CS1231S: Discrete Structures
Tutorial #5: Relations & Partial Orders
Answers

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1. Let S be the set of all strings over the alphabet $\mathcal{A} = \{s, u\}$, i.e. an element of S is a sequence of characters, each of which is either s or u. Examples of elements of S are: ε (the empty string), $s, u, sus, usssuu$, and $susussssssu$.

   Define a relation R on S by the following: $\forall a, b \in S\ \big(aRb \iff len(a) \leq len(b)\big)$ where $len(x)$ denotes the length of x, i.e. the number of characters in x.

   Is R antisymmetric? Prove or disprove it.

   > Let R be a relation on a set A.
   > R is **antisymmetric** iff $\forall x, y \in A\ (xRy \land yRx \Rightarrow x = y)$.

   Answer:
   R is not antisymmetric. Counterexample: $s\ R\ u$ and $u\ R\ s$ but $s \neq u$.

2. Consider the "divides" relation on each of the following sets of integers. For each of them, draw a Hasse diagram, and find all minimal, maximal, smallest and largest elements.

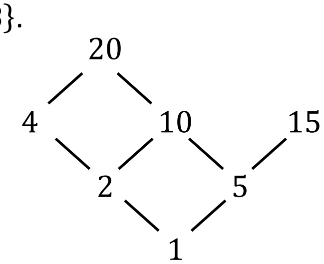
   (a) $A = \{1, 2, 4, 5, 10, 15, 20\}$.

   (b) $B = \{2, 3, 4, 6, 8, 9, 12, 18\}$.

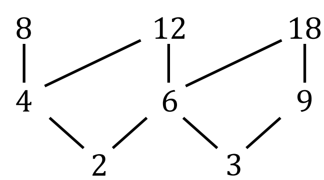
   > Let a set A be partially ordered with respect to a relation $\preccurlyeq$ and $c \in A$.
   > - c is a **maximal element** of A iff $\forall x \in A\ (c \preccurlyeq x \Rightarrow c = x)$.
   > - c is a **minimal element** of A iff $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$.
   > - c is the **largest element** of A iff $\forall x \in A\ (x \preccurlyeq c)$.
   > - c is the **smallest element** of A iff $\forall x \in A\ (c \preccurlyeq x)$.

   Answers:

   (a) Minimal: 1.
   Maximal: 15 and 20.
   Smallest: 1.
   Largest: None.



   (b) Minimal: 2 and 3.
   Maximal: 8, 12 and 18.
   Smallest: None.
   Largest: None.



3. Let $\mathcal{P}(A)$ denote the power set of set A. Prove that the binary relation $\subseteq$ on $\mathcal{P}(A)$ is a partial order.

   Answer:
   Proof:
   1. (Reflexivity) Take any $S \in \mathcal{P}(A)$,
      1.1. $S \subseteq S$ by the definition of subset.
      1.2. Hence $\subseteq$ is reflexive.
   2. (Antisymmetry) Take any $S, T \in \mathcal{P}(A)$,
      2.1. Suppose $S \subseteq T$ and $T \subseteq S$.
      2.2. Then $S = T$ by the definition of set equality.
      2.3. Hence $\subseteq$ is antisymmetric.

3. (Transitivity)
   3.1. $\subseteq$ is transitive by Theorem 6.2.1.
4. Therefore $\subseteq$ on $\mathcal{P}(A)$ is a partial order.

> **Theorem 6.2.1**
> For all sets A, B, C,
> $A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$.

4. Let $B = \{0,1\}$ and define the binary relation R on $B \times B$ as follows:

$$\forall (a,b),(c,d) \in B \times B \, \big((a,b)\ R\ (c,d) \Leftrightarrow (a \le c) \wedge (b \le d)\big).$$

(a) Prove that R is a partial order.

(b) Draw the Hasse diagram for R.

(c) Find the maximal, largest, minimal and smallest elements.

(d) Is $(B \times B, R)$ well-ordered?

