Student Number: ______________________________

NATIONAL UNIVERSITY OF SINGAPORE

MA1101R - Linear Algebra I

(Semester 1 : AY2014/2015)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number clearly in the space provided at the top of this page.** This booklet (and only this booklet) will be collected at the end of the examination.

2. Please write your matriculation/student number only. Do not write your name.

3. This examination paper contains **SIX** questions and comprises **NINETEEN** printed pages.

4. Answer **ALL** questions.

5. This is a CLOSED BOOK (with helpsheet) examination.

6. You are allowed to use two A4 size helpsheets.

7. You may use scientific calculators. However, you should lay out systematically the various steps in the calculations)

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
5	
6	
Total	

Question 1 [15 marks]

(a) [6 marks]

Let $\boldsymbol{A}$ be a 4×5 matrix such that its row echelon form is

$$\boldsymbol{R} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) Write down a basis for the row space of $\boldsymbol{A}$.

(ii) Extend the basis found in (i) to a basis for $\mathbb{R}^5$. (Just write down the additional vectors.)

(iii) Find a basis for the nullspace of $\boldsymbol{A}$. Show your working.

Show your working below.

(i) $\{(1,1,0,1,0),(0,0,1,1,0),(0,0,0,0,1)\}$

(ii) $\{(0,1,0,0,0),(0,0,0,1,0)\}$.

(There are many possible answers. Common ones are $\{(0,\oplus,*,*,*),(0,0,0,\otimes,*)\}$ where $\oplus,\otimes$ are non-zero.)

(iii) Let x_1,x_2,x_3,x_4,x_5 be the variables in the homogeneous system $\boldsymbol{Ax}=\boldsymbol{0}$.

To find the general solution of the system, set $x_2=t, x_4=s$.

Then $x_5=0, x_3=-s, x_1=-t-s$.

So $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -t-s \\ t \\ -s \\ s \\ 0 \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

A basis for nullspace of $\boldsymbol{A}$ is $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}\right\}$.

Question 1

(b) [6 marks]

Let

$$\boldsymbol{B} = \begin{pmatrix} x & x(x-1) & 0 \\ 0 & x-1 & (x-1)(x+1) \\ 0 & 0 & x+1 \end{pmatrix}.$$

Find all values of x such that:
(i) rank($\boldsymbol{B}$)=1; (ii) rank($\boldsymbol{B}$)=2; (iii) rank($\boldsymbol{B}$)=3.

Justify your answer.

Show your working below.

When $x \neq 0, 1, -1$, rank($\boldsymbol{B}$)=3 (as $\boldsymbol{B}$ is in row echelon form with 3 pivot columns).

When $x = 0$, then $\boldsymbol{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$. So rank($\boldsymbol{B}$)=2.

When $x = 1$, then $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. So rank($\boldsymbol{B}$)=2.

When $x = -1$, then $\boldsymbol{B} = \begin{pmatrix} -1 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So rank($\boldsymbol{B}$)=2.

(i) rank($\boldsymbol{B}$)=1: no such x.

(ii) rank($\boldsymbol{B}$)=2: $x = 0, 1$ or -1.

(iii) rank($\boldsymbol{B}$)=3: $x \neq 0, 1$ and -1.

Question 1

(c) [3 marks]

Give an example of a 3×4 matrix $\boldsymbol{C}$ with no identical columns such that the column space of $\boldsymbol{C}$ is span $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

(You are not required to justify your answer.)

What is the nullity of $\boldsymbol{C}$?

Show your working below.

There are plenty of examples, the condition is that all columns of $\boldsymbol{C}$ must be (distinct) scalar multiples of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

For example, $\boldsymbol{C} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$.

Nullity$(\boldsymbol{C}) = 4-$ rank$(\boldsymbol{C}) = 4 - 1 = 3$.

Question 2 [15 marks]

(a) [6 marks]

Let $V = \{(x, y, z, w) \mid x = y + z, w = 2y\}$ be a subspace of $\mathbb{R}^4$.

