Question 1 **[20 marks]**

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ a & b & c & d \end{pmatrix}$ be a 4×4 matrix where a, b, c, d are some real numbers.

(i) (4 marks) Find $\det \boldsymbol{A}$ and write down the condition in terms of a, b, c, d such that the homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$ has non-trivial solutions.

(ii) (4 marks) Let $S = \{(a, b, c, d) \mid \boldsymbol{Ax} = \boldsymbol{0} \text{ has only the trivial solution}\}$. Is S a subspace of $\mathbb{R}^4$? Why?

(iii) (4 marks) Given $\operatorname{rank} \boldsymbol{A} = 3$, find the general solution of $\boldsymbol{Ax} = \boldsymbol{0}$. Show your working.

(iv) (4 marks) Given that $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\boldsymbol{A}$, find the condition satisfied by a, b, c, d.

(v) (4 marks) If a, b, c, d are all equal, find a basis for the column space of $\boldsymbol{A}$ in terms of a. Explain how you derive your answer.

(i) $\det \boldsymbol{A} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ a & b & c & d \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ b & c & d \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ a & b & c \end{vmatrix}$

$= \left(\begin{vmatrix} 1 & 1 \\ c & d \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ b & d \end{vmatrix} \right) - \left(- \begin{vmatrix} 0 & 1 \\ a & c \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ a & b \end{vmatrix} \right) = d - c + b - a.$

So the system $\boldsymbol{Ax} = \boldsymbol{0}$ has non-trivial solutions if and only if $d - c + b - a = 0$.

(ii) $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution if and only if $\det \boldsymbol{A} \neq 0$.
From (i), we can rewrite the set notation of S as $\{(a, b, c, d) \mid d - c + b - a \neq 0\}$.
Let $\boldsymbol{u} = (1, 0, 0, 0)$ and $\boldsymbol{v} = (0, 1, 0, 0)$. Both vectors belong to S.
Then $\boldsymbol{u} + \boldsymbol{v} = (1, 1, 0, 0) \notin S$.
So S does not satisfied the closure property and hence is not a subspace of $\mathbb{R}^4$.

(iii) Since the first three rows of $\boldsymbol{A}$ are linearly independent, in order that $\operatorname{rank} \boldsymbol{A} = 3$, the last row (a, b, c, d) is "redundant" and hence a row echelon form of $\boldsymbol{A}$ is $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.
Denote the four variables of $\boldsymbol{Ax} = \boldsymbol{0}$ by x, y, z, w, by back substitution, we get the general solution

$$w = t, z = -t, y = t, x = -t \quad \text{for} \quad t \in \mathbb{R}.$$

In matrix form, this is given by $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$.

(iv) $\boldsymbol{A} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ a+b+c+d \end{pmatrix}$.

If $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\boldsymbol{A}$, we must have $a+b+c+d = 2$.

(v) In this case, $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ a & a & a & a \end{pmatrix}$.

Note that the last row $(a, a, a, a) = a(1, 0, 0, 1) + a(0, 1, 1, 0)$ is a linear combination of first and third row.

So a row echelon form of $\boldsymbol{A}$ is $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Hence we can take any three columns of $\boldsymbol{A}$ as the basis for the column space.

In particular, we can take the basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ a \end{pmatrix} \right\}$.

Question 2a [12 marks]

Let $S = \{(1,1,2,0),(2,2,4,0),(0,0,1,3),(1,1,3,3),(1,1,1,-3)\}$ and $V = \mathrm{span}(S)$.

(i) (4 marks) Find a basis S' for V such that $S' \subseteq S$ and write down $\dim V$.

(ii) (4 marks) Is $V = \mathrm{span}\{(2,2,5,3),(2,2,3,-3),(1,1,0,-6)\}$? Justify your answer.

(iii) (4 marks) Let $W = \{(x,y,z,w) \mid x - y + z - w = 0\}$. Find $W \cap V$. Give your answer as a linear span.

