MA2001 LINEAR ALGEBRA

VECTOR SPACES

Goh Jun Le / Wang Fei

gohjunle@nus.edu.sg / matwf@nus.edu.sg

Department of Mathematics
Office: S17-06-25 / S17-06-16
Tel: 6601-1355 / 6516-2937

Row and Column Spaces

- **Definition.** Let $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$.
  - Let $\boldsymbol{r}_i = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix}$ denote the ith row of $\boldsymbol{A}$.
    - Then $\boldsymbol{r}_i \in \mathbb{R}^n$ and $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \vdots \\ \boldsymbol{r}_m \end{pmatrix}$.

  The **row space** of $\boldsymbol{A}$ is the vector space spanned by the rows of $\boldsymbol{A}$:
  - $\text{span}\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m\}$.

  It is a subspace of $\mathbb{R}^n$.

Row and Column Spaces

- **Definition.** Let $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$.
  - Let $\boldsymbol{c}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$ denote the jth column of $\boldsymbol{A}$.
    - Then $\boldsymbol{c}_j \in \mathbb{R}^m$ and $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \cdots & \boldsymbol{c}_n \end{pmatrix}$.

  The **column space** of $\boldsymbol{A}$ is the vector space spanned by the columns of $\boldsymbol{A}$:
  - $\text{span}\{\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n\}$.

  It is a subspace of $\mathbb{R}^m$.

Examples

- Let $\boldsymbol{A}$ be a matrix.
  - The row space of $\boldsymbol{A} =$ the column space of $\boldsymbol{A}^{\mathrm{T}}$.
  - The column space of $\boldsymbol{A} =$ the row space of $\boldsymbol{A}^{\mathrm{T}}$.
- Let $\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. The rows of $\boldsymbol{A}$ are
  - $\boldsymbol{r}_1 = \begin{pmatrix} 2 & -1 & 0 \end{pmatrix}$,
  - $\boldsymbol{r}_2 = \begin{pmatrix} 1 & -1 & 3 \end{pmatrix}$,
  - $\boldsymbol{r}_3 = \begin{pmatrix} -5 & 1 & 0 \end{pmatrix}$,
  - $\boldsymbol{r}_4 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$.

  The row space of $\boldsymbol{A}$ is $\operatorname{span}\{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4\} \subseteq \mathbb{R}^3$.
  - One checks that it has dimension 3.

Examples

- Let $\boldsymbol{A}$ be a matrix.
  - The row space of $\boldsymbol{A} =$ the column space of $\boldsymbol{A}^{\mathrm{T}}$.
  - The column space of $\boldsymbol{A} =$ the row space of $\boldsymbol{A}^{\mathrm{T}}$.
- Let $\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. The columns of $\boldsymbol{A}$ are
  - $\boldsymbol{c}_1 = \begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix}, \quad \boldsymbol{c}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{c}_3 = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}.$

  The column space of $\boldsymbol{A}$ is $\operatorname{span}\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\} \subseteq \mathbb{R}^4$.
  - One checks that it has dimension 3.

Notation

- Recall that every vector $\boldsymbol{v} = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ can be identified as a row vector or a column vector.
  - If it is viewed as a row vector $\begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix}$,
    - then we write $(c_1, c_2, \ldots, c_n)$.
  - If it is viewed as a column vector $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$,
    - then we write $(c_1, c_2, \ldots, c_n)^{\mathrm{T}}$.
- **Example.** Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.
  - $\boldsymbol{r}_1 = (1, 2, 3)$, $\boldsymbol{r}_2 = (4, 5, 6)$.
  - $\boldsymbol{c}_1 = (1, 4)^{\mathrm{T}}$, $\boldsymbol{c}_2 = (2, 5)^{\mathrm{T}}$, $\boldsymbol{c}_3 = (3, 6)^{\mathrm{T}}$.

Row Equivalence

- Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices of the same size.
  - $\boldsymbol{A}$ and $\boldsymbol{B}$ are **row equivalent** if one can be obtained from another by a series of elementary row operations.
    - $\boldsymbol{A} \to \boldsymbol{A}_1 \to \boldsymbol{A}_2 \to \cdots \to \boldsymbol{A}_k \to \boldsymbol{A}_{k-1} \to \boldsymbol{B}$.
- **Theorem.** Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices of the same size.
  - Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent.
    - Then $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same row spaces.
- **Remark.** Let $\boldsymbol{R}$ be a row-echelon form of $\boldsymbol{A}$.
  - Then the row space of $\boldsymbol{A}$ = the row space of $\boldsymbol{R}$.

  (Q3.26) The nonzero rows of $\boldsymbol{R}$ are linearly independent.
  - Nonzero rows of $\boldsymbol{R}$ form a basis for the row space of $\boldsymbol{A}$.
  - The number of nonzero rows of $\boldsymbol{R}$ is the dimension of the row space of $\boldsymbol{A}$.

