1. For each of the following $m \times n$ matrices,

(i) find a basis for the row space and a basis for the column space;

(ii) extend the basis for the row space in (i) to a basis for $\mathbb{R}^n$;

(iii) extend the basis for the column space in (i) to a basis for $\mathbb{R}^m$;

(iv) find a basis for the nullspace;

(v) find the rank and nullity of the matrix and hence verify the Dimension Theorem for Matrices; and

(vi) determine if the matrix has full rank.

(a) $\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 0 & 5 & 2 \\ 2 & 1 & 0 & 3 & 0 \\ -1 & 3 & 0 & 2 & 2 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}$, (b) $\boldsymbol{B} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ -1 & 3 & 6 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix}$,

(c) $\boldsymbol{C} = \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix}$, (d) $\boldsymbol{D} = \begin{pmatrix} 1 & 4 & 5 & 8 \\ -1 & 4 & 3 & 0 \\ 2 & 0 & 2 & 1 \end{pmatrix}$.

(c) $\boldsymbol{C} = \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix}$,

i) Into row-echelon form

$$\xrightarrow{\substack{R_2-2R_1 \\ R_3-R_1 \\ R_4-3R_1}} \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 0 & 0 & -6 & 1 & -2 \\ 0 & 0 & -6 & 1 & -2 \\ 0 & 0 & -6 & 1 & -2 \end{pmatrix} \xrightarrow{\substack{R_3-R_1 \\ R_4-R_1}} \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 0 & 0 & -6 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Non zero rows form the basis for the row space :

$\{(2,1,4,1,2), (0,0,-6,1,-2)\}$

Corresponding pivot colums form the basis for the column space

$\{(2,4,2,6)^T, (4,2,-2,6)^T\}$

ii) Extend the basis for the row space for $\mathbb{R}^5$

$\{(2,1,4,1,2), (0,0,-6,1,-2), (0,0,0,1,0), (0,1,0,0,0), (0,0,0,0,1)\}$

is a basis for $\mathbb{R}^5$

iii) $\begin{pmatrix} 2 & 4 & 2 & 6 \\ 4 & 2 & -2 & 6 \end{pmatrix} \xrightarrow{R_2-2R_1} \begin{pmatrix} 2 & 4 & 2 & 6 \\ 0 & -6 & -6 & -6 \end{pmatrix}$

So, $\{(2,4,2,6)^T, (4,2,-2,6)^T, (0,0,1,0)^T, (0,0,0,1)^T)$ is a basis for $\mathbb{R}^4$

iv)
$$\begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 0 & 0 & -6 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \frac{1}{2} & 2 & \frac{1}{2} & 1 \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & \frac{1}{2} & 0 & \frac{5}{6} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left(-\frac{1}{2}s-\frac{5}{6}t-\frac{1}{3}z,\ s,\ \frac{1}{6}t-\frac{1}{3}z,\ t,\ z\right)$$

$$s\left(-\frac{1}{2}, 1, 0, 0\right)+t\left(-\frac{5}{6}, 0, \frac{1}{6}, 1, 0\right)+z\left(\frac{1}{3}, 0, -\frac{1}{3}, 0, 1\right)$$

$$\left\{\left(-\frac{1}{2}, 1, 0, 0, 0\right)^T, \left(-\frac{5}{6}, 0, \frac{1}{6}, 1, 0\right)^T, \left(-\frac{1}{3}, 0, -\frac{1}{3}, 0, 1\right)^T\right\}$$

is the basis for the null space

because vectors in null space are viewed as column vectors

v) $\operatorname{rank}(C)=2$ and $\operatorname{nulity}(C)=3$

$\operatorname{rank}(C)+\operatorname{nulity}(C)=5$

$=$ number of column in C.

vi) no, $\operatorname{rank}(C) \leq \min(4,5)$

9. Let $\boldsymbol{A}$ be a 3×4 matrix. Suppose that $x_1 = 1$, $x_2 = 0$, $x_3 = -1$, $x_4 = 0$ is a solution to a non-homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{b}$ and that the homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$ has a general solution $x_1 = t - 2s$, $x_2 = s + t$, $x_3 = s$, $x_4 = t$ where s, t are arbitrary parameters.

(a) Find a basis for the nullspace of $\boldsymbol{A}$ and determine the nullity of $\boldsymbol{A}$.

(b) Find a general solution for the system $\boldsymbol{Ax} = \boldsymbol{b}$.

(c) Write down the reduced row-echelon form of $\boldsymbol{A}$.

(d) Find a basis for the row space of $\boldsymbol{A}$ and determine the rank of $\boldsymbol{A}$.