(i) Let $\boldsymbol{B} = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 1 \\ 0 & 0 & 3 & 3 & -3 \end{pmatrix} \xrightarrow{GJE} \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

So columns 1 and 3 of $\boldsymbol{B}$ are linearly independent.

Hence $S' = \{(1,1,2,0),(0,0,1,3)\}$ form a basis for V and $\dim V = 2$.

(ii) Stack the two spanning set of vectors column-wise as an augmented matrix as follow:

$$\left(\begin{array}{cc|ccc} 1 & 0 & 2 & 2 & 1 \\ 1 & 0 & 2 & 2 & 1 \\ 2 & 1 & 5 & 3 & 0 \\ 0 & 3 & 3 & -3 & -6 \end{array}\right) \xrightarrow{GJE} \left(\begin{array}{cc|ccc} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This represents a consistent system, hence

$$\mathrm{span}\{(2,2,5,3),(2,2,3,-3),(1,1,0,-6)\} \subseteq V - - - - - (1)$$

Flipping the two sets of vectors around:

$$\left(\begin{array}{ccc|cc} 2 & 2 & 1 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 2 & 1 \\ 3 & -3 & -6 & 0 & 3 \end{array}\right) \xrightarrow{GJE} \left(\begin{array}{ccc|cc} 1 & 0 & -3/4 & 1/4 & 1/2 \\ 0 & 1 & -5/4 & 1/4 & -1/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This again represents a consistent system, hence

$$V \subseteq \mathrm{span}\{(2,2,5,3),(2,2,3,-3),(1,1,0,-6)\} - - - - - (2)$$

By (1) and (2), we get $V = \mathrm{span}\{(2,2,5,3),(2,2,3,-3),(1,1,0,-6)\}$.

(iii) Observe that the vector $(1,1,3,3) \in V$ satisfies the equation $x - y + z - w = 0$ and hence it belongs to W.

Hence $W \cap V$ is non-trivial and this implies $\dim W \cap V \geq 1$.

On the other hand, the vector $(1,1,2,0) \in V$ does not satisfy the equation $x-y+z-w = 0$ and hence it does not belong to W.

Hence $W \cap V$ is a proper subset of V and this implies $\dim W \cap V < \dim V = 2$.

So we conclude that $\dim W \cap V = 1$.

Since $(1,1,3,3) \in W \cap V$, we have $W \cap V = \mathrm{span}\{(1,1,3,3)\}$.

Question 2b **[8 marks]**

Let W be a subspace of $\mathbb{R}^n$ and $W^\perp = \{\boldsymbol{w} \in \mathbb{R}^n \mid \boldsymbol{w} \cdot \boldsymbol{v} = 0 \text{ for all } \boldsymbol{v} \in W\}$.

(i) (2 marks) Show that $W \cap W^\perp = \{\mathbf{0}\}$.

(ii) (6 marks) Show that every vector $\boldsymbol{v} \in \mathbb{R}^n$ can be written uniquely as $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ where $\boldsymbol{v}_1 \in W$ and $\boldsymbol{v}_2 \in W^\perp$.

(You may assume in part (ii) that W and $W^\perp$ are associated to the row space and nullspace of certain matrix.)

(i) Let $\boldsymbol{x} \in W \cap W^\perp$.
Then $\boldsymbol{x} \in W$ and $\boldsymbol{x} \in W^\perp$.
So $\boldsymbol{x} \cdot \boldsymbol{x} = 0 \Rightarrow \boldsymbol{x} = \mathbf{0}$.
Hence we conclude that $W \cap W^\perp = \{\mathbf{0}\}$.

