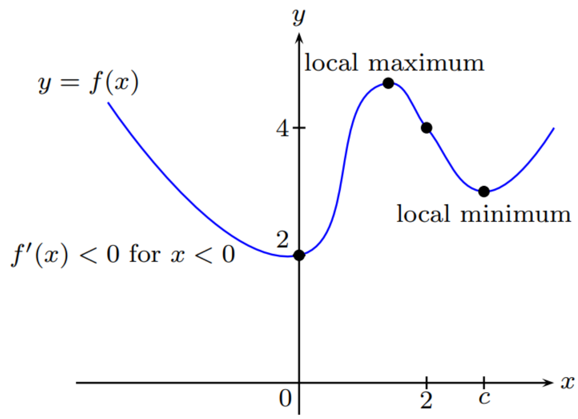
Solutions to Tutorial 3

MA1521 Calculus for Computing

1. An example of the graph of f is sketched below. Note that $f'(x) < 0$ means f is decreasing for $x < 0$.



2. Note that $f(x) = \sec x + \tan x$, where $x \in (0, 2\pi)$, is not defined at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

   $f'(x) = \sec x \tan x + \sec^2 x = \frac{1+\sin x}{\cos^2 x}$. Then $f'(x) > 0 \Leftrightarrow 1 + \sin x > 0 \Leftrightarrow \sin x > -1$, which is true for all $x \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}\}$. Thus f is increasing in the intervals $(0, \pi/2)$, $(\pi/2, 3\pi/2)$ and $(3\pi/2, 2\pi)$.

3. Note that f is defined on $[-5, 9]$. We can read off from the graph that $f(-5) = 6, f(-2) = -1, f(1) = 3, f(2) = -4, f(6) = 8, f(9) = -6$. From the graph and by comparing the above values at the critical points and the endpoints, we deduce that f has a local maximum at $x = 1, 6$; a local minimum at $x = -2, 2$ and also f attains the absolute maximum at $x = 6$, and the absolute minimum at $x = 9$.

4. **(a)** $y = \dfrac{x+1}{x^2+1}, x \in [-3, 3]$.

   $y' = \dfrac{2-(x+1)^2}{(x^2+1)^2}$ and $y' = 0$ if $x = -1 \pm \sqrt{2}$.

   So critical points are $x = -1 \pm \sqrt{2}$.

   Now, for $y' > 0$, we need $2 - (x+1)^2 > 0$. That means

   $$(\sqrt{2} - (x+1))(\sqrt{2} + (x+1)) > 0.$$

   Solving this, we see that $y' > 0$ if and only if $-1 - \sqrt{2} < x < -1 + \sqrt{2}$. We have thus shown the following:

$$y' \begin{cases} < 0 & \text{if } -3 \le x < -1-\sqrt{2}, \\ = 0 & \text{if } x = -1-\sqrt{2}, \\ > 0 & \text{if } -1-\sqrt{2} < x < -1+\sqrt{2}, \\ = 0 & \text{if } x = -1+\sqrt{2}, \\ < 0 & \text{if } -1+\sqrt{2} < x \le 3. \end{cases}$$

y is decreasing only if $y' < 0$. Hence y is decreasing if $x \in [-3, -1-\sqrt{2}]$; or $x \in [-1+\sqrt{2}, 3]$. On the other hand, $y' > 0$ if and only y is increasing. Therefore, y is increasing if $x \in [-1-\sqrt{2}, -1+\sqrt{2}]$.

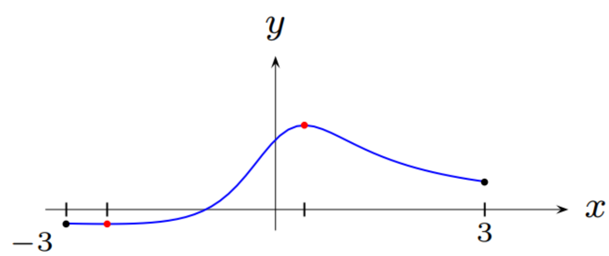
Applying the first derivative test, we have a local minimum at $x = -1-\sqrt{2}$, with value $y(-1-\sqrt{2}) = -\frac{1}{2(\sqrt{2}+1)}$, and a local maximum at $x = -1+\sqrt{2}$, with value $y(-1+\sqrt{2}) = \frac{1}{2(\sqrt{2}-1)}$.

