CS2040S
Data Structures and Algorithms

Hashing! (Part 2)

Puzzle of the Week:

You throw a dice repeatedly until you get a 6.

Conditioned on the event that all throws gave even numbers, what is the expected number of throws (including the throw giving 6)?

Plan: this week and next

Second day of hashing

- Analysis of chaining
  - Some probability and expectations
- Hashing in Java
- Designing Hash Functions
- Open Addressing

Review: Symbol Table Abstract Data Type

Which of the following is *not* typically a symbol table operation?

1. insert(key, data)
2. delete(key)
3. successor(key)
4. search(key)
5. None of the above.

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Review: Symbol Table Abstract Data Type

Which of the following is *not* typically a symbol table operation?

1. insert(key, data)
2. delete(key)
3. successor(key)
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5. None of the above.

Abstract Data Types

Symbol Table

```
public interface  SymbolTable
```

`void`	`insert(Key k, Value v)`	*insert (k,v) into table*
`Value`	`search(Key k)`	*get value paired with k*
`void`	`delete(Key k)`	*remove key k (and value)*
`boolean`	`contains(Key k)`	*is there a value for k?*
`int`	`size()`	*number of (k,v) pairs*

Note: no successor / predecessor queries.

Direct Access Tables

Attempt #1: Use a table, indexed by keys.

0	null
1	null
2	item1
3	null
4	null
5	item3
6	null
7	null
8	item2
9	null

Universe U={0..9} of size $m = 10$.

(key, value)

(2, item1)
(8, item2)
(5, item3)

Assume keys are distinct.

Direct Access Tables

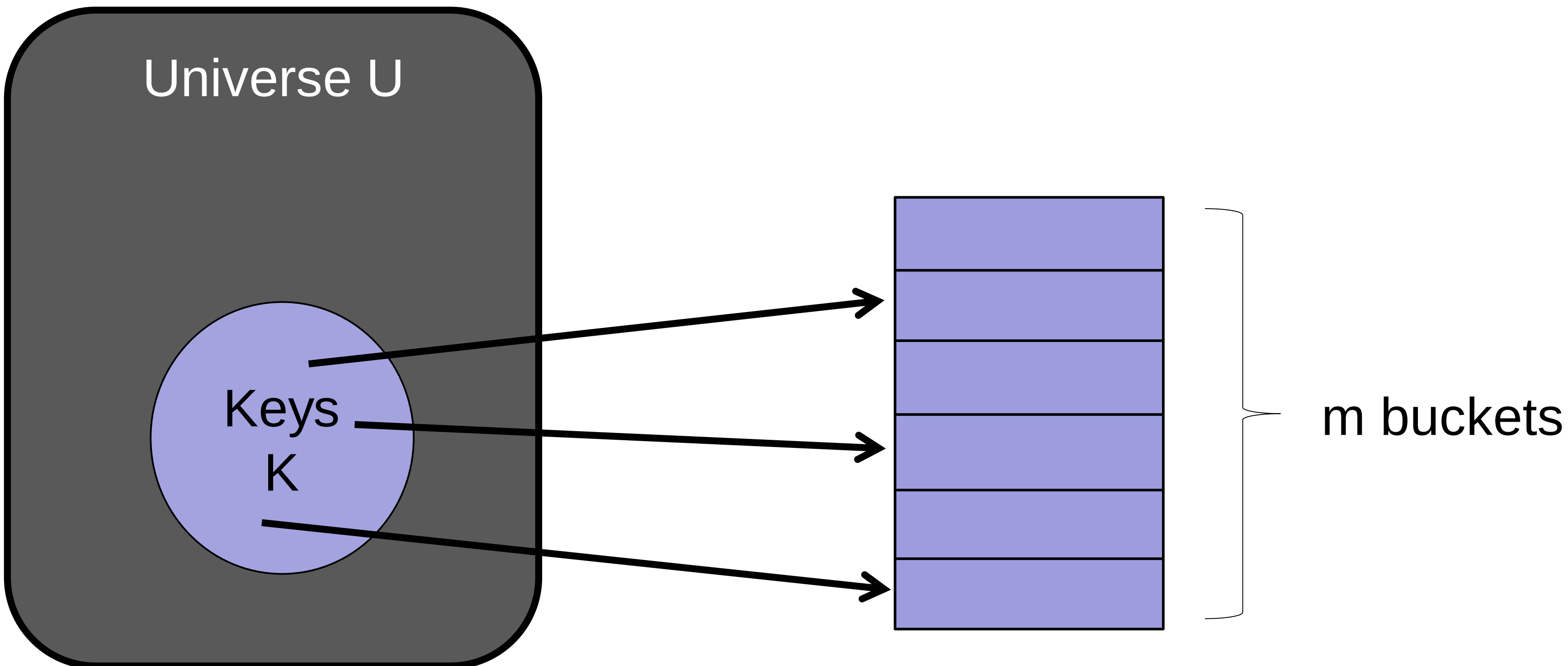
Problems:

- Too much space
  - If keys are integers, then table-size > 4 billion
- What if keys are not integers?
  - Where do you put the key/value “**(hippopotamus, bob)**”?
  - Where do you put 3.14159…?

Hash Functions

Problem:

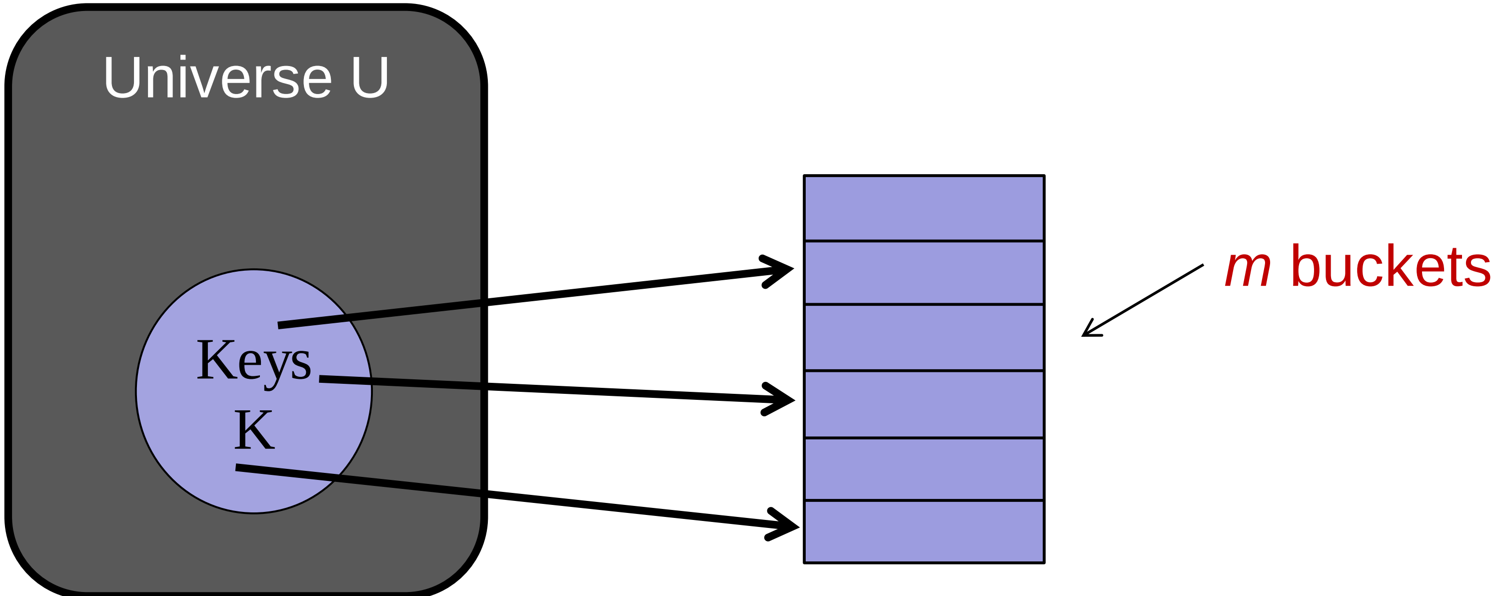
- Huge universe U of possible keys.
- Smaller number n of actual keys.
- How to map n keys to $m \approx n$ buckets?



Hash Functions

Define hash function $h : U \rightarrow \{1..m\}$

– Store key k in bucket $h(k)$.



Hash Functions

Collisions:

- We say that two distinct keys k_1 and k_2 collide if:

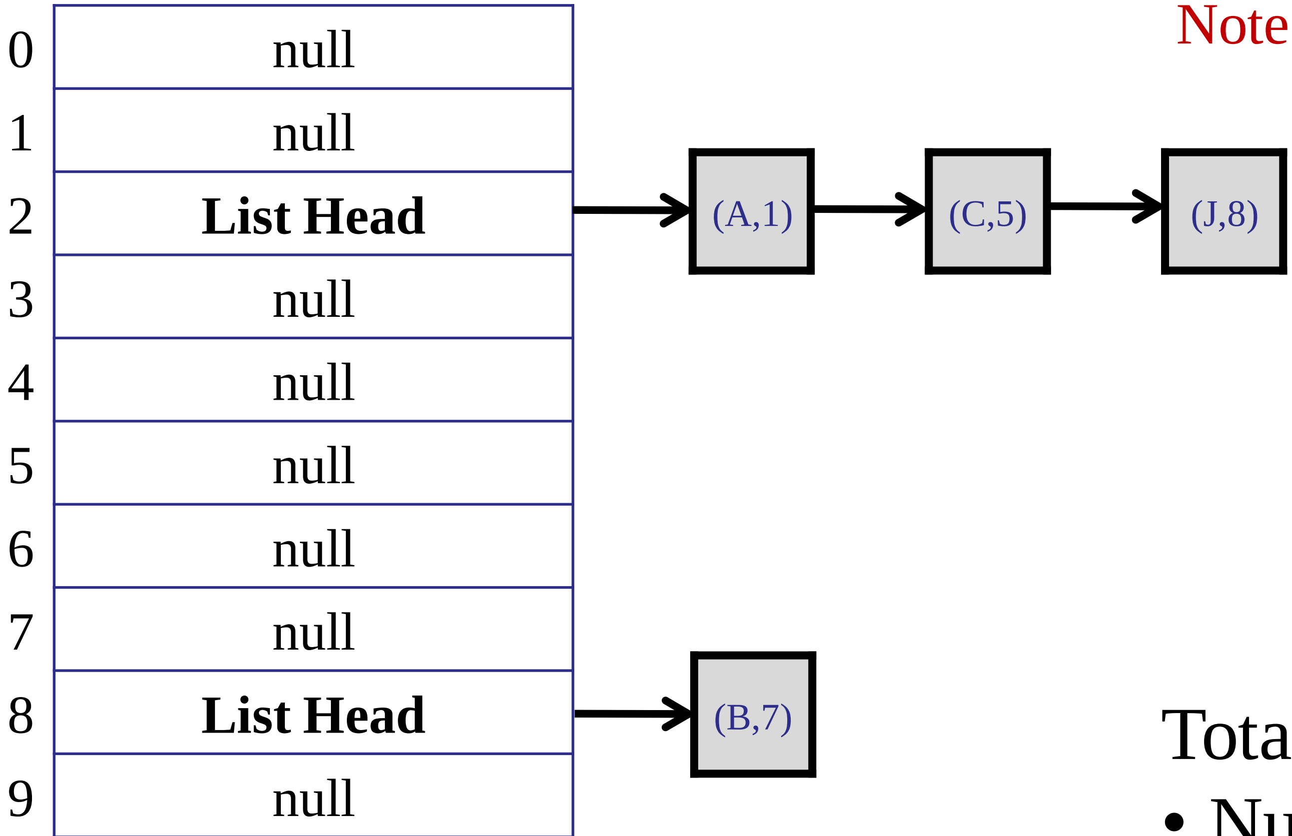
$$h(k_1) = h(k_2)$$

- Unavoidable!
  - The table size is smaller than the universe size.
  - The pigeonhole principle says:
    - There must exist two keys that map to the same bucket.
    - Some keys must collide!

Chaining

Each bucket contains a linked list of (key, val) pairs.

Note: h(A) == h(C) == h(J)

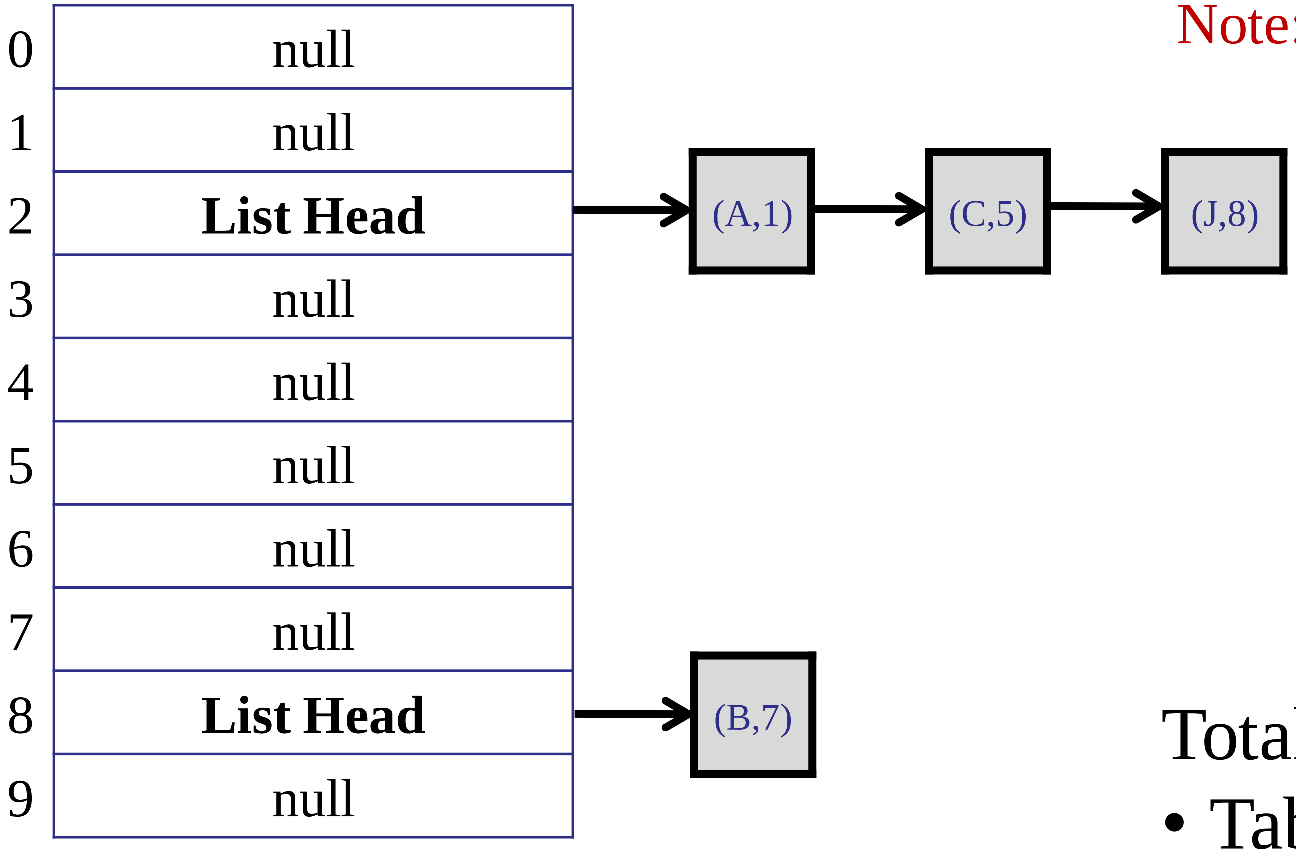


Total space:

- Number buckets: m
- Number entries: n

Chaining

Each bucket contains a linked list of (key, val) pairs.



