MATLAB LESSON 2: VECTOR SPACES AND REDUCED ROW-ECHELON FORM

ABSTRACT. In this laboratory session, we will learn how to use the `rref` command to better understand and solve problems related to concepts in vector space, including linear combinations, linear spans, linear independence, bases and dimensions.

Type `format rat`. Throughout the entire worksheet, we will use the rational format to read the entries of matrices.

1. Linear Combinations

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. A vector $\boldsymbol{u} \in \mathbb{R}^n$ is a **linear combination** of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ if there exist numbers $c_1, c_2, \ldots, c_k$ such that

$$\boldsymbol{u} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k.$$

View each $\boldsymbol{u}_i$ and $\boldsymbol{v}$ as column vectors, and write $\boldsymbol{A} = \begin{pmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_k\end{pmatrix}$. Then $\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ if and only if the linear system $\boldsymbol{Ax} = \boldsymbol{v}$ is consistent.

For example, let $\boldsymbol{u}_1 = (1,0,1,2,3)$, $\boldsymbol{u}_2 = (2,1,-1,1,0)$, $\boldsymbol{u}_3 = (1,1,-2,-1,-3)$ and $\boldsymbol{u}_4 = (1,2,3,1,1)$. To see whether $\boldsymbol{u} = (2,0,0,1,0)$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$:

(i) Input $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$ and $\boldsymbol{u}$ as column vectors in MATLAB. For example,

```
>>  u1 = [1; 0; 1; 2; 3]
u1 =  1
      0
      1
      2
      3
```

(ii) Define the 5×4 matrix $\boldsymbol{A}$ whose columns are $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ and $\boldsymbol{u}_4$:

```
>>  A = [u1 u2 u3 u4]
A =  1    2     1    1
     0    1     1    2
     1   -1    -2    3
     2    1    -1    1
     3    0    -3    1
```

(iii) Find the reduced row-echelon form of the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{u})$ to check the consistency of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{u}$:

```
>> rref([A u])
ans =  1    0   -1    0    0
       0    1    1    0    0
       0    0    0    1    0
       0    0    0    0    1
       0    0    0    0    0
```

Since the last column of the reduced row-echelon form of $(\boldsymbol{A} \mid \boldsymbol{u})$ is pivot, the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{u}$ is inconsistent. Therefore, $\boldsymbol{u}$ is not a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$.

Repeat the same argument for $\boldsymbol{v} = (-2, -1, 1, -1, 0)$.

```
>> v = [-2; -1; 1; -1; 0]
v =  -2
     -1
     1
     -1
     0
>> rref([A v])
ans =  1    0   -1    0    0
       0    1    1    0   -1
       0    0    0    1    0
       0    0    0    0    0
       0    0    0    0    0
```

Since the last column of the reduced row-echelon form of $(\boldsymbol{A} \mid \boldsymbol{v})$ is non-pivot, the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{v}$ is consistent. Therefore, $\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$.

2. Linear Independence

Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a subset of vectors in $\mathbb{R}^n$. Then S is said to be **linearly independent** if the linear system $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}$ has only the trivial solution $c_1 = c_2 = \cdots = c_k = 0$.

View each $\boldsymbol{v}_i$ and $\boldsymbol{0}$ as column vectors, and write $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_k \end{pmatrix}$. Then S is linearly independent if and only if the homogeneous linear system $\boldsymbol{B}\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution.

For example, let $\boldsymbol{v}_1 = (1, 0, 2, 0, 3)$, $\boldsymbol{v}_2 = (1, 1, 0, 2, 2)$, $\boldsymbol{v}_3 = (1, -3, 8, -6, 6)$, $\boldsymbol{v}_4 = (1, 2, 3, 4, 1)$, $\boldsymbol{v}_5 = (0, -1, 1, -2, 1)$, $\boldsymbol{v}_6 = (1, 1, 1, 1, 1)$.

