Department of Mathematics

National University of Singapore

(2022/23) Semester I MA1521 Calculus for Computing Tutorial 2

(1) Evaluate

(a) $\displaystyle\lim_{x\to 0}\frac{2x\sin(3x)}{\tan^2(4x)}$

(b) $\displaystyle\lim_{x\to 3}\left(\frac{\tan(2\ln(x-2))}{3\ln(x-2)}\right)^2$

(c) $\displaystyle\lim_{x\to 1}\frac{x^2-4x+3}{\tan(x^2-x)}$

Ans. (a) $\frac{3}{8}$, (b) $\frac{4}{9}$, (c) -2.

(2) Suppose we know $3^x > x^4$ for any $x \geq 12$. Find the following limits.

(a) $\displaystyle\lim_{x\to\infty}\frac{(3^x+1)}{x^2}$.

(b) $\displaystyle\lim_{x\to\infty}\frac{x^3}{(3^x+1)}$.

Ans. (a) ∞, (b) 0.

(3) Find the first derivatives of the following functions.

(a) $y=\dfrac{ax+b}{cx+d}$, (b) $y=\sin^n x\,\cos(mx)$,

(c) $y=e^{x^2+x^3}$, (d) $y=x^3-4(x^2+e^2+\ln 2)$,

(e) $y=\left(\dfrac{\sin\theta}{\cos\theta-1}\right)^2$ (f) $y=t\tan(2\sqrt{t})+7$,

(g) $r=\sin(\theta+\sqrt{\theta+1})$, (h) $s=\dfrac{4}{\cos x}+\dfrac{1}{\tan x}$.

(i) $s=\sin^{-1}(x^2-1)$ (j) $s=\tan^{-1}(e^x+2\sqrt{x})$

Ans. (a) $y'=\dfrac{ad-bc}{(cx+d)^2}$, (b) $y'=n\sin^{n-1}x\cos x\cos mx-m\sin^n x\sin mx$,
(c) $y'=e^{x^2+x^3}(2x+3x^2)$, (d) $y'=3x^2-8x$,
(e) $y'=-2\sin\theta(\cos\theta-1)^{-2}$, (f) $y'=\sqrt{t}\sec^2(2\sqrt{t})+\tan(2\sqrt{t})$,
(g) $r'=\frac{2\sqrt{\theta+1}+1}{2\sqrt{\theta+1}}\cos(\theta+\sqrt{\theta+1})$, (h) $s'=4\tan x\sec x-\csc^2 x$,

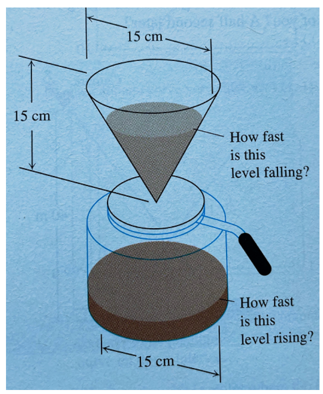
$$\text{(i) } s' = \frac{x}{\sqrt{1-(x^2-1)^2}}, \quad \text{(j) } s' = \frac{e^x + x^{-\frac{1}{2}}}{1+(e^x+2\sqrt{x})^2)}.$$

(4) Coffee is drained from a conical filter into a cylindrical coffeepot at the rate of 10 cm^3/min.

(a) How fast is the level in the pot rising when the coffee in the cone is 5 cm. deep?

(b) How fast is the level in the cone falling then?

(Volume of cone: $\frac{1}{3}\times$ base area $\times$ height)



Ans. (a) $\frac{8}{45\pi}$ cm/min, (b) $\frac{8}{5\pi}$ cm/min.

(5) For the following functions, find y' and y''.

(a) $x^{2/3} + y^{2/3} = a^{2/3}$, $0 < x < a$, $0 < y$,

(b) $y = (\sin x)^{\sin x}$, $0 < x < \frac{\pi}{2}$,

(c) $x = a\cos t$, $y = a\sin t$.

Ans. (a) $y' = -\sqrt{(\frac{a}{x})^{2/3} - 1}$, $\quad y'' = \dfrac{a^{2/3}}{3x^{4/3}\sqrt{a^{2/3}-x^{2/3}}}$.

(b) $y' = (\sin x)^{\sin x}(1+\ln\sin x)\cos x$,

$y'' = (\sin x)^{\sin x}\left[(1+\ln\sin x)^2\cos^2 x + \dfrac{\cos^2 x}{\sin x} - (1+\ln\sin x)\sin x\right]$.

(c) $y' = -\cot t$, $y'' = -\dfrac{1}{a\sin^3 t}$.

Further Exercises

(1) Suppose a rain drop evaporates in such a way that it maintains a spherical shape. Given that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$ and its surface area is $A = 4\pi r^2$, if the radius changes in time, show that $V' = Ar'$. If the rate of

evaporation V' is proportional to the surface area, show that the radius changes at constant rate.

(2) Let n be a positive integer. Show that

$$\frac{d^n}{dx^n} \ln \frac{1-x}{1+x} = -(n-1)! \left(\frac{1}{(1-x)^n} - \frac{(-1)^n}{(1+x)^n} \right),$$

for $-1 < x < 1$.

(3) A light is at the top of a pole 80 feet high. A ball is dropped from the same height (80 feet) from a point 20 feet from the light. Assuming that the ball falls according the law $s = 16t^2$ (where s is the distance travelled in feet and t is measured in seconds), how fast is the shadow of the ball moving along the ground one second later.

Ans. 200 ft/sec.