Row Equivalence

- **Proof.** It suffices to show that if $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ using a single elementary row operation, then $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same row spaces.
  - Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n$ matrices.
    - Suppose $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{r}_1 \\ \vdots \\ \boldsymbol{r}_i \\ \vdots \\ \boldsymbol{r}_m \end{pmatrix} \xrightarrow{cR_i} \boldsymbol{B} = \begin{pmatrix} \boldsymbol{r}_1 \\ \vdots \\ c\boldsymbol{r}_i \\ \vdots \\ \boldsymbol{r}_m \end{pmatrix}$.

    $\boldsymbol{r}_1, \dots, c\boldsymbol{r}_i, \dots, \boldsymbol{r}_m \in \operatorname{span}\{\boldsymbol{r}_1, \dots, \boldsymbol{r}_i, \dots, \boldsymbol{r}_m\}$.

    - row space of $\boldsymbol{B} \subseteq$ row space of $\boldsymbol{A}$.

    $\boldsymbol{r}_1, \dots, \boldsymbol{r}_i = \frac{1}{c}(c\boldsymbol{r}_i), \dots, \boldsymbol{r}_m \in \operatorname{span}\{\boldsymbol{r}_1, \dots, c\boldsymbol{r}_i, \dots, \boldsymbol{r}_m\}$

    - row space of $\boldsymbol{A} \subseteq$ row space of $\boldsymbol{B}$.

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Row Equivalence

- **Proof.** It suffices to show that if $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ using a single elementary row operation, then $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same row spaces.
  - Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n$ matrices.
    - Suppose $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{r}_1 \\ \vdots \\ \boldsymbol{r}_i \\ \vdots \\ \boldsymbol{r}_j \\ \vdots \\ \boldsymbol{r}_m \end{pmatrix} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B} = \begin{pmatrix} \boldsymbol{r}_1 \\ \vdots \\ \boldsymbol{r}_j \\ \vdots \\ \boldsymbol{r}_i \\ \vdots \\ \boldsymbol{r}_m \end{pmatrix}$.

$$\begin{aligned} \text{row space of } \boldsymbol{A} &= \operatorname{span}\{\boldsymbol{r}_1, \dots, \boldsymbol{r}_i, \dots, \boldsymbol{r}_j, \dots, \boldsymbol{r}_m\} \\ &= \operatorname{span}\{\boldsymbol{r}_1, \dots, \boldsymbol{r}_j, \dots, \boldsymbol{r}_i, \dots, \boldsymbol{r}_m\} \\ &= \text{row space of } \boldsymbol{B}. \end{aligned}$$

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Row Equivalence

- **Proof.** It suffices to show that if $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ using a single elementary row operation, then $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same row spaces.
  - Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n$ matrices.
    - Suppose $\boldsymbol{A} \xrightarrow{R_i + cR_j} \boldsymbol{B}$.

    The ith row of $\boldsymbol{B}$ becomes $\boldsymbol{r}_i + c\boldsymbol{r}_j \in \operatorname{span}\{\boldsymbol{r}_i, \boldsymbol{r}_j\}$.

$$\begin{aligned}\text{row space of } \boldsymbol{B} &= \operatorname{span}\{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_i + c\boldsymbol{r}_j, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_m\} \\ &\subseteq \operatorname{span}\{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_m\} \\ &= \text{row space of } \boldsymbol{A}.\end{aligned}$$

    $\boldsymbol{r}_i = (\boldsymbol{r}_i + c\boldsymbol{r}_j) + (-c)\boldsymbol{r}_j \in \operatorname{span}\{\boldsymbol{r}_i + c\boldsymbol{r}_j, \boldsymbol{r}_j\}$.

$$\begin{aligned}\text{row space of } \boldsymbol{A} &= \operatorname{span}\{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_m\} \\ &\subseteq \operatorname{span}\{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_i + c\boldsymbol{r}_j, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_m\} \\ &= \text{row space of } \boldsymbol{B}.\end{aligned}$$

Examples

- Let $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 4 \\ \frac{1}{2} & 1 & 2 \end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}$.
  - One checks: $\boldsymbol{A} \xrightarrow{R_1 \leftrightarrow R_3} \boldsymbol{B} \xrightarrow{2R_1} \boldsymbol{C} \xrightarrow{R_1 + (-1)R_2} \boldsymbol{D}$.
  - Then $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ have the same row space.
    - $\operatorname{span}\{(0,0,1), (0,2,4), (\frac{1}{2}, 1, 2)\}$
      $= \operatorname{span}\{(1,0,0), (0,2,4), (0,0,1)\}$.

    Note that $\boldsymbol{D}$ is in row-echelon form.

