CS2040S – Data Structures and Algorithms

Lecture 12 – Balancing Act ~ AVL Tree

chongket@comp.nus.edu.sg

National University of Singapore
School of Computing

Outline

Binary Search Tree (BST): **A Quick Revision**

The Importance of a **Balanced** BST

- To keep **h** = O(log **N**)

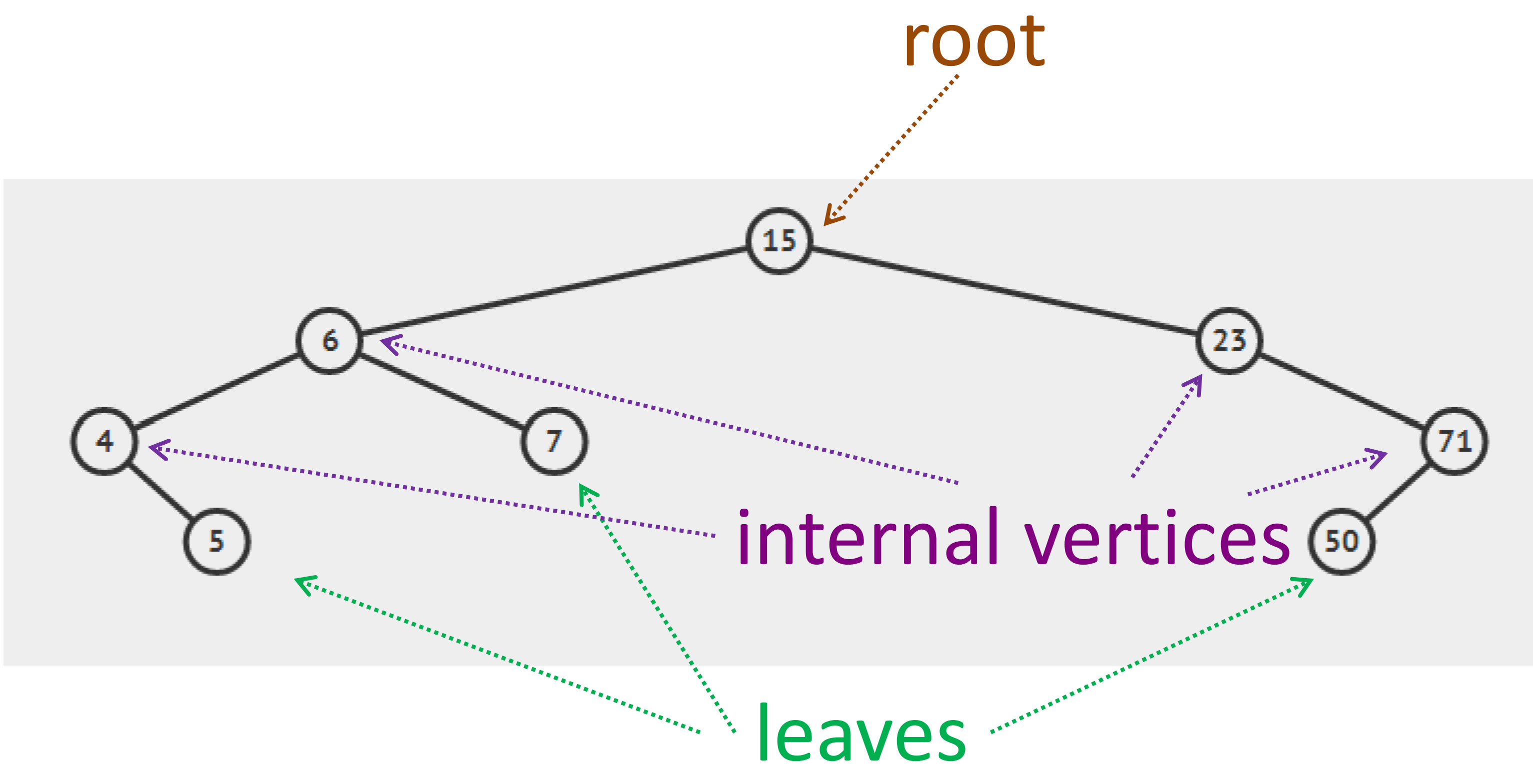
Adelson-Velskii Landis (AVL) Tree

- Principle of "Height-Balanced"
- Keeping AVL Tree balanced via rotations

Reference in CP4 book: Section 2.3.3

BST Web-based Review

https://visualgo.net/**bst**



Binary Search Tree: Summary

Operations that **modify** the BST (*dynamic* data structure):

- insert: O(**h**)
- delete: O(**h**)

Query operations (the BST structure remains the same):

- search: O(**h**)
- findMin, findMax: O(**h**)
- predecessor, successor: O(**h**)
- inorder traversal: O(**N**) – the only one that does not depend on **h**
  - PS: We also have preorder and postorder traversals for tree structure
- select/rank: ? (we have not discuss this yet)

More BST Attributes: Height and Size

Two more attributes at each BST vertex: Height and Size

- Height: #edges on the path from this vertex to deepest leaf
- Size: #vertices of the subtree rooted at this vertex

These values are recursively defined/computed:

- x.height = -1 (if x is an empty tree)
- x.height = max(x.left.height, x.right.height) + 1 (all other cases)

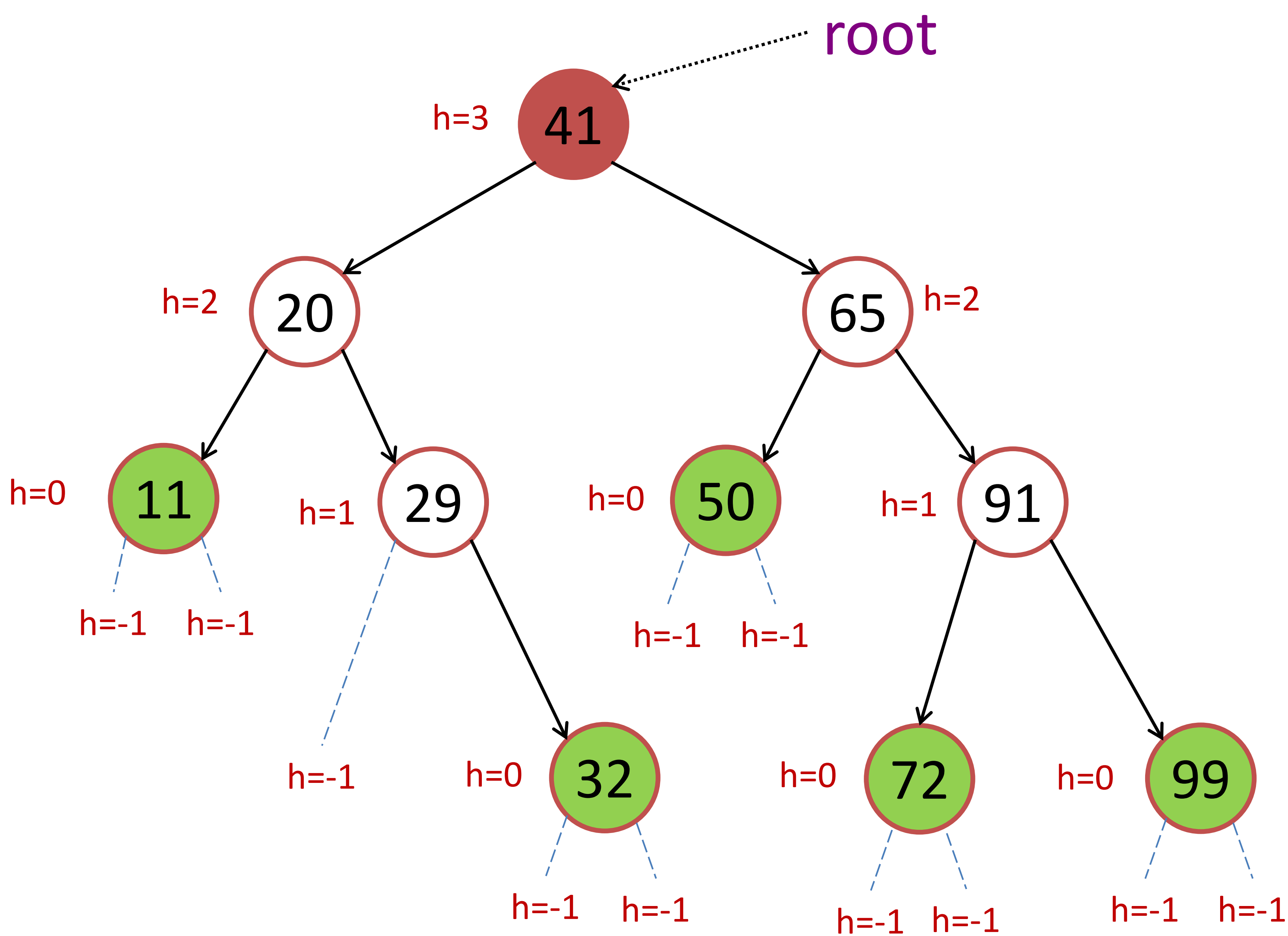
find the max of left and right child + 1(include the root)

- x.size = 0 (if x is an empty tree)
- x.size = x.left.size + x.right.size + 1 (all other cases)

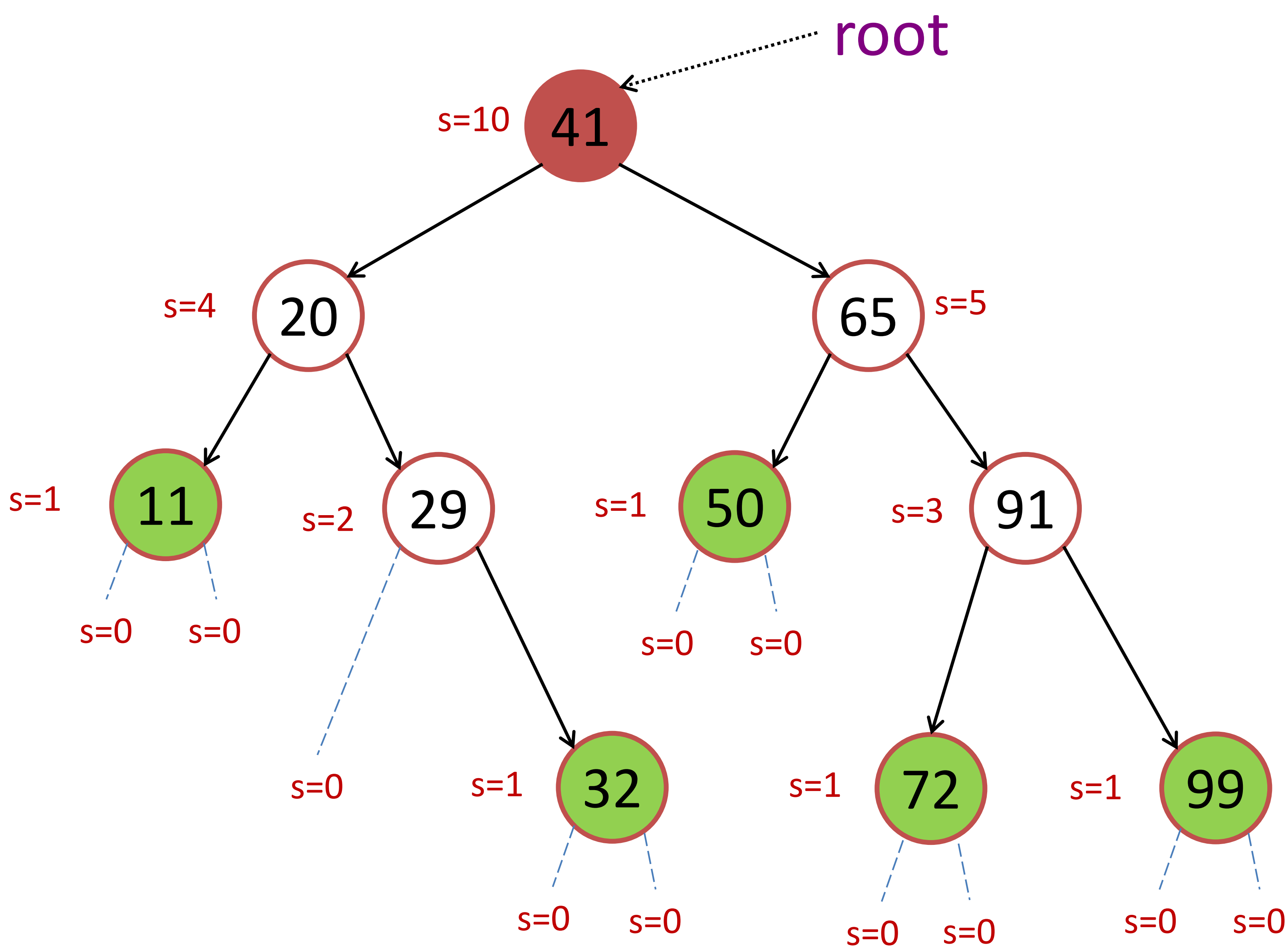
The height of the BST is thus: root.height

The size of the BST is thus: root.size

Binary Search Trees: Height (h)