(i) Write down an explicit form of a general vector in V.

(ii) Express V in linear span form.

(iii) Write down a basis for V and $\dim V$.

Show your working below.

(i) One can directly write down $(y + z, y, z, 2y)$ with $y, z \in \mathbb{R}$.

Alternatively, one can also solve the two equations to get $(\frac{1}{2}t + s, \frac{1}{2}t, s, t)$ with $s, t \in \mathbb{R}$.

(There are many ways to express the general vector in V.)

(ii) For any $\boldsymbol{v} \in V$,

$\boldsymbol{v} = (y + z, y, z, 2y) = y(1, 1, 0, 2) + z(1, 0, 1, 0)$.

So $V = \text{span}\{(1, 1, 0, 2), (1, 0, 1, 0)\}$.

(There are many possible answers depending on the choice of the explicit form of the general vector.)

(iii) Basis for V : $\{(1, 1, 0, 2), (1, 0, 1, 0)\}$ with $\dim V = 2$.

(Again the basis may vary according to the answers obtained in (i) and (ii).)

Question 2

(b) [6 marks]

Let $S = \{\boldsymbol{u}, \boldsymbol{v}\}$ and $T = \{\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} + 2\boldsymbol{v}\}$ be two bases for a vector space U.

(i) Find the transition matrix from T to S.

(ii) Find the transition matrix from S to T.

(iii) Given the coordinate vector of $\boldsymbol{w} \in U$ with respect to T is $[\boldsymbol{w}]_T = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find $[\boldsymbol{w}]_S$.

Show your working below.

(i) $[\boldsymbol{u} - \boldsymbol{v}]_S = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$[\boldsymbol{u} + \boldsymbol{2v}]_S = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

So the transition matrix from T to S is $\boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$.

(ii) The transition matrix from S to T is $\boldsymbol{P}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$.

(iii) $[\boldsymbol{w}]_S = P[\boldsymbol{w}]_T = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$.

Question 2

(c) [3 marks]

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ be a basis for $\mathbb{R}^4$.

Suppose $U_1 = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $U_2 = \text{span}\{\boldsymbol{u}_3, \boldsymbol{u}_4\}$. Is it possible that

$$U_1 \cup U_2 = \mathbb{R}^4?$$

Justify your answer.

Show your working below.

No.
We have $\boldsymbol{u}_3 \notin U_1$ but $\boldsymbol{u}_1 \in U_1$. So $\boldsymbol{u}_1 + \boldsymbol{u}_3 \notin U_1$.
We have $\boldsymbol{u}_3 \in U_2$ but $\boldsymbol{u}_1 \notin U_2$. So $\boldsymbol{u}_1 + \boldsymbol{u}_3 \notin U_2$.
Hence $\boldsymbol{u}_1 + \boldsymbol{u}_3 \notin U_1 \cup U_2$.
So $U_1 \cup U_2 \neq \mathbb{R}^4$.
(Other justification possible.)

Question 3 [15 marks]

(a) [6 marks]

Find the least squares solutions of the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

and hence find the smallest possible value of $\|\boldsymbol{Ax} - \boldsymbol{b}\|$ from among all $\boldsymbol{x} \in \mathbb{R}^2$.

Show your working below.

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

$$\boldsymbol{A}^T\boldsymbol{b} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Solving $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$:

$$\boldsymbol{x} = \tfrac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{4}{3} \end{pmatrix}$$

So the least squares solution is $\begin{pmatrix} \frac{2}{3} \\ \frac{4}{3} \end{pmatrix}$.

The smallest $\|\boldsymbol{Ax} - \boldsymbol{b}\|$ is when $\boldsymbol{x} = \begin{pmatrix} \frac{2}{3} \\ \frac{4}{3} \end{pmatrix}$:

$$= \left\| \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{4}{3} \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{4}{3} \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} \right\| = \frac{1}{\sqrt{3}}.$$

Question 3

(b) [4 marks]

Let $\boldsymbol{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

(i) Find an orthogonal basis $\{\boldsymbol{u}', \boldsymbol{v}'\}$ for $V = \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ such that $\boldsymbol{u} = \boldsymbol{u}'$.

