Solutions to Tutorial 1

MA1521 Calculus for Computing

1. (a) Since f is a rational function, the maximal domain of f consists of all real numbers x such that the denominator of f is nonzero. We can factorize the denominator of f over real numbers as $(4+x^2)(2-x)(4+2x+x^2)(1-x)(1+x)(1+x^2)$ whose roots are $-1,1,2$. Thus the maximal domain of f is $\mathbb{R}\setminus\{-1,1,2\}$.

   (b) Given $g(x)=\sqrt{2-\ln(x-3)}$, it is defined for x satisfying $2-\ln(x-3)\geq 0$ and $x-3>0$ since ln is defined for positive real numbers and the square root function is defined for nonnegative real numbers. Now $2-\ln(x-3)\geq 0$ and $x-3>0 \Leftrightarrow \ln(x-3)\leq 2$ and $3<x \Leftrightarrow x-3\leq e^2$ and $3<x \Leftrightarrow 3<x\leq 3+e^2$. Thus the maximal domain of g is $(3,3+e^2]$.

   (c) Given $h(x)=\frac{\ln(\sqrt{16-2x}+1)}{\sqrt{\ln x}-1}$, it is defined for x such that $\ln x\geq 0$ and $\sqrt{\ln x}-1\neq 0$ and $16-2x\geq 0$. That is $x\geq 1$ and $x\neq e$ and $8\geq x \Leftrightarrow 1\leq x\leq 8$ and $x\neq e$.

2. For x inside the intervals, $(-\infty,-5),(-5,-1),(-1,1)$ and $(1,\infty)$, the function f is defined by a constant, a polynomial, a constant or a rational function, respectively. Thus f is continuous at each point inside these open intervals. We just have to examine the continuity of f at the endpoints of these intervals.

   $\lim\limits_{x\to-5^-} f(x)=2$, $\lim\limits_{x\to-5^+} f(x)=\lim\limits_{x\to-5^+} x^2-1=24$. As $\lim\limits_{x\to-5^-} f(x)\neq\lim\limits_{x\to-5^+} f(x)$, $\lim\limits_{x\to-5} f(x)$ does not exists, and f is not continuous at $x=-5$.

   $\lim\limits_{x\to-1^-} f(x)=\lim\limits_{x\to-1^-} x^2-1=0$, $\lim\limits_{x\to-1^+} f(x)=\lim\limits_{x\to-1^+} 0=0$. Thus $\lim\limits_{x\to-1} f(x)=0=f(0)$. Therefore, f is continuous at $x=-1$.

   $\lim\limits_{x\to1^+} f(x)=\lim\limits_{x\to1^+}\dfrac{1}{x-1}=+\infty$. Thus $\lim\limits_{x\to1} f(x)$ does not exist. Therefore, f is not continuous at $x=1$.

   Consequently, the points of discontinuity of f are at $x=-5,1$.

3. f is continuous at $x=4 \Rightarrow \lim\limits_{x\to4} f(x)=f(4)\Rightarrow \lim\limits_{x\to4^-} f(x)=f(4)\Rightarrow \lim\limits_{x\to4^-} p\sqrt{x}\Rightarrow 2p=7\Rightarrow p=\frac{7}{2}$.

   f is continuous at $x=4\Rightarrow \lim\limits_{x\to4^+} f(x)=f(4)\Rightarrow \lim\limits_{x\to4^+} q(x-2)^2+5\Rightarrow 4q+5=7\Rightarrow q=\dfrac{1}{2}$.

   $\lim\limits_{x\to6} f(x)$ exists $\Rightarrow \lim\limits_{x\to6^-} f(x)=\lim\limits_{x\to6^+} f(x)\Rightarrow \lim\limits_{x\to6^-}\dfrac{1}{2}(x-2)^2+5=\lim\limits_{x\to6^+}\dfrac{r}{x-5}\Rightarrow 13=r$.

4. (a) $\lim\limits_{x\to1}\dfrac{4+x}{2-x}=\dfrac{4+1}{2-1}=5$.

   (b) $\lim\limits_{x\to2}\dfrac{4-x^2}{x^2-3x+2}=\lim\limits_{x\to2}\dfrac{(2+x)(2-x)}{(x-1)(x-2)}=\lim\limits_{x\to2}\dfrac{-(2+x)}{x-1}=-4$.

(c) $\lim_{x\to -2} \dfrac{4-x^2}{\sqrt{x^2-x-2}-\sqrt{2-x}}$

$= \lim_{x\to -2} \dfrac{4-x^2}{\sqrt{x^2-x-2}-\sqrt{2-x}}$

$= \lim_{x\to -2} \dfrac{4-x^2}{\sqrt{x^2-x-2}-\sqrt{2-x}} \dfrac{\sqrt{x^2-x-2}+\sqrt{2-x}}{\sqrt{x^2-x-2}+\sqrt{2-x}}$

