Student Number: ____________________________

NATIONAL UNIVERSITY OF SINGAPORE

MA1101R - Linear Algebra I

(Semester 2 : AY2014/2015)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number clearly in the space provided at the top of this page.** This booklet (and only this booklet) will be collected at the end of the examination.

2. Please write your matriculation/student number only. Do not write your name.

3. This examination paper contains **FOUR** questions and comprises **NINETEEN** printed pages.

4. Answer **ALL** questions.

5. This is a CLOSED BOOK (with helpsheet) examination.

6. You are allowed to use two A4 size helpsheets.

7. You may use scientific calculators. However, you should lay out systematically the various steps in the calculations)

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
Total	

Question 1 [25 marks]

(a) [15 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

(i) Is $\boldsymbol{A}$ invertible? Justify your answer.

(ii) Find elementary matrices $\boldsymbol{E_1}, \boldsymbol{E_2}, \cdots, \boldsymbol{E_k}$ such that $\boldsymbol{A} = \boldsymbol{E_k} \cdots \boldsymbol{E_2} \boldsymbol{E_1} \boldsymbol{R}$ where $\boldsymbol{R}$ is a matrix in row-echelon form.

(iii) Find a matrix $\boldsymbol{P}$ that orthogonally diagonalizes $\boldsymbol{A}$ and determine $\boldsymbol{P}^T \boldsymbol{A} \boldsymbol{P}$. (You may assume that the characteristic polynomial for $\boldsymbol{A}$ is $(\lambda+1)^2(\lambda-2)$.)

Show your working below.

(i)

$$\det(\boldsymbol{A}) = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 + 1 = 2 \neq 0$$

Since $\det(\boldsymbol{A}) \neq 0$, it is invertible.

(ii)

$$\boldsymbol{A} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} = \boldsymbol{R}.$$

Thus if we choose

$$\boldsymbol{E_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \boldsymbol{E_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \boldsymbol{E_3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we will have $\boldsymbol{A} = \boldsymbol{E_3} \boldsymbol{E_2} \boldsymbol{E_1} \boldsymbol{R}$.

(iii) Since $(\lambda+1)^2(\lambda-2) = 0$ if and only if $\lambda = -1$ or $\lambda = 2$, $\boldsymbol{A}$ has eigenvalues -1 and 2.

Consider E_{-1} and solve the homogeneous linear system $(-\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$.

$$\left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - R_1} \left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus a general solution to the linear system is

$$\begin{cases} x = -s - t \\ y = s \\ z = t, \quad s, t \in \mathbb{R} \end{cases}$$

More working space for Question 1(a)

A basis for E_{-1} is $\{(-1,1,0),(-1,0,1)\}$. Since this is not an orthogonal basis, we apply Gram-Schmidt Process. Let

$$\begin{aligned} \boldsymbol{u_1} &= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \boldsymbol{u_2} &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} \end{aligned}$$

Normalizing, we have the following orthonormal basis for E_{-1}:

$$\left\{ \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix} \right\}.$$

Consider E_2 and solve the homogeneous linear system $(2\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$.

$$\left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array} \right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Thus a general solution to the linear system is

$$\begin{cases} x = s \\ y = s \\ z = s, \quad s \in \mathbb{R} \end{cases}$$

An orthonormal basis for E_2 is

$$\left\{ \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \right\}.$$

Thus an orthogonal matrix $\boldsymbol{P}$ that orthogonally diagonalizes $\boldsymbol{A}$ can be

$$\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \text{ and } \boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Question 1

(b) [10 marks]

Let $\boldsymbol{B} = \begin{pmatrix} -1 & k & 2 \\ -3 & 2 & 1 \\ k & 0 & 1 \end{pmatrix}$ where k is a real number.

(i) Compute $\det(\boldsymbol{B})$ in terms of k.

(ii) Find all values of k such that $\boldsymbol{Bx} = \mathbf{0}$ has only the trivial solution. Justify your answer.

