NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 2, 2022/2023

MA2001 Linear Algebra **Homework Assignment 1**

1. Consider the following linear system

$$\begin{cases} 4x_1 + 2x_2 + 4x_3 = 0 \\ 5x_1 + 4x_2 \phantom{{}+ 4x_3} = 1 \\ 4x_1 + x_2 + 2x_3 = 5. \end{cases}$$

(i) In every row-echelon form of the augmented matrix corresponding to the given linear system, how many pivot columns are there?

(ii) How many arbitrary parameters are needed to describe a general solution for the given linear system?

(iii) How many solutions are there for the given linear system?

(iv) Find the solution of the linear system.

Solution. The augmented matrix is $\boldsymbol{A} = \left(\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 5 & 4 & 0 & 1 \\ 4 & 1 & 2 & 5 \end{array}\right)$. Apply Gaussian elimination:

$$\boldsymbol{A} \xrightarrow[R_3 - R_1]{R_2 - \frac{5}{4}R_1} \left(\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 0 & \frac{3}{2} & -5 & 1 \\ 0 & -1 & -2 & 5 \end{array}\right) \xrightarrow{R_3 + \frac{2}{3}R_2} \left(\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 0 & \frac{3}{2} & -5 & 1 \\ 0 & 0 & -\frac{16}{3} & \frac{17}{3} \end{array}\right) = \boldsymbol{R}.$$

All row-echelon form of $\boldsymbol{A}$ have the same pivot and non-pivot columns. The answer is on $\boldsymbol{R}$.

(i) $\boldsymbol{R}$ has 3 pivot columns, which are the columns corresponding to the variables x_1, x_2, x_3.

(ii) The system has 0 arbitrary parameter.

(iii) Note that the last column is non-pivot. The system has a unique solution.

(iv) One may solve the system using back-substitution, or apply Gauss-Jordan elimination.

$$\boldsymbol{R} \xrightarrow[-\frac{3}{16}R_3]{\substack{\frac{1}{4}R_1 \\ \frac{2}{3}R_2}} \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -\frac{10}{3} & \frac{2}{3} \\ 0 & 0 & 1 & -\frac{17}{16} \end{array}\right) \xrightarrow[R_2 + \frac{10}{3}R_3]{R_1 - R_3} \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & \frac{17}{16} \\ 0 & 1 & 0 & -\frac{23}{8} \\ 0 & 0 & 1 & -\frac{17}{16} \end{array}\right) \xrightarrow{R_1 - \frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{2} \\ 0 & 1 & 0 & -\frac{23}{8} \\ 0 & 0 & 1 & -\frac{17}{16} \end{array}\right).$$

So the solution is $x_1 = \frac{5}{2}$, $x_2 = -\frac{23}{8}$, $x_3 = -\frac{17}{16}$. □

2. Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ 3 & d \end{pmatrix}$, where a, b, d are real constants. If $\boldsymbol{A}^{\mathrm{T}} = -\boldsymbol{A}$, find the value of $a + b + d$.

Solution. $\boldsymbol{A}^{\mathrm{T}} = -\boldsymbol{A} \Leftrightarrow \begin{pmatrix} a & 3 \\ b & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -3 & -d \end{pmatrix} \Leftrightarrow \begin{cases} a = -a \\ 3 = -b \\ b = -3 \\ d = -d \end{cases} \Leftrightarrow \begin{cases} a = 0 \\ b = -3 \\ d = 0. \end{cases}$ So $a + b + d = -3$. □

3. Let $\boldsymbol{A} = \begin{pmatrix} -1 & 2 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$.

(i) Find $\det(\boldsymbol{A})$.

(ii) Find $\det(\boldsymbol{A} - 2\boldsymbol{I})$.

(iii) Find $\mathbf{adj}(\boldsymbol{A})$.

(iv) Find $\boldsymbol{A}^{-1}$.