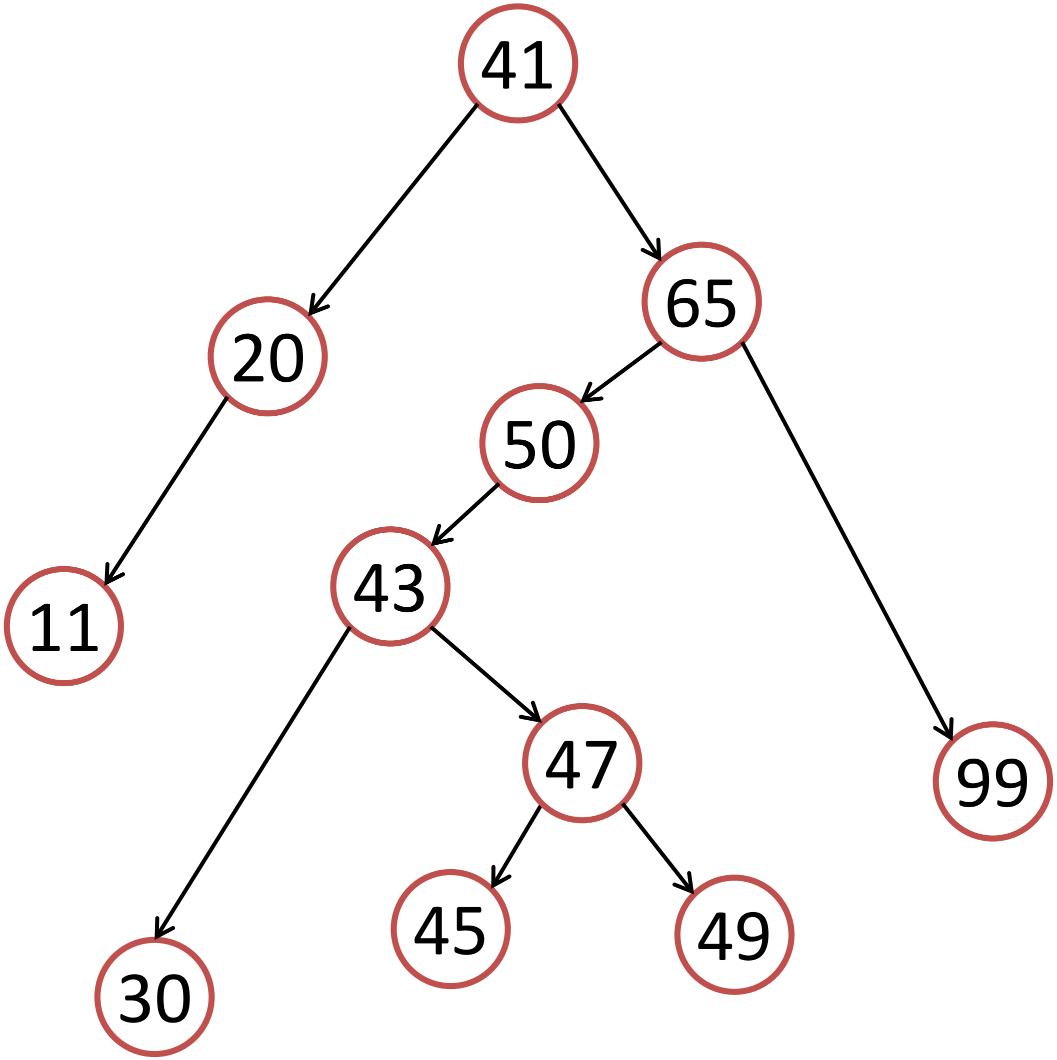


Binary Search Trees: Size (s)



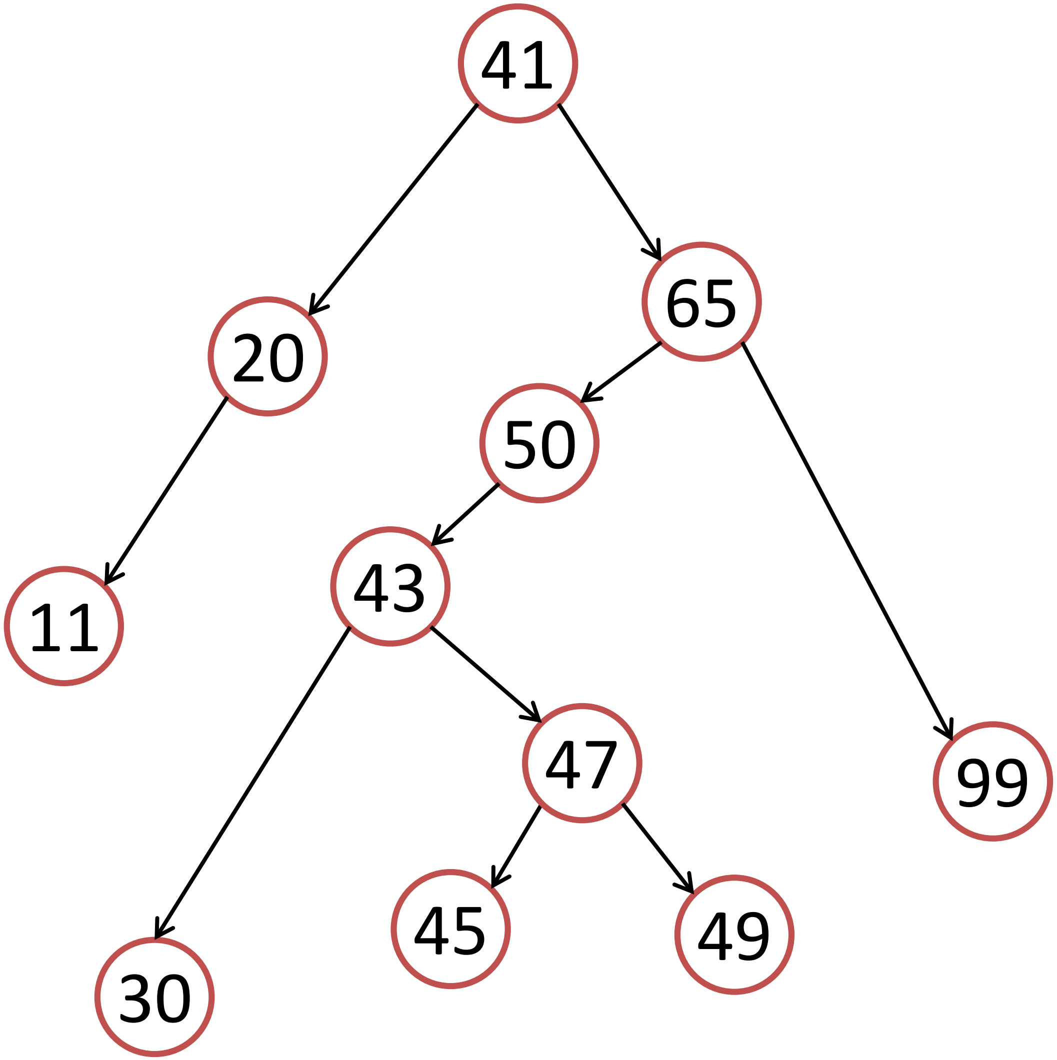
The height of this tree is?

1. 2
2. 4
3. 5
4. 6
5. 7
6. 42



The size of this tree is?

1. 10
2. 11
3. 12
4. 13
5. 14
6. 15

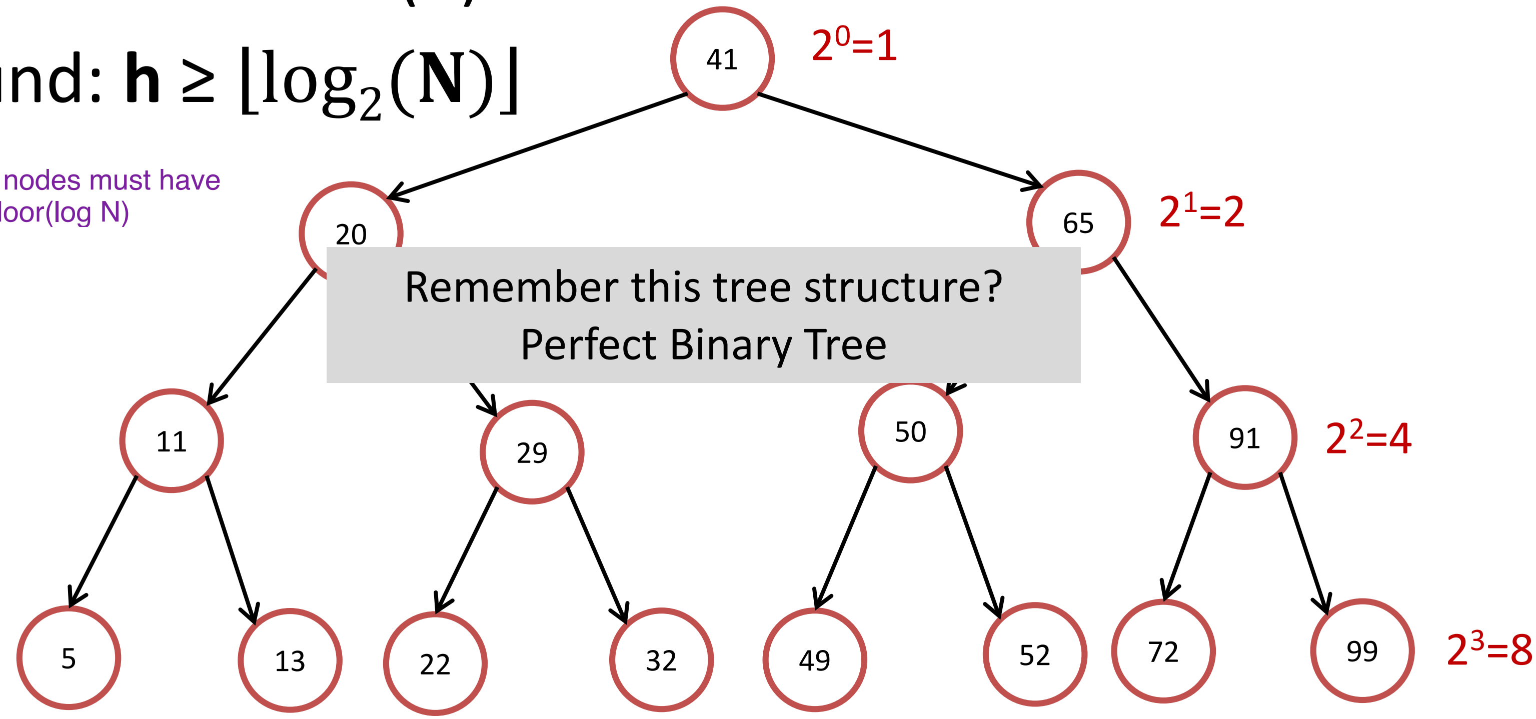


The Importance of Being Balanced

Most operations take O(**h**) time

Lower bound: $\mathbf{h} \geq \lfloor \log_2(\mathbf{N}) \rfloor$

perfect binary tree of n nodes must have smallest height which floor(log N)



$N = 1 + 2 + 4 + \ldots + 2^h = 2^0 + 2^1 + 2^2 + \ldots + 2^h$

$= 2^{h+1} - 1 < 2^{h+1}$ (sum of geometric progression)

$\log_2(N) < \log_2(2^{h+1}) \Rightarrow \log_2(N) < (h+1) * \log_2(2) \Rightarrow h > \log_2(N)-1$

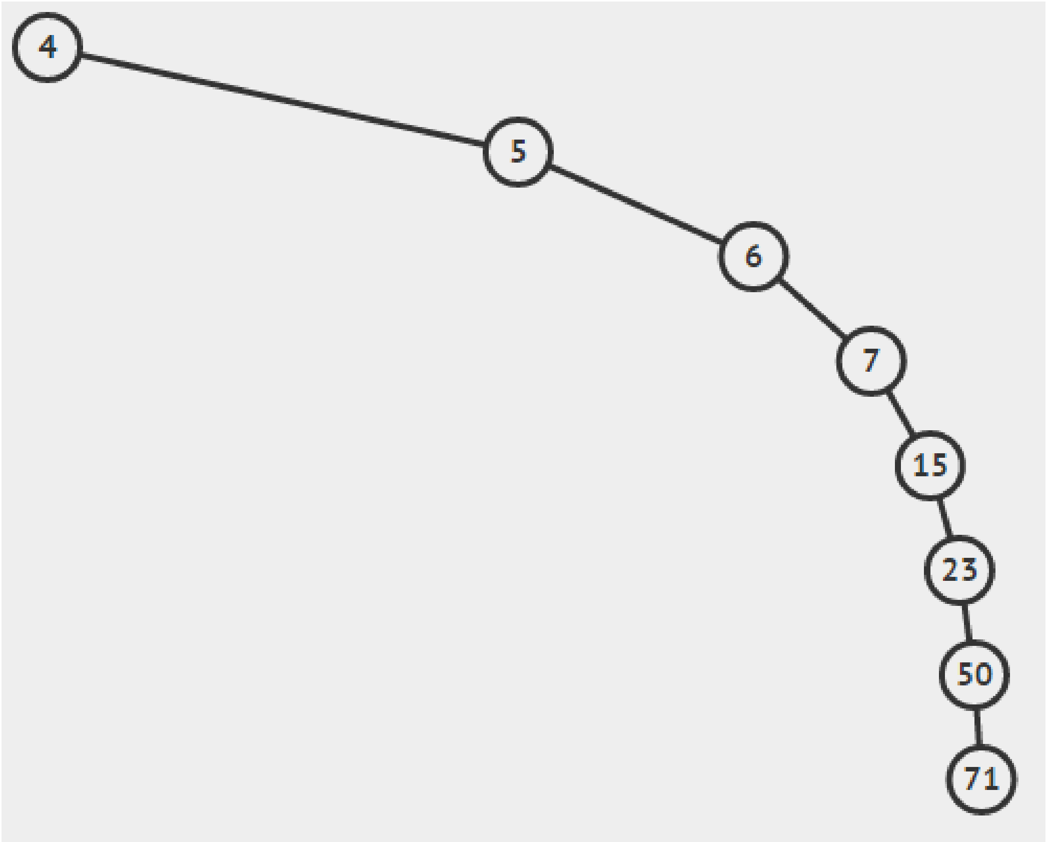
$\Rightarrow h \geq \lfloor \log_2(\mathbf{N}) \rfloor$

