CS2040S
Data Structures and Algorithms

Last weekend…

https://hacknroll2023.devpost.com/project-gallery

Welcome!

ARCHIPELAGO
is open

How to Search!

Algorithm Analysis

- Big-O Notation
- Model of computation

Searching

Peak Finding

- 1-dimension
- 2-dimensions

Admin

Archipelago

- With some browsers/devices don’t redirect properly from the link.
- So… 404 error
- No workaround yet… try different browser/device?

Coursemology

- Safari V.14:
- Misplaced title bar
- Cannot access assignment 2.
- Use different browser or upgrade safari.

Admin

Tutorial and Recitations

DO NOT make any changes directly on ModReg.

All changes have to go through Coursemology appeal.

- If you change directly, you risk being dropped from the class.
- Please do not e-mail me or the module staff directly if you are unhappy with your slot.

Admin

Tutorial and Recitations

- Current assignment available on ModReg.
- 99% / 97% got one of their first choices.
- But 16 (tutorial) and 15 (recitation) still do not have slots.

Admin

Tutorial and Recitation Registration

- There is an appeal form available on Coursemology
- Swaps (where both parties agree) are okay.
- Otherwise, we will allow changes into available sessions.
- (If a session has no space, we will not move you there.)

Admin

Tutorial Availability

Much availability:

- Wed. 3-5pm
- Wed. 4-6pm

Overfull:

- Tues. 12-2pm
- Tues. 2-4pm

Some availability:

- Wed. 11-1pm
- Tues. 3-5pm

At capacity:

- Tues. 1-3pm.
- Wed. 9-11am

Admin

Recitation Availability

Availability:

- Thurs. 2-3pm
- Thurs. 3-4pm
- Fri. 11-12pm

Overfull:

- Thurs. 1-2pm

Some availability:

- Thur. 12-1pm
- Fri. 1-2pm

How to Search!

Algorithm Analysis

- Big-O Notation
- Model of computation

Searching

Peak Finding

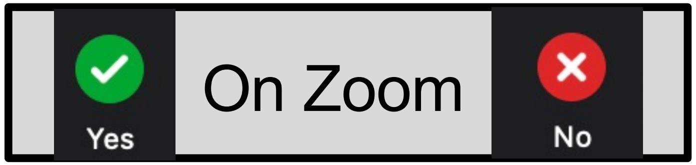
- 1-dimension
- 2-dimensions

Algorithm Analysis

Warm up: which takes longer?

```
void pushAll(int k) {
    for (int i=0;
            i<= 100*k;
            i++)
    {
        stack.push(i);
    }
}
```

```
void pushAdd(int k) {
    for (int i=0; i<= k; i++)
    {
        for (int j=0; j<= k; j++){
                stack.push(i+j);
        }
    }
}
```



Algorithm Analysis

Warm up: which takes longer?

```
void pushAll(int k) {
   for (int i=0;
         i<= 100*k;
         i++)
   {
      stack.push(i);
   }
}
```

```
void pushAdd(int k) {
   for (int i=0; i<= k; i++)
   {
      for (int j=0; j<= k; j++){
            stack.push(i+j);
      }
   }
}
```

$100k$ push operations

k^2 push operations

Which grows faster?

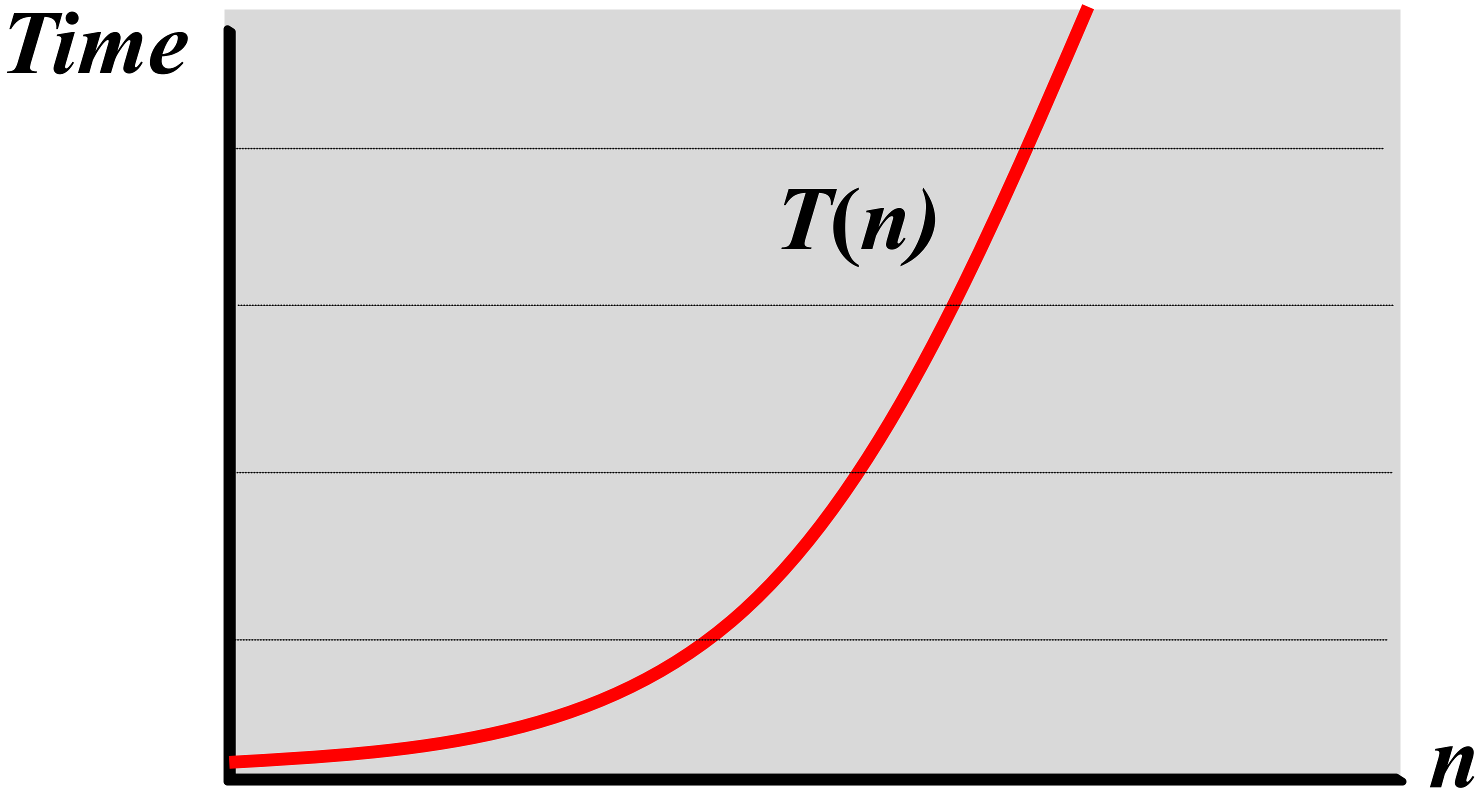
$T(k) = 100k$	$T(k) = k^2$
$T(0) = 0$	$T(0) = 0$
$T(1) = 100$	$T(1) = 1$
$T(100) = 10{,}000$	$T(100) = 10{,}000$
$T(1000) = 100{,}000$	$T(1000) = 1{,}000{,}000$

Always think of big input

Big-O Notation

How does an algorithm scale?

- For large inputs, what is the running time?
- $T(n)$ = running time on inputs of size n



Big-O Notation

Definition: $T(\mathrm{n}) = \mathrm{O}(f(n))$ if T grows no faster than f

$\boldsymbol{T(n) = \mathrm{O}(f(n))}$ if:

- there exists a constant $c > 0$
- there exists a constant $n_0 > 0$

such that for all $n > n_0$:

$$\boldsymbol{T(n) \leq c\, f(n)}$$

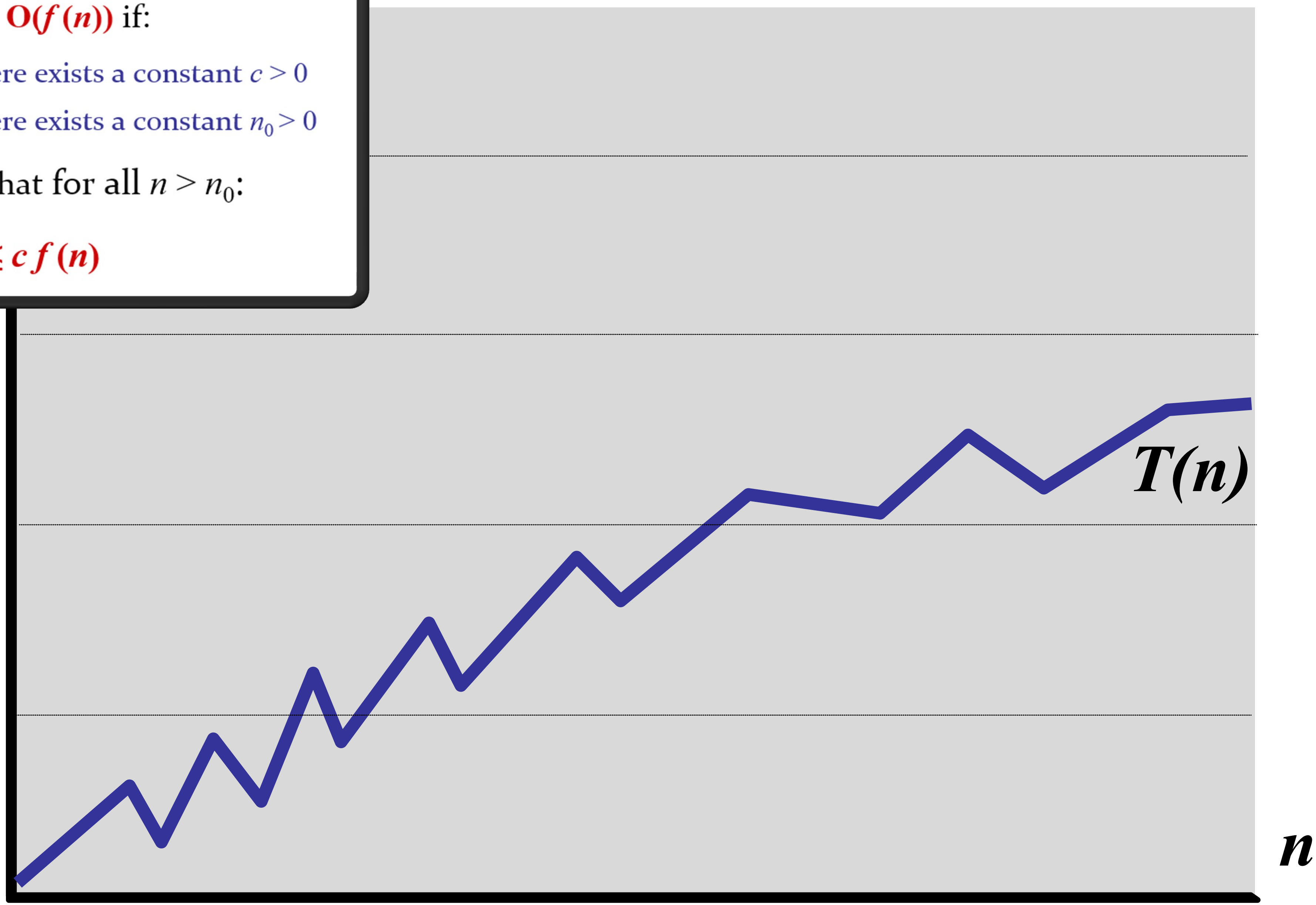
Big-O Notation

$T(n) = O(f(n))$ if:

- there exists a constant $c > 0$
- there exists a constant $n_0 > 0$

such that for all $n > n_0$:

$T(n) \leq c\, f(n)$



Big-O Notation

$T(n) = O(f(n))$ if:

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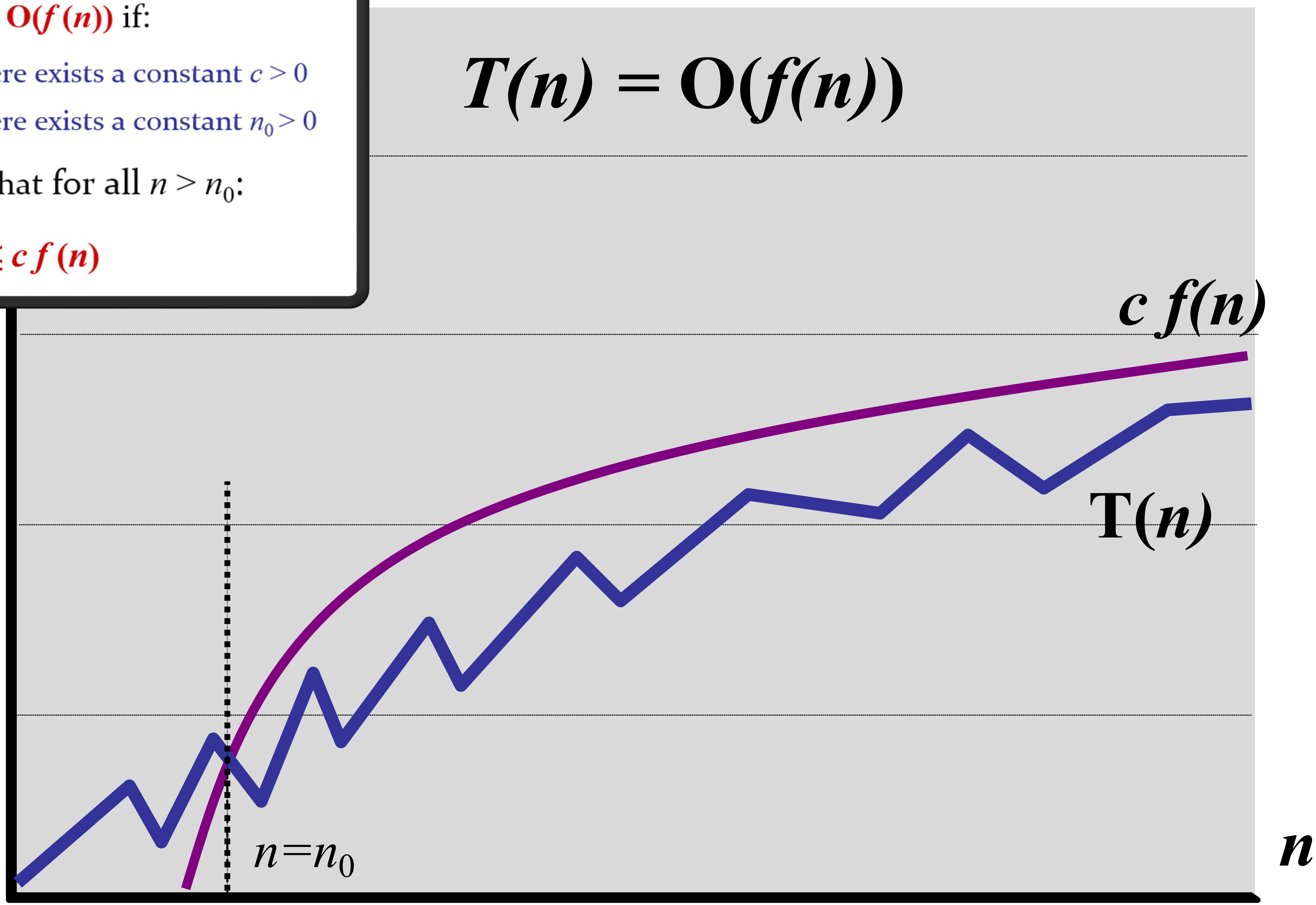


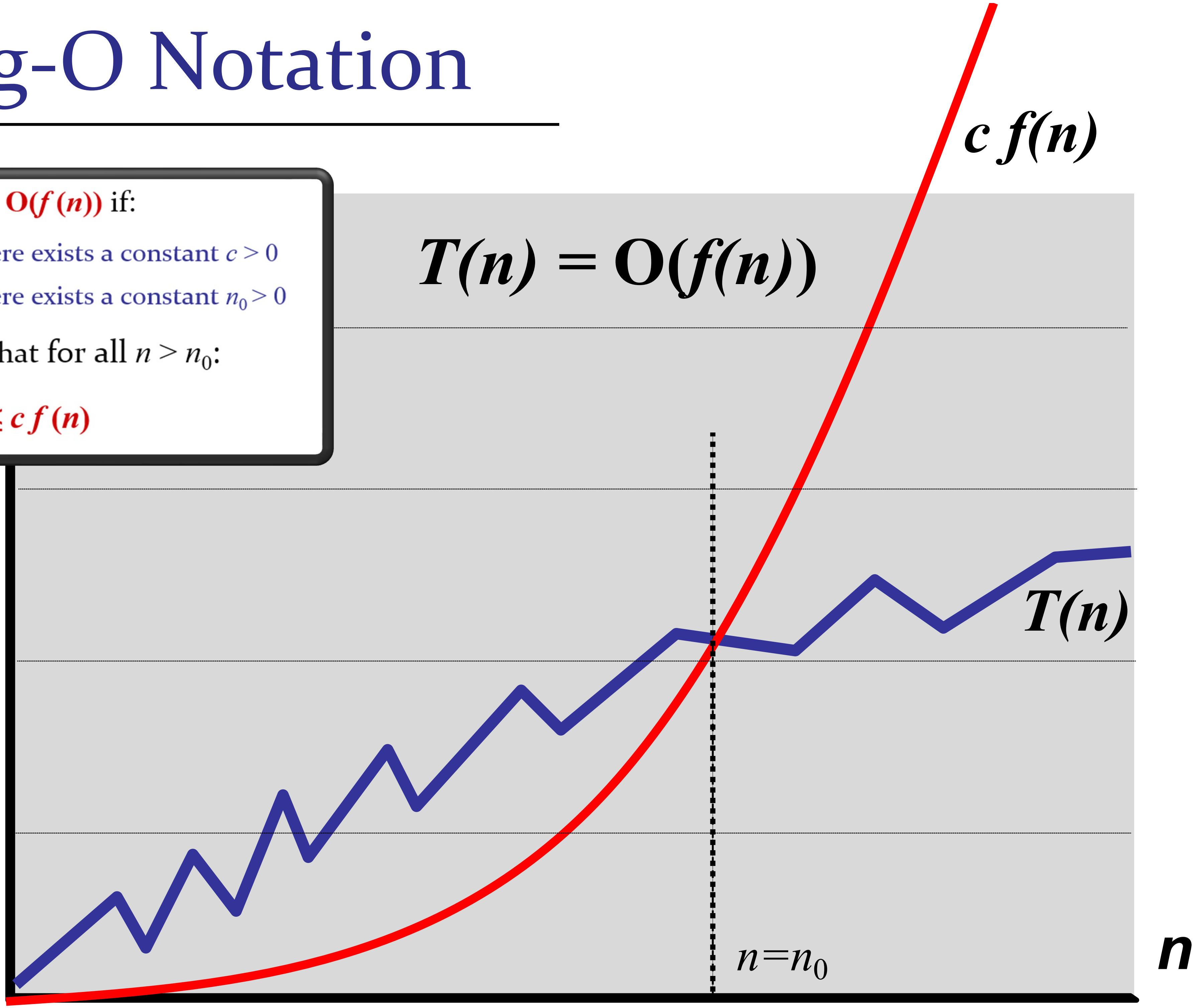
Big-O Notation

$T(n) = O(f(n))$ if:

- there exists a constant $c > 0$
- there exists a constant $n_0 > 0$

such that for all $n > n_0$:

$T(n) \leq c\, f(n)$



Big-O Notation

Example proof: T(n) = O(n²)

$$T(n) = 4n^2 + 24n + 16$$

Example

$T(n)$	big-O
$T(n) = 1000n$	$T(n) = O(n)$
$T(n) = 1000n$	$T(n) = O(n^2)$
$T(n) = n^2$	$T(n) \neq O(n)$
$T(n) = 13n^2 + n$	$T(n) = O(n^2)$

Not tight

Big-O Notation

Definition: $T(\mathrm{n}) = \mathrm{O}(f(n))$ if T grows no faster than f

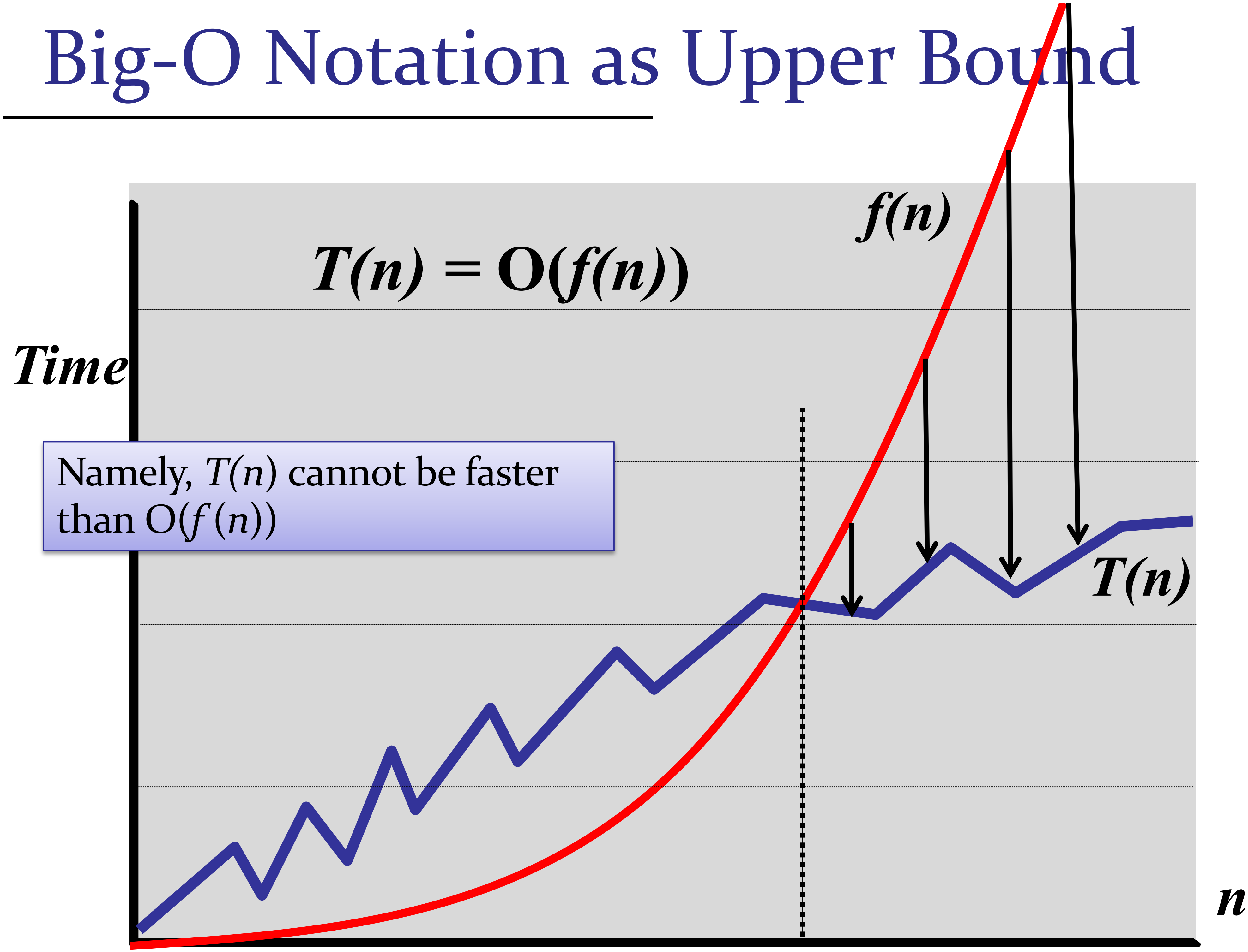
$\boldsymbol{T(n) = \mathrm{O}(f(n))}$ if:

- there exists a constant $c > 0$
- there exists a constant $n_0 > 0$

such that for all $n > n_0$:

$$\boldsymbol{T(n) \leq c\, f(n)}$$

Big-O Notation as Upper Bound

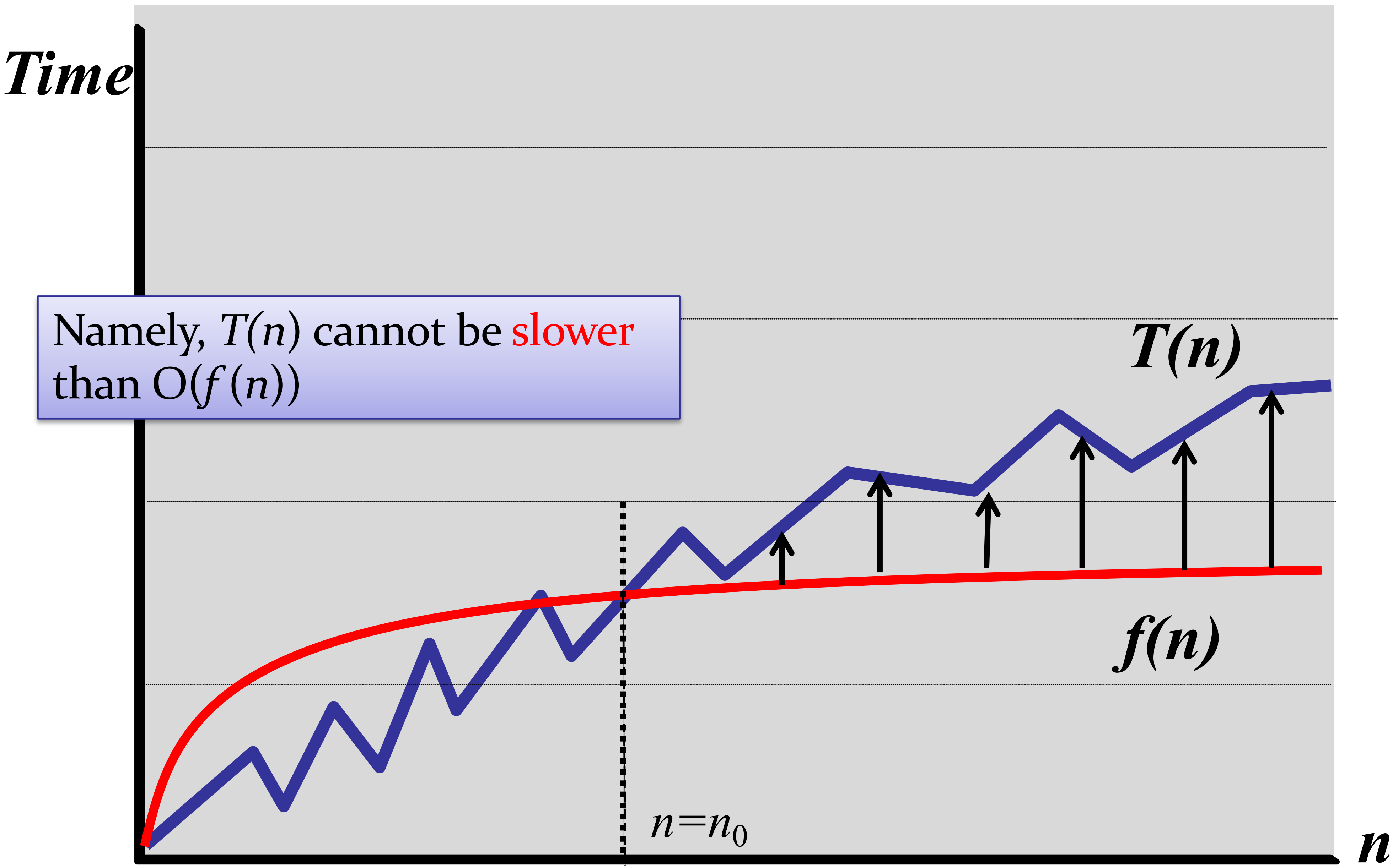


Namely, $T(n)$ cannot be faster than $O(f(n))$

How about Lower bound?

How about Lower bound?

Namely, $T(n)$ cannot be slower than $O(f(n))$



Big-O Notation

Definition: $T(n) = \Omega(f(n))$ if T grows no slower than f

$\boldsymbol{T(n) = \Omega(f(n))}$ if:

- there exists a constant $c > 0$
- there exists a constant $n_0 > 0$

such that for all $n > n_0$:

$$\boldsymbol{T(n) \geq c\, f(n)}$$

Example

$T(n)$	Asymptotic
$T(n) = 1000n$	$T(n) = \Omega(1)$
$T(n) = n$	$T(n) = \Omega(n)$
$T(n) = n^2$	$T(n) = \Omega(n)$
$T(n) = 13n^2 + n$	$T(n) = \Omega(n^2)$

Big-O Notation

Exercise:

True or false:

"$f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$"

Prove that your claim is correct using the definitions of O and Ω or by giving an example.

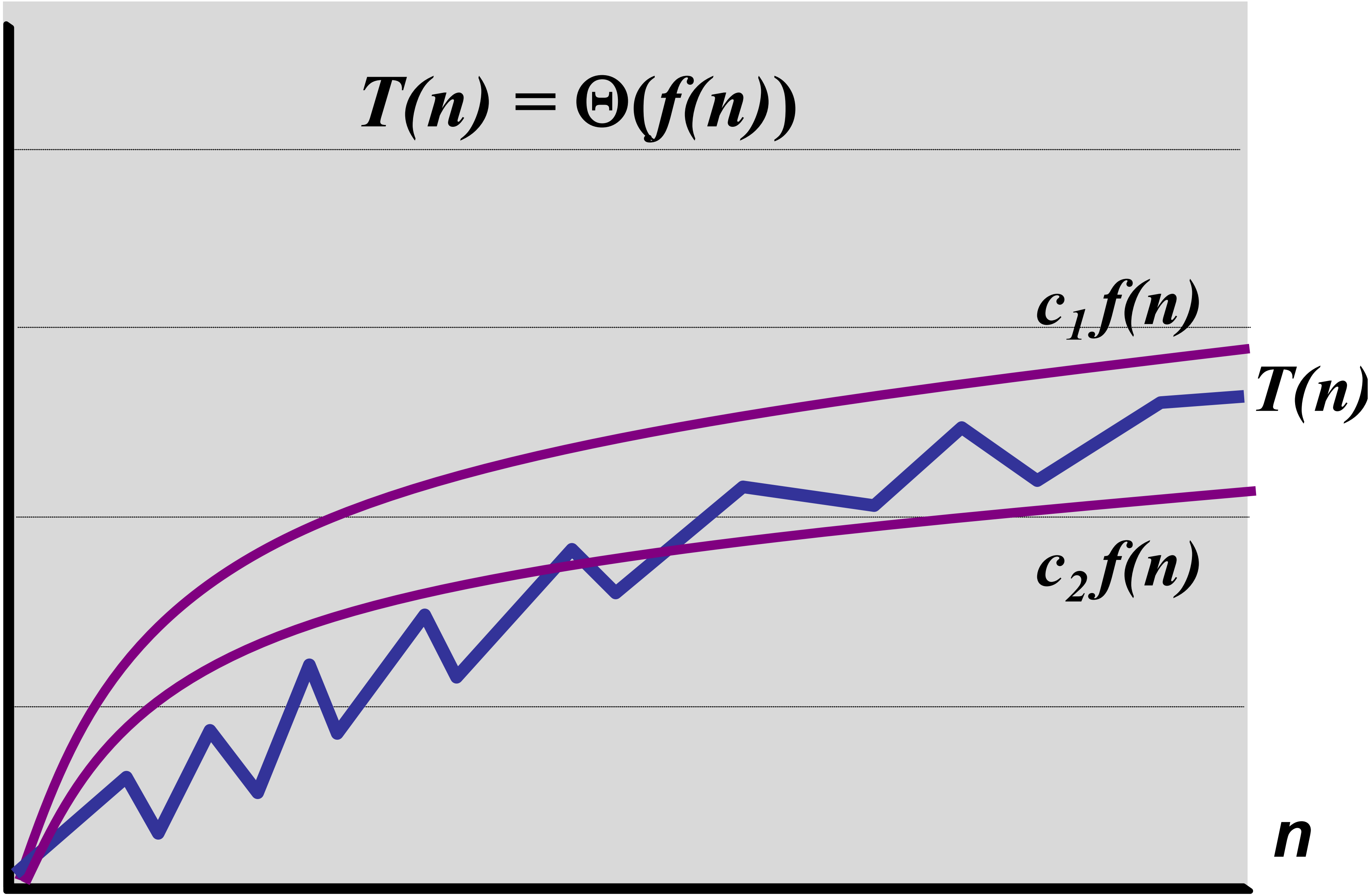
Big-O Notation

Definition: $T(n) = \Theta(f(n))$ if T grows at the same rate as f

$\boldsymbol{T(n) = \Theta(f(n))}$ if and only if:

- $T(n) = O(f(n))$ *and*
- $T(n) = \Omega(f(n))$

Big-O Notation



Example

$T(n)$	big-O
$T(n) = 1000n$	$T(\mathrm{n}) = \Theta(n)$
$T(n) = n$	$T(n) \neq \Theta(1)$
$T(n) = 13n^2 + n$	$T(n) = \Theta(n^2)$
$T(n) = n^3$	$T(n) \neq \Theta(n^2)$

Big-O Notation

Some simple rules for most cases...

Big-O Notation

Order or size:

Function	Name
5	Constant
loglog(n)	double log
log(n)	logarithmic
$log^2(n)$	Polylogarithmic
n	linear
nlog(n)	log-linear
n^3	polynomial
$n^3log(n)$	
n^4	polynomial
2^n	exponential
2^{2n}	
n!	factorial

Big-O Notation

Rules:

If $T(n)$ is a polynomial of degree k then:

$$T(n) = O(n^k)$$

Example:

$$10n^5 + 50n^3 + 10n + 17 = O(n^5)$$

Big-O Notation

Rules:

If $T(n) = \mathrm{O}(f(n))$ and $S(n) = \mathrm{O}(g(n))$ then

$$\boldsymbol{T(n) + S(n) = \mathrm{O}(f(n) + g(n))}$$

Example:

$$10n^2 = \mathrm{O}(n^2)$$

$$5nlog(n) = \mathrm{O}(nlog(n))$$

$$10n^2 + 5nlog(n) = \mathrm{O}(n^2 + nlog(n)) = \mathrm{O}(n^2)$$

Big-O Notation

Rules:

If $T(n) = O(f(n))$ and $S(n) = O(g(n))$ then:

$$\boldsymbol{T(n)*S(n) = O(f(n)*g(n))}$$

Example:

$10n^2 = O(n^2)$

$5n = O(n)$

$(10n^2)(5n) = 50n^3 = O(n*n^2) = O(n^3)$

Why don’t you try a few?

$$4n^2\log(n) + 8n + 16 = ?$$

1. $O(\log n)$
2. $O(n)$
3. $O(n\log n)$
4. $O(n^2\log n)$
5. $O(2^n)$

$$2^{2n} + 2^n + 2 =$$

1. $O(n)$
2. $O(n^6)$
3. $O(2^n)$
4. $O(2^{2n})$
5. $O(n^n)$

$\log(n!) =$

1. $O(\log n)$
2. $O(n)$
3. $O(n \log n)$
4. $O(n^2)$
5. $O(2^n)$

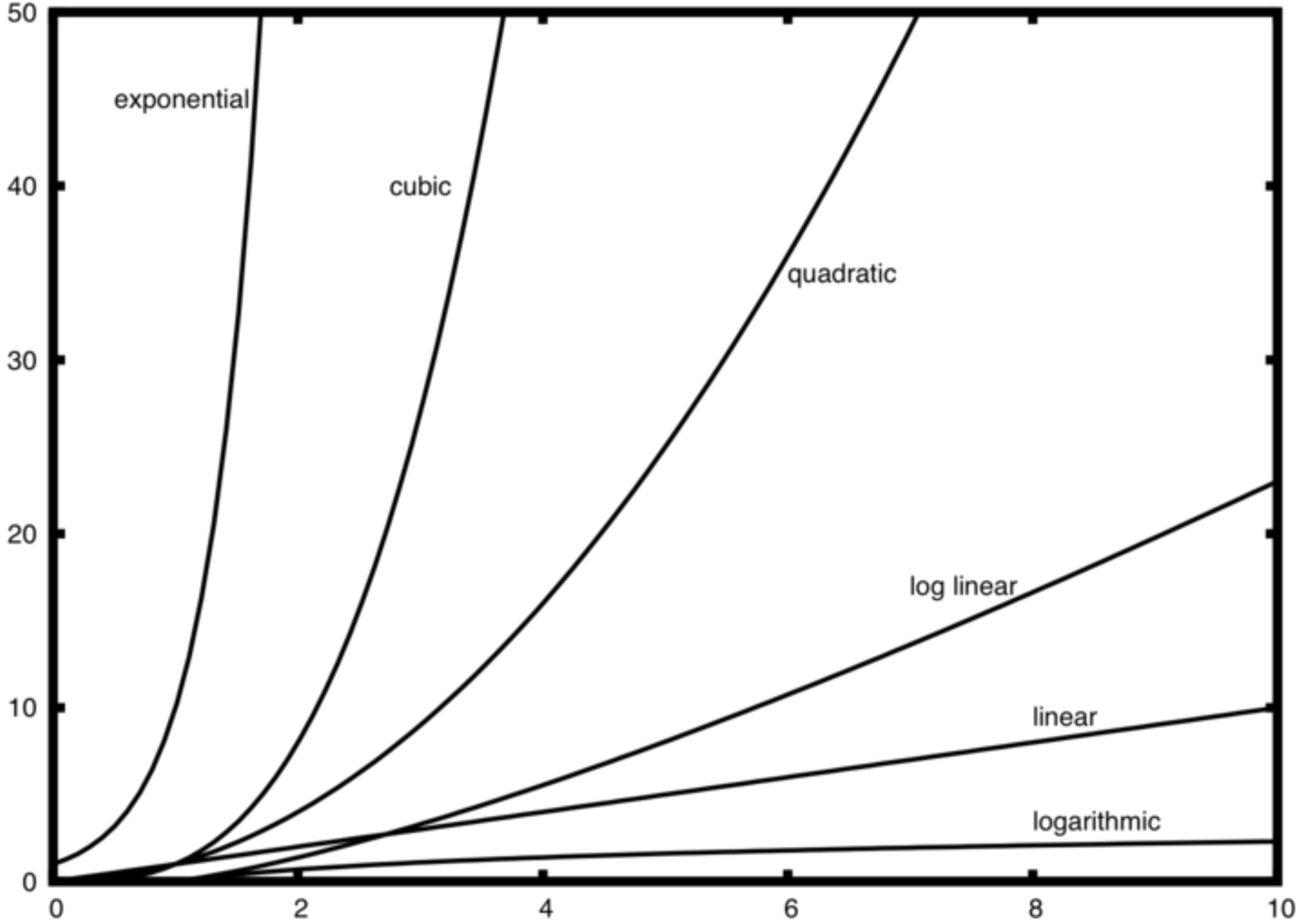
$\log(n!) =$

1. $O(\log n)$
2. $O(n)$
3. $O(n \log n)$
4. $O(n^2)$
5. $O(2^n)$

Hint: Sterling's Approximation

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

In General



50
40
30
20
10
0
0
2
4
6
8
10
exponential
cubic
quadratic
log linear
linear
logarithmic

Model of Computation?