    - $\{(1,0,0), (0,2,4), (0,0,1)\}$ is a basis for the row space of $\boldsymbol{D}$ (or of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$).
    - The row space has dimension 3.

Examples

- Let $\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$.
  - A row-echelon form $\boldsymbol{R} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

  Then $\boldsymbol{A}$ and $\boldsymbol{R}$ has the same row space.
  - $\boldsymbol{R}$ has 3 nonzero rows.
  - Dimension of the row space of $\boldsymbol{A}$ (and of $\boldsymbol{R}$) is 3.
  - Basis $\{(2, 2, -1, 0, 1), (0, 0, \frac{3}{2}, -3, \frac{3}{2}), (0, 0, 0, 3, 0)\}$.

Row Operations to Columns

- Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices.
  - Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_n \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{d}_1 & \cdots & \boldsymbol{d}_n \end{pmatrix}$.

  Note that there exist elementary matrices $\boldsymbol{E}_i$ such that
  - $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{B}$.

  $\boldsymbol{M} = \boldsymbol{E}_k \cdots \boldsymbol{E}_1$ is invertible and $\boldsymbol{M}\boldsymbol{A} = \boldsymbol{B}$.
  - $\boldsymbol{M}\boldsymbol{c}_1 = \boldsymbol{d}_1, \ldots, \boldsymbol{M}\boldsymbol{c}_n = \boldsymbol{d}_n$.

  Suppose that $a_1\boldsymbol{c}_1 + \cdots + a_n\boldsymbol{c}_n = \boldsymbol{c}_j$. Then

$$\begin{aligned} \boldsymbol{d}_j = \boldsymbol{M}\boldsymbol{c}_j &= \boldsymbol{M}(a_1\boldsymbol{c}_1 + \cdots + a_n\boldsymbol{c}_n) \\ &= a_1\boldsymbol{M}\boldsymbol{c}_1 + \cdots + a_n\boldsymbol{M}\boldsymbol{c}_n \\ &= a_1\boldsymbol{d}_1 + \cdots + a_n\boldsymbol{d}_n. \end{aligned}$$

  The linear relation on columns is preserved by elementary row operations.

Properties

- **Theorem.** Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices.
  - If there is a linear relation among a given set of columns of $\boldsymbol{A}$,
    - then the same linear relation exists among the corresponding set of columns of $\boldsymbol{B}$.
  - A given set of columns of $\boldsymbol{A}$ is linearly independent

    $\Leftrightarrow$ the corresponding set of columns of $\boldsymbol{B}$ is linearly independent.
  - A given set of columns of $\boldsymbol{A}$ is a basis for the column space of $\boldsymbol{A}$

    $\Leftrightarrow$ the corresponding set of columns of $\boldsymbol{B}$ is a basis for the column space of $\boldsymbol{B}$.

Examples

- Let $\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$.
  - Row-echelon form $\boldsymbol{R} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

  In $\boldsymbol{R}$, the 1st, 3rd and 4th columns are pivot columns.
  - So they form a basis for the column space of $\boldsymbol{R}$.

  Then the 1st, 3rd and 4th columns of $\boldsymbol{A}$ form

  a basis for the column space of $\boldsymbol{A}$.

Examples

- **Problem.** How to find a basis for a vector space V?
- Let $V = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6\}$.
  - $\boldsymbol{v}_1 = (1,2,2,1)$, $\boldsymbol{v}_2 = (3,6,6,3)$, $\boldsymbol{v}_3 = (4,9,9,5)$,
  - $\boldsymbol{v}_4 = (-2,-1,-1,1)$, $\boldsymbol{v}_5 = (5,8,9,4)$, $\boldsymbol{v}_6 = (4,2,7,3)$.
- View each $\boldsymbol{v}_i$ as a row vector and form a matrix.
  - $\begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \\ \boldsymbol{v}_4 \\ \boldsymbol{v}_5 \\ \boldsymbol{v}_6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 3 & 6 & 6 & 3 \\ 4 & 9 & 9 & 5 \\ -2 & -1 & -1 & 1 \\ 5 & 8 & 9 & 4 \\ 4 & 2 & 7 & 3 \end{pmatrix} \xrightarrow{\text{G.E.}} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$
    - V has a basis $\{(1,2,2,1), (0,1,1,1), (0,0,1,1)\}$.
    - $\dim(V) = 3$.