Solution. (i) $\det(\boldsymbol{A}) \xlongequal{R_2 + R_1} \begin{vmatrix} -1 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & -1 \end{vmatrix} \xlongequal{R_3 - \frac{1}{2}R_2} \begin{vmatrix} -1 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -\frac{3}{2} \end{vmatrix} = (-1)(2)(-\tfrac{3}{2}) = 3.$

(ii) $\boldsymbol{A} - 2\boldsymbol{I} = \begin{pmatrix} -3 & 2 & -1 \\ 1 & -2 & 2 \\ 0 & 1 & -3 \end{pmatrix}$. So

$$\det(\boldsymbol{A} - 2\boldsymbol{I}) \xlongequal{R_2 + \frac{1}{3}R_1} \begin{vmatrix} -3 & 2 & -1 \\ 0 & -\frac{4}{3} & \frac{5}{3} \\ 0 & 1 & -3 \end{vmatrix} \xlongequal{R_3 + \frac{3}{4}R_2} \begin{vmatrix} -3 & 2 & -1 \\ 0 & -\frac{4}{3} & \frac{5}{3} \\ 0 & 0 & -\frac{7}{4} \end{vmatrix} = (-3)(-\tfrac{4}{3})(-\tfrac{7}{4}) = -7.$$

(iii) $\mathbf{adj}(\boldsymbol{A}) = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} -1 & -1 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$

(iv) $\boldsymbol{A}^{-1} = \dfrac{1}{\det(\boldsymbol{A})} \mathbf{adj}(\boldsymbol{A}) = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}.$ □

4. Let $\boldsymbol{A}$ be a square matrix such that $\det(\boldsymbol{A}) = 2$. Find the number of solutions of the linear system $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$.

Solution. Since $\det(\boldsymbol{A}) = 2 \neq 0$, $\boldsymbol{A}$ is an invertible matrix. Hence, $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ has a unique (the trivial) solution. □

5. Consider the following linear system:

$$\begin{cases} ax + \ \ y + \ \ z = a^3 \\ \ \ x + ay + \ \ z = 1 \\ \ \ x + \ \ y + az = a, \end{cases}$$

where a is a real constant. Find the conditions on a such that

(i) the system has no solution;

(ii) the system has a unique solution;

(iii) the system has infinitely many solutions.

Solution. Consider the augmented matrix and apply Gaussian elimination.

$$\left(\begin{array}{ccc|c} a & 1 & 1 & a^3 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & a \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & a & 1 & 1 \\ a & 1 & 1 & a^3 \\ 1 & 1 & a & a \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - aR_1} \left(\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1-a^2 & 1-a & a^3-a \\ 0 & 1-a & a-1 & a-1 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1-a & a-1 & 1-a \\ 0 & 1-a^2 & 1-a & a^3-a \end{array}\right)$$

$$\xrightarrow{R_3 - (1+a)R_2} \left(\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1-a & a-1 & a-1 \\ 0 & 0 & (1-a)(2+a) & (1-a)^2(1+a) \end{array}\right).$$

(i) If $a = -2$, the matrix is reduced to $\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 3 & -3 & -3 \\ 0 & 0 & 0 & 9 \end{array}\right)$, and the linear system has no solution.

(ii) If $a \neq 1$ and $a \neq -2$, then linear the system has a unique solution.

(iii) If $a = 1$, the matrix is reduced to $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$, and the linear system has infinitely many solutions (with 2 arbitrary parameters). $\square$

6. Let $\boldsymbol{A} = \begin{pmatrix} b & b & 1 \\ a & -a & 0 \\ a-2 & a+1 & 0 \end{pmatrix}$. Find the condition on a and b such that $\boldsymbol{A}$ is invertible.

Solution. Expand along the 3rd column:

$$\det(\boldsymbol{A}) = \begin{vmatrix} a & -a \\ a-2 & a+1 \end{vmatrix} = a(a+1) - (a-2)(-a) = 2a^2 - a.$$

Then

$$\boldsymbol{A} \text{ is invertible} \Leftrightarrow \det(\boldsymbol{A}) = 2a^2 - a \neq 0 \Leftrightarrow a \neq 0 \text{ and } a \neq 1/2. \qquad \square$$

7. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of order 3 such that

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{B}.$$

Find a sequence of 3 elementary row operations which, when applied to $\boldsymbol{A}$, yields $\boldsymbol{B}$.