The Importance of Being Balanced

Most operations take O(**h**) time

Upper bound: **h** ≤ **N**-1 ➔ **h** < **N**

Remember this tree structure?
The worst case for BST…



The Importance of Being Balanced

Most operations take O(**h**) time

Combined bound: $\lfloor \log_2(\mathbf{N}) \rfloor \leq \mathbf{h} < \mathbf{N}$

$\log_2$(**N**) versus **N** in picture (revisited with larger numbers):

N = 500

$\log_2$(**N**) ~ 9 After learning CS2040S ☺

N = 1 000

$\log_2$(**N**) ~ 10 After learning CS2040S ☺

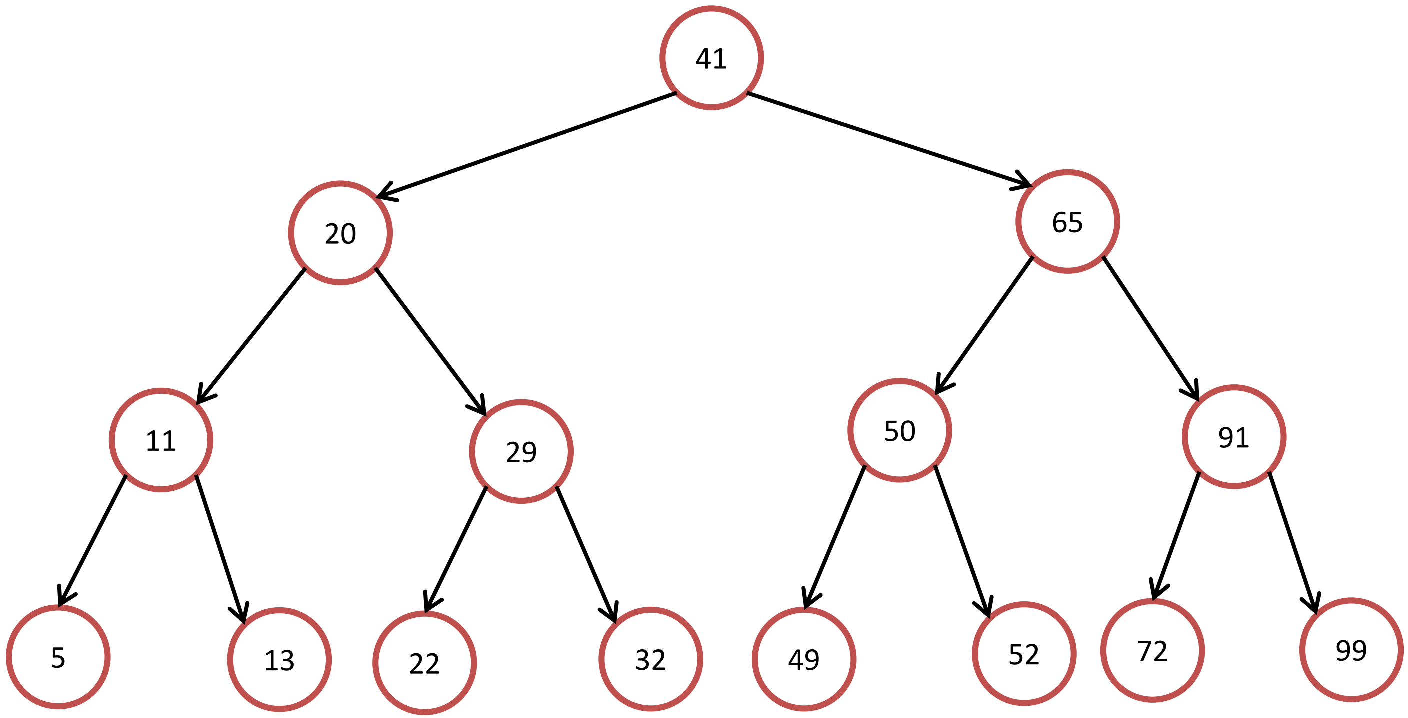
We say a BST is balanced if **h** = O(log **N**), i.e. c * log **N**

On a balanced BST, all operations run in O(log **N**) time

The Importance of Being Balanced

Example of a perfectly balanced BST: always has odd num of nodes

This is hard to achieve though…



The Importance of Being Balanced

How to get a balanced tree:

- Define a good property of a tree
- Show that if the good property holds, then the tree is **balanced** $h = O(\log n)$
- After every insert/delete, make sure the good property still holds
  - If not, fix it!

Adelson-Velskii & Landis, 1962 (~57 years ago… :O)

Can be a little bit frustrating if you are not comfortable with recursion

Hang on…

AVL TREES

AVL Trees [Adelson-Velskii & Landis 1962]

Step 1: Augment (i.e. add more information)

In every vertex x, we also store its height: **x.height**

(Note that x already has: **x.left**, **x.right**, **x.parent**, and **x.key**)

During insertion and deletion, we *also* update **height**:

recursive method call

```
insert(x, v)
  // ... same as before ...
  x.height = max(x.left.height, x.right.height) + 1
```

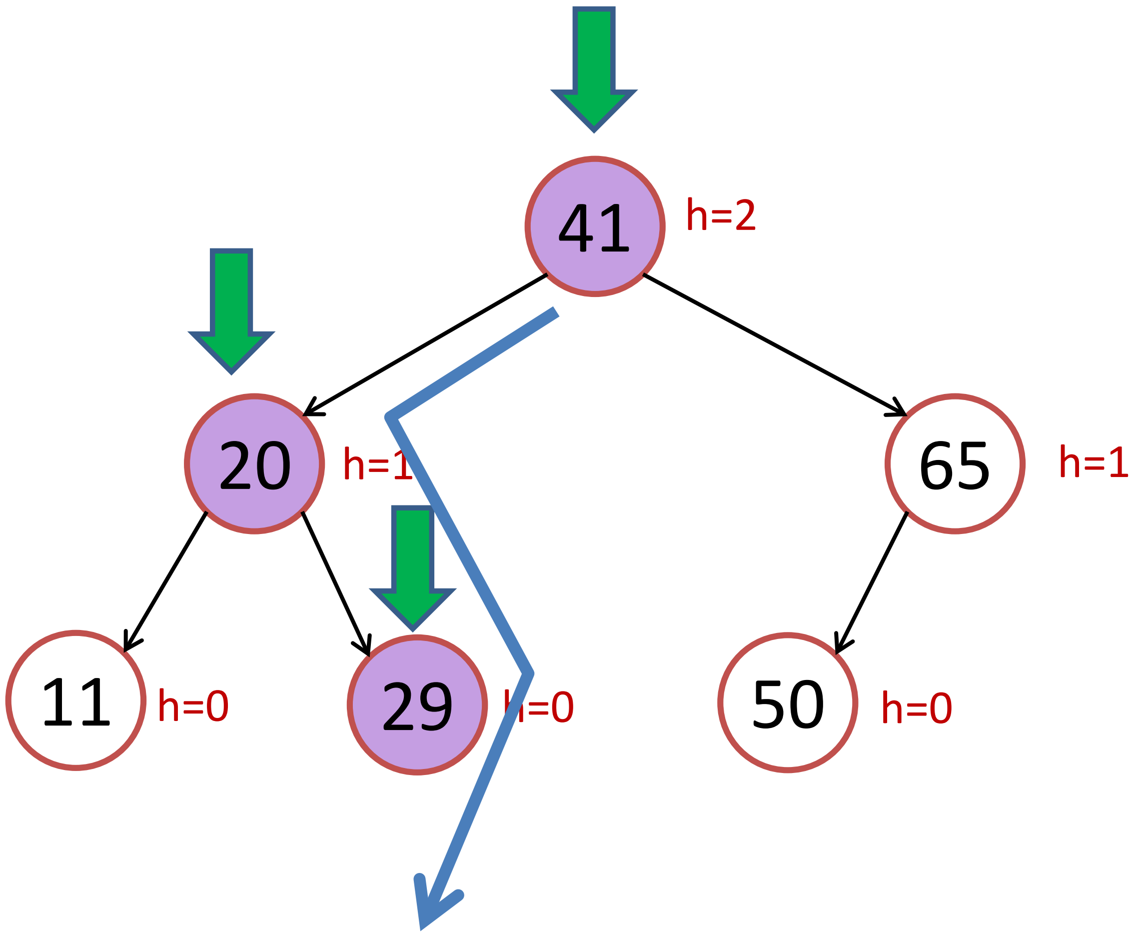
before we exit, we just add this line -> recursively update the height as we unwind the recursion

```
// update height during deletion too (same as above)
// update on attribute size can be done similarly
```

Binary Search Trees

Height of empty trees are ignored in this illustration (all -1)

insert(27)

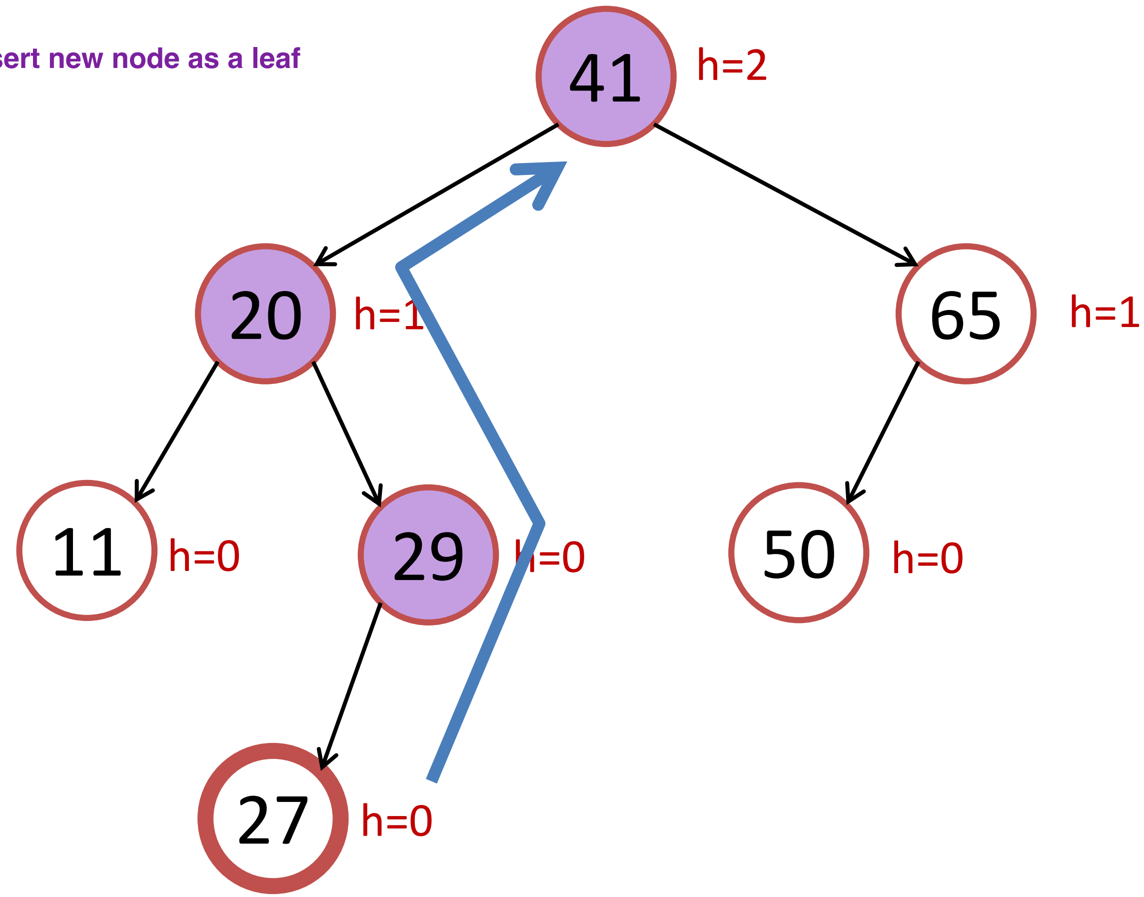


Height information during insertion/deletion is not shown in VisuAlgo (yet)

Binary Search Trees

insert(27)