(ii) Let $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_k\}$ be a basis for W and let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \vdots \\ \boldsymbol{r}_k \end{pmatrix}$ be a $k \times m$ matrix with i-th row equal to $\boldsymbol{r}_i$.
Then W is the row space of $\boldsymbol{A}$ and $W^\perp$ is the nullspace of $\boldsymbol{A}$.
Let $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_h\}$ be a basis for $W^\perp$.
Then by dimension theorem, $k + h = rank(\boldsymbol{A}) + nullity(\boldsymbol{A}) = n$.
To show that $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_k, \boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_h\}$ is a basis for $\mathbb{R}^n$, we just need to show the set is linearly independent:

$$a_1\boldsymbol{r}_1 + \cdots + a_k\boldsymbol{r}_k + b_1\boldsymbol{s}_1 + \cdots + b_h\boldsymbol{s}_h = \mathbf{0}.$$

This can be rewritten as

$$a_1\boldsymbol{r}_1 + \cdots + a_k\boldsymbol{r}_k = -b_1\boldsymbol{s}_1 - \cdots - b_h\boldsymbol{s}_h \ (*)$$

Since LHS of $(*)$ belongs to W and RHS of $(*)$ belongs to $W^\perp$, both sides belong to $W \cap W^\perp = \{\mathbf{0}\}$.
Hence $a_1\boldsymbol{r}_1 + \cdots + a_k\boldsymbol{r}_k = \mathbf{0}$ implies $a_1 = \cdots = a_k = 0$
and $b_1\boldsymbol{s}_1 + \cdots + b_h\boldsymbol{s}_h = \mathbf{0}$ implies $b_1 = \cdots = b_h = 0$.
Therefore we conclude that $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_k, \boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_h\}$ is linearly independent.
For any $\boldsymbol{v} \in \mathbb{R}^n$, $v = c_1\boldsymbol{r}_1 + \cdots + c_k\boldsymbol{r}_k + d_1\boldsymbol{s}_1 + \cdots + d_h\boldsymbol{s}_h = \boldsymbol{v}_1 + \boldsymbol{v}_2$ where $\boldsymbol{v}_1 = c_1\boldsymbol{r}_1 + \cdots + c_k\boldsymbol{r}_k \in W$ and $\boldsymbol{v}_2 = d_1\boldsymbol{s}_1 + \cdots + d_h\boldsymbol{s}_h \in W^\perp$.
Furthermore, the decomposition $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ is unique:
Suppose $\boldsymbol{v} = \boldsymbol{u}_1 + \boldsymbol{u}_2$ where $\boldsymbol{u}_1 \in W$ and $\boldsymbol{u}_2 \in W^\perp$.
Then $\boldsymbol{v}_1 + \boldsymbol{v}_2 = \boldsymbol{u}_1 + \boldsymbol{u}_2 \Rightarrow \boldsymbol{v}_1 - \boldsymbol{u}_1 = \boldsymbol{u}_2 - \boldsymbol{v}_2 \ (**)$.
Like before, LHS of $(**)$ belongs to W and RHS of $(**)$ belongs to $W^\perp$.
So $\boldsymbol{v}_1 - \boldsymbol{u}_1 = \mathbf{0} \Rightarrow \boldsymbol{v}_1 = \boldsymbol{u}_1$ and $\boldsymbol{u}_2 - \boldsymbol{v}_2 = \mathbf{0} \Rightarrow \boldsymbol{v}_2 = \boldsymbol{u}_2$.

Question 3a **[14 marks]**

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.

(i) (4 marks) Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where $\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{u}_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}$.

Show that S is an orthogonal basis for the column space V of $\boldsymbol{A}$.

(ii) (2 marks) Normalise S to get an orthonormal basis $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ for V.

(iii) (4 marks) Find the least squares solutions of $\boldsymbol{Ax} = \boldsymbol{b}$.

(iv) (4 marks) Extend the basis T in part (ii) to an orthonormal basis $T' = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ for $\mathbb{R}^4$ without using Gram-Schmidt.

(i) Direct checking: $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = 0, \boldsymbol{u}_1 \cdot \boldsymbol{u}_3 = 0, \boldsymbol{u}_2 \cdot \boldsymbol{u}_3 = 0$.

So S is an orthogonal set and hence it is linearly independent.

Denote the four columns of $\boldsymbol{A}$ by $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4$.