(iii) Find all values of k such that the solution space of $\boldsymbol{Bx} = \mathbf{0}$ has dimension at least 1. Justify your answer.

(iv) What is the smallest possible value of $\text{rank}(\boldsymbol{B})$? Justify your answer.

(v) Are there values of k such that the solution space of $\boldsymbol{B}^T\boldsymbol{x} = \mathbf{0}$ is a plane in $\mathbb{R}^3$ that contains the origin? Justify your answer.

Show your working below.

(i) By cofactor expansion,

$$\det(\boldsymbol{B}) = k\begin{vmatrix} k & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} -1 & k \\ -3 & 2 \end{vmatrix} = k(k-4) + (-2+3k) = k^2 - k - 2 = (k-2)(k+1).$$

(ii) $\boldsymbol{Bx} = \mathbf{0}$ has only the trivial solution if and only if $\boldsymbol{B}$ is invertible. Since $\det(\boldsymbol{B}) = (k-2)(k+1)$, we see that $\boldsymbol{B}$ is invertible for all $k \in \mathbb{R}$, $k \neq -1, 2$.

(iii) Note that the solution space of $\boldsymbol{Bx} = \mathbf{0}$ has dimension 0 if and only if $\boldsymbol{Bx} = \mathbf{0}$ has only the trivial solution which is equivalent to $\boldsymbol{B}$ being invertible. Thus the values of k such that the solution space of $\boldsymbol{Bx} = \mathbf{0}$ has dimension at least 1 are $k = -1$ and 2.

(iv) For $k \neq -1, 2$, $\text{rank}(\boldsymbol{B}) = 3$ since $\boldsymbol{B}$ is invertible. When $k = -1$, $\boldsymbol{B} = \begin{pmatrix} -1 & -1 & 2 \\ -3 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$. As the rows of $\boldsymbol{B}$ are not all multiples of each other, we conclude that $\text{rank}(\boldsymbol{B}) = 2$. Similarly, when $k = 2$, $\boldsymbol{B} = \begin{pmatrix} -1 & 2 & 2 \\ -3 & 2 & 1 \\ -2 & 0 & 1 \end{pmatrix}$ and $\text{rank}(\boldsymbol{B}) = 2$. Thus the smallest possible value for $\text{rank}(\boldsymbol{B})$ is 2 for all k.

(v) No. Note that $\text{rank}(\boldsymbol{B}) = \text{rank}(\boldsymbol{B}^T)$ and we have determined that $\text{rank}(\boldsymbol{B}) \geq 2$ for all values of k and so $\text{rank}(\boldsymbol{B}^T) \geq 2$ for all values of k. By dimension theorem for matrices, $\text{nullity}(\boldsymbol{B}^T) + \text{rank}(\boldsymbol{B}^T) = 3$ so $\text{nullity}(\boldsymbol{B}^T) \leq 1$. This implies that the dimension of the solution space of $\boldsymbol{B}^T\boldsymbol{x} = \mathbf{0}$ is at most 1 and thus can never be a plane in $\mathbb{R}^3$ which has dimension 2.

More working space for Question 1(b)

Question 2 [25 marks]

(a) [11 marks]

Let $V = \{(a, a, a, 0) \mid a \in \mathbb{R}\}$.

(i) Find a basis for V and determine $\dim(V)$.

(ii) Find a subspace W of $\mathbb{R}^4$ such that $\dim(W) = 3$ and $\dim(W \cap V) = 1$. Justify your answer.

(iii) Let $U = \{\boldsymbol{u} \in \mathbb{R}^4 \mid \boldsymbol{u} \cdot \boldsymbol{v} = 0 \text{ for all } \boldsymbol{v} \in V\}$. Find a basis for and determine the dimension of U.

Show your working below.

(i)

$$V = \{(a, a, a, 0) \mid a \in \mathbb{R}\} = \{a(1, 1, 1, 0) \mid a \in \mathbb{R}\} = \text{span}\{(1, 1, 1, 0)\}.$$

Thus $\{(1, 1, 1, 0)\}$ is a basis for V and $\dim(V) = 1$.