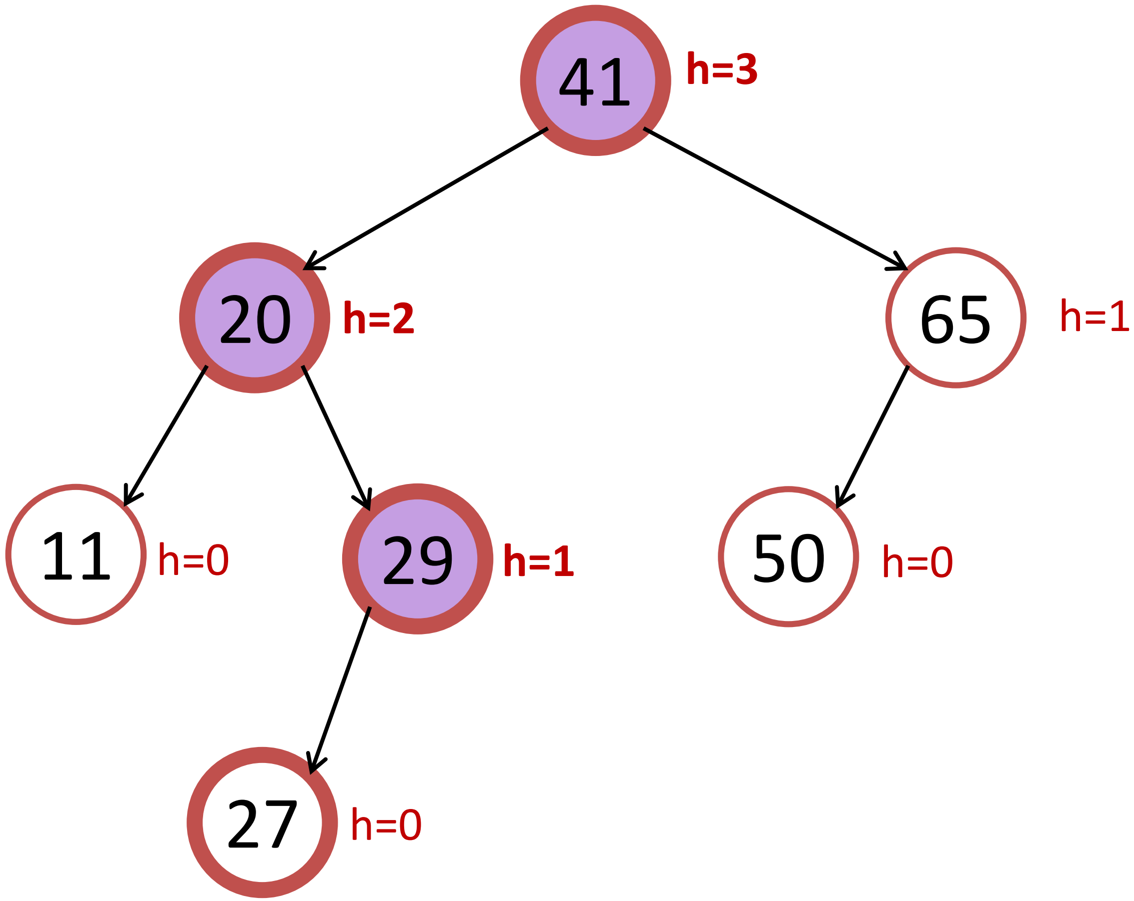
insertion: always insert new node as a leaf



Binary Search Trees

insert(27)

Notice that only vertices along the insertion path may have their height attribute updated...



AVL Trees [Adelson-Velskii & Landis 1962]

AVL Property

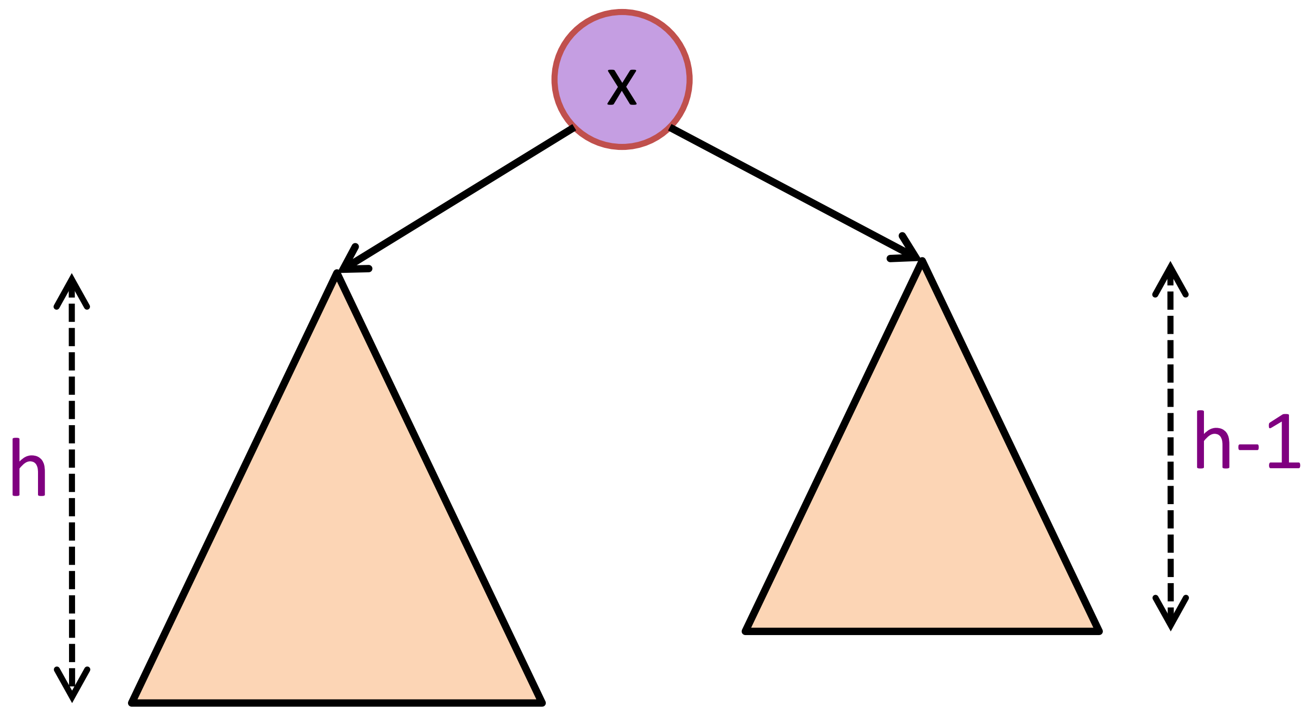
Step 2: Define Invariant (something that will not change)

A vertex x is said to be <u>height-balanced</u> if:

$|x.left.height - x.right.height| \leq 1$

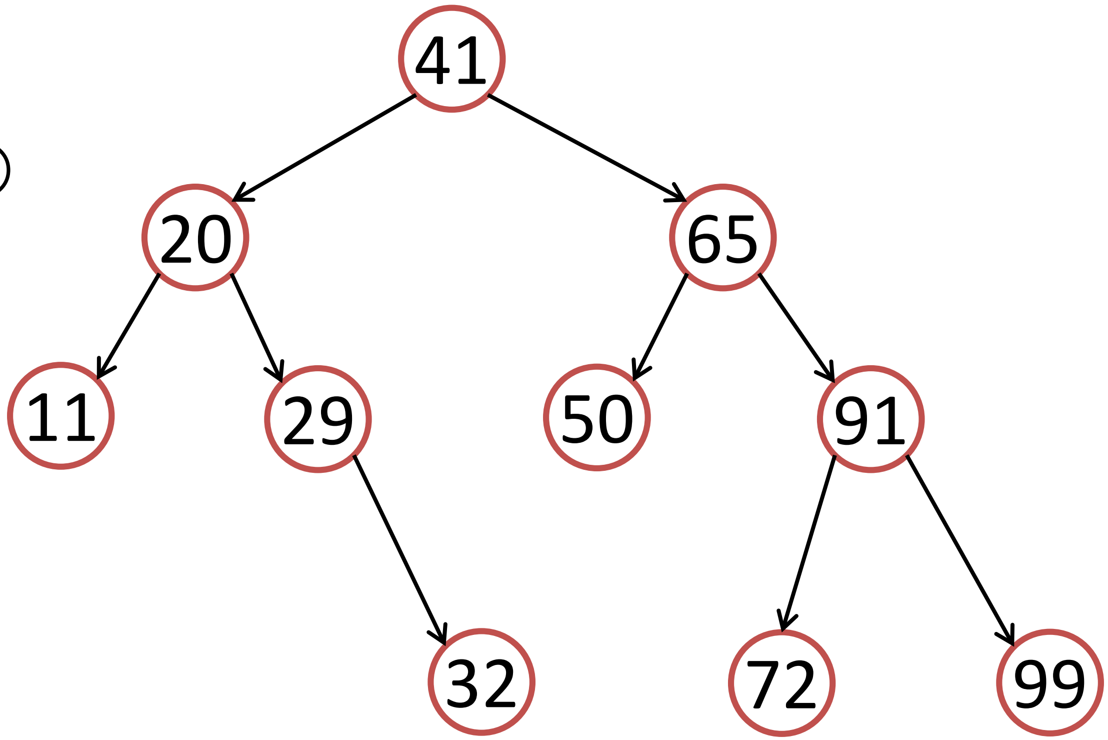
A binary search tree is said to be <u>height balanced</u> if:

every vertex in the tree is height-balanced



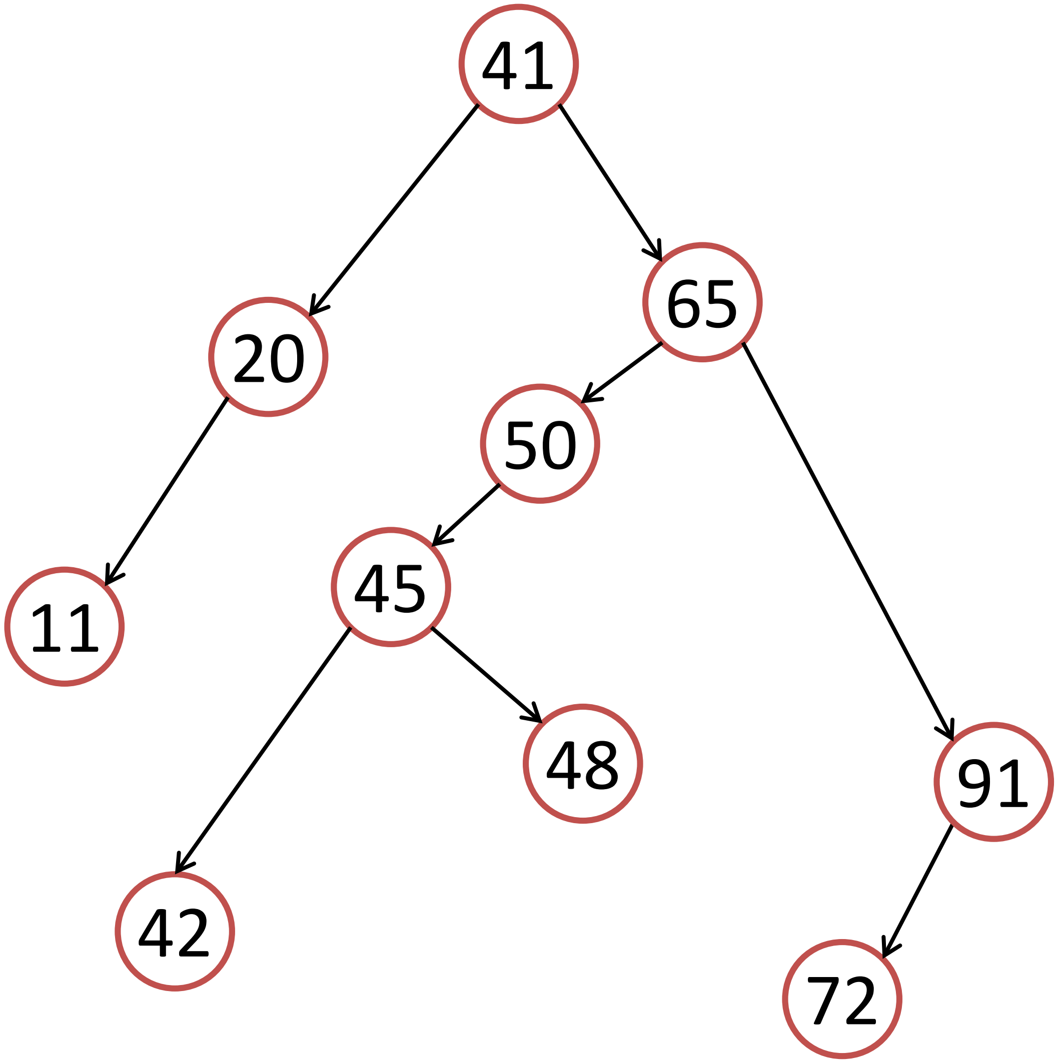
Is this tree height-balanced according to AVL?

1. Yes
2. No
3. I am confused... ☹



Is this tree height-balanced according to AVL?