Different ways to "compute":

- Sequential (RAM) model of computation
- Parallel (PRAM, BSP, Map-Reduce)
- Circuits
- Turing Machine
- Counter machine
- Word RAM model
- Quantum computation
- Etc.

Model of Computation

Sequential Computer

- One thing at a time

- All operations take constant time

  Addition, subtraction, multiplication, comparison

Algorithm Analysis

Example:

```
void sum(int k, int[] intArray) {
   int total=0;
   for (int i=0; i<= k; i++){
       total = total + intArray[i];
   }
   return total;
}
```

1 assignment

1 assignment
k+1 comparisons
k increments

k array access
k addition
k assignment

1 return

Total: 1 + 1 + (k+1) + 3k + 1 = 4k+4 = O(k)

Algorithm Analysis

What is the cost of this operation?

Example:

```
void sum(int k, int[] intArray) {
   int total=0;
   String name="Stephanie";
   for (int i=0; i<= k; i++){
        total = total + intArray[i];
        name = name + "?"
   }
   return total;
}
```

Not 1!
Not constant!
Not k!

Moral: all costs are not 1.

Rules

Loops

cost = (# iterations)x(max cost of one iteration)

```
int sum(int k, int[] intArray) {
   int total=0;
   for (int i=0; i<= k; i++){
       total = total + intArray[i];
   }
   return total;
}
```

Rules

Nested Loops

cost = (# iterations)(max cost of one iteration)

```
int sum(int k, int[] intArray) {
    int total=0;
    for (int i=0; i<= k; i++){
        for (int j=0; j<= k; j++){
            total = total + intArray[i];
        }
    }
    return total;
}
```

Rules

Sequential statements

cost = (cost of first) + (cost of second)

```
int sum(int k, int[] intArray) {
   for (int i=0; i<= k; i++)
        intArray[i] = k;
   for (int j =0; j<= k; j++)
        total = total + intArray[i];
   return total;
}
```

Rules

if / else statements

cost = max(cost of first, cost of second)

<= (cost of first) + (cost of second)

```
void sum(int k, int[] intArray) {
   if (k > 100)
       doExpensiveOperation();
   else
       doCheapOperation();
   return;
}
```

Rules

For recursive function calls.....

Recurrences

$$T(n) = 1 + T(n - 1) + T(n - 2)$$
$$= O(2^n)$$

$T(n\text{-}1)$ $T(n\text{-}1)$

```
int fib(int n) {
   if (n <= 1)
       return n;
   else
       return fib(n-1) + fib(n-2);
}
```

What is the running time?

```
for (int i = 0; i<n; i++)
  for (int j = 0; j<i; j++)
    store[i] = i + j;
```

1. $O(1)$
2. $O(n)$
3. $O(n \log n)$
4. $O(n^2)$
5. $O(n^2 \log n)$
6. $O(2^n)$

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Today: Divide and Conquer!

Algorithm Analysis

- Big-O Notation
- Model of computation

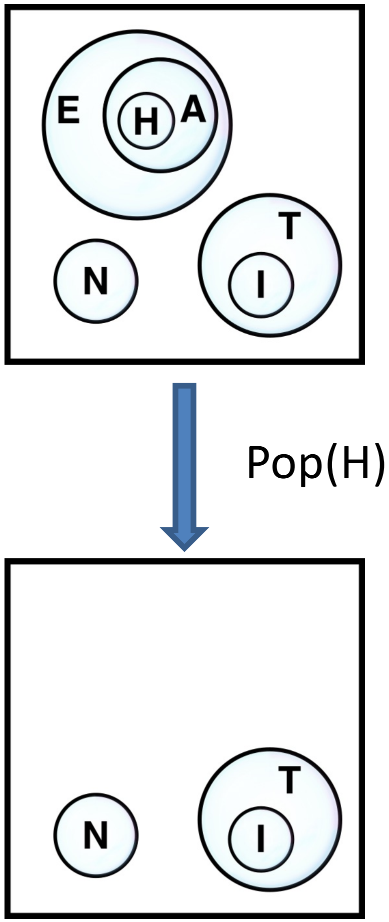
Searching

Peak Finding

- 1-dimension
- 2-dimensions

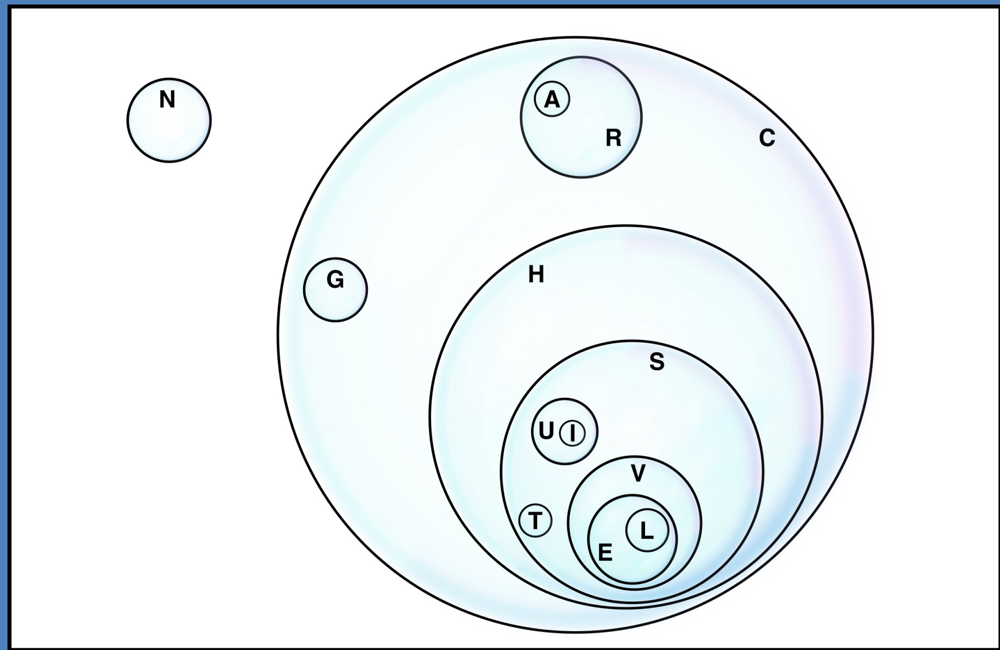
Puzzle of the Week: Bubbly

(Courtesy: MIT Puzzle Hunt, 2019)



- Two player game
- Players alternate popping bobbles
- The last player to pop a bubble wins.

Find the best first move.



Today: Divide and Conquer!

Algorithm Analysis

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Peak Finding

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Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

Search for *k* in array A.

Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

Search for 17 in array A.

Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

Search for 17 in array A.

– Find middle element: 7

Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

Search for 17 in array A.

- Find middle element: 7
- Compare 17 to middle element: 17 > 7

Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

Search for 17 in array A.

- Find middle element: 7
- Compare 17 to middle element: 17 > 7

Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

Search for 17 in array A.

- Find middle element: 7
- Compare 17 to middle element: 17 > 7
- Recurse on right half

Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

Search for 17 in array A.

- Find middle element
- Compare 17 to middle element
- Recurse

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Binary Search

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Search for 17 in array A.

- Find middle element
- Compare 17 to middle element
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Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

Search for 17 in array A.

- Find middle element
- Compare 17 to middle element
- Recurse

Problem Solving: Reduce the Problem

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

Reduce-and-Conquer:

- Start with n elements to search.
- Eliminate half of them.
- End with $n/2$ elements to search.
- Repeat.

How hard is binary search?

How many of you think you could write a correct implementation of binary search, right now?

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By Jon Bentley

programming pearls

WRITING CORRECT PROGRAMS

Jon Bentley

The Challenge of Binary Search
Even with the best of designs, every now and then a programmer has to write subtle code. This column is about one problem that requires particularly careful code: binary search. After defining the problem and sketching an algorithm to solve it, we'll use principles of program verification in several stages as we develop the program.

Most programmers think that with the above description in hand, writing the code is easy; they're wrong. The only way you'll believe this is by putting down this column right now, and writing the code yourself. Try it.

programming pearls

By Jon Bentley

I've given this problem as an in-class assignment in courses at Bell Labs and IBM. The professional programmers had one hour (sometimes more) to convert the above description into a program in the language of their choice; a high-level pseudocode was fine. At the end of the specified time, almost all the programmers reported that they had correct code for the task. We would then take 30 minutes to examine their code, which the programmers did with test cases. In many different classes and with over a hundred programmers, the results varied little: 90 percent of the programmers found bugs in their code (and I wasn't always convinced of the correctness of the code in which no bugs were found).

I found this amazing: only about 10 percent of professional programmers were able to get this small program right. But they aren't the only ones to find this task difficult. In the history in Section 6.2.1 of his *Sorting and Searching*, Knuth points out that while the first binary search was published in 1946, the first published binary search without bugs did not appear until 1962.

Jon Bentley

How hard is binary search?

How many of you think you could write a correct implementation of binary search, right now?

Try it yourself!

Missions

Problem Sets | Lecture Trainings | Optional Practice

Title	Experience Points	Requirement For	Start At	
Guess the Number (Binary Search)	200		17 Jan 18:00	Resume Submissions
WiFi (Binary Search)	200		19 Jan 18:00	Attempt Submissions

Binary Search (buggy)

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

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is open

```
Search(A, key, n)
    begin = 0
    end = n
    while begin != end do:
        if  key < A[(begin+end)/2]  then
              end = (begin+end)/2 - 1
        else begin = (begin+end)/2
    return A[begin]
```

Bug 1

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

```
Search(A, key, n)
    begin = 0
    end = n
    while begin != end do:
        if  key < A[(begin+end)/2] then
                end = (begin+end)/2 - 1
        else begin = (begin+end)/2
    return A[end]
```

array out of bounds!

Bug 1

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

```
Search(A, key, n)
    begin = 0
    end = n
    while begin != end do:
        if  key < A[(begin+end)/2] then
            end = (begin+end)/2 - 1
        else begin = (begin+end)/2
    return A[end]
```

array out of bounds!

(Can't happen because of other bugs…)

Bug 1

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

```
Search(A, key, n)
    begin = 0
    end = n-1
    while begin != end do:
        if  key < A[(begin+end)/2] then
              end = (begin+end)/2 - 1
        else begin = (begin+end)/2
    return A[end]
```

Bug 2

Sorted arra

2	4	4

Example: search(7)

- begin = 0, end = 1
- mid = (0+1)/2 = 0
- key >= A[mid] ➔ begin = 0

5	10

```
Search(A, k
    begin = 0
    end = n-1
    while begin != end do:
        if  key < A[(begin+end)/2] then
            end = (begin+end)/2 - 1
        else begin = (begin+end)/2
    return A[end]
```

May not terminate!

round down

Bug 2

Sorted arra

2	4	4

Example: search(2)

- begin = 0, end = 1
- mid = (0+1)/2 = 0
- key < A[mid] → end = 0 – 1 = -1

5	10

```
Search(A, k
    begin = 0
    end = n-1
    while begin != end do:
        if  key < A[(begin+end)/2] then
                end = (begin+end)/2 - 1
        else begin = (begin+end)/2
    return A[end]
```

end < begin

subtract?

Bug 3

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

```
Search(A, key, n)
    begin = 0
    end = n-1
    while begin != end do:
        if  key < A[(begin+end)/2] then
              end = (begin+end)/2 - 1
        else begin = (begin+end)/2
    return A[end]
```

Useful return value?

Binary Search

Specification:

- Returns element if it is in the array.
- Returns “null” if it is not in the array.

Alternate Specification:

- Returns index if it is in the array.
- Returns -1 if it is not in the array.

Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

```
int search(A, key, n)
    begin = 0
    end = n-1
    while begin < end do:
        if  key <= A[(begin+end)/2]  then
              end = (begin+end)/2
        else begin = 1+(begin+end)/2
    return (A[begin]==key) ? begin : -1
```

Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

less-than-or-equal

```
int search(A, key, n)
    begin = 0
    end = n-1
    while begin < end do:
        if  key <= A[(begin+end)/2]  then
            end = (begin+end)/2
        else begin = 1+(begin+end)/2
    return (A[begin]==key) ? begin : -1
```

Binary Search

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strictly greater than

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Array of out bounds?

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```

Array of out bounds?
No: division rounds down.

Bug 4

Sorted array: `A[0..n-1]`

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What if begin > MAX_INT/2?

Binary Search

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```

What if begin > MAX_INT/2?

Overflow error: begin+end > MAX_INT

Binary Search

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int search(A, key, n)
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```

Moral of the Story

Easy algorithms are *hard* to write correctly.

Binary search is 9 lines of code.

If you can't write 9 correct lines of code, how do you expect to write thousands of lines of bug-free code??

Precondition and Postcondition

Precondition:

- Fact that is true when the function begins.
- Usually important for it to work correctly.

Postcondition:

- Fact that is true when the function ends.
- Usually useful to show that the computation was done correctly.

Binary Search

ARCHIPELAGO

is open

Sorted array: `A[0..n-1]`

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```

What are useful preconditions and postconditions?

Binary Search

Functionality:

- If element is in the array, return index of element.
- If element is not in array, return -1.

Binary Search

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- If element is in the array, return index of element.
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Preconditions:

- Array is of size n
- Array is sorted

Good practice:
Validate pre-conditions when possible.

Binary Search

Functionality:

- If element is in the array, return index of element.
- If element is not in array, return -1.

Preconditions:

- Array is of size n
- Array is sorted

You can usually check this directly.

Binary Search

Functionality:

- If element is in the array, return index of element.
- If element is not in array, return -1.

Preconditions:

- Array is of size n
- Array is sorted

Should we do input validation to make sure array is sorted??

Yes or No on Zoom.

Binary Search

Functionality:

- If element is in the array, return index of element.
- If element is not in array, return -1.

Preconditions:

- Array is of size n
- Array is sorted

Should we do input validation to make sure array is sorted??
NO! Too slow!

Binary Search

Functionality:

- If element is in the array, return index of element.
- If element is not in array, return -1.

Preconditions:

- Array is of size n
- Array is sorted

Postcondition:

- If element is in the array: `A[begin] = key`

Invariants

Invariant:

- relationship between variables that is always true.

Invariants

Invariant:

- relationship between variables that is always true.

Loop Invariant:

- relationship between variables that is true at the beginning (or end) of each iteration of a loop.

Binary Search

ARCHIPELAGO
is open

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```

What are useful invariants?

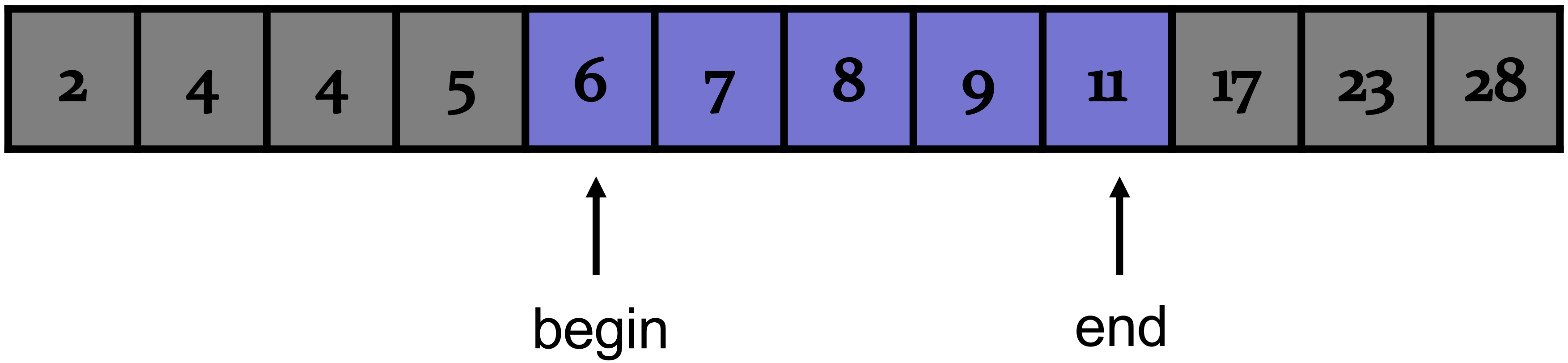
Binary Search

Loop invariant:

- `A[begin] ≤ key ≤ A[end]`

Interpretation:

- The key is in the range of the array



Binary Search

Loop invariant:

- `A[begin] ≤ key ≤ A[end]`

Interpretation:

- The key is in the range of the array

Validation (in debug mode; disable for production?):

```
if ((A[begin] > key) or (A[end] < key))
    System.out.println("error");
```

Binary Search

Sorted array: `A[0..n-1]`

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```

Is the loop invariant always true?

Yes or No on Zoom.

Binary Search

Sorted array: `A[0..n-1]`

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```

To enforce invariant, we would need an extra step. Or we can refine the invariant.

Binary Search

n	2	4	4	5	6	7	8	9	11	17	23	28
n/2	2	4	4	5	6	7	8	9	11	17	23	28
n/4	2	4	4	5	6	7	8	9	11	17	23	28
n/8	2	4	4	5	6	7	8	9	11	17	23	28

Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

Iteration 1: (end – begin) = n

Iteration 2: (end – begin) = $n/2$

Iteration 3: (end – begin) = $n/4$

…

Iteration k: (end – begin) <= $n/2^k$

Another invariant!

$n/2^k = 1 \quad \Rightarrow \quad k = \log(n)$

Key Invariants:

Correctness:

- `A[begin] ≤ key ≤ A[end]`

Performance:

- `(end-begin) <= n/2`^{k} in iteration k.

Binary Search

Sorted array: A[0..n-1]

2	4	4	5	6	7	8	9	11	17	23	28

Not just for searching arrays:

- Assume a complicated function:

  int complicatedFunction(int s)

- Assume the function is always increasing:

  complicatedFunction(i) < complicatedFunction(i+1)

- Find the minimum value j such that:

  complicatedFunction(j) > 100

A problem...

Tutorial allocation

A problem...

Tutorial allocation

Tutorials
(in order
of tutor
preference)

T1

T2

T3

T4

T5

T6

T7

T8

A problem...

Tutorial allocation

Tutorials
(in order
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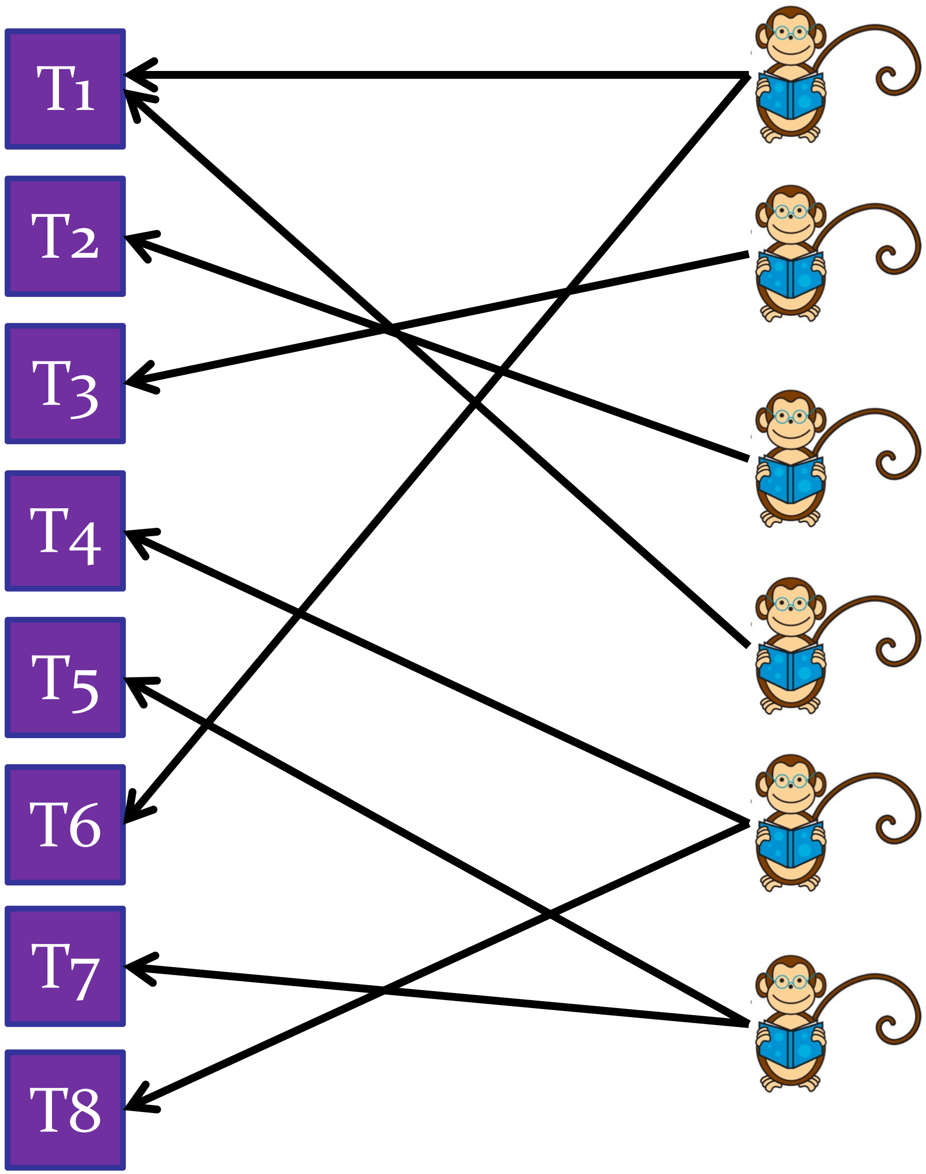
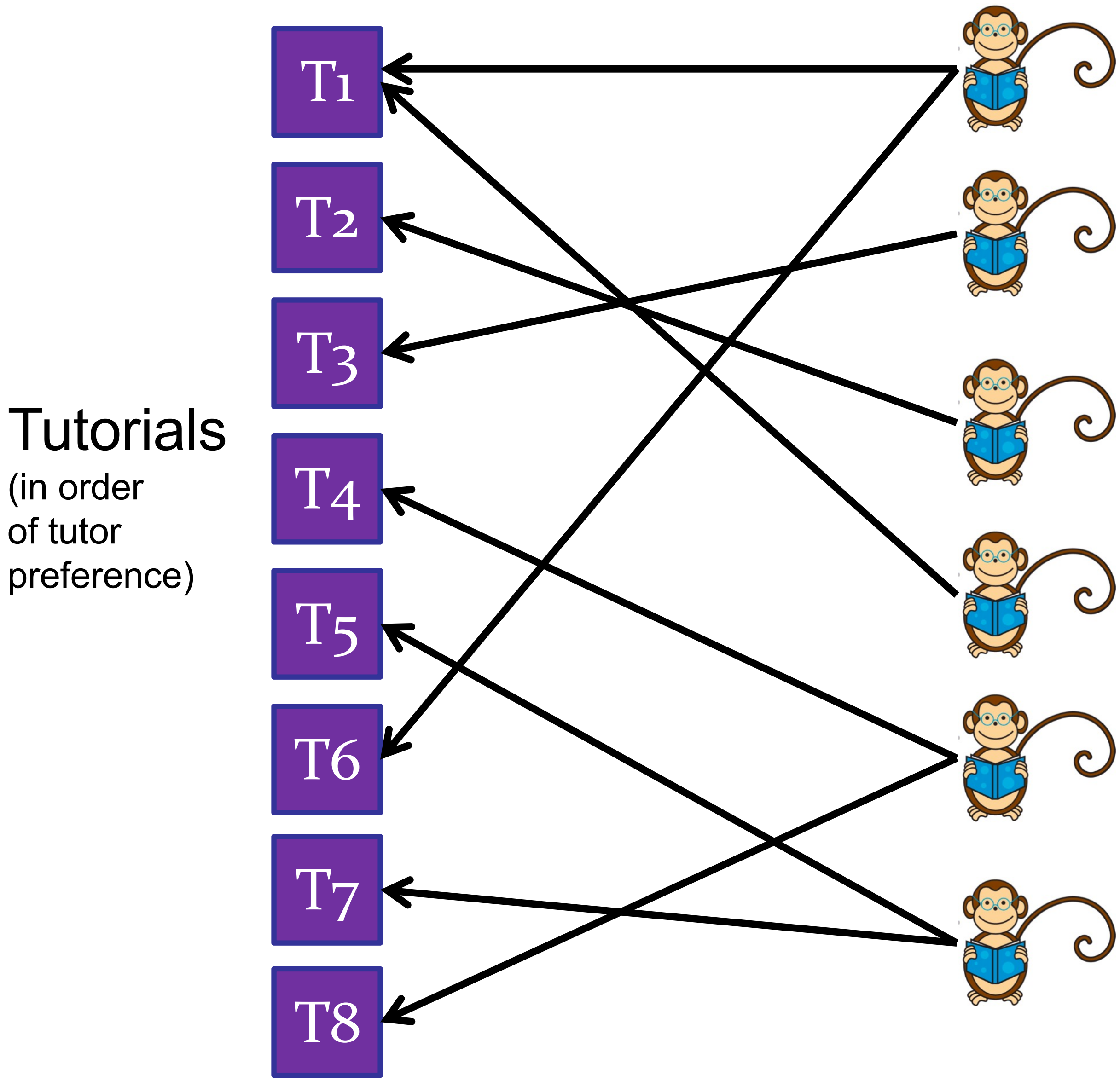
preference)



Students want certain tutorials.

