CS1231S Assignment #2

AY2022/23 Semester 1

Deadline: Monday, 31 October 2022, 1:00pm

ANSWERS

IMPORTANT: Please read the instructions below

This is a graded assignment worth 10% of your final grade. There are **six questions** (excluding question 0) with a total score of 40 marks. Please work on it by yourself, not in a group or in collaboration with anybody. Anyone found committing plagiarism (submitting other's work as your own), or sending your answers to others, or other forms of academic dishonesty will be penalised with a straight zero for the assignment, and possibly an F grade for the module.

You are to submit your assignment to **Canvas > Assignments** before the deadline.

Your answers may be typed or handwritten. Make sure that it is legible (for example, don't use very light pencil or ink if it is handwritten, or font size smaller than 11 if it is typed) or marks may be deducted.

You are to submit **a SINGLE pdf file**, where each page is A4 size. Do not submit files in other formats.

You may test out your submission folder before the deadline, but make sure you remove any test files you have submitted earlier.

Late submission will NOT be accepted. We will set the closing time of the submission folders to slightly later than 1pm to give you a few minutes of grace, but in your mind, you should treat **1pm** as the deadline. If you think you might be too busy on the day of the deadline, please submit earlier. Also, avoid submitting in the last minute; the system may get sluggish due to overload and you will miss the deadline.

Note the following as well:

- Name your pdf file with your **Student Number**. Your student number begins with 'A' (eg: A0234567X). (Do not mix up your student number with your NUSNET-id which begins with 'e'.)
- At the top of the first page of your submission, write your **Name** and **Tutorial Group**.
- To keep the submitted file short, please submit your answers **without** including the questions.
- As this is an assignment given well ahead of time, we expect you to work on it early. You should submit **polished work**, not answers that are untidy or appear to have been done in a hurry, for example, with scribbling and cancellation all over the places. Marks may be deducted for untidy work.

To combine all pages into a single pdf document for submission, you may find the following scanning apps helpful if you intend to scan your handwritten answers:

* for Android: https://fossbytes.com/best-android-scanner-apps/

* for iphone: https://www.switchingtomac.com/tutorials/ios-tutorials/the-best-ios-scanner-apps-to-scan-documents-images/

If you need any clarification about this assignment, please do NOT email us or post on telegram, but post on the **Canvas > Discussions> Assignments** forum so that all queries are at one place and everybody can read the answers to the queries.

Note: Do not use any methods that have not been covered in class. When using a theorem or result that has appeared in class (lectures or tutorials), please quote the theorem number/name, the lecture and slide number, or the tutorial number and question number in that tutorial, failing which marks may be deducted. Remember to use numbering and give justification for important steps in your proof, or marks may be deducted.

Question 0. (2 marks)
Check that …

- you have submitted a pdf file with your Student Number as the filename. [1 mark]
- you have written both your name and tutorial group number (eg: T08A) at the top of the first page of your file. [1 mark]

Question 1. Structural induction (4 marks)

The set S is defined recursively as follows:

(1) $(0,0) \in S$, $(1,1) \in S$. (base clause)

(2) If $(n-1, x) \in S$ and $(n-2, y) \in S$ then $(n, x+y) \in S$. (recursion clause)

(3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

(a) Is $(2,2) \in S$? Explain. [1 mark]

(b) Is $(10,55) \in S$? Explain [1 mark]

(c) What is S? [2 marks]

Answers:

(a) False. Since $(0,0) \in S$ and $(1,1) \in S$, then for $n = 2$, $(2,1) \in S$ by the recursion clause.

(b) True. The following are members of S:
$(0,0), (1,1), (2,1), (3,2), (4,3), (5,5), (6,8), (7,13), (8,21), (9,34), (10,55)$.

(c) $S = \{(n, F_n) : n \in \mathbb{N}$ and F_n the n^{th} Fibonacci number$\}$.

