CS2040S
Data Structures and Algorithms

Welcome!

ARCHIPELAGO

is open

How to Search!

Algorithm Analysis

- Big-O Notation
- Model of computation

Searching

Peak Finding

- 1-dimension
- 2-dimensions

Admin: Tutorial/Recitations

Please do not modify your tutorial/recitation sections directly on ModReg!

All changes much go through us.

We are trying hard to keep sections well-balanced.

Admin: Tutorial/Recitations

Allocation Appeals:

- Almost done
- < 10 students left

Please respond to emails requesting your schedule.

Please finalize any swaps/appeals that you would like to make.

Problem Set 2

Released on Monday

Due next week (after CNY)

FAQ:

- If you can't see it, you are probably using Safari V.14. Try a difference browser.
- If you have questions, ask on the Coursemology forum.
- Think carefully about the different possible inputs.
- Think carefully about the strange corner cases.
- Private test cases are for the purpose of evaluation (i.e., we will not release them or tell you what they are); they may include hints. It may be the same of them are testing very hard cases.

Problem Set Policies

1. No resubmission.

- Tutors only have time to grade once!

2. Almost no unsubmission.

- Please do not submit until you are ready to have it graded.
- In extreme cases, can ask tutor for unsubmission, with very good reason.
- If tutor deems that you have entirely misunderstood the question, they may unsubmit for you.

3. As much feedback as you want.

- Tutors will help you to understand what you got wrong, look at any fixes that you make, and help you to learn.

4. Tutors can grant rare *short* extensions.

- Ask your tutor if you need an extension for a very good reason.

How to Search!

Algorithm Analysis

- Big-O Notation
- Model of computation

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Peak Finding

- 1-dimension
- 2-dimensions

Last time...

Binary Search

- Simple, ubiquitous algorithm.
- Surprisingly easy to add bugs.
- Some ideas for avoiding bugs:
  - Problem specification
  - Preconditions
  - Postconditions
  - Invariants / loop invariants
  - Validate (when feasible)

Sorted array: A[0..n-1]

2	4	4	5	6	7	8	9	11	17	23	28

```
int search(A, key, n)
    begin = 0
    end = n-1
    while begin < end do:
        mid = begin + (end-begin)/2;
        if  key <= A[mid] then
            end = mid
        else begin = mid+1
    return (A[begin]==key) ? begin : -1
```

Binary Search

Sorted array: A[0..n-1]

2	4	4	5	6	7	8	9	11	17	23	28

Not just for searching arrays:

- Assume a complicated function:

  int complicatedFunction(int s)

- Assume the function is always increasing:

  complicatedFunction(i) < complicatedFunction(i+1)

- Find the minimum value j such that:

  complicatedFunction(j) > 100

A problem...

Tutorial allocation

A problem...

Tutorial allocation

Tutorials
(in order
of tutor
preference)

T1

T2

T3

T4

T5

T6

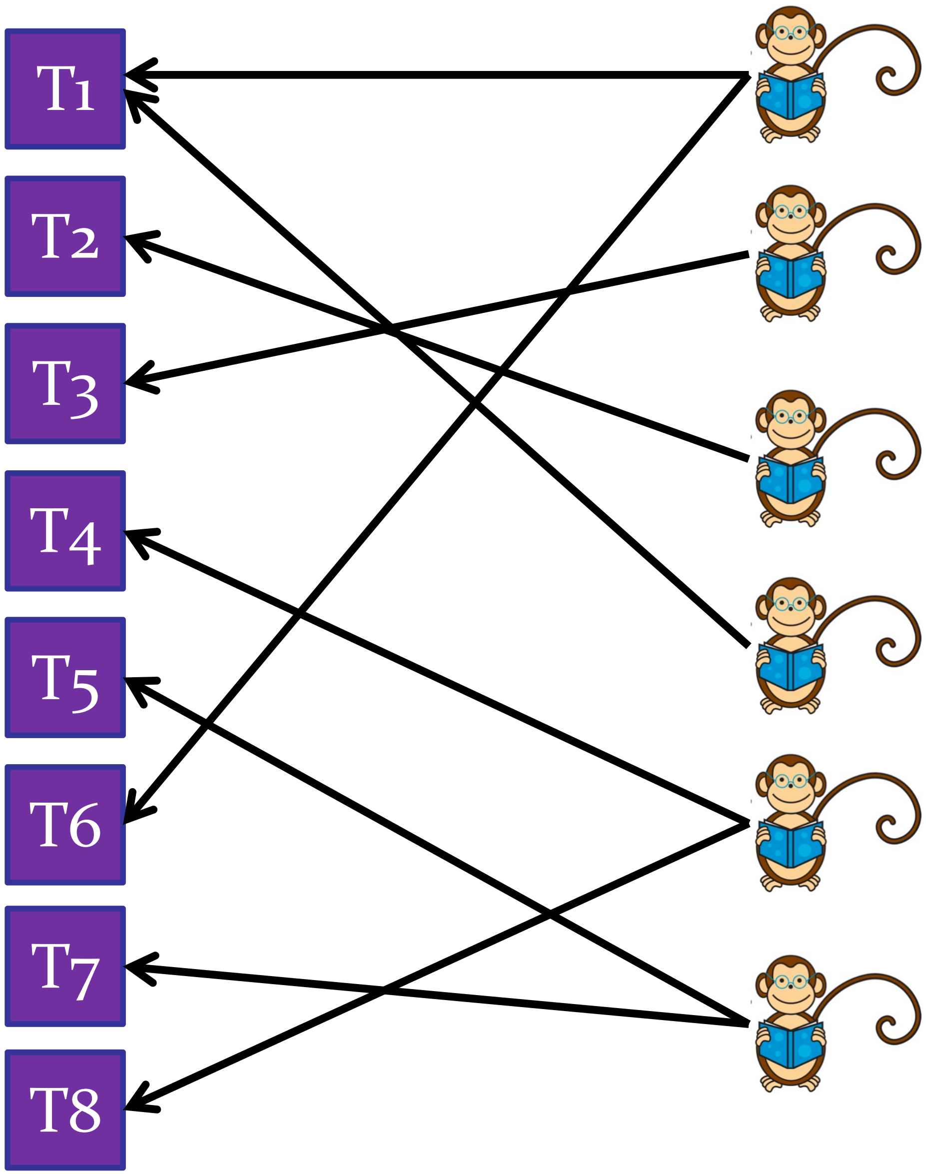
T7

T8

A problem...

Tutorial allocation

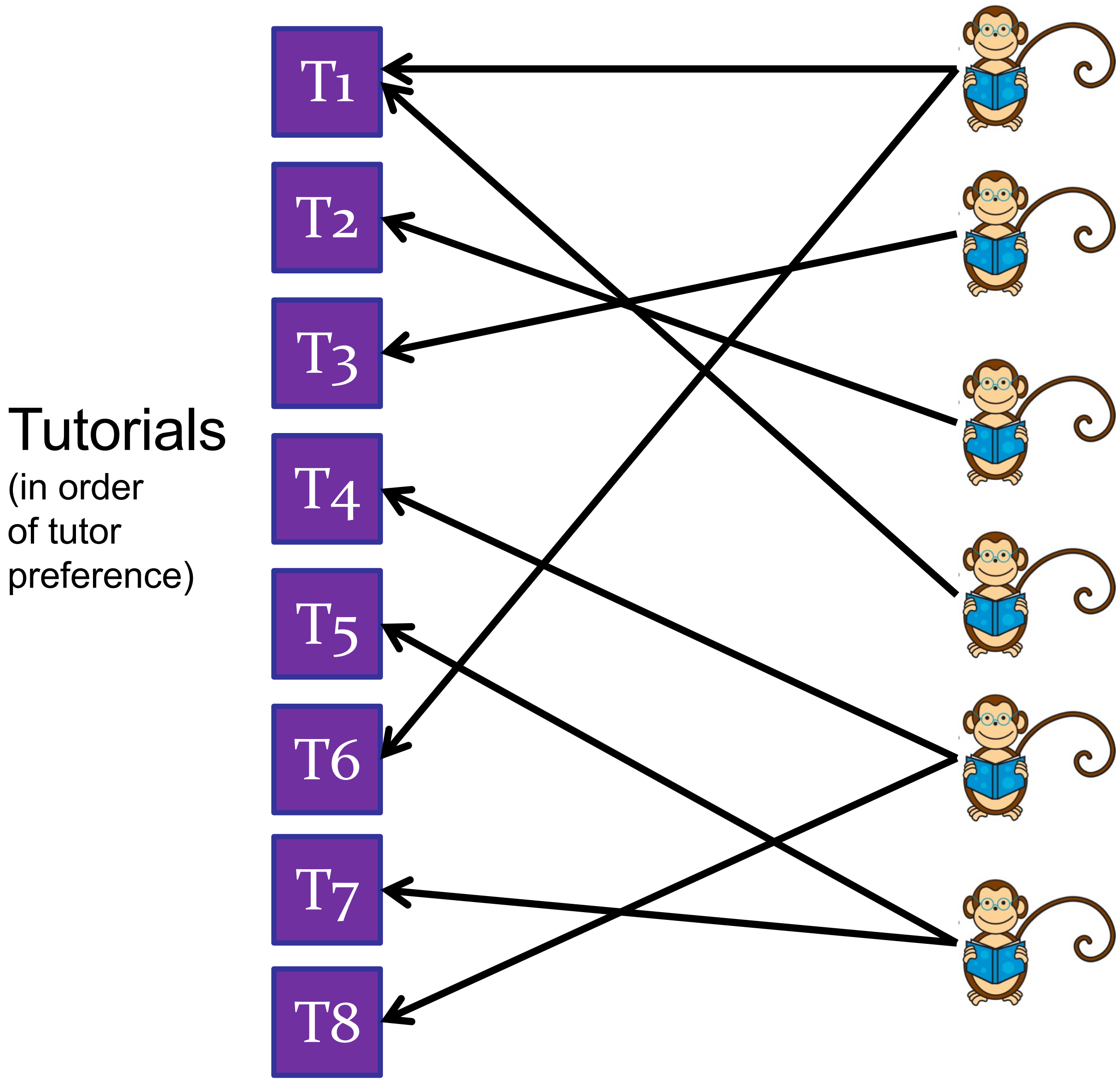
Tutorials
(in order
of tutor
preference)



Students want
certain tutorials.

A problem…

Tutorial allocation



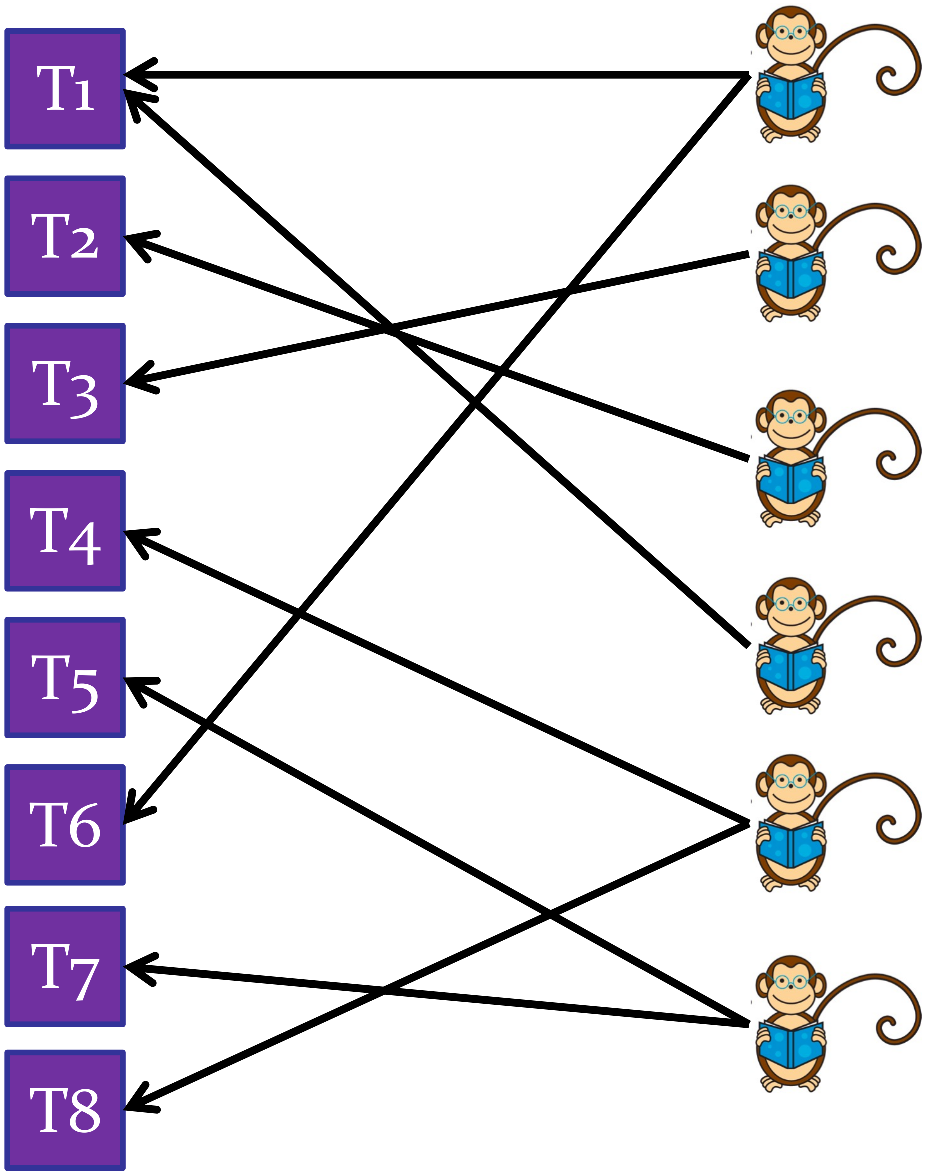
Students want certain tutorials.

