Department of Mathematics

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(2022/23) Semester I MA1521 Calculus for Computing Tutorial 8

1. In an electric circuit, the voltage of V volts (V), current of I amperes (A), and resistance of R ohms (Ω) are governed by Ohm's Law: $V = I \times R$.
   (i) If the resistance is fixed at 15 Ω, how fast is the current increasing with respect to voltage?
   (ii) If the voltage is fixed at 120 V, how fast is the current increasing with respect to resistance at the instant when resistance is 20 Ω?
   (iii) If the resistance is slowly increasing as the resistor heats up, how is the current changing at the moment when $R = 400\Omega$, $I = 0.08$A, $dV/dt = -0.01$ V/s and $dR/dt = 0.03$ Ω/s?

   Ans. (i) ≈ 0.0667 A/V, (ii) *decreasing* at 0.3 A/Ω, (iii) decreasing at 3.1×10^{-5} A/s.

2. Find the directional derivative of $f(x, y) = xe^{2y-x}$ at $P(-2, -1)$ in each of the following the directions

   (i) $\dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$, (ii) $3\mathbf{i} + 4\mathbf{j}$.

   Find the direction that gives the *largest possible* directional derivative of f at P.

   Ans. (i) $-\sqrt{2}/2$; (ii) $-7/5$; in the direction of $\nabla f(-2, -1) = f_x(-2, -1)\mathbf{i} + f_y(-2, -1)\mathbf{j} = 3\mathbf{i} - 4\mathbf{j}$.

3. Let $f(x, y, z) = \sin(xyz)$ and $P = (\frac{1}{2}, \frac{1}{3}, \pi)$.
   (i) Find the rate of change of f at P in the direction $\mathbf{u} = \dfrac{1}{\sqrt{3}}\mathbf{i} - \dfrac{1}{\sqrt{3}}\mathbf{j} + \dfrac{1}{\sqrt{3}}\mathbf{k}$.
   (ii) Suppose P moves 0.1 unit along $\mathbf{u}$ in part (i). How much will the value of f have changed?

   Ans. (i) $\frac{1}{12}(1 - \pi)$; (ii) decreases by ≈ 0.01785.

4. Find the local maximum and minimum values and saddle points (if any) of each of the following functions.
   (i) $f(x, y) = \ln(x^2y) - xy - 2x$, where $x > 0$, $y > 0$

(ii) $g(x,y) = xy(1-x-y)$

(iii) $h(x,y) = x^2 + y^2 + x^{-2}y^{-2}$, where $x \neq 0,\ y \neq 0$

Ans. (i) $f(1/2,2) = -\ln 2 - 2$ is a local maximum, (ii) $(0,0)$, $(1,0)$, $(0,1)$ are saddle points,

$g(1/3,1/3) = 1/27$ is a local maximum, (iii) $h(\pm1,\pm1) = h(\pm1,\mp1) = 3$ are local minima.

Further Exercises

1. The temperature at a point (x, y) on a plane is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature (experienced by the bug) rising on the bug's path after 3 seconds?

   Ans. $2\mathrm{C}^\circ/s$.

2. Let $f(x, y, z) = xy^2z^3$ and $P = (1, -2, 1)$. Find the rate of change of f at P in the direction $\mathbf{u} = \langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$. (That is to find $D_{\mathbf{u}}f(P)$).

   Ans. $\frac{20}{\sqrt{3}}$.

3. Let $f(x, y) = \frac{1}{2}\cos(\frac{x}{2}) + \sin(\frac{y}{4})$, where $-6\pi < x < 2\pi, -2\pi < y < 6\pi$. Find all the points at which f has a local maximum, a local minimum or a saddle point.

   Ans. f has a local maximum at $(0, 2\pi)$, a saddle point at $(-2\pi, 2\pi)$ and a local maximum at $(-4\pi, 2\pi)$.