(i) Input $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_6$ and $\boldsymbol{w} = \mathbf{0}$ as column vectors in MATLAB. For example,

```
>> v1 = [1; 0; 2; 0; 3]
v1 =  1
      0
      2
      0
      3
```

(ii) Define the 5×6 matrix $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 & \boldsymbol{v}_5 & \boldsymbol{v}_6 \end{pmatrix}$.

```
>> B = [v1 v2 v3 v4 v5 v6]
B =  1    1    1    1    0    1
     0    1   -3    2   -1    1
     2    0    8    3    1    1
     0    2   -6    4   -2    1
     3    2    6    1    1    1
```

(iii) Find the reduced row-echelon form of the augmented matrix $(\boldsymbol{B} \mid \mathbf{0})$ of the homogeneous linear system $\boldsymbol{B}\boldsymbol{x} = \mathbf{0}$. (Recall that $\boldsymbol{w} = \mathbf{0}$ is defined in Step (i).)

```
>> rref([B w])
ans =  1    0    4    0    4/5    0    0
       0    1   -3    0   -3/5    0    0
       0    0    0    1   -1/5    0    0
       0    0    0    0    0      1    0
       0    0    0    0    0      0    0
```

Since the last column is non-pivot and the 3rd and 5th columns are non-pivot, the homogeneous linear system $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ has infinitely many non-trivial solutions (with 2 arbitrary parameters). As a conclusion, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6\}$ is a linearly dependent set.

Note that if $\boldsymbol{R}$ is the reduced row-echelon form of $\boldsymbol{B}$, then $(\boldsymbol{R} \mid \mathbf{0})$ is the reduced row-echelon form of $(\boldsymbol{B} \mid \mathbf{0})$, and vice versa. Therefore, we can drop the last zero column and simply check whether the reduced row echelon form $\boldsymbol{R}$ of $\boldsymbol{B}$ has non-pivot columns:

```
>> R = rref(B)
R =  1    0    4    0    4/5    0
     0    1   -3    0   -3/5    0
     0    0    0    1   -1/5    0
     0    0    0    0    0      1
     0    0    0    0    0      0
```

Since the 3rd and 5th columns of the reduced row-echelon form of $\boldsymbol{B}$ are non-pivot, the homogeneous linear system $\boldsymbol{Bx} = \boldsymbol{0}$ has infinitely many non-trivial solutions. We also conclude that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6\}$ is a linearly dependent set.

3. Redundant Vectors

Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a set of vectors in $\mathbb{R}^n$, and let $V = \text{span}(S)$. If S is linearly independent, then S is a **basis** for V. If S is linearly dependent, then some of the vectors in S are redundant to generate the vector space V.

For example, let $\boldsymbol{v}_1 = (1,0,2,0,3)$, $\boldsymbol{v}_2 = (1,1,0,2,2)$, $\boldsymbol{v}_3 = (1,-3,8,-6,6)$, $\boldsymbol{v}_4 = (1,2,3,4,1)$, $\boldsymbol{v}_5 = (0,-1,1,-2,1)$, $\boldsymbol{v}_6 = (1,1,1,1,1)$ as in Section 2.

View each $\boldsymbol{v}_i$ as column vectors, and write $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 & \boldsymbol{v}_5 & \boldsymbol{v}_6 \end{pmatrix}$. We have found the reduced row-echelon form $\boldsymbol{R}$ of $\boldsymbol{B}$:

```
>>  R

R =  1     0     4      0     4/5    0

     0     1     -3    0     -3/5    0

     0     0     0      1     -1/5    0

     0     0     0      0       0    1

     0     0     0      0       0    0
```

Since the 3rd and 5th columns of $\boldsymbol{R}$ are non-pivot; $\boldsymbol{v}_3$ and $\boldsymbol{v}_5$ are redundant vectors to span V, i.e., $V = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4, \boldsymbol{v}_6\}$.

