CS1231S: Discrete Structures

Tutorial #7: Mathematical Induction and Recursion

Answers

In writing Mathematical Induction proofs, please follow the format shown in class.

1. Prove by induction that for all $n \in \mathbb{Z}^+$,

$$\mathbf{1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)}.$$

Answer:

1. For each $n \in \mathbb{Z}^+$, let $P(n) \equiv 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.
2. (Basis step) $P(1)$ is true because $1^2 = 1 = \frac{1}{6} \times 1 \times (1+1) \times (2 \times 1 + 1)$.
3. (Induction step)
   - 3.1. Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true, i.e. $1^2 + 2^2 + \cdots + k^2 = \frac{1}{6}n(k+1)(2k+1)$
   - 3.2. Then $1^2 + 2^2 + \cdots + k^2 + (k+1)^2$
   - 3.3. $= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$ by the induction hypothesis
   - 3.4. $= \frac{1}{6}(k+1)\big(k(2k+1) + 6(k+1)\big)$
   - 3.5. $= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$
   - 3.6. $= \frac{1}{6}(k+1)(k+2)(2k+3)$
   - 3.7. $= \frac{1}{6}(k+1)\big((k+1)+1\big)(2(k+1)+1)$ by basic algebra
   - 3.8. Thus $P(k+1)$ is true.
4. Therefore, $\forall n \in \mathbb{Z}^+ \; P(n)$ is true by MI.

2. Let $x \in \mathbb{R}_{\geq -1}$. Prove by induction that $\mathbf{1 + nx \leq (1+x)^n}$ for all $n \in \mathbb{Z}^+$.

Answer:

1. For each $n \in \mathbb{Z}^+$, let $P(n) \equiv (1 + nx \leq (1+x)^n)$.
2. (Basis step) $P(1)$ is true because $1 + 1x = 1 + x = (1+x)^1$.
3. (Induction step)
   - 3.1. Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true, i.e. $1 + kx \leq (1+x)^k$.
   - 3.2. Then $(1+x)^{k+1}$
   - 3.3. $= (1+x)^k(1+x)$
   - 3.4. $\geq (1+kx)(1+x)$ by the induction hypothesis
   - 3.5. $= 1 + (k+1)x + kx^2$
   - 3.6. $\geq 1 + (k+1)x$ as $k \geq 1$ and $x^2 \geq 0$
   - 3.7. Thus $P(k+1)$ is true.
4. Therefore, $\forall n \in \mathbb{Z}^+ \; P(n)$ is true by MI.

3. In Lecture #5, we claim that given any set A, $|\boldsymbol{\mathcal{P}(A)}| = \mathbf{2^n}$, where $\mathcal{P}(A)$ denotes the power set of A and $|A| = n$. Prove by induction on n that this claim is true by using the argument in Lecture #5.

Answer:

1. For each $n \in \mathbb{N}$, let $P(n) \equiv (|\mathcal{P}(A)| = 2^n)$ where A is any n-element set.
2. (Basis step) $P(0)$ is true because $|\mathcal{P}(\emptyset)| = |\{\emptyset\}| = 1 = 2^0$ as $\mathcal{P}(\emptyset) = \{\emptyset\}$ and $|\emptyset| = 0$.
3. (Induction step)
    - 3.1. Let $k \in \mathbb{N}$ such that $P(k)$ is true, i.e., $|\mathcal{P}(X)| = 2^k$ where X is any k-element set.
    - 3.2. Let A be a $k+1$-element set.
    - 3.3. Since $k \geq 0$, there is at least one element in A. Pick $z \in A$.
    - 3.4. The subsets of A can be split into 2 groups: those that contain z and those that don't.
    - 3.5. Those subsets that do not contain z are the same as the subsets of $A \setminus \{z\}$, which has a cardinality of k, and hence $|\mathcal{P}(A \setminus \{z\})| = 2^k$ by the induction hypothesis.
    - 3.6. Those subsets that contain z can be matched up one for one with those subsets that do not contain z by unioning $\{z\}$ to the latter.
    - 3.7. Hence there is an equal number of subsets that contain z and subsets that don't.
    - 3.8. Hence $|\mathcal{P}(A)| = 2^k + 2^k = 2^{k+1}$.
    - 3.9. Thus $P(k+1)$ is true.
4. Therefore, $\forall n \in \mathbb{N}\ P(n)$ is true by MI.