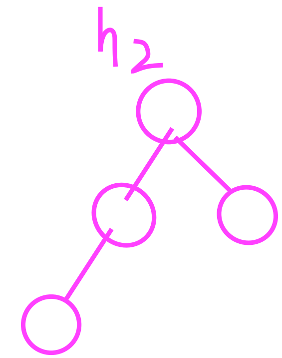
1. Yes
2. No
3. I am confused… ☹



Height-Balanced Trees

Claim:

A height-balanced tree with **N** vertices has height **h** < 2 * $\log_2$(**N**)

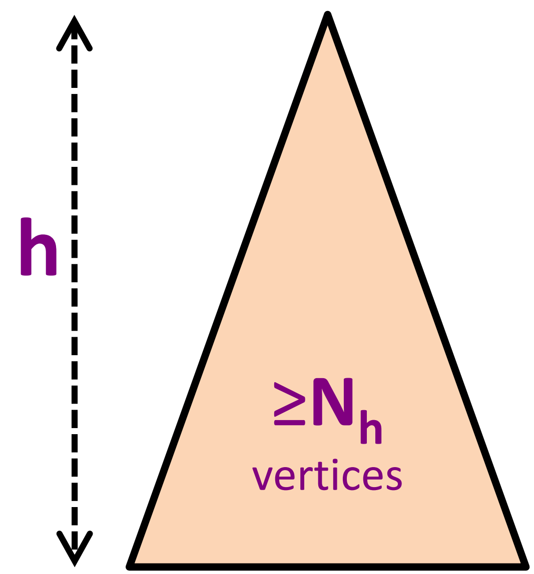


Proof (do **not** be scared):

Let N_h be the minimum number of vertices in a height-balanced tree of height **h**

The actual number of vertices $N \geq N_h$

any tree of height h has some N vertices which must be >= Nh



if want smallest tree of height 2, then the left and right subtree must also be of the smallest(in terms of no of nodes) tree of a particular height

since the allowance is of 1 between the two subtrees of h2, we can have one subtree with h-1 and another with h-2(instead of both h-1) -> so the smallest tree of h2 has 4 nodes as compared to 5

Height-Balanced Trees

Proof:

Let $\mathbf{N_h}$ be the minimum number of vertices in a height-balanced tree of height h

$\mathbf{N_h} = 1 + \mathbf{N_{h-1}} + \mathbf{N_{h-2}}$

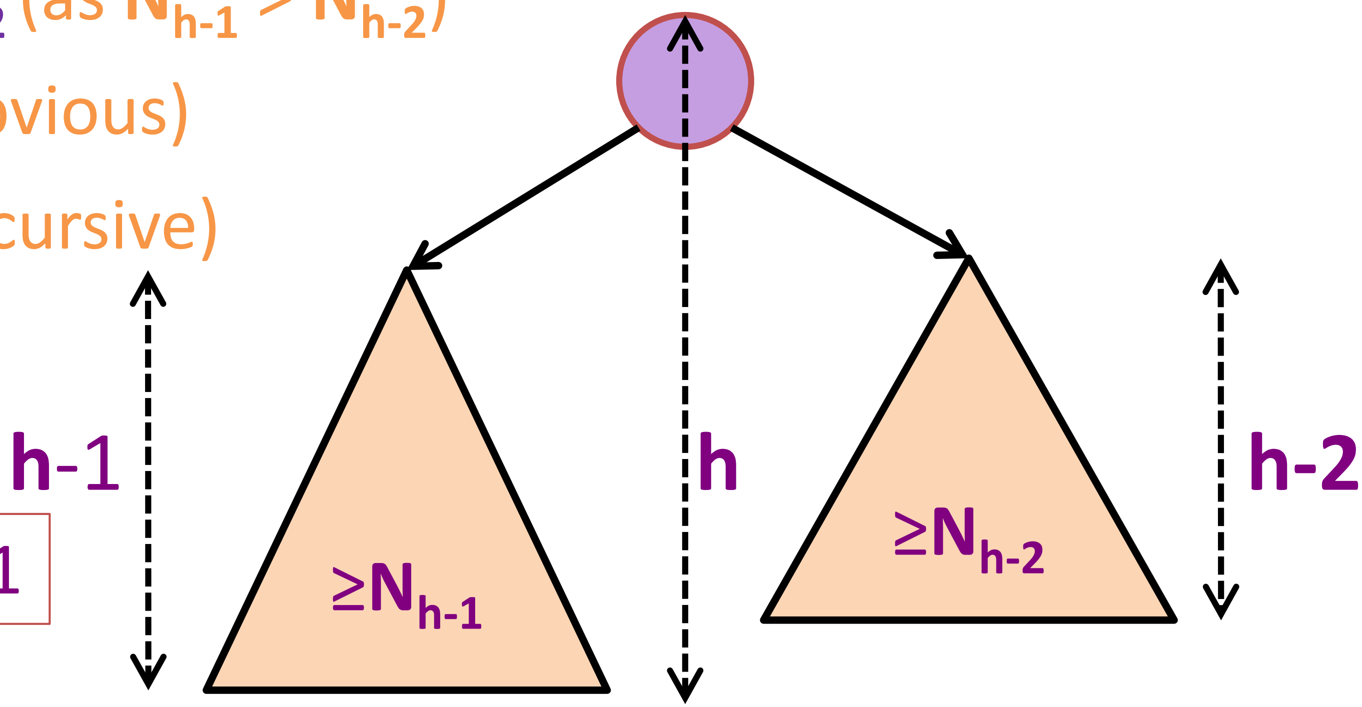
$\mathbf{N_h} > 1 + 2\mathbf{N_{h-2}}$ (as $\mathbf{N_{h-1}} > \mathbf{N_{h-2}}$)

$\mathbf{N_h} > 2\mathbf{N_{h-2}}$ (obvious)

$= 4\mathbf{N_{h-4}}$ (recursive)

$= 8\mathbf{N_{h-6}}$

$= \dots$



Base case: $\mathbf{N_0} = 1$

Height-Balanced Trees

Proof:

Let N_h be the minimum number of vertices in a height-balanced tree of height h

N >= Nn > 2^h/2

N > 2^h/2

logN > h/2

2logN > h

h < 2 lgN

$N_h = 1 + N_{h-1} + N_{h-2}$

$N_h > 1 + 2N_{h-2}$

$$\left.\begin{aligned} N_h &> 2N_{h-2} \\ &> 4N_{h-4} \\ &> 8N_{h-6} \\ &> \ldots \end{aligned}\right\}$$

As each step we reduce **h** by 2,
Then we need to do this step **h**/2 times
to reduce **h** (assume **h** is even) to 0

Base case: $N_0 = 1$

$N_h > 2^{h/2} N_0$

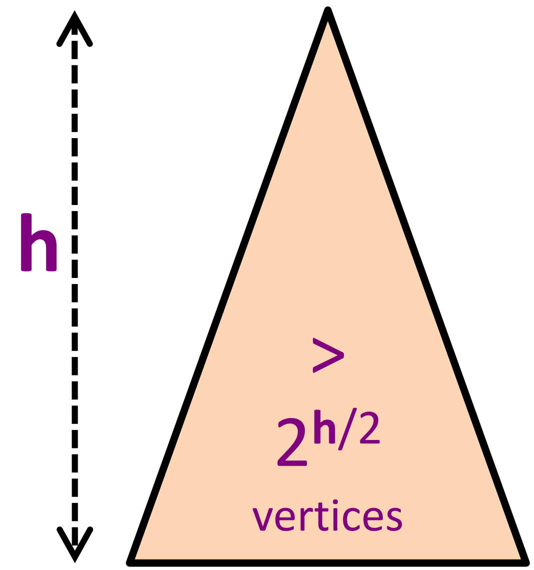
$N_h > 2^{h/2}$

Height-Balanced Trees

Claim:

A height-balanced tree is balanced,
i.e. has height $\mathbf{h} = O(\log(\mathbf{N}))$

We have shown that: $\mathbf{N_h} > 2^{\mathbf{h}/2}$ and $\mathbf{N} \geq \mathbf{N_h}$



$\mathbf{N} \geq \mathbf{N_h} > 2^{\mathbf{h}/2}$

$\mathbf{N} > 2^{\mathbf{h}/2}$

$\log_2(\mathbf{N}) > \log_2(2^{\mathbf{h}/2})$ ($\log_2$ on both side)

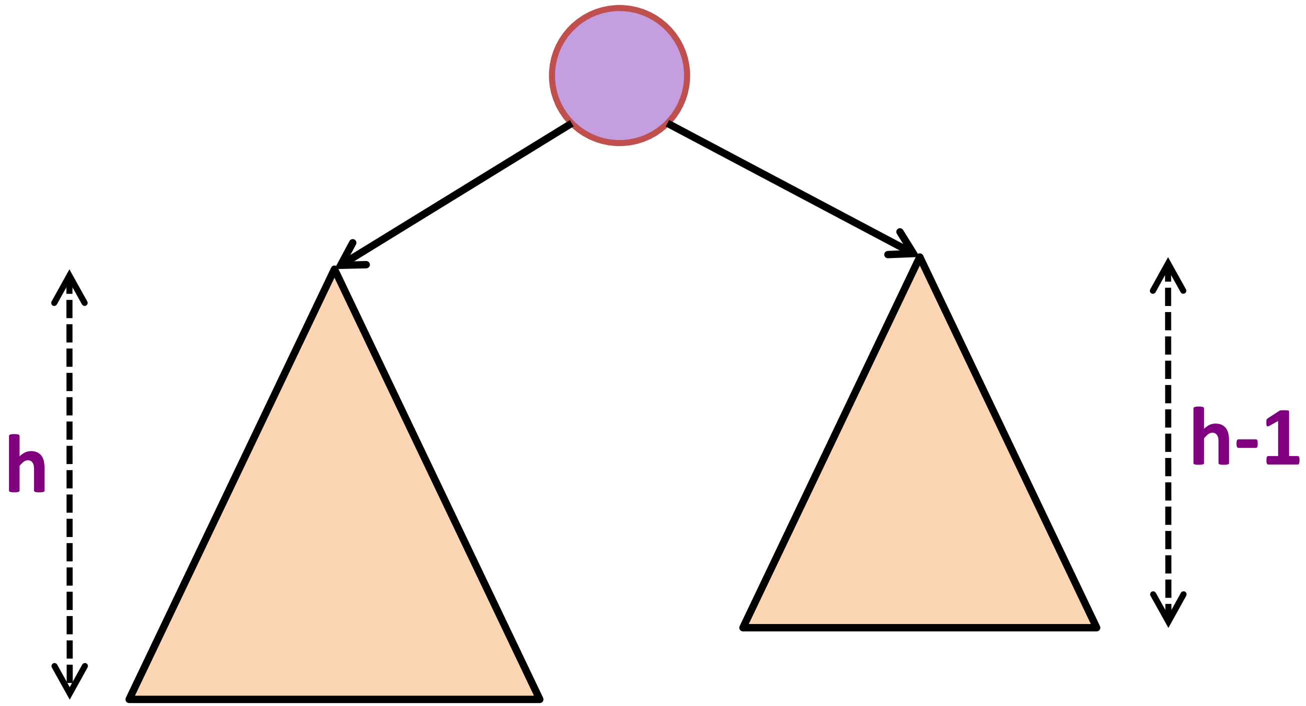
$\log_2(\mathbf{N}) > \mathbf{h}/2$ (formula simplification)

$2 * \log_2(\mathbf{N}) > \mathbf{h}$ or $\mathbf{h} < 2 * \log_2(\mathbf{N})$

$\mathbf{h} = O(\log(\mathbf{N}))$

AVL Trees [Adelson-Velskii & Landis 1962]

Step 3: Show how to maintain height-balance



Insertion to an AVL Tree

insert(37)

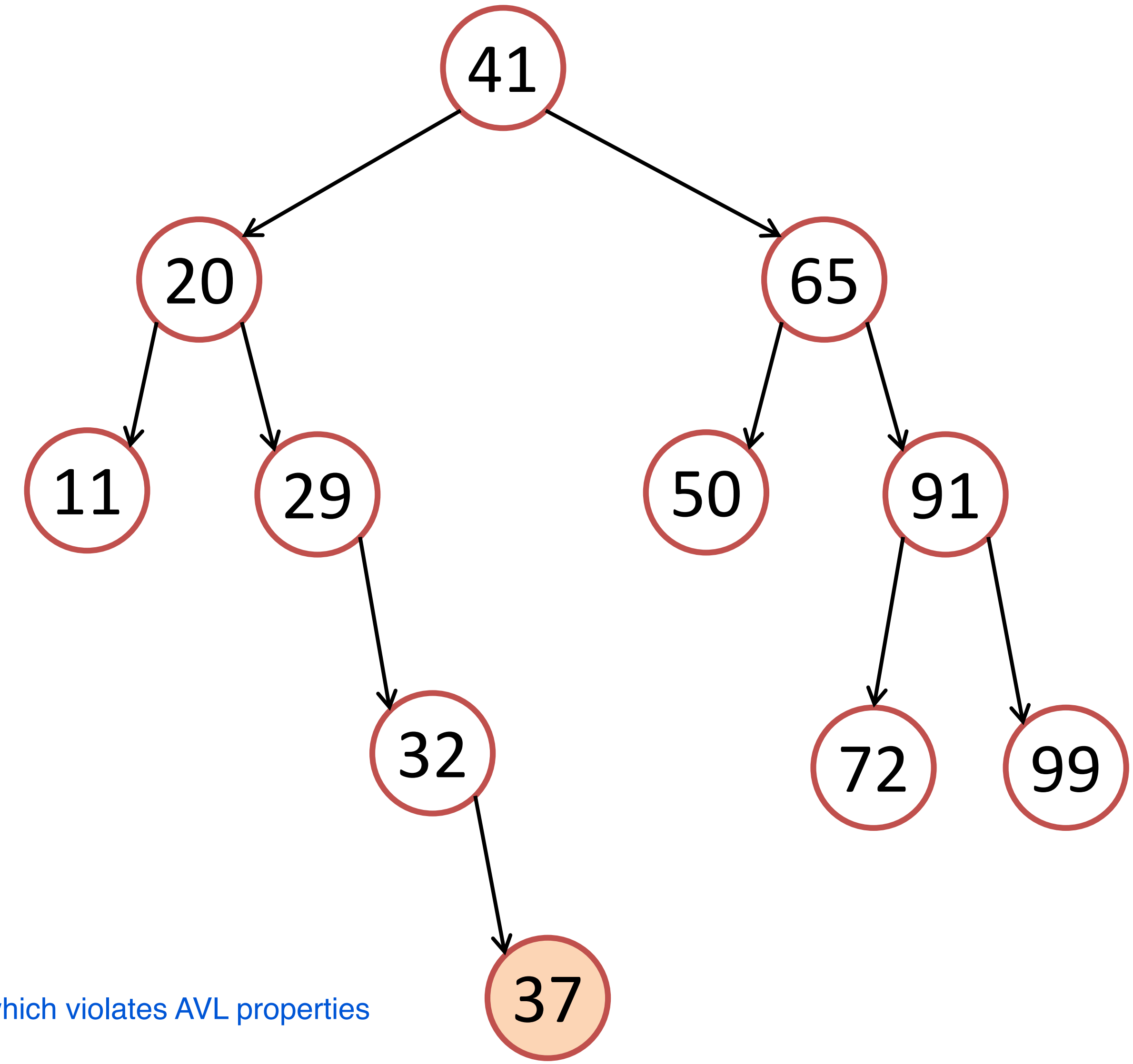
Initially balanced

But no longer balanced after Inserting 37

Need to rebalance!