A problem...

Tutorial allocation



Students want certain tutorials.

We want each tutorial to have < 18 students..

A problem...

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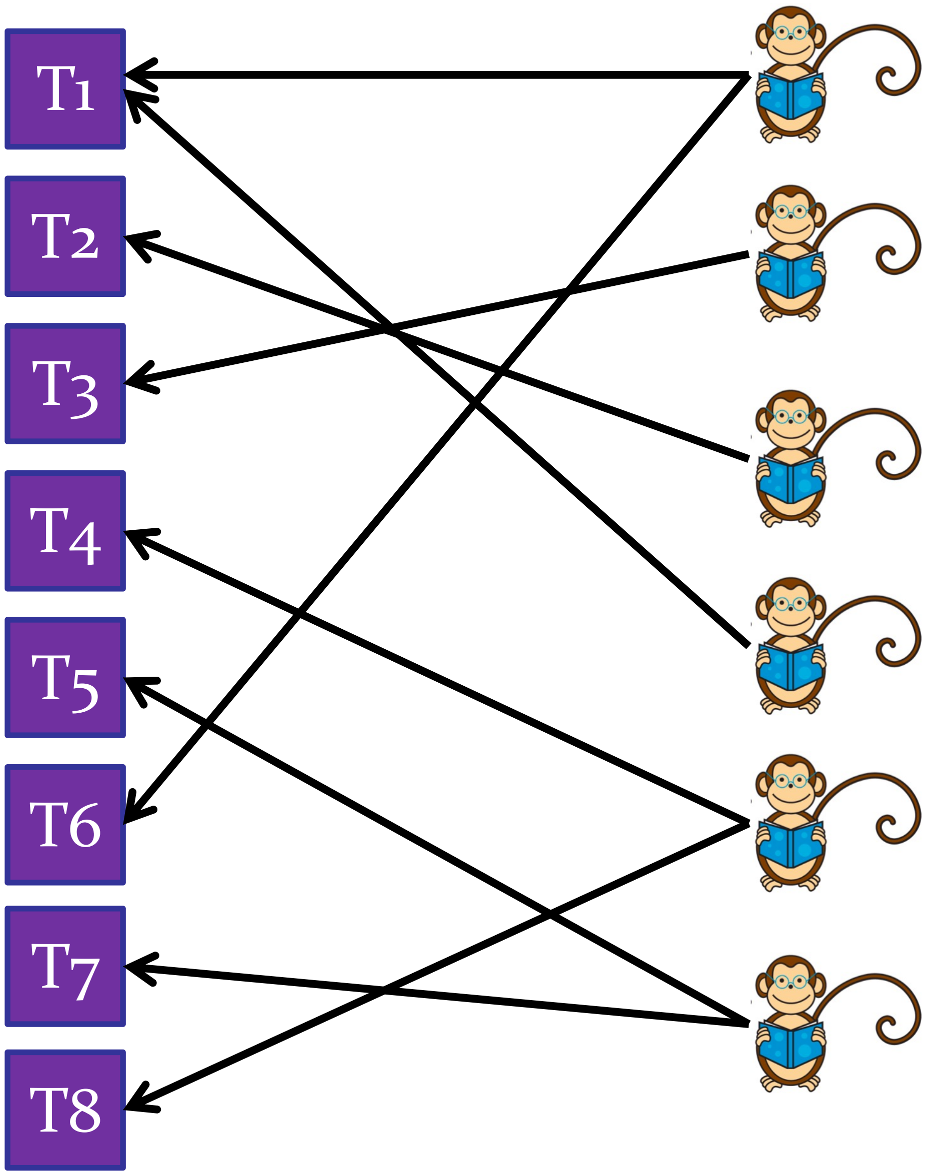

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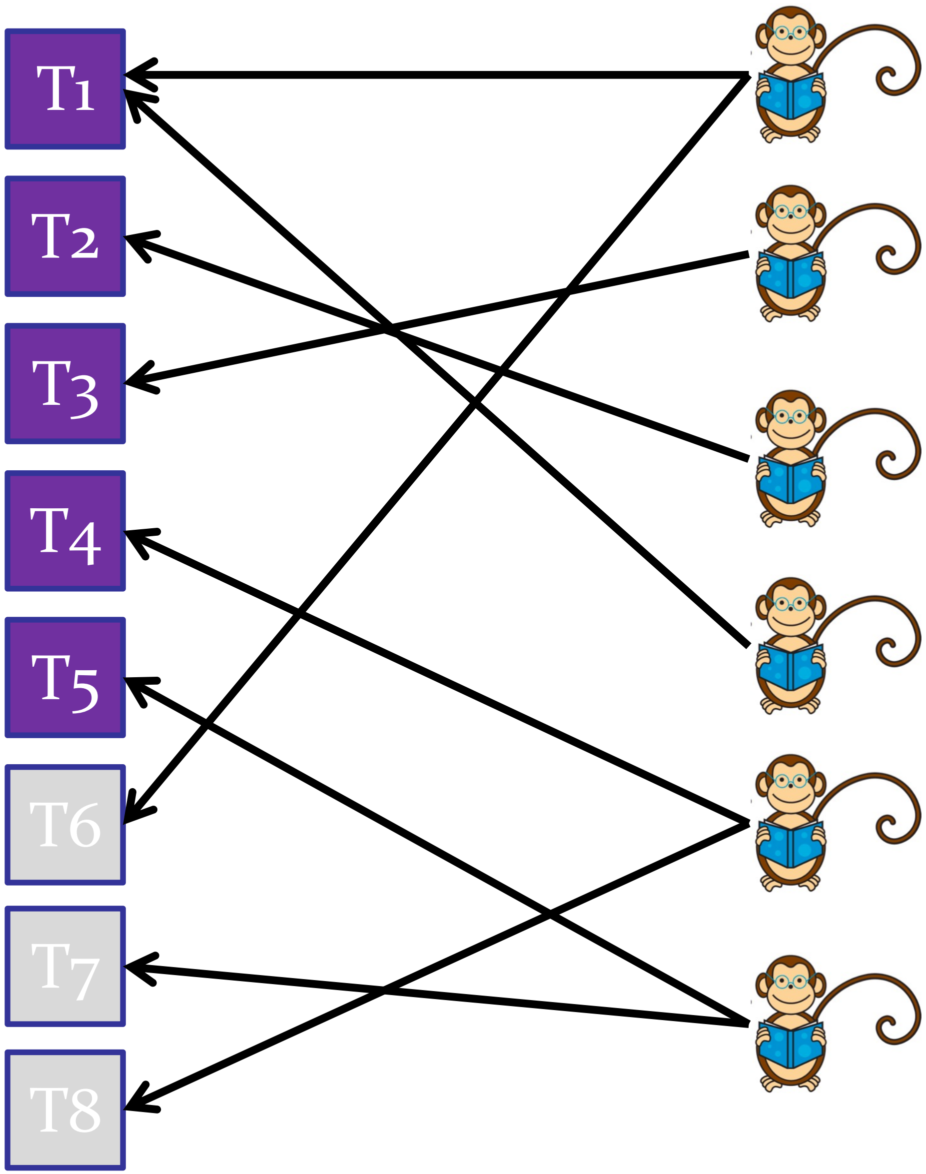

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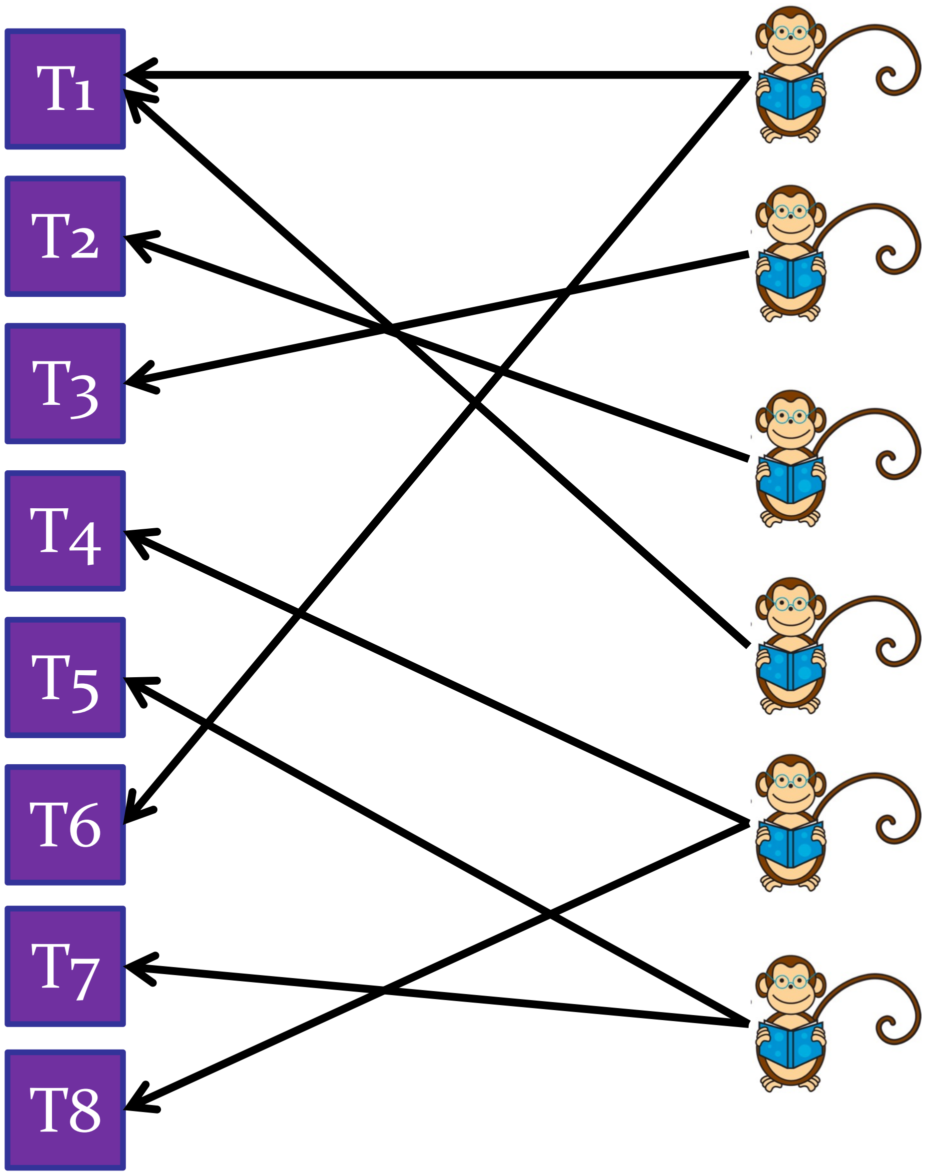
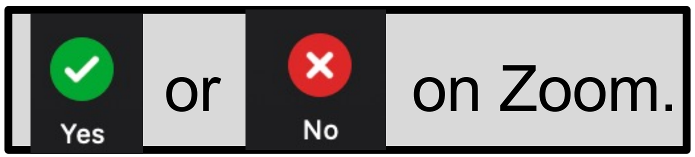