4. Let a be an odd integer. Prove by induction that $\mathbf{2^{n+2} \mid a^{2^n} - 1}$ for all $n \in \mathbb{Z}^+$.
Here you may use without proof the fact that the product of any two consecutive integers is even.
(Note that $a^{b^c} = a^{(b^c)}$ by convention.)

Answer:

1. For each $n \in \mathbb{Z}^+$, let $P(n) \equiv 2^{n+2} \mid a^{2^n} - 1$.
2. (Basis step)
    - 2.1. $a = 2p + 1$ for some integer p by the definition of odd integers.
    - 2.2. Then $a^{2^1} - 1 = a^2 - 1 = (a-1)(a+1) = (2p+1-1)(2p+1+1) = 4p(p+1)$.
    - 2.3. Now $p(p+1)$ is even (given by the question), so $p(p+1) = 2m$ for some integer m by the definition of even integers.
    - 2.4. Hence, $a^{2^1} - 1 = 4(2m) = 8m = 2^3 m$.
    - 2.5. So $2^{1+2} \mid a^{2^1} - 1$ and hence $P(1)$ is true.
3. (Induction step)
    - 3.1. Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true, i.e., $2^{k+2} \mid a^{2^k} - 1$.
    - 3.2. So $a^{2^k} - 1 = 2^{k+2} m$ for some integer m by the definition of divisibility.
    - 3.3. Then $a^{2^{k+1}} - 1 = a^{2^k \times 2} - 1$
    - 3.4. $= \left(a^{2^k}\right)^2 - 1$
    - 3.5. $= (a^{2^k} - 1)(a^{2^k} + 1)$
    - 3.6. $= (2^{k+2} m)\big((2^{k+2} m + 1) + 1\big)$ by line 3.2
    - 3.7. $= 2^{k+3} m (2^{k+1} m + 1)$
    - 3.8. Thus $2^{k+3} m \mid a^{2^{k+1}} - 1$ and so $P(k+1)$ is true.
4. Therefore, $\forall n \in \mathbb{Z}^+\ P(n)$ is true by MI.

5. Prove by induction that

$$\forall \boldsymbol{n} \in \mathbb{Z}_{\geq 8}\ \exists \boldsymbol{x}, \boldsymbol{y} \in \mathbb{N}\ (\boldsymbol{n} = \boldsymbol{3x} + \boldsymbol{5y}).$$

(In other words, any integer-valued transaction of at least \$8 can be carried out using only \$3 and \$5 notes.)

Answer:

1. For each $n \in \mathbb{Z}_{\geq 8}$, let $P(n) \equiv \exists x, y \in \mathbb{N}\ (n = 3x + 5y)$.
2. (Basis step) $P(8)$ is true as $8 = 3(1) + 5(1)$.
3. (Induction step)
   - 3.1. Let $k \in \mathbb{Z}_{\geq 8}$ such that $P(k)$ is true.
   - 3.2. Find $x, y \in \mathbb{N}$ such that $k = 3x + 5y$.
   - 3.3. Case 1: $y > 0$.
     - 3.3.1. Then $k + 1 = (3x + 5y) + 1$ by the choice of x, y.
     - 3.3.2. $= 3(x + 2) + 5(y - 1)$.
     - 3.3.3. $y - 1 \in \mathbb{N}$ as $y > 0$.
     - 3.3.4. As $x + 2 \in \mathbb{N}$ and $y - 1 \in \mathbb{N}$, so $P(k + 1)$ is true.
   - 3.4. Case 2: $y = 0$.
     - 3.4.1. Then $k = 3x + 5(0) = 3x$.
     - 3.4.2. $\therefore x = k/3 \geq 8/3$ as $k \geq 8$.
     - 3.4.3. $\therefore x \geq 3$ as $x \in \mathbb{N}$.
     - 3.4.4. Thus $k + 1 = 3x + 1 = 3(x - 3) + 5(2)$.
     - 3.4.5. As $x - 3 \in \mathbb{N}$ and $2 \in \mathbb{N}$, So $P(k + 1)$ is true.
   - 3.5. Hence $P(k + 1)$ is true for all cases.
4. Therefore, $\forall n \in \mathbb{N}\ P(n)$ is true by MI.