Then we have $\boldsymbol{u}_1 = \boldsymbol{c}_1, \boldsymbol{u}_2 = -3\boldsymbol{c}_1 + 2\boldsymbol{c}_2, \boldsymbol{u}_3 = -\boldsymbol{c}_2 + \boldsymbol{c}_3$.

Hence $S \subseteq V$ (the column space of $\boldsymbol{A}$).

Check that $\operatorname{rank}(\boldsymbol{A}) = 3$. So $\dim V = 3$.

Hence S is an orthogonal basis for V.

(ii) $\boldsymbol{v}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{v}_2 = \frac{1}{\sqrt{12}}\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{v}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}$

(iii) We solve $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$.

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 4 & 6 & 6 & 9 \\ 6 & 12 & 12 & 16 \\ 6 & 12 & 18 & 24 \\ 9 & 16 & 24 & 33 \end{pmatrix} \text{ and } \boldsymbol{A}^T\boldsymbol{b} = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

$$(\boldsymbol{A}^T\boldsymbol{A} \mid \boldsymbol{A}^T\boldsymbol{b}) \xrightarrow{GJE} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1/2 & -1 \\ 0 & 0 & 1 & 4/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

By back substitution, we get the general solution:

$$w = t, z = \frac{1}{3} - \frac{4}{3}t, y = -1 + \frac{1}{2}t, x = 1 - t.$$

So the least squares solutions of $\boldsymbol{Ax} = \boldsymbol{b}$ are:

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 - t \\ -1 + \frac{1}{2}t \\ \frac{1}{3} - \frac{4}{3}t \\ t \end{pmatrix}.$$

(iv) Let $\boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \\ \frac{1}{3} \\ 0 \end{pmatrix}$ be one of the least squares solutions in (iii).

Then $\boldsymbol{p} = \boldsymbol{Av} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ is the projection of $\boldsymbol{b}$ onto V.

Hence $\boldsymbol{p} - \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ is orthogonal to V, and hence to $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

So we can take $\boldsymbol{v}_4 = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$.

Question 3b **[6 marks]**

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ be two orthonormal bases for a proper subspace V of $\mathbb{R}^n$.

Let $\boldsymbol{C} = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_m \end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_m \end{pmatrix}$ be matrices formed using the basis vectors of S and T as their columns respectively.

Determine whether the following are true or false. Justify your answers.

(i) (2 marks) $\boldsymbol{C}$ and $\boldsymbol{D}$ are orthogonal matrices.

(ii) (2 marks) If the reduced row echelon form of $(\boldsymbol{D} \mid \boldsymbol{C})$ is given by $(\boldsymbol{I} \mid \boldsymbol{P})$, then $\boldsymbol{P}$ is the transition matrix from S to T.

(iii) (2 marks) $\boldsymbol{C}^T\boldsymbol{D}$ is the transition matrix from T to S.

(i) False. The sizes of $\boldsymbol{C}$ and $\boldsymbol{D}$ are $n \times m$, so they are non-square matrices, and hence cannot be orthogonal matrices.

(ii) False. The size of $\boldsymbol{P}$ is $n \times m$, So it is a non-square matrix, and hence cannot be a transition matrix.

(iii) True.

$$\boldsymbol{C}^T\boldsymbol{D} = \begin{pmatrix} \boldsymbol{u}_1^T \\ \boldsymbol{u}_2^T \\ \vdots \\ \boldsymbol{u}_m^T \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_m \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}_1 \cdot \boldsymbol{v}_1 & \boldsymbol{u}_1 \cdot \boldsymbol{v}_2 & \cdots & \boldsymbol{u}_1 \cdot \boldsymbol{v}_m \\ \boldsymbol{u}_2 \cdot \boldsymbol{v}_1 & \boldsymbol{u}_2 \cdot \boldsymbol{v}_2 & \cdots & \boldsymbol{u}_2 \cdot \boldsymbol{v}_m \\ \vdots & & & \vdots \\ \boldsymbol{u}_m \cdot \boldsymbol{v}_1 & \boldsymbol{u}_m \cdot \boldsymbol{v}_2 & \cdots & \boldsymbol{u}_m \cdot \boldsymbol{v}_m \end{pmatrix}$$

which is the transition matrix from T to S.