Note: h(A) == h(C) == h(J)

Total space: $O(m + n)$
- Table size: m
- Linked list size: n

Hashing with Chaining

Operations:

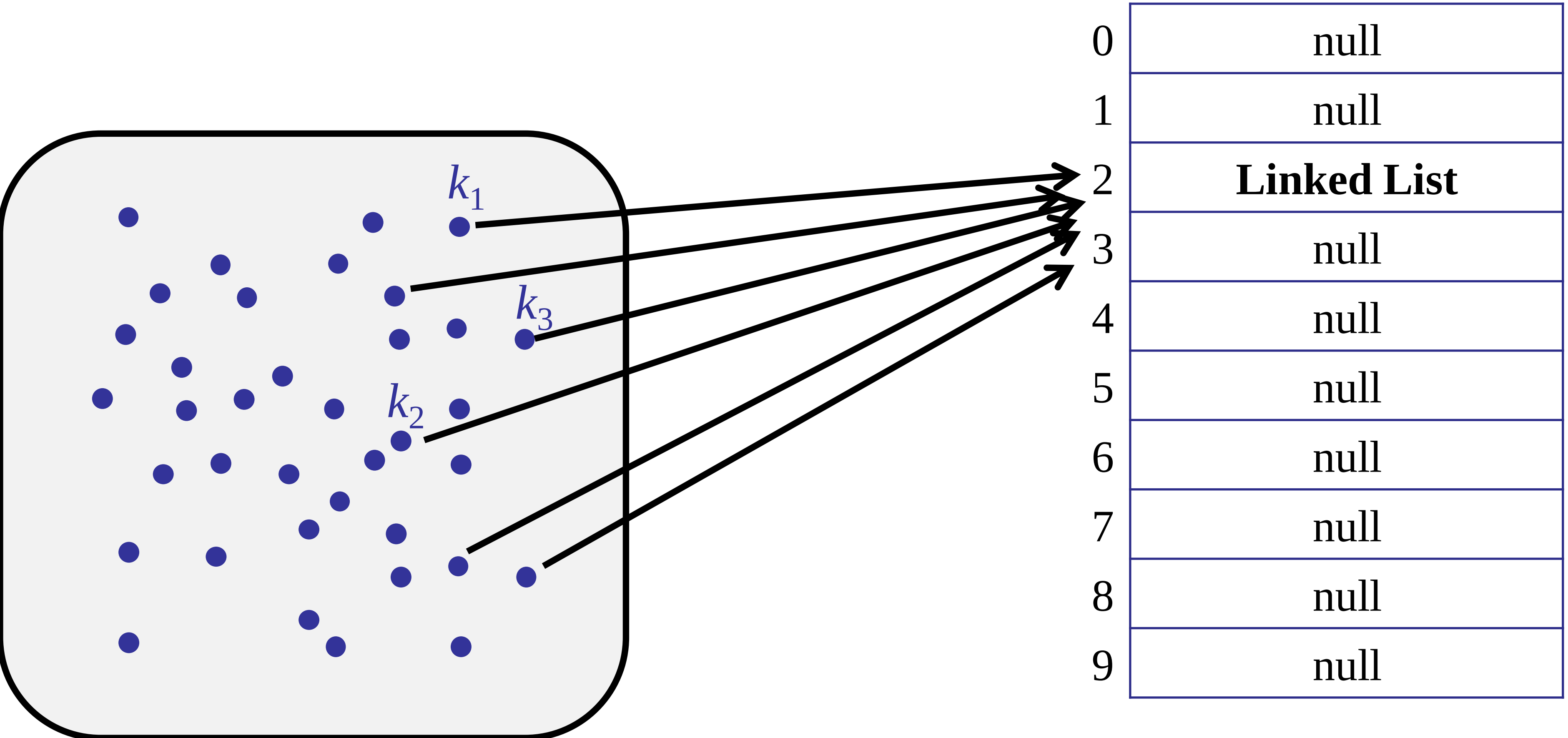
- insert(key, value)
  - Calculate h(key)
  - Lookup h(key) and add (key,value) to the linked list.

- search(key)
  - Calculate h(key)
  - Search for (key,value) in the linked list.

Hashing with Chaining

What if all keys hash to the same bucket!

- Worst-case search costs O(n)
- Oh no!



Let's be optimistic today.

The <u>Simple Uniform Hashing</u> Assumption

- Every key is equally likely to map to every bucket.
- Keys are mapped independently.

Assume hash function has this property, even if it may not!

Intuition:

- Each key is put in a random bucket.
- Then, as long as there are enough buckets, we won't get too many keys in any one bucket.

Why don’t we just insert each key into a random bucket (instead of using h)?

Searching would be very slow. How do you find the item?

Let's be optimistic today.

The Simple Uniform Hashing Assumption

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Intuition:

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- Then, as long as there are enough buckets, we won't get too many keys in any one bucket.

Let's be optimistic today.

The Simple Uniform Hashing Assumption

- Assume:
  - n items
  - m buckets
- Define: load(hash table) = n/m

  = average # items / bucket.

- **Claim**: Expected search time = O(1) + load(hash table)

hash function + array access

linked list traversal

Probability Theory

Set of outcomes for $X = (e_1, e_2, e_3, \ldots, e_k)$:

- $\Pr(e_1) = p_1$
- $\Pr(e_2) = p_2$
- …
- $\Pr(e_k) = p_k$

Expected outcome:

$$E[X] = e_1p_1 + e_2p_2 + \ldots + e_kp_k$$

Probability Theory

Linearity of Expectation:

- $E[A + B] = E[A] + E[B]$

Example:

- A = # heads in 2 coin flips
- B = # heads in 2 coin flips
- A + B = # heads in 4 coin flips

$E[A+B] = E[A] + E[B] = 1 + 1 = 2$

Let's be optimistic today.

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  - m buckets
- Define: load(hash table) = n/m

  = average # items / bucket.

- **Claim**: Expected search time = $O(1)$ + load(hash table)

hash function + array access

linked list traversal

A little more probability

Indicator random variables

$X(i, j) = 1$ if i'th key is put in bucket j

$= 0$ otherwise

With SUHA, Pr(X(i, j) == 1) = ?

1. 1/m
2. 1/n
3. 1/(m+n)
4. m/n
5. n/m
6. log(n)

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Let's be optimistic today.

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- Every key is equally likely to map to every bucket.
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A little probability

Indicator random variables

$X(i, j) = 1$ if i'th key is put in bucket j

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$\mathbf{Pr}(X(i, j) == 1) = 1/m$

A little probability

Indicator random variables

$X(i, j) = 1$ if item i is put in bucket j

$\quad\quad\quad = 0$ otherwise

$\mathbf{Pr}(X(i, j) == 1) = 1/m$

$\mathbf{E}(X(i, j)) = \mathbf{??}$

A little probability

Indicator random variables

$X(i, j) = 1$ if item i is put in bucket j

$= 0$ otherwise

$\mathbf{Pr}(X(i, j)==1) = 1/m$

$\mathbf{E}(X(i, j)) = \mathbf{Pr}(X(i, j)==1)*1 + \mathbf{Pr}(X(i, j)==0)*0$

$= \mathbf{Pr}(X(i, j)==1)$

$= 1/m$

A little probability

Indicator random variables

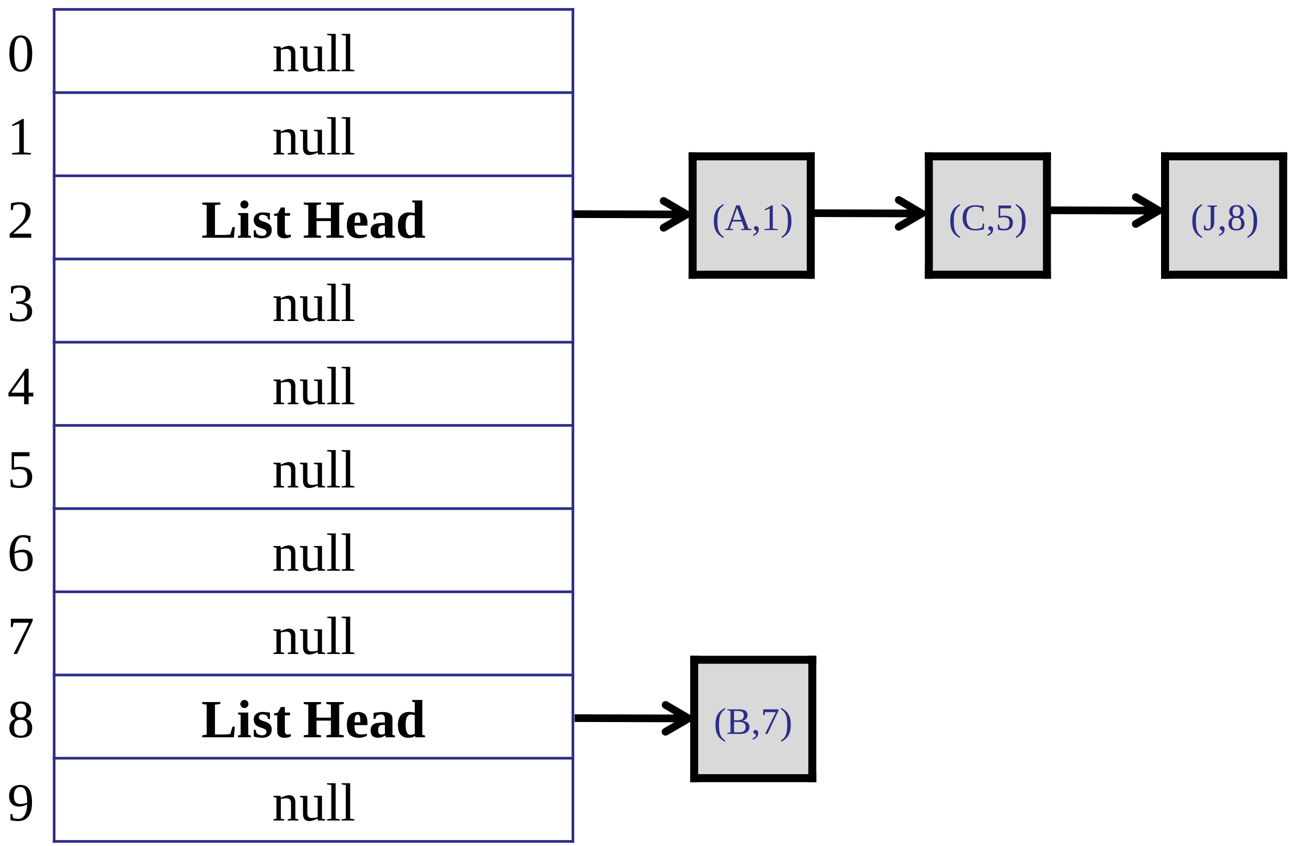
$X(i, j) = 1$ if i'th key is put in bucket j

$= 0$ otherwise

$\Sigma_i X(i, b) =$ number of items in bucket b

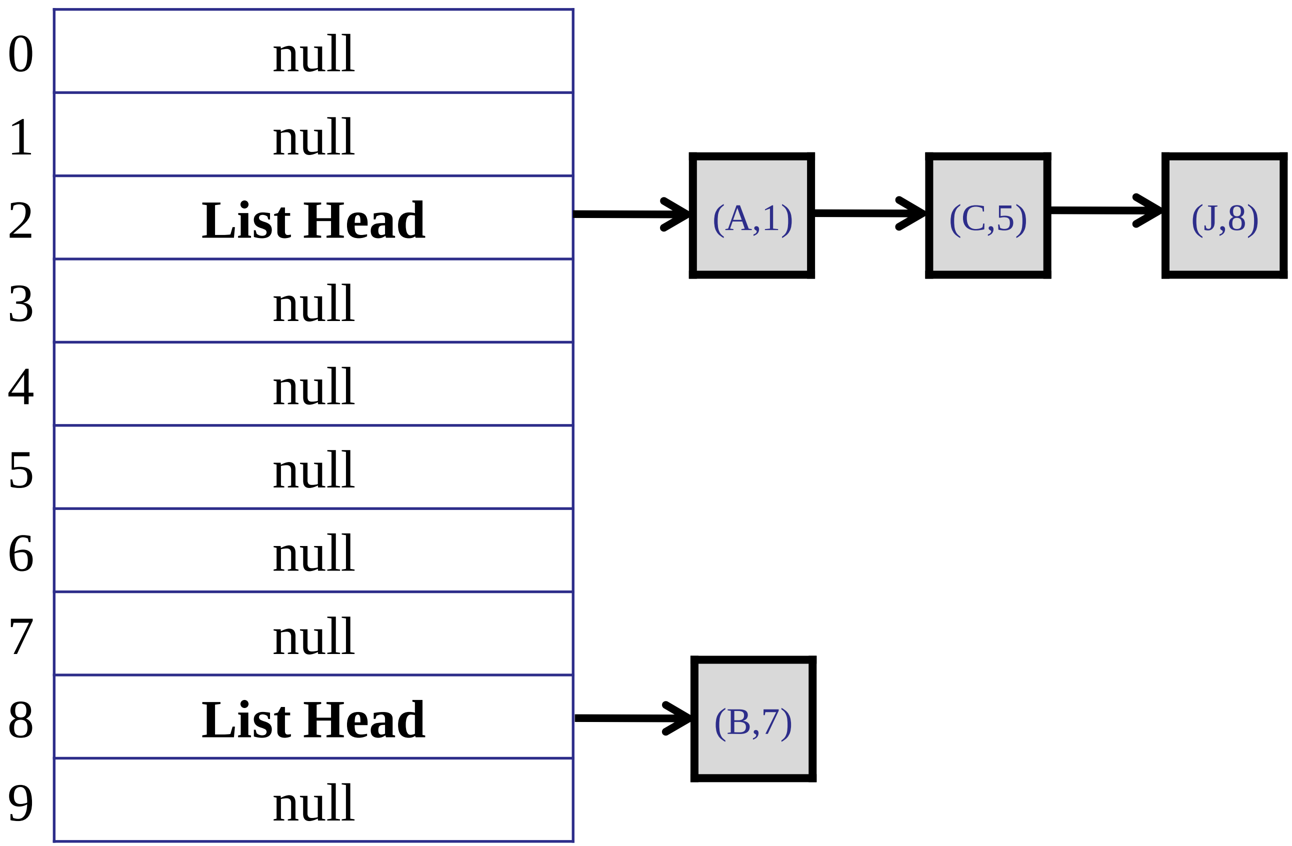
A little probability

Each item contributes `1’ to the bucket it is in..



