10. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n$ and $n \times p$ matrices respectively.

(a) Suppose the homogeneous linear system $\boldsymbol{Bx} = \boldsymbol{0}$ has infinitely many solutions. How many solutions does the system $\boldsymbol{ABx} = \boldsymbol{0}$ have?

(b) Suppose $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution. Can we tell how many solutions are there for $\boldsymbol{ABx} = \boldsymbol{0}$? $\hookrightarrow$ means $x = 0$

a) Let $x = u$ be any solution to the system $Bx = 0$. Then $ABu = A0 = 0$.
The system $ABx = 0$ has at least as many solutions as the system $Bx = 0$.
Thus it has infinitely many solutions

b) No take an example where $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

When $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, both $Bx = 0$ and $ABx = 0$ have only trivial solution
$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$
When $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $Bx = 0$ has only trivial solution, but
$ABx = 0$ has infinitely many solutions.

21. Given that $\boldsymbol{A}$ is a 3×3 matrix such that

$$\boldsymbol{A}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{A}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Find a matrix $\boldsymbol{X}$ such that

$$\boldsymbol{AX} = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 0 & 4 \\ 1 & 0 & 7 \end{pmatrix}.$$

(Hint: Write $\boldsymbol{X} = (\boldsymbol{x_1} \quad \boldsymbol{x_2} \quad \boldsymbol{x_3})$ where $\boldsymbol{x_i}$ is the ith column of $\boldsymbol{X}$.)

$X = (x_1 \ x_2 \ x_3)$
$AX = (Ax_1 \ Ax_2 \ Ax_3)$

$Ax_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $Ax_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $Ax_3 = \begin{pmatrix} 4 \\ 4 \\ 7 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ from $AX = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 0 & 4 \\ 1 & 0 & 7 \end{pmatrix}$

1. $x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ from $A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

2. $x_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ as any $A\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and we are finding a particular x

3. $Ax_3 = \begin{pmatrix} 4 \\ 4 \\ 7 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 4A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3A\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = A\left(4\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = A\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$
hence $x_3 = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$

Therefore $X = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

22. Prove Remark 1.1.10:

Show that a linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has either no solution, only one solution or infinitely many solutions.

(Hint: Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ has two different solutions $\boldsymbol{u}$ and $\boldsymbol{v}$. Use $\boldsymbol{u}$ and $\boldsymbol{v}$ to construct infinitely many other solutions.)

Since 0 solution ✓
1 solution ✓
We just have to show >1 solution

Suppose $Ax = b$ has two different solutions u and v.
$Au = b$ $Av = b$ where $u \neq v$
Then for all $t \in \mathbb{R}$
$A(tu + (1-t)v)$
$= tAu + (1-t)Av$
$= tb + (1-t)b = b$
Therefore $tu + (1-t)v$ is also a solution of $Ax = b$.
Since $t_1 u + (1-t_1)v \neq t_2 u + (1-t_2)v$ whenever $t_1 \neq t_2$,
There are infinitely many solutions

24. Determine which of the following statements are true. Justify your answer.

(a) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonal matrices of the same size, then $\boldsymbol{AB} = \boldsymbol{BA}$.

(b) If $\boldsymbol{A}$ is a square matrix, then $\frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}})$ is symmetric.

(c) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices of the same size, $(\boldsymbol{A} + \boldsymbol{B})^2 = \boldsymbol{A}^2 + \boldsymbol{B}^2 + 2\boldsymbol{AB}$.

(d) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric matrices of the same size, then $\boldsymbol{A} - \boldsymbol{B}$ is symmetric.

(e) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric matrices of the same size, then $\boldsymbol{AB}$ is symmetric.

(f) If $\boldsymbol{A}$ is a square matrix such that $\boldsymbol{A}^2 = \boldsymbol{0}$, then $\boldsymbol{A} = \boldsymbol{0}$.

(g) If $\boldsymbol{A}$ is a matrix such that $\boldsymbol{AA}^{\mathrm{T}} = \boldsymbol{0}$, then $\boldsymbol{A} = \boldsymbol{0}$.

b) If $\boldsymbol{A}$ is a square matrix, then $\frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}})$ is symmetric. True

Let $B = \frac{1}{2}(A + A^T)$. B is symmetric iff $B = B^T$

$$\begin{aligned} B^T &= \left(\tfrac{1}{2}(A+A^T)\right)^T \\ &= \tfrac{1}{2}(A+A^T)^T \\ &= \tfrac{1}{2}[A^T + (A^T)^T] \\ &= \tfrac{1}{2}(A^T + A) \\ &= B \end{aligned}$$

Thus $\frac{1}{2}(A+A^T)$ is symmetric

c) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices of the same size, $(\boldsymbol{A} + \boldsymbol{B})^2 = \boldsymbol{A}^2 + \boldsymbol{B}^2 + 2\boldsymbol{AB}$. False

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$(A+B)^2 = A^2 + AB + BA + B^2$ as matrix multiplication is not commutative. ie $AB \neq BA$

f) If $\boldsymbol{A}$ is a square matrix such that $\boldsymbol{A}^2 = \boldsymbol{0}$, then $\boldsymbol{A} = \boldsymbol{0}$. False

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

g) If $\boldsymbol{A}$ is a matrix such that $\boldsymbol{AA}^{\mathrm{T}} = \boldsymbol{0}$, then $\boldsymbol{A} = \boldsymbol{0}$. True

The (i,i) entry of AA^T: $a_{i1}a_{i1} + a_{i2}a_{i2} \cdots = \sum_{i=1}^{n} a_{ik}^2$

So $AA^T = 0$ implies that $a_{ik} = 0$ for all i and k. $A = 0$

27. (a) Give three examples of 2×2 matrices $\boldsymbol{A}$ such that $\boldsymbol{A}^2 = \boldsymbol{A}$.

(b) Let $\boldsymbol{A}$ be a square matrix such that $\boldsymbol{A}^2 = \boldsymbol{A}$. Show that $\boldsymbol{I} + \boldsymbol{A}$ is invertible and $(\boldsymbol{I} + \boldsymbol{A})^{-1} = \frac{1}{2}(2\boldsymbol{I} - \boldsymbol{A})$.

a) $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

as $A^2 = A$

b) $(I+A)\left[\frac{1}{2}(2I - A)\right] = \frac{1}{2}(I+A)(2I-A) = \frac{1}{2}(2I + A - A^2) = I$

$I+A$ is invertable and its inverse is $\frac{1}{2}(2I - A)$ by the def of inverse of square matrix

* Note that to show B is a inverse of A, $AB = I$ and $BA = I$

In this case A is a square matrix, this is not needed.