Solution to Exam of MA1521, AY2021-22 S2.

1. An equation of the plane passing through the three points $(3,-1,2)$, $(8,2,4)$ and $(-1,-2,-3)$ has the form $ax+by+cz=42$. Find the value of $a+b+c$.

   Solution. The vectors $\langle 8,2,4\rangle-\langle 3,-1,2\rangle=\langle 5,3,2\rangle$, and $\langle -1,-2,-3\rangle-\langle 3,-1,2\rangle=\langle -4,-1,-5\rangle$ are parallel to the plane. Thus a normal vector to the plane is $\langle 5,3,2\rangle\times\langle -4,-1,-5\rangle=\langle -13,17,7\rangle$. Therefore, an equation of the plane is given by $\langle x+1,y+2,z+3\rangle\cdot\langle -13,17,7\rangle=0$ which simplifies to $13x-17y-7z=42$. Thus $a+b+c=13-17-7=-11$.

   Alternatively, one may substitute the three given points into the given equation of the plane $ax+by+cz=42$ and solve for a,b,c to obtain $a=13,b=-17,c=-7$.

2. For each of the following series, determine whether it converges or diverges. Justify your answers.

   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{n!(2n)!}{(3n)!}$,

   (b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^3}$.

   Solution. (a)

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{\frac{(n+1)!(2n+2)!}{(3n+3)!}}{\frac{n!(2n)!}{(3n)!}}\right| &= \lim_{n\to\infty}\frac{(n+1)(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)}\\ &= \lim_{n\to\infty}\frac{(1+\frac{1}{n})(2+\frac{2}{n})(2+\frac{1}{n})}{(3+\frac{3}{n})(3+\frac{2}{n})(3+\frac{1}{n})}\\ &= \frac{4}{27}<1.\end{aligned}$$

   By ratio test, the series converges.

   (b)

$$\begin{aligned}\int_2^{\infty}\frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty}\int_2^{b}\frac{1}{x(\ln x)^3}\,dx &= \lim_{b\to\infty}\left[-\frac{1}{2(\ln x)^2}\right]_2^b\\ &= \lim_{b\to\infty}\left[-\frac{1}{2(\ln b)^2}+\frac{1}{2(\ln 2)^2}\right]=\frac{1}{2(\ln 2)^2}.\end{aligned}$$

   By integral test, the series converges. ■

3. Let a be a constant and let $f(x)=\dfrac{ax+a+1}{x^2-x-2}$. It is known that $f^{(7)}(0)=0$, where $f^{(7)}(0)$ denotes the seventh derivative of f evaluated at the point $x=0$. By finding the Maclaurin series of f, determine the value of a.

   Solution We first find the Maclaurin series of f.

$$\frac{ax+a+1}{x^2-x-2} = \frac{ax+a+1}{(x-2)(x+1)}$$

$$= \tfrac{1}{3}(ax+a+1)(\tfrac{1}{x-2} - \tfrac{1}{x+1}) = -\tfrac{1}{3}(ax+a+1)(\tfrac{1}{2}\tfrac{1}{1-\frac{x}{2}} + \tfrac{1}{1+x})$$

$$= -\tfrac{1}{3}(ax+a+1)\Big((\tfrac{1}{2} + \tfrac{x}{4} + \tfrac{x^2}{8} + \tfrac{x^3}{16} + \tfrac{x^4}{32} + \tfrac{x^5}{64} + \tfrac{x^6}{128} + \tfrac{x^7}{256} + \cdots) \\ + (1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \cdots)\Big)$$

$$= -\tfrac{1}{3}(ax+a+1)\Big(\cdots + (\tfrac{1}{128}+1)x^6 + (\tfrac{1}{256}-1)x^7 + \cdots\Big)$$

$$= -\tfrac{1}{3}(ax+a+1)\Big(\cdots + \tfrac{129}{128}x^6 - \tfrac{255}{256}x^7 + \cdots\Big).$$

The coefficient of x^7 is $-\frac{1}{3}(-(a+1)\times\frac{255}{256} + a\times\frac{129}{128}) = \frac{85-a}{256}$. Since $\frac{f^{(7)}(0)}{7!} = \frac{85-a}{256} = 0$, we have $\frac{85-a}{256} = 0$ so that $a = 85$.