A little probability

What is the expected cost for search?



search(D)

search(J)

Unsuccessful Search

Suppose search is for key k, and the hash table does not contain k.

Let $b = h(k)$.

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$$\mathbf{E}\,(\text{chain length at bucket b})$$
$$= \mathbf{E}\,(\textstyle\sum_i X(i, b))$$

Unsuccessful Search

Suppose search is for key k, and the hash table does not contain k.

Let $b = h(k)$.

Linearity of expectation:
E(A + B) = E(A) + E(B)

$$\mathbf{E}\,(\text{chain length at bucket b})$$
$$= \mathbf{E}\,(\Sigma_i \, X(i, b))$$
$$= \Sigma_i \, \mathbf{E}(X(i,b))$$

Unsuccessful Search

Suppose search is for key k, and the hash table does not contain k.

Let $b = h(k)$.

$$\mathbf{E}\,(\text{chain length at bucket b})$$
$$= \mathbf{E}\,(\Sigma_i \, X(i, b))$$
$$= \Sigma_i \, \mathbf{E}(X(i,b))$$
$$= \Sigma_i \, 1/m = n/m = \alpha.$$

Successful Search

Suppose search is for the t'th inserted key, k_t.

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$$= 1 + \textstyle\sum_{i \neq t} \mathbf{E}(\mathrm{X(i, b)})$$

Successful Search

Suppose search is for the t'th inserted key, k_t.

Let $b = h(k_t)$. We know that the b'th bucket contains at least one key, k_t.

E (chain length at bucket b)

$$\leq 1 + \mathbf{E}\left(\sum_{i \neq t} X(i, b)\right)$$

$$= 1 + \sum_{i \neq t} \mathbf{E}(X(i, b))$$

$$= 1 + \sum_{i \neq t} 1/m$$

Successful Search

Suppose search is for the t'th inserted key, k_t.

Let $b = h(k_t)$. We know that the b'th bucket contains at least one key, k_t.

$$
\begin{aligned}
\mathbf{E}\,(\text{chain length at bucket b})& \\
&\leq 1 + \mathbf{E}\,\left(\textstyle\sum_{i \neq t} \mathrm{X}(\mathrm{i}, \mathrm{b})\right) \\
&= 1 + \textstyle\sum_{i \neq t} \mathbf{E}(\mathrm{X}(\mathrm{i}, \mathrm{b})) \\
&= 1 + \textstyle\sum_{i \neq t} 1/\mathrm{m} \\
&= 1 + (n-1)/m \leq 1 + \alpha
\end{aligned}
$$

Let's be optimistic today.

The Simple Uniform Hashing Assumption

- Assume:
  - n items
  - $m = \Omega(n)$ buckets, e.g., $m = 2n$

- Expected search time $= O(1) + n/m$

$$= O(1)$$

Hashing with Chaining

Searching:

- Expected search time = 1 + n/m = O(1), with SUHA
- Worst-case search time = O(n)

Inserting:

- Worst-case insertion time = O(1)

Hashing with Chaining

What is the expected *maximum* chain length, with SUHA?

Hashing with Chaining

What is the expected *maximum* chain length, with SUHA?

– Analogy:
  - Throw n balls in m = n bins.
  - What is the maximum number of balls in a bin?

Cost: O(log n)

Hashing with Chaining

What is the expected *maximum* chain length, with SUHA?

- Analogy:
  - Throw n balls in m = n bins.
  - What is the maximum number of balls in a bin?

Cost: $\Theta(\log n / \log\log n)$

Hashing: Recap

Problem: coping with large universe of keys

- Number of possible keys is very, very large.
- Direct Access Table takes too much space

Hash functions

- Use hash function to map keys to buckets.
- Sometimes, keys collide (inevitably!)
- Use linked list to store multiple keys in one bucket.

Analyze performance with simple uniform hashing.

- Expected number of keys / bucket is O(n/m) = O(1).

Today

- Analysis of chaining
- **Java hashing**
- Designing hash functions
- Collision resolution: open addressing
- Table (re)sizing

Symbol Tables in Java

Symbol Tables in Java

java.util.Map

public interface java.util.Map<Key, Value>

void	clear()	*removes all entries*
boolean	containsKey(Object k)	*is k in the map?*
boolean	containsValue(Object v)	*is v in the map?*
Value	get(Object k)	*get value for k*
Value	put(Key k, Value v)	*adds (k,v) to table*
Value	remove(Object k)	*remove mapping for k*
int	size()	*number of entries*

Note: no successor / predecessor queries.

Symbol Tables in Java

java.util.Map

- Parameterized by key and value.
- Not necessarily comparable

```
public interface   java.util.Map<Key, Value>

         void   clear()                       removes all entries
      boolean   containsKey(Object k)         is k in the map?
      boolean   containsValue(Object v)       is v in the map?
        Value   get(Object k)                 get value for k
        Value   put(Key k, Value v)           adds (k,v) to table
        Value   remove(Object k)              remove mapping for k
          int   size()                        number of entries
```

Note: no successor / predecessor queries.

Symbol Tables in Java

java.util.Map

• Search by key.

public interface java.util.Map<Key, Value>

void	clear()	*removes all entries*
boolean	containsKey(Object k)	*is k in the map?*
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Value	remove(Object k)	*remove mapping for k*
int	size()	*number of entries*

Note: no successor / predecessor queries.

Symbol Tables in Java

java.util.Map

- Search by key.
- Search by value.

(May not be efficient.)

```
public interface  java.util.Map<Key, Value>
```

`void`	`clear()`	*removes all entries*
`boolean`	`containsKey(Object k)`	*is k in the map?*
`boolean`	`containsValue(Object v)`	*is v in the map?*
`Value`	`get(Object k)`	*get value for k*
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`Value`	`remove(Object k)`	*remove mapping for k*
`int`	`size()`	*number of entries*

Note: no successor / predecessor queries.

Symbol Tables in Java

java.util.Map

• Can use any Object as key?

```
public interface  java.util.Map<Key, Value>
```

void	clear()	*removes all entries*
boolean	containsKey(Object k)	*is k in the map?*
boolean	containsValue(Object v)	*is v in the map?*
Value	get(Object k)	*get value for k*
Value	put(Key k, Value v)	*adds (k,v) to table*
Value	remove(Object k)	*remove mapping for k*
int	size()	*number of entries*

Note: no successor / predecessor queries.

Symbol Tables in Java

java.util.Map

• Put new (key, value) in table.

`public interface java.util.Map<Key, Value>`

`void`	`clear()`	*removes all entries*
`boolean`	`containsKey(Object k)`	*is k in the map?*
`boolean`	`containsValue(Object v)`	*is v in the map?*
`Value`	`get(Object k)`	*get value for k*
`Value`	`put(Key k, Value v)`	*adds (k,v) to table*
`Value`	`remove(Object k)`	*remove mapping for k*
`int`	`size()`	*number of entries*

Note: no successor / predecessor queries.

Map Interface in Java

java.util.Map<Key, Value>

- No duplicate keys allowed.
- No *mutable* keys
  - If you use an *object* as a key, then you can’t modify that object later.

Symbol Table

What time does this plane depart at?

Key Mutability

```
SymbolTable<Time, Plane> t =
           new SymbolTable<Time, Plane>();

Time  t1 = new Time(9:00);
Time  t2 = new Time(9:15);

t.insert(t1, "SQ0001");
t.insert(t2, "SQ0002");

t1.setTime(10:00);

x = new Time(9:00);
t.search(x);
```

Symbol Table

Moral: Keys should be immutable.

Examples: Integer, String

Key Mutability

```
SymbolTable<Time, Plane> t =
                  new SymbolTable<Time, Plane>();

Time   t1 = new Time(9:00);
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t1.setTime(10:00);

x = new Time(9:00);
t.search(x);
```

Symbol Tables in Java

java.util.Map

	`public interface`	**`java.util.Map<Key, Value>`**
`Set<Map.Entry<Key, Value>`	`entrySet()`	*set of all mappings*
`Set<Key>`	`keySet()`	*set of all keys*
`Collection<Value>`	`values()`	*collection of all values*

Note: not sorted

not necessarily efficient to work with these sets/collections.

What is wrong here?

Example:

There is a bug here!

```
Map<String, Integer> ageMap = new Map<String, Integer>();

ageMap.put(“Alice”, 32);
ageMap.put(“Bernice”, 84);
ageMap.put(“Charlie”, 7);

Integer age = ageMap.get(“Alice”)
```

- Key-type: String
- Value-type: Integer

What is wrong here?

Example:

Map is an interface!
Cannot instantiate an interface.

```
Map<String, Integer> ageMap = new Map<String, Integer>();

ageMap.put("Alice", 32);
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Integer age = ageMap.get("Alice")
```

- Key-type: String
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Map Class in Java

Example: HashMap

```
Map<String, Integer> ageMap = new HashMap<String, Integer>();

ageMap.put("Alice", 32);
ageMap.put("Bernice", 84);
ageMap.put("Charlie", 7);

Integer age = ageMap.get("Alice");
System.out.println("Alice's age is: " + age + ".");
```

- Key-type: String
- Value-type: Integer

Map Class in Java

Example: HashMap

```
Map<String, Integer> ageMap = new HashMap<String, Integer>();

ageMap.put("Alice", 32);
ageMap.put("Bernice", null);
ageMap.put("Charlie", 7);

Integer age = ageMap.get("Bob");
if (age==null){
    System.out.println("Bob's age is unknown.");
}
```

- Returns “null” when key is not in map.
- Returns “null” when value is null.

Map Classes in Java

HashMap

Symbol Table

- containsKey
- containsValue
- entrySet
- get
- isEmpty
- keySet
- put
- putAll
- remove
- values

TreeMap

Dictionary

- containsKey
- containsValue
- entrySet
- get
- isEmpty
- keySet
- put
- putAll
- remove
- values

Map Classes in Java

HashMap

Symbol Table

TreeMap

Dictionary

- ceilingEntry
- ceilingKey
- descendingKeySet
- firstEntry
- firstKey
- floorEntry
- floorKey
- headMap
- higherEntry
- higherKey
- … (and more)

Hashing in Java

How does your program know which hash function to use?

```
HashMap<MyFoo, Integer> hmap = new ...
MyFoo foo = new MyFoo();


hmap.put(foo, 8);
```

Java Hash Functions

Every object supports the method:

```
int hashCode()
```

Java Object

Every class implicitly extends Object

public class	Object	
Object	clone()	*creates a copy*
boolean	**equals(Object obj)**	***is obj equal to this?***
void	finalize()	*used by garbage collector*
Class	getClass()	*returns class*
int	**hashCode()**	***calculates hash code***
void	notify()	*wakes up a waiting thread*
void	notifyAll()	*wakes up all waiting threads*
String	**toString()**	***returns string representation***
void	wait(…)	*wait until notified*

Hashing in Java

How does your program know which hash function to use?

```
HashMap<MyFoo, Integer> hmap = new ...
MyFoo foo = new MyFoo();


int hash = foo.hashCode();


hmap.put(foo, 8);
```

Java Hash Functions

Every object supports the method:

```
int hashCode()
```

Rules:

- Always returns the same value, if the object hasn't changed.
- If two objects are equal, then they return the same hashCode.

No random hashcodes!

Is it legal for every object to return 32?

Java Hash Functions

Every object supports the method:

```
int hashCode()
```

Rules:

- Always returns the same value, if the object hasn't changed.
- If two objects are equal, then they return the same hashCode.

Is it *legal* for every object to return 32? (YES)

Java Hash Functions

Every object supports the method:

```
int hashCode()
```

Default Java implementation:

- hashCode returns the memory location of the object
- Every object hashes to a different location

Must implement/override `hashCode()` for your class.

Java Library Classes

`Integer`

`Long`

`String`

Integer

```
public int hashCode() {
   return value;
}
```

Rules:

- Always returns the same value, if the object hasn't changed.
- If two objects are equal, then they return the same hashCode.

Note: hashcode is always a 32-bit integer.

Note: every 32-bit integer gets a unique hashcode.

What do you do for smaller hash tables?
Can there be collisions?

Long

Collision can happen!

```
public int hashCode() {
   return (int)(value ^ (value >>> 32));
}
```

32 bits 32 bits

hash(0 1 1 0 0 1 0 1 1 0 0 1 1 1 0 0 0 0 1 0 1 0 0 0 0 1 1 0 0 1 0 0)

0 1 1 0 0 1 0 1 1 0 0 1 1 1 0 0

XOR 0 0 1 0 1 0 0 0 0 1 1 0 0 1 0 0

0 1 0 0 1 1 0 1 1 1 1 1 1 0 0 0

String

```
public int hashCode() {
    int h = hash; // only calculate hash once
    if (h == 0 && count > 0) {  // empty = 0
        int off = offset;
        char val[] = value;
        int len = count;
        for (int i = 0; i < len; i++) {
            h = 31*h + val[off++];
        }
        hash = h;
    }
    return h;
}
```

String

HashCode calculation:

```
hash = s[0]*31^(n-1) +
       s[1]*31^(n-2) +
       s[2]*31^(n-3) +
       ... +
       s[n-2]*31 +
       s[n-1]
```

Why did they choose 31?

String

HashCode calculation:

```
hash = s[0]*31^(n-1) +
       s[1]*31^(n-2) +
       s[2]*31^(n-3) +
       ... +
       s[n-2]*31 +
       s[n-1]
```

Why did they choose 31? Prime, 2^5-1

Creating a new class

```
public class Pair {
  private int first;
  private int second;

  Pair(int a, int b){
     first = a;
     second = b;
  }
}
```

Creating a new class

```
public void testPair() {

  HashMap<Pair, Integer> htable =
          new HashMap<Pair, Integer>();

  Pair one = new Pair(20, 40);
  htable.put(one, 7);

  Pair two = new Pair(20, 40);
  int question = htable.get(two);
}
```

htable.get(new Pair(20, 40)) == ?