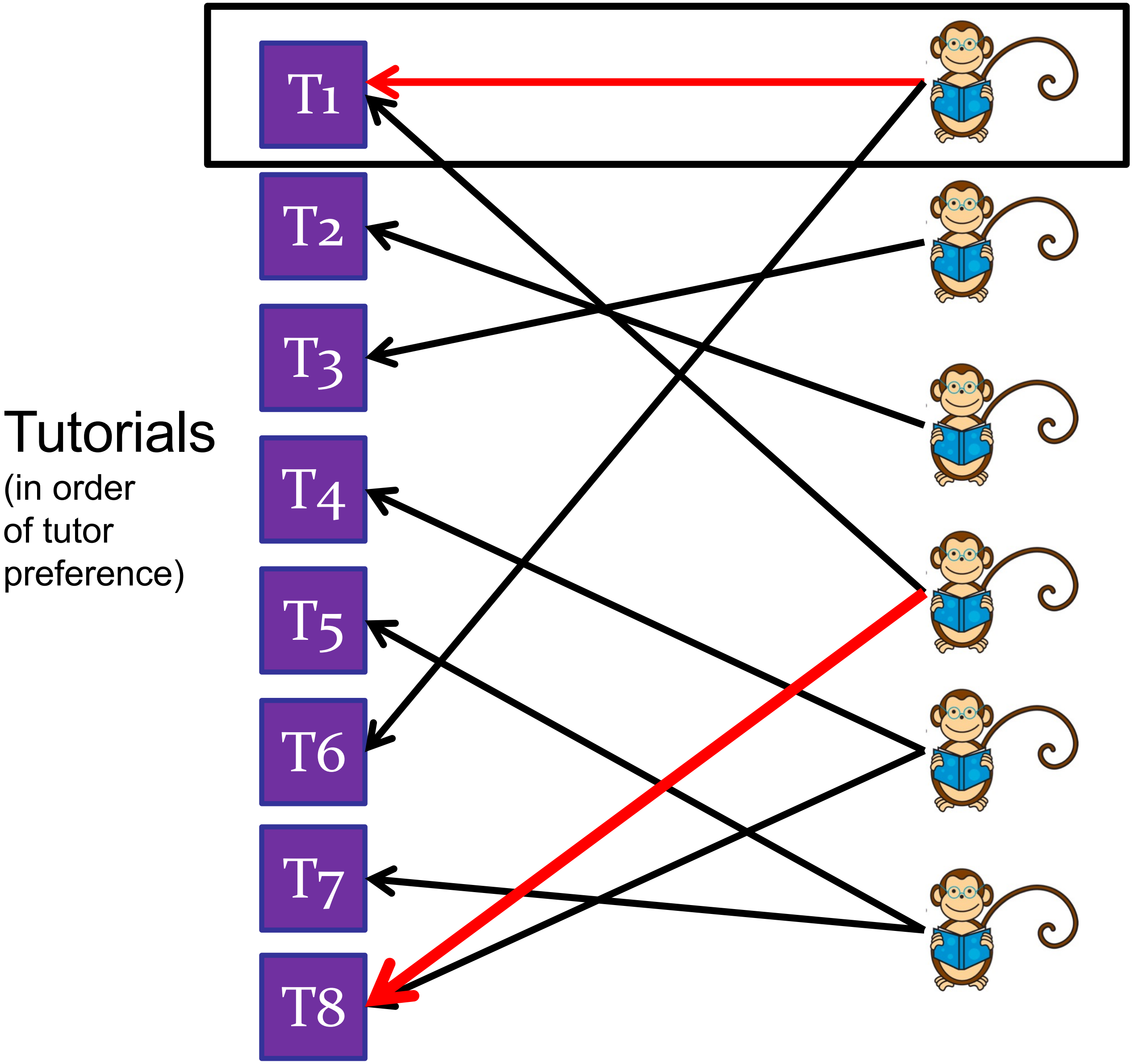
Can we do
greedy allocation?

First, fill T1.
Then fill T2.
Then fill T3.
…
Stop when all
students are
allocated



A problem…

Tutorial allocation



Can we do greedy allocation?

NO

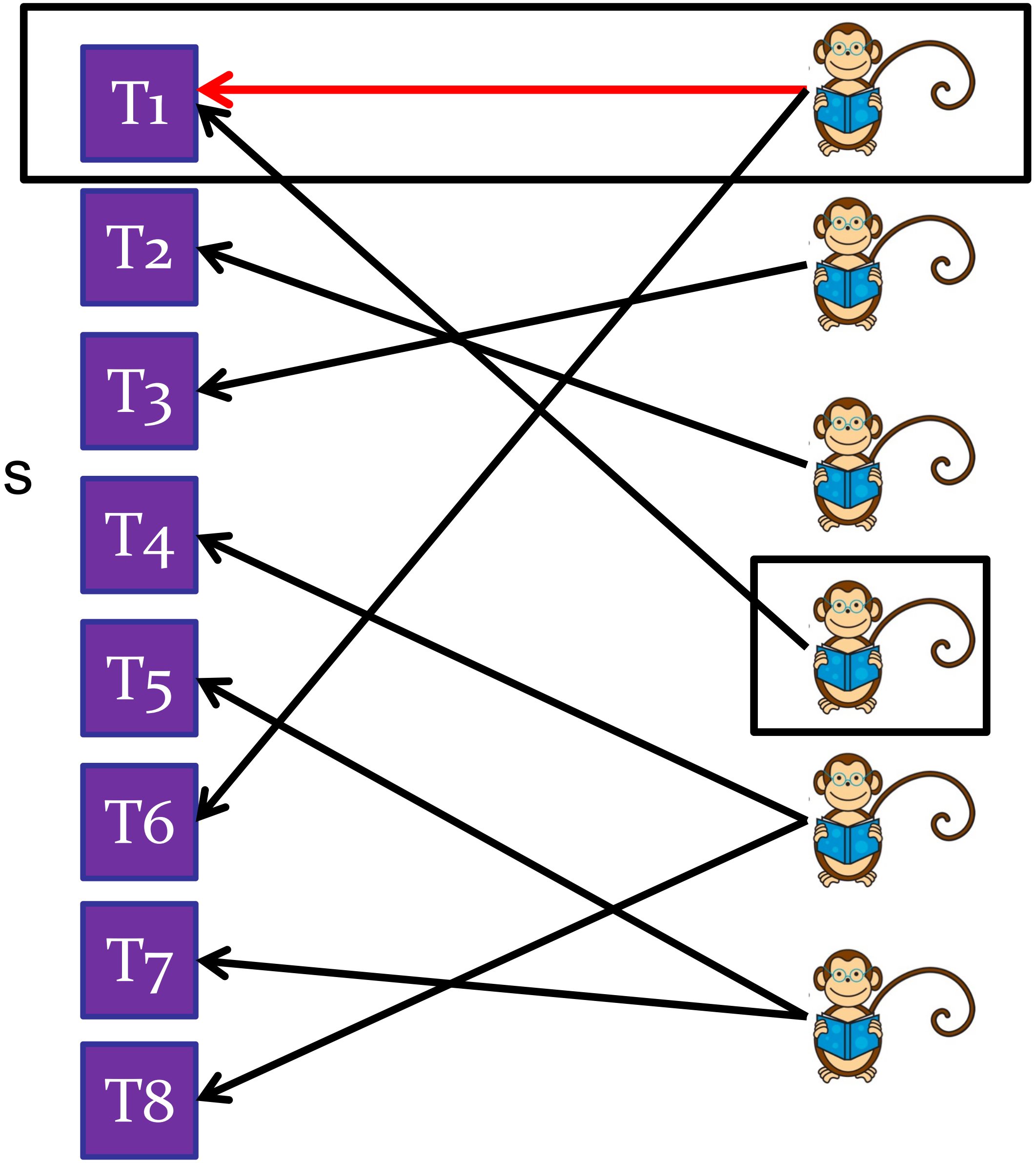
Assume max tutorial size is 1.

Assign student 1 to tutorial 1.

Now we need all 8 tutorials.

A problem…

Tutorial allocation



Tutorials
(in order
of tutor
preference)

Can we do
greedy allocation?

NO

Assume max
tutorial size is 1.

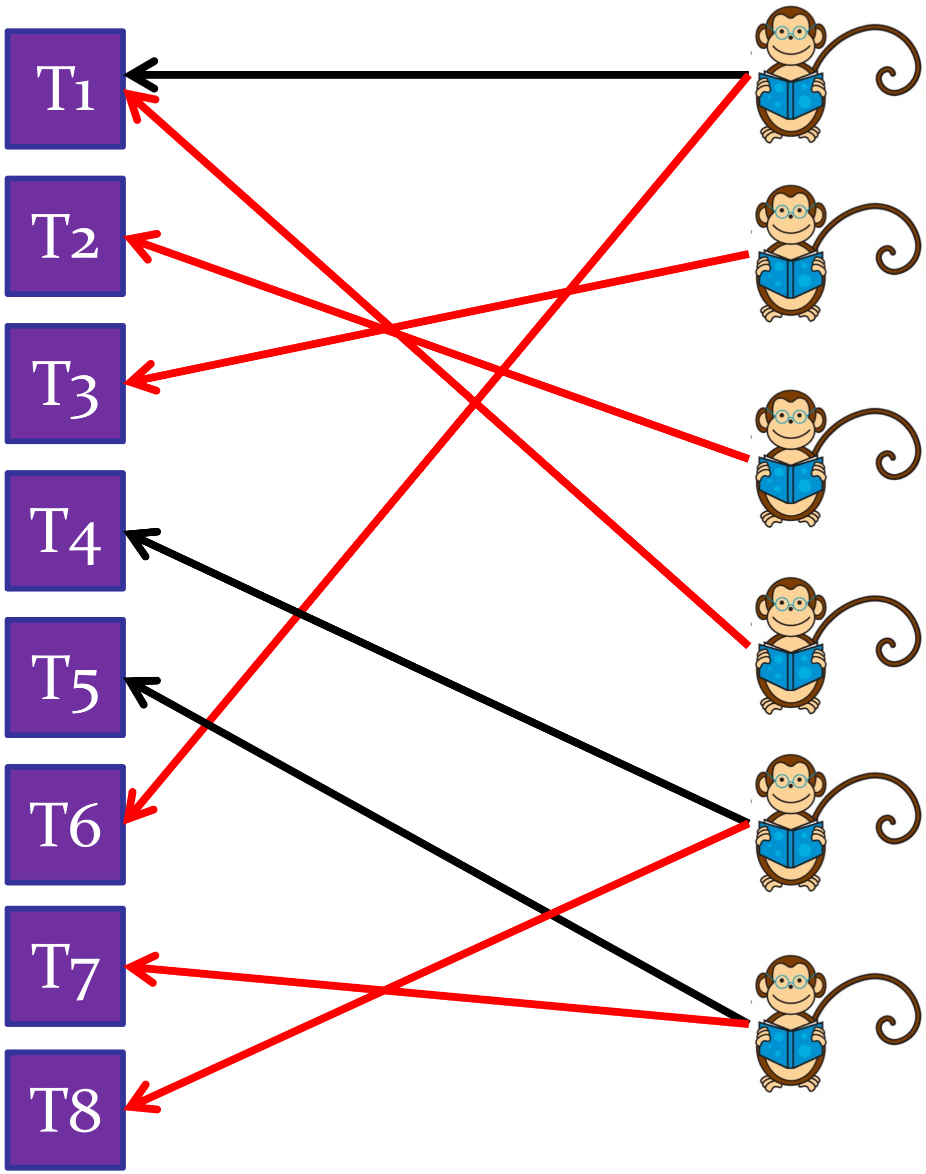
Assign student 1 to
tutorial 1.

Now one student
has no feasible
allocation!

A problem…

Tutorial allocation

Tutorials
(in order
of tutor
preference)



Assume we can solve allocation problem:

Given a fixed set of tutorials and a fixed set of students, find an allocation where every student has a slot.

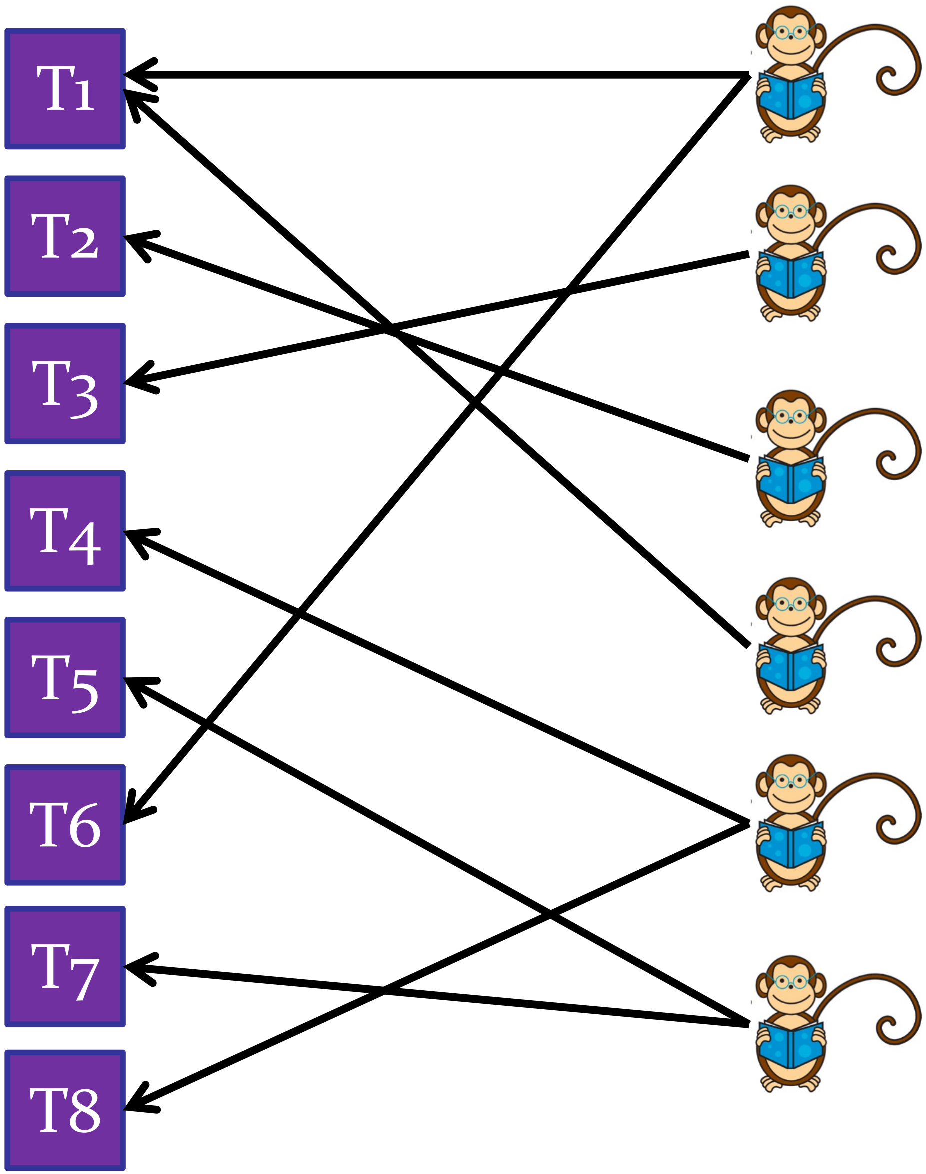
Warning:

- may be > 18 students in a slot!
- minimizes max students in a slot.