At $x = -3$, $y(-3) = -\frac{1}{5}$ and at $x = 3$, $y(3) = \frac{2}{5}$. Since

$$-\frac{1}{2(\sqrt{2}+1)} < -\frac{1}{5} < \frac{2}{5} < \frac{1}{2(\sqrt{2}-1)},$$

the absolute minimum value is $\displaystyle\min_{x\in[-3,3]} y = -\frac{1}{2(\sqrt{2}+1)}$ at $x = -1-\sqrt{2}$

and the absolute maximum value is $\displaystyle\max_{x\in[-3,3]} y = \frac{1}{2(\sqrt{2}-1)}$ at $x = -1+\sqrt{2}$.



$y = \frac{x+1}{x^2+1}$.

(b) $y = (x-1)\sqrt[3]{x^2}$, $x \in (-\infty, \infty)$.

A better way to understand the term $\sqrt[3]{x^2}$ is that, one first take the square of x and then take the posible root of x^2. Hence, $\sqrt[3]{x^2} \ge 0$.

It then follows that $\lim_{x\to-\infty}(x-1)\sqrt[3]{x^2} = -\infty$. Note that $x-1$ is going to $-\infty$ and $\sqrt[3]{x^2}$ is going to ∞. Clearly, $\lim_{x\to\infty}(x-1)\sqrt[3]{x^2} = \infty$.

Next, we compute y'.

$$y' = x^{2/3} + \frac{2}{3}(x-1)x^{-1/3} = \frac{5x-2}{3x^{1/3}}$$

and $y' = 0$ if $x = \frac{2}{5}$.

Note that y' does not exist at $x = 0$.

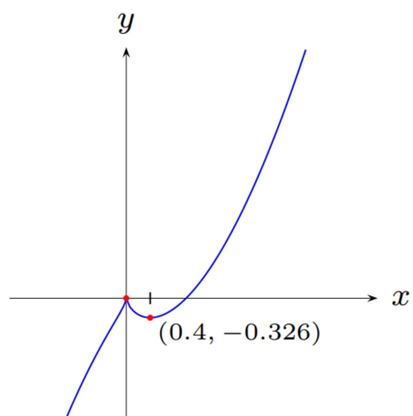
So the critical points are $x = 0$ and $x = \frac{2}{5}$.

$$y' \begin{cases} > 0 & \text{if } x < 0, \\ \text{does not exist} & \text{if } x = 0, \\ < 0 & \text{if } 0 < x < \frac{2}{5}, \\ = 0 & \text{if } x = \frac{2}{5}, \\ > 0 & \text{if } x > \frac{2}{5}. \end{cases}$$

Hence y is increasing in $(-\infty, 0]$, decreasing in $[0, \frac{2}{5}]$, and increasing in $[\frac{2}{5}, \infty)$.

So local maximum value is $y(0) = 0$ and local minimum value is $y(\frac{2}{5}) = -\frac{3}{5}(\frac{2}{5})^{2/3}$.

Since $\lim_{x\to-\infty} y = -\infty$, $\lim_{x\to\infty} y = \infty$, so there are no absolute extrema.



$y = (x-1)x^{\frac{2}{3}}$.

5. Let x be the distance between B and C. Suppose the energy that it takes to fly over land is 1 unit per km, then it will take 1.4 unit per km to fly over water.

   The total energy is given by the function

$$f(x) = 1.4\sqrt{5^2 + x^2} + (13 - x), \quad x > 0.$$

   Then

$$f'(x) = \frac{1.4x - \sqrt{5^2 + x^2}}{\sqrt{5^2 + x^2}}, \quad x > 0.$$

   Solving $f'(x) = 0$, we have $x = 5.103$ and the First Derivative Test shows that this point is an absolute minimum.

   Alternatively, regard f as defined on $[0, 13]$. Then $f(0) = 20, f(5.103) = 17.90, f(13) = 19.50$. Thus the absolute minimum value is 19.50.

6. (a) $\lim\limits_{x\to\pi/2} \dfrac{1-\sin x}{1+\cos 2x} = \lim\limits_{x\to\pi/2} \dfrac{-\cos x}{-2\sin 2x} = \lim\limits_{x\to\pi/2} \dfrac{\sin x}{-4\cos 2x} = \dfrac{1}{4}.$

(b) $\lim\limits_{x\to 0} \dfrac{\ln(\cos ax)}{\ln(\cos bx)} = \lim\limits_{x\to 0} \dfrac{\dfrac{-a\sin ax}{\cos ax}}{\dfrac{-b\sin bx}{\cos bx}} = \lim\limits_{x\to 0} \dfrac{a\sin ax\ \cos bx}{b\sin bx\ \cos ax} = \lim\limits_{x\to 0} \dfrac{a^2 \frac{\sin ax}{ax}\ \cos bx}{b^2 \frac{\sin bx}{bx}\ \cos ax} = \dfrac{a^2}{b^2}.$

(c) In order to apply L'Hopital's rule , one must express the limit

$$\lim_{x\to\infty} x\tan\frac{1}{x}$$

in the form of $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)}$ such that either $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = \pm\infty$ or $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = 0$.