1. 1
2. 7
3. 11
4. null

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Creating a new class

```
Pair one = new Pair(20, 40);
Pair two = new Pair(20, 40);

one.hashCode() != two.hashCode()
```

Creating a new class

```
Pair one = new Pair(20, 40);
Pair two = new Pair(20, 40);
htable.put(one, "first item");
```

```
htable.get(one) → "first item"

htable.get(two) → null
```

Creating a new class

```
public class Pair {
  private int first;
  private int second;

  Pair(int a, int b){
     first = a;
     second = b;
  }

  int hashCode(){
     return (first ^ second);
  }
}
```

Creating a new class

```
Pair one = new Pair(20, 40);
Pair two = new Pair(20, 40);
htable.put(one, “first item”);

htable.get(one) → “first item”
htable.get(two) → null

one.equals(two) → false
```

Java Hash Functions

Every object supports the method:

```
int hashCode()
```

Rules:

- Always returns the same value, if the object hasn't changed.
- If two objects are equal, then they return the same hashCode.
- **Must redefine .equals to be consistent with hashCode.**

Creating a new class

```
Pair one = new Pair(20, 20);
Pair two = new Pair(20, 20);
htable.put(one, "first item");

htable.get(one) => "first item"

htable.get(two) => null
```

Java Hash Functions

Every object supports the method:

```
boolean equals(Object o)
```

Rules:

- **Reflexive**: x.equals(x) ➔ true
- **Symmetric**: x.equals(y) == y.equals(x)
- **Transitive**: x.equals(y), y.equals(z) ➔ x.equals(z)
- **Consistent**: always returns the same answer
- **Null is null**: x.equals(null) ➔ false

Java Hash Functions

Every object supports the method:

```
boolean equals(Object o)
```

```
boolean equals(Object p){
    if (p == null) return false;
    if (p == this) return true;

    if (!(p instanceOf Pair)) return false;
    Pair pair = (Pair)p;

    if (pair.first != first) return false;
    if (pair.second != second) return false;
    return true;
}
```

Java HashMap

```
public V get(Object key) {
    if (key == null) return getForNullKey();
    int hash = hash(key.hashCode());
    for (Entry<K,V> e = table[indexFor(hash,table.length)];
         e != null;
         e = e.next)
    {
        Object k;
        if (e.hash==hash &&((k=e.key)==key)||key.equals(k)))
            return e.value;
    }
    return null;
}
```

Java HashMap

```
public V get(Object key) {
    if (key == null) return getForNullKey();
    int hash = hash(key.hashCode());
    for (Entry<K,V> e = table[indexFor(hash,table.length)];
         e != null;
         e = e.next)
    {
        Object k;
        if (e.hash==hash &&((k=e.key)==key)||key.equals(k)))
            return e.value;
    }
    return null;
}
```

Java HashMap

```
public V get(Object key) {
    if (key == null) return getForNullKey();
    int hash = hash(key.hashCode());
    for (Entry<K,V> e = table[indexFor(hash,table.length)];
         e != null;
         e = e.next)
    {
       Object k;
       if (e.hash==hash &&((k=e.key)==key)||key.equals(k)))
          return e.value;
    }
    return null;
}
```

Java checks if the key is equal to the item in the hash table before returning it!

Today

- Analysis of chaining
- Java hashing
- **Designing hash functions**
- Collision resolution: open addressing
- Table (re)sizing

Designing Hash Functions

Goal: find a hash function whose values *look* random.

- Similar to pseudorandom generators:
  - When you use Java random, there is no real randomness.
  - Instead, it generates a sequence of numbers that looks random.
- For every hash function, some set of keys is bad!
- If you know the keys in advance, you can choose a hash function that is always good!
  - But if you change the keys, then it might be bad again.

Designing Hash Functions

Two common hashing techniques…

- Division Method
- Multiplication Method

Designing Hash Functions

Division Method

- $h(k) = k \bmod m$
  - For example: if m=7, then h(17) = 3
  - For example: if m=20, then h(100) = 0
  - For example: if m=20, then h(97) = 17

- Two keys k_1 and k_2 collide when:

$$k_1 = k_2 \bmod m$$

- Collision unlikely if keys are random.

Designing Hash Functions

Division Method

– (Bad) idea: choose $m = 2^x$

Very fast to calculate $k \bmod m$ via shifts

Recall: 001001 >> 1 = 00100

001001 >> 2 = 0010

001001 >> 3 = 001

Designing Hash Functions

Division Method

- (Bad) idea: choose $m = 2^x$

  Very fast to calculate k mod m via shifts:

  $$k \bmod 2^x = k - ((k >> x) << x)$$

Designing Hash Functions

Division Method

– (Bad) idea: choose $m = 2^x$

Very fast to calculate $k \bmod m$ via shifts

– Problem: Regularity

- Input keys are often regular
- Assume input keys are even.
- Then $h(k) = k \bmod m$ is even!

$$k \bmod m + i(m) = k$$

even

even

Definition of mod function:
i is any constant.

Designing Hash Functions

Division Method

- Assume k and m have common divisor d.

$$k \bmod m + \underbrace{i * m}_{\text{divisible by } d} = \underbrace{k}_{\text{divisible by } d}$$

- Implies that $h(k) = k \bmod m$ is divisible by d.

If d is a divisor of m and every key k, then what percentage of the table is used?

1. $1/d$
2. $1/k$
3. $1/m$
4. d/n
5. m/n
6. d/m

Designing Hash Functions

Division Method

– Assume k and m have common divisor d.

$$k \bmod m + i*m = k$$

divisible by d

– Implies that h(k) is divisible by d.

– If all keys are divisible by d, then you only use 1 out of every d slots

0	**A**
1	null
2	null
d = 3	**B**
4	null
5	null
2d =6	**C**
7	null
8	null
3d =9	**D**

Designing Hash Functions

Division Method

- $h(k) = k \bmod m$
- Choose m = prime number
  - Not too close to a power of 2.
  - Not too close to a power of 10.
- Division method is popular (and easy), but not always the most effective.
- Division is slow.

Designing Hash Functions

Two common hashing techniques…

- Division Method
- Multiplication Method

Designing Hash Functions

Multiplication Method

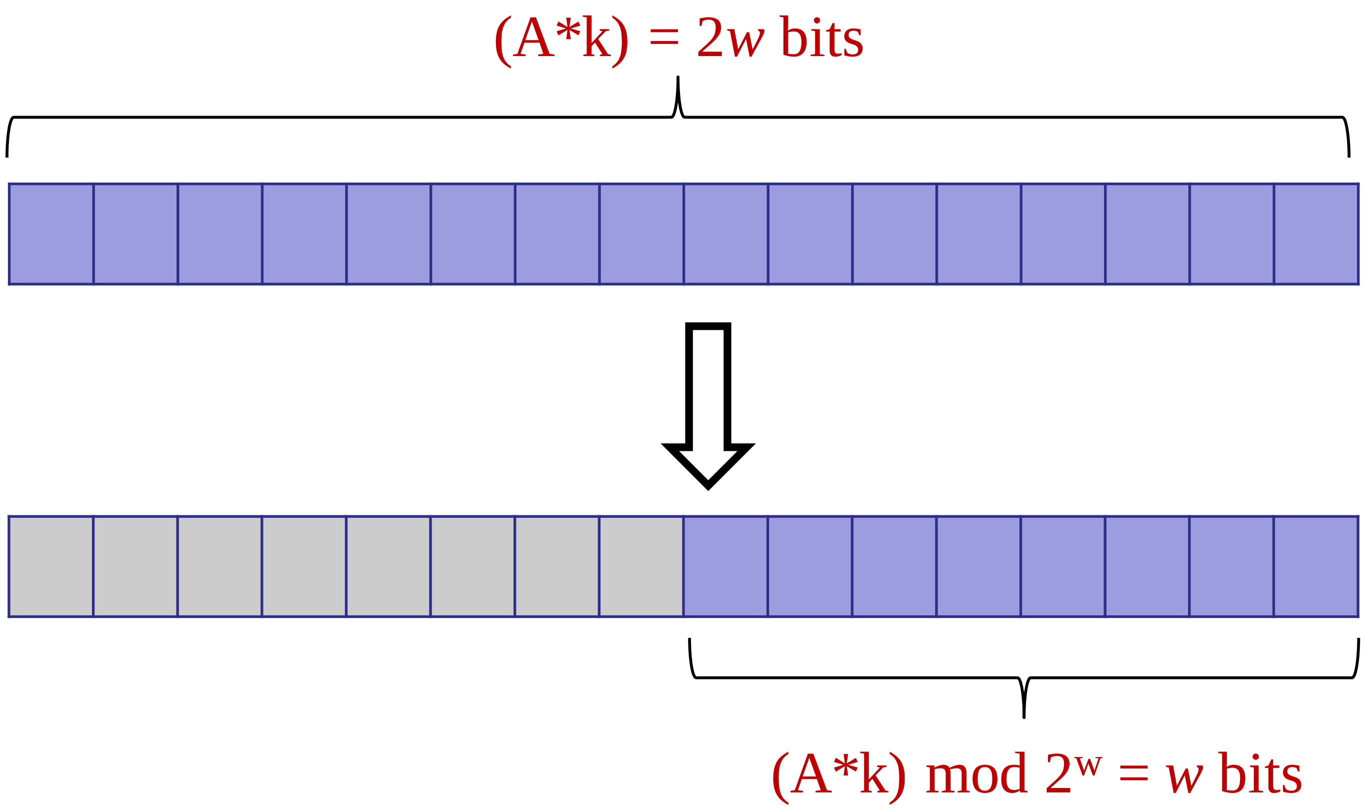
- Fix table size: $m = 2^r$, for some constant r.
- Fix word size: w, size of a key in bits.
- Fix (odd) constant A.

$$h(k) = (Ak) \bmod 2^w \gg (w - r)$$

Designing Hash Functions

Multiplication Method

– Given m, w, r, A: $h(k) = (Ak) \bmod 2^w \gg (w - r)$

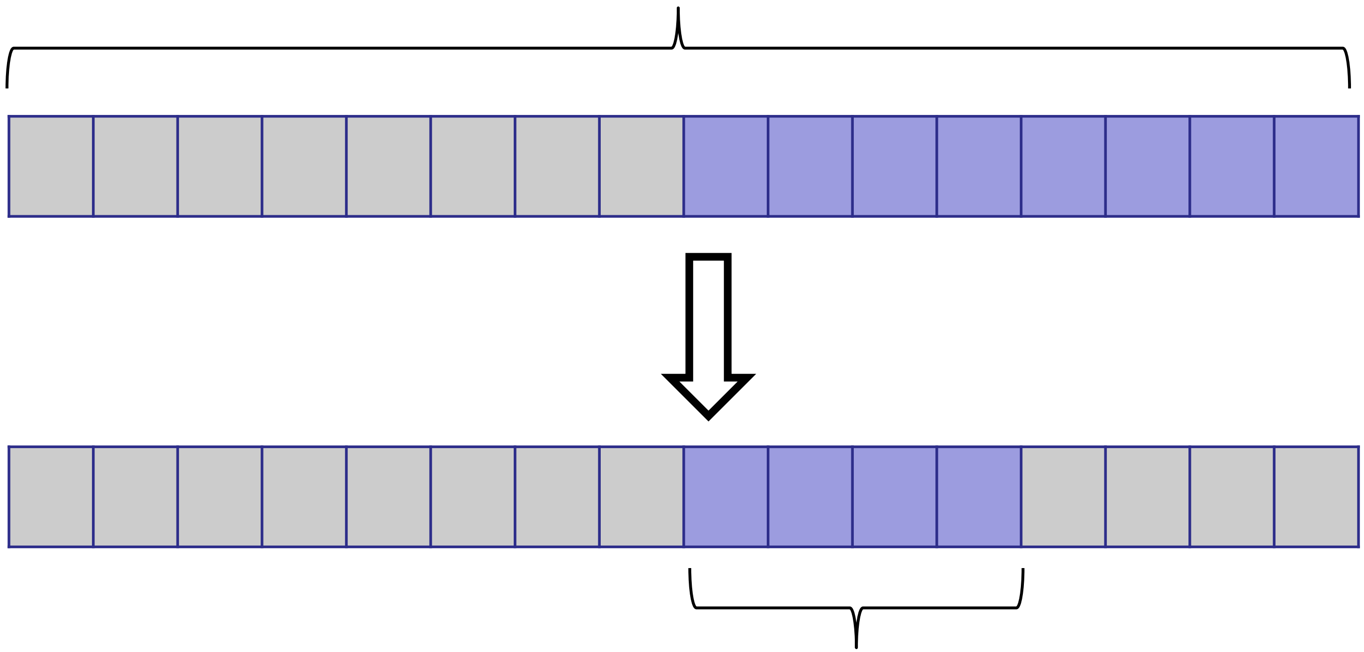


Designing Hash Functions

Multiplication Method

– Given m, w, r, A: $h(k) = (Ak) \bmod 2^w \gg (w - r)$

(A*k) mod 2^w = w bits



(A*k) mod 2^w >> (w–r) = r bits

Designing Hash Functions

Multiplication Method

- Faster than Division Method
  - Multiplication, shifting faster than division
- Works reasonably well when A is an odd integer > 2^{w-1}
  - Odd: if it is even, then lose at least one bit's worth
  - Big enough: use all the bits in A.

