = -\dfrac{120}{20^2} = -0.3$ A/Ω.

   (iii) By Chain rule applied to $I = \frac{V}{R}$,

   $$\frac{dI}{dt} = \frac{\partial I}{\partial V}\frac{dV}{dt} + \frac{\partial I}{\partial R}\frac{dR}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{V}{R^2}\frac{dR}{dt}.$$

   Since $R = 400$, $I = 0.08$, $V = 32$, $\dfrac{dV}{dt} = -0.01$, $\dfrac{dR}{dt} = 0.03$, so

   $$\frac{dI}{dt} = \frac{1}{400}(-0.01) - \frac{(0.08)(400)}{400^2}(0.03) = -3.1 \times 10^{-5}.$$

2. $f_x = e^{2y-x} + xe^{2y-x}(-1) = e^{2y-x}(1-x)$ and $f_y = 2xe^{2y-x}$.

   So $f_x(-2,-1) = 3$ and $f_y(-2,-1) = -4$.

   (i) $\mathbf{u} = \dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$ is a unit vector. Thus,

   $$D_{\mathbf{u}} f(-2,-1) = 3 \times \frac{1}{\sqrt{2}} - 4 \times \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

   (ii) The unit vector in the direction of $3\mathbf{i} + 4\mathbf{j}$ is $\mathbf{u} = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j}$. Thus,

   $$D_{\mathbf{u}} f(-2,-1) = 3 \times \frac{3}{5} - 4 \times \frac{4}{5} = -\frac{7}{5}.$$

   Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ be a unit vector. (That is $\|\mathbf{u}\| = 1$.) Then

   $$\begin{aligned} D_{\mathbf{u}} f(-2,-1) &= f_x(-2,-1) \times a + f_y(-2,-1) \times b \\ &= (f_x(-2,-1)\mathbf{i} + f_y(-2,-1)\mathbf{j}) \cdot (a\mathbf{i} + b\mathbf{j}) \\ &= \|f_x(-2,-1)\mathbf{i} + f_y(-2,-1)\mathbf{j}\| \, \|\mathbf{u}\| \cos\theta \end{aligned}$$

   where θ is the angle between $f_x(-2,-1)\mathbf{i} + f_y(-2,-1)\mathbf{j}$ and $\mathbf{u}$.

   Since the largest value of $\cos\theta$ is 1 which occurs when $\theta = 0$, this means that the largest possible value of $D_{\mathbf{u}} f(-2,-1)$ occurs when $\mathbf{u}$ is in the same direction as $\nabla f(-2,-1) = f_x(-2,-1)\mathbf{i} + f_y(-2,-1)\mathbf{j} = 3\mathbf{i} - 4\mathbf{j}$.

3. $f_x = yz\cos(xyz)$, $\quad f_y = xz\cos(xyz)$ $\quad$ and $\quad f_z = xy\cos(xyz)$.

So $f_x(\frac{1}{2}, \frac{1}{3}, \pi) = \frac{\sqrt{3}}{6}\pi$, $\quad f_y(\frac{1}{2}, \frac{1}{3}, \pi) = \frac{\sqrt{3}}{4}\pi$ $\quad$ and $\quad f_z(\frac{1}{2}, \frac{1}{3}, \pi) = \frac{\sqrt{3}}{12}$.

(i) Let $\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$.

Thus, the rate of change of f at P in the direction $\mathbf{u}$ is given by

$$D_{\mathbf{u}}f\left(\frac{1}{2}, \frac{1}{3}, \pi\right) = \frac{\sqrt{3}}{6}\pi \times \frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{4}\pi \times \left(-\frac{1}{\sqrt{3}}\right) + \frac{\sqrt{3}}{12} \times \frac{1}{\sqrt{3}} = \frac{1}{12}(1-\pi).$$

(ii) The change in the value of f:

$$\triangle f \approx D_{\mathbf{u}}f\left(\frac{1}{2}, \frac{1}{3}, \pi\right) \times \triangle t = \frac{1}{12}(1-\pi) \times 0.1 \approx -0.01785.$$

So the value of f will *decrease* by about 0.01785 unit.