Question 4a **[12 marks]**

Let $\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$.

(i) (4 marks) Find the characteristic polynomial and all the eigenvalues of $\boldsymbol{C}$. Show your working.

(ii) (4 marks) Find a basis for each eigenspace of $\boldsymbol{C}$. Show your working.

(iii) (4 marks) Find a matrix $\boldsymbol{P}$ that orthogonally diagonalizes $\boldsymbol{C}$ and write down the corresponding diagonal matrix $\boldsymbol{D}$. Explain how your answers are derived.

(i) $\det(x\boldsymbol{I} - \boldsymbol{C}) = \begin{vmatrix} x-1 & 0 & 0 & -1 \\ 0 & x-2 & -2 & 0 \\ 0 & -2 & x-2 & 0 \\ -1 & 0 & 0 & x-1 \end{vmatrix} = (x-1)\begin{vmatrix} x-2 & -2 & 0 \\ -2 & x-2 & 0 \\ 0 & 0 & x-1 \end{vmatrix} + \begin{vmatrix} 0 & x-2 & -2 \\ 0 & -2 & x-2 \\ -1 & 0 & 0 \end{vmatrix}$

$= (x-1)^2[(x-2)^2-4] - [(x-2)^2-4] = (x^2-2x)(x^2-4x) = x^2(x-2)(x-4)$

So the eigenvalues of $\boldsymbol{C}$: 0, 2 and 4.

(ii) For $\lambda = 0$:

$$\left(\begin{array}{cccc|c} -1 & 0 & 0 & -1 & 0 \\ 0 & -2 & -2 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \end{array}\right) \xrightarrow{GJE} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

By back substitution, we get the general solution: $w = t, z = s, y = -s, x = -t$.
So the eigenspace E_0 for $\lambda = 0$ is $\{(-t, -s, s, t)^T \mid s, t \in \mathbb{R}\}$ and a basis for E_0 is

$$\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

For $\lambda = 2$:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{GJE} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

By back substitution, we get the general solution: $w = t, z = 0, y = 0, x = t$.
So the eigenspace E_2 for $\lambda = 2$ is $\{(t, 0, 0, t)^T \mid t \in \mathbb{R}\}$ and a basis for E_2 is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

For $\lambda = 4$:

$$\left(\begin{array}{cccc|c} 3 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ -1 & 0 & 0 & 3 & 0 \end{array}\right) \xrightarrow{GJE} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

By back substitution, we get the general solution: $w = 0, z = t, y = t, x = 0$.
So the eigenspace E_4 for $\lambda = 4$ is $\{(0, s, s, 0)^T \mid s \in \mathbb{R}\}$ and a basis for E_4 is

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(iii) The four eigenvectors in the bases for the eigenspaces:

$$\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

are orthogonal.
We normalise these vectors to get an orthogonal matrix

$$\boldsymbol{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

that orthogonally diagonalise $\boldsymbol{C}$ to give the diagonal matrix $\boldsymbol{D} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$.

Question 4b **[8 marks]**

Let $\boldsymbol{M}$ is an $n \times n$ matrix such that $\boldsymbol{M}^2 = \boldsymbol{M}$ and both 0 and 1 are eigenvalues of $\boldsymbol{M}$.

(i) (4 marks) Show that the column space of $\boldsymbol{M}$ is the eigenspace $\boldsymbol{E}_1$ associated to eigenvalue 1.