We want each tutorial to have < 18 students..

A problem...

Tutorial allocation

Tutorials
(in order
of tutor
preference)



Students want certain tutorials.

We want each tutorial to have < 18 students..

How many tutorials do we need to run?

A problem...

Tutorial allocation

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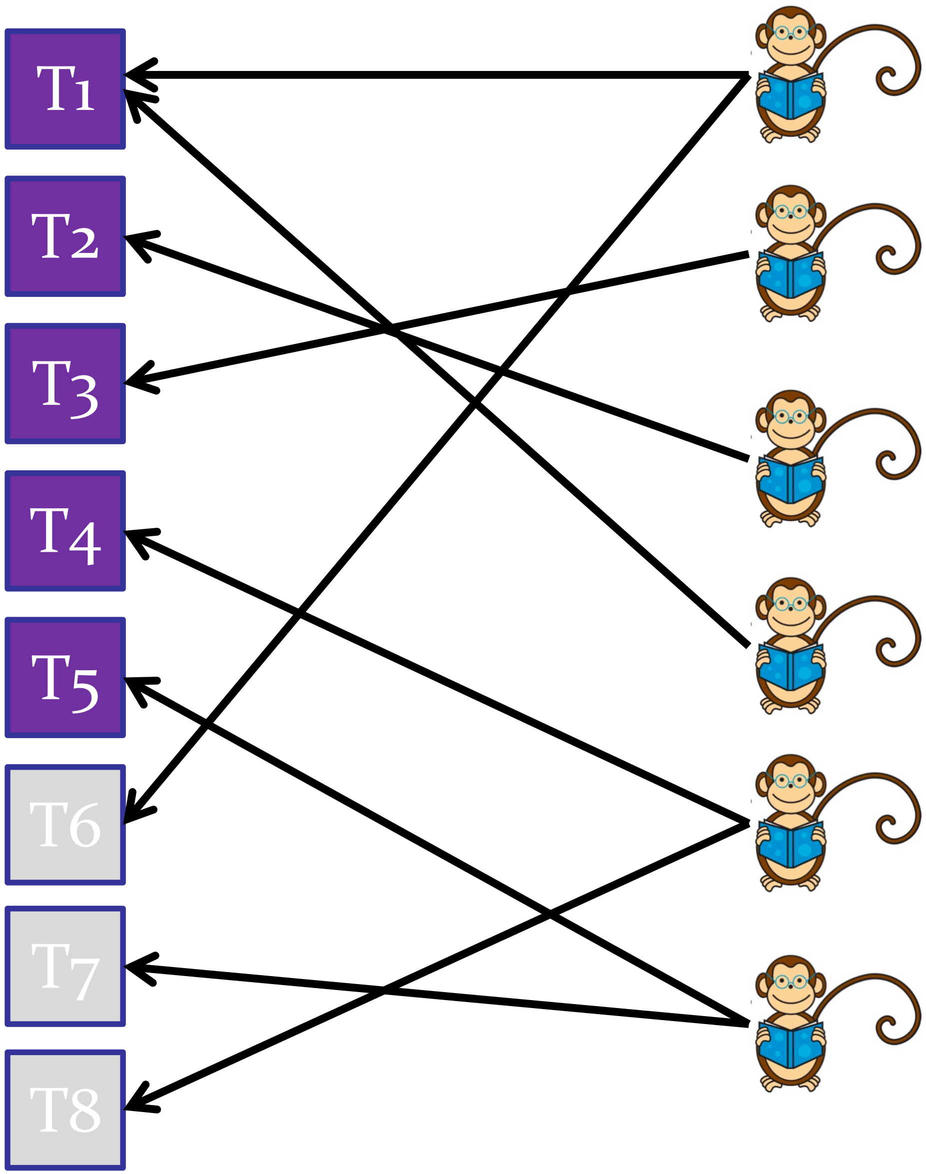

Students want certain tutorials.

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A problem...

Tutorial allocation

Tutorials
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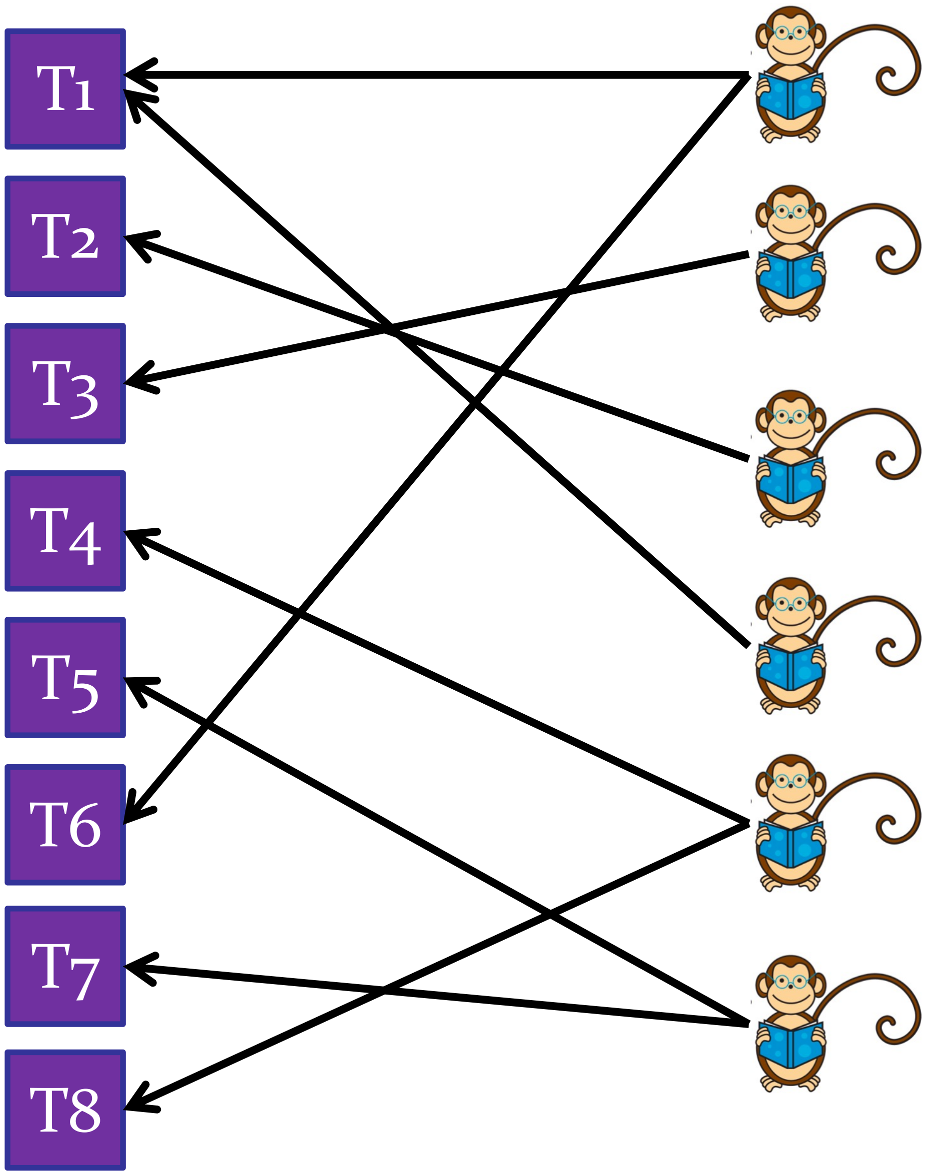
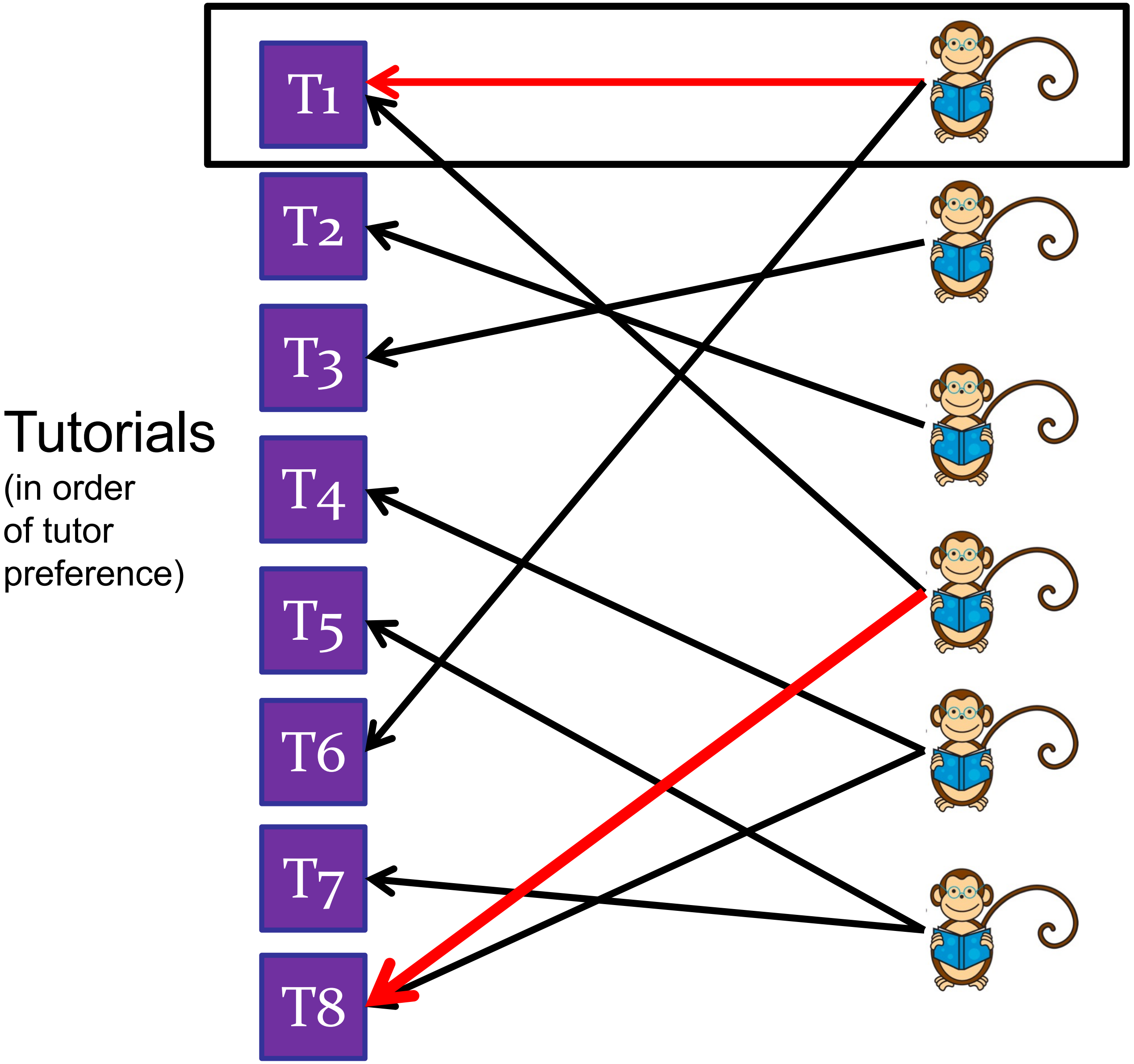


Can we do
greedy allocation?

First, fill T1.
Then fill T2.
Then fill T3.
…
Stop when all
students are
allocated

A problem...

Tutorial allocation



Can we do
greedy allocation?

NO

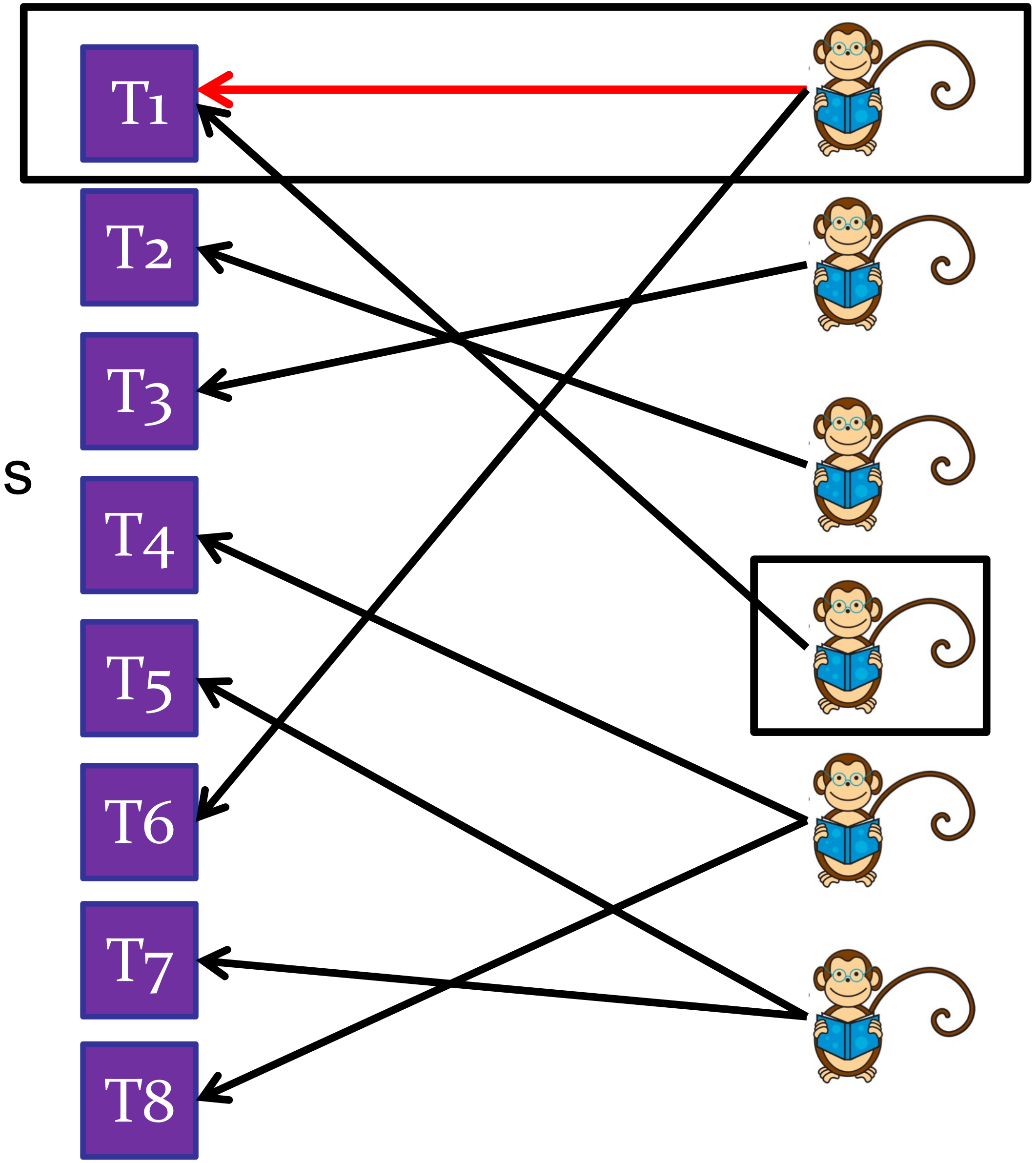
Assume max
tutorial size is 1.