Designing Hash Functions

Two common hashing techniques…

- Division Method
- Multiplication Method

Other common techniques:

- Tabulation hashing (very fast, good uniformity)
- Zobrist hashing (good for board games)

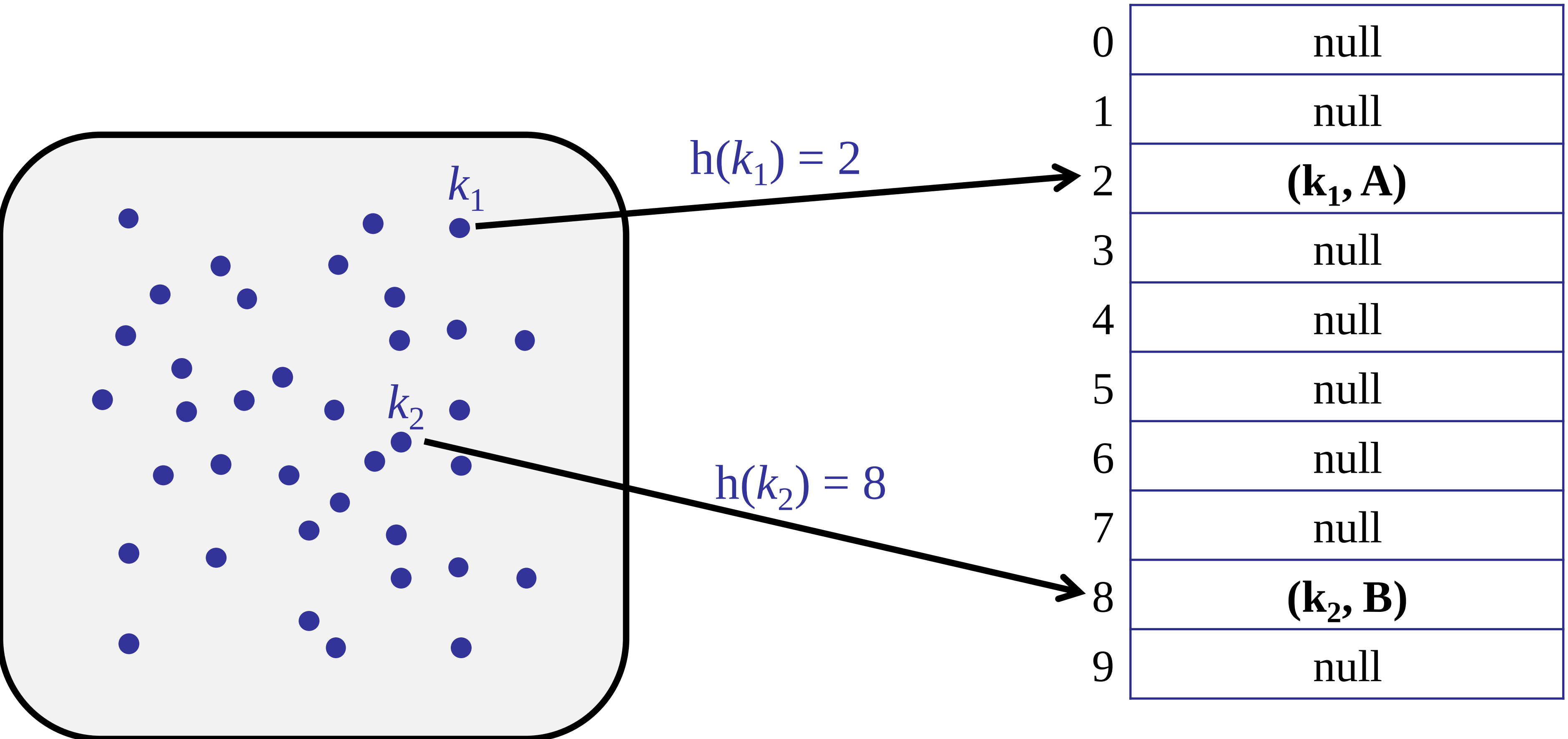
Today

- Analysis of chaining
- Java hashing
- Designing hash functions
- **Collision resolution: open addressing**
- Table (re)sizing

Review

Hash Tables

- Store each item from the symbol table in a table.
- Use hash function to map each key to a bucket.



Resolving Collisions

- Basic problem:
  - What to do when two items hash to the same bucket?
- Solution 1: Chaining
  - Insert item into a linked list.
- Solution 2: Open Addressing
  - Find another free bucket.

Open Addressing

Advantages:

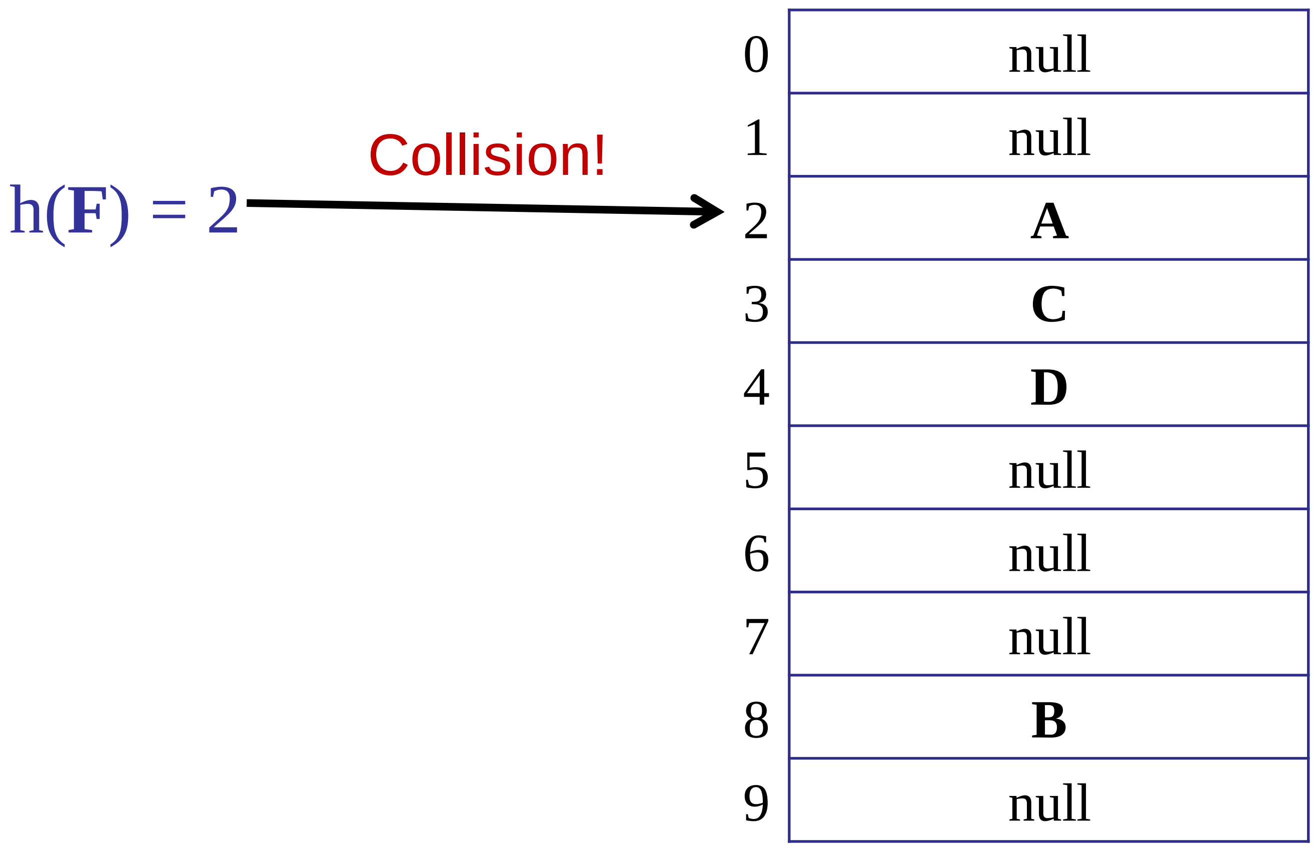
- No linked lists!
- All data directly stored in the table.
- One item per slot.

0	null
1	null
2	**A**
3	null
4	null
5	null
6	null
7	null
8	**B**
9	null

Open Addressing

On collision:

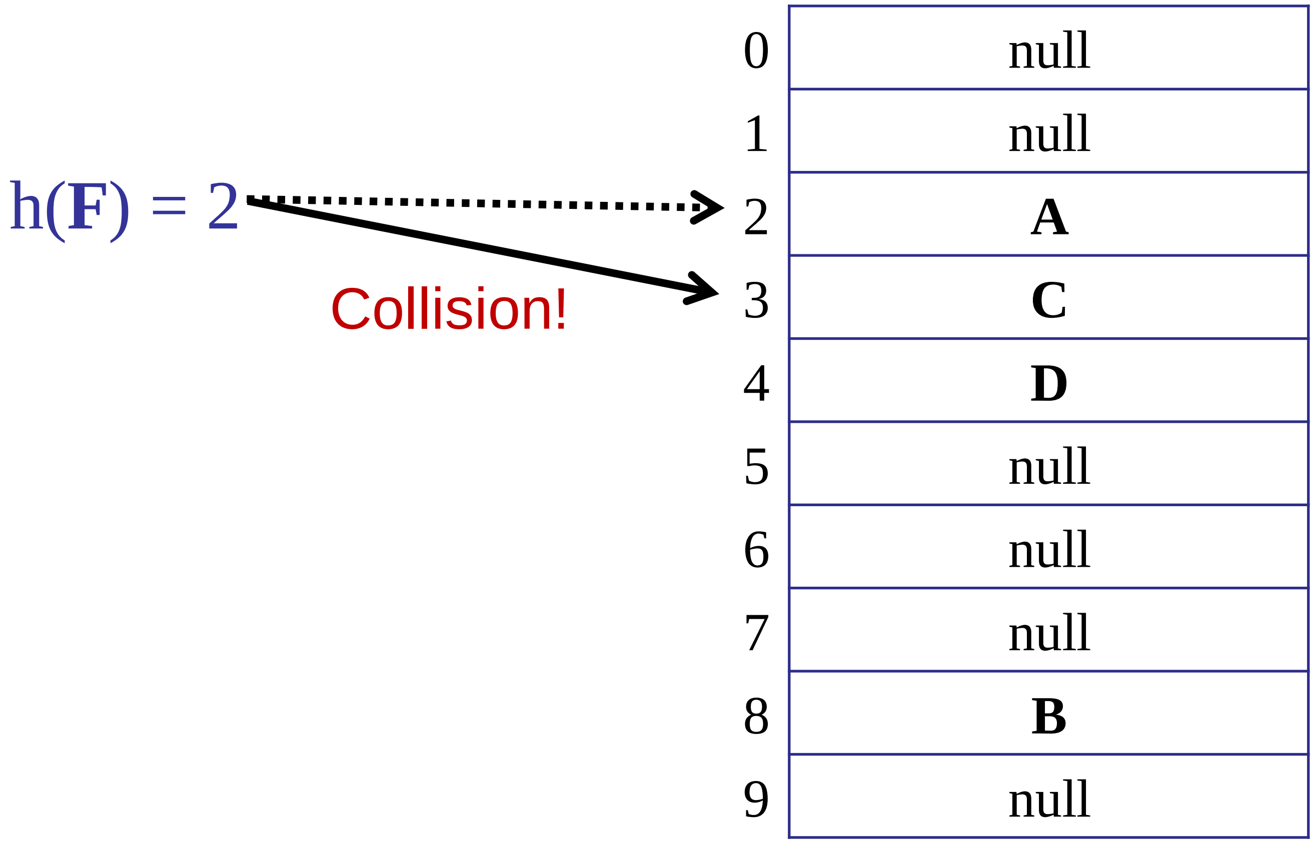
Probe a sequence of buckets until you find an empty one.



Open Addressing

On collision:

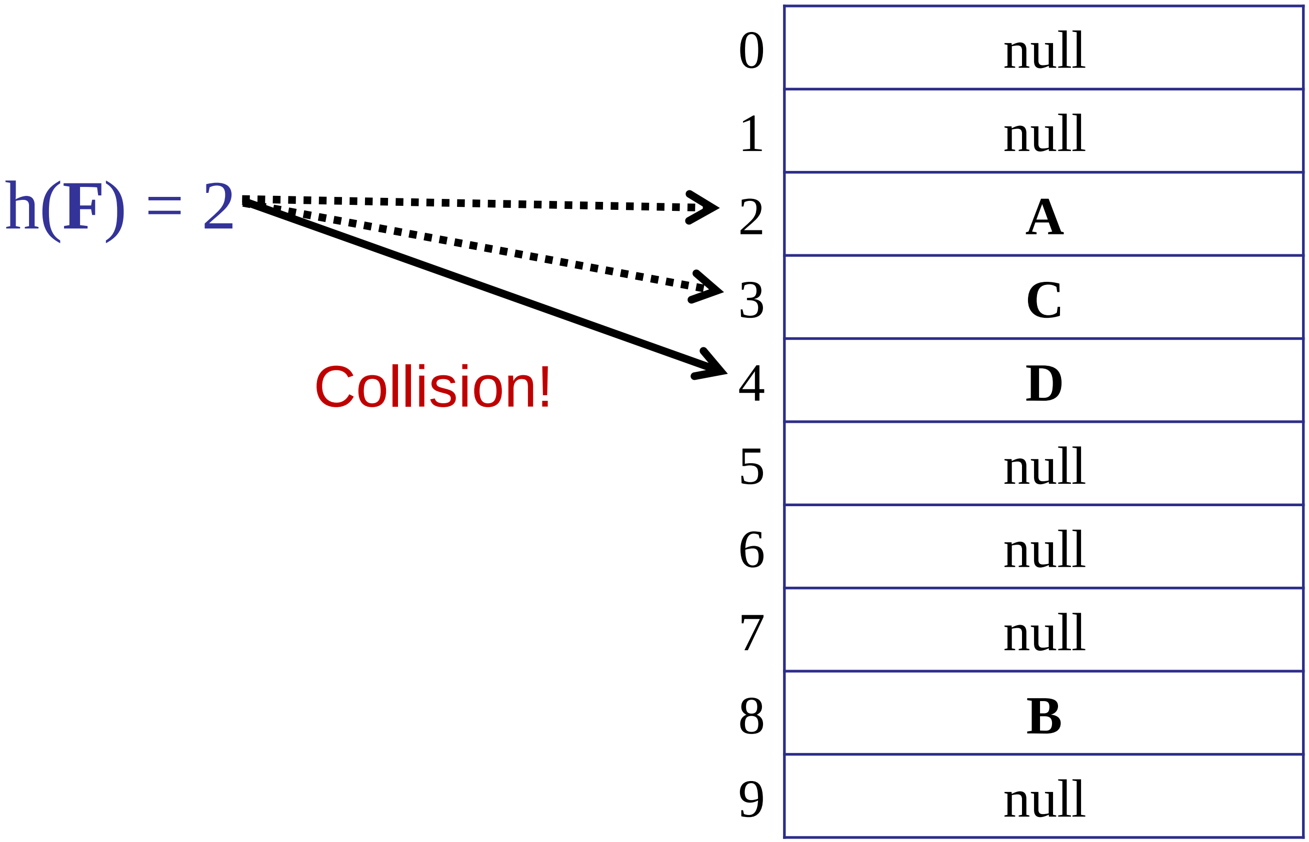
Probe a sequence of buckets until you find an empty one.



