Solutions to Tutorial 2

MA1521 Calculus for Computing

1. (a) $\lim_{x\to 0}\frac{2x\sin(3x)}{\tan^2(4x)} = \lim_{x\to 0}\frac{6}{16}\frac{\frac{\sin(3x)}{3x}}{(\frac{\tan(4x)}{4x})^2} = \frac{3}{8}$. Here we use the results that $\lim_{x\to 0}\frac{\sin(3x)}{3x} = 1$ and $\lim_{x\to 0}\frac{\tan(4x)}{4x} = 1$.

   (b) $\lim_{x\to 3}\left(\frac{\tan(2\ln(x-2))}{3\ln(x-2)}\right)^2 = \left(\frac{2}{3}\lim_{x\to 3}\frac{\tan(2\ln(x-2))}{2\ln(x-2)}\right)^2 = \frac{4}{9}$.

   (c) $\lim_{x\to 1}\frac{x^2-4x+3}{\tan(x^2-x)} = \lim_{x\to 1}\frac{(x-1)(x-3)}{\tan(x(x-1))} = \lim_{x\to 1}\frac{x(x-1)}{\tan(x(x-1))}\frac{x-3}{x} = -2$.

2. (a) By the given inequality, we have $\frac{3^x+1}{x^2} > \frac{x^4}{x^2} = x^2$ for $x \ge 12$. As $\lim_{x\to\infty} x^2 = +\infty$, we have $\lim_{x\to\infty}\frac{3^x+1}{x^2} = +\infty$.

   (b) By the given inequality, we have $0 \le \frac{x^3}{3^x+1} < \frac{x^3}{x^4+1} < \frac{x^3}{x^4} = \frac{1}{x}$ for $x \ge 12$. Since $\lim_{x\to\infty}\frac{1}{x} = 0$, we have $\lim_{x\to\infty}\frac{x^3}{3^x+1} = 0$ by squeeze theorem.

3. (a) By quotient rule, $y' = \frac{a(cx+d)-c(ax+b)}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}$.

   (b) Using product rule and chain rule, we have $y' = n\sin^{n-1}x\cos x\cos mx - m\sin^n x\sin mx$.

   (c) By chain rule, $y' = e^{x^2+x^3}(2x+3x^2)$.

   (d) Note that e^2 and $\ln 2$ are constants. Thus $y' = 3x^2-8x$.

   Similarly, we obtain the derivatives in (e)-(j) as follow.

   (e) $y' = -2\sin\theta(\cos\theta-1)^{-2}$. (quotient and chain rule)

   (f) $y' = \sqrt{t}\sec^2(2\sqrt{t}) + \tan(2\sqrt{t})$. (product and chain rule)

   (g) $r' = \frac{2\sqrt{\theta+1}+1}{2\sqrt{\theta+1}}\cos(\theta+\sqrt{\theta+1})$. (chain rule)

   (h) $s' = 4\tan x\sec x - \csc^2 x$. (quotient rule)

   (i) $r' = \frac{2x}{\sqrt{1-(x^2-1)^2}}$. ($\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$)

   (j) $s' = \frac{e^x+x^{-\frac{1}{2}}}{1+(e^x+2\sqrt{x})^2}$. ($\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$)

4. Let $V_c(t)$ be the volume of coffee in the cone at time t and $V_p(t)$ be the volume of coffee in the pot at time t.

   Note that the rate of volume change in the pot $\dfrac{dV_p}{dt}$ is equal to the negative of the rate of volume change in the cone $\dfrac{dV_c}{dt}$. Since the volume of coffee in the cone is decreasing at the rate of 10 cm^3/min, we take $\dfrac{dV_c}{dt} = -10$. Thus, $\dfrac{dV_p}{dt} = -\dfrac{dV_c}{dt} = -(-10) = 10$.

   Let $h_c(t)$ be the level of coffee in the cone at time t and $h_p(t)$ be the level of coffee in the pot at time t.

   (a) We have $V_p = \text{base area} \times h_p = \frac{225}{4}\pi h_p$.

   Then $\dfrac{dV_p}{dt} = \dfrac{225}{4}\pi\dfrac{dh_p}{dt} \Rightarrow 10 = \dfrac{225}{4}\pi\dfrac{dh_p}{dt} \Rightarrow \dfrac{dh_p}{dt} = \dfrac{8}{45\pi}$.

   Therefore, the level in the pot is rising at the rate of $\frac{8}{45\pi}$ cm/min. Note that we do not need the value of $h_c = 5$.

   (b) $V_c = \dfrac{1}{3}\text{base area} \times h_c = \dfrac{1}{3}\pi r^2 h_c = \dfrac{1}{3}\pi(\dfrac{h_c}{2})^2 h_c = \dfrac{\pi h_c^3}{12}$.

   Note that the base radius r of the circular coffee surface in the cone is half that of the height h_c.

   Then $\dfrac{dV_c}{dt} = \dfrac{\pi h_c^2}{4}\dfrac{dh_c}{dt} \Rightarrow -10 = \dfrac{\pi 5^2}{4}\dfrac{dh_c}{dt} \Rightarrow \dfrac{dh_c}{dt} = -\dfrac{8}{5\pi}$.

   Thus the level in the cone is falling at the rate of $\frac{8}{5\pi}$ cm/min.