Question 2. Mathematical induction (6 marks)

Given a non-zero integer x, prove by induction that x^2 is a factor of

$$(x+1)^n - nx - 1 \text{ for all integers } n \geq 2.$$

Remember to follow the structure introduced in class, that is, define the predicate, solve the basis step, state the induction hypothesis, and solve the induction step.

Answer:

1. Let $P(n)$ be the proposition that given a non-zero integer x, $x^2 \mid ((x+1)^n - nx - 1)$ for all integers $n \geq 2$.
2. **(Basis step)** $n = 2$.
   2.1. $(x+1)^2 - 2x - 1 = x^2 + 2x + 1 - 2x - 1 = x^2 \cdot 1$
   2.2. As $x^2 \mid ((x+1)^2 - 2x - 1)$, hence $P(2)$ is true.
3. **(Induction hypothesis)** Assume $P(k)$ is true for $k \geq 2$, that is, $(x+1)^k - kx - 1 = mx^2$ for some integer m. (by the definition of divisibility)

4. **(Induction step)** To prove $P(k+1)$
   4.1. $(x+1)^{k+1} - (k+1)x - 1$
   4.2. $= (x+1)^k(x+1) - kx - x - 1$ (by basic algebra)
   4.3. $= (x+1)[(x+1)^k - 1] - kx$ (by basic algebra)
   4.4. $= (x+1)[((x+1)^k - kx - 1) + kx + 1 - 1] - kx$ (by basic algebra)
   4.5. $= (x+1)(mx^2 + kx) - kx$ (from line 3)
   4.6. $= (x+1)mx^2 + (x+1)kx - kx$ (by basic algebra)
   4.7. $= (x+1)mx^2 + kx^2$ (by basic algebra)
   4.8. $= [(x+1)m + k]x^2$ (by basic algebra)
   4.9. $= px^2$ (where $p = (x+1)m + k \in \mathbb{Z}$ by closure of integers under $\times$ and $+$)
   4.10. So $x^2 \mid (x+1)^{k+1} - (k+1)x - 1$ and hence $P(k+1)$ is true.
5. Therefore, $P(n)$ is true for all integers $n \geq 2$ by MI.

Question 3. Functions (8 marks)

(a) Let $A = \{x \in \mathbb{R} : x \neq 1\}$ and define the function $f : A \to \mathbb{R}$ by

$$f(x) = \frac{x+1}{x-1}.$$

Is f a bijection? Prove or disprove it. [4 marks]

(b) Let $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \neq 2\}$ and define the function $g : S \to \mathbb{R} \times \mathbb{R}$ by

$$g(x, y) = \left(\frac{y+2}{x-2}, \frac{1}{x-2}\right).$$

Is g injective? Prove or disprove it. Is g surjective? Prove or disprove it. [4 marks]

Answers:

(a) f is not surjective:
Counterexample: $f(x) = 1$.
There is no pre-image of 1 as $\frac{x+1}{x-1} \neq 1$ for all values of $x \in \mathbb{R}$ and $x \neq 1$.

Therefore f is not a bijection.

(b) g is injective:
1. Let $(x_1, y_1), (x_2, y_2) \in S$ such that $g(x_1, y_1) = g(x_2, y_2)$.
2. Then $\left(\frac{y_1+2}{x_1-2}, \frac{1}{x_1-2}\right) = \left(\frac{y_2+2}{x_2-2}, \frac{1}{x_2-2}\right)$ by the definition of g.
3. So $\frac{y_1+2}{x_1-2} = \frac{y_2+2}{x_2-2}$ and $\frac{1}{x_1-2} = \frac{1}{x_2-2}$ by equality of ordered pairs (Lecture 5 slide 16).
   3.1. $\frac{1}{x_1-2} = \frac{1}{x_2-2} \Rightarrow x_1 = x_2$ by basic algebra.
   3.2. Since $x_1 = x_2$, from $\frac{y_1+2}{x_1-2} = \frac{y_2+2}{x_2-2}$ we have $y_1 + 2 = y_2 + 2 \Rightarrow y_1 = y_2$ by basic algebra.
4. Since $(x_1, y_1) = (x_2, y_2)$, g is injective.

g is not surjective:
There is no pre-image of $(0,0)$ as $\frac{1}{x-2} \neq 0$ for all values of $x \in \mathbb{R}$ and $x \neq 2$.