(ii) Find the projection of $\boldsymbol{w}$ onto the subspace V.

Show your working below.

(i) $\boldsymbol{u}' = \boldsymbol{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$$\boldsymbol{v}' = \boldsymbol{v} - \left(\frac{\boldsymbol{v} \cdot \boldsymbol{u}}{\boldsymbol{u} \cdot \boldsymbol{u}}\right) \boldsymbol{u}$$

$$= \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} - \left(\frac{-1}{3}\right) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}.$$

(Any non-zero scalar multiple of $\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ can be the answer for $\boldsymbol{v}'$.)

(ii)

$$\begin{aligned} \text{proj}_V \boldsymbol{w} &= \left(\frac{\boldsymbol{w} \cdot \boldsymbol{u}'}{\|\boldsymbol{u}'\|^2}\right) \boldsymbol{u}' + \left(\frac{\boldsymbol{w} \cdot \boldsymbol{v}'}{\|\boldsymbol{v}'\|^2}\right) \boldsymbol{v}' \\ &= \frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \end{aligned}$$

Question 3

(c) [5 marks]

Suppose the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

Show that the set $\left\{\frac{1}{\sqrt{2}}\boldsymbol{v}_1 - \frac{1}{\sqrt{2}}\boldsymbol{v}_2, \frac{1}{\sqrt{2}}\boldsymbol{v}_1 + \frac{1}{\sqrt{2}}\boldsymbol{v}_2, \boldsymbol{v}_3\right\}$ is also an orthonormal basis for $\mathbb{R}^3$.

Show your working below.

Just need to check (i) pairwise orthogonality; (ii) norm equals 1.

(i) $\left(\frac{1}{\sqrt{2}}\boldsymbol{v}_1 - \frac{1}{\sqrt{2}}\boldsymbol{v}_2\right) \cdot \left(\frac{1}{\sqrt{2}}\boldsymbol{v}_1 + \frac{1}{\sqrt{2}}\boldsymbol{v}_2\right) = \frac{1}{2}\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 - \frac{1}{2}\boldsymbol{v}_2 \cdot \boldsymbol{v}_2 = \frac{1}{2} - \frac{1}{2} = 0$
(since $\boldsymbol{v}_1, \boldsymbol{v}_2$ have norm 1).
$\left(\frac{1}{\sqrt{2}}\boldsymbol{v}_1 - \frac{1}{\sqrt{2}}\boldsymbol{v}_2\right) \cdot \boldsymbol{v}_3 = \frac{1}{\sqrt{2}}\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 - \frac{1}{\sqrt{2}}\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 0$
(since $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are orthogonal to each other).
Similarly,
$\left(\frac{1}{\sqrt{2}}\boldsymbol{v}_1 + \frac{1}{\sqrt{2}}\boldsymbol{v}_2\right) \cdot \boldsymbol{v}_3 = \frac{1}{\sqrt{2}}\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 + \frac{1}{\sqrt{2}}\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 0.$