4. (i) Let $f(x,y) = \ln(x^2y) - xy - 2x = 2\ln x + \ln y - xy - 2x$, where $x > 0$, $y > 0$.

We have $f_x = \frac{2}{x} - y - 2$ and $f_y = \frac{1}{y} - x$. Also, $f_{xx} = -\frac{2}{x^2}$, $f_{xy} = -1$ and $f_{yy} = -\frac{1}{y^2}$. Then $f_y = 0$ implies that $x = \frac{1}{y}$, and substitution into $f_x = 0$ gives $2y - y - 2 = 0$, i.e. $y = 2$. So $x = 1/2$. Thus, the only critical point is $(1/2, 2)$.

Now $D(1/2,2) = (-8)(-1/4) - 1^2 = 1 > 0$, $\; f_{xx}(1/2,2) = -8 < 0$. By the second derivative test, f has a local maximum at $(1/2, 2)$ with value $f(1/2,2) = -\ln 2 - 2$.

(ii) Let $g(x,y) = xy(1-x-y)$.

We have $g_x = y - 2xy - y^2$, $g_y = x - x^2 - 2xy$, $g_{xx} = -2y$, $g_{yy} = -2x$ and $g_{xy} = 1 - 2x - 2y$.

Then $g_x = 0$ implies $y = 0$ or $y = 1 - 2x$. Substituting $y = 0$ into $g_y = 0$ gives $x = 0$ or $x = 1$. Substituting $y = 1 - 2x$ into $g_y = 0$ gives $x = 0$ or $x = 1/3$.

Thus the critical points are $(0,0), (1,0), (0,1)$ and $(1/3, 1/3)$.

Now $D(0,0) = D(1,0) = D(0,1) = -1$; $D(1/3,1/3) = 1/3$ and $g_{xx}(1/3,1/3) = -2/3 < 0$.

By the second derivative test, g has a saddle point at $(0,0), (1,0)$ and $(0,1)$; and g has a local maximum at $(1/3, 1/3)$ with value $g(1/3,1/3) = 1/27$.

(iii) Let $h(x,y) = x^2 + y^2 + x^{-2}y^{-2}$.

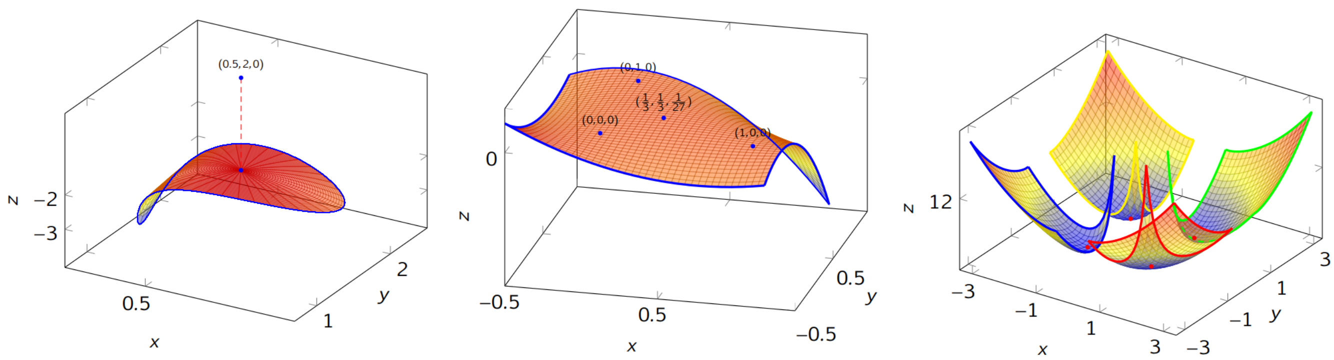
We have $h_x = 2x - 2x^{-3}y^{-2}$, $h_y = 2y - 2x^{-2}y^{-3}$, $h_{xx} = 2 + 6x^{-4}y^{-2}$, $h_{yy} = 2 + 6x^{-2}y^{-4}$ and $h_{xy} = 4x^{-3}y^{-3}$.