Examples

- **Problem.** How to find a basis for a vector space V?
- Let $V = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6\}$.
  - $\boldsymbol{v}_1 = (1,2,2,1)$, $\boldsymbol{v}_2 = (3,6,6,3)$, $\boldsymbol{v}_3 = (4,9,9,5)$,
  - $\boldsymbol{v}_4 = (-2,-1,-1,1)$, $\boldsymbol{v}_5 = (5,8,9,4)$, $\boldsymbol{v}_6 = (4,2,7,3)$.
- View each $\boldsymbol{v}_j$ as a column vector and form a matrix.
  - $\begin{pmatrix} 1 & 3 & 4 & -2 & 5 & 4 \\ 2 & 6 & 9 & -1 & 8 & 2 \\ 2 & 6 & 9 & -1 & 9 & 7 \\ 1 & 3 & 5 & 1 & 4 & 3 \end{pmatrix} \xrightarrow{\text{G.E.}} \begin{pmatrix} 1 & 3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
    - The 1st, 3rd, 5th columns of row-echelon form are pivot columns.
      - They form a basis for the column space of the row-echelon form.
    - $\{\boldsymbol{v}_1, \boldsymbol{v}_3, \boldsymbol{v}_5\}$ is a basis for V.
    - $\dim(V) = 3$.

Examples

- **Problem.** Find a basis for a vector space $V = \operatorname{span}(S)$.

  Method 1: View each $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in S$ as a row vector.

  - Find a row-echelon form $\boldsymbol{R}$ of $\begin{pmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_m \end{pmatrix}$.
  - Then the nonzero rows of $\boldsymbol{R}$ is a basis for V.

  Method 2: View each $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in S$ as a column vector.

  - Find a row-echelon form $\boldsymbol{R}'$ of $\begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_m \end{pmatrix}$.
  - Find the pivot columns of $\boldsymbol{R}'$.
  - Then the corresponding $\boldsymbol{v}_j$ form a basis for V.
- Using column vectors, we can **select** a basis from a given spanning set of a vector space.

Examples

- **Problem.** Let S be a linearly independent subset of $\mathbb{R}^n$.
  - How to extend S to a basis for $\mathbb{R}^n$.
- $S = \{(1, 4, -2, 5, 1), (2, 9, -1, 8, 2), (2, 9, -1, 9, 3)\}$.
  - $\begin{pmatrix} 1 & 4 & -2 & 5 & 1 \\ 2 & 9 & -1 & 8 & 2 \\ 2 & 9 & -1 & 9 & 3 \end{pmatrix} \xrightarrow{\text{G.E.}} \begin{pmatrix} 1 & 4 & -2 & 5 & 1 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$.
    - S is linearly independent.
    - The 1st, 2nd, 4th columns of row-echelon form are pivot.
      - Add rows to row-echelon form such that all columns are pivot.
      - $\begin{pmatrix} 1 & 4 & -2 & 5 & 1 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.
    - $S \cup \{(0, 0, 1, 0, 0), (0, 0, 0, 0, 1)\}$ is a basis for $\mathbb{R}^5$.

Consistency of Linear System

- Consider the linear system
  - $$\begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -2 \\ 3 \end{pmatrix}.$$

  This is equivalent to
  - $$x \begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix} + y \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -2 \\ 3 \end{pmatrix}.$$

  The system is consistent

  $$\Leftrightarrow \begin{pmatrix} -1 \\ 4 \\ -2 \\ 3 \end{pmatrix} \text{ is a l.comb. of } \begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}.$$

Consistency

- **Theorem.** Let $\boldsymbol{A}$ be an $m \times n$ matrix.
  - The column space of $\boldsymbol{A}$ is $\{\boldsymbol{A}\boldsymbol{v} \mid \boldsymbol{v} \in \mathbb{R}^n\}$.
  - The linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent

    $\Leftrightarrow$ $\boldsymbol{b}$ lies in the column space of $\boldsymbol{A}$.
- **Proof.** Let $\boldsymbol{c}_j$ be the jth column of $\boldsymbol{A}$.

$$\begin{aligned}
& \boldsymbol{w} \in \text{column space of } \boldsymbol{A} \\
\Leftrightarrow\ & \boldsymbol{w} \in \operatorname{span}\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\} \\
\Leftrightarrow\ & \boldsymbol{w} = v_1 \boldsymbol{c}_1 + \cdots + v_n \boldsymbol{c}_n \text{ for some } v_1, \ldots, v_n \in \mathbb{R} \\
\Leftrightarrow\ & \boldsymbol{w} = \begin{pmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\
\Leftrightarrow\ & \boldsymbol{w} = \boldsymbol{A}\boldsymbol{v} \text{ for some } \boldsymbol{v} = (v_1, \ldots, v_n)^{\mathrm{T}} \in \mathbb{R}^n.
\end{aligned}$$

Consistency

- **Theorem.** Let $\boldsymbol{A}$ be an $m \times n$ matrix.
  - The column space of $\boldsymbol{A}$ is $\{\boldsymbol{A}\boldsymbol{v} \mid \boldsymbol{v} \in \mathbb{R}^n\}$.
  - The linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent

    $\Leftrightarrow$ $\boldsymbol{b}$ lies in the column space of $\boldsymbol{A}$.
- **Proof.** For any $\boldsymbol{b} \in \mathbb{R}^m$.

$$\begin{aligned}&\text{The linear system } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \text{ is consistent}\\ \Leftrightarrow\ & \boldsymbol{A}\boldsymbol{v} = \boldsymbol{b} \text{ for some } \boldsymbol{v} \in \mathbb{R}^m\\ \Leftrightarrow\ & \boldsymbol{b} \text{ lies in the column space of } \boldsymbol{A}.\end{aligned}$$

Ranks

Rank

- Let $\boldsymbol{A}$ be an $m \times n$ matrix.
  - The row space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^n$.
  - The column space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^m$.