(ii) (4 marks) Show that $\boldsymbol{M}$ is diagonalizable

(i) Let $\boldsymbol{v} \in E_1$ (the eigenspace associated to eigenvalue 1).
Then $\boldsymbol{M}\boldsymbol{v} = \boldsymbol{v}$ which implies $\boldsymbol{v}$ belongs to the column space of $\boldsymbol{M}$.
Hence $E_1 \subseteq$ column space of $\boldsymbol{M}$ (1).
Let $\boldsymbol{v} \in$ column space of $\boldsymbol{M}$. Then $\boldsymbol{v} = \boldsymbol{M}\boldsymbol{w}$ for some $\boldsymbol{w} \in \mathbb{R}^n$.
So $\boldsymbol{v} = \boldsymbol{M}^2\boldsymbol{w} = \boldsymbol{M}(\boldsymbol{M}\boldsymbol{w}) = \boldsymbol{M}\boldsymbol{v}$.
This implies $\boldsymbol{v} \in E_1$.
Hence column space of $\boldsymbol{M} \subseteq E_1$ (2).
By (1) and (2), we conclude that column space of $\boldsymbol{M} = E_1$

(ii) From (i), $\dim E_1 = \dim(\text{column space of } \boldsymbol{M}) = \text{rank } \boldsymbol{M}$.
On the other hand, the eigenspace E_0 associated to eigenvalue 0 is the nullspace of $\boldsymbol{M}$.
So $\dim E_0 = \dim(\text{nullspace of } \boldsymbol{M}) = \text{nullity } \boldsymbol{M}$.
By Dimension Theorem, $\dim E_1 + \dim E_0 = \text{rank } \boldsymbol{M} + \text{nullity } \boldsymbol{M} = n$.
This implies there are n linearly independent eigenvectors of $\boldsymbol{M}$, and hence $\boldsymbol{M}$ is diagonalizable.

Question 5a **[12 marks]**

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix}, \; T\begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}2\\0\\4\end{pmatrix}, \; T\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}0\\2\\4\end{pmatrix}.$$

(i) (3 marks) Find the standard matrix of T. Show how your answer is derived.

(ii) (3 marks) Find the kernel of T. Give your answer as a linear span.

(iii) (3 marks) Find the largest possible subspace V of $\mathbb{R}^3$ such that every vector $\boldsymbol{v} \in V$ maps to itself under T. Explain how your answer is derived.

(iv) (3 marks) Are there any vector $\boldsymbol{v} \in \mathbb{R}^3$ such that $T(\boldsymbol{v}) = \begin{pmatrix}1\\2\\0\end{pmatrix}$? Justify your answer.

(i) Let $\boldsymbol{A}$ be the standard matrix of T.

$$\boldsymbol{A}\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix}, \; \boldsymbol{A}\begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}2\\0\\4\end{pmatrix}, \; \boldsymbol{A}\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}0\\2\\4\end{pmatrix}.$$

By stacking, we have: $\boldsymbol{A}\begin{pmatrix}0 & 1 & 1\\1 & 0 & 2\\2 & 2 & 0\end{pmatrix} = \begin{pmatrix}0 & 2 & 0\\1 & 0 & 2\\2 & 4 & 4\end{pmatrix}$

$$\Rightarrow \boldsymbol{A} = \begin{pmatrix}0 & 2 & 0\\1 & 0 & 2\\2 & 4 & 4\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\1 & 0 & 2\\2 & 2 & 0\end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix}4 & -2 & 1\\0 & 3 & 0\\8 & 2 & 2\end{pmatrix}.$$

(ii) $\ker T = \text{nullspace } \boldsymbol{A}$:

$$\frac{1}{3}\left(\begin{array}{ccc|c}4 & -2 & 1 & 0\\0 & 3 & 0 & 0\\8 & 2 & 2 & 0\end{array}\right) \xrightarrow{GJE} \left(\begin{array}{ccc|c}1 & 0 & 1/4 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right)$$

By back substitution, we get the general solutions: $z = t, y = 0, x = -\frac{1}{4}t$.
So $\ker T = \text{span}\{(-1/4, 0, 1)\}$.

(iii) The largest possible subspace V of $\mathbb{R}^3$ such that $T(\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v} \in V$ is E_1, the eigenspace of $\boldsymbol{A}$ associated to eigenvalue 1.

From the given conditions as well as part (ii), we know $\boldsymbol{A}$ has three distinct eigenvalues $1, 2, 0$. So each eigenspace has dimension 1.