Moreover, by observing the entries in the 3rd column $\begin{pmatrix} 4 \\ -3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and 5th column $\begin{pmatrix} 4/5 \\ -3/5 \\ -1/5 \\ 0 \\ 0 \end{pmatrix}$ of $\boldsymbol{R}$,

$$\boldsymbol{v}_3 = 4\boldsymbol{v}_1 - 3\boldsymbol{v}_2 \quad \text{and} \quad \boldsymbol{v}_5 = \tfrac{4}{5}\boldsymbol{v}_1 - \tfrac{3}{5}\boldsymbol{v}_2 - \tfrac{1}{5}\boldsymbol{v}_4.$$

Verify the above relations:

```
>>  v3 - (4*v1 - 3*v2)

ans =  0

       0

       0

       0
```

```
          0

>>  v5 - (4/5*v1 - 3/5*v2 - 1/5*v4)

ans =   -1/18014398509481984

          0

          0

          0

          -1/2251799813685248
```

The 1^{st} and 5^{th} entries are supposed to be 0. The nonzero values displayed are due to rounding errors.

4. Linear Spans

Let $S = \{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\}$ and $T = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_l\}$ be subsets of vectors in $\mathbb{R}^n$.

Let $U = \text{span}(S)$ and $V = \text{span}(V)$. Then

(i) $U \subseteq V$ if and only if every vector in S is a linear combination of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_l$.
(ii) $V \subseteq U$ if and only if every vector in T is a linear combination of $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k$.

For example, let $U = \text{span}\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\}$ and $V = \text{span}\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4\}$, where

$$\boldsymbol{c}_1 = (1,1,2,2,3), \quad \boldsymbol{c}_2 = (1,0,2,0,3), \quad \boldsymbol{c}_3 = (1,1,1,1,1),$$

and

$$\boldsymbol{d}_1 = (3,2,5,3,7), \quad \boldsymbol{d}_2 = (0,0,1,1,2), \quad \boldsymbol{d}_3 = (2,2,1,1,0), \quad \boldsymbol{d}_4 = (1,-1,3,-1,5).$$

(i) Input $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ and $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$ as column vectors in MATLAB. For example,

```
>>  c1 = [1; 1; 2; 2; 3]

c1 =   1

       1

       2

       2

       3
```

(ii) Form the matrices $\boldsymbol{C} = \begin{pmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \boldsymbol{c}_3 \end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3 & \boldsymbol{d}_4 \end{pmatrix}$. For example,

```
>>  C = [c1 c2 c3]

C =   1     1     1

      1     0     1

      2     2     1

      2     0     1

      3     3     1
```

(iii) In order to check whether $V \subseteq U$, we shall check if each $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$ is a linear combination of $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$; i.e., if the linear systems $\boldsymbol{C}\boldsymbol{x} = \boldsymbol{d}_i$, $i = 1, 2, 3, 4$, are consistent.

One may check one by one. Alternatively, consider $(\boldsymbol{C} \mid \boldsymbol{D}) = (\boldsymbol{C} \mid \boldsymbol{d}_1 \mid \boldsymbol{d}_2 \mid \boldsymbol{d}_3 \mid \boldsymbol{d}_4)$:

```
>>  rref([C D])
ans =  1    0    0    1    1    -1    0
       0    1    0    1    0     0    2
       0    0    1    1   -1     3   -1
       0    0    0    0    0     0    0
       0    0    0    0    0     0    0
```

The columns corresponding to $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$ are all non-pivot. So $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$ are linear combinations of $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$. In fact, observing the entries of the column vectors,

$$\boldsymbol{d}_1 = \boldsymbol{c}_1 + \boldsymbol{c}_2 + \boldsymbol{c}_3, \quad \boldsymbol{d}_2 = \boldsymbol{c}_1 - \boldsymbol{c}_3, \quad \boldsymbol{d}_3 = -\boldsymbol{c}_1 + 3\boldsymbol{c}_3, \quad \boldsymbol{d}_4 = 2\boldsymbol{c}_2 - \boldsymbol{c}_3.$$

Hence, $V \subseteq U$.

