Solutions to Tutorial 6

MA1521 Calculus for Computing

1. (a) Since $\lim_{n\to\infty} \cos\frac{1}{n} = 1 \neq 0$, the series is divergent by the divergence test or nth-term test.

   (b) We apply the integral test. $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{\ln x}\right]_2^b = \frac{1}{\ln 2}$. Thus the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converges.

   (c) As $\displaystyle\lim_{n\to\infty}\sqrt[n]{\left|\sin^n(\frac{1}{\sqrt{n}})\right|} = \lim_{n\to\infty}\left|\sin(\frac{1}{\sqrt{n}})\right| = 0 < 1$, the series $\displaystyle\sum_{n=1}^{\infty}\sin^n(\frac{1}{\sqrt{n}})$ converges by the root test.

   (d) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{1+n^2}}$ is an alternating series. For $n \geq 1$, $\displaystyle\frac{1}{\sqrt{1+n^2}} \geq \frac{1}{\sqrt{1+(n+1)^2}}$.

   Also $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{1+n^2}} = 0$. By the alternating series test, $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{1+n^2}}$ is convergent.

2. (a) Let $u_n = (-1)^n\frac{(x+2)^n}{n}$.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{n+1}\cdot\frac{n}{(x+2)^n}\right| = |x+2|.$$

   By ratio test, the power series converges absolutely for $|x+2| < 1$ and diverges for $|x+2| > 1$.

   Therefore the radius of convergence is 1.

   (b) Let $u_n = \frac{(3x)^n}{n!}$.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(n+1)!}\cdot\frac{n!}{(3x)^n}\right| = \lim_{n\to\infty}\left|\frac{3x}{n+1}\right| = 0.$$

   Since the limit is less than 1 for any x, so the radius of convergence is ∞.

   (c) Let $u_n = (nx)^n$.

$$\lim_{n\to\infty}\sqrt[n]{|u_n|} = \lim_{n\to\infty}\sqrt[n]{|(nx)^2|} = \lim_{n\to\infty} n|x| = \infty,$$

   for all $x \neq 0$. Therefore by root test, the radius of convergence is 0.

(d) Let $u_n = \frac{(4x-5)^{2n+1}}{n^{3/2}}$.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(4x-5)^{2n+3}}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{(4x-5)^{2n+1}}\right| = |4x-5|^2.$$

By ratio test, the power series converges absolutely for $|4x-5|^2 < 1 \Leftrightarrow |4x-5| < 1 \Leftrightarrow |x-\frac{5}{4}| < \frac{1}{4}$, and diverges for $|x-\frac{5}{4}| > \frac{1}{4}$.

Therefore, the radius of convergence is $\dfrac{1}{4}$.

3. Let $z \in (-R, R)$. Since the radius of convergence of $\displaystyle\sum_{n=1}^{\infty} a_n x^n$ is R, the series $\displaystyle\sum_{n=1}^{\infty} a_n z^n$ converges absolutely. That is $\displaystyle\sum_{n=1}^{\infty} |a_n z^n|$ converges. Consider the series $\displaystyle\sum_{n=1}^{\infty} b_n z^n$. We have $|b_n z^n| = |b_n||z^n| \le |a_n||z^n| = |a_n z^n|$ for all n. By comparison test, $\displaystyle\sum_{n=1}^{\infty} |b_n z^n|$ converges. Thus the radius of convergence of $\displaystyle\sum_{n=1}^{\infty} b_n x^n$ is at least R.

4. (a) $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}2^{n+1}}{a_n 2^n}\right| = 2 \times \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2$. Thus the radius of convergence of $\displaystyle\sum_{n=1}^{\infty} a_n 2^n x^n$ is $\frac{1}{2}$.

   (b) First rewrite the power series as $\displaystyle\sum_{n=1}^{\infty} a_n(-1)^n x^{2n} = \sum_{n=1}^{\infty} a_n(-1)^n (x^2)^n$.

   Then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}(-1)^{n+1}(x^2)^{n+1}}{a_n(-1)^n(x^2)^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x|^2 = |x|^2$. It follows by ratio test that the given series converges absolutely for $|x|^2 < 1$ and diverges for $|x|^2 > 1$. Or equivalently, the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. Therefore, the radius of convergence of the given power series is 1.

5. (a)

$$\frac{x}{1-x} = x\left(\frac{1}{1-x}\right) = x\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}, \quad |x| < 1.$$

   (b) Let $f(x) = \frac{1}{x^2}$.

   Then $f'(x) = -\dfrac{2}{x^3}$, $f''(x) = \dfrac{3!}{x^4}$, ... and in general $f^{(n)}(x) = (-1)^n \dfrac{(n+1)!}{x^{n+2}}$.

Therefore, $f^{(n)}(1) = (-1)^n(n+1)!$.

