25. For each of the sets S_1 to S_6 in Question 3.9, determine whether the set is linearly independent

$S_1 = \{(1,1,-1), (-2,2,1)\}$.

$S_2 = \{(1,1,-1), (-2,-2,2)\}$.

$S_3 = \{(1,1,-1), (-2,2,1), (1,5,-2)\}$.

$S_4 = \{(1,1,-1), (-2,2,1), (4,0,3)\}$.

$S_5 = \{(1,1,-1), (-2,2,1), (1,5,-2)\,(0,8,-2)\}$.

$S_6 = \{(1,1,-1), (-2,2,1), (4,0,3), (2,6,-3)\}$.

S_1 and S_2 are linearly independent while S_2, S_3, S_5, S_6 are linearly dependent

Example: Let $S_1 = \{(1,1,-1), (-2,2,1)\}$

$$c_1(1,1,-1) + c_2(-2,2,1) = 0$$

$$\begin{pmatrix} 1 & -2 \\ 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 1 & 2 \\ -1 & 1 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2-R_1} \begin{pmatrix} 1 & -2 \\ 0 & 4 \\ 0 & -1 \end{pmatrix} \xrightarrow{R_3+\frac{1}{4}R_2} \begin{pmatrix} 1 & -2 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$$

All the columns are pivot. The system has only the trivial solution.
Therefore, S_1 is a linearly independent set.

27. In Question 3.13, suppose $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent vectors in $\mathbb{R}^n$. Determine which of the sets S_1 to S_5 are linearly independent.

$S_1 = \{\boldsymbol{u}, \boldsymbol{v}\}$, $S_2 = \{\boldsymbol{u}-\boldsymbol{v}, \boldsymbol{v}-\boldsymbol{w}, \boldsymbol{w}-\boldsymbol{u}\}$, $S_3 = \{\boldsymbol{u}-\boldsymbol{v}, \boldsymbol{v}-\boldsymbol{w}, \boldsymbol{u}+\boldsymbol{w}\}$,
$S_4 = \{\boldsymbol{u}, \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}\}$, $S_5 = \{\boldsymbol{u}+\boldsymbol{v}, \boldsymbol{v}+\boldsymbol{w}, \boldsymbol{u}+\boldsymbol{w}, \boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}\}$.

Checking S_1:

$au + bv = 0 \iff au + bv + 0w = 0$

only trivial solution

Since u, v, w are linearly independent, we have $a=0, b=0$

Thus S_1 is linearly independent

Checking S_2: $a(u-v) + b(v-w) + c(w-u) = 0$

When $a=1, b=1, c=1$

$u-v+v-w+w-u = 0$ hence the system has a non trivial solution.

Therefore S_2 is linearly dependent.

Checking S_3: $a(u-v) + b(v-w) + c(u+w) = 0$

$au - va + bv - wb + cu + cw = 0$

$u(a+c) + v(-a+b) + w(-b+c) = 0$

Since u, v, w are linearly independent

$$\begin{cases} a + c = 0 \\ -a + b = 0 \\ -b + c = 0 \end{cases} \quad \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$a=0, b=0, c=0$

The system has only the trivial solution.
Thus S_3 is linearly independent

29. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\mathbb{R}^3$ such that $V = \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ and $W = \text{span}\{\boldsymbol{u}, \boldsymbol{w}\}$ are planes in $\mathbb{R}^3$. Find $V \cap W$ if

(a) $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent.

(b) $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are not linearly independent.

a) If u, v, w are linearly independent, then the two planes intersect at the line spanned by u and hence $V \cap W = \text{Span}(u)$

b) V and W are planes in $\mathbb{R}^3$. So, u, v are linearly independent and u, v are linearly independent.
If u, v, w are linearly dependent, then u, v, w must lie on the same plane and hence $V = W = V \cap W$

30. (All vectors in this question are written as column vectors.) Let $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ be vectors in $\mathbb{R}^n$ and $\boldsymbol{P}$ a square matrix or order n.

(a) Show that if $\boldsymbol{Pu_1}, \boldsymbol{Pu_2}, \ldots, \boldsymbol{Pu_k}$ are linearly independent, then $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ are linearly independent.

(b) Suppose $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ are linearly independent.

(i) Show that if $\boldsymbol{P}$ is invertible, then $\boldsymbol{Pu_1}, \boldsymbol{Pu_2}, \ldots, \boldsymbol{Pu_k}$ are linearly independent.

(ii) If $\boldsymbol{P}$ is not invertible, are $\boldsymbol{Pu_1}, \boldsymbol{Pu_2}, \ldots, \boldsymbol{Pu_k}$ linearly independent?

a) $c_1 u_1 + c_2 u_2 + \cdots + c_k u_k = 0$

$\Rightarrow P(c_1 u_1 + c_2 u_2 + \cdots + c_k u_k) = P0$

$\Rightarrow c_1 Pu_1 + c_2 P u_2 + \cdots c_k P u_k = 0$

Since $Pu_1 + Pu_2 + \cdots Pu_k$ are linearly independent,
$c_1, c_2, \cdots, c_k = 0$. Thus $u_1, u_2, \cdots, u_k$ are linearly independent

b) i) $P(c_1 u_1 + c_2 u_2 + \cdots + c_k u_k) = 0$

Since P is invertable, $c_1 u_1 + c_2 u_2 + \cdots + c_k u_k = 0$
Since $u_1 \cdots u_k$ are linearly independent, $c_1 \cdots c_k = 0$
Therefore $Pu_1, Pu_2 \cdots Pu_k$ are linearly independent.

ii) No, conclusion

Let $u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

If $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then Pu_1 and Pu_2 are linearly independent.

If $P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then Pu_1 and Pu_2 are linearly dependent.