But how?

diff between left and right of 29 = 2 which violates AVL properties



"Infinite more" examples in VisuAlgo…

Balance Factor (bf(x))

AVL property: balance factor either 1, 0 or -1

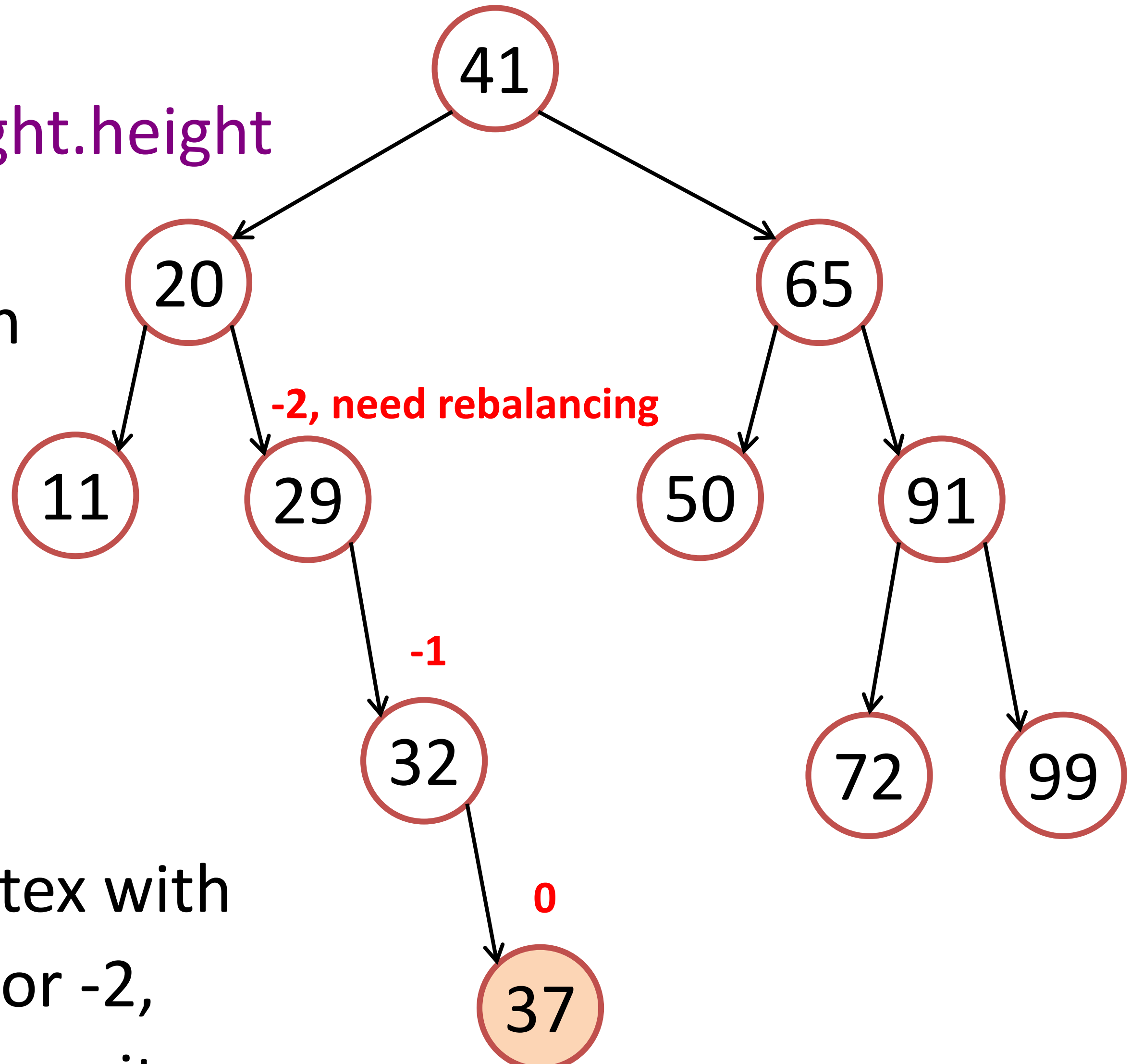
balance factor of a particular node

bf(x) =

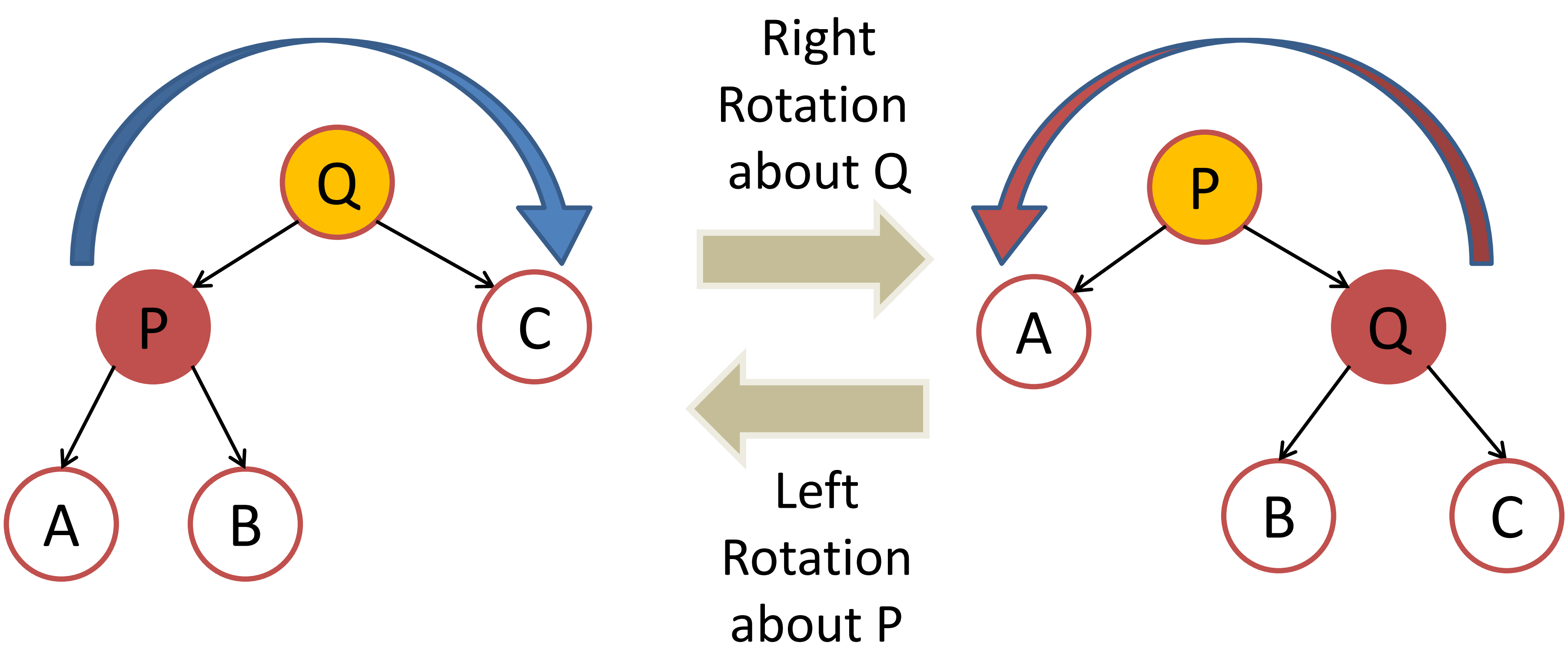
x.left.height - x.right.height

From the insertion point, check the balance factor of each vertex up to the root

Once we have vertex with balance factor +2 or -2, we have to rebalance it



Tree Rotations (1)



Right Rotation

- Need Q to have a left child P
- Make Q right child of P
- Other manipulations …

Left Rotation

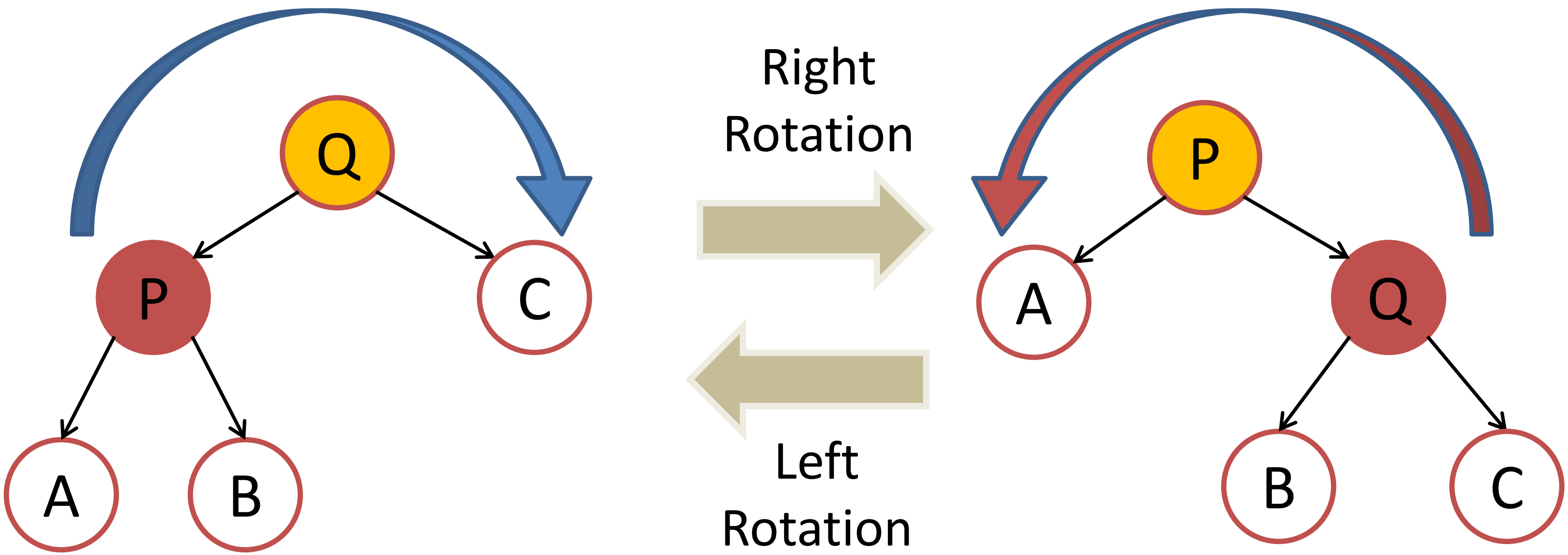
- Need P to have a right child Q
- Make P left child of Q
- Other manipulations …

Tree Rotations (2)

P < B < Q

P < B < Q

rotation maintains bst property of maintaining order of the nodes



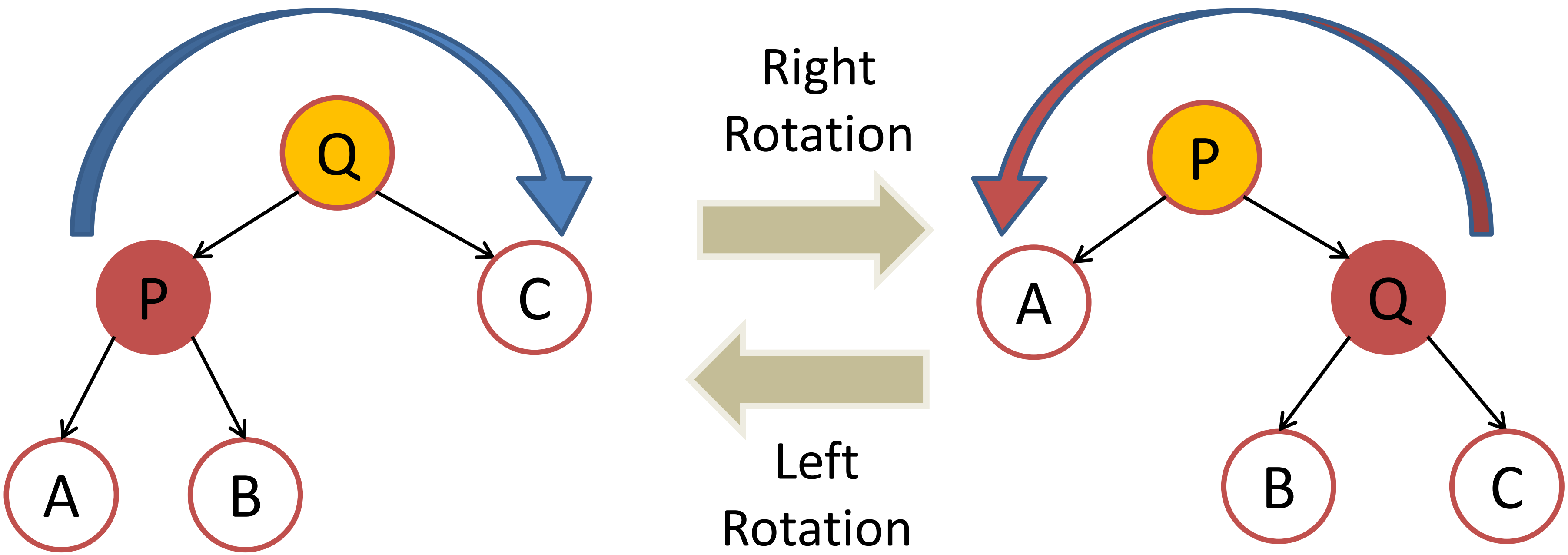
Right Rotation

- Need Q to have a left child P
- Make Q right child of P
- Make B (right child of P) left child of Q

