# Set Theory

Membership of a set : $\in$

Cardinality of a set : $|S|$ the number of elements in $S$

| Symbol | Meaning |
|---|---|
| $\mathbb{N}$ | The set of all natural numbers {0, 1, 2, 3,...}* |
| $\mathbb{Z}$ | The set of all integers |
| $\mathbb{Q}$ | The set of all rational numbers |
| $\mathbb{R}$ | The set of all real numbers |
| $\mathbb{C}$ | The set of all complex numbers |
| $\mathbb{Z}^+$ | The set of all positive integers |
| $\mathbb{Z}^-$ | The set of all negative integers |
| $\mathbb{Z}_{\ge 0}$ | The set of all non-negative integer |

$\mathbb{Q}^+, \mathbb{Q}^-, \mathbb{Q}_{\ge 0}$, $\mathbb{R}^+, \mathbb{R}^-, \mathbb{R}_{\ge 0}$, etc. are defined similarly.

*: In this module we define the set $\mathbb{N}$ to include zero.

⚠ Zero is neither positive nor negative.

Set-Roster Notation : Write all of its element between braces. $\{1, 2, 3\}$
Order and duplicates Do NOT MATTER
$\{9, 8, 7\} = \{7, 9, 8, 7\}$

Set-Builder Notation : $\{x \in U : P(x)\}$ or $\{x \in U \mid P(x)\}$

"The set of all $x$ in $U$ such that $P(x)$"

Replacement Notation : $\{t(x) : x \in A\}$ or $\{t(x) \mid x \in A\}$

"The set of all $t(x)$ where $x \in A$"

Subset : $A \subseteq B$ iff $\forall x\, (x \in A \Rightarrow x \in B)$

Proper subset : $A \subsetneq B \Leftrightarrow \exists x\, (x \in A \wedge x \notin B)$

Theorem 6.2.4 : An empty set is a subset of every set $\emptyset \subseteq A$ for all sets $A$

Singleton : A set with exactly one element

Ordered Pair : $(x, y)$ . $(a, b) = (c, d) \Leftrightarrow (a = c) \wedge (b = d)$

Ordered n-tuple : $(x_1, x_2, \cdots, x_n)$

Cartesian Product : $A \times B = \{(a, b) : a \in A \wedge b \in B\}$

Set equality : $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$

$A = B \Leftrightarrow \forall x\, (x \in A \Leftrightarrow x \in B)$

Let $A$ and $B$ be subset of a universal set $U$

Union : $A \cup B = \{x \in U : x \in A \vee x \in B\}$

Intersection : $A \cap B = \{x \in U : x \in A \wedge x \in B\}$

Difference : $B \setminus A = \{x \in U : x \in B \wedge x \notin A\}$

Complement : $\bar{A} = \{x \in U \mid x \notin A\}$

Intervals of real numbers : $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
$[a, b] = \{x \in \mathbb{R} : a \le x \le b\}$

$\bigcup_{i=0}^{n} A_i = A_0 \cup A_1 \cup \cdots \cup A_n$ , $\bigcap_{i=0}^{n} A_i = A_0 \cap A_1 \cap \cdots \cap A_n$

Disjoint : $A \cap B = \emptyset$

Mutually disjoint : $A_i \cap A_j = \emptyset$ whenever $i \neq j$

Theorem 4.4.1 The Quotient-Remainder Theorem : Given any int $n$ and pos int $d$, there exist unique int $q$ and $r$ such that
$n = dq + r$ and $0 \le r < d$

Power Sets : Given a set $A$, $\mathcal{P}(A)$ is the set of all subsets of $A$
$A = \{x, y\}$ $\mathcal{P}(A) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$

Theorem 6.3.1 : $|\mathcal{P}(A)| = 2^{|A|}$

Theorem 6.2.1 Inclusion of intersection $A \cap B \subseteq A$ $A \cap B \subseteq B$

Inclusion of Union $A \subseteq A \cup B$ $B \subseteq A \cup B$

Transitive Property of Subsets $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$

## Procedural Version of Set Definition

Let $X$ and $Y$ be subsets of a universal set $U$ and suppose $a$ and $b$ are elements of $U$.

1. $a \in X \cup Y \Leftrightarrow a \in X \vee a \in Y$
2. $a \in X \cap Y \Leftrightarrow a \in X \wedge a \in Y$
3. $a \in X - Y \Leftrightarrow a \in X \wedge a \notin Y$
4. $a \in \bar{X} \Leftrightarrow a \notin X$
5. $(a, b) \in X \times Y \Leftrightarrow a \in X \wedge b \in Y$

## Theorem 6.2.2 Set identities

Let all sets referred to below be subsets of a universal se

1. *Commutative Laws*: For all sets $A$ and $B$,
   (a) $A \cup B = B \cup A$ and (b) $A \cap B = B \cap A$.
2. *Associative Laws*: For all sets $A, B$ and $C$,
   (a) $(A \cup B) \cup C = A \cup (B \cup C)$ and (b) $(A \cap B) \cap C =$
3. *Distributive Laws*: For all sets $A, B$ and $C$,
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
4. *Identity Laws*: For all sets $A$,
   (a) $A \cup \emptyset = A$ and (b) $A \cap U = A$.
5. *Complement Laws*: For all sets $A$,
   (a) $A \cup \bar{A} = U$ and (b) $A \cap \bar{A} = \emptyset$.
6. *Double Complement Law*: For all sets $A$,
   $\bar{\bar{A}} = A$.
7. *Idempotent Laws*: For all sets $A$,
   (a) $A \cup A = A$ and (b) $A \cap A = A$.
8. *Universal Bound Laws*: For all sets $A$,
   (a) $A \cup U = U$ and (b) $A \cap \emptyset = \emptyset$.
9. *De Morgan's Laws*: For all sets $A$ and $B$,
   (a) $\overline{A \cup B} = \bar{A} \cap \bar{B}$ and (b) $\overline{A \cap B} = \bar{A} \cup \bar{B}$.
10. *Absorption Laws*: For all sets $A$ and $B$,
    (a) $A \cup (A \cap B) = A$ and (b) $A \cap (A \cup B) = A$.
11. *Complements of $U$ and $\emptyset$*:
    (a) $\bar{U} = \emptyset$ and (b) $\bar{\emptyset} = U$.
12. *Set Difference Law*: For all sets $A$ and $B$,
    $A \setminus B = A \cap \bar{B}$.

To prove set $A = B$, show $A \subseteq B$ and $B \subseteq A$