Solution. It is given that $\boldsymbol{B} \xrightarrow{R_1 - R_2} \bullet \xrightarrow{R_1 \leftrightarrow R_3} \bullet \xrightarrow{2R_3} \boldsymbol{A}$. Hence,

$$\boldsymbol{A} \xrightarrow{\frac{1}{2}R_3} \bullet \xrightarrow{R_1 \leftrightarrow R_3} \bullet \xrightarrow{R_1 + R_2} \boldsymbol{B}.$$ □

8. Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{B}$ be an $n \times p$ matrix. Prove that if $\boldsymbol{A}$ has a zero row, then $\boldsymbol{AB}$ has a zero row.

Proof. Let $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$. Assume that the i^{th} row of $\boldsymbol{A}$ is a zero row, i.e., $a_{ik} = 0$ for all $k = 1, \ldots, n$. Then for any $j = 1, \ldots, p$, the (i, j)-entry of $\boldsymbol{AB}$ is

$$\sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj} = 0.$$

Hence, the i^{th} row of $\boldsymbol{AB}$ is also a zero row. □

9. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of order n. Prove that if $\boldsymbol{A}$ is singular, then $\boldsymbol{AB}$ is singular.

Proof. If $\boldsymbol{A}$ is singular, then $\det(\boldsymbol{A}) = 0$. Hence, $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B}) = 0$, and it follows that $\boldsymbol{AB}$ is singular.

Alternatively, assume that $\boldsymbol{AB}$ is invertible with inverse $\boldsymbol{C}$, then $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{I}$, which contradicts the assumption that $\boldsymbol{A}$ is singular. Hence, $\boldsymbol{AB}$ is singular. □

10. Write down a nonzero 3×3 matrix $\boldsymbol{A}^2 = \boldsymbol{0}$.

Solution. For example, let $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with $a \neq 0$. Then $\boldsymbol{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. □

11. Give an example of a 2×2 matrix which is not the product of elementary matrices.

Solution. A square matrix is singular $\Leftrightarrow$ it is not the product of elementary matrices.

Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\boldsymbol{A}$ is singular $\Leftrightarrow$ $\det(\boldsymbol{A}) = ad - bc = 0$. For example, $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. □

12. Prove that if $\boldsymbol{A}$ and $\boldsymbol{B}$ be a symmetric matrices of the same order, then $\boldsymbol{A} + \boldsymbol{B} = (\boldsymbol{A} + \boldsymbol{B})^{\mathrm{T}}$.

Proof. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric, then $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$ and $\boldsymbol{B}^{\mathrm{T}} = \boldsymbol{B}$. Hence,

$$(\boldsymbol{A} + \boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}} = \boldsymbol{A} + \boldsymbol{B}.$$

Alternatively, let $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$, then $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$. So

$$\begin{aligned}
(i,j)\text{-entry of } (\boldsymbol{A}+\boldsymbol{B})^{\mathrm{T}} &= (j,i)\text{-entry of } \boldsymbol{A}+\boldsymbol{B} \\
&= a_{ji} + b_{ji} = a_{ij} + b_{ij} \\
&= (i,j)\text{-entry of } \boldsymbol{A}+\boldsymbol{B}.
\end{aligned}$$

Hence, $(\boldsymbol{A}+\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}+\boldsymbol{B}$. □

13. Let k be a scalar and $\boldsymbol{A}$ be a nonzero matrix such that $\boldsymbol{A}^{\mathrm{T}} = k\boldsymbol{A}$. Prove that

(i) $\boldsymbol{A}$ is a square matrix.

(ii) $k = 1$ or $k = -1$.

Proof. (i) Suppose $\boldsymbol{A}$ is an $m \times n$ matrix. Then $\boldsymbol{A}^{\mathrm{T}}$ is $n \times m$ and $k\boldsymbol{A}$ is $m \times n$. It follows that $m = n$, i.e., $\boldsymbol{A}$ is a square matrix.

(ii) $\boldsymbol{A} = (\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = (k\boldsymbol{A})^{\mathrm{T}} = k\boldsymbol{A}^{\mathrm{T}} = k(k\boldsymbol{A}) = k^2\boldsymbol{A}$, i.e., $(1-k^2)\boldsymbol{A} = \mathbf{0}$.

Since $\boldsymbol{A} \neq \mathbf{0}$, we must have $1 - k^2 = 0$, i.e., $k = 1$ or $k = -1$. □