Open Addressing

On collision:

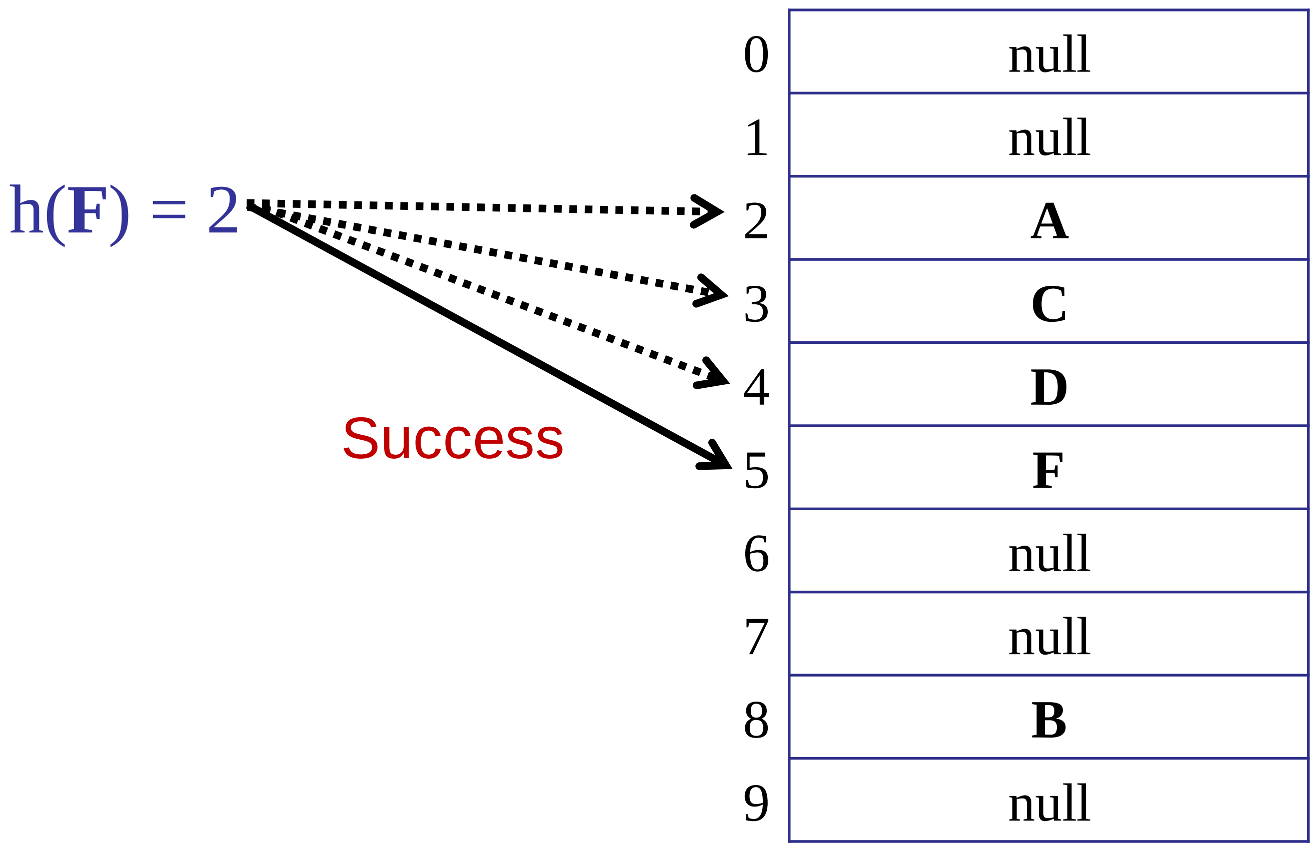
Probe a sequence of buckets until you find an empty one.



Open Addressing

On collision:

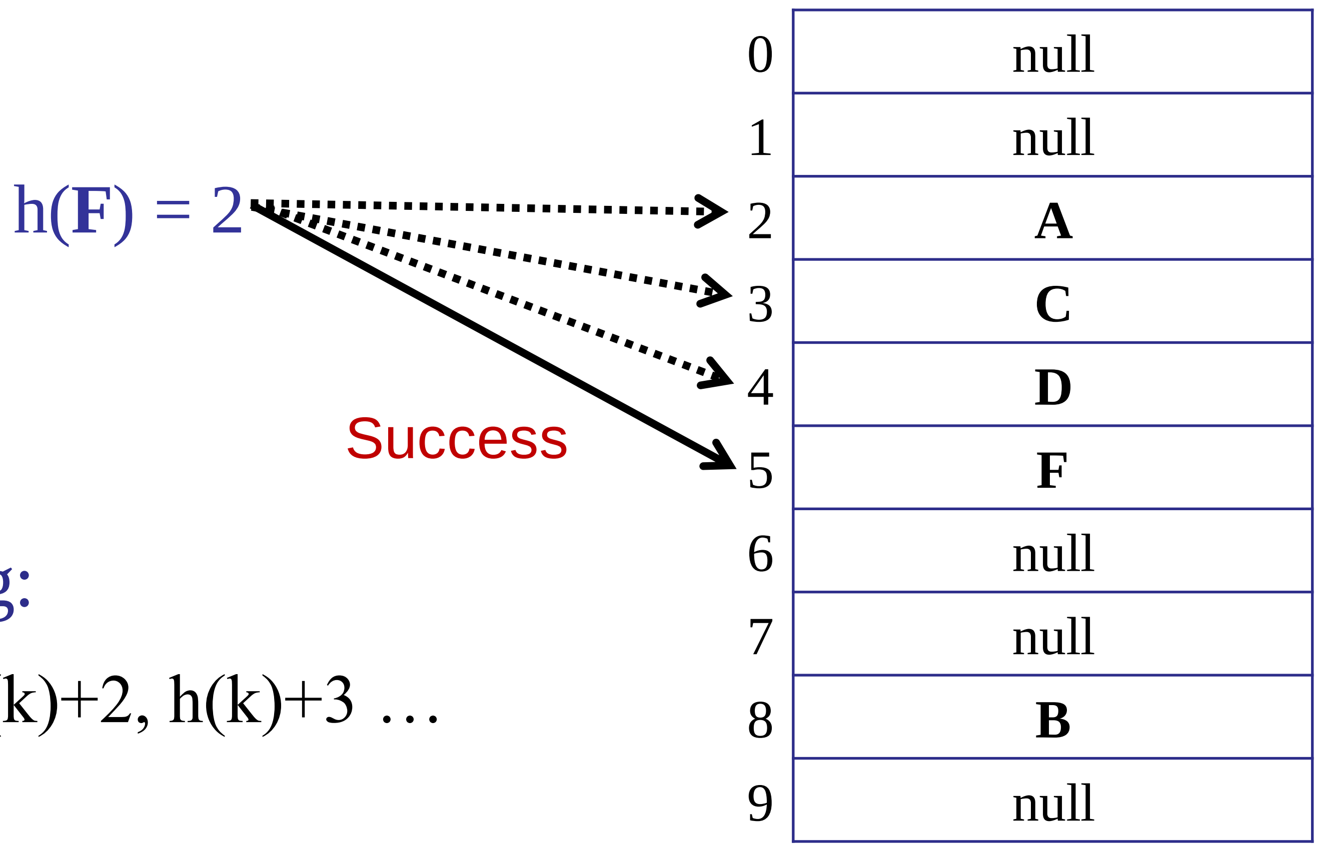
Probe a sequence of buckets until you find an empty one.



Open Addressing

On collision:

Probe a sequence of buckets until you find an empty one.



Linear Probing:

- h(k)+1, h(k)+2, h(k)+3 …

Open Addressing

Hash Function re-defined:

$h(key, i) : U \rightarrow \{1..m\}$

Two parameters:

- key : the thing to map
- i : number of collisions

Open Addressing

Hash Function re-defined:

$h(\text{key}, i) : U \rightarrow \{1..m\}$

Example: Linear Probing

- $h(k, 1)$ = hash of key k
- $h(k, 2) = h(k, 1) + 1$
- $h(k, 3) = h(k, 1) + 2$
- $h(k, 4) = h(k, 1) + 3$
- …
- $h(k, i) = h(k, 1) + i \bmod m$

0	null
1	null
2	**A**
3	**C**
4	**D**
5	**F**
6	null
7	null
8	**B**
9	null

Open Addressing

Hash Function re-defined:

$h(key, i) : U \rightarrow \{1..m\}$

Example: Weird Probing

- $h(k, 1) = 4$
- $h(k, 2) = 1$
- $h(k, 3) = 8$
- $h(k, 4) = 5$

0	null
1	**G**
2	**A**
3	**C**
4	**D**
5	null
6	null
7	null
8	**B**
9	null

Open Addressing

```
hash-insert(key, data)
int i = 1;
while (i <= m) {                                   // Try every bucket
        int bucket = h(key, i);
        if (T[bucket] == null){                    // Found an empty bucket
                T[bucket] = {key, data};           // Insert key/data
                return success;                    // Return
        }
        i++;
}
throw new TableFullException();                    // Table full!
```

Open Addressing

Hash Function re-defined:

$h(key, i) : U \rightarrow \{1..m\}$

search(key)

- $h(key, 1) = 4$
- $h(key, 2) = 1$
- $h(key, 3) = 8$
- $h(key, 4) = 5$

0	null
1	**G**
2	**A**
3	**C**
4	**D**
5	key
6	null
7	null
8	**B**
9	null

Open Addressing

```
hash-search(key)
int i = 1;
while (i <= m) {
    int bucket = h(key, i);
    if (T[bucket] == null)  // Empty bucket!
        return key-not-found;
    if (T[bucket].key == key)  // Full bucket.
            return T[bucket].data;
    i++;
}
return key-not-found;  // Exhausted entire table.
```

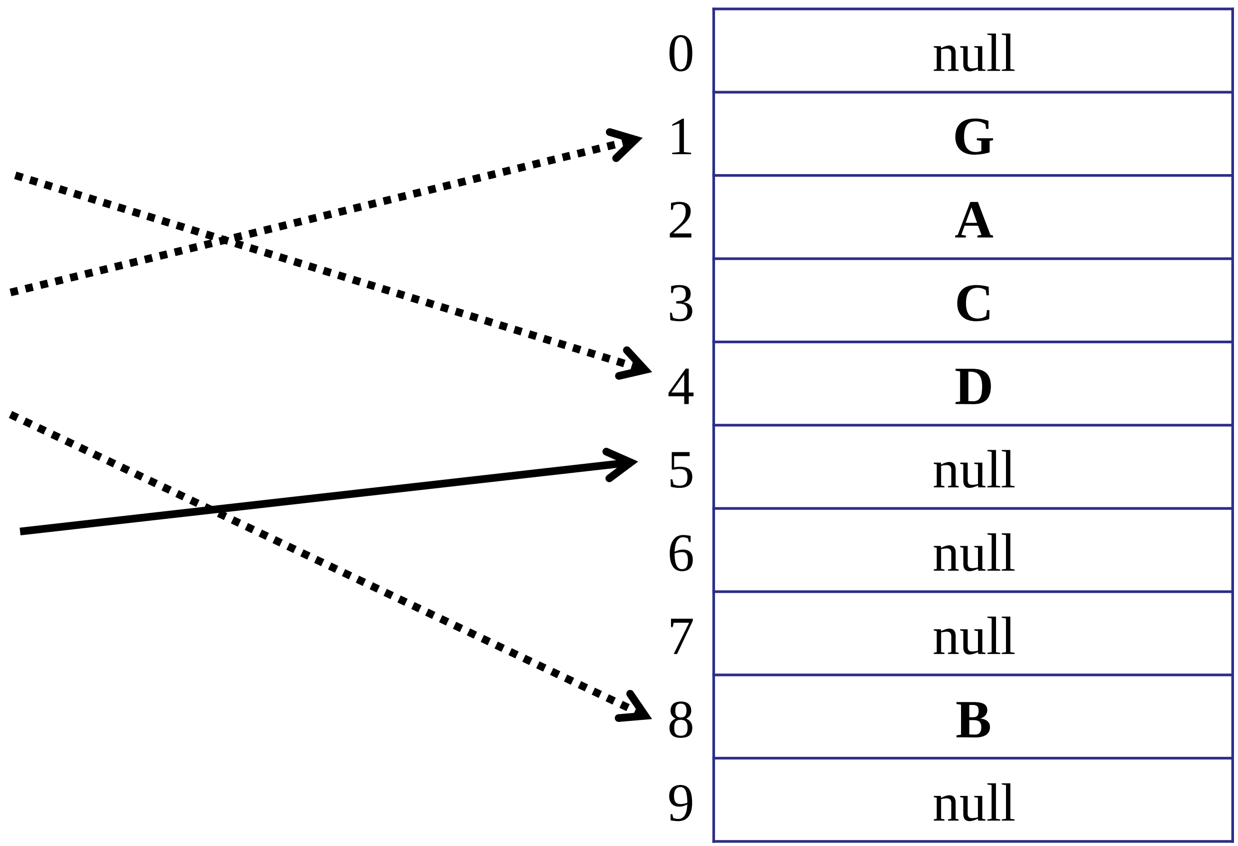
Open Addressing

Hash Function re-defined:

$h(key, i) : U \rightarrow \{1..m\}$

search(key)

- $h(key, 1) = 4$
- $h(key, 2) = 1$
- $h(key, 3) = 8$
- $h(key, 4) = 5$



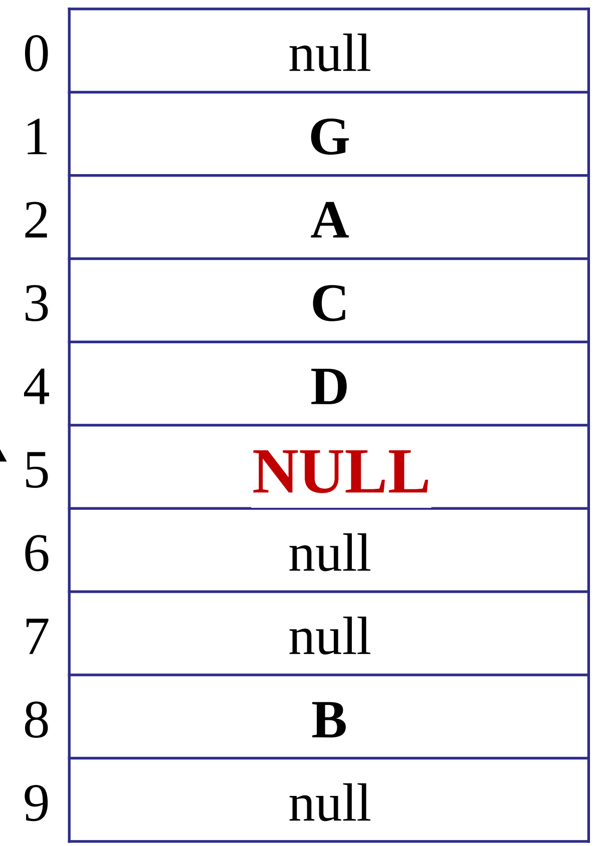
Open Addressing

Hash Function re-defined:

$h(key, i) : U \rightarrow \{1..m\}$

delete(key)

- Find key to delete
- Remove it from table.
- Set bucket to null.