(ii) $\left(\frac{1}{\sqrt{2}}\boldsymbol{v}_1 - \frac{1}{\sqrt{2}}\boldsymbol{v}_2\right) \cdot \left(\frac{1}{\sqrt{2}}\boldsymbol{v}_1 - \frac{1}{\sqrt{2}}\boldsymbol{v}_2\right) = \frac{1}{2}\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \frac{1}{2}\boldsymbol{v}_2 \cdot \boldsymbol{v}_2 = \frac{1}{2} - 0 + \frac{1}{2} = 1.$
So $\left\|\frac{1}{\sqrt{2}}\boldsymbol{v}_1 - \frac{1}{\sqrt{2}}\boldsymbol{v}_2\right\| = 1.$
$\left(\frac{1}{\sqrt{2}}\boldsymbol{v}_1 + \frac{1}{\sqrt{2}}\boldsymbol{v}_2\right) \cdot \left(\frac{1}{\sqrt{2}}\boldsymbol{v}_1 + \frac{1}{\sqrt{2}}\boldsymbol{v}_2\right) = \frac{1}{2}\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 + \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \frac{1}{2}\boldsymbol{v}_2 \cdot \boldsymbol{v}_2 = \frac{1}{2} + 0 + \frac{1}{2} = 1.$
So $\left\|\frac{1}{\sqrt{2}}\boldsymbol{v}_1 + \frac{1}{\sqrt{2}}\boldsymbol{v}_2\right\| = 1.$
$\|\boldsymbol{v}_3\| = 1$ is given.

Question 4 [15 marks]

(a) [6 marks]

Let $\boldsymbol{A}$ be the matrix $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(i) Find all the eigenvalues of $\boldsymbol{A}$. Explain how you get your answer.

(ii) Find a basis for the eigenspace of $\boldsymbol{A}$ associated with each of the eigenvalues. Show your working.

Show your working below.

(i) Since $\boldsymbol{A}$ is an upper triangular matrix, the eigenvalues are just the diagonal entries of $\boldsymbol{A}$, namely 1 and 2.

(ii) Eigenspace for $\lambda = 1$ (E_1):

$$(\boldsymbol{I} - \boldsymbol{A} \mid \boldsymbol{0}) = \left(\begin{array}{cccc|c} 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = s, x_4 = t, x_3 = -t, x_2 = 0.$

So $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ -t \\ t \end{pmatrix} = s \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$.

So a basis for E_1 is $\{(1,0,0,0)^T, (0,0,-1,1)^T\}$.

Eigenspace for $\lambda = 2$ (E_2):

$$(2\boldsymbol{I} - \boldsymbol{A} \mid \boldsymbol{0}) = \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$x_2 = s, x_3 = t, x_4 = 0, x_1 = s.$

So $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} s \\ s \\ t \\ 0 \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

So a basis for E_2 is $\{(1,1,0,0)^T, (0,0,1,0)^T\}$.

Question 4

(b) [4 marks]

Suppose $\boldsymbol{B}$ is a 2×2 matrix such that

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \boldsymbol{B} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}.$$

Find a matrix $\boldsymbol{C}$ such that $\boldsymbol{C}^2 = \boldsymbol{B}$.

Explain how you obtain your answer.

Show your working below.

Since $\boldsymbol{C}$ is a square root of $\boldsymbol{B}$, the eigenvalues of $\boldsymbol{C}$ must be square roots of eigenvalues of $\boldsymbol{B}$ with the same corresponding eigenvectors.
So we may set

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \boldsymbol{C} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Note that the diagonal entries on the right hand side can be ± 2 and ± 1.)
So

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}.$$

Question 4

(c) [5 marks]

Let $\boldsymbol{M}$ be a non-invertible 3×3 symmetric matrix such that

$$\boldsymbol{M}\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}2\\2\\0\end{pmatrix} \quad \text{and} \quad \boldsymbol{M}\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}-1\\1\\-1\end{pmatrix}.$$

What are the eigenvalues of $\boldsymbol{M}$?

Write down a basis for $\mathbb{R}^3$ consisting entirely of eigenvectors of $\boldsymbol{M}$.

Justify your answers.

Show your working below.

$\boldsymbol{M}$ is not invertible, so we deduce that 0 is an eigenvalue of $\boldsymbol{M}$.

From $\boldsymbol{M}\begin{pmatrix}1\\1\\0\end{pmatrix}=2\begin{pmatrix}1\\1\\0\end{pmatrix}$, we get 2 is an eigenvalue of $\boldsymbol{M}$ with eigenvector $\boldsymbol{v}_1=\begin{pmatrix}1\\1\\0\end{pmatrix}$.