Then $h_x = 0$ implies that $2x^4y^2 - 2 = 0$ or $y^2 = x^{-4}$. Note that neither x nor y can be zero. Now $h_y = 0$ implies that $2x^2y^4 - 2 = 0$, and with $y^2 = x^{-4}$ this implies $2x^{-6} - 2 = 0$ or $x^6 = 1$, that is $x = \pm 1$. If $x = 1$, then $y = \pm 1$. If $x = -1$, then $y = \pm 1$.

So the critical points are $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$.

Now $D(\pm 1, \pm 1) = D(\pm 1, \mp 1) = 64 - 16 > 0$ and h_{xx} is always greater than zero.

By the second derivative test, h has a local minimum at $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$ with the same value $h(\pm 1, \pm 1) = h(\pm 1, \mp 1) = 3$.



(i) $f(x,y) = \ln(x^2y) - xy - 2x$

(ii) $g(x,y) = xy(1-x-y)$

(iii) $h(x,y) = x^2 + y^2 + x^{-2}y^{-2}$

Solutions to Further Exercises

1. The temperature in Celsius experienced by the bug along its path after t seconds is given by $C(t) \equiv T(x(t), y(t))$.
$\dfrac{dC}{dt} = \dfrac{\partial T}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial T}{\partial y}\dfrac{dy}{dt} = T_x(x,y)x'(t) + T_y(x,y)y'(t) = T_x(x,y)\dfrac{1}{2\sqrt{1+t}} + T_y(x,y)(\dfrac{1}{3})$.
At $t = 3$, we have $x = 2, y = 3$ and $x'(3) = \frac{1}{4}, y'(3) = \frac{1}{3}$.
Hence, $\dfrac{dC}{dt}\Big|_{t=3} = T_x(2,3)x'(3) + T_y(2,3)y'(3) = (4)(\dfrac{1}{4}) + 3(\dfrac{1}{3}) = 2$°C per second. This is the rate of change of the temperature as experienced by the bug on its path.

2. Let $f(x,y,z) = xy^2z^3$. Then $\nabla f(x,y,z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$. Hence, $\nabla f(1,-2,1) = \langle 4,-4,12\rangle$. Let $\mathbf{u} = \langle \dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\rangle$. Thus, the rate of change of f at P in the direction $\mathbf{u}$ is given by $\langle 4,-4,12\rangle \cdot \mathbf{u} = \dfrac{20}{\sqrt{3}}$.

3. $f_x = -\frac{1}{4}\sin(\frac{x}{2})$, $f_y = \frac{1}{4}\cos(\frac{y}{4})$. Then $f_x = 0, f_y = 0 \Leftrightarrow \sin(\frac{x}{2}) = 0, \cos(\frac{y}{4}) = 0 \Leftrightarrow \frac{x}{2} = m\pi, \frac{y}{4} = n\pi + \frac{\pi}{2} \Leftrightarrow x = 2m\pi, y = 2(2n+1)\pi$, where $m, n \in \mathbb{Z}$.
Within the ranges $-6\pi < x < 2\pi, -2\pi < y < 6\pi$, the solutions are $x = 0, -2\pi, -4\pi$ and $y = 2\pi$. Thus there are 3 critical points: $(0,2\pi), (-2\pi,2\pi), (-4\pi,2\pi)$.
We use the second derivative test to determine the nature of these critical points. We have $f_{xx} = -\frac{1}{8}\cos(\frac{x}{2}), f_{yy} = -\frac{1}{16}\sin(\frac{y}{4}), f_{xy} = 0$. Thus $D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = \frac{1}{128}\cos(\frac{x}{2})\sin(\frac{y}{4})$.
As $D(0,2\pi) = \frac{1}{128} > 0$, $f_{xx}(0,2\pi) = -\frac{1}{8} < 0$, f has a local maximum at $(0,2\pi)$.
As $D(-2\pi,2\pi) = \frac{1}{128} < 0$, f has a saddle point at $(-2\pi,2\pi)$.
As $D(-4\pi,2\pi) = \frac{1}{128} > 0$, $f_{xx}(-4\pi,2\pi) = -\frac{1}{8} < 0$, f has a local maximum at $(-4\pi,2\pi)$.