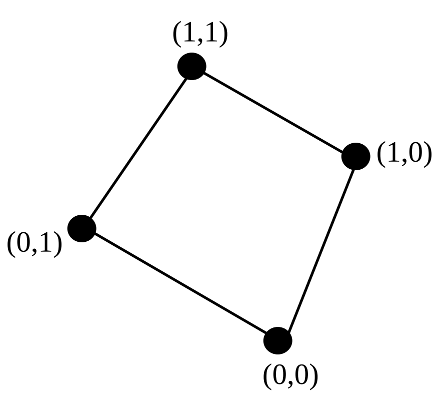
> Let R be a relation on a set A.
> - R is **reflexive** iff $\forall x \in A\ (xRx)$.
> - R is **antisymmetric** iff $\forall x, y \in A\ (xRy \wedge yRx \Rightarrow x = y)$.
> - R is **transitive** iff $\forall x, y, z \in A\ (xRy \wedge yRz \Rightarrow xRz)$.

Answers:

(a) Proof:

1. (Reflexivity) Take any $(a,b) \in B \times B$,
   1.1. $a \le a$ and $b \le b$.
   1.2. So $(a,b)\ R\ (a,b)$ by the definition of R.
   1.3. Hence R is reflexive.
2. (Antisymmetry) Take any $(a,b),(c,d) \in B \times B$,
   2.1. Suppose $(a,b)\ R\ (c,d)$ and $(c,d)\ R\ (a,b)$.
   2.2. Then $a \le c, b \le d, c \le a$ and $d \le b$ by the definition of R.
   2.3. Then $a = c$ and $b = d$ by the antisymmetry of $\le$.
   2.4. So $(a,b) = (c,d)$ by equality of ordered pairs.
   2.5. Hence R is antisymmetric.
3. (Transitivity) Take any $(a,b),(c,d),(e,f) \in B \times B$,
   3.1. Suppose $(a,b)\ R\ (c,d)$ and $(c,d)\ R\ (e,f)$.
   3.2. Then $a \le c, b \le d, c \le e$ and $d \le f$ by the definition of R.
   3.3. Then $a \le e$ and $b \le f$ by the transitivity of $\le$.
   3.4. So $(a,b)\ R\ (e,f)$ by the definition of R.
   3.5. Hence R is transitive.
4. Therefore R on $B \times B$ is a partial order.

(b)



(c) Maximal and largest: (1,1);
Minimal and smallest: (0,0).

(d) No. It is not even a total order, because (0,1) and (1,0) are not comparable.

5. Let R be a binary relation on a non-empty set A. Let $x, y \in A$. Define a relation S on A by

$$x\ S\ y \Leftrightarrow x = y \vee x\ R\ y \text{ for all } x, y \in A.$$

Show that:

(a) S is reflexive;

(b) $R \subseteq S$; and

(c) if S' is another reflexive relation on A and $R \subseteq S'$, then $S \subseteq S'$.

What is this relation S called? (Hint: Refer to Transitive Closure in Lecture 6).

Answers:

Let R be a relation on a set A. R is **reflexive** iff $\forall x \in A\ (xRx)$.

(a) 1. Let $x \in A$.
2. $x = x$, so $x\ S\ x$ by the definition of S.
3. Therefore, S is reflexive.

(b) 1. Suppose $(x, y) \in R$, that is, $x\ R\ y$.
2. So $x\ S\ y$ by the definition of S.
3. So $(x, y) \in S$.
4. Therefore, $R \subseteq S$ by the definition of $\subseteq$.

(c) 1. Suppose $(x, y) \in S$.
2. Then $x\ S\ y$, which means $x = y \vee x\ R\ y$ by the definition of S.
3. Case 1: $x = y$
   3.1. Then $x\ S'\ y$ since S' is reflexive.
   3.2. So $(x, y) \in S'$.
4. Case 2: $x\ R\ y$
   4.1. Then $(x, y) \in R \subseteq S'$.
   4.2. Then $(x, y) \in S'$.
5. In all cases, $(x, y) \in S'$.
6. Therefore, $S \subseteq S'$.

The relation S is called the reflexive closure of R. It is the smallest relation on A that is reflexive and contains R as a subset.

6. Let R be a binary relation on a set A.
We have defined antisymmetry in class: R is **antisymmetric** iff

$$\forall x, y \in A\ (x\ R\ y \wedge y\ R\ x \Rightarrow x = y).$$

We define asymmetry here. R is **asymmetric** iff

$$\forall x, y \in A\ (x\ R\ y \Rightarrow y \not R\ x).$$

(a) Find a binary relation on A that is both asymmetric and antisymmetric.