Left Rotation

- Need P to have a right child Q
- Make P left child of Q
- Make B (left child of Q) right child of P

Tree Rotations (3)



Rotations maintain ordering of keys

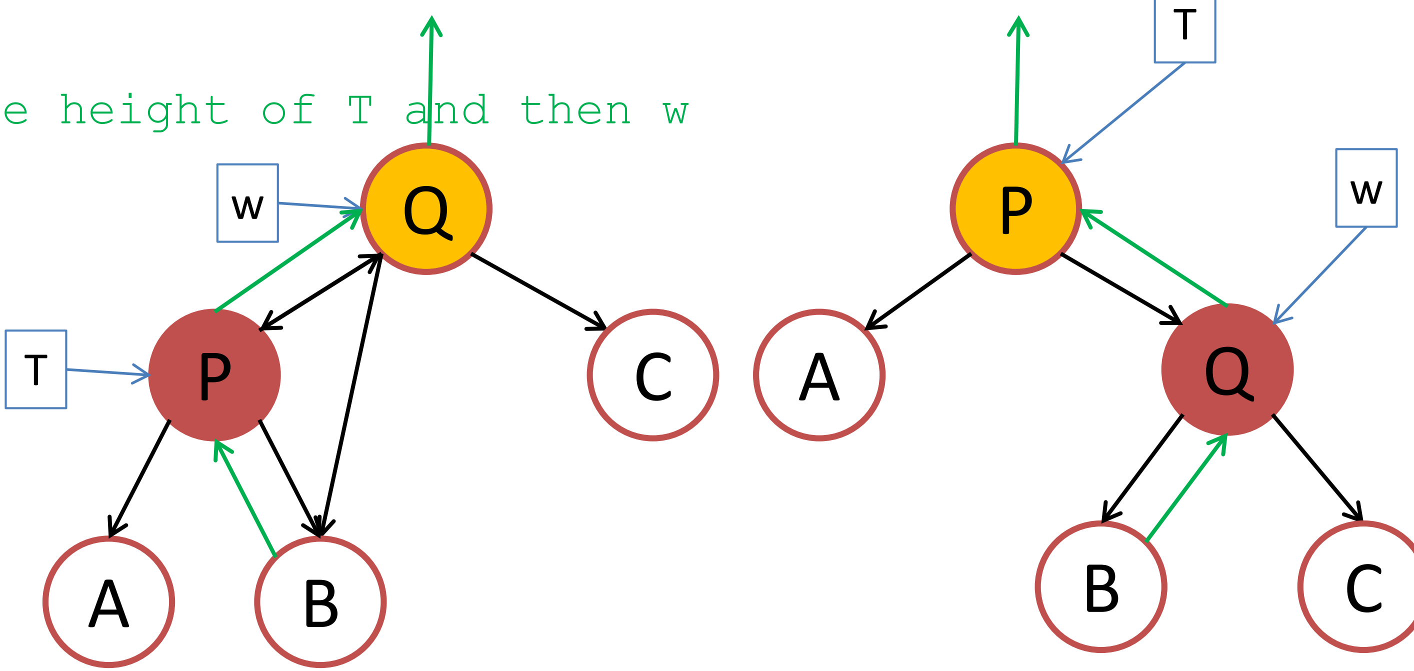
$\Rightarrow$ Maintains BST property *(see vertex B where $P \leq B \leq Q$)*

Tree Rotations Pseudo Code → O(1)


```
BSTVertex rotateLeft(BSTVertex T) // pre-req: T.right != null
   BSTVertex w = T.right
   w.parent = T.parent
   T.parent = w
   T.right = w.left
   if (w.left != null) w.left.parent = T
   w.left = T
   // Update the height of T and then w
   return w
```

need to check if it is null

`rotateRight` is the mirrored version of this pseudocode



This slide is can be confusing without the animation

Four Possible Cases

means it is left heavy

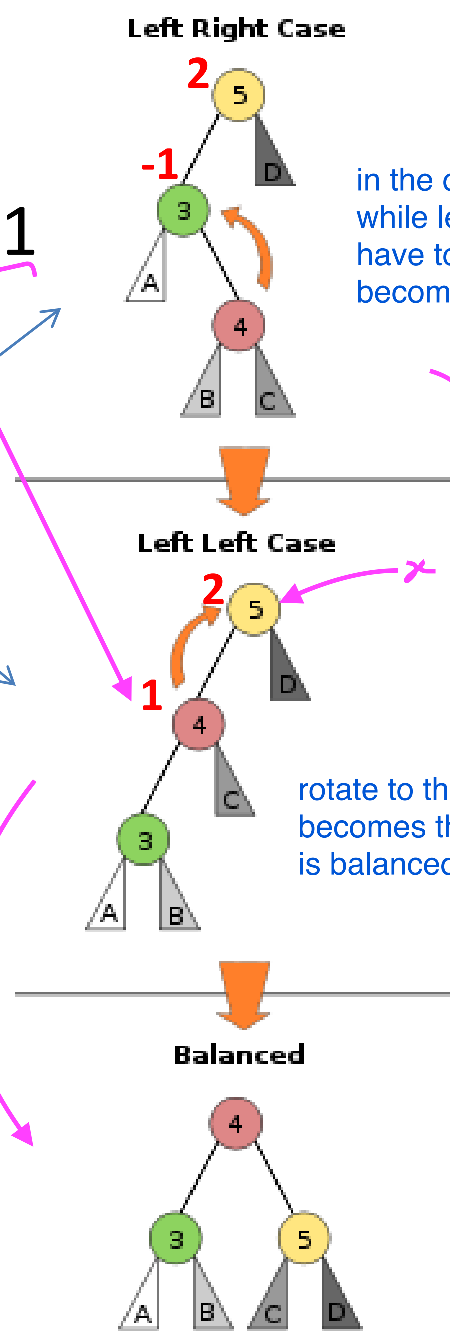
bf(x) = +2 and 0 <= bf(x.left) <= 1

  rightRotate(x)

bf(x) = +2 and bf(x.left) = -1

  leftRotate(x.left)

  rightRotate(x)



in the other case, node is left heavy while left node is right heavy, so we have to rotate it twice so that it can become balanced

1x

x

rotate to the right once so that 4 becomes the root and the tree is balanced

2x

Pictures from Wikipedia

Four Possible Cases

bf(x) = -2 and -1 <= bf(x.right) <= 0

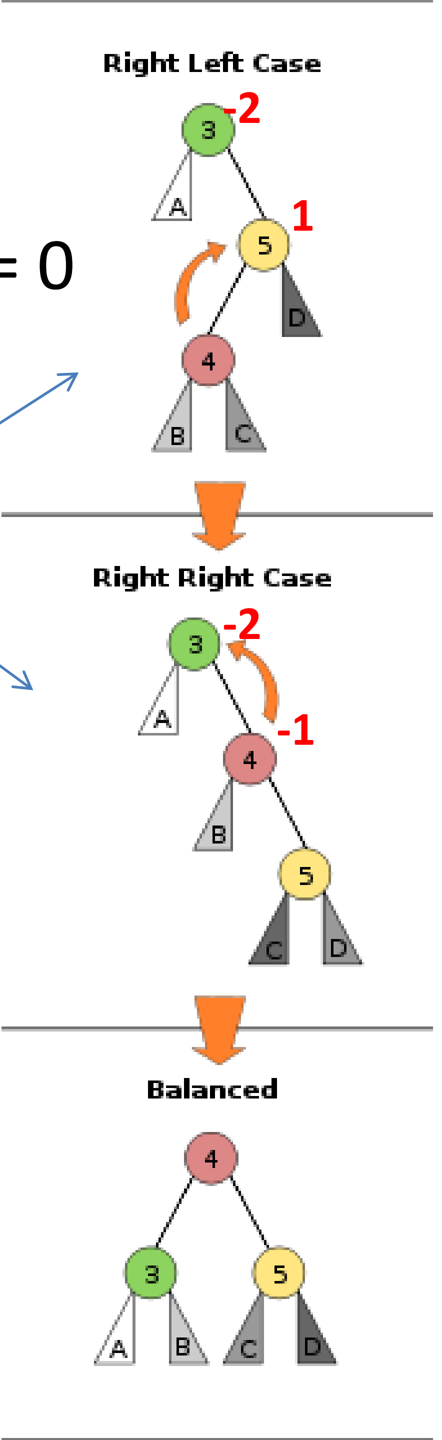
leftRotate(x)

bf(x) = -2 and bf(x.right) = 1

rightRotate(x.right)

leftRotate(x)

4 cases
1. left left
2. left right
3. right right
4. right left



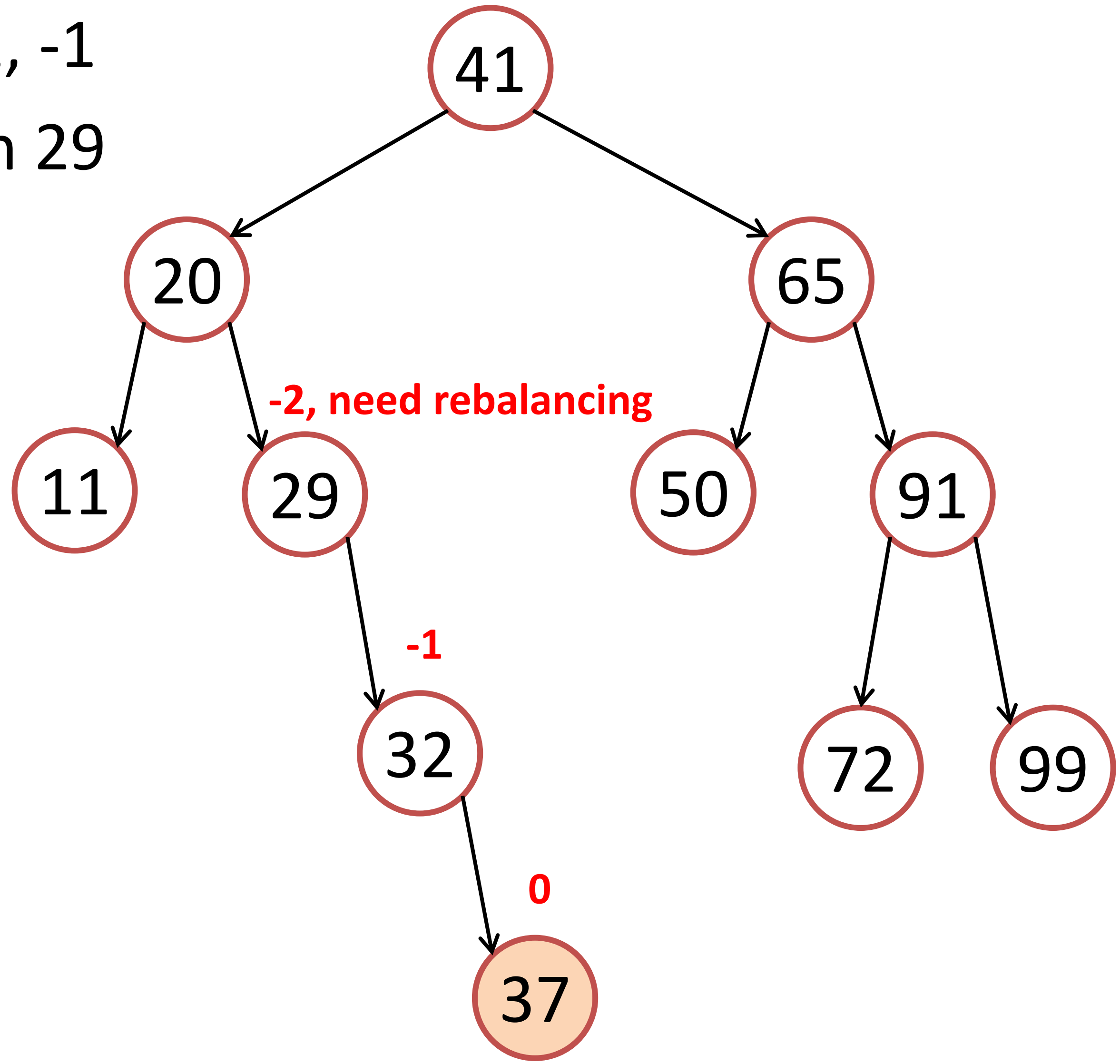
Pictures from Wikipedia

Rebalancing (1)