Thus the Taylor series of f at $x = 1$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n.$$

(c)

$$\begin{aligned}\frac{x}{1+x} &= \frac{(1+x)-1}{1+x} = 1 - \frac{1}{1+x} = 1 - \frac{1}{1+(x+2)-2} \\ &= 1 + \frac{1}{1-(x+2)} = 1 + \sum_{n=0}^{\infty}(x+2)^n = 2 + \sum_{n=1}^{\infty}(x+2)^n, \text{ for } |x+2| < 1.\end{aligned}$$

6. (i) We have

$$xe^x = x\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}.$$

Note that the radius of convergence of this power series is infinite. Integrating both sides from 0 to 1, we have

$$\int_0^1 xe^x\,dx = \sum_{n=0}^{\infty}\int_0^1 \frac{x^{n+1}}{n!}\,dx = \sum_{n=0}^{\infty}\frac{1}{n!(n+2)}.$$

On the other hand,

$$\int_0^1 xe^x\,dx = [xe^x]_0^1 - \int_0^1 e^x\,dx = e - (e-1) = 1.$$

So we conclude $S = \sum_{n=0}^{\infty} \frac{1}{n!(n+2)} = 1$.

(ii) We have

$$\frac{e^x - 1}{x} = \frac{\sum_{n=1}^{\infty}\frac{x^n}{n!}}{x} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty}\frac{x^n}{(n+1)!}.$$

Note that the radius of convergence of this power series is infinite. Differentiating both sides with respect to x, we have

$$\frac{xe^x - (e^x - 1)}{x^2} = \sum_{n=1}^{\infty}\frac{nx^{n-1}}{(n+1)!} = \sum_{n=0}^{\infty}\frac{(n+1)x^n}{(n+2)!} = \sum_{n=0}^{\infty}\frac{x^n}{n!(n+2)}.$$

The result now follows by setting $x = 1$.

Solutions to Further Exercises

1. The first term of the geometric series is $a = 1$ and the common ratio is $r = -\dfrac{(x-3)}{2}$.

   So the sum of the series is

   $$\frac{a}{1-r} = \frac{1}{1+(x-3)/2} = \frac{2}{x-1},$$

   provided $\left|\dfrac{(x-3)}{2}\right| < 1 \Leftrightarrow 1 < x < 5$.

2. First we calculate

   $\frac{1}{2}\int_0^1 t^{n-1}(1-t)^2\,dt$
   $= \frac{1}{2}\int_0^1 t^{n-1}\left(1-2t+t^2\right)dt$
   $= \frac{1}{2}\left[\frac{1}{n}t^n - \frac{2}{n+1}t^{n+1} + \frac{1}{n+2}t^{n+2}\right]_0^1$
   $= \frac{1}{n(n+1)(n+2)}$.

   Next we apply this result to $n = 1, 3, 5, 7, \cdots$. We find that

   $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \frac{1}{5\cdot 6\cdot 7} + \frac{1}{7\cdot 8\cdot 9} + \cdots$
   $= \frac{1}{2}\int_0^1 t^{1-1}(1-t)^2\,dt + \frac{1}{2}\int_0^1 t^{3-1}(1-t)^2\,dt + \frac{1}{2}\int_0^1 t^{5-1}(1-t)^2\,dt + \frac{1}{2}\int_0^1 t^{7-1}(1-t)^2\,dt + \cdots$
   $= \frac{1}{2}\int_0^1 (1-t)^2\left(1+t^2+t^4+t^6+\cdots\right)dt$
   $= \frac{1}{2}\int_0^1 (1-t)^2\left(\frac{1}{1-t^2}\right)dt$
   $= \frac{1}{2}\int_0^1 \frac{1-t}{1+t}\,dt$
   $= -\frac{1}{2}\int_0^1 \frac{1+t-2}{1+t}\,dt$
   $= -\frac{1}{2}\int_0^1 \left(1-\frac{2}{1+t}\right)dt$
   $= \ln 2 - \frac{1}{2}$.

3. Ramanujan's series is

   $$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!(1103+26390n)}{(n!)^4(396)^{4n}}.$$

   When $n = 0$, we only have the term $\frac{\sqrt{8}\times 1103}{9801}$. Thus $\pi \approx \frac{9801}{\sqrt{8}\times 1103} = 3.141592$, of which 6 digits after the decimal point are correct.

   When $n = 0$ and 1, we have a sum of two terms:

   $$\frac{\sqrt{8}}{9801}\left(1103 + \frac{4!(1103+26390)}{(396)^4}\right) = \frac{\sqrt{8}}{9801}\left(1103 + \frac{27493}{1024635744}\right) = \frac{\sqrt{8}\times 1130173253125}{9801\times 1024635744}.$$

Thus $\pi \approx \frac{9801\times 1024635744}{\sqrt{8}\times 1130173253125} = 3.14159265358979$, of which 14 digits after the decimal point are correct.

We apply the ratio test.

$$\lim_{n\to\infty}\left|\frac{\frac{(4(n+1))!(1103+26390(n+1))}{((n+1)!)^4(396)^{4(n+1)}}}{\frac{(4n)!(1103+26390n)}{(n!)^4(396)^{4n}}}\right| = \lim_{n\to\infty}\frac{(4n+4)(4n+3)(4n+2)(4n+1)(1103+26390(n+1))}{(n+1)^4(1103+26390n)(396)^4} = \frac{4^4}{(396)^4} = \frac{1}{99^4} < 1.$$

Thus the series converges absolutely.