Alternative answer:

1. For each $n \in \mathbb{Z}_{\geq 8}$, let $P(n) \equiv \exists x, y \in \mathbb{N}\ (n = 3x + 5y)$.
2. (Basis step)
   - 2.1. $P(8)$ is true because $8 = 3(1) + 5(1)$.
   - 2.2. $P(9)$ is true because $9 = 3(3) + 5(0)$.
   - 2.3. $P(10)$ is true because $10 = 3(0) + 5(2)$.
3. (Induction step)
   - 3.1. Let $k \in \mathbb{Z}_{\geq 8}$ such that $P(8), P(9), \cdots, P(k + 2)$ are true.
   - 3.2. Apply $P(k)$ to find $x, y \in \mathbb{N}$ such that $k = 3x + 5y$.
   - 3.3. Then $k + 3 = (3x + 5y) + 3$ by the choice of x, y.
   - 3.4. $= 3(x + 1) + 5y$ where $x + 1, y \in \mathbb{N}$.
   - 3.5. Hence $P(k + 3)$ is true.
4. Therefore, $\forall n \in \mathbb{N}\ P(n)$ is true by Strong MI.

6. Prove by induction that every positive integer can be written as a sum of *distinct* non-negative integer powers of 2, i.e.,

$$\forall \boldsymbol{n} \in \mathbb{Z}^+\ \exists \boldsymbol{l} \in \mathbb{Z}^+\ \exists \boldsymbol{i_1}, \boldsymbol{i_2}, \cdots, \boldsymbol{i_l} \in \mathbb{N}\ (\boldsymbol{i_1} < \boldsymbol{i_2} < \cdots < \boldsymbol{i_l} \wedge \boldsymbol{n} = \boldsymbol{2^{i_1}} + \boldsymbol{2^{i_2}} + \cdots + \boldsymbol{2^{i_l}}).$$

Answer:

1. For each $n \in \mathbb{Z}^+$, let $P(n)$ be the proposition
   "$\exists l \in \mathbb{Z}^+\ \exists i_1, i_2, \cdots, i_l \in \mathbb{N}\ (i_1 < i_2 < \cdots < i_l \wedge n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_l})$"
2. (Basis step) $P(1)$ is true as $1 = 2^0$.
3. (Induction step)
   - 3.1. Let $k \in \mathbb{Z}^+$ such that $P(1), P(2), \cdots, P(k)$ are true.
   - 3.2. Find $m \in \mathbb{Z}$ such that $k + 1 = 2m$ or $k + 1 = 2m + 1$. (This is possible because $k + 1$ is either even or odd, by lecture #1 assumption 1.)
   - 3.3. Note that $2m \leq k + 1$ as $k + 1 = 2m$ or $k + 1 = 2m + 1$;
   - 3.4. $\leq k + k$ as $k \geq 1$;
   - 3.5. $= 2k$.
   - 3.6. So $m \leq k$.
   - 3.7. Also, $2m + 1 \geq k + 1$ as $k + 1 = 2m$ or $k + 1 = 2m + 1$;
   - 3.8. So $2m \geq k \geq 1$ or $m \geq \frac{1}{2}$ or $m \geq 1$ as $m \in \mathbb{Z}$ and 1 is the smallest integer $\geq \frac{1}{2}$.
   - 3.9. By lines 3.6 and 3.8, $1 \leq m \leq k$, and so $P(m)$ is true by the induction hypothesis.
   - 3.10. Apply $P(m)$ to find $l \in \mathbb{Z}^+$ and $i_1, i_2, \cdots, i_l \in \mathbb{N}$ such that
     $$i_1 < i_2 < \cdots < i_l \quad \text{and} \quad m = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_l}.$$
   - 3.11. Case 1: $k + 1 = 2m$.
     - 3.11.1. Then $k + 1 = 2(2^{i_1} + 2^{i_2} + \cdots + 2^{i_l})$
     - 3.11.2. $= 2^{i_1+1} + 2^{i_2+1} + \cdots + 2^{i_l+1}$.
     - 3.11.3. Also, $i_1 + 1 < i_2 + 1 < \cdots < i_l + 1$ as $i_1 < i_2 < \cdots < i_l$.
     - 3.11.4. So $P(k + 1)$ is true.
   - 3.12. Case 1: $k + 1 = 2m + 1$.
     - 3.12.1. Then $k + 1 = 2(2^{i_1} + 2^{i_2} + \cdots + 2^{i_l}) + 1$
     - 3.12.2. $= 2^0 + 2^{i_1+1} + 2^{i_2+1} + \cdots + 2^{i_l+1}$.
     - 3.12.3. Also, $0 < i_1 + 1 < i_2 + 1 < \cdots < i_l + 1$ as $0 \leq i_1 < i_2 < \cdots < i_l$.
     - 3.12.4. So $P(k + 1)$ is true.
   - 3.13. Hence $P(k + 1)$ is true for all cases.
4. Therefore, $\forall n \in \mathbb{Z}^+\ P(n)$ is true by Strong MI.

Alternative answer (not for the faint-hearted):

1. For each $n \in \mathbb{Z}^+$, let $P(n)$ be the proposition
   "$\exists l \in \mathbb{Z}^+ \; \exists i_1, i_2, \cdots, i_l \in \mathbb{N} \; (i_1 < i_2 < \cdots < i_l \; \wedge n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_l})$"
2. (Basis step) $P(1)$ is true as $1 = 2^0$.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true.
   3.2. Apply this assumption to obtain $l \in \mathbb{Z}^+$ and $i_1, i_2, \cdots, i_l \in \mathbb{N}$ such that
   $$i_1 < i_2 < \cdots < i_l \quad \text{and} \quad k = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_l}.$$
   3.3. Since $k < 2^k$, we know k is an element of the set $\mathbb{N} \setminus \{i_1, i_2, \cdots, i_l\}$.
   3.4. By the Well-Ordering Principle, this set has a minimum, say m.
   3.5. Find $j \in \{0, 1, \cdots, l\}$ such that $i_1 < i_2 < \cdots < i_j < m < i_{j+1} < \cdots < i_l$.
   3.6. The minimality of m tells us $0, 1, 2, \cdots, m-1 \in \{i_1, i_2, \cdots, i_l\}$.
   3.7. Thus $0, 1, 2, \cdots, m-1 \in \{i_1, i_2, \cdots, i_j\}$ by the choice of j.
   3.8. The choice of j also tell us $i_1, i_2, \cdots, i_j \in \{0, 1, 2, \cdots, m-1\}$.
   3.9. From these, we deduce that $\{i_1, i_2, \cdots, i_j\} = \{0, 1, 2, \cdots, m-1\}$.
   3.10. Now, $k + 1 = 1 + 2^{i_1} + 2^{i_2} + \cdots + 2^{i_l}$ by line 3.2;
   3.11. $= 2^0 + 2^0 + 2^1 + 2^2 + \cdots + 2^{m-1} + 2^{i_j+1} + \cdots + 2^{i_l}$ by line 3.9;
   3.12. $= 2^1 + 2^1 + 2^2 + \cdots + 2^{m-1} + 2^{i_j+1} + \cdots + 2^{i_l}$
   3.13. $= 2^2 + 2^2 + \cdots + 2^{m-1} + 2^{i_j+1} + \cdots + 2^{i_l}$
   3.14. $= \cdots = 2^{m-1} + 2^{m-1} + 2^{i_j+1} + \cdots + 2^{i_l}$
   3.15. $= 2^m + 2^{i_j+1} + \cdots + 2^{i_l}$
   3.16. So $P(k+1)$ is true.
4. Therefore, $\forall n \in \mathbb{Z}^+ \; P(n)$ is true by MI.