(b) Find a binary relation on A that is not asymmetric but antisymmetric.

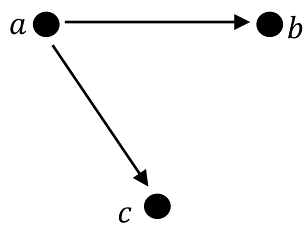
(c) Find a binary relation on A that is asymmetric but not antisymmetric.

(d) Find a binary relation on A that is neither asymmetric nor antisymmetric.

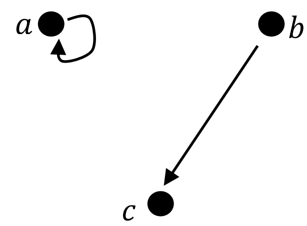
Answers:

Let $A = \{a, b, c\}$.

(a) $R = \{(a, b), (a, c)\}$.

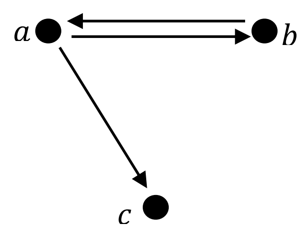


(b) $R = \{(a, a), (b, c)\}$



(c) No solution.
Every asymmetric relation is antisymmetric (see proof below).

(d) $R = \{(a, b), (b, a), (a, c)\}$



Proof: (Every asymmetric relation is antisymmetric.)

1. Take any binary relation R on a set A.
2. Suppose R is asymmetric.
   - 2.1. Then $\forall x, y \in A\ (x\ R\ y \Rightarrow y \not R\ x)$ by the definition of asymmetry.
   - 2.2. $\equiv \forall x, y \in A\ (x \not R\ y \vee y \not R\ x)$ by the implication law.
   - 2.3. $\Rightarrow \forall x, y \in A\ ((x \not R\ y \vee y \not R\ x) \vee x = y)$ by generalization.
   - 2.4. $\equiv \forall x, y \in A\ ({\sim}(x\ R\ y \wedge y\ R\ x) \vee x = y)$ by the De Morgan's law.
   - 2.5 $\equiv \forall x, y \in A\ (x\ R\ y \wedge y\ R\ x \Rightarrow x = y)$ by the implication law.
3. Line 2.5 is the definition of antisymmetry, hence R is antisymmetric.

We see that asymmetry property forces the antisymmetry property to be vacuously true.

7. Consider a set A and a total order $\preccurlyeq$ on A. Show that all minimal elements are smallest.

Answer:

1. Let $c \in A$ that is minimal with respect to $\preccurlyeq$.
2. Pick any $x \in A$.
3. As $\preccurlyeq$ is a total order, either $x \preccurlyeq c$ or $c \preccurlyeq x$.
4. Case 1: Suppose $x \preccurlyeq c$.
   - 4.1. Then $x = c$ by the minimality of c.
   - 4.2. So $c \preccurlyeq x$ by the reflexibility of $\preccurlyeq$.
5. Case 2: Suppose $c \preccurlyeq x$.
   - 5.1. Then $c \preccurlyeq x$.
6. In all cases, $c \preccurlyeq x$, i.e. c is the smallest element.

> Let a set A be partially ordered with respect to a relation $\preccurlyeq$ and $c \in A$.
>
> - c is a **minimal element** of A iff $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$.
> - c is the **smallest element** of A iff $\forall x \in A\ (c \preccurlyeq x)$.

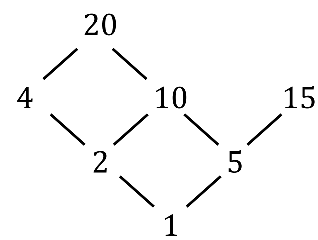
8. **Definitions.** Consider a partial order $\preccurlyeq$ on a set A and let $a, b \in A$.

- We say a, b are **comparable** iff $a \preccurlyeq b$ or $b \preccurlyeq a$.
- We say a, b are **compatible** iff there exists $c \in A$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

In question 3, you are given the "divides" relation on $A = \{1, 2, 4, 5, 10, 15, 20\}$. List out the pairs of distinct elements in A that are (a) comparable; (b) compatible. Use the notation $\{x, y\}$ to represent the pair of elements x and y.