Assign student 1 to
tutorial 1.

Now we need all 8
tutorials.

A problem…

Tutorial allocation



Tutorials
(in order of tutor preference)

Can we do greedy allocation?

NO

Assume max tutorial size is 1.

Assign student 1 to tutorial 1.

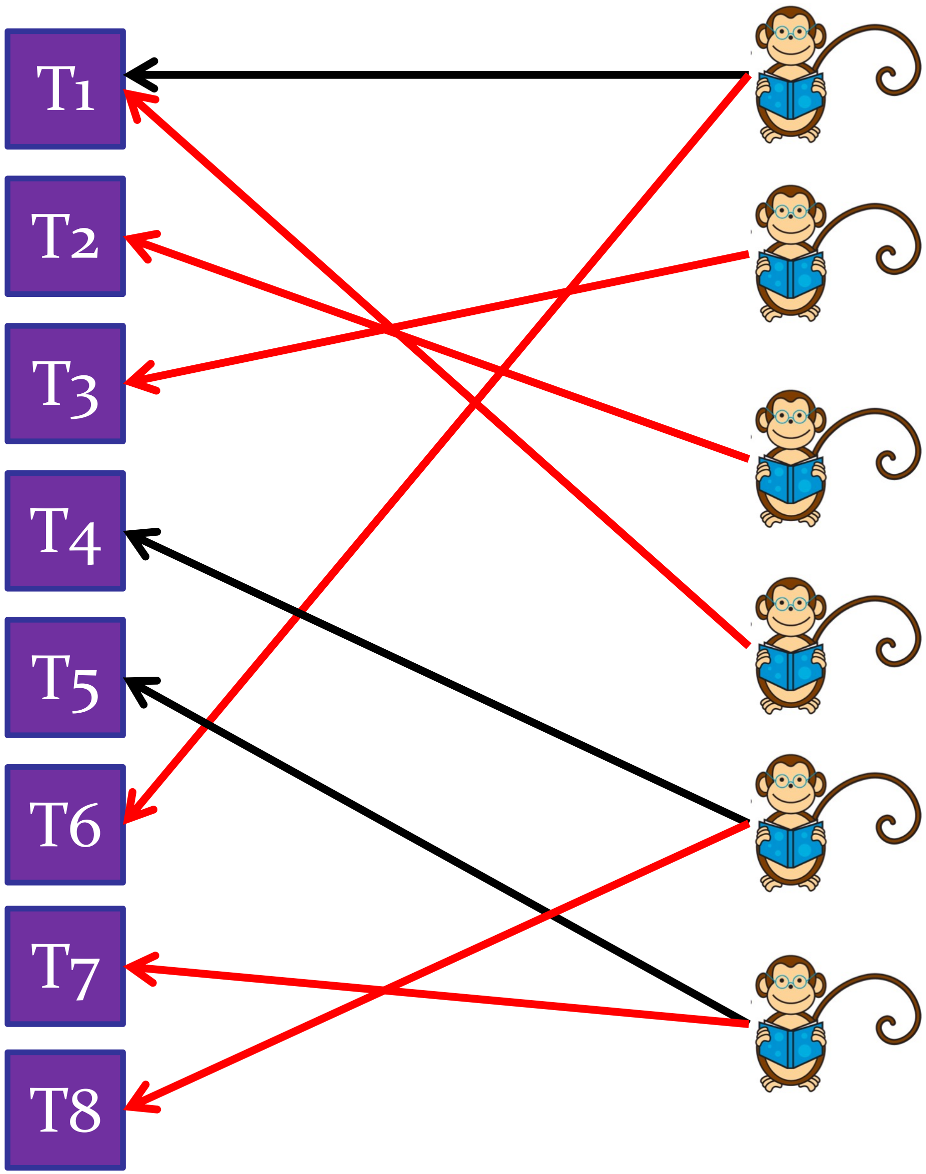
Now one student has no feasible allocation!

A problem...

Example of decomposing an algorithm into parts!

Tutorial allocation

Tutorials
(in order of tutor preference)



Assume we can solve allocation problem:

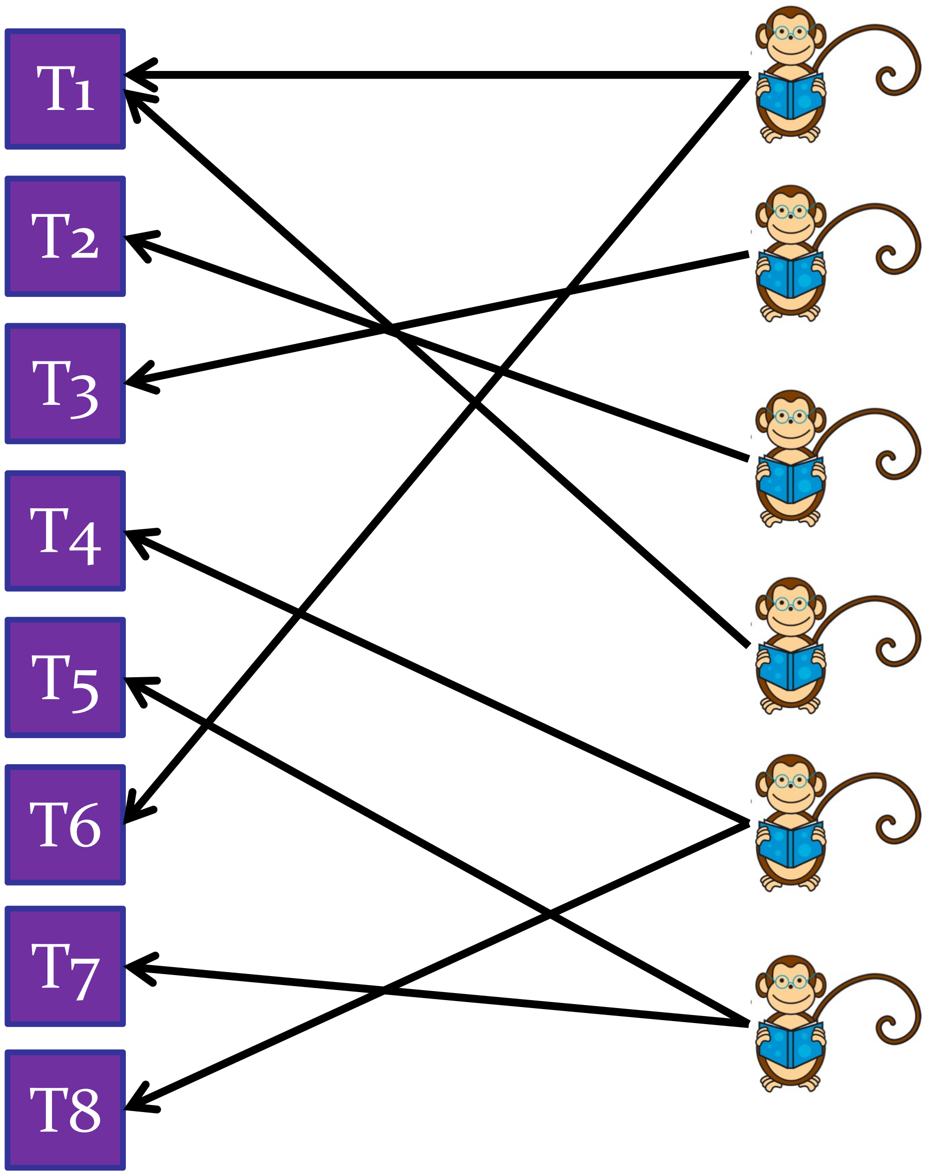
Given a fixed set of tutorials and a fixed set of students, find an allocation where every student has a slot.

Assumptions:

- may be > 18 students in a slot!
- minimizes max students in a slot.

A problem…

Tutorial allocation



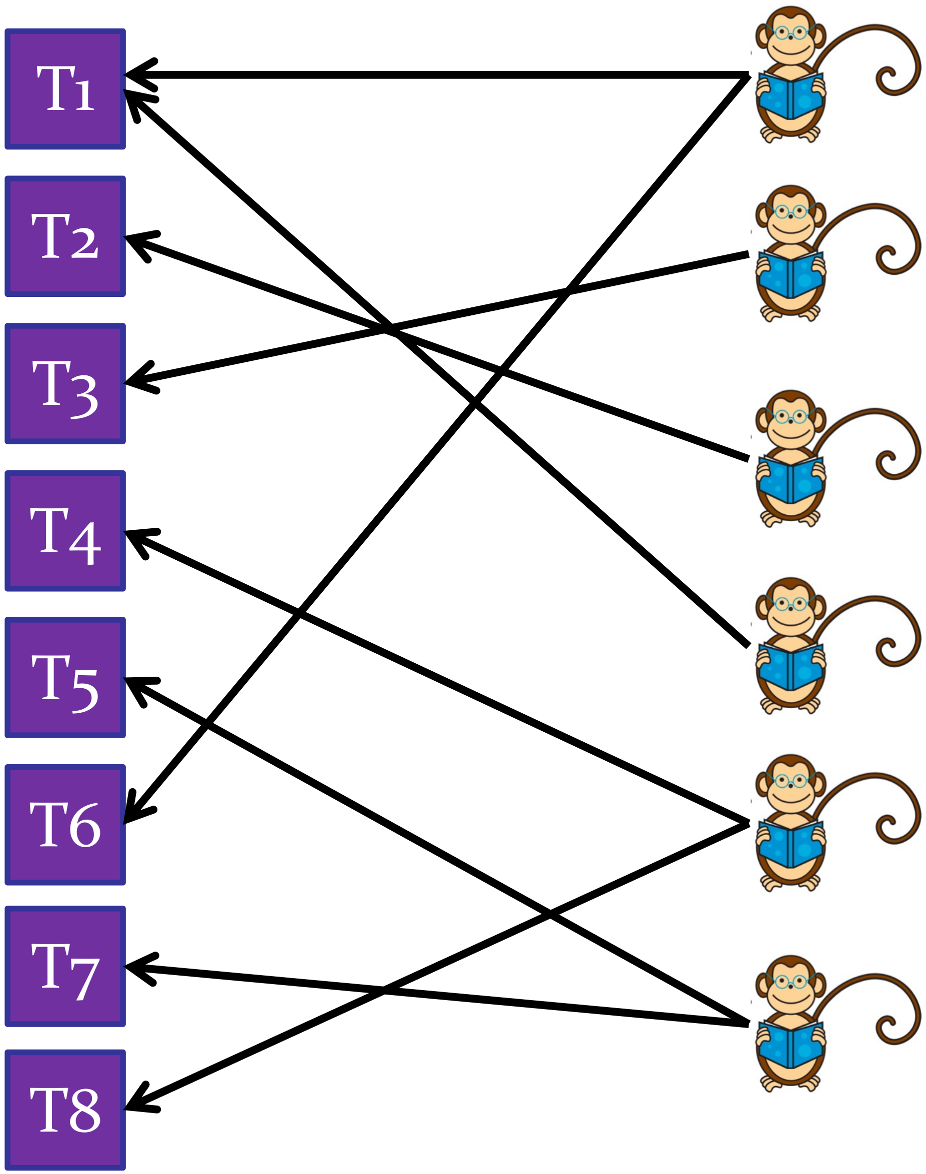
Tutorials
(in order
of tutor
preference)

How to find minimum number of tutorials that we need to open to ensure: no tutorial has more than 18 students.