$= \lim_{x\to -2} \dfrac{(4-x^2)(\sqrt{x^2-x-2}+\sqrt{2-x})}{x^2-x-2-(2-x)}$

$= \lim_{x\to -2} \dfrac{(4-x^2)(\sqrt{x^2-x-2}+\sqrt{2-x})}{x^2-4}$

$= \lim_{x\to -2} \dfrac{(2+x)(2-x)(\sqrt{x^2-x-2}+\sqrt{2-x})}{(x+2)(x-2)}$

$= \lim_{x\to -2} -(\sqrt{x^2-x-2}+\sqrt{2-x}) = -4.$

(d) $\lim_{x\to 1} \dfrac{3-\sqrt{x+8}}{\sqrt{x+3}-\sqrt{5-x}}$

$= \lim_{x\to 1} \dfrac{3-\sqrt{x+8}}{\sqrt{x+3}-\sqrt{5-x}} \dfrac{\sqrt{x+3}+\sqrt{5-x}}{\sqrt{x+3}+\sqrt{5-x}}$

$= \lim_{x\to 1} \dfrac{(3-\sqrt{x+8})(\sqrt{x+3}+\sqrt{5-x})}{(x+3)-(5-x)}$

$= \lim_{x\to 1} \dfrac{(3-\sqrt{x+8})(\sqrt{x+3}+\sqrt{5-x})}{2(x-1)}$

$= \lim_{x\to 1} \dfrac{(3+\sqrt{x+8})(3-\sqrt{x+8})(\sqrt{x+3}+\sqrt{5-x})}{2(3+\sqrt{x+8})(x-1)}$

$= \lim_{x\to 1} \dfrac{(9-(x+8))(\sqrt{x+3}+\sqrt{5-x})}{2(3+\sqrt{x+8})(x-1)}$

$= \lim_{x\to 1} \dfrac{-(x-1)(\sqrt{x+3}+\sqrt{5-x})}{2(3+\sqrt{x+8})(x-1)}$

$= \lim_{x\to 1} \dfrac{-(\sqrt{x+3}+\sqrt{5-x})}{2(3+\sqrt{x+8})} = -\dfrac{1}{3}.$

(e) $\lim_{x\to 1} \dfrac{x^2-1}{(x-1)^2} = \lim_{x\to 1} \dfrac{(x+1)(x-1)}{(x-1)^2} = \lim_{x\to 1} \dfrac{x+1}{x-1}$. This limit does not exist, or undefined as the numerator tends to a nonzero number but the denominator tends to 0.

5. (a) $\lim_{x\to\infty} \sqrt{\dfrac{9x^{10}+3x-1}{(x^2+3x+5)^3(2x-5)^4}} = \lim_{x\to\infty} \sqrt{\dfrac{9x^{10}+3x-1}{16x^{10}+\cdots}} = \sqrt{\dfrac{9}{16}} = \dfrac{3}{4}.$

(b) $\lim_{x\to -\infty} \dfrac{\sqrt{9x^{10}+3x-1}}{(1+2x)^5} = \lim_{x\to -\infty} \dfrac{\sqrt{9x^{10}+3x-1}}{-\sqrt{(1+2x)^{10}}}$ since the term $(1+2x)^5$ is negative for $x \to -\infty$.

Thus $\lim\limits_{x\to-\infty}\dfrac{\sqrt{9x^{10}+3x-1}}{(1+2x)^5}=\lim\limits_{x\to-\infty}-\sqrt{\dfrac{9x^{10}+3x-1}{(1+2x)^{10}}}=\lim\limits_{x\to-\infty}-\sqrt{\dfrac{9x^{10}+3x-1}{2^{10}x^{10}+\cdots}}=-\dfrac{3}{32}$.

(c) Note that the term $(1+2x)^2(x^2+x-1)$ is positive as $x\to-\infty$. Thus

$$\lim_{x\to-\infty}\frac{\sqrt{9x^{10}+3x-1}}{(1+2x)^2(x^2+x-1)}=\lim_{x\to-\infty}\sqrt{\frac{9x^{10}+3x-1}{(1+2x)^4(x^2+x-1)^2}}=\lim_{x\to-\infty}\sqrt{\frac{9x^{10}+3x-1}{2^4x^8+\cdots}}=\infty.$$

6. By continuity of f and g, $4=\lim\limits_{x\to3}[2f(x)-g(x)]=2\lim\limits_{x\to3}f(x)-\lim\limits_{x\to3}g(x)=2f(3)-g(3)=2(2)-g(3)$. Thus $g(3)=0$.

Solutions to Further Exercises

1. First $|4x+13|=|4x+12+1|\leq|4x+12|+1=4|x+3|+1$. Since $|x+3|<\frac{1}{2}$, it follows that $|4x+13|<4(\frac{1}{2})+1=3$.

2. (a) Given $f(x)=\frac{x+1}{x-2}$. For f to be defined, we only require $x\neq2$. Thus the domain of f is $\{x\in\mathbb{R}\mid x\neq2\}$, or we may also write $\mathbb{R}\setminus\{2\}$.

   (b) Set $\frac{x+1}{x-2}=1$. Then $x+1=x-2$ giving $1=-2$, a contradiction. Therefore, there is no value of x such that $f(x)=1$.

   (c) Set $\frac{x+1}{x-2}=c$. Solving x in terms of c, we obtain $x=\frac{2c+1}{c-1}$. (Note that from this we also see that $c\neq1$.

   (d) Part (ii) implies that the range of f is a subset of $\mathbb{R}\setminus\{1\}$ while part (iii) implies that $\mathbb{R}\setminus\{1\}$ lies in the range of f. Consequently, the range of g is $\mathbb{R}\setminus\{1\}$.