From $\boldsymbol{A}\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix}$, we know that $E_1 = \text{span}\left\{\begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$.

(iv) No. This is the same as saying $\begin{pmatrix}1\\2\\0\end{pmatrix} \notin$ column space of $\boldsymbol{A}$.

$$\frac{1}{3}\left(\begin{array}{ccc|c} 4 & -2 & 1 & 3 \\ 0 & 3 & 0 & 6 \\ 8 & 2 & 2 & 0 \end{array}\right) \xrightarrow{GJE} \left(\begin{array}{ccc|c} 1 & 0 & 1/4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Since the above system is inconsistent, we conclude that $T(\boldsymbol{v}) \neq \begin{pmatrix}1\\2\\0\end{pmatrix}$ for any vector $\boldsymbol{v} \in \mathbb{R}^3$.

Question 5b [8 marks]

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and $R(T)$ is the range of T.
Denote $T^1 = T$ and $T^{k+1} = T \circ T^k$ for all (integers) $k \geq 1$.

(i) (2 marks) Show that $R(T^{k+1}) \subseteq R(T^k)$ for all $k \geq 1$.

(ii) (6 marks) Suppose T^m is the zero transformation for some $m > n$. Show that T^n must be the zero transformation. (Note that T itself need not be the zero transformation.)
Hint: Show that if $R(T^k) = R(T^{k+1})$ for some $k \geq 1$, then $R(T^k) = R(T^h)$ for all $h \geq k$.

(i) For any $k \geq 1$, let $\boldsymbol{v} \in R(T^{k+1})$. So $\boldsymbol{v} = T^{k+1}(\boldsymbol{w}) = T^k(T(\boldsymbol{w}))$ for some $\boldsymbol{w} \in \mathbb{R}^n$.
Hence $\boldsymbol{v} \in R(T^k)$.
This implies $R(T^{k+1}) \subseteq R(T^k)$.

(ii) First of all, we show that if $R(T^k) = R(T^{k+1})$, then $R(T^k) = R(T^{k+2})$.
Let $\boldsymbol{v} \in R(T^k)$. Then $\boldsymbol{v} \in R(T^{k+1})$. So $\boldsymbol{v} = T^{k+1}(\boldsymbol{w}) = T(T^k(\boldsymbol{w}))$ for some $\boldsymbol{w} \in \mathbb{R}^n$.
Let $\boldsymbol{u} = T^k(\boldsymbol{w}) \in R(T^k) = R(T^{k+1})$. So $\boldsymbol{u} = T^{k+1}(\boldsymbol{x})$ for some $\boldsymbol{x} \in \mathbb{R}^n$.
So $\boldsymbol{v} = T(\boldsymbol{u}) = T(T^{k+1}(\boldsymbol{x})) = T^{k+2}(\boldsymbol{x}) \in R(T^{k+2})$.
Hence we have $R(T^k) \subseteq R(T^{k+2})$.
On the other hand, by (i), $R(T^{k+2}) \subseteq R(T^{k+1}) = R(T^k)$.
So we conclude $R(T^k) = R(T^{k+2})$.
Inductively, if we have $R(T^k) = R(T^{k+1})$, then $R(T^k) = R(T^h)$ for all $h \geq k$.
Now let p be the smallest integer such that $R(T^P) = R(T^{p+1})$. Then

$$R(T) \supsetneq R(T^2) \supsetneq \cdots \supsetneq R(T^p) = R(T^{p+1}) = \cdots = R(T^m) = \{\mathbf{0}\}.$$

This implies

$$n \geq rank(T) > rank(T^2) > \cdots > rank(T^p) = rank(T^{p+1}) = \cdots = rank(T^m) = 0$$

since T^m is the zero transformation.
Since we have a strictly decreasing sequence from rank(T) to rank(T^p), and rank(T^p) $= 0$, this implies $p \leq n$.
Consequently, rank$(T^n) = 0 \Rightarrow R(T^n) = \{\mathbf{0}\} \Rightarrow T^n$ is the zero transformation.