What is wrong with delete?

1. Search may fail to find an element.
2. The table will have gaps in it.
3. Space is used inefficiently.
4. If the key is inserted again, it may end up in a different bucket.

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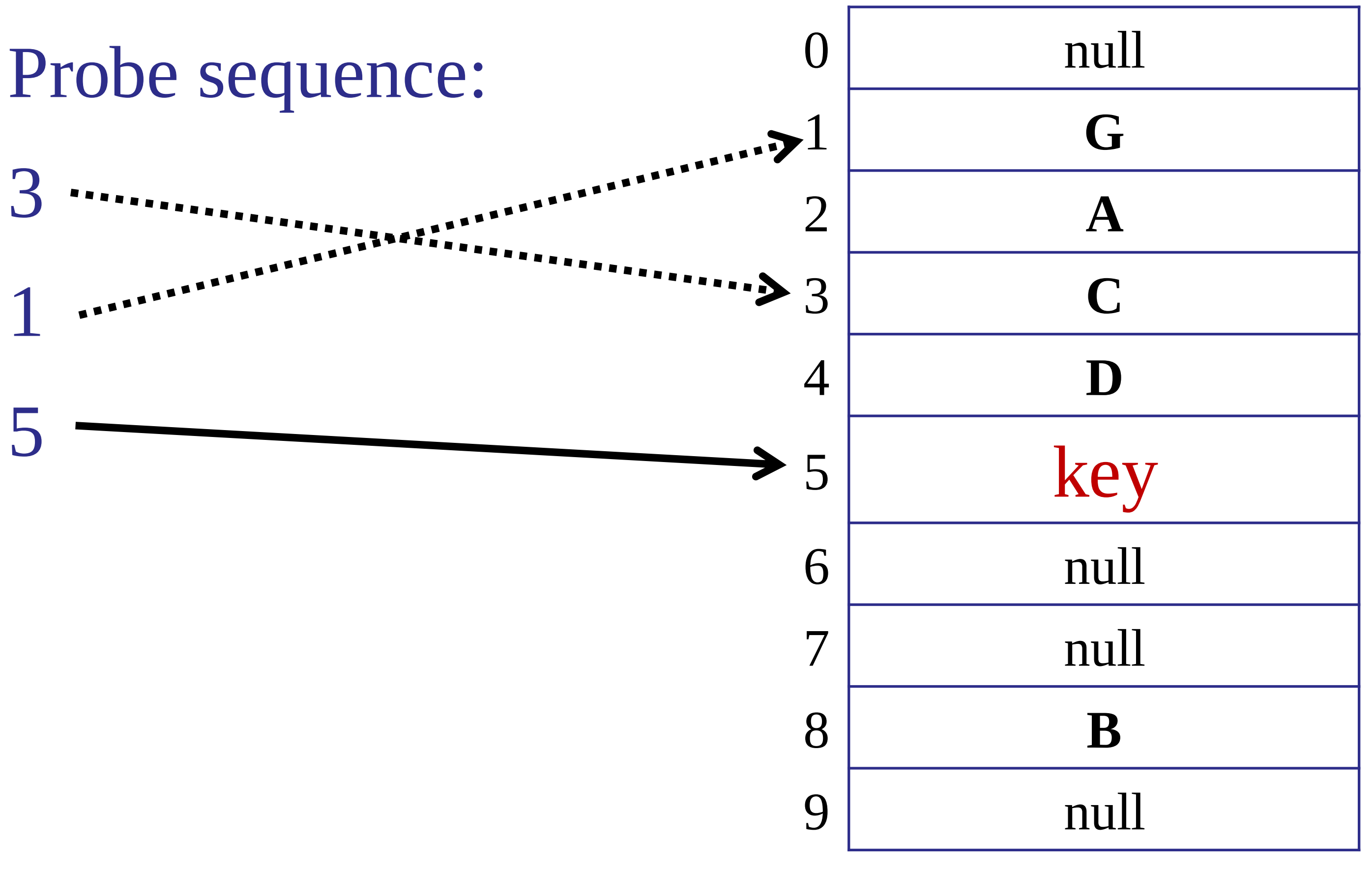
is open

Open Addressing

insert(key)

Probe sequence:



Open Addressing

insert(key)

delete(G)

0	null
1	**G → NULL**
2	**A**
3	**C**
4	**D**
5	key
6	null
7	null
8	**B**
9	null

Open Addressing

insert(key)

delete(G)

search(key)

0	null
1	**NULL**
2	**A**
3	**C**
4	**D**
5	key
6	null
7	null
8	**B**
9	null

Open Addressing

insert(key)

delete(G)

search(key)

Probe sequence:

3

1

5

0	null
1	**NULL**
2	**A**
3	**C**
4	**D**
5	key
6	null
7	null
8	**B**
9	null

Open Addressing

insert(key)

delete(G)

search(key)

Probe sequence:

3

1

Not found!

0	null
1	**NULL**
2	**A**
3	**C**
4	**D**
5	key
6	null
7	null
8	**B**
9	null

Open Addressing

Hash Function re-defined:

$h(key, i) : U \rightarrow \{1..m\}$

delete(key)

- Find key to delete
- Remove it from table.
- Set bucket to DELETED.

(Tombstone value.)

0	null
1	**G**
2	**A**
3	**C**
4	**D**
5	**DELETED**
6	null
7	null
8	**B**
9	null

Open Addressing

insert(key)

delete(G)

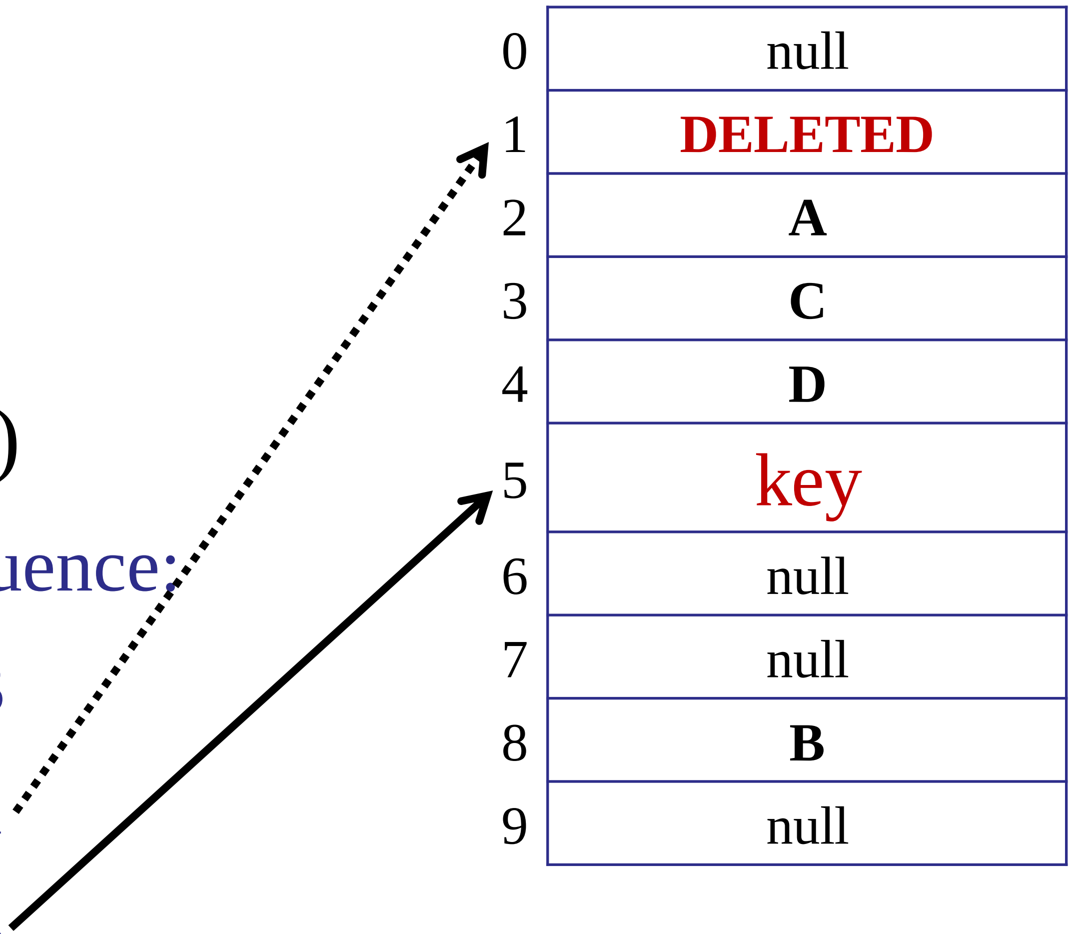
search(key)

Probe sequence:

3

1

5



What happens when an insert finds a DELETED cell?

1. Overwrite the deleted cell.
2. Continue probing.
3. Fail.

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Hash Functions

Two properties of a good hash function:

1. h(*key*, *i*) enumerates all possible buckets.
   - For every bucket *j*, there is some *i* such that:

     $$h(key, i) = j$$

   - The hash function is permutation of $\{1..m\}$.
   - For linear probing: true!

What goes wrong if the sequence is not a permutation?

1. Search incorrectly returns key-not-found.
2. Delete fails.
3. Insert puts a key in the wrong place
4. Returns table-full even when there is still space left.

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Hash Functions

Two properties of a good hash function:

2. Simple Uniform Hashing Assumption

   Every key is equally likely to be mapped to every bucket, independently of every other key.

For h(*key*, 1)?

For every h(*key*, *i*)?

Hash Functions

Two properties of a good hash function:

2. Uniform Hashing Assumption

   Every key is equally likely to be mapped to every ***permutation***, independent of every other key.

   n! permutations for probe sequence: e.g.,

   - 1 2 3 4
   - 1 2 4 3
   - 1 4 2 3
   - 1 4 3 2
   - …

Hash Functions

Two properties of a good hash function:

2. Uniform Hashing Assumption

   Every key is equally likely to be mapped to every ***permutation***, independent of every other key.

n! permutations for probe sequence: e.g.,

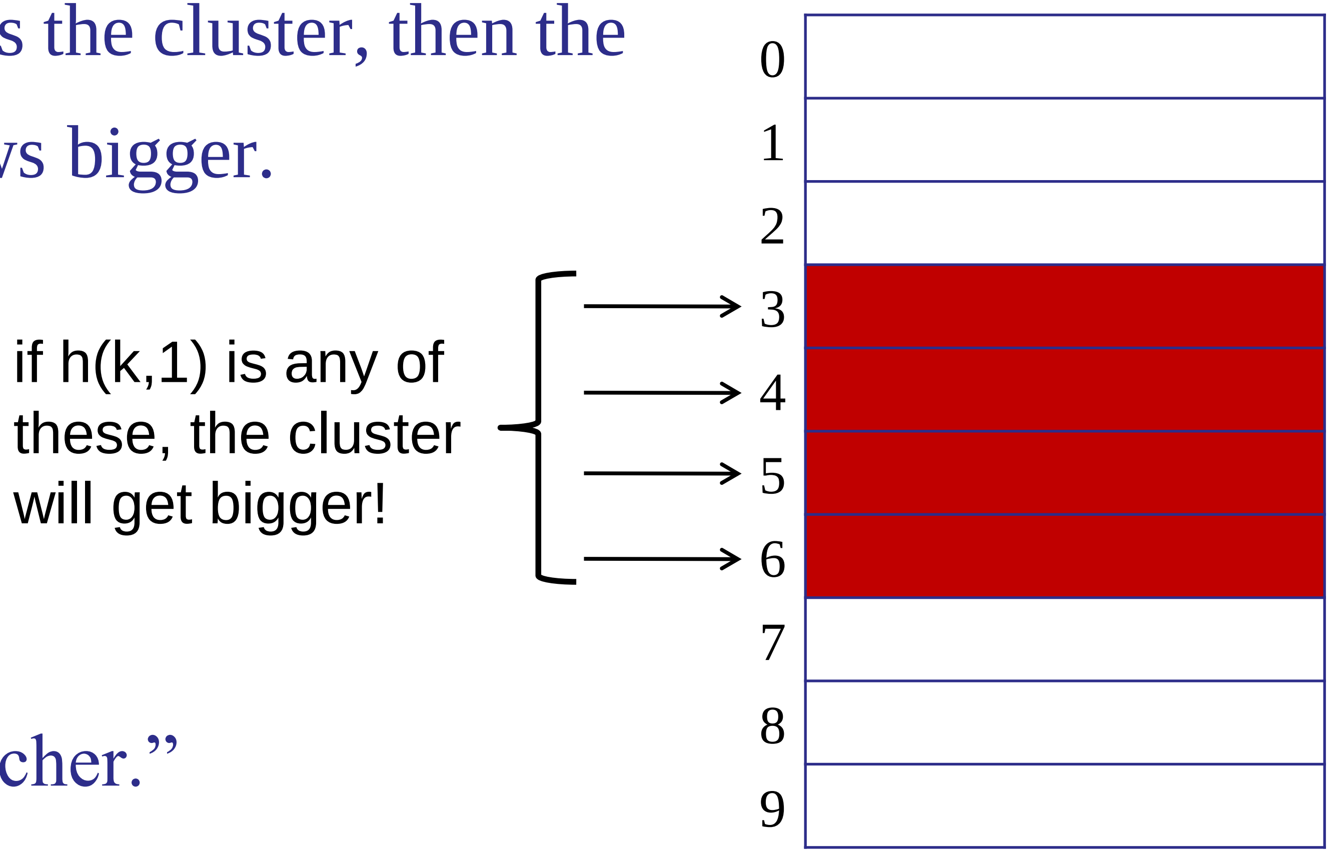
- 1 2 3 4 Pr(1/m)
- 1 2 4 3 Pr(0)
- 1 4 2 3 Pr(0)
- 1 4 3 2 Pr(0)
- …

NOT Linear Probing

Linear Probing

Problem with linear probing: ***clusters***

- If there is a cluster, then there is a higher probability that the next h(k) will hit the cluster.
- If h(k,1) hits the cluster, then the cluster grows bigger.



- "Rich get richer."