Alternate Solution. By resolving $\frac{ax+a+1}{x^2-x-2}$ into partial fractions, we have for $|x| < 1$,

$$f(x) = \frac{ax+a+1}{x^2-x-2} = \frac{1}{3}\left(\frac{3a+1}{x-2} - \frac{1}{x+1}\right) = -\frac{1}{3}\left(\frac{3a+1}{2}\frac{1}{1-\frac{x}{2}} + \frac{1}{1+x}\right)$$

$$= -\tfrac{1}{3}\Big(\tfrac{3a+1}{2}(1 + \tfrac{x}{2} + \cdots + \tfrac{x^7}{2^7} + \cdots) + (1 - x + \cdots - x^7 + \cdots)\Big).$$

The coefficient of x^7 is $-\frac{1}{3}(\frac{3a+1}{2^8} - 1)$. Thus $f^{(7)}(0) = 0$ implies that$-\frac{1}{3}(\frac{3a+1}{2^8} - 1) = 0 \Leftrightarrow a = \frac{1}{3}(2^8 - 1) = 85$. ■

4. Let $f(x,y)$ be a differentiable function defined on $\mathbb{R}^2$. It is known that the directional derivative of f at $(1,2)$ along the direction $3\mathbf{i}+4\mathbf{j}$ is 15 and along the direction of $-3\mathbf{i}+4\mathbf{j}$ is 9. Find the directional derivative of f at $(1,2)$ in the direction of $24\mathbf{i}+7\mathbf{j}$.

   Solution We are given $\langle f_x(1,2), f_y(1,2)\rangle\cdot\langle\frac{3}{5},\frac{4}{5}\rangle = 15$ and $\langle f_x(1,2), f_y(1,2)\rangle\cdot\langle-\frac{3}{5},\frac{4}{5}\rangle = 9$.

   That is $3f_x(1,2)+4f_y(1,2) = 75$ and $-3f_x(1,2)+4f_y(1,2) = 45$. Solving for $f_x(1,2)$ and $f_y(1,2)$, we obtain $f_x(1,2) = 5$ and $f_y(1,2) = 15$.

   Thus the directional derivative of f at $(1,2)$ in the direction of $24\mathbf{i}+7\mathbf{j}$ is $\langle 5,15\rangle\cdot\langle\frac{24}{25},\frac{7}{25}\rangle = \frac{225}{25} = 9$. ■

5. Let $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$. Find all critical points of f. For each critical point, determine whether f has a local maximum, a local minimum or a saddle point.

   Solution First $f_x(x,y) = 3x^2 + 3y^2 - 6x$ and $f_y(x,y) = 6xy - 6y$. Solving the equations $3x^2 + 3y^2 - 6x = 0$ and $6y(x-1) = 0$, we obtain the critical points of f which are $(0,0), (2,0), (1,1), (1,-1)$.

   For the second partial derivatives of f, we find that $f_{xx}(x,y) = 6x-6$, $f_{yy}(x,y) = 6x-6$ and $f_{xy}(x,y) = 6y$ so that $D(x,y) = (6x-6)^2 - (6y)^2 = 36((x-1)^2 - y^2)$.

At $(0,0)$, $D(0,0) = 36 > 0$ and $f_{xx}(0,0) = -6 < 0$. Thus f has a local maximum at $(0,0)$.

At $(2,0)$, $D(2,0) = 36 > 0$ and $f_{xx}(2,0) = 6 > 0$. Thus f has a local minimum at $(0,0)$.

At $(1,1)$, $D(1,1) = -36 < 0$. Thus f has a saddle point at $(1,1)$.

At $(1,-1)$, $D(1,-1) = -36 < 0$. Thus f has a saddle point at $(1,-1)$. ■

6. Let a denote a positive constant. It is known that the graph of

$$y^2 = x^2a^3 - 3x^3a^2 + 3x^4a - x^5$$

has a loop which bounds a region R in the xy-plane and the area of R is $\frac{8}{35}a^{\frac{7}{2}}$. Suppose

$$\iint_R (x-4)\,dA = 0.$$

Determine the value of a.