(iv) In order to check whether $U \subseteq V$, we shall check if each $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ is a linear combination of $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$. Similarly, consider $(\boldsymbol{D} \mid \boldsymbol{C}) = (\boldsymbol{D} \mid \boldsymbol{c}_1 \mid \boldsymbol{c}_2 \mid \boldsymbol{c}_3)$:

```
>>  rref([D C])
ans =  1    0    0    2        0      1      0
       0    1    0    -9/2    3/2    -2     1/2
       0    0    1    -5/2    1/2    -1     1/2
       0    0    0    0        0      0      0
       0    0    0    0        0      0      0
```

The columns corresponding to $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ are all non-pivot. So $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ are linear combinations of $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$. In fact,

$$\boldsymbol{c}_1 = \tfrac{3}{2}\boldsymbol{d}_2 + \tfrac{1}{2}\boldsymbol{d}_3, \quad \boldsymbol{c}_2 = \boldsymbol{d}_1 - 2\boldsymbol{d}_2 - \boldsymbol{d}_3, \quad \boldsymbol{c}_3 = \tfrac{1}{2}\boldsymbol{d}_2 + \tfrac{1}{2}\boldsymbol{d}_3.$$

Hence, $U \subseteq V$. We conclude that $U = V$.

Suppose we use the same U and V except $\boldsymbol{d}_4$ is replaced by $\boldsymbol{e}_4 = (1, -1, 3, -1, 0)$.

(i) Input $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ and $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{e}_4$ as column vectors. In fact, we just need to define $\boldsymbol{e}_4$:

```
>>  e4 = [1; -1; 3; -1; 0]
e4 =  1
     -1
      3
     -1
      0
```

(ii) Form the matrices $\boldsymbol{C} = \begin{pmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \boldsymbol{c}_3 \end{pmatrix}$ and $\boldsymbol{E} = \begin{pmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3 & \boldsymbol{e}_4 \end{pmatrix}$. Indeed, we just need to define $\boldsymbol{E}$:

```
>>  E = [d1 d2 d3 e4]
E =  3     0     2     1
     2     0     2    -1
     5     1     1     3
     3     1     1    -1
     7     2     0     0
```

(iii) Check the consistency of $\boldsymbol{Cx} = \boldsymbol{d}_i$, $i = 1,2,3$, and $\boldsymbol{Cx} = \boldsymbol{e}_4$:

```
>>  rref([C E])
ans =  1     0     0     1     1    -1     0
       0     1     0     1     0     0     0
       0     0     1     1    -1     3     0
       0     0     0     0     0     0     1
       0     0     0     0     0     0     0
```

Since the column corresponding to $\boldsymbol{e}_4$ is pivot, the system $\boldsymbol{Cx} = \boldsymbol{e}_4$ is inconsistent; so $\boldsymbol{e}_4 \notin \operatorname{span}\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\} = U$. Consequently, $V \not\subseteq U$.

(iv) Check the consistency of $\boldsymbol{Ex} = \boldsymbol{c}_i$, $i = 1,2,3$.

```
>>  rref([E C])
ans =  1     0     0     0     0      1      0
       0     1     0     0     3/2   -2      1/2
       0     0     1     0     1/2   -1      1/2
       0     0     0     1     0      0      0
       0     0     0     0     0      0      0
```

Since the columns corresponding to $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ are all non-pivot, the systems $\boldsymbol{Ex} = \boldsymbol{c}_i$, $i = 1,2,3$, are all consistent; so $\boldsymbol{c}_i \in \operatorname{span}\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{e}_4\} = V$, $i = 1,2,3$. Consequently, $U \subseteq V$.

5. Base and Dimensions

Let S be a subset of vectors in $\mathbb{R}^n$. Then S is a **basis** for a vector space V if (i) $V = \operatorname{span}(S)$ and (ii) S is linearly independent. For this case, the number of vectors in S, $|S|$, is called the **dimension** of V, denoted by $\dim(V)$.

5.1. Find Basis from Generating Set.

Let $S = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4\}$, where

$$\boldsymbol{g}_1 = (1,1,1,1,1), \quad \boldsymbol{g}_2 = (1,-1,2,3,0), \quad \boldsymbol{g}_3 = (-1,-3,0,1,-2), \quad \boldsymbol{g}_4 = (0,1,1,-1,-1).$$

Let $V = \text{span}(S)$. Then S is a basis for V if and only if S is linearly independent.