  Let $\boldsymbol{R}$ be a row-echelon form of $\boldsymbol{A}$.
  - nonzero rows of $\boldsymbol{R}$ form a basis for row space of $\boldsymbol{A}$.
    - $\dim$ of row space of $\boldsymbol{A} =$ no. of nonzero rows of $\boldsymbol{R}$.
  - The columns in $\boldsymbol{A}$ which correspond to the pivot columns in $\boldsymbol{R}$ form a basis for the column space of $\boldsymbol{A}$.
    - $\dim$ of column space of $\boldsymbol{A} =$ no. of pivot columns of $\boldsymbol{R}$.

    Recall that

$$\begin{aligned}\text{no. of nonzero rows of } \boldsymbol{R} &= \text{no. of pivot points of } \boldsymbol{R}\\ &= \text{no. of pivot columns of } \boldsymbol{R}\end{aligned}$$

Rank

- **Theorem.** Let $\boldsymbol{A}$ be a matrix. Then
  - the dimension of the row space of $\boldsymbol{A}$

    $=$ the dimension of the column space of $\boldsymbol{A}$.
- **Definition.** Let $\boldsymbol{A}$ be a matrix.
  - The dimension of the row (or column) space of $\boldsymbol{A}$

    is called the **rank** of $\boldsymbol{A}$, denoted by $\operatorname{rank}(\boldsymbol{A})$.
- **Remarks.** Let $\boldsymbol{A}$ be an $m \times n$ matrix.
  - $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}^{\mathrm{T}})$.
  - $\operatorname{rank}(\boldsymbol{A}) = 0 \Leftrightarrow \boldsymbol{A} = \boldsymbol{0}$.
  - $\operatorname{rank}(\boldsymbol{A}) \le m$ and $\operatorname{rank}(\boldsymbol{A}) \le n$.
    - $\operatorname{rank}(\boldsymbol{A}) \le \min\{m, n\}$.
    - $\boldsymbol{A}$ is called **full rank** if $\operatorname{rank}(\boldsymbol{A}) = \min\{m, n\}$.
  - A square matrix $\boldsymbol{A}$ is of full rank $\Leftrightarrow$ $\boldsymbol{A}$ is invertible.

Examples

- Let $\boldsymbol{C} = \begin{pmatrix} 2 & 0 & 3 & -1 & 8 \\ 2 & 1 & 1 & -2 & 5 \\ -4 & -3 & 0 & 5 & -7 \end{pmatrix}$.
  - A row-echelon form $\boldsymbol{R} = \begin{pmatrix} 2 & 0 & 3 & -1 & 8 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.
  - Row space of $\boldsymbol{C}$ has a basis
    - $\{(2, 0, 3, -1, 8), (0, 1, -2, -1, -3)\}$.

    Column space of $\boldsymbol{C}$ has a basis
    - $\{(2, 2, -4)^{\mathrm{T}}, (0, 1, -3)^{\mathrm{T}}\}$.

  Then $\operatorname{rank}(\boldsymbol{C}) = 2$. In particular, $\boldsymbol{C}$ is not of full rank.

Rank & Consistency of Linear System

- Let $\boldsymbol{Ax} = \boldsymbol{b}$ be a linear system.
  - Let $\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$ be the columns of $\boldsymbol{A}$.

$$\begin{aligned}
&\boldsymbol{Ax} = \boldsymbol{b} \text{ is consistent} \\
\Leftrightarrow\ & \boldsymbol{b} \in \operatorname{span}\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\} \\
\Leftrightarrow\ & \operatorname{span}\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\} = \operatorname{span}\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n, \boldsymbol{b}\} \\
\Leftrightarrow\ & \dim \operatorname{span}\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\} = \dim \operatorname{span}\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n, \boldsymbol{b}\} \\
\Leftrightarrow\ & \operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A} \mid \boldsymbol{b}).
\end{aligned}$$

  Alternatively, let $\boldsymbol{R}$ be a row-echelon form of $\boldsymbol{A}$.
  - Then a row-echelon form of $(\boldsymbol{A} \mid \boldsymbol{b})$ is $(\boldsymbol{R} \mid \boldsymbol{b}')$.