7. Let $F_0, F_1, F_2, \cdots$ be the Fibonacci sequence. Show that $\boldsymbol{F_{n+4} = 3F_{n+2} - F_n}$ for all $n \in \mathbb{N}$.

Answer:

1. $F_{n+4} = F_{n+3} + F_{n+2}$ by the definition of F_{n+4};
2. $= (F_{n+2} + F_{n+1}) + F_{n+2}$ by the definition of F_{n+3};
3. $= 2F_{n+2} + F_{n+1}$
4. $= 3F_{n+2} - F_{n+2} + F_{n+1}$
5. $= 3F_{n+2} - (F_{n+1} + F_n) + F_{n+1}$ by the definition of F_{n+2};
6. $= 3F_{n+2} - F_n$

Note: Not all questions need to be solved with mathematical induction, unless the question explicitly states so. For this question, a simple direct proof like this suffices.

8. Let $F_0, F_1, F_2, \cdots$ be the Fibonacci sequence. Show by induction that for for all $n \in \mathbb{N}$,

$$\boldsymbol{F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n.}$$

Answer:

1. For each $n \in \mathbb{N}$, let $P(n)$ be the proposition
   "$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$"
2. (Basis step)
   2.1. Since $F_0 = 0$ and $F_1 = 1$,
   $F_{0+1}^2 - F_{0+1}F_0 - F_0^2 = 1^2 - (1 \times 0) - 0^2 = 1 = (-1)^0$.
   2.2. So $P(0)$ is true.
3. (Induction step)
   3.1. Let $k \in \mathbb{N}$ such that $P(k)$ is true, i.e., $F_{k+1}^2 - F_{k+1}F_k - F_k^2 = (-1)^k$.
   3.2. Then $F_{(k+1)+1}^2 - F_{(k+1)+1}F_{k+1} - F_{k+1}^2$
   3.3. $= F_{k+2}^2 - F_{k+2}F_{k+1} - F_{k+1}^2$
   3.4. $= (F_{k+1} + F_k)^2 - (F_{k+1} + F_k)F_{k+1} - F_{k+1}^2$ by the definition of F_{k+2};
   3.5. $= F_{k+1}^2 + 2F_{k+1}F_k + F_k^2 - F_{k+1}^2 - F_kF_{k+1} - F_{k+1}^2$
   3.6. $= -(F_{k+1}^2 - F_kF_{k+1} - F_k^2)$
   3.7. $= -(-1)^k$ by the induction hypothesis;
   3.8. $= (-1)^{k+1}$.
   3.9. Thus $P(k+1)$ is true.
4. Therefore, $\forall n \in \mathbb{N}\ P(n)$ is true by MI.

9. Let $a_0, a_1, a_2 \cdots$ be the sequence satisfying

$$a_0 = 0, \quad a_1 = 2, \quad a_2 = 7, \quad \text{and} \quad a_{n+3} = a_{n+2} + a_{n+1} + a_n$$

for all $n \in \mathbb{N}$. Prove by induction that $\boldsymbol{a_n < 3^n}$ for all $n \in \mathbb{N}$.