This is a case of -2, -1

Do left rotate on 29

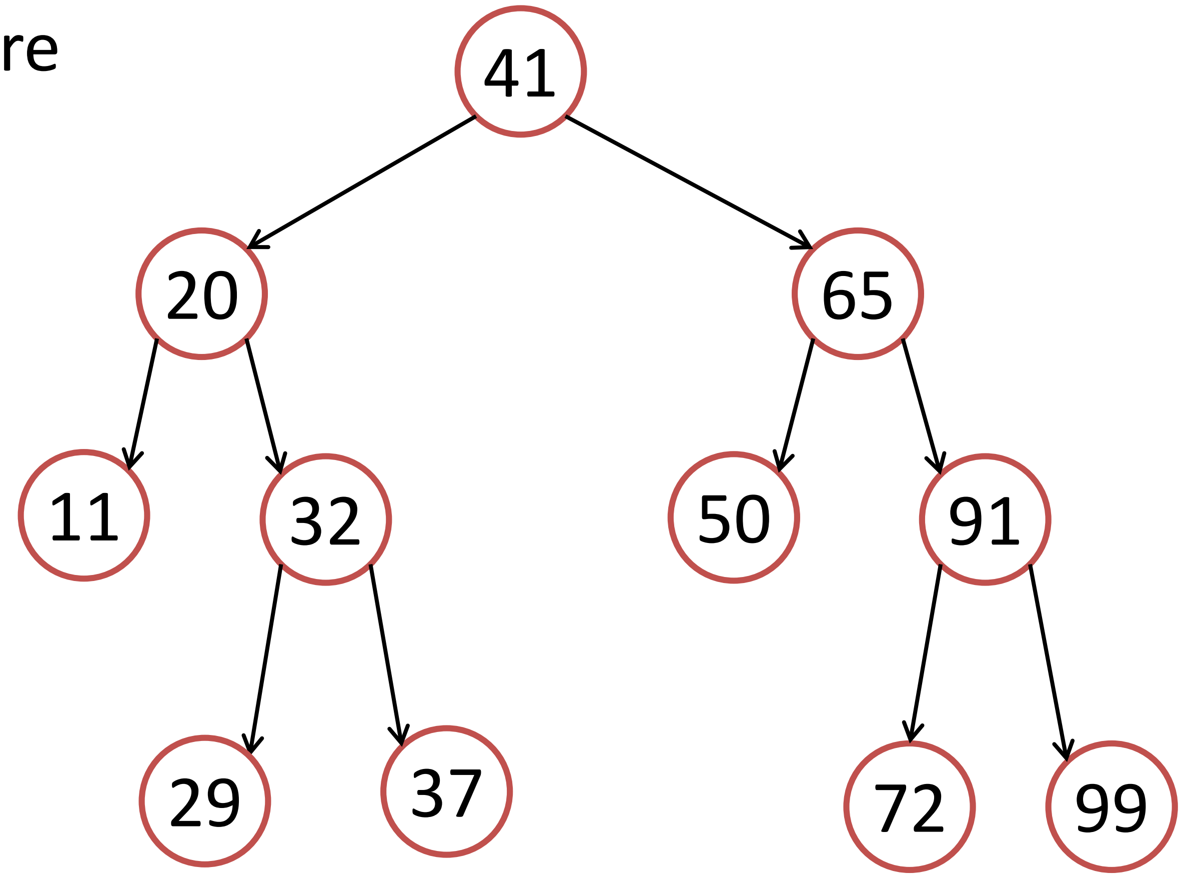
right right case
just do left rotate about 29



“Infinite more” examples in VisuAlgo…

Rebalancing (2)

Now all vertices are balanced again



"Infinite more" examples in [VisuAlgo AVL Tree Visualization]

Insertion to an AVL Tree

Summary:

- Just insert the key as in normal BST
- Walk up the AVL tree from the insertion point to root:
  - At every step, update height & check balance factor
  - If a certain vertex is out-of-balance (+2 or -2), use rotations to rebalance
    - During insertion to an AVL tree, you can only trigger one of the four possible rebalancing cases as shown earlier once !

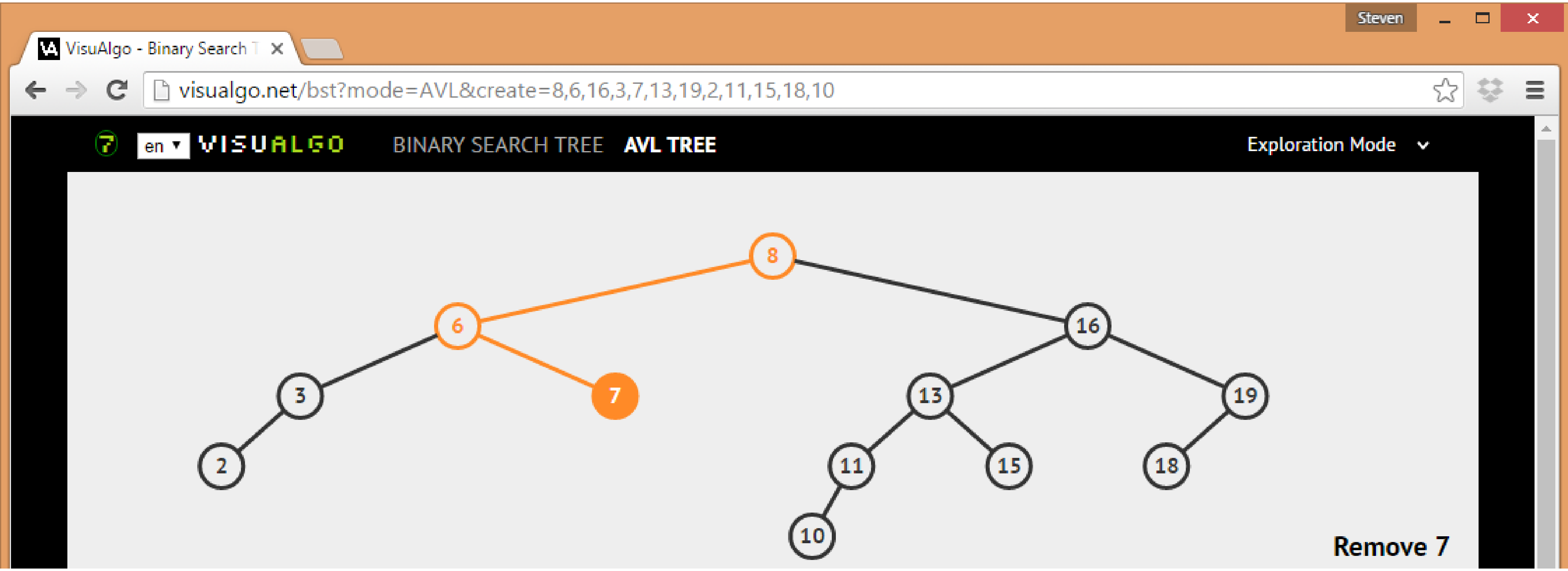
Deletion from an AVL Tree

Deletion is quite similar to Insertion:

- Just delete the key as in normal BST
- Walk up the AVL tree from the deletion point to root:
  - At every step, update height & check balance factor
  - If a certain vertex is out-of-balance (+2 or -2), use rotations to rebalance
    - The main difference compared to insertion into AVL tree is that you may trigger one of the four possible rebalancing cases several times, up to **h** = log **n** times :O, see this example (next slide)

AVL Tree Web-based Review

Create an AVL Tree using 8,6,16,3,7,13,19,2,11,15,18,10



Try **Remove (Delete)** vertex 7, it triggers **two (more than one)** rebalancing actions

Then try various **Insert operations** and notice that at most it will only trigger one (out of the four cases) of rebalancing actions

Balanced Search Trees

Many different flavors of balanced search trees

- AVL trees (Adelson-Velskii & Landis, 1962)
  - Discussed in this lecture…
- B-trees / 2-3-4 trees (Bayer & McCreight, 1972)
- BB[α] trees (Nievergelt & Reingold, 1973)
- Red-black trees (see CLRS 13)
- Splay trees (Sleator and Tarjan, 1985) ← Next Topic !
- Skip Lists (Pugh, 1989)
- Treaps (Seidel and Aragon, 1996)

Now, after we learn balanced BST

No	Operation	Unsorted Array	Sorted Array	bBST
1	Search(age)	O(**N**)	O(log **N**)	**O(log N)**
2	Insert(age)	O(1)	O(**N**)	**O(log N)**
3	FindOldest()	O(**N**)	O(1)	**O(log N)**
4	ListSortedAges()	O(**N** log **N**)	O(**N**)	O(**N**)
5	NextOlder(age)	O(**N**)	O(log **N**)	**O(log N)**
6	Remove(age)	O(**N**)	O(**N**)	**O(log N)**
7	GetMedian()	O(**N** log **N**)	O(1)	**????**
8	NumYounger(age)	O(**N** log **N**)	O(log **N**)	**????**

NumYounger(age) = rank(age)-1

Now, how to get rank(v) efficiently?

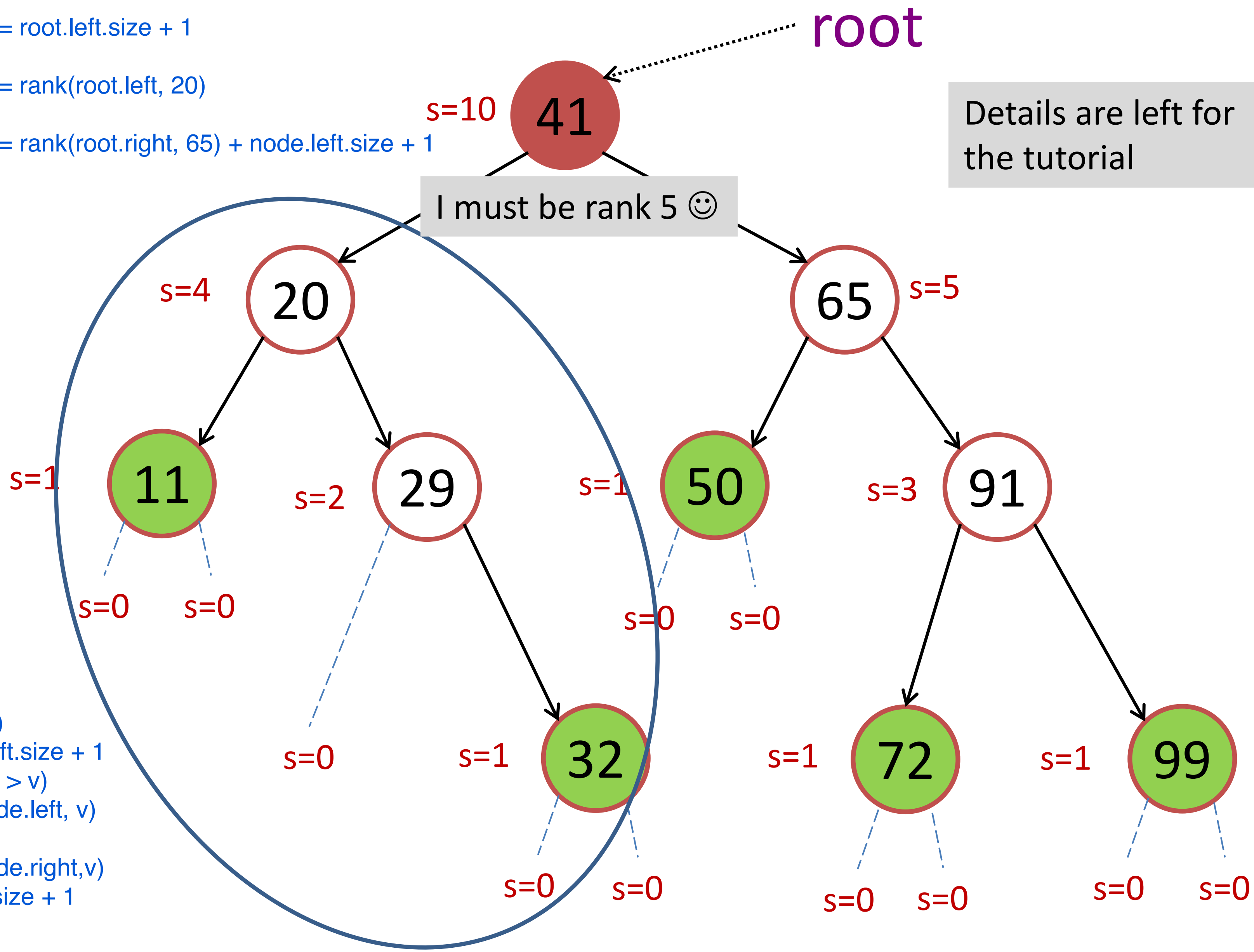
–

Binary Search Trees: Size (s)

rank(root, 41) = root.left.size + 1

rank(root, 20) = rank(root.left, 20)

rank(root, 65) = rank(root.right, 65) + node.left.size + 1




```
rank(node, v)
  if(node.key ==v)
    return node.left.size + 1
  else if(node.key > v)
    return rank(node.left, v)
  else
    return rank(node.right,v)
       + node.left.size + 1
```

Balanced BST

Summary:

- The Importance of Being Balanced
- Height Balanced Trees – AVL Trees
- Tree Rotations to re-balance the Tree