A problem…

Tutorial allocation

Tutorials
(in order
of tutor
preference)



Observation:

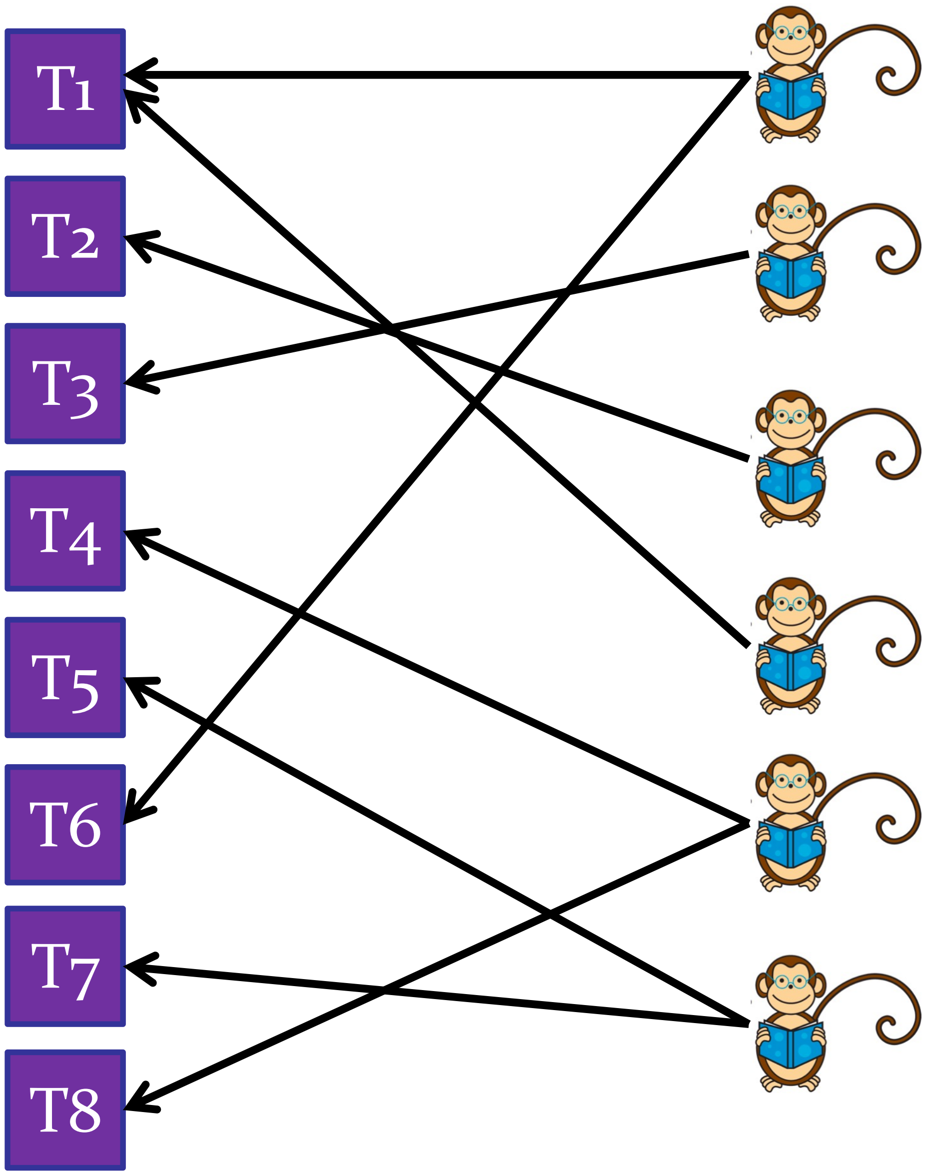
Number of students in BIGGEST tutorial only decreases as number of tutorials increases.

Monotonic function of number of tutorials!

A problem...

Tutorial allocation

Tutorials
(in order
of tutor
preference)



Monotonic function of number of tutorials!

⇓

Binary Search

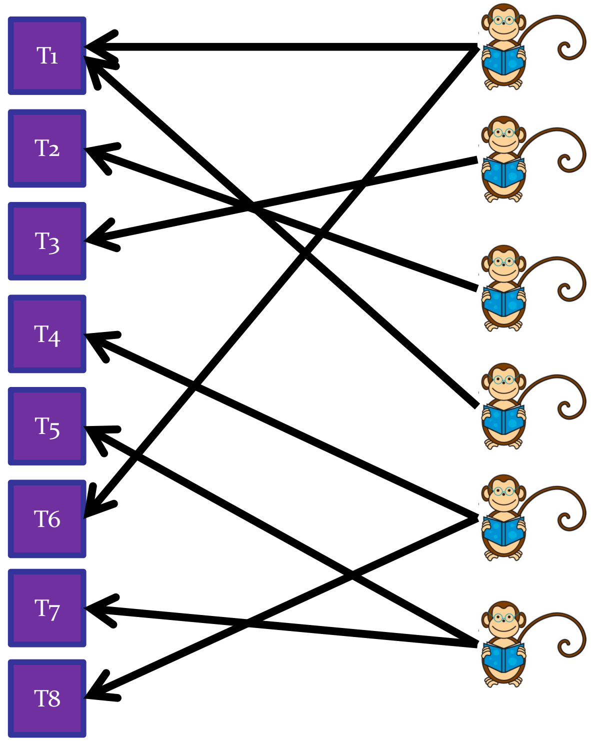
A problem…

Tutorial allocation

Solution:

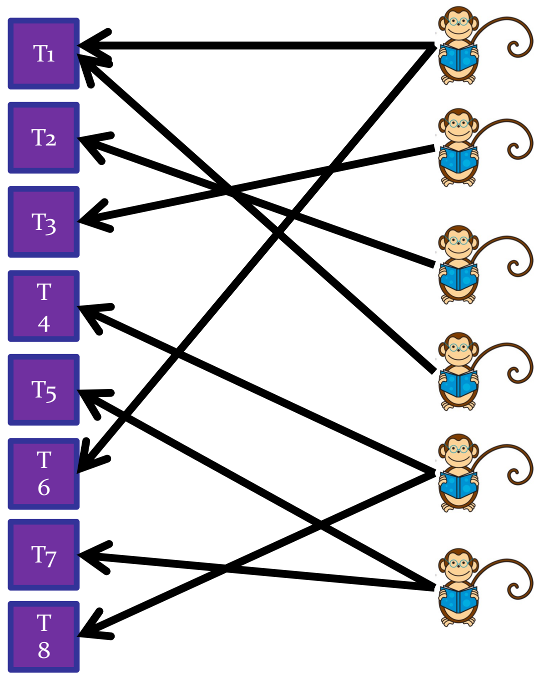
Binary Search

Define:



MaxStudents(x) = number of students in most crowded tutorial, if we offer x tutorials.

Binary Search



MaxStudents(x) = number of students in most crowded tutorial, if we offer x tutorials.

```
Search(n)
    begin = 0
    end = n-1
    while begin < end do:
        mid = begin + (end-begin)/2;
        if  MaxStudents(mid) <= 18 then
              end = mid
        else begin = mid+1
    return begin
```

Binary Search

Sorted array: A[0..n-1]

2	4	4	5	6	7	8	9	11	17	23	28

Not just for searching arrays:

- Assume a complicated function:

  int complicatedFunction(int s)

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How to Search!

Algorithm Analysis

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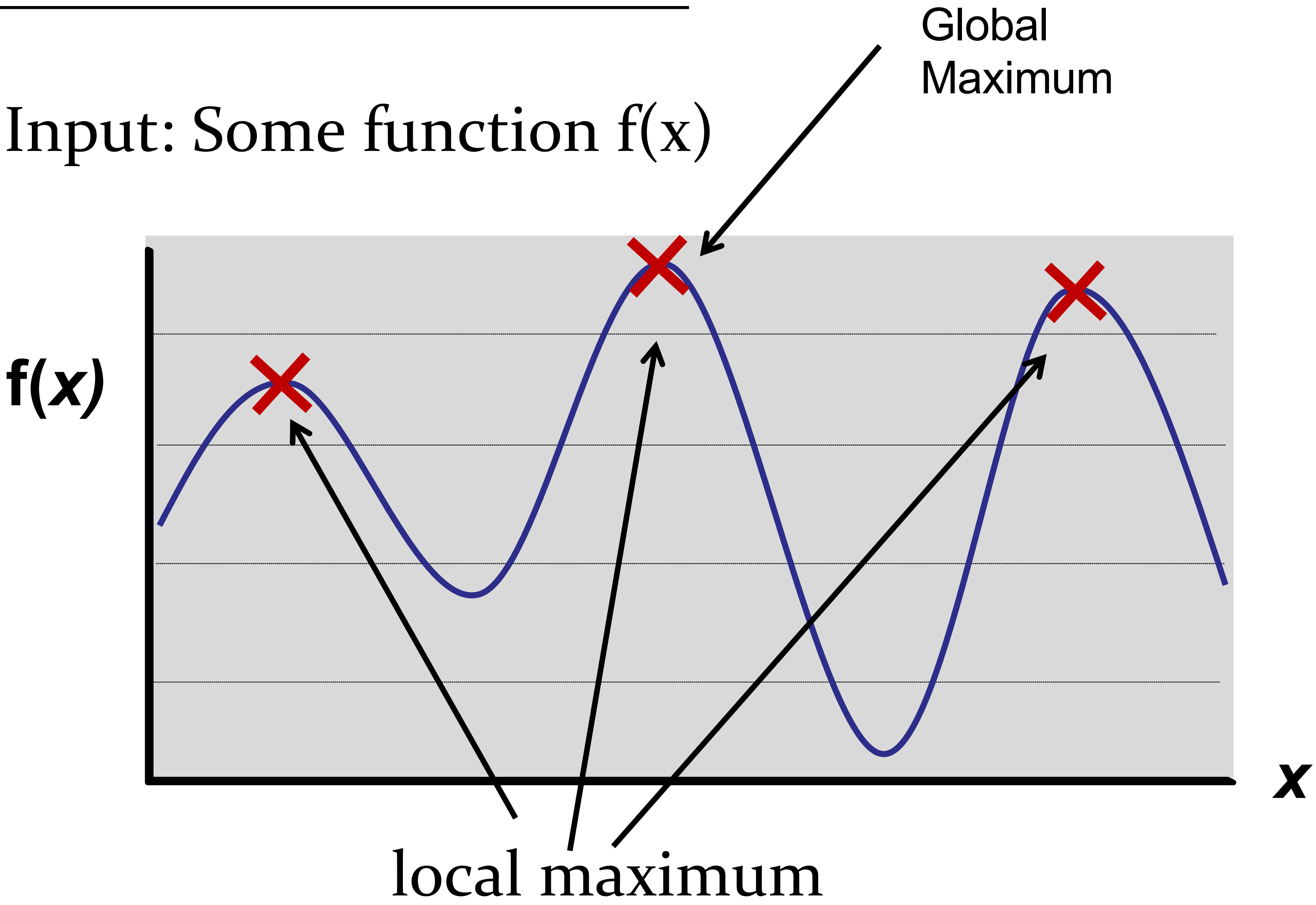
Peak Finding

- 1-dimension
- 2-dimensions

Peak Finding

Input: Some function f(x)



Peak Finding

Global Maximum for Optimization problems:

- Find a good solution to a problem.
- Find a design that uses less energy.
- Find a way to make more money.
- Find a good scenic viewpoint.
- Etc.

Why local maximum?

- Finds a *good enough* solution.
- Local maxima are close to the global maximum?
- Much, much faster.

Global Maximum

Input: Array $A[0..n-1]$

Output: global maximum element in A

How long to find a global maximum?

Input: Arbitrary array $A[0..n-1]$

Output: maximum element in A

1. $O(\log n)$
2. $O(n)$
3. $O(n \log n)$
4. $O(n^2)$
5. $O(2^n)$

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Global Maximum

Unsorted array: `A[0..n-1]`

7	4	9	2	11	6	23	4	28	8	17	5

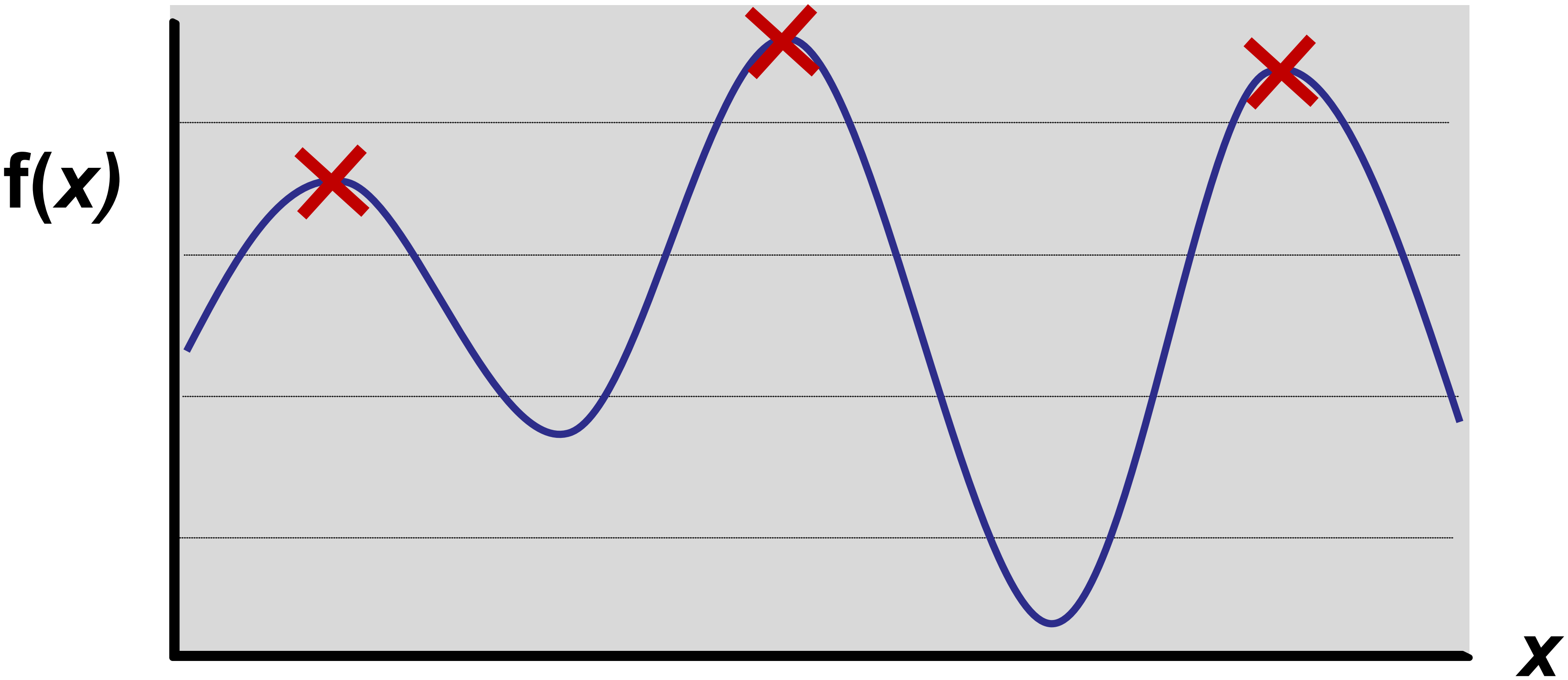
```
FindMax(A,n)
    max = A[1]
    for i = 1 to n do:
        if (A[i]>max) then max=A[i]
```

Too slow!