Solution. First $y^2 = x^2a^3 - 3x^3a^2 + 3x^4a - x^5 = x^2(a-x)^3$. Thus $y = 0 \Leftrightarrow x = 0$ or a. That means the loop is within the range for $x = 0$ to $x = a$. The upper and lower curves bounding the loop have equations given by $y = x(a-x)^{\frac{3}{2}}$ and $y = -x(a-x)^{\frac{3}{2}}$ respectively.

Next we have

$$\iint_R (x-4)\,dA = 0 \Leftrightarrow \iint_R x\,dA = 4\iint_R dA = 4 \times \text{area of } R = \frac{32}{35}a^{\frac{7}{2}}.$$

$$\iint_R x\,dA = \int_0^a \int_{-x(a-x)^{\frac{3}{2}}}^{x(a-x)^{\frac{3}{2}}} x\,dy\,dx = 2\int_0^a x^2(a-x)^{\frac{3}{2}}\,dx = 2\int_0^a (a-(a-x))^2(a-x)^{\frac{3}{2}}\,dx$$

$$= 2\int_0^a (a^2 - 2a(a-x) + (a-x)^2)(a-x)^{\frac{3}{2}}\,dx = 2\int_0^a a^2(a-x)^{\frac{3}{2}} - 2a(a-x)^{\frac{5}{2}} + (a-x)^{\frac{7}{2}}\,dx$$

$$= \left[-\frac{4a^2}{5}(a-x)^{\frac{5}{2}} + \frac{8a}{7}(a-x)^{\frac{7}{2}} - \frac{4}{9}(a-x)^{\frac{9}{2}}\right]_0^a = \frac{4}{5}a^{\frac{9}{2}} - \frac{8}{7}a^{\frac{9}{2}} + \frac{4}{9}a^{\frac{9}{2}} = \frac{32}{315}a^{\frac{9}{2}}.$$

Thus $\frac{32}{315}a^{\frac{9}{2}} = \frac{32}{35}a^{\frac{7}{2}} \Leftrightarrow a^{\frac{9}{2}} = 9a^{\frac{7}{2}} \Leftrightarrow a^{\frac{7}{2}}(a-9) = 0 \Leftrightarrow a = 9$ or 0. Since $a > 0$, we have $a = 9$. ■

7. Find the surface area of the portion of the surface $z = 6y^2 + \sqrt{323}x$ lying above the triangular region R in the xy-plane with vertices at $(0,0), (0,2)$ and $(2,2)$.

Solution First we have $z_x = \sqrt{323}, z_y = 12y$. Thus

Surface area $= \iint_R \sqrt{1 + z_x^2 + z_y^2}\,dA = \iint_R \sqrt{1 + 323 + (12y)^2}\,dA$

$= \iint_R \sqrt{324 + 144y^2}\,dA = 6\iint_R \sqrt{9 + 4y^2}\,dA = 6\int_0^2 \int_0^y \sqrt{9 + 4y^2}\,dx\,dy$

$= 6\int_0^2 \left[x\sqrt{9+4y^2}\right]_{x=0}^{x=y} dy = 6\int_0^2 y\sqrt{9+4y^2}\,dy = \left[\frac{1}{2}(9+4y^2)^{\frac{3}{2}}\right]_0^2$

$= \frac{1}{2}(25^{\frac{3}{2}} - 9^{\frac{3}{2}}) = \frac{1}{2}(125 - 27) = 49.$ ■

8. Find the volume of the solid bounded by the paraboloid $z = 150+(2x-1)^2+4y^2$, the cylinder $x^2 + y^2 = 9$ and the xy-plane. Express your answer in terms of π.

Solution Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 3^2\}$. The volume of the solid is

$$\iint_D 150 + (2x-1)^2 + 4y^2\, dA.$$

Changing to polar coordinates, we have

$$\begin{aligned}\iint_D 150 + (2x-1)^2 + 4y^2\, dA &= \int_0^{2\pi}\int_0^3 (150 + (2r\cos\theta - 1)^2 + 4(r\sin\theta)^2) r\, dr d\theta \\ &= \int_0^{2\pi}\int_0^3 (150 - 4r\cos\theta + 1 + 4r^2) r\, dr d\theta \\ &= \int_0^{2\pi}\int_0^3 151r - 4r^2\cos\theta + 4r^3\, dr d\theta \\ &= \int_0^{2\pi}\left[\tfrac{151r^2}{2} - \tfrac{4}{3}r^3\cos\theta + r^4\right]_{r=0}^{r=3} d\theta \\ &= \int_0^{2\pi} \tfrac{1521}{2} - 36\cos\theta\, d\theta \\ &= \left[\tfrac{1521\theta}{2} - 36\sin\theta\right]_0^{2\pi} = 1521\pi.\end{aligned}$$