$$\begin{aligned}
\boldsymbol{Ax} = \boldsymbol{b} \text{ is consistent} &\Leftrightarrow \boldsymbol{b}' \text{ is non-pivot} \\
&\Leftrightarrow \operatorname{rank}(\boldsymbol{R}) = \operatorname{rank}(\boldsymbol{R} \mid \boldsymbol{b}') \\
&\Leftrightarrow \operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A} \mid \boldsymbol{b}).
\end{aligned}$$

Example

- $\begin{cases} 2x - y = 1 \\ x - y + 3z = 0 \\ -5x + y = 0 \\ x + z = 0 \end{cases}$ $\qquad \boldsymbol{Ax} = \boldsymbol{b}$.
  - $\left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 1 & -1 & 3 & 0 \\ -5 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\text{G.E.}} \left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & -\frac{1}{2} & 3 & -\frac{1}{2} \\ 0 & 0 & -9 & 4 \\ 0 & 0 & 0 & \frac{7}{9} \end{array}\right)$.
    - $(\boldsymbol{A} \mid \boldsymbol{b}) \to (\boldsymbol{R} \mid \boldsymbol{b}')$.
  - $\operatorname{rank}(\boldsymbol{A}) = 3$ but $\operatorname{rank}(\boldsymbol{A} \mid \boldsymbol{b}) = 4$.
    - So the system is inconsistent.
- **Remark.** In general,
  - $\operatorname{rank}(\boldsymbol{A}) \leq \operatorname{rank}(\boldsymbol{A} \mid \boldsymbol{b}) \leq \operatorname{rank}(\boldsymbol{A}) + 1$.

Properties

- Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{B}$ be an $n \times p$ matrix.
  - Column space of $\boldsymbol{A} = \{\boldsymbol{Au} \mid \boldsymbol{u} \in \mathbb{R}^n\}$.
  - Column space of $\boldsymbol{AB} = \{\boldsymbol{ABv} \mid \boldsymbol{v} \in \mathbb{R}^p\}$.

  Let $\boldsymbol{w} \in$ column space of $\boldsymbol{AB}$. Then
  - $\boldsymbol{w} = \boldsymbol{ABv}$ for some $\boldsymbol{v} \in \mathbb{R}^p$.

  Let $\boldsymbol{u} = \boldsymbol{Bv}$. Then $\boldsymbol{u} \in \mathbb{R}^n$ and
  - $\boldsymbol{w} = \boldsymbol{Au} \in$ column space of $\boldsymbol{A}$.

  Therefore, column space of $\boldsymbol{AB} \subseteq$ column space of $\boldsymbol{A}$.

$$\begin{aligned}\text{row space of } \boldsymbol{AB} &= \text{column space of } (\boldsymbol{AB})^{\mathrm{T}} \\ &= \text{column space of } \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} \\ &\subseteq \text{column space of } \boldsymbol{B}^{\mathrm{T}} \\ &= \text{row space of } \boldsymbol{B}.\end{aligned}$$

Properties

- **Theorem.** Let $\boldsymbol{A}$ be an $m \times n$ matrix, and $\boldsymbol{B}$ be an $n \times p$ matrix. Then
  - column space of $\boldsymbol{AB} \subseteq$ column space of $\boldsymbol{A}$;
  - row space of $\boldsymbol{AB} \subseteq$ row space of $\boldsymbol{B}$.

  In particular,
  - $\operatorname{rank}(\boldsymbol{AB}) \leq \operatorname{rank}(\boldsymbol{A})$;
  - $\operatorname{rank}(\boldsymbol{AB}) \leq \operatorname{rank}(\boldsymbol{B})$.

  That is, $\operatorname{rank}(\boldsymbol{AB}) \leq \min\{\operatorname{rank}(\boldsymbol{A}), \operatorname{rank}(\boldsymbol{B})\}$.
- **Questions**:
  - When $\operatorname{rank}(\boldsymbol{AB}) = \operatorname{rank}(\boldsymbol{A})$?
  - When $\operatorname{rank}(\boldsymbol{AB}) = \operatorname{rank}(\boldsymbol{B})$?

Nullspace and Nullity

- **Definition.** Let $\boldsymbol{A}$ be an $m \times n$ matrix.
  - The **nullspace** of $\boldsymbol{A}$ is the solution space of $\boldsymbol{Ax} = \boldsymbol{0}$:
    - $\{\boldsymbol{v} \in \mathbb{R}^n \mid \boldsymbol{Av} = \boldsymbol{0}\}$.