(i) Input $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4$ as column vectors in MATLAB.

(ii) Define the matrix $\boldsymbol{G} = \begin{pmatrix} \boldsymbol{g}_1 & \boldsymbol{g}_2 & \boldsymbol{g}_3 & \boldsymbol{g}_4 \end{pmatrix}$:

```
>>  G = [g1 g2 g3 g4]
G =  1    1    -1    0
     1   -1    -3    1
     1    2     0    1
     1    3     1   -1
     1    0    -2   -1
```

(iii) Find the reduced row-echelon form of $\boldsymbol{G}$:

```
>>  rref(G)
ans =  1    0    -2    0
       0    1     1    0
       0    0     0    1
       0    0     0    0
       0    0     0    0
```

The 1^{st}, 2^{nd} and 4^{th} columns are pivot, while the 3^{rd} column is non-pivot. We conclude that

(i) $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_4\}$ is a basis for V.

(ii) $\dim(V) = 3$.

Moreover, by observing the entries of the 3^{rd} column, $\boldsymbol{g}_3 = -2\boldsymbol{g}_1 + \boldsymbol{g}_2$.

5.2. **Check Basis.**

If $V = \text{span}(S)$, in order to check whether a set of vectors T is a basis for V, we shall verify:

(i) T is linearly independent;

(ii) $V \subseteq \text{span}(T)$, i.e., every vector in S is a linear combination of vectors in T; and

(iii) $\text{span}(T) \subseteq V$, i.e., every vector in T is a linear combination of vectors in S.

We use the set S and vector space V as in Section 5.1:

Let $V = \text{span}(S)$, where $S = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4\}$,

$$\boldsymbol{g}_1 = (1,1,1,1,1), \quad \boldsymbol{g}_2 = (1,-1,2,3,0), \quad \boldsymbol{g}_3 = (-1,-3,0,1,-2), \quad \boldsymbol{g}_4 = (0,1,1,-1,-1).$$

Set $T = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3\}$, where

$$\boldsymbol{h}_1 = (2,0,3,4,1), \quad \boldsymbol{h}_2 = (1,0,3,2,-1), \quad \boldsymbol{h}_3 = (1,2,2,0,0).$$

In the following, we check whether T is a basis for V:

(i) Input $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$ as column vectors, and define $\boldsymbol{H} = \begin{pmatrix} \boldsymbol{h}_1 & \boldsymbol{h}_2 & \boldsymbol{h}_3 \end{pmatrix}$. Find the reduced row-echelon form of $\boldsymbol{H}$:

```
>> rref(H)
ans =  1    0    0
       0    1    0
       0    0    1
       0    0    0
       0    0    0
```

Since the columns are all pivot, $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$ are linearly independent.

(ii) In order to check the consistency of $\boldsymbol{H}\boldsymbol{x} = \boldsymbol{g}_i$, $i = 1, \ldots, 4$, we find the reduced row-echelon form of $(\boldsymbol{H} \mid \boldsymbol{g}_1 \mid \boldsymbol{g}_2 \mid \boldsymbol{g}_3 \mid \boldsymbol{g}_4) = (\boldsymbol{H} \mid \boldsymbol{G})$:

```
>> rref([H G])
ans =  1    0    0    1/2     1/2    -1/2    -1/2
       0    1    0    -1/2    1/2     3/2     1/2
       0    0    1    1/2    -1/2    -3/2     1/2
       0    0    0      0       0       0       0
       0    0    0      0       0       0       0
```

Since the columns corresponding to $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4$ are all non-pivot, each $\boldsymbol{g}_i$ is a linear combination of $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$. Hence, $V = \text{span}(S) \subseteq \text{span}(T)$.