Time Complexity: $O(n)$

Peak (Local Maximum) Finding

Input: Some function f(x)



Output: A local maximum

Peak Finding

Input: Some ~~function~~ array A[0..n-1]

7	4	9	2	11	6	23	4	6	8	17	21

Output: a local maximum in A

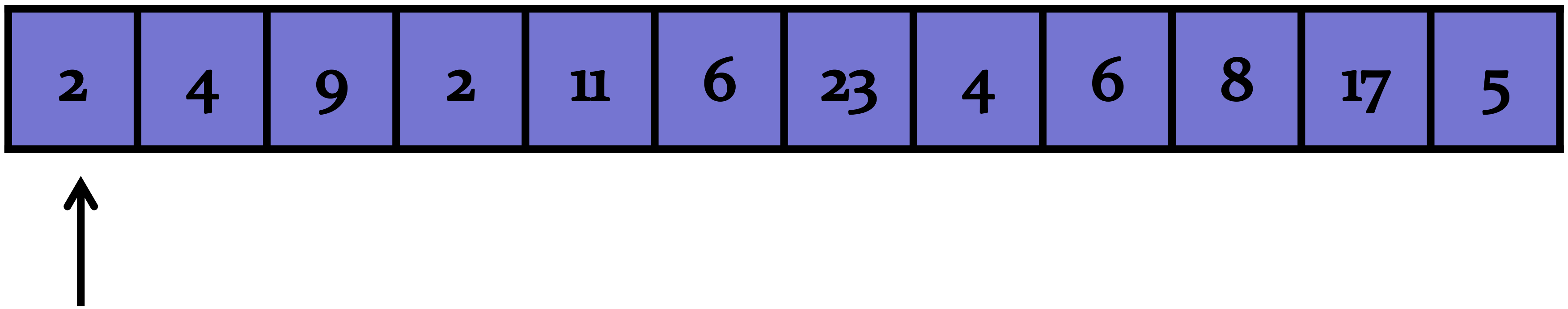
`A[i-1]` $\leq$ `A[i]` **and** `A[i+1]` $\leq$ `A[i]`

Assume that

`A[-1] = A[n] = -MAX_INT`

Peak Finding: Algorithm 1

Input: Some array `A[0..n-1]`

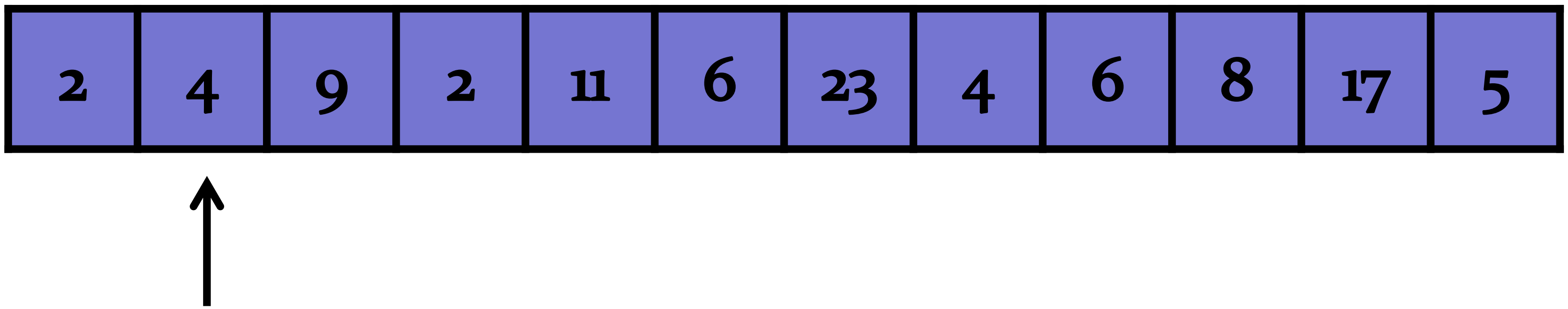


FindPeak

- Start from A[1]
- Examine every element
- Stop when you find a peak.

Peak Finding: Algorithm 1

Input: Some array `A[0..n-1]`



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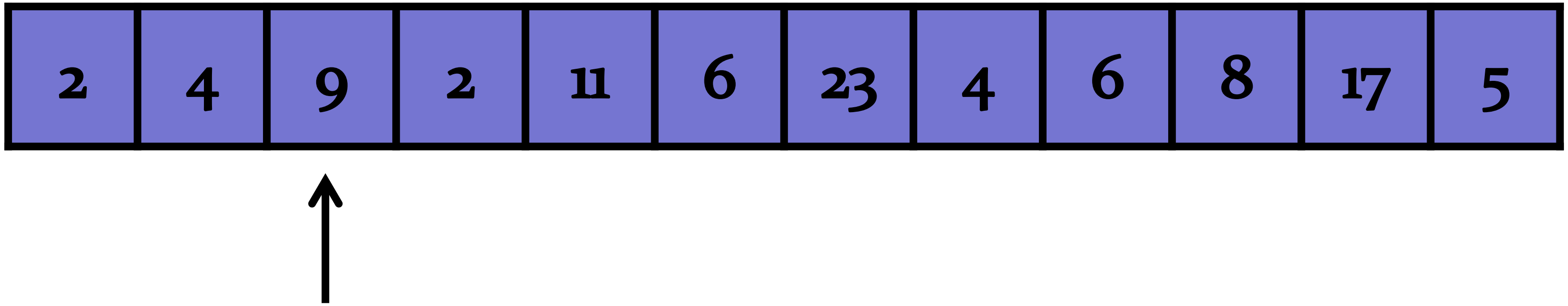
Peak Finding: Algorithm 1

Input: Some array `A[0..n-1]`



FindPeak

- Start from A[1]
- Examine every element
- Stop when you find a peak.

Peak Finding: Algorithm 1

Input: Some array `A[0..n-1]`

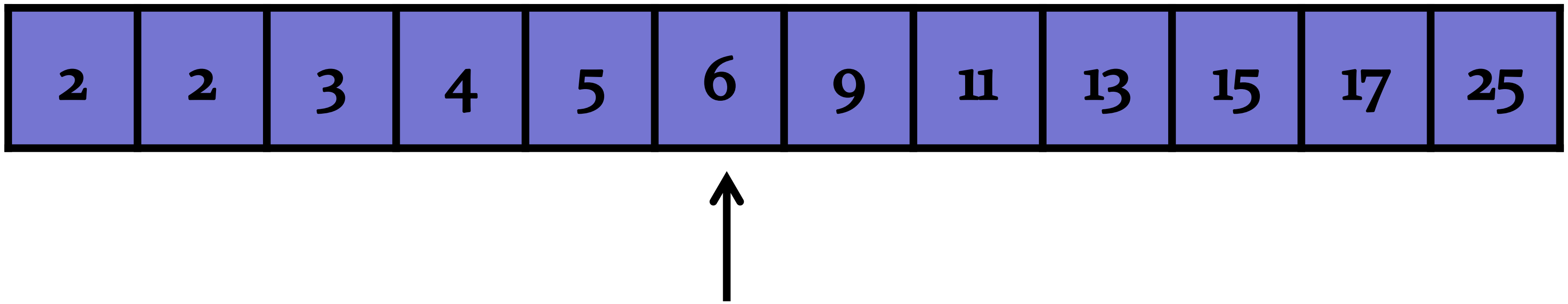
2	4	9	2	11	6	23	4	6	8	17	5

Running time: n

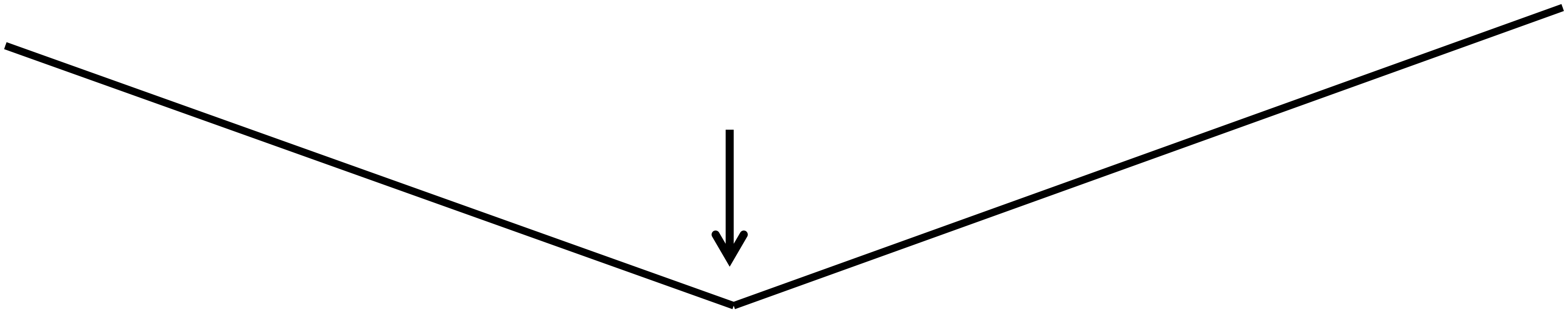
Simple improvement?

Peak Finding: Algorithm 1

Input: Some array `A[0..n-1]`



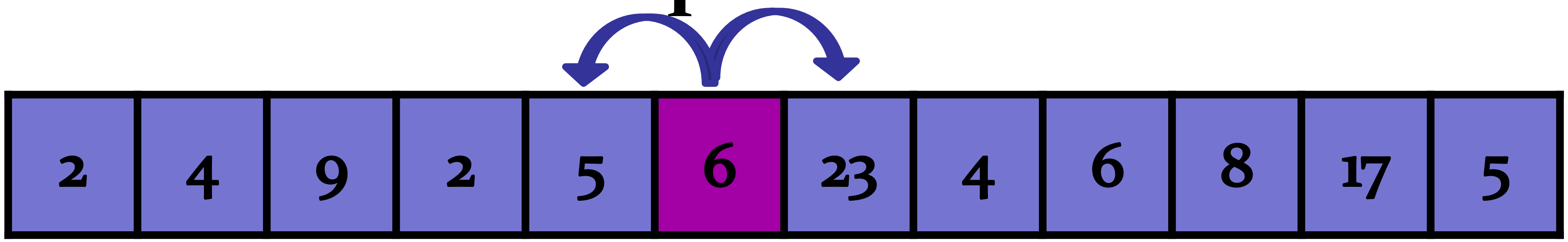
Start in the middle!