Answers:

Hasse diagram



(a) Comparable:

$\{1,2\}$, $\{1,5\}$, $\{1,4\}$, $\{1,10\}$, $\{1,15\}$, $\{1,20\}$,

$\{2,4\}$, $\{2,10\}$, $\{2,20\}$, $\{5,10\}$, $\{5,15\}$, $\{5,20\}$,

$\{4,20\}$ and $\{10,20\}$.

Not comparable: {2,5}, {2,15}, {5,4}, {4,10}, {4,15}, {10,15} and {15,20}.

(b) Compatible:

$\{1,2\}$, $\{1,5\}$, $\{1,4\}$, $\{1,10\}$, $\{1,15\}$, $\{1,20\}$,

$\{2,4\}$, $\{2,5\}$, $\{2,10\}$, $\{2,20\}$, $\{5,4\}$, $\{5,10\}$, $\{5,15\}$, $\{5,20\}$,

$\{4,10\}$, $\{4,20\}$ and $\{10,20\}$.

Not compatible: {2,15}, {4,15}, {10,15} and {15,20}.

9. Let $A = \{a, b, c, d\}$. Consider the following partial order on A:

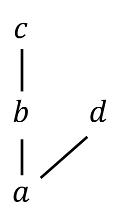
$$R = \{(a,a),\ (a,b),\ (a,c),\ (a,d),\ (b,b),\ (b,c),\ (c,c),\ (d,d)\}.$$

(a) Draw a Hasse diagram of R..

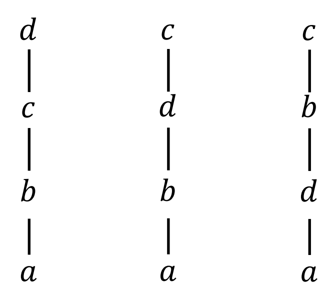
(b) Draw Hasse diagrams of all the linearizations of R.

Answers:

(a)



(b)



10. For each of the following statements, state whether it is true or false and justify your answer.

(a) In all partially ordered sets, any two comparable elements are compatible.

(b) In all partially ordered sets, any two compatible elements are comparable.

Let $\preccurlyeq$ be a partial order $\preccurlyeq$ on a set A and let $a, b \in A$.
- We say a, b are **comparable** iff $a \preccurlyeq b$ or $b \preccurlyeq a$.
- We say a, b are **compatible** iff there exists $c \in A$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

Answers:

(a) True. Proof:

1. Let $a, b \in A$ such that a and b are comparable.
2. Then either $a \preccurlyeq b$ or $b \preccurlyeq a$ by the definition of comparability.
3. Case 1: $a \preccurlyeq b$
   - 3.1. Let $c = b$.
   - 3.2. Then $a \preccurlyeq b = c$ by assumption and
     $b \preccurlyeq b = c$ by the reflexivity of $\preccurlyeq$.
   - 3.3. Hence a and b are compatible by the definition of compatibility.
4. Case 2: $b \preccurlyeq a$
   - 4.1. Let $c = a$.
   - 4.2. Then $b \preccurlyeq a = c$ by assumption and
     $a \preccurlyeq a = c$ by the reflexivity of $\preccurlyeq$.
   - 4.3. Hence a and b are compatible by the definition of compatibility.
5. In all cases, a and b are compatible.

Note that the 2 cases are symmetrical and hence very similar (just switch the roles of a and b). We may use WLOG (without loss of generality) to cover one of the cases. (Use WLOG with care! Make sure that the cases are indeed similar before you use WLOG.)

1. Let $a, b \in A$ such that a and b are comparable.
2. Then either $a \preccurlyeq b$ or $b \preccurlyeq a$ by the definition of comparability.
3. WLOG, let $a \preccurlyeq b$.
   - 3.1. Let $c = b$.
   - 3.2. Then $a \preccurlyeq b = c$ by assumption and
     $b \preccurlyeq b = c$ by the reflexivity of $\preccurlyeq$.
   - 3.3. Hence a and b are compatible by the definition of compatibility.
4. Hence a and b are compatible.

(b) False. Consider the poset $(\mathbb{Z}^+, |)$ where $|$ is the "divides" relation. 2 and 3 are compatible as $2 \mid 6$ and $3 \mid 6$. However, 2 and 3 are not comparable as $2 \nmid 3$ and $3 \nmid 2$.