  The dimension of the nullspace is called the **nullity** of $\boldsymbol{A}$, denoted by $\text{nullity}(\boldsymbol{A})$.
- **Notation.** From now on, unless otherwise stated,
  - vectors in **nullspace** are viewed as **column** vectors.
- **Remarks.** Let $\boldsymbol{R}$ be a row-echelon form of $\boldsymbol{A}$.
  - $\boldsymbol{Ax} = \boldsymbol{0} \Leftrightarrow \boldsymbol{Rx} = \boldsymbol{0}$.
  - nullspace of $\boldsymbol{A}$ = nullspace of $\boldsymbol{R}$.

$$\begin{aligned}\text{nullity}(\boldsymbol{A}) &= \text{nullity}(\boldsymbol{R}) \\ &= \text{no. of non-pivot columns of } \boldsymbol{R}.\end{aligned}$$

Examples

- $\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$.
  - $(\boldsymbol{A} \mid \boldsymbol{0}) \xrightarrow{\text{G.-J.E.}} \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$.
  - $\boldsymbol{Ax} = \boldsymbol{0} \Leftrightarrow \boldsymbol{x} = \begin{pmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$.
  - nullspace $= \text{span}\{(-1, 1, 0, 0, 0)^{\mathrm{T}}, (-1, 0, -1, 0, 1)^{\mathrm{T}}\}$.
    - $\text{nullity}(\boldsymbol{A}) = 2$. Note that $\text{rank}(\boldsymbol{A}) = 3$.

Examples

- $\boldsymbol{B} = \begin{pmatrix} 2 & 1 & -5 & 1 \\ -1 & -1 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix}$.
  - $(\boldsymbol{B} \mid \boldsymbol{0}) \xrightarrow{\text{G.-J.E.}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{7}{9} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{4}{9} & 0 \end{array}\right)$.
  - $\boldsymbol{Bx} = \boldsymbol{0} \Leftrightarrow \boldsymbol{x} = \begin{pmatrix} \frac{7}{9}t \\ -\frac{1}{3}t \\ \frac{4}{9}t \\ t \end{pmatrix} = t \begin{pmatrix} \frac{7}{9} \\ -\frac{1}{3} \\ \frac{4}{9} \\ 1 \end{pmatrix}, t \in \mathbb{R}$.
  - nullspace of $\boldsymbol{B} = \text{span}\{(\frac{7}{9}, -\frac{1}{3}, \frac{4}{9}, 1)^{\mathrm{T}}\}$.
    - $\text{nullity}(\boldsymbol{B}) = 1$. Note that $\text{rank}(\boldsymbol{B}) = 3$.

Dimension Theorem

- **Theorem.** Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then
  - $\text{rank}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{A}) = n$.

  Proof. Let $\boldsymbol{R}$ be a row-echelon form of $\boldsymbol{A}$.

$$\begin{aligned} & \text{rank}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{A}) \\ = \ & \text{rank}(\boldsymbol{R}) + \text{nullity}(\boldsymbol{R}) \\ = \ & \text{no. of pivot columns of } \boldsymbol{R} + \text{no. of non-pivot columns of } \boldsymbol{R} \\ = \ & \text{no. of columns of } \boldsymbol{R} = n. \end{aligned}$$

- **Example.** $\boldsymbol{0}_{m \times n}\boldsymbol{v} = \boldsymbol{0}$ for all $\boldsymbol{v} \in \mathbb{R}^n$.
  - nullspace of $\boldsymbol{0}_{m \times n} = \mathbb{R}^n$.
    - $\text{nullity}(\boldsymbol{0}_{m \times n}) = \dim(\mathbb{R}^n) = n$.
  - row space of $\boldsymbol{0}_{m \times n} = \{\boldsymbol{0}\} \subseteq \mathbb{R}^n$,
    column space of $\boldsymbol{0}_{m \times n} = \{\boldsymbol{0}\} \subseteq \mathbb{R}^m$.
    - $\text{rank}(\boldsymbol{0}_{m \times n}) = 0$.