(ii) Let $W = \text{span}\{(1, 1, 1, 0), (0, 1, 0, 0), (0, 0, 1, 0)\}$. Since $V = \text{span}\{(1, 1, 1, 0)\}$, we have $V \subseteq W$ and $W \cap V = V$ and thus $\dim(W \cap V) = 1$. On the other hand,

$$a(1, 1, 1, 0) + b(0, 1, 0, 0) + c(0, 0, 1, 0) = (0, 0, 0, 0) \Rightarrow a = b = c = 0.$$

Thus $\{(1, 1, 1, 0), (0, 1, 0, 0), (0, 0, 1, 0)\}$ is a linearly independent set and $\dim(W) = 3$.

(iii) Since $V = \text{span}\{(1, 1, 1, 0)\}$, $\boldsymbol{u} = (u_1, u_2, u_3, u_4) \in U$ if and only if $(u_1, u_2, u_3, u_4) \cdot (1, 1, 1, 0) = 0$, which implies $u_1 + u_2 + u_3 = 0$. Solving, we have

$$\begin{cases} u_1 &= -s - t \\ u_2 &= s \\ u_3 &= t \\ u_4 &= r, \quad s, t, r \in \mathbb{R}. \end{cases}$$

So

$$U = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

A basis for U is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and $\dim(U) = 3$.

More working space for Question 2(a)

Continue on last page if you need more writing space. $\cdots - 8-$

Question 2

(b) [14 marks]

Suppose $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ and $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ are two different bases for $\mathbb{R}^3$. Let

$$\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 2 \end{pmatrix}$$

be the transition matrix from S to T.

(i) Write down the coordinate vectors $[\boldsymbol{u_1}]_T$, $[\boldsymbol{u_2}]_T$ and $[\boldsymbol{u_3}]_T$.

(ii) Suppose

$$\boldsymbol{v_1} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \boldsymbol{v_3} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}, \quad \boldsymbol{u_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Find vectors $\boldsymbol{v_2}, \boldsymbol{u_1}, \boldsymbol{u_3}$.

(iii) Let $\boldsymbol{w} = (-2, 1, 1)$. You may assume that $[\boldsymbol{w}]_S = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. Compute $[\boldsymbol{w}]_T$.

(iv) Use your answer in (iii) to verify that your answer for $\boldsymbol{v_2}$ in (ii) is correct.

Show your working below.

(i)

$$[\boldsymbol{u_1}]_T = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad [\boldsymbol{u_2}]_T = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad [\boldsymbol{u_3}]_T = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

(ii) From the third column of $\boldsymbol{P}$, we see that $\boldsymbol{u_3} = 2\boldsymbol{v_3}$. Thus

$$\boldsymbol{u_3} = 2\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -6 \end{pmatrix}.$$

From the second column of $\boldsymbol{P}$, we see that $\boldsymbol{u_2} = \boldsymbol{v_1} + \boldsymbol{v_2} - \boldsymbol{v_3}$. Thus

$$\boldsymbol{v_2} = \boldsymbol{u_2} - \boldsymbol{v_1} + \boldsymbol{v_3} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix}.$$

From the first column of $\boldsymbol{P}$, we see that $\boldsymbol{u_1} = \boldsymbol{v_1} + 2\boldsymbol{v_2} - 3\boldsymbol{v_3}$. Thus

$$\boldsymbol{u_1} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}.$$

More working space for Question 2(b)

(iii)

$$[\boldsymbol{w}]_T = \boldsymbol{P}\,[\boldsymbol{w}]_S = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}.$$

(iv) Since $[\boldsymbol{w}]_T = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$, we check (using $\boldsymbol{v_2}$ obtained in (i)) that

$$\boldsymbol{w} = 0\boldsymbol{v_1} + \boldsymbol{v_2} - 2\boldsymbol{v_3} = \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

which is consistent with the vector $\boldsymbol{w}$ given in (ii). Thus our answer for $\boldsymbol{v_2}$ in (i) is correct.