From $\boldsymbol{M}\begin{pmatrix}1\\-1\\1\end{pmatrix}=-1\begin{pmatrix}1\\-1\\1\end{pmatrix}$,

we get -1 is an eigenvalue of $\boldsymbol{M}$ with eigenvector $\boldsymbol{v}_2=\begin{pmatrix}1\\-1\\1\end{pmatrix}$.

So all the eigenvalues of $\boldsymbol{M}$ are $2, -1, 0$.

To find the required basis, we need to find an eigenvector associated to the eigenvalue 0.

Let such a vector be $v_3=\begin{pmatrix}a\\b\\c\end{pmatrix}$.

Since $\boldsymbol{M}$ is symmetric, the eigenvectors associated with distinct eigenvalues are orthogonal to each other.

So $\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = 0$ which gives $a+b=0$ and hence $a=-b$.

Also $\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 0$ which gives $a-b+c=0$ and hence $c=b-a=2b$.

So $v_3=\begin{pmatrix}-b\\b\\2b\end{pmatrix}=b\begin{pmatrix}-1\\1\\2\end{pmatrix}$. We may simply let it be $\begin{pmatrix}-1\\1\\2\end{pmatrix}$.

Hence a basis consisting entirely of eigenvectors can be given by

$$\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\2\end{pmatrix}\right\}$$

Question 5 [15 marks]

(a) [6 marks]

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ x+y \\ x-y \end{pmatrix}.$$

(i) Write down the standard matrix for T.

(ii) Find the kernel of T. Show your working.

(iii) Suppose $S : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation with standard matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$. Write down the formula for the composition $S \circ T$.

Show your working below.

(i) $\begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$.

(ii) $\begin{pmatrix} 2x \\ x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ giving $x = 0, y = 0$.

So $\ker(T) = \{\mathbf{0}\}$.

(iii) The standard matrix of $S \circ T$ is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}.$$

So $S \circ T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$S \circ T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x+y \\ 2y \end{pmatrix}.$$

Question 5

(b) [4 marks]

Given that $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation, P is a plane in $\mathbb{R}^3$ given by the equation $x + y + z = 0$, and ℓ is a line in $\mathbb{R}^3$ given by the set $\{(t, t, t) \mid t \in \mathbb{R}\}$.

Suppose F maps the plane P onto the line ℓ and maps the line ℓ to the origin.

Show that the linear transformation F^2 (i.e. $F \circ F$) is the zero transformation.

Show your working below.

Take a basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ for the plane P.
Take the basis $\{\boldsymbol{u}_3\}$ for the line ℓ where $\boldsymbol{u}_3 = (1, 1, 1)$.
Note that $\boldsymbol{u}_3$ is a normal vector to the plane P, and hence $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a basis for $\mathbb{R}^3$.
Now $F(\boldsymbol{u}_1) = k_1\boldsymbol{u}_3$ (since F maps P to ℓ)
and $F^2(\boldsymbol{u}_1) = k_1F(\boldsymbol{u}_3) = 0$ (since F maps ℓ to $\mathbf{0}$).
Similarly $F(\boldsymbol{u}_2) = k_2\boldsymbol{u}_3$
and $F^2(\boldsymbol{u}_2) = k_2F(\boldsymbol{u}_3) = 0$.
Also $F^2(\boldsymbol{u}_3) = 0$.
Since F^2 maps a basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ for $\mathbb{R}^3$ to $\mathbf{0}$, it maps every vector in $\mathbb{R}^3$ to $\mathbf{0}$.
Hence F^2 must be the zero transformation.

Question 5

(c) [5 marks]

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation.

Suppose $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a basis for $\mathbb{R}^3$, and $\{T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), T(\boldsymbol{u}_3)\}$ spans $\mathbb{R}^2$.

Show that the standard matrix of T is of full rank.

Show your working below.