Question 4. Cardinality (9 marks)

State whether each of the following statements is true or false. If it is true, prove it. If it is false, provide a counterexample. (Note that the proofs expected here are short. As usual, you should provide justification for important steps.)

You may quote this (if necessary) without proof: Any subset of a finite set is finite.

You may also use this equality: $A = (A \setminus B) \cup (A \cap B)$.
(Proof: $(A \setminus B) \cup (A \cap B) = (A \cap \bar{B}) \cup (A \cap B) = A \cap (\bar{B} \cup B) = A \cap U = A$.)

(a) A and B are sets.
If A is countably infinite and B is finite, then $A \setminus B$ is countably infinite. [3 marks]

(b) A and B are sets.
If A and B are countably infinite, then $A \cap B$ is countably infinite. [3 marks]

(c) Suppose A, B and C are sets such that $A \subseteq B \subseteq C$. If A and C are countably infinite, then B is countably infinite too. [3 marks]

Answers:

(a) **True.**
1. $A \setminus B$ is countable because $A \setminus B$ is a subset of A. (Theorem 7.4.3. Any subset of any countable set is countable).
2. To show that $A \setminus B$ is infinite, we prove by contradiction, that is, suppose $A \setminus B$ is finite.
   - 2.1. Note that $A \cap B$ is finite since B is finite.
   - 2.2. Then $A = (A \setminus B) \cup (A \cap B)$ is finite (Tutorial 8 Q3), which contradicts that A is infinite.
   - 2.3. Therefore $A \setminus B$ is infinite.
3. Since $A \setminus B$ is countable and infinite (from 1 and 2), therefore $A \setminus B$ is countably infinite.

(b) **False.** Many possible counterexamples.
$\mathbb{Z}^+$ is countably infinite. $\mathbb{Z}^-$ is countably infinite (define bijection $f: \mathbb{Z}^+ \to \mathbb{Z}^-$ by $f(x) = -x$).
But $\mathbb{Z}^+ \cap \mathbb{Z}^- = \emptyset$, which is finite.

(c) **True.**
1. Since A is infinite, so B is infinite too. (By the given statement: Any subset of a finite set is finite. Because otherwise, if B is finite, then A would be finite, contradicting that A is infinite.)
2. Since C is countable, so B is countable too. (Theorem 7.4.3. Any subset of any countable set is countable.)
3. Therefore B is countably infinite (from lines 1 and 2).

Question 5. Counting (5 marks)

Contract bridge is a 4-player card game played with a standard deck of 52 cards. All cards are evenly distributed to the players, so each player has 13 cards in their hand.

It is often useful to count the *High Card Points (HCP)* in one's hand at the start of the game. This is done by summing up the HCP of all the cards in one's hand. An Ace has 4 HCP, a King has 3 HCP, a Queen has 2 HCP and a Jack has 1 HCP, and all other cards have 0 HCP.

Dueet is playing contract bridge with 3 other friends. How many ways are possible for her hand to have:

(a) 1 HCP? [1 mark] (b) 2 HCP? [2 marks] (c) 2 or fewer HCP? [2 marks]

You are to show your working. Your final answers must be single numbers.

Answers:

There are 36 cards with no HCP.