Worst-case: n/2

Peak Finding: Algorithm 2

Reduce-and-Conquer

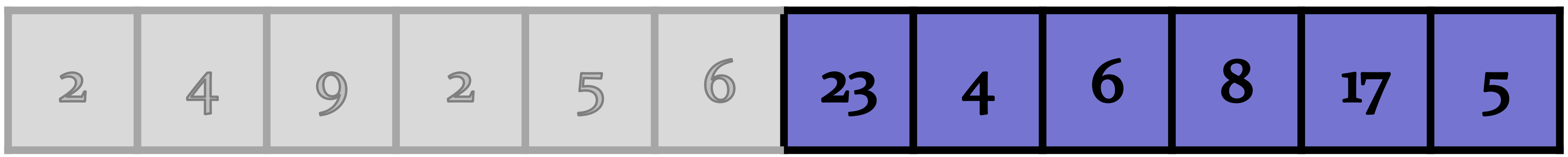


Start in the middle

$5 < 6$? ← OK

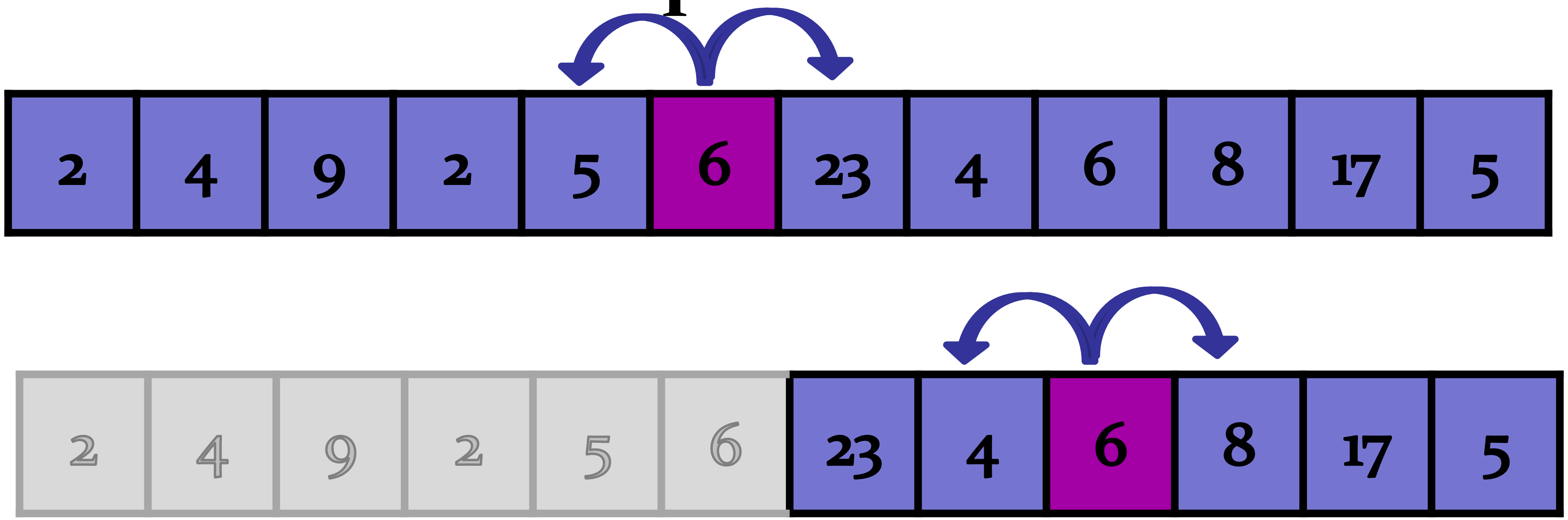
$6 > 23$? ← NO

Recurse!



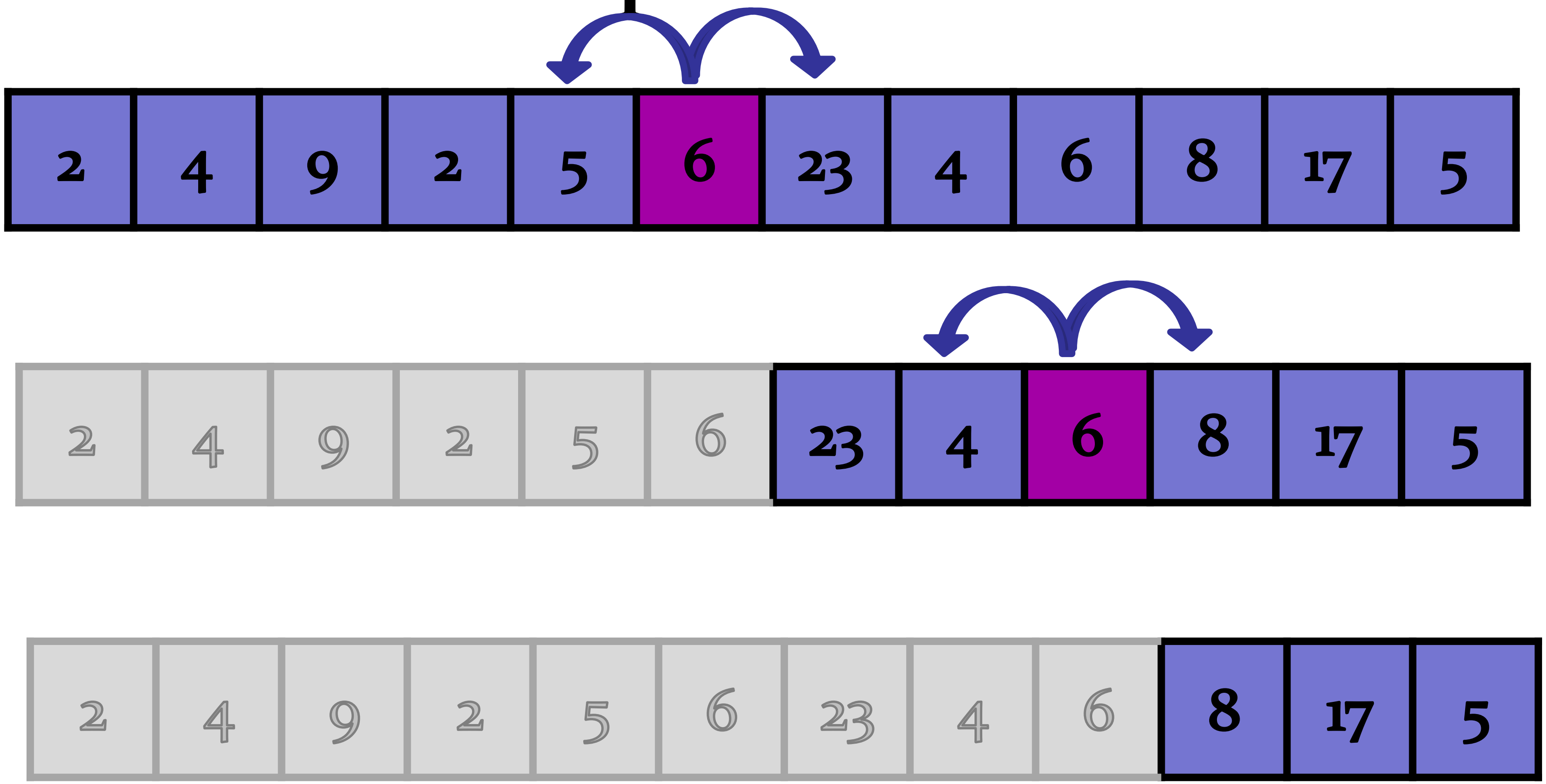
Peak Finding: Algorithm 2

Reduce-and-Conquer



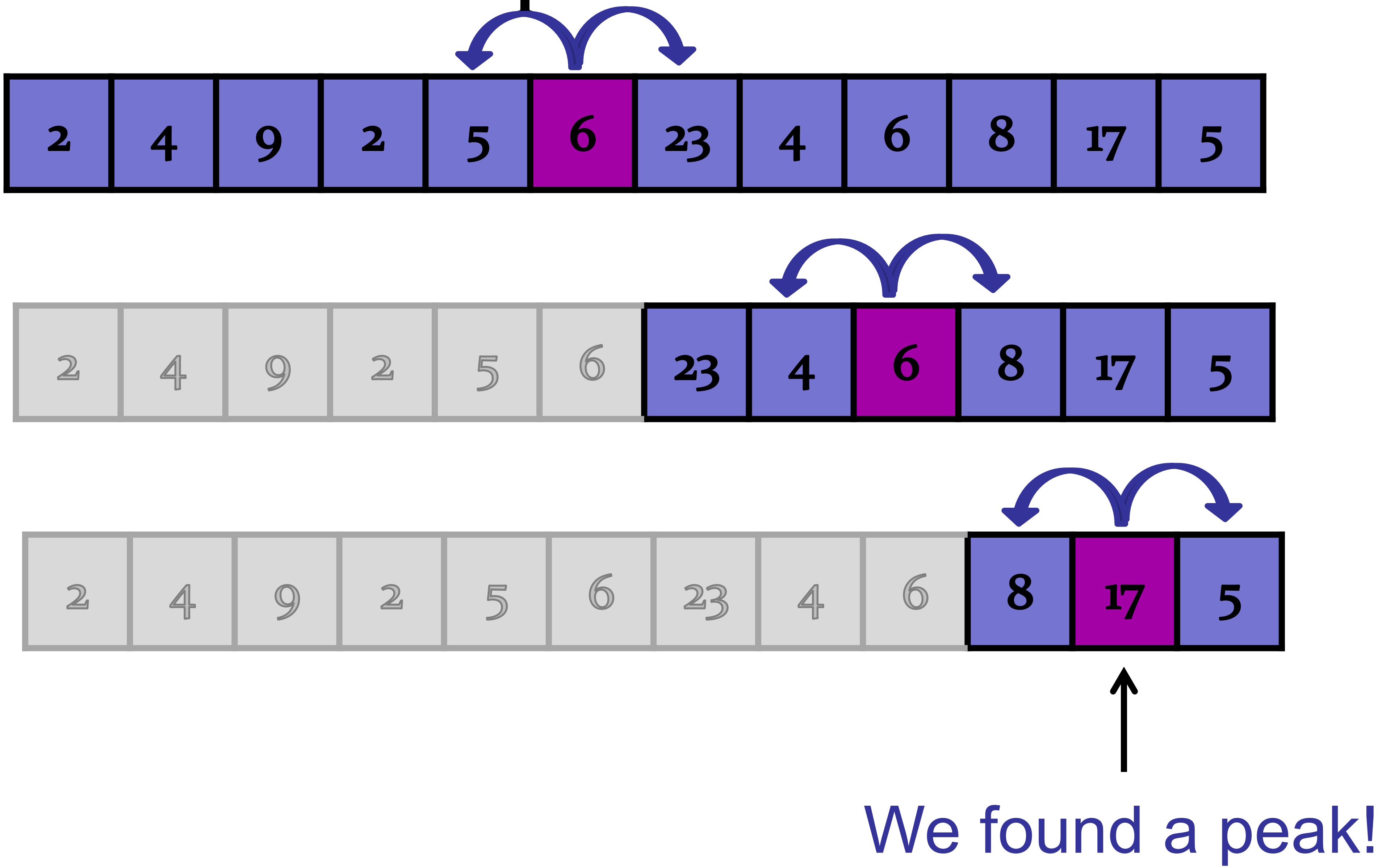
Peak Finding: Algorithm 2

Divide-and-Conquer



Peak Finding: Algorithm 2

Divide-and-Conquer



Peak Finding

Input: Some array $A[0..n-1]$

2	4	9	2	11	6	23	4	6	8	17	5

FindPeak(A, n)

if $A[n/2]$ is a peak **then return** $n/2$

else if $A[n/2+1] > A[n/2]$ **then**

Search for peak in right half.

else if $A[n/2-1] > A[n/2]$ **then**

Search for peak in left half.

Peak Finding

Input: Some array A[0..n-1]

2	4	9	2	11	6	23	4	6	8	17	5

FindPeak(A, n)

    if A[n/2] is a peak **then return** n/2

    else if A[n/2+1] > A[n/2] **then**

        FindPeak (A[n/2+1..n], n/2)

    else if A[n/2–1] > A[n/2] **then**

        FindPeak (A[1..n/2-1], n/2)

Peak Finding

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Is this correct?

2	4	9	2	11	6	23	4	6	8	17	5

Should this be >=?

```
FindPeak(A, n)
    if A[n/2] is a peak then return n/2
    else if A[n/2+1] > A[n/2] then
        FindPeak (A[n/2+1..n], n/2)
    else if A[n/2–1] > A[n/2] then
        FindPeak (A[1..n/2-1], n/2)
```

Missing else
condition?

Peak Finding

Should this be >=? No: recurse on the larger half.

2	4	9	2	11	6	23	4	6	8	17	5

FindPeak(A, n)

if A[n/2] is a peak **then return** n/2

else if A[n/2+1] > A[n/2] **then**

FindPeak (A[n/2+1..n], n/2)

else if A[n/2–1] > A[n/2] **then**

FindPeak (A[1..n/2-1], n/2)

Peak Finding

Missing else condition? No: else we have found a peak!

2	4	9	2	11	6	23	4	6	8	17	5

FindPeak(A, n)

    if A[n/2] is a peak **then return** n/2

    else if A[n/2+1] > A[n/2] **then**

        FindPeak (A[n/2+1..n], n/2)

    else if A[n/2–1] > A[n/2] **then**

        FindPeak (A[1..n/2-1], n/2)

Peak Finding

Missing else condition? No: else we have found a peak!

2	4	9	2	11	6	23	4	6	8	17	5

```
FindPeak(A, n)
    if A[n/2+1] > A[n/2] then
        FindPeak (A[n/2+1..n], n/2)
    else if A[n/2–1] > A[n/2] then
        FindPeak (A[1..n/2-1], n/2)
    else A[n/2] is a peak; return n/2
```

Peak Finding

Key property → invariant:

If we recurse in the right half, then there exists a peak in the right half.

2	4	9	2	5	6	23	4	6	8	17	5

Peak Finding

Key property:

- If we recurse in the right half, then there exists a peak in the right half.