Question 3 [25 marks]

(a) [10 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 & 1 \end{pmatrix}$.

(i) Find a basis for each of the row space and column space of $\boldsymbol{A}$ and state its rank.

(ii) Extend the basis for the row space of $\boldsymbol{A}$ in part (i) to a basis for $\mathbb{R}^5$.

(iii) Is it possible to find a full rank 5×3 matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{0}$? Justify your answer.

Show your working below.

(i)

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 & 1 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

From the r.e.f of $\boldsymbol{A}$, a basis for the row space of $\boldsymbol{A}$ is $\{(1,1,0,1,0),(0,1,2,-1,1),(0,0,0,1,1)\}$.

A basis for the column space is:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}.$$

The rank of $\boldsymbol{A}$ is 3.

(ii) We can extend the basis for the row space in (i) into the basis

$$\{(1,1,0,1,0),(0,1,2,-1,1),(0,0,0,1,1),(0,0,1,0,0),(0,0,0,0,1)\}$$

for $\mathbb{R}^5$.

(iii) No. A full rank 5×3 matrix $\boldsymbol{B}$ will have rank 3, which means it has three linearly independent columns $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$. But $\boldsymbol{AB} = \boldsymbol{0}$ means $\boldsymbol{Ab}_i = 0$ for $i = 1,2,3$. i.e. $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$ belong to the nullspace of $\boldsymbol{A}$. But the nullity of $\boldsymbol{A}$ is $5 - \texttt{rank}(\boldsymbol{A}) = 2$. This means the nullspace of $\boldsymbol{A}$ cannot have more than 2 linearly independent vectors. This contradicts $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$ are linearly independent.

More working space for Question 3(a)

Question 3

(b) [9 marks]

Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ with $\boldsymbol{v}_1 = (1, 1, 0, 1), \boldsymbol{v}_2 = (0, 1, 1, -1), \boldsymbol{v}_3 = (1, -1, 1, 0)$.

(i) Show that S is an orthogonal set.

(ii) Let $\boldsymbol{w} = (5, -2, 2, 3)$. Find the projection of $\boldsymbol{w}$ onto the subspace $V = \text{span}(S)$. Does $\boldsymbol{w}$ belong to V?

(iii) Without performing Gaussian elimination, can you tell whether the system

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 2 \\ 3 \end{pmatrix}$$

has no solution, exactly one solution, or infinitely many solutions? Why?

Show your working below.

(i) Check orthogonality:
$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 1 \times 0 + 1 \times 1 + 0 \times 1 + 1 \times (-1) = 0.$
$\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = 1 \times 1 + 1 \times (-1) + 0 \times 1 + 1 \times 0 = 0.$
$\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 0 \times 1 + 1 \times (-1) + 1 \times 1 + (-1) \times 0 = 0.$

(ii) By (i), S is an orthogonal basis for $V = \text{span}(S)$. So the projection of $\boldsymbol{w}$ onto V is given by:
$\frac{(5,-2,2,3)\cdot(1,1,0,1)}{(1,1,0,1)\cdot(1,1,0,1)}(1, 1, 0, 1) + \frac{(5,-2,2,3)\cdot(0,1,1,-1)}{(0,1,1,-1)\cdot(0,1,1,-1)}(0, 1, 1, -1) + \frac{(5,-2,2,3)\cdot(1,-1,1,0)}{(1,-1,1,0)\cdot(1,-1,1,0)}(1, -1, 1, 0)$
$= 2(1, 1, 0, 1) - (0, 1, 1, -1) + 3(1, -1, 1, 0) = (5, -2, 2, 3).$

Since the projection of $\boldsymbol{w}$ onto V is equal to $\boldsymbol{w}$ itself, this implies $\boldsymbol{w} \in V$.