Answer:

1. For each $n \in \mathbb{N}$, let $P(n) \equiv a_n < 3^n$.
2. (Basis step)
   $P(0), P(1), P(2)$ are true as $a_0 = 0 < 1 = 3^0$, $a_1 = 2 < 3 = 3^1$, and $a_2 = 7 < 9 = 3^2$.
3. (Induction step)
   3.1. Let $k \in \mathbb{N}$ such that $P(0), P(1), \cdots, P(k+2)$ are true.
   3.2. $P(k), P(k+1), P(k+2)$ are true means $a_k < 3^k, a_{k+1} < 3^{k+1}, a_{k+2} < 3^{k+2}$.
   3.3. $a_{k+3} = a_{k+2} + a_{k+1} + a_k$ by the definition of a_{k+3};
   3.4. $< 3^{k+2} + 3^{k+1} + 3^k$ by the induction hypothesis;
   3.5. $< 3^{k+2} + 3^{k+2} + 3^{k+2}$
   3.6. $= 3(3^{k+2}) = 3^{k+3}$.
   3.7. Thus $P(k+1)$ is true.
4. Therefore, $\forall n \in \mathbb{N}\ P(n)$ is true by Strong MI.

10. Define a set S recursively as follows.

    (1) $2 \in S$. (base clause)

    (2) If $x \in S$, then $3x \in S$ and $x^2 \in S$. (recursion clause)

    (3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

    Which of the numbers $0, 6, 15, 16, 36$ are in S? Which are not?

    Answer:

    Structural induction over S. To prove that $\forall n \in S\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

    (basis step) show that $P(2)$ is true; and

    (induction step) show that $\forall n \in S\ \big(P(n) \Rightarrow P(3x) \wedge P(x^2)\big)$ is true.

    - We know $0 \notin S$ because all $n \in S$ satisfy $n \geq 2$, as one can show by structural induction over S as follows.
        1. For each $n \in S$, let $P(n) \equiv n \geq 2$.
        2. (Basis step) $P(2)$ is true because $2 \geq 2$.
        3. (Induction step)
            - 3.1. Let $x \in S$ such that $P(x)$ is true, i.e., that $x \geq 2$.
            - 3.2. Then $3x \geq 3 \times 2 = 6 \geq 2$ and $x^2 \geq 2^2 = 4 \geq 2$.
            - 3.3. So $P(3x)$ and $P(x^2)$ are both true.
        4. Hence $\forall n \in S\ P(n)$ is true by structural induction over S.
    - $2 \in S$ by the base clause.
      $\therefore$ $6 \in S$ by the recursion clause with $n = 2$ and the previous line.
      $\therefore$ $36 \in S$ by the recursion clause with $n = 6$ and the previous line.
    - $2 \in S$ by the base clause.
      $\therefore$ $4 \in S$ by the recursion clause with $n = 2$ and the previous line.
      $\therefore$ $16 \in S$ by the recursion clause with $n = 4$ and the previous line.
    - We know $15 \notin S$ because no $n \in S$ is odd, as one can show by structural induction over S.

11. Let $A = \{1,2,3,4,5\}$ and $B = \{1,3,5,7,9\}$. Define a set S recursively as follows.

    (1) $A, B \in S$. (base clause)

    (2) If $X, Y \in S$, then $X \cap Y \in S$ and $X \cup Y \in S$ and $X \setminus Y \in S$ (recursion clause)

    (3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

    For each of the following sets, determine whether it is in S, and use one sentence to explain your answer.

    (a) $C = \{2,4,7,9\}$.

    (b) $D = \{2,3,4,5\}$.

Answer:

Structural induction over S. To prove that $\forall X \in S\ P(X)$ is true, where each $P(X)$ is a proposition, it suffices to:

(basis step) show that $P(A)$ and $P(B)$ are true; and

(induction step) show that

$$\forall X, Y \in S\ \big(P(X) \wedge P(Y) \Rightarrow P(X \cap Y) \wedge P(X \cup Y) \wedge P(X \setminus Y)\big)$$

is true.