■

9. Let $y(x)$ be the solution of the differential equation

$$x\frac{dy}{dx} + y = xy^3,\ 0 < x < \frac{8}{7},$$

satisfying $y(\frac{1}{2}) = \frac{4}{3}$ and $y(x) > 0$ for $0 < x < \frac{8}{7}$. Find the exact value of $y(1)$.

Solution. Rewrite the equation in the standard form: $\frac{dy}{dx} + \frac{y}{x} = y^3$. This is a Bernoulli's equation with $n = 3$. Let $z = y^{1-3} = y^{-2}$. Then $z' = -2y^{-3}y'$. The equation can be written as $-\frac{1}{2}y^3z' + \frac{y}{x} = y^3 \Leftrightarrow -\frac{1}{2}y^2z' + \frac{1}{x} = y^2 \Leftrightarrow z' - \frac{2y^{-2}}{x} = -2 \Leftrightarrow z' - \frac{2z}{x} = -2$.

This is a linear equation with an integrating factor $e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$. Multiplying throughout the above equation by $\frac{1}{x^2}$, we obtain $\frac{1}{x^2}z' - \frac{2z}{x^3} = -\frac{2}{x^2}$. That is $(\frac{z}{x^2})' = -\frac{2}{x^2}$. Integrating, we obtain $\frac{z}{x^2} = \frac{2}{x} + C \Leftrightarrow z = 2x + Cx^2 \Leftrightarrow \frac{1}{y^2} = 2x + Cx^2$.

As $y(\frac{1}{2}) = \frac{4}{3}$, we have $(\frac{3}{4})^2 = 2(\frac{1}{2}) + C(\frac{1}{2})^2 \Leftrightarrow \frac{9}{16} = 1 + \frac{C}{4} \Leftrightarrow C = -\frac{7}{4}$. Therefore, $\frac{1}{y^2} = 2x - \frac{7}{4}x^2 \Leftrightarrow \frac{1}{y^2} = \frac{x(8-7x)}{4}$.

Since $y(x) > 0$, we conclude that $y = \frac{2}{\sqrt{x(8-7x)}}$. Consequently, $y(1) = 2$. ■

10. At time $t = 0$, a tank contains 400 grams of salt dissolved in 4 litres of water. Assume that water containing 40 grams of salt per litre is entering the tank at a rate of $\ln(4)$ litres per minute and that the well-mixed solution is draining from the tank at the same rate. Find the amount of salt in grams in the tank at time $t = 4$ minutes. Give your answer correct to two decimal places.

Solution. First note that the volume of the solution remains constant which is 4 litres. Let Q be the amount of salt in grams at time t in minutes. The concentration of salt in the solution at time t is $\frac{Q}{4}$ grams per litre. Suppose at time $t+dt$, the amount of salt is $Q+dQ$. Then

$$dQ = \text{salt input} - \text{salt output} = 40 \times \ln 4 \times dt - \ln 4 \times \frac{Q}{4} \times dt.$$

That is

$$\frac{dQ}{dt} = \frac{\ln 4}{4}(160 - Q).$$

Thus

$$\int \frac{dQ}{160 - Q} = \frac{\ln 4}{4} \int dt,$$

so that

$$\ln|160 - Q| = -\frac{\ln 4}{4} t + C.$$

That is $Q = 160 + Ae^{-\frac{\ln 4}{4}t}$, or $Q = 160 + A \times 2^{-\frac{t}{2}}$, for some constant A.

When $t = 0$, we have $Q = 400$. Thus $400 = 160 + A$ so that $A = 240$. Consequently,

$$Q = 160 + 240 \times 2^{-\frac{t}{2}}.$$

Therefore, $Q(4) = 160 + 240 \times 2^{-2} = 220$ grams. ■