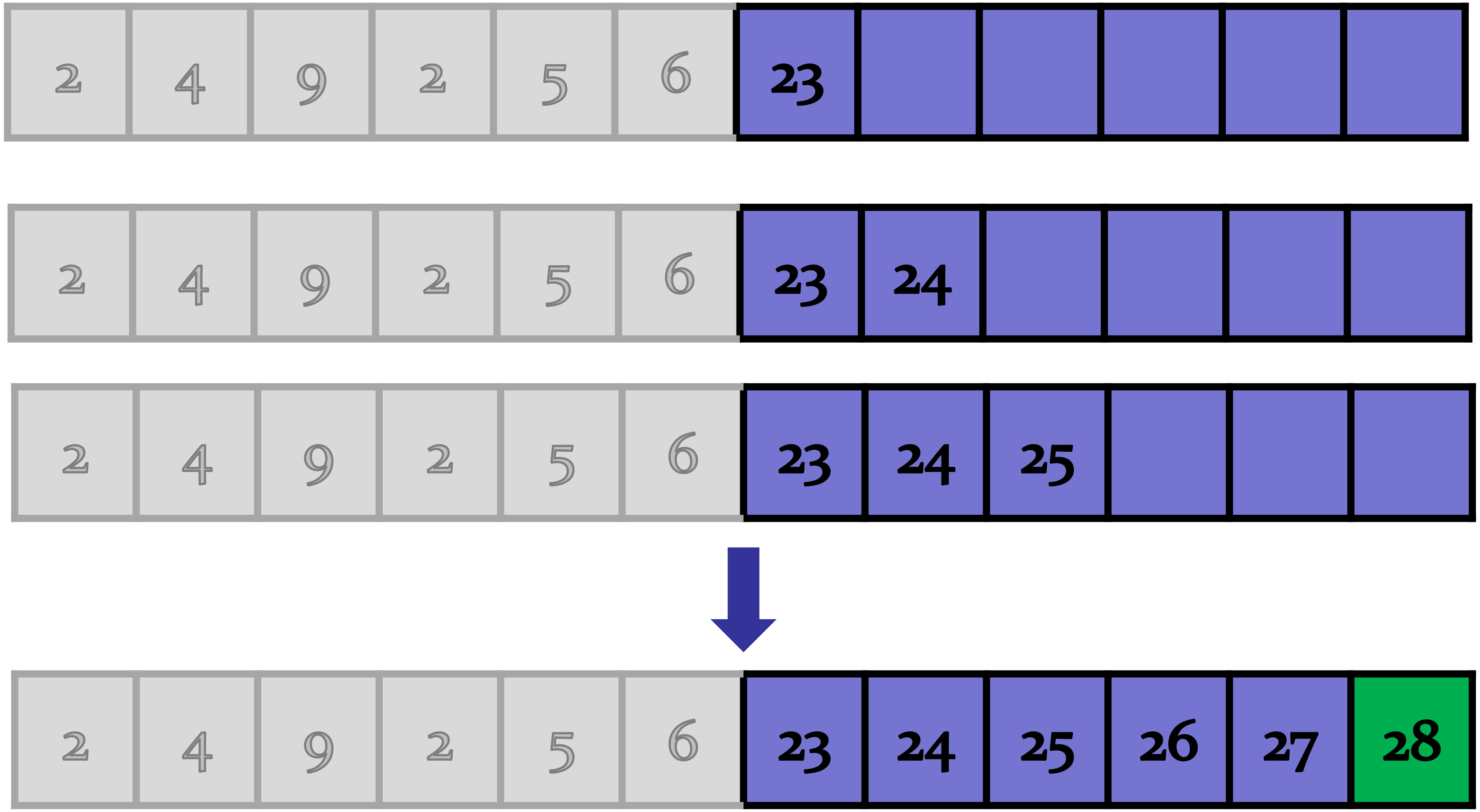
Explanation:

- Assume there is "no peak" in the right half.
- Given: A[middle] < A[middle + 1]
- Since no peaks, A[middle+1] < A[middle+2]
- Since no peaks, A[middle+2] < A[middle+3]
- …
- Since no peaks, A[n-1] < A[n] ⟵ PEAK!!

Peak Finding

Recurse on right half, since 23 > 6.

Assume no peaks in right half.



Peak Finding

Key property:

- If we recurse in the right half, then there exists a peak in the right half.

Explanation:

- Assume there is "no peak" in the right half.
- Because we recursed right: A[middle] < A[middle + 1]
- Since no peaks, A[middle+1] < A[middle+2]
- Since no peaks, A[middle+2] < A[middle+3]
- …
- Since no peaks, A[n-1] < A[n] ⟵ PEAK!!

Peak Finding

Key property:

- If we recurse in the right half, then there exists a peak in the right half.

Induction:

- Assume there is “no peak” in the right half.
- Inductive hypothesis:

  For all (j > middle): A[j-1] < j

Peak Finding

Key property:

- If we recurse in the right half, then there exists a peak in the right half.

Induction:

- Assume there is “no peak” in the right half.
- Inductive hypothesis:

  For all (j > middle): A[j-1] < j

- Base case: j = middle+1

  Because we recursed on the right half, we know that A[middle] < A[middle + 1].

Peak Finding

Key property:

- If we recurse in the right half, then there exists a peak in the right half.

Induction:

- Assume there is "no peak" in the right half.
- Inductive hypothesis:

  For all (j > middle): A[j-1] < j
- Induction: j > middle+1

  By induction, A[j-2] <= A[j-1].

  If A[j-1] >= A[j], then A[j-1] is a peak ➔ contradiction.

Peak Finding

Key property:

- If we recurse in the right half, then there exists a peak in the right half.

Induction:

- Assume there is “no peak” in the right half.
- Inductive hypothesis:

  For all (j > middle): $A[j-1] < j$

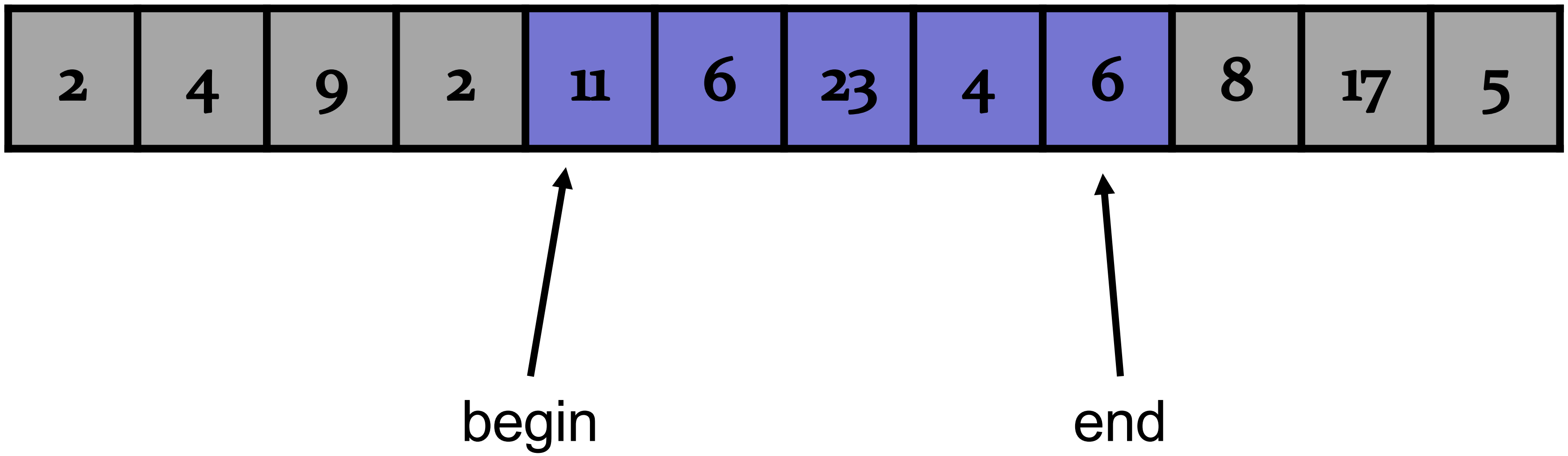
- Conclusion: $A[n-2] < A[n-1]$

  ➔ $A[n-1]$ is a peak ➔ contradiction.

Key Invariants:

Correctness:

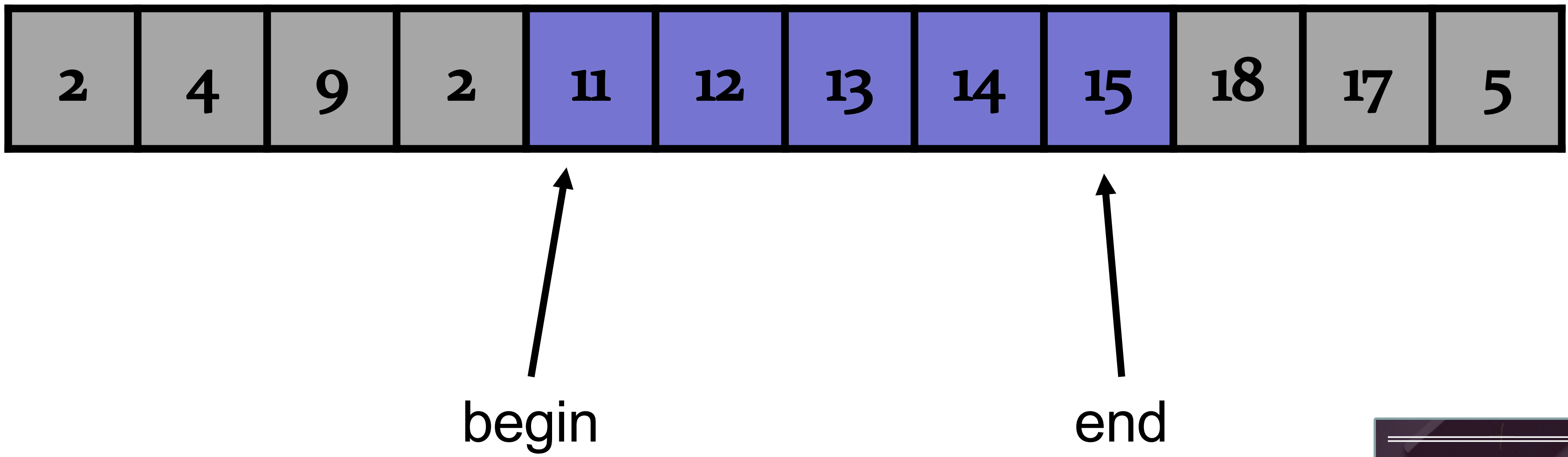
There exists a peak in the range [begin, end].



Key Invariants:

Is this good enough to prove the algorithm works?

There exists a peak in the range [begin, end].



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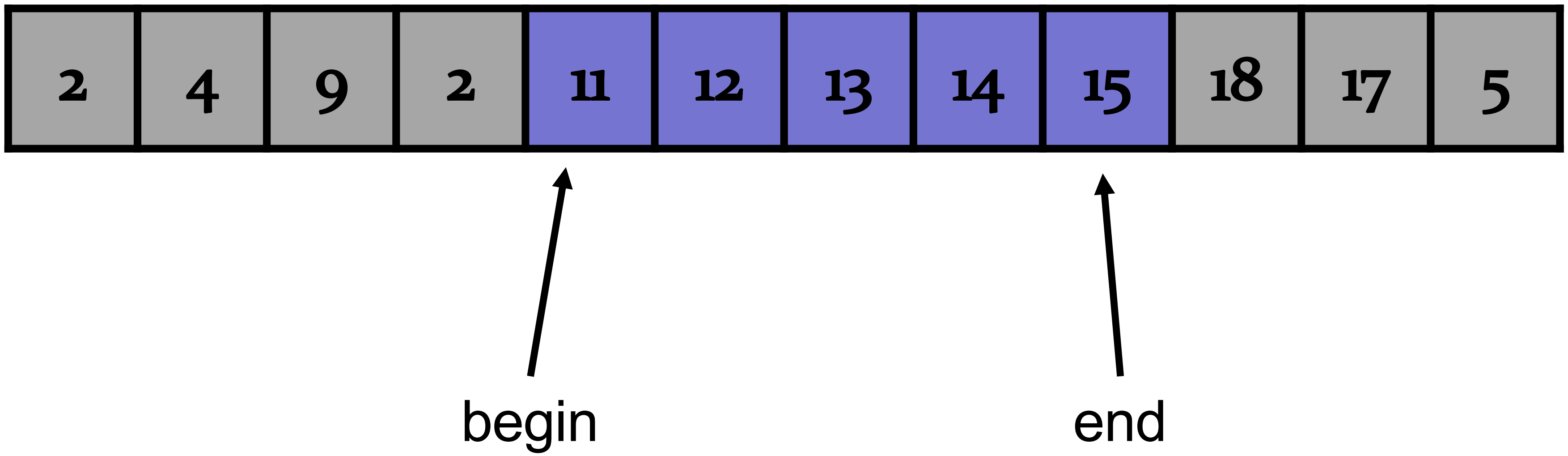
Key Invariants:

Not good enough to prove the algorithm works!

There exists a peak in the range [begin, end].



Key Invariants:

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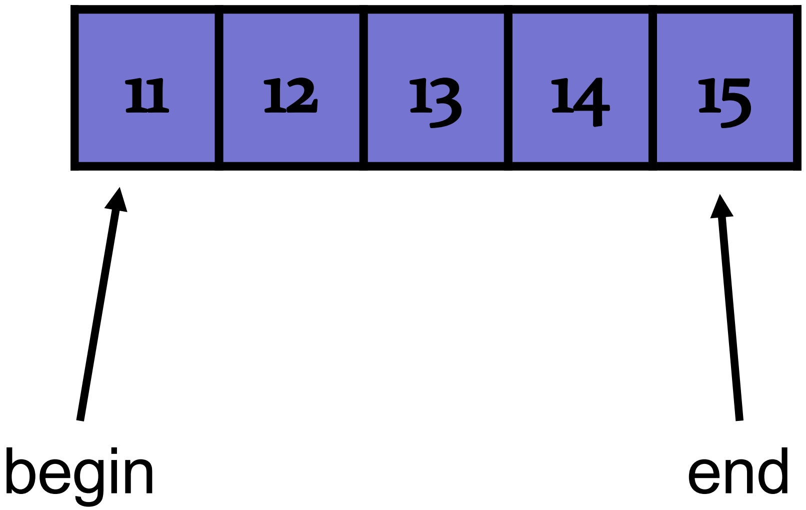
There exists a peak in the range [begin, end].



Key Invariants:

Not good enough to prove the algorithm works!

There exists a peak in the range [begin, end].