5. (a) We are given $x^{2/3} + y^{2/3} = a^{2/3}$. Differentiating the equality with respect to x, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0.$$

   Since $0 < x < a$ and $0 < y$, we have

$$\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}} = -\frac{\sqrt{a^{2/3} - x^{2/3}}}{x^{1/3}} = -\sqrt{(\frac{a}{x})^{2/3} - 1},$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}\frac{1}{\sqrt{(\frac{a}{x})^{2/3} - 1}}(-\frac{2}{3})a^{2/3}x^{-5/3} = \frac{a^{2/3}}{3x^{5/3}\sqrt{(\frac{a}{x})^{2/3} - 1}}$$

$$= \frac{a^{2/3}}{3x^{4/3}\sqrt{a^{2/3} - x^{2/3}}}.$$

(b) $y = (\sin x)^{\sin x}$, $0 < x < \frac{\pi}{2}$, so $\sin x > 0$.

$$\begin{aligned}
\ln y &= \sin x \ln \sin x, \quad \frac{y'}{y} = \cos x \ln \sin x + \cos x, \quad y' = y(1 + \ln \sin x)\cos x, \\
y'' &= y'(1 + \ln \sin x)\cos x + y\left[(1 + \ln \sin x)(-\sin x) + \frac{\cos^2 x}{\sin x}\right] \\
&= y(1 + \ln \sin x)^2 \cos^2 x + y\left[\frac{\cos^2 x}{\sin x} - (1 + \ln \sin x)\sin x\right].
\end{aligned}$$

Hence

$$\begin{aligned}
y' &= (\sin x)^{\sin x}(1 + \ln \sin x)\cos x, \\
y'' &= (\sin x)^{\sin x}\left[(1 + \ln \sin x)^2 \cos^2 x + \frac{\cos^2 x}{\sin x} - (1 + \ln \sin x)\sin x\right].
\end{aligned}$$

(c) $x = a\cos t$, $y = a\sin t$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{-a\sin t} = -\cot t, \\
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(-\cot t)}{\frac{dx}{dt}} = \frac{\frac{1}{\sin^2 t}}{-a\sin t} = -\frac{1}{a\sin^3 t}.
\end{aligned}$$

Solutions to Further Exercises

1. Let $'$ denotes differentiation with respect to time. As the rate of evaporation V' is proportional to the surface area, we have $V'/A = k$ is a constant, Thus
   $V = \frac{4}{3}\pi r^3 \Rightarrow V' = \frac{4}{3}\pi 3r^2 r' = Ar' \Rightarrow r' = V'/A = k$ is a constant.

2. First we have

$$\begin{aligned}
\frac{d}{dx}\ln(1-x) &= \frac{1}{1-x} \\
\frac{d^2}{dx^2}\ln(1-x) &= -\frac{1}{(1-x)^2} \\
\frac{d^3}{dx^3}\ln(1-x) &= -\frac{2}{(1-x)^3} \\
\frac{d^4}{dx^4}\ln(1-x) &= -\frac{2\times 3}{(1-x)^4} \\
\vdots &= \vdots \\
\frac{d^n}{dx^n}\ln(1-x) &= -\frac{(n-1)!}{(1-x)^n}.
\end{aligned}$$

   Similarly,

$$\frac{d^n}{dx^n}\ln(1+x) = -(-1)^n\frac{(n-1)!}{(1+x)^n}.$$

   Thus

$$\begin{aligned}\frac{d^n}{dx^n}\ln\frac{1-x}{1+x} &= \frac{d^n}{dx^n}\ln(1-x) - \frac{d^n}{dx^n}\ln(1+x)\\ &= -\frac{(n-1)!}{(1-x)^n} + (-1)^n\frac{(n-1)!}{(1+x)^n}\\ &= -(n-1)!\left(\frac{1}{(1-x)^n} - \frac{(-1)^n}{(1+x)^n}\right).\end{aligned}$$

Alternatively, one can prove the formula by induction on n.

3. By similar triangles,

$$\frac{y}{80} = \frac{x-20}{x},$$

and so

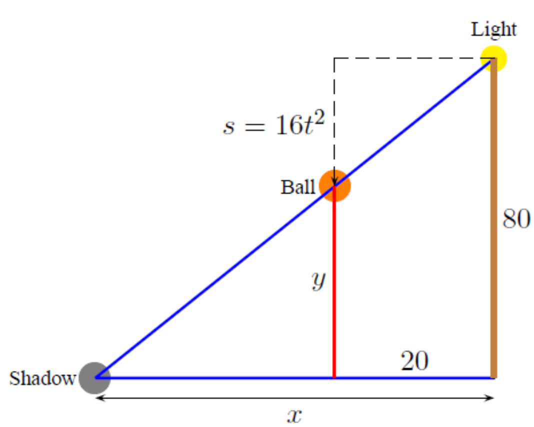
$$x = \frac{1600}{80-y}.$$

Thus,

$$\frac{dx}{dy} = \frac{1600}{(80-y)^2}.$$

On the other hand, $y = 80 - 16t^2$, and so

$$\frac{dy}{dt} = -32t.$$



Hence

$$\frac{dx}{dt} = \frac{dx}{dy}\frac{dy}{dt} = \frac{1600}{(80-y)^2}(-32t).$$

When $t = 1$, we have $80 - y = 16$. Thus

$$\frac{dx}{dt} = \frac{dx}{dy}\frac{dy}{dt} = \frac{1600}{(16)^2}(-32) = -200.$$

That is, the shadow is moving at 200 feet per second. The negative sign says that x is decreasing, i.e. the shadow is moving towards the right.