(iii) Yes. The system has exactly one solution. First of all, observe that the 3 columns of the coefficient matrix of the system are precisely $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ and the constant matrix is $\boldsymbol{w}$. From (ii), we know $\boldsymbol{w}$ belongs to the column space $V = \text{span}(S)$ of the matrix, and hence the system is consistent. Furthermore, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ is a basis for V, hence $\boldsymbol{w}$ can only be expressed as a linear combination of this basis in exactly one way. Hence the system has exactly one solution.

More working space for Question 3(b)

Question 3

(c) [6 marks]

(i) Let $\boldsymbol{A}$ be a 2×3 matrix and $\boldsymbol{b}$ a 2×1 column vector. Suppose

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 5 \end{pmatrix} \quad \text{and} \quad \boldsymbol{A}^T\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Find the least squares solutions of $\boldsymbol{Ax} = \boldsymbol{b}$.

(ii) True or false: Given any 2×3 matrix $\boldsymbol{M}$ and 2×1 column vector $\boldsymbol{c}$, the linear system $\boldsymbol{Mx} = \boldsymbol{c}$ always has infinitely many least squares solutions. Justify your answer.

Show your working below.

(i) To find the least squares solution of $\boldsymbol{Ax} = \boldsymbol{b}$, we can solve:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

By Gaussian elimination:

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 4 & 2 & 2 \\ 2 & 2 & 5 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 4 & 2 & 2 \\ 0 & 2 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Solving we get

$$\begin{cases} x = 1 - 2t \\ y = \frac{1}{2} - \frac{1}{2}t \\ z = t \end{cases} \quad \text{with} \quad t \in \mathbb{R}.$$

(ii) True. If $\boldsymbol{M}$ is 2×3, the rank of $\boldsymbol{M}$ is at most 2, and hence the nullity of $\boldsymbol{M}$ is at least 1. This implies the nullity of $\boldsymbol{M}^T\boldsymbol{M}$ is at least 1. Hence $\boldsymbol{M}^T\boldsymbol{M}$ is not invertible.

Since $\boldsymbol{M}^T\boldsymbol{Mx} = \boldsymbol{M}^T\boldsymbol{c}$ is always consistent, this means it will have infinitely many solutions. i.e. $\boldsymbol{Mx} = \boldsymbol{c}$ always has infinitely many least squares solutions.

More working space for Question 3(c)

Question 4 [25 marks]

(a) [21 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & -3 \\ 3 & -3 & 1 \end{pmatrix}$ and $\boldsymbol{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

(i) Compute $\boldsymbol{Av}_1$, $\boldsymbol{Av}_2$ and $\boldsymbol{Av}_3$.

(ii) Write down all the eigenvalues of $\boldsymbol{A}$.

(iii) For each eigenvalue of $\boldsymbol{A}$, write down a basis for the corresponding eigenspace.

(iv) Diagonalize the matrix $\boldsymbol{A}$.

(v) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation with standard matrix $\boldsymbol{A}$. Find the range $R(T)$ and kernel $\ker(T)$ of T. Justify your answers.

(vi) Write down the equation of the plane P in the xyz-space that is not transformed to a different plane* under the linear transformation T in part (v).
(* This means for any vector $\boldsymbol{v}$ on plane P, $T(\boldsymbol{v})$ would still be a vector on P.)

(vii) Find a linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$(S \circ T)(\boldsymbol{v}_1) = 4\boldsymbol{v}_1, \quad (S \circ T)(\boldsymbol{v}_2) = 4\boldsymbol{v}_2, \quad (S \circ T)(\boldsymbol{v}_3) = -4\boldsymbol{v}_3.$$

(You may give your answer in terms of the standard matrix of S.)

Show your working below.

(i) $\boldsymbol{Av}_1 = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix} = -2\boldsymbol{v}_1$, $\boldsymbol{Av}_2 = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix} = 4\boldsymbol{v}_2$ and $\boldsymbol{Av}_3 = \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix} = 4\boldsymbol{v}_3$.

(ii) From (i), all the eigenvalues of $\boldsymbol{A}$ are -2 and 4.