(a) $A \setminus B = \{2,4\}$ and $B \setminus A = \{7,9\}$.
$C \in S$ because $C = \{2,4,7,9\} = (A \setminus B) \cup (B \setminus A)$.

(b) $D \neq S$ because $1 \notin D$ and $3 \in D$, but one can show by structural induction that

$$\forall X \in S\ (1 \in X \Leftrightarrow 3 \in X).$$

(Proof shown below.)

1. For each $X \in S$, let $P(X) \equiv 1 \in X \Leftrightarrow 3 \in X$**.**
2. (Basis step)
   - 2.1. As $1,3 \in A$ and $1,3 \in B$, we know $1 \in A \Leftrightarrow 3 \in A$ and $1 \in B \Leftrightarrow 3 \in B$.
   - 2.2. So $P(A)$ an $P(B)$ are true.
3. (Induction step)
   - 3.1. Let $X, Y \in S$ such that $P(X)$ and $P(Y)$ are true, i.e.,
     $1 \in X \Leftrightarrow 3 \in X$ and $1 \in Y \Leftrightarrow 3 \in Y$.
   - 3.2. Case 1: If $1,3 \in X$ and $1,3 \in Y$, then
     - 3.2.1. $1,3 \in X \cap Y$ and $1,3 \in X \cup Y$ and $1,3 \notin X \setminus Y$;
     - 3.2.2. So $1 \in X \cap Y \Leftrightarrow 3 \in X \cap Y$ and $1 \in X \cup Y \Leftrightarrow 3 \in X \cup Y$ and $1 \in X \setminus Y \Leftrightarrow 3 \in X \setminus Y$.
     - 3.2.3. Thus $P(X \cap Y)$ and $P(X \cup Y)$ and $P(X \setminus Y)$ are all true.
   - 3.3. Case 2: If $1,3 \in X$ and $1,3 \notin Y$, then
     - 3.3.1. $1,3 \notin X \cap Y$ and $1,3 \in X \cup Y$ and $1,3 \in X \setminus Y$;
     - 3.3.2. So $1 \in X \cap Y \Leftrightarrow 3 \in X \cap Y$ and $1 \in X \cup Y \Leftrightarrow 3 \in X \cup Y$ and $1 \in X \setminus Y \Leftrightarrow 3 \in X \setminus Y$.
     - 3.3.3. Thus $P(X \cap Y)$ and $P(X \cup Y)$ and $P(X \setminus Y)$ are all true.
   - 3.4. Case 3: If $1,3 \notin X$ and $1,3 \in Y$, then
     - 3.4.1. $1,3 \notin X \cap Y$ and $1,3 \in X \cup Y$ and $1,3 \notin X \setminus Y$;
     - 3.4.2. So $1 \in X \cap Y \Leftrightarrow 3 \in X \cap Y$ and $1 \in X \cup Y \Leftrightarrow 3 \in X \cup Y$ and $1 \in X \setminus Y \Leftrightarrow 3 \in X \setminus Y$.
     - 3.4.3. Thus $P(X \cap Y)$ and $P(X \cup Y)$ and $P(X \setminus Y)$ are all true.
   - 3.5. Case 4: If $1,3 \notin X$ and $1,3 \notin Y$, then
     - 3.5.1. $1,3 \notin X \cap Y$ and $1,3 \notin X \cup Y$ and $1,3 \notin X \setminus Y$;
     - 3.5.2. So $1 \in X \cap Y \Leftrightarrow 3 \in X \cap Y$ and $1 \in X \cup Y \Leftrightarrow 3 \in X \cup Y$ and $1 \in X \setminus Y \Leftrightarrow 3 \in X \setminus Y$.
     - 3.5.3. Thus $P(X \cap Y)$ and $P(X \cup Y)$ and $P(X \setminus Y)$ are all true.
   - 3.6. Hence $P(X \cap Y)$ and $P(X \cup Y)$ and $P(X \setminus Y)$ are all true in all cases.
4. It follows that $\forall X \in S\ P(X)$ is true by structural induction over S.