(iii) In order to check the consistency of $\boldsymbol{G}\boldsymbol{x} = \boldsymbol{h}_i$, $i = 1, 2, 3$, we find the reduced row-echelon form of $(\boldsymbol{G} \mid \boldsymbol{h}_1 \mid \boldsymbol{h}_2 \mid \boldsymbol{h}_3) = (\boldsymbol{G} \mid \boldsymbol{H})$:

```
>> rref([G H])
ans =  1    0    -2    0    1    0    1
       0    1    1    0    1    1    0
       0    0    0    1    0    1    1
       0    0    0    0    0    0    0
       0    0    0    0    0    0    0
```

Since the columns corresponding to $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$ are all non-pivot, each $\boldsymbol{h}_i$ is a linear combination of $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4$. Hence, $\text{span}(T) \subseteq \text{span}(S)$.

Therefore, we conclude that T is a basis for V.

5.3. **Find Coordinate Vector.**

Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a basis for a vector space V. Then every vector in V can be uniquely represented as a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$. Precisely, for any $\boldsymbol{v} \in V$, there exist unique numbers $c_1, c_2, \ldots, c_k \in \mathbb{R}$ such that

$$\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k.$$

Then the column vector $(c_1, c_2, \ldots, c_k)$ is called the **coordinate vector** of $\boldsymbol{v}$ relative to S, denoted by $(\boldsymbol{v})_S$.

Using the same definition as in Sections 5.1 and 5.2, $T = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3\}$ is a basis for V. In the following, we find the coordinate vector of $\boldsymbol{h} = (-1, -3, 0, 1, -2)$ relative to S.

(i) Input $\boldsymbol{h}$ as a column vector in MATLAB.

```
>>  h = [-1; -3; 0; 1; -2]
h =   -1
      -3
      0
      1
      -2
```

(ii) Solve the linear system $\boldsymbol{Hx} = \boldsymbol{h}$ (recall that $\boldsymbol{H} = \begin{pmatrix}\boldsymbol{h}_1 & \boldsymbol{h}_2 & \boldsymbol{h}_3\end{pmatrix}$).

```
>>  rref([H h])
ans =   1    0    0    -1/2
        0    1    0    3/2
        0    0    1    -3/2
        0    0    0    0
        0    0    0    0
```

Observing the entries in the column corresponding to $\boldsymbol{h}$, we obtain $\boldsymbol{h} = -\frac{1}{2}\boldsymbol{h}_1 + \frac{3}{2}\boldsymbol{h}_2 - \frac{3}{2}\boldsymbol{h}_3$. Hence, $(\boldsymbol{h})_T = \left(-\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}\right)$.

6. Practices

1. Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5, \boldsymbol{u}_6\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6\}$, where

$$\begin{aligned} \boldsymbol{u}_1 &= (1,1,1,1,1,1), & \boldsymbol{u}_2 &= (1,0,1,0,1,0), & \boldsymbol{u}_3 &= (1,1,1,0,0,0), \\ \boldsymbol{u}_4 &= (1,1,0,0,1,1), & \boldsymbol{u}_5 &= (1,1,0,1,1,0), & \boldsymbol{u}_6 &= (1,0,0,1,0,0), \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{v}_1 &= (1,-1,2,0,1,2), & \boldsymbol{v}_2 &= (-1,2,0,1,2,1), & \boldsymbol{v}_3 &= (2,-3,2,-1,-1,1), \\ \boldsymbol{v}_4 &= (0,1,2,1,-1,2), & \boldsymbol{v}_5 &= (1,2,1,-1,2,0), & \boldsymbol{v}_6 &= (1,3,3,0,1,2). \end{aligned}$$

(i) Determine the relation between $\text{span}(S)$ and $\text{span}(T)$, i.e., whether (a) $\text{span}(S) \subseteq \text{span}(T)$ and (b) $\text{span}(T) \subseteq \text{span}(S)$.

(ii) Let $V = \text{span}(T)$. Find a basis T' for V consisting of vectors in T. Express the redundant vectors in T as linear combinations of vectors in T'. Moreover, write down their coordinate vectors with respect to T'.

(iii) Determine whether S is a basis for $\mathbb{R}^6$.