A problem...

Tutorial allocation

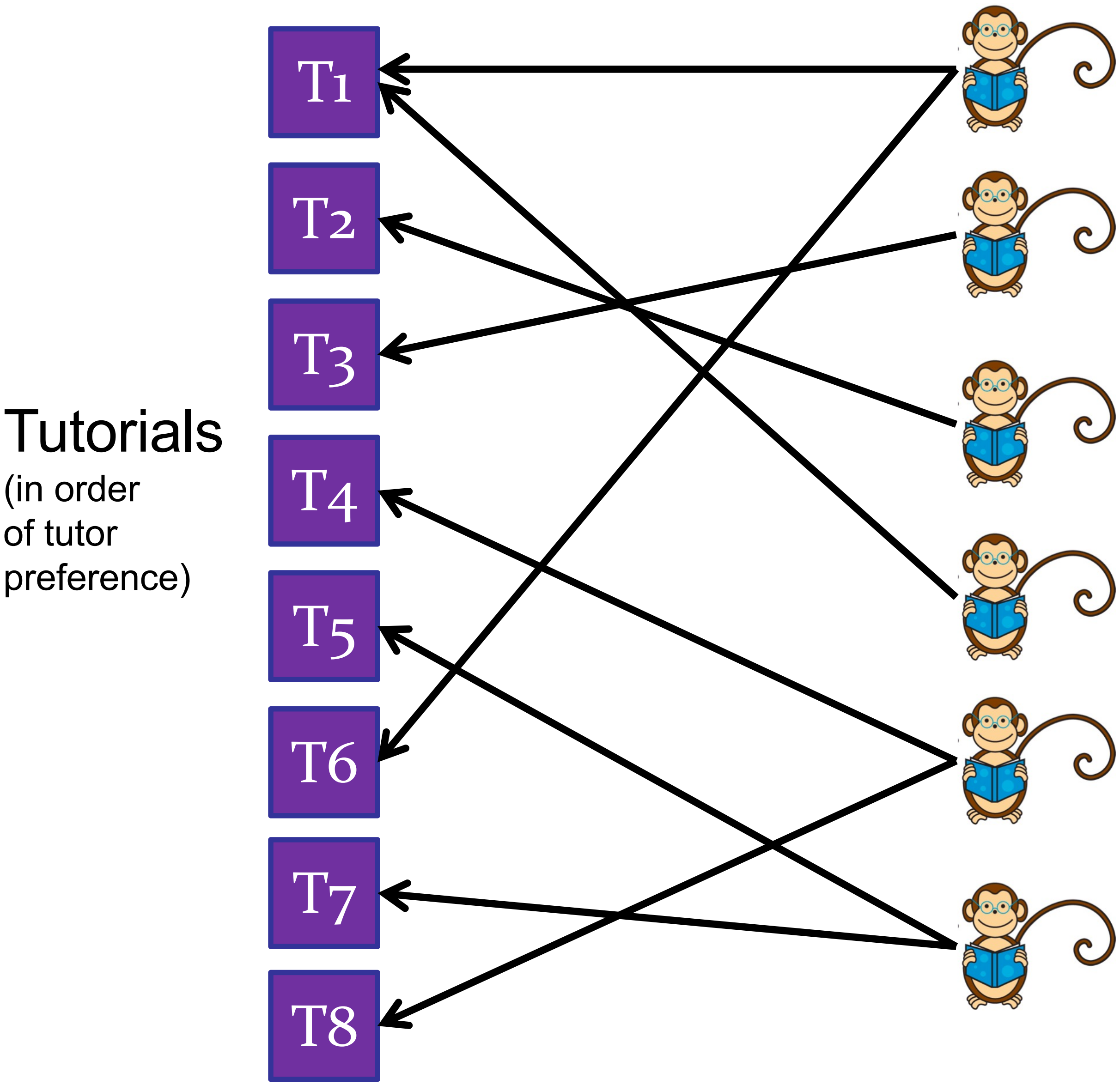
Tutorials
(in order
of tutor
preference)



How to find minimum number of tutorials that we need to open to ensure: no tutorial has more than 18 students.

A problem...

Tutorial allocation



Observation:

Number of students in BIGGEST tutorial only decreases as number of tutorials increases.

Monotonic function of number of tutorials!

A problem...

Tutorial allocation

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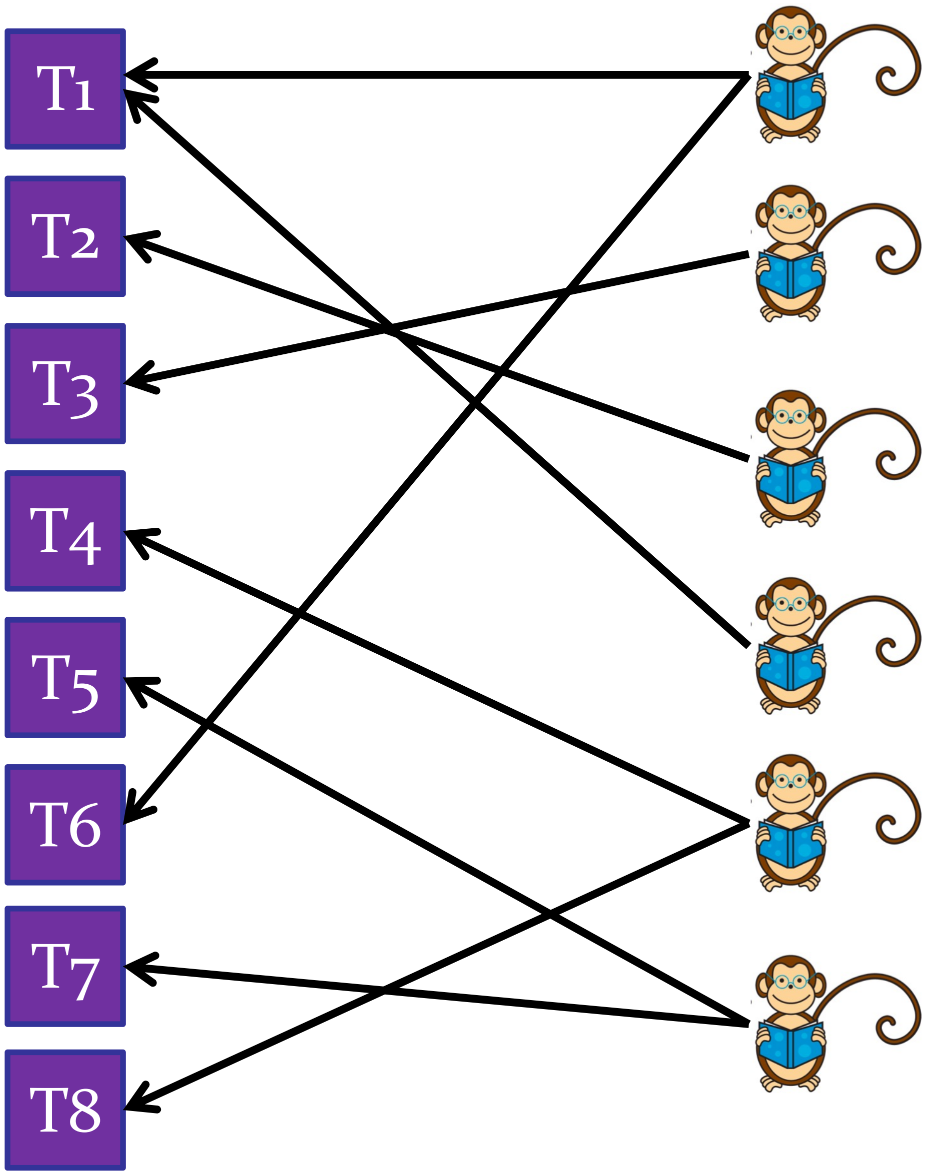
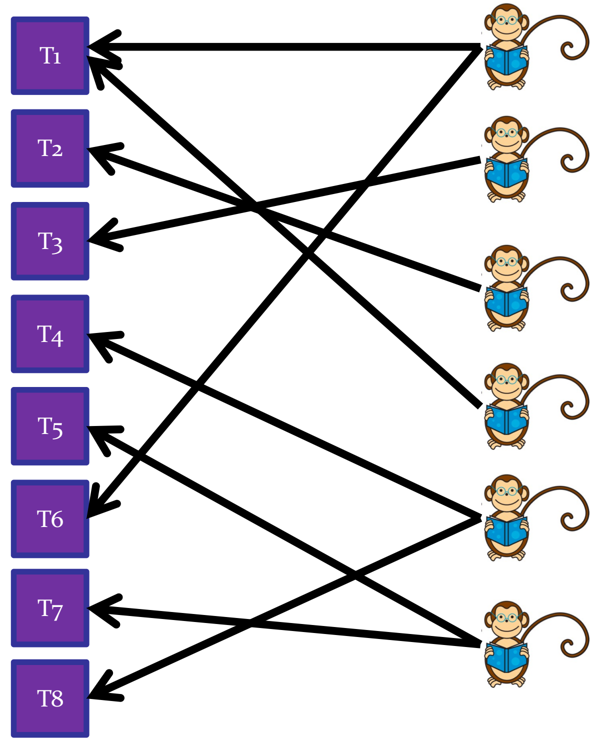


Monotonic function of number of tutorials!

Binary Search

A problem…

Tutorial allocation



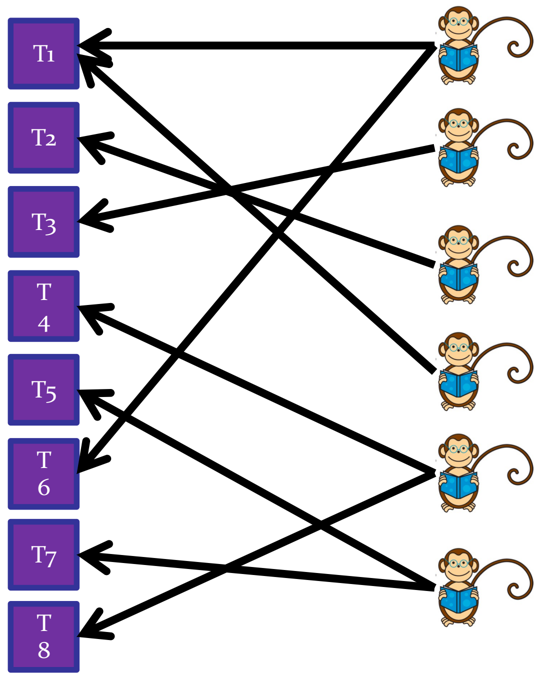
Solution:

Binary Search

Define:

MaxStudents(x) = number of students in most crowded tutorial, if we offer x tutorials.

Binary Search



MaxStudents(x) = number of students in most crowded tutorial, if we offer x tutorials.

```
Search(n)
    begin = 0
    end = n-1
    while begin < end do:
        mid = begin + (end-begin)/2;
        if  MaxStudents(mid) <= 18 then
                end = mid
        else begin = mid+1
    return begin
```

Binary Search

Sorted array: A[0..n-1]

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- Assume a complicated function:

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- Assume the function is always increasing:

  complicatedFunction(i) < complicatedFunction(i+1)

- Find the minimum value j such that:

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Today: Divide and Conquer!

Algorithm Analysis

- Big-O Notation
- Model of computation

Searching

Peak Finding

- 1-dimension
- 2-dimensions