Let the standard matrix of T be $\boldsymbol{A}$, a 2×3 matrix.
Let $\boldsymbol{P} = (\boldsymbol{u}_1 \;\; \boldsymbol{u}_2 \;\; \boldsymbol{u}_3)$, i.e. the 3×3 matrix formed by the 3 column vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.
$\boldsymbol{P}$ is invertible since $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ are linearly independent.
Now let $\boldsymbol{Q} = (T(\boldsymbol{u}_1) \;\; T(\boldsymbol{u}_2) \;\; T(\boldsymbol{u}_3)) = (\boldsymbol{A}\boldsymbol{u}_1 \;\; \boldsymbol{A}\boldsymbol{u}_2 \;\; \boldsymbol{A}\boldsymbol{u}_3) = \boldsymbol{A}(\boldsymbol{u}_1 \;\; \boldsymbol{u}_2 \;\; \boldsymbol{u}_3) = \boldsymbol{A}\boldsymbol{P}$.
Since $\{T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), T(\boldsymbol{u}_3)\}$ spans $\mathbb{R}^2$, we have $\text{rank}(\boldsymbol{Q}) = 2$.
i.e. $\text{rank}(\boldsymbol{A}\boldsymbol{P}) = 2$, which implies $\text{rank}(\boldsymbol{A}) = 2$, since $\boldsymbol{P}$ is invertible.
Hence $\boldsymbol{A}$ is full rank.

Question 6 [15 marks]

Determine whether each of the following parts is true or false. Justify your answer.

(a) [3 marks]

If $\boldsymbol{A}$ is a square matrix, then its row space is equal to its column space.

Show your working below.

False.

Example $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

The row space is $\text{span}\{(1,0)\}$ while the column space is $\text{span}\{(1,1)\}$.

(b) [3 marks]

The set $W = \{(a,b,c,abc) \mid a,b,c \in \mathbb{R}\}$ is not a subspace of $\mathbb{R}^4$.

Show your working below.

True.

W is not closed under vector addition, and hence not a subspace of $\mathbb{R}^4$.

For example, take $(1,1,0,0)$ and $(0,0,1,0) \in W$.

The sum $(1,1,1,0) \notin W$.

(W is also not closed under scalar multiplication.)

Question 6

(c) [3 marks]

Let $\boldsymbol{u}, \boldsymbol{v}$ be non-zero vectors in some vector space.
If $\text{span}\{\boldsymbol{u}\} \cap \text{span}\{\boldsymbol{v}\} = \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$, then $\text{span}\{\boldsymbol{u}\} = \text{span}\{\boldsymbol{v}\}$.

Show your working below.

True.
$\text{span}\{\boldsymbol{u}\} \cap \text{span}\{\boldsymbol{v}\} \subseteq \text{span}\{\boldsymbol{u}\} \subseteq \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ (i).
$\text{span}\{\boldsymbol{u}\} \cap \text{span}\{\boldsymbol{v}\} \subseteq \text{span}\{\boldsymbol{v}\} \subseteq \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ (ii).
Since the two sets on the extreme left and right of both (i) and (ii) are equal, the intermediate set must also be equal to these sets.
Hence $\text{span}\{\boldsymbol{u}\} = \text{span}\{\boldsymbol{v}\}$.

(d) [3 marks]

There is no 3×3 matrix of rank 2 with only 1 eigenvalue.

Show your working below.

False.
Example $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.
The matrix has only one eigenvalue 0 and has rank 2.

Question 6

(e) [3 marks]

Let $\boldsymbol{A}$ be a 3×2 matrix with two columns $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$, and $\boldsymbol{b}$ is a non-zero 3×1 column vector orthogonal to $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$. Then the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is inconsistent.

Show your working below.

True.
Since $\boldsymbol{b}$ is orthogonal to $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$, so $\boldsymbol{b} \notin \mathrm{span}\{\boldsymbol{c}_1, \boldsymbol{c}_2\}$.
This implies $\boldsymbol{b}$ is not in the column space of $\boldsymbol{A}$.
Hence $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is inconsistent.