Examples

- Find $\mathrm{rank}(\boldsymbol{A})$, $\mathrm{nullity}(\boldsymbol{A})$ and $\mathrm{nullity}(\boldsymbol{A}^{\mathrm{T}})$:
  - Suppose $\boldsymbol{A}$ is 3×4 and $\mathrm{rank}(\boldsymbol{A}) = 3$.
    - $\mathrm{nullity}(\boldsymbol{A}) = 4 - \mathrm{rank}(\boldsymbol{A}) = 4 - 3 = 1.$
    - $\boldsymbol{A}^{\mathrm{T}}$ is 4×3 and $\mathrm{rank}(\boldsymbol{A}^{\mathrm{T}}) = \mathrm{rank}(\boldsymbol{A}) = 3.$
      - $\mathrm{nullity}(\boldsymbol{A}^{\mathrm{T}}) = 3 - \mathrm{rank}(\boldsymbol{A}^{\mathrm{T}}) = 3 - 3 = 0.$
  - Suppose $\boldsymbol{A}$ is 7×5 and $\mathrm{nullity}(\boldsymbol{A}) = 3$.
    - $\mathrm{rank}(\boldsymbol{A}) = 5 - \mathrm{nullity}(\boldsymbol{A}) = 5 - 3 = 2.$
    - $\boldsymbol{A}^{\mathrm{T}}$ is 5×7 and $\mathrm{rank}(\boldsymbol{A}^{\mathrm{T}}) = \mathrm{rank}(\boldsymbol{A}) = 2.$
      - $\mathrm{nullity}(\boldsymbol{A}^{\mathrm{T}}) = 7 - \mathrm{rank}(\boldsymbol{A}^{\mathrm{T}}) = 7 - 2 = 5.$
  - Suppose $\boldsymbol{A}$ is 3×2 and $\mathrm{nullity}(\boldsymbol{A}^{\mathrm{T}}) = 3$.
    - $\boldsymbol{A}^{\mathrm{T}}$ is 2×3 and $\mathrm{rank}(\boldsymbol{A}^{\mathrm{T}}) = 3 - \mathrm{nullity}(\boldsymbol{A}^{\mathrm{T}}) = 0.$
    - $\mathrm{rank}(\boldsymbol{A}) = \mathrm{rank}(\boldsymbol{A}^{\mathrm{T}}) = 0.$
      - $\mathrm{nullity}(\boldsymbol{A}) = 2 - \mathrm{rank}(\boldsymbol{A}) = 2 - 0 = 2.$

Inhomogeneous Linear System

- Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent. Fix a solution $\boldsymbol{v}$.
  - For any vector $\boldsymbol{u}$,

$$\begin{aligned} \boldsymbol{u} \text{ is a solution to } \boldsymbol{Ax} = \boldsymbol{b} &\Leftrightarrow \boldsymbol{Au} = \boldsymbol{b} \\ &\Leftrightarrow \boldsymbol{Au} - \boldsymbol{b} = \boldsymbol{0} \\ &\Leftrightarrow \boldsymbol{A}(\boldsymbol{u} - \boldsymbol{v}) = \boldsymbol{0} \\ &\Leftrightarrow \boldsymbol{u} - \boldsymbol{v} \in \text{nullspace of } \boldsymbol{A} \\ &\Leftrightarrow \boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w},\ \boldsymbol{w} \in \text{nullspace of } \boldsymbol{A} \end{aligned}$$

  Let M be the solution set of $\boldsymbol{Ax} = \boldsymbol{b}$. Then
  - $M = \{\boldsymbol{v} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{nullspace of } \boldsymbol{A}\}$.

  Using the notation in Question 3.18,
  - $M = \boldsymbol{v} + W$, where W is the nullspace of $\boldsymbol{A}$.

Inhomogeneous Linear System

- **Theorem.** Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution $\boldsymbol{v}$.
  - The solution set of $\boldsymbol{Ax} = \boldsymbol{b}$ is
    - $\{\boldsymbol{v} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{nullspace of } \boldsymbol{A}\}$.

    A general solution of $\boldsymbol{Ax} = \boldsymbol{b}$ is
    - (a particular solution of $\boldsymbol{Ax} = \boldsymbol{b}$)

      $+$ (a general solution of $\boldsymbol{Ax} = \boldsymbol{0}$).
- **Remark.** In particular, suppose $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent.
  - $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution

    $\Leftrightarrow$ $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution
    $\Leftrightarrow$ nullspace of $\boldsymbol{A}$ is $\{\boldsymbol{0}\}$
    $\Leftrightarrow$ $\text{nullity}(\boldsymbol{A}) = 0$
    $\Leftrightarrow$ $\text{rank}(\boldsymbol{A}) =$ no. of columns of $\boldsymbol{A}$.

Example

- Let $\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -3 \end{pmatrix}$.
  - We have found that the nullspace of $\boldsymbol{A}$ is
    - $\text{span}\{(-1, 1, 0, 0, 0)^{\mathrm{T}}, (-1, 0, -1, 0, 1)^{\mathrm{T}}\}$

    $\boldsymbol{Ax} = \boldsymbol{0}$ has solution set (solution space)
    - $\left\{ s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \middle| \, s, t \in \mathbb{R} \right\}$.
  - One verifies that $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is a solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

Example

- Let $\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -3 \end{pmatrix}$.
    - We have found that the nullspace of $\boldsymbol{A}$ is
        - $\operatorname{span}\{(-1, 1, 0, 0, 0)^{\mathrm{T}}, (-1, 0, -1, 0, 1)^{\mathrm{T}}\}$
    - $\boldsymbol{Ax} = \boldsymbol{b}$ has solution set
        - $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \middle| \, s, t \in \mathbb{R} \right\}$.
- **Note.** The solution set of the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is a vector space $\Leftrightarrow$ $\boldsymbol{b} = \boldsymbol{0}$.

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