(a) 1 Jack: $\binom{4}{1}\binom{36}{12} = \mathbf{5006710800.}$

(b) Case 1 (2 Jacks): $\binom{4}{2}\binom{36}{11} = 3604831776.$

Case 2 (1 Queen): $\binom{4}{1}\binom{36}{12} = 5006710800.$

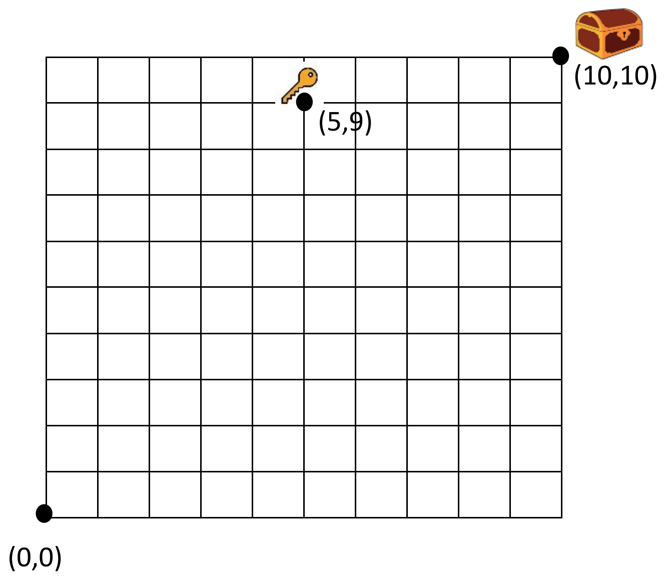
Total: $\mathbf{8611542576.}$

(c) 0 HCP: $\binom{36}{13} = 2310789600.$

Total: |0 HCP| + |1 HCP| + |2 HCP| = $\mathbf{15929042976.}$

Question 6. Counting (6 marks)

Aiken has been transported into a mysterious and somewhat chaotic reality show Octopus Game which promises a gazillion dollars if you win and a painful fate otherwise. Aiken has survived in the game so far and all that stands between him and the prize chest is a final maze, where the prize chest is located at the top right. He has been told, however, that just to make him earn his prize, he can only move up or right. The maze is a 10-by-10 coordinate grid, with the bottom-left corner being (0,0) and the top-right corner being (10,10), as shown in the figure below.



(a) How many ways can Aiken reach the prize chest from the bottom-left corner (0,0) where he is forced to start? [1 mark]

(b) Realising that it may be too easy for Aiken, the Octopus Game host puts a further obstacle for Aiken, somewhere within that maze is the key to the prize chest. Aiken is told that the key is at (5,9). How many ways can Aiken pick up the key first and then reach the chest? [2 marks]

(c) Frustrated that you are helping Aiken compute his options really fast, the Octopus Game host throws a final gauntlet – he removes the old key and puts two new keys at (7,3) and (8,7) instead. Of the two new keys, only one of them unlocks the treasure chest and the other is a fake. How many ways can Aiken pick up both the keys and then reach the chest? [3 marks]

You are to show your working. Your final answers must be single numbers.

Answers:

Let's define $W\big((a,b),(a+m,b+n)\big)$ to be the number of ways to get from point (a,b) to point $(a+m,b+n)$. Then $W\big((a,b),(a+m,b+n)\big) = \binom{m+n}{m}$ or $\binom{m+n}{n}$. (This problem is similar to filling a string of $m+n$ characters with m R's (for 'right') and n U's (for 'up').)

(a) $W\big((0,0),(10,10)\big) = \binom{20}{10} = \mathbf{184756}$.

(b) $W\big((0,0),(5,9)\big) \times W\big((5,9),(10,10)\big) = \binom{14}{5} \times \binom{6}{5} = 2002 \times 6 = \mathbf{12012}$.

(c) Note that (7,3) must be visited before (8,7).

$W\big((0,0),(7,3)\big) \times W\big((7,3),(8,7)\big) \times W\big((8,7),(10,10)\big)$

$= \binom{10}{7} \times \binom{5}{1} \times \binom{5}{2} = 120 \times 5 \times 10 = \mathbf{6000}$.

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