11	12	13	14	15

Run peak finding algorithm ➔ returns 15

Key Invariants:

Not good enough to prove the algorithm works!

There exists a peak in the range [begin, end].

2	4	9	2	11	12	13	14	15	18	17	5

Run peak finding algorithm → returns 15

But 15 is NOT a peak!

If the recursive call finds a peak, is it still a peak after the recursive call returns?

Key Invariants:

Correctness:

1. There exists a peak in the range [begin, end].

2. Every peak in[begin, end] is a peak in [1, n].

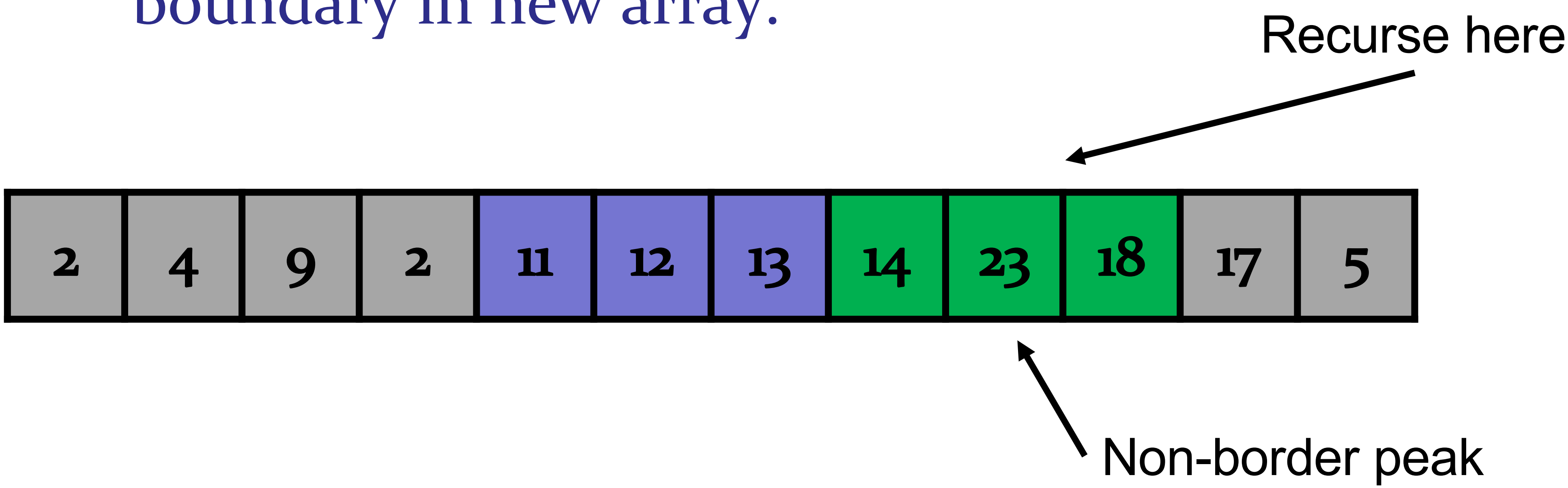
Peak Finding

Key property:

- If we recurse in the right half, then every peak in the right half is a peak in the array.

Proof: use the invariant (inductively)

- Immediately true for every peak that is not at a boundary in new array.



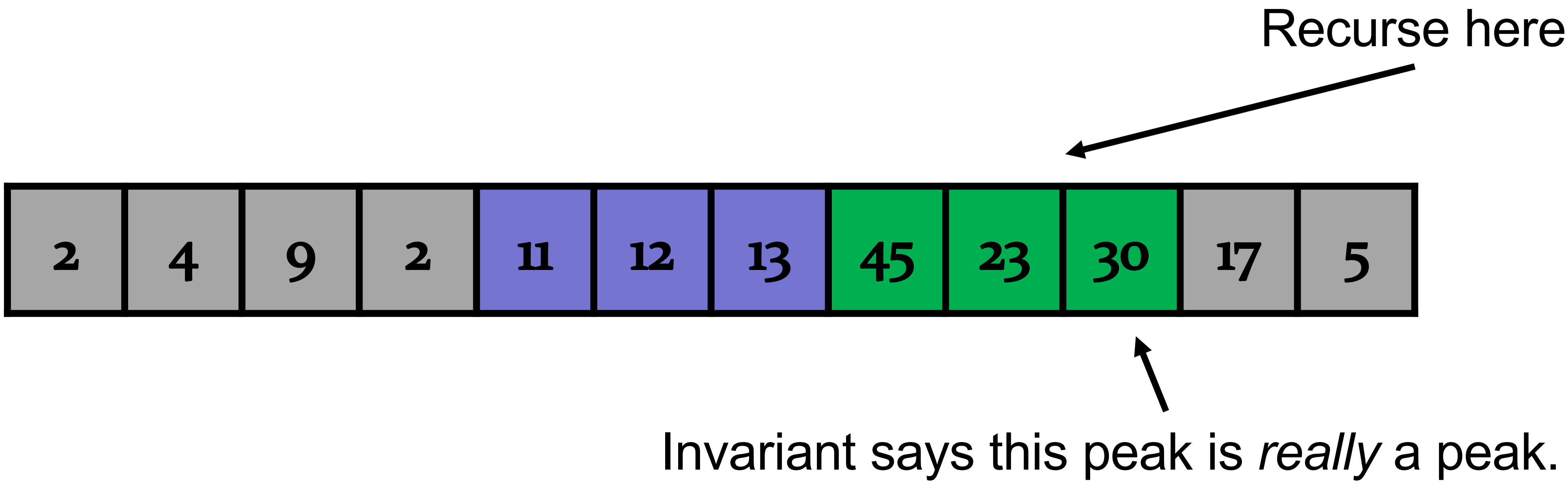
Peak Finding

Key property:

- If we recurse in the right half, then every peak in the right half is a peak in the array.

Proof: use the invariant (inductively)

- True by invariant for current array.



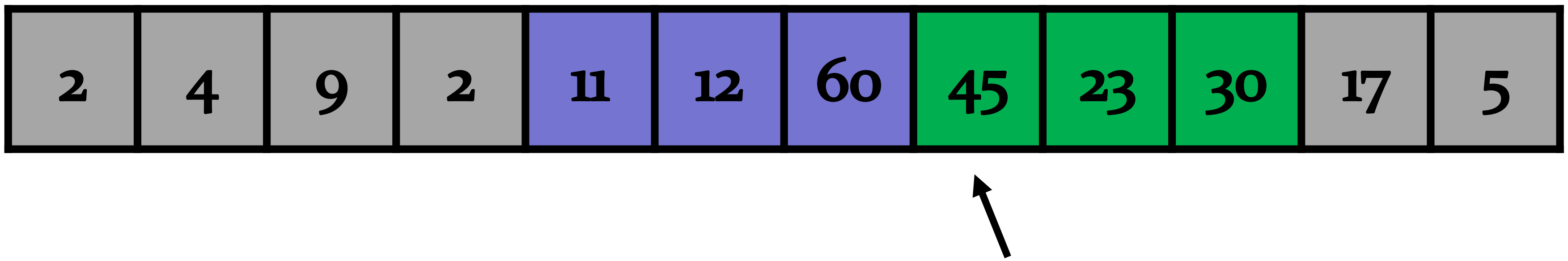
Peak Finding

Key property:

- If we recurse in the right half, then every peak in the right half is a peak in the array.

Proof: use the invariant (inductively)

- If 45 is a peak in the new array but not the old array, then we would not recurse on the right side.
  ➔ If left edge is a peak in new array, then it is a peak.



If 45 is a peak in right half and we recurse on right half, then it is a peak.

Key Invariants:

Correctness:

1. There exists a peak in the range [begin, end].

2. Every peak in[begin, end] is a peak in [1, n].

Peak Finding

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is open

Running time?

2	4	9	2	11	6	23	4	6	8	17	5

FindPeak(A, n)

    if A[n/2] is a peak **then return** n/2

    else if A[n/2+1] > A[n/2] **then**

        Search for peak in right half.

    else if A[n/2–1] > A[n/2] **then**

        Search for peak in left half.

Peak Finding

Running time:

Time for comparing A[n/2] with neighbors

Time to find a peak in an array of size n

Recursion

$$T(n) = T(n/2) + \theta(1)$$

Peak Finding

Running time:

Time for comparing A[n/2] with neighbors

Time to find a peak in an array of size n

Recursion

$$T(n) = T(n/2) + \theta(1)$$

Unrolling the recurrence:

$$T(n) = \theta(1) + \theta(1) + \ldots + \theta(1) = O(\log n)$$

Peak Finding

Unrolling the recurrence:

Rule:
T(X) = T(X/2) + O(1)

$$T(n) = T(n/2) + \theta(1)$$

$$= T(n/4) + \theta(1) + \theta(1)$$

$$= T(n/8) + \theta(1) + \theta(1) + \theta(1)$$

$$\ldots$$

$$\ldots$$

$$= T(1) + \theta(1) + \ldots + \theta(1) =$$

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Peak Finding

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$$\cdots$$

$$\cdots$$

$$= T(1) + \theta(1) + \ldots + \theta(1) =$$

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Number of times you can divide n by 2 until you reach 1.

Peak Finding

How many times can you divide a number n in half before you reach 1?

$$\underbrace{2 \text{ x } 2 \text{ x} \ldots \text{x } 2}_{\log(n)} = 2^{\log(n)} = n$$

Note: I always assume $\log = \log_2$

$$O(\log_2 n) = O(\log n)$$

Peak Finding

Running time:

Time to find a peak in an array of size n

Recursion

Time for comparing A[n/2] with neighbors

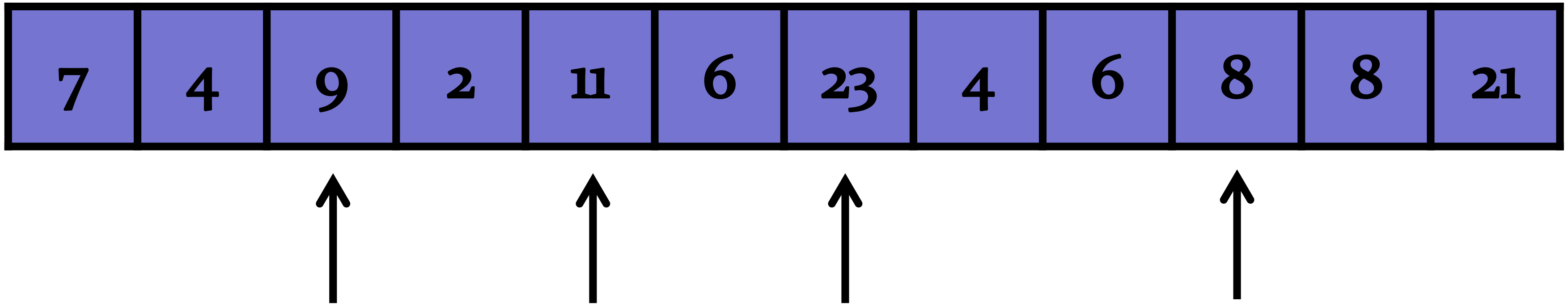
$$T(n) = T(n/2) + \theta(1)$$

Unrolling the recurrence:

$$T(n) = \underbrace{\theta(1) + \theta(1) + \ldots + \theta(1)}_{\log(n)} = O(\log n)$$

Peak Finding

Input: Some array A[0..n-1]



Output: a local maximum in A

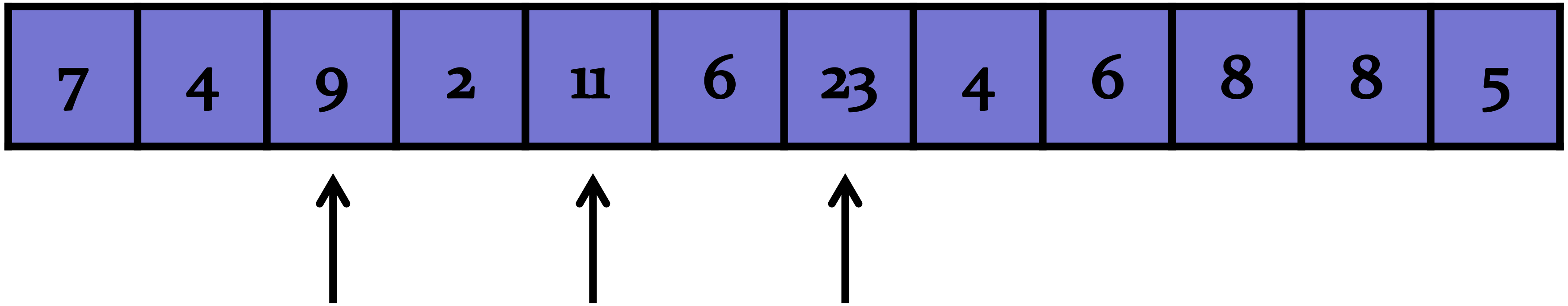
`A[i-1]` $\le$ `A[i]` **and** `A[i+1]` $\le$ `A[i]`

Assume that

`A[-1] = A[n] = -MAX_INT`

Steep Peaks

Input: Some array A[0..n-1]



Output: a local maximum in A

`A[i-1]<A[i]` **and** `A[i+1]<A[i]`

Assume that

`A[-1] = A[n] = -MAX_INT`

Steep Peaks

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is open

Input: Some array A[0..n-1]

7	4	9	2	11	6	23	4	6	8	8	5

Output: a local maximum in A

`A[i-1]` **<** `A[i]` **and** `A[i+1]` **<** `A[i]`

Can we find *steep* peaks efficiently (in O(log n) time) using the same approach?