(e) Do we have enough information for us to find the column space of $\boldsymbol{A}$?

a)
$$x = s\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore \{(-2,1,1,0)^T, (1,1,0,1)^T\}$ is the basis for the nullspace of A.

The nulity is 2

b) The general Solution is the Sum of the null Solution and the Particular solution

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

or

$x_1 = 1+t-2s$

$x_2 = s+t$

$x_3 = -1+s$

$x_4 = t$

c) $A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

d) $\{(1,0,2,-1), (0,1,-1,-1)\}$ is the basis of the row space

$\text{rank}(A) = 2$

e) No

10. Let $\boldsymbol{A} = (\boldsymbol{a_1} \quad \boldsymbol{a_2} \quad \boldsymbol{a_3} \quad \boldsymbol{a_4} \quad \boldsymbol{a_5})$ be a 4×5 matrix such that the columns $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, $\boldsymbol{a_3}$ are linearly independent while $\boldsymbol{a_4} = \boldsymbol{a_1} - 2\boldsymbol{a_2} + \boldsymbol{a_3}$ and $\boldsymbol{a_5} = \boldsymbol{a_2} + \boldsymbol{a_3}$.

(a) Determine the reduced row-echelon form of $\boldsymbol{A}$. (Hint: The linear relations between columns will not be changed by row operations. In this question, the fifth column of $\boldsymbol{A}$ is the sum of the second and the third columns of $\boldsymbol{A}$. Then the fifth column of the reduced row-echelon form $\boldsymbol{R}$ is still the sum of the second and the third columns of $\boldsymbol{R}$.)

(b) Find a basis for the row space of $\boldsymbol{A}$ and a basis for the column space of $\boldsymbol{A}$.

a) Let R be the reduced row-echelon form of A. Since a_1, a_2, a_3 are linearly independent, the first three columns of R are linearly independent. Thus the first three columns of R must be $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Together with the information given:

$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

b) Blue: row space

Yellow: column space

22. Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{P}$ an $m \times m$ matrix.

(a) If $\boldsymbol{P}$ is invertible, show that $\text{rank}(\boldsymbol{PA}) = \text{rank}(\boldsymbol{A})$.

(b) Give an example such that $\text{rank}(\boldsymbol{PA}) < \text{rank}(\boldsymbol{A})$.

(c) Suppose $\text{rank}(\boldsymbol{PA}) = \text{rank}(\boldsymbol{A})$. Is it true that $\boldsymbol{P}$ must be invertible? Justify your answer.

a) Since P is invertible, we can write $P = E_n \cdots E_1$ where E_i are elementary matrices. So, $PA = E_n \cdots E_1 A$ and A are row equivalent matrices. They have the same row space. Thus

$\text{Rank}(PA) = \dim(\text{the row space of } PA)$

$= \dim(\text{the row space of } A)$

$= \text{Rank}(A)$

b) $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

c) No $P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

25. Let $\boldsymbol{A}$ be an $m \times n$ matrix.

(a) Show that the nullspace of $\boldsymbol{A}$ is equal to the nullspace of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$.

(b) Show that nullity$(\boldsymbol{A})$ = nullity$(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})$ and rank$(\boldsymbol{A})$ = rank$(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})$.

(c) Is it true that nullity$(\boldsymbol{A})$ = nullity$(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})$? Justify your answer.

(d) Is it true that rank$(\boldsymbol{A})$ = rank$(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})$? Justify your answer.

a) Proving nullspace of $A \subseteq$ nullspace of $A^T A$

Let u be any vector in the nullspace of A .ie $Au = 0$
Then, $A^T A u = A^T 0 = 0$. So u is also a vector in the nullspace of $A^T A$.
Therefore the nullspace of A is a subspace of the nullspace of $A^T A$.

Proving nullspace of $A^T A \supseteq$ nullspace of A

Let v be any vector in the nullspace of $A^T A$. ie $A^T A v = 0$
Suppose $Av = (b_1, b_2 \cdots b_m)^T$. Then

$$(Av)^T (Av) = v^T A^T Av = v^T 0 = 0$$
$$\Rightarrow b_1^2 + b_2^2 + \cdots + b_m^2 = 0$$
$$\Rightarrow b_1 = b_2 = \cdots = b_m = 0$$

That is, $Av = 0$. So v is also a vector in the null space of A.
The nullspace of $A^T A$ is a subspace of the nullspace of A.

Hence nullspace A = nullspace $A^T A$.

b) By a) nulity (A) = nulity $(A^T A)$
By the dimension theorem of matrices, rank(A) = n - nulity(A) = n - nulity$(A^T A)$ = rank$(A^T A)$

c) No $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

d) Yes by b) rank(A) = rank(A^T) = rank$((A^T)^T A^T)$ = rank(AA^T)