That is why we should write

$$\lim_{x\to\infty} x\tan\frac{1}{x} = \lim_{x\to\infty} \frac{\tan(x^{-1})}{x^{-1}} = \lim_{x\to\infty} \frac{-x^{-2}\sec^2(x^{-1})}{-x^{-2}} = \lim_{x\to\infty} \cos^{-2}(x^{-1}) = 1.$$

(d) $\lim\limits_{x\to 0+} x^a \ln x = \lim\limits_{x\to 0+} \dfrac{\ln x}{x^{-a}} = \lim\limits_{x\to 0+} \dfrac{\dfrac{1}{x}}{-ax^{-a-1}} = \lim\limits_{x\to 0+} \dfrac{x^a}{-a} = 0.$

(e) $\lim\limits_{x\to 1} \ln x^{\frac{1}{1-x}} = \lim\limits_{x\to 1} \dfrac{\ln x}{1-x} = \lim\limits_{x\to 1} \dfrac{\dfrac{1}{x}}{-1} = -1.$ So $\lim_{x\to 1}\ x^{\frac{1}{1-x}} = e^{-1}$.

(f) Using (4d) we have

$$\lim_{x\to 0+} \ln x^{\sin x} = \lim_{x\to 0+} \sin x \ln x = \lim_{x\to 0+} \frac{\sin x}{x}\cdot x\ln x = \lim_{x\to 0+} \frac{\sin x}{x} \lim_{x\to 0+} x\ln x = 0.$$

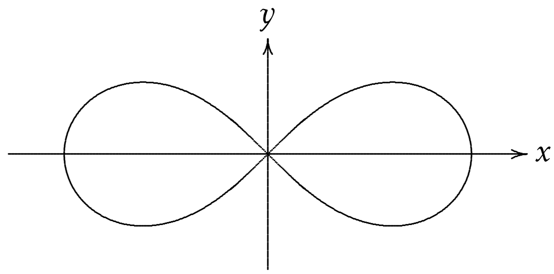
So $\lim\limits_{x\to 0+} x^{\sin x} = e^0 = 1.$

Solutions to Further Exercises

1. Differentiating both sides of the equation with respect to x, we have

$$2(2)(x^2+y^2)(2x+2yy') = 25(2x-2yy').$$

Thus, $y' = \dfrac{x(25-4x^2-4y^2)}{y(25+4x^2+4y^2)}$ and $y'(3) = -9/13$. The equation of the tangent line to the curve at $(3,1)$ is given by $y-1 = -\frac{9}{13}(x-3)$, or equivalently $9x+13y = 40$.



2. (a) Applying L'Hôpital's Rule 3 times, we have

$$\lim_{x\to\infty}\frac{x^3}{3^x}=\lim_{x\to\infty}\frac{3x^2}{3^x\ln(3)}=\lim_{x\to\infty}\frac{6x}{3^x\ln(3)^2}=\lim_{x\to\infty}\frac{6}{3^x\ln(3)^3}=0.$$

(b) $$\lim_{x\to 0}\ln\left[\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}\right]=\lim_{x\to 0}\frac{\ln(\frac{\sin x}{x})}{x^2}=\lim_{x\to 0}\frac{(\frac{x}{\sin x})\cdot\frac{x\cos x-\sin x}{x^2}}{2x}$$
$$=\frac{1}{2}\lim_{x\to 0}\frac{x}{\sin x}\lim_{x\to 0}\frac{x\cos x-\sin x}{x^3}=\frac{1}{2}\lim_{x\to 0}\frac{\cos x-x\sin x-\cos x}{3x^2}$$
$$=-\frac{1}{6}\lim_{x\to 0}\frac{\sin x}{x}=-\frac{1}{6}.$$

So $\lim_{x\to 0}(\frac{\sin x}{x})^{\frac{1}{x^2}}=e^{-1/6}$.

3. $$\lim_{x\to 1}\frac{\sqrt{x}-1}{g(x)-g(1)}=\lim_{x\to 1}\frac{x-1}{g(x)-g(1)}\frac{\sqrt{x}-1}{x-1}\frac{\sqrt{x}+1}{\sqrt{x}+1}=\lim_{x\to 1}\frac{1}{\frac{g(x)-g(1)}{x-1}}\frac{1}{\sqrt{x}+1}=\frac{1}{2g'(1)}.$$