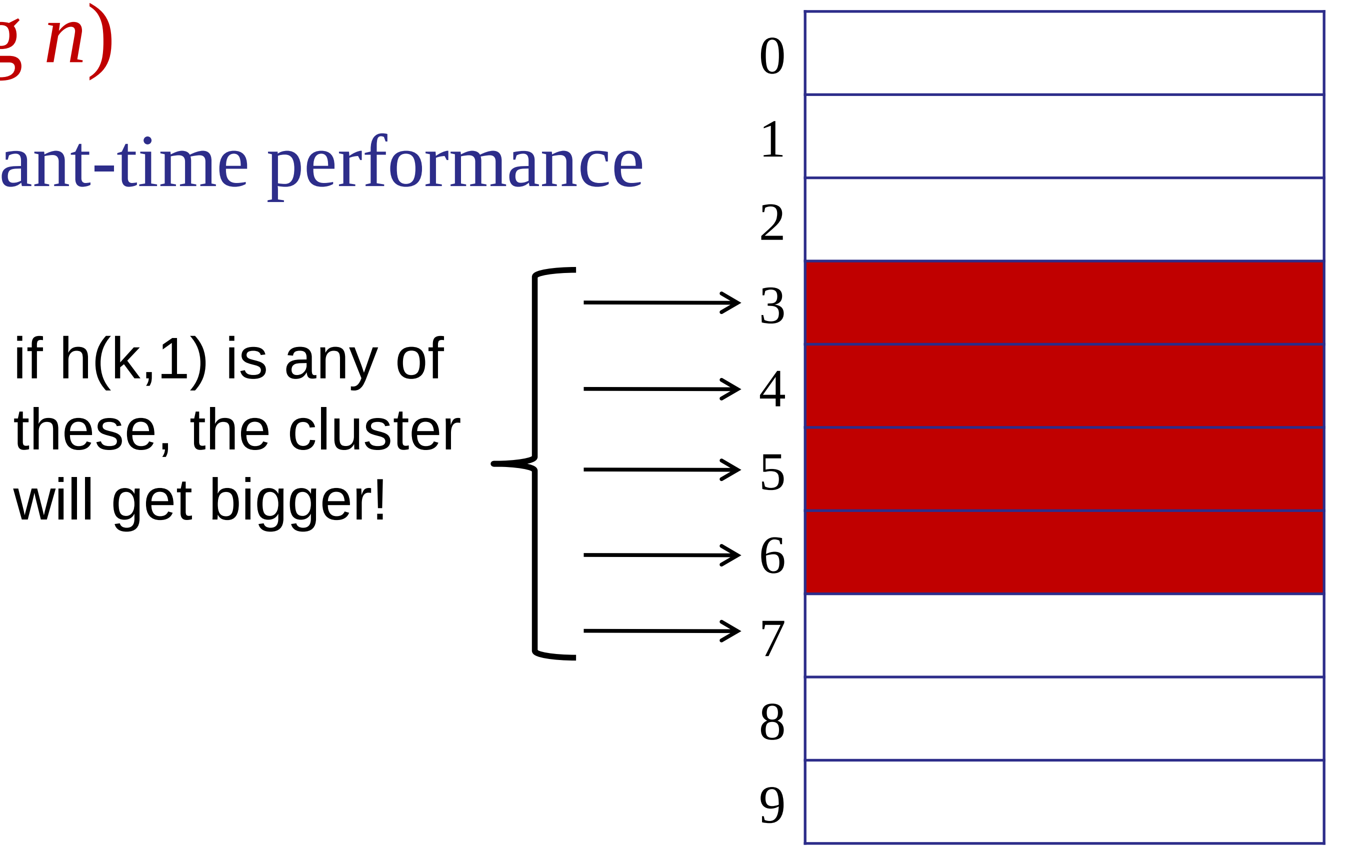
Linear Probing

Problem with linear probing: ***clusters***

- If the table is 1/4 full, then there will be clusters of size:

$$\theta(\log n)$$

- Ruins constant-time performance



Linear probing

In practice, linear probing is faster!

- Why? Caching!
- It is *cheap* to access nearby array cells.
  - Example: access T[17]
  - Cache loads: T[10..50]
  - Almost 0 cost to access T[18], T[19], T[20], …
- If the table is 1/4 full, then there will be clusters of size: $\theta(\log n)$
  - Cache may hold entire cluster!
  - No worse than wacky probe sequence.

That conversation again…

Professor (for the last 30 years):

"Linear probing is bad because it leads to clusters and bad performance. We need uniform hashing."

Punk in the front row:

"But I ran some experiments and linear probing seems really fast."

Professor:

"Maybe your experiments were too small, or just weren't very well done. Let me prove to you that uniform hashing is good."

Punk in the front row goes and starts a billion dollar startup doing high performance data processing.

Student sitting next to punk in the front row goes to grad school and proves that linear probing really is faster.

Open Addressing

Properties of a good hash function:

2. Uniform Hashing Assumption

   Every key is equally likely to be mapped to every ***permutation***, independent of every other key.

n! permutations for probe sequence: e.g.,

- 1 2 3 4
- 1 2 4 3
- 1 4 2 3
- 1 4 3 2
- …

Double Hashing

- Start with two ordinary hash functions:

$$\mathrm{f}(k),\ \mathrm{g}(k)$$

- Define new hash function:

$$\mathrm{h}(k, i) \ = \ \mathrm{f}(k) + i\cdot\mathrm{g}(k) \quad \mathrm{mod}\ m$$

- Note:
  - Since f(k) is good, f(k, 1) is “almost” random.
  - Since g(k) is good, the probe sequence is “almost” random.

Double Hashing

Hash function

$$h(k, i) \;=\; f(k) + i \cdot g(k) \mod m$$

Claim: if $g(k)$ is relatively prime to m, then $h(k, i)$ hits all buckets.

– Assume not: then for some distinct $i, j < m$:

$$f(k) + i \cdot g(k) = f(k) + j \cdot g(k) \mod m$$

$\Rightarrow i \cdot g(k) = j \cdot g(k) \mod m$

$\Rightarrow (i - j) \cdot g(k) = 0 \mod m$

$\Rightarrow$ g(k) not relatively prime to m, since $(i\text{-}j \neq 0 \bmod m)$

Double Hashing

Hash function

$$h(k, i) = f(k) + i \cdot g(k) \mod m$$

Claim: if g(k) is relatively prime to m, then $h(k, i)$ hits all buckets.

Example: if $(m = 2^r)$, then choose $g(k)$ odd.

Performance of Open Addressing

If (m==n), what is the expected insert time, under uniform hashing assumption?

1. O(1)
2. O(log n)
3. O(n)
4. O(n^2)
5. None of the above.

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is open

Performance of Open Addressing

- Chaining:
  - When (m==n), we can still add new items to the hash table.
  - We can still search efficiently.

- Open addressing:
  - When (m==n), the table is full.
  - We cannot insert any more items.
  - We cannot search efficiently.

Performance of Open Addressing

Define:

- Load $\alpha = n / m$

Average # items / bucket

- Assume $\alpha < 1$.

Performance of Open Addressing

Define:

- Load $\alpha = n / m$ Average # items / bucket
- Assume $\alpha < 1$.

Claim: Type equation here.

For n items, in a table of size m, assuming *uniform hashing*, the expected cost of an operation is:

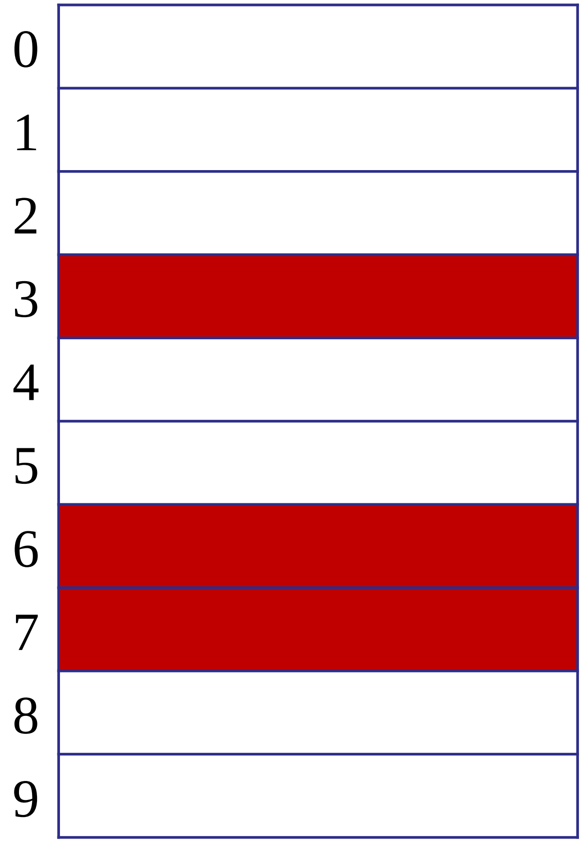
$$\frac{1}{1-\alpha}$$

Example: if (α=90%), then E[# probes] = 10

Performance of Open Addressing

Proof of Claim:

- First probe: probability that first bucket is full is: n/m

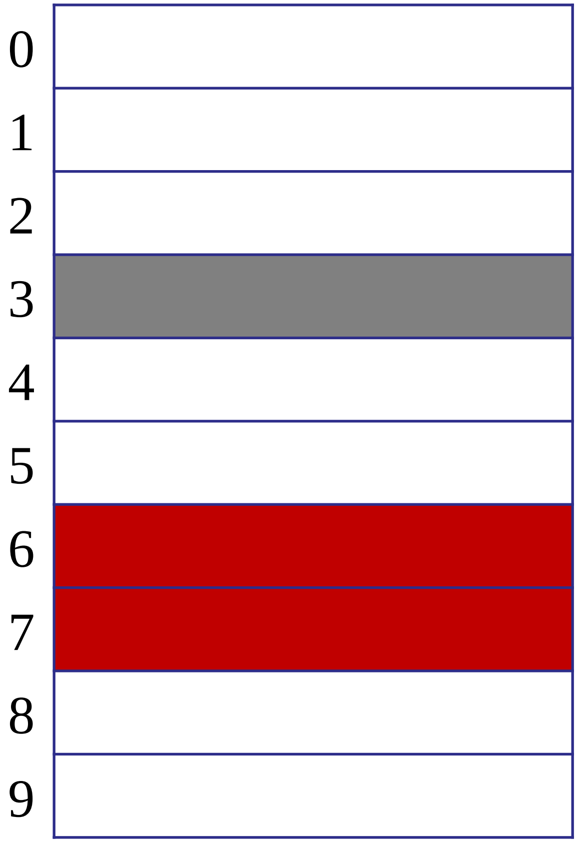


Performance of Open Addressing

Proof of Claim:

- First probe: probability that first bucket is full is: n/m

- Second probe: if first bucket is full, then the probability that the second bucket is also full: $(n - 1) / (m - 1)$



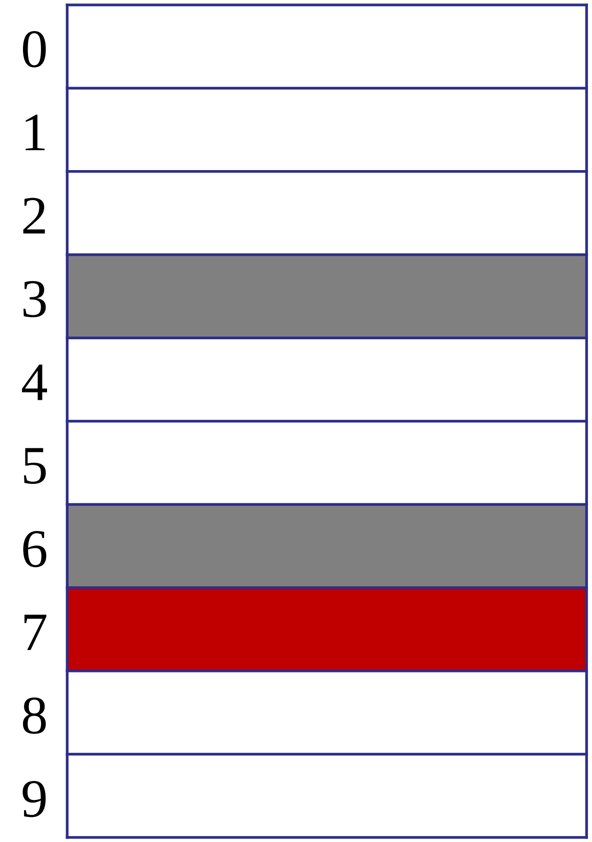
Performance of Open Addressing

Proof of Claim:

- First probe: probability that first bucket is full is: n/m

- Second probe: if first bucket is full, then the probability that the second bucket is also full: $(n - 1) / (m - 1)$

- Third probe: probability is full: $(n - 2) / (m - 2)$



Performance of Open Addressing

Proof of Claim:

– Expected cost:

$$1+\frac{n}{m}\left(\boxed{\text{Expected cost of remaining probes}}\right)$$

First probe

Probability
of collision
on first probe

Performance of Open Addressing

Proof of Claim:

– Expected cost:

$$1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(\boxed{\text{Expected cost of remaining probes}}\right)\right)$$

First probe

Probability of collision on first probe

Probability of collision on second probe

Performance of Open Addressing

Proof of Claim:

– Expected cost:

$$1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(\boxed{\text{Expected cost of remaining probes}}\right)\right)$$

– Note that for small i:

$$\frac{n-i}{m-i}\approx\frac{n}{m}=\alpha$$

Performance of Open Addressing

Proof of Claim:

- Expected cost:

$$1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(1+\frac{n-2}{m-2}\left(\cdots\right)\right)\right)$$

Expected cost of remaining probes

$$\approx 1+\alpha\left(1+\alpha\left(1+\alpha\left(\cdots\right)\right)\right)$$

Performance of Open Addressing

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$$= 1+\alpha+\alpha^2+\alpha^3+\cdots$$

$$= \frac{1}{1-\alpha}$$

Performance of Open Addressing

Define:

- Load $\alpha = n / m$
- Assume $\alpha < 1$.

Average # items / bucket

Claim:

For n items, in a table of size m, assuming *uniform hashing*, the expected cost of an operation is:

$$\frac{1}{1-\alpha}$$

Example: if (α=90%), then E[# probes] = 10

Advantages…

Open addressing:

- Saves space
  - Empty slots vs. linked lists.
- Rarely allocate memory
  - No new list-node allocations.
- Better cache performance
  - Table all in one place in memory
  - Fewer accesses to bring table into cache.
  - Linked lists can wander all over the memory.

Disadvantages…

Open addressing:

– More sensitive to choice of hash functions.
  - Clustering is a common problem.
  - See issues with linear probing.

– More sensitive to load.
  - Performance degrades badly as $\alpha \rightarrow 1$.

Disadvantages…

Open addressing:

- Performance degrades badly as $\alpha \rightarrow 1$.