(iii) The eigenspace corresponding to -2 has a basis $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

The eigenspace corresponding to 4 has a basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$.

(iv) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}^{-1}$.

(v) Since $\boldsymbol{A}$ is invertible (0 is not an eigenvalue of $\boldsymbol{A}$), the column space of $\boldsymbol{A}$ is $\mathbb{R}^3$ and the nullspace of $\boldsymbol{A}$ is the zero space.

Hence we can conclude that $R(T) = \mathbb{R}^3$, and the $\ker(T) = \{0\}$.

More working space for Question 4(a)

(vi) We observe that the eigenspace of $\boldsymbol{A}$ corresponding to the eigenvalue 4 is $\text{span}\{\boldsymbol{v}_2, \boldsymbol{v}_3\}$, which represents a plane P. Furthermore, every vector $\boldsymbol{v}$ on this plane P satisfies $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v} = 4\boldsymbol{v}$ which is again a vector on P. Hence it is not transformed by T to a different plane.

We can easily solve for the equation of this plane: $-x + y + z = 0$.

(vii) Suppose the standard matrix of S is $\boldsymbol{B}$. Then we have

$$\boldsymbol{BAv}_1 = 4\boldsymbol{v}_1, \quad \boldsymbol{BAv}_2 = 4\boldsymbol{v}_2, \quad \boldsymbol{BAv}_3 = -4\boldsymbol{v}_3.$$

We see that $\boldsymbol{BA}$ has same eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ with eigenvalues $4, 4, -4$.

By letting $\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$, we have

$$\begin{aligned}
\boldsymbol{BA} &= \boldsymbol{P} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{pmatrix} \boldsymbol{P}^{-1} \\
&= \boldsymbol{P} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \boldsymbol{P}^{-1} \\
&= \boldsymbol{P} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \boldsymbol{P}^{-1}\boldsymbol{P} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \boldsymbol{P}^{-1} \\
&= \left[\boldsymbol{P} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \boldsymbol{P}^{-1} \right] \boldsymbol{A}
\end{aligned}$$

Hence we can take $\boldsymbol{B} = \boldsymbol{P} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \boldsymbol{P}^{-1}$ and S be the linear transformation with this standard matrix $\boldsymbol{B}$.

More working space for Question 4(a)

Question 4

(b) [4 marks]

Prove the following:
Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be two $n \times n$ matrices. Suppose $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ is a set of linearly independent eigenvectors for both $\boldsymbol{M}$ and $\boldsymbol{N}$. Then $\boldsymbol{MN} = \boldsymbol{NM}$.

Show your working below.

Let $\boldsymbol{P}$ be the invertible matrix $(\boldsymbol{v}_1 \ \ \boldsymbol{v}_2 \ \ \cdots \ \ \boldsymbol{v}_n)$.

From the given condition, we know $\boldsymbol{M}$ and $\boldsymbol{N}$ are both diagonalizable matrices, and are diagonalizable by $\boldsymbol{P}$.

Then $\boldsymbol{M} = \boldsymbol{P}\boldsymbol{D}_1\boldsymbol{P}^{-1}$ and $\boldsymbol{N} = \boldsymbol{P}\boldsymbol{D}_2\boldsymbol{P}^{-1}$ for some diagonal matrices $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$.

Hence
$\boldsymbol{MN} = \boldsymbol{P}\boldsymbol{D}_1\boldsymbol{P}^{-1}\boldsymbol{P}\boldsymbol{D}_2\boldsymbol{P}^{-1} = \boldsymbol{P}\boldsymbol{D}_1\boldsymbol{D}_2\boldsymbol{P}^{-1} = \boldsymbol{P}\boldsymbol{D}_2\boldsymbol{D}_1\boldsymbol{P}^{-1} = \boldsymbol{P}\boldsymbol{D}_2\boldsymbol{P}^{-1}\boldsymbol{P}\boldsymbol{D}_1\boldsymbol{P}^{-1} = \boldsymbol{NM}$.

END OF PAPER