Solutions to Tutorial 11

MA1521 Calculus for Computing

1. (a) The equation $xy' + (1+x)y = e^{-x},\ x > 0$, is a linear DE.

$$\begin{aligned}
& xy' + (1+x)y = e^{-x} \\
\Leftrightarrow\ & y' + (\frac{1}{x} + 1)y = \frac{1}{x}e^{-x} \\
\Leftrightarrow\ & xe^x(y' + (\frac{1}{x}+1)y) = xe^x \cdot \frac{1}{x}e^{-x} \quad \text{(an integrating factor is } xe^x = e^{\int \frac{1}{x}+1\,dx}\text{).} \\
\Leftrightarrow\ & (xye^x)' = 1 \\
\Leftrightarrow\ & xye^x = x + C \\
\Leftrightarrow\ & y = e^{-x}(1 + \frac{C}{x}).
\end{aligned}$$

(b) The equation $y' - (1 + \frac{3}{x})y = x + 2,\ y(1) = e - 1,\ x > 0$, is a linear DE. An integrating factor is $e^{-\int 1+\frac{3}{x}\,dx} = x^{-3}e^{-x}$. Multiplying the given DE by this integrating factor, we have $(yx^{-3}e^{-x})' = (x^{-2} + 2x^{-3})e^{-x}$.
Thus, $yx^{-3}e^{-x} = \int (x^{-2} + 2x^{-3})e^{-x}\,dx = \int x^{-2}e^{-x}\,dx + \int 2x^{-3}e^{-x}\,dx = \int x^{-2}e^{-x}\,dx - x^{-2}e^{-x} - \int x^{-2}e^{-x}\,dx = -x^{-2}e^{-x} + C$, using integration by parts.
Therefore, $y = x^3e^x[-x^{-2}e^{-x} + C] = -x + Cx^3e^x$. Since $y(1) = e - 1$, we have $e - 1 = -1 + Ce$ so that $C = 1$. Consequently, $y = -x + x^3e^x$.

(c) The equation $y' + y + \frac{x}{y} = 0$, or equivalently, $y' + y = -xy^{-1}$ is a Bernoulli's equation with $n = -1$. Using the substitution $u = y^2$, the equation can be transformed into $u' + 2u = -2x$. Multiplying by the integrating factor e^{2x}, we have $(ue^{2x})' = -2xe^{2x}$. Thus $ue^{2x} = \int -2xe^{2x}\,dx = (\frac{1}{2} - x)e^{2x} + C$ using integration by parts. Therefore, $u = \frac{1}{2} - x + Ce^{-2x}$. That is $y^2 = \frac{1}{2} - x + Ce^{-2x}$.

(d) The equation $2xyy' + (x-1)y^2 = x^2e^x$, or equivalently, $y' + \frac{1}{2}(1 - \frac{1}{x})y = \frac{1}{2}xe^xy^{-1}$, is a Bernoulli's equation with $n = -1$. Using the substitution $u = y^2$, the equation can be transformed into $u' + (1 - \frac{1}{x})u = xe^x$. Multiplying by the integrating factor $\frac{e^x}{x}$, we have $(\frac{e^xu}{x})' = e^{2x}$. Integrating, $\frac{e^xu}{x} = \frac{1}{2}e^{2x} + C$. Therefore, $u = xe^{-x}(\frac{1}{2}e^{2x} + C) = \frac{1}{2}xe^x + Cxe^{-x}$. That is $y^2 = \frac{1}{2}xe^x + Cxe^{-x}$.

2. (a) To solve $y' = \frac{1-2y-4x}{1+y+2x}$, we let $z = y + 2x$. Then $y' = z' - 2$. Thus the DE can be written as $z' - 2 = \frac{1-2z}{1+z}$, which can be simplifies to $z' = \frac{3}{1+z}$. That is $\int (1+z)\,dz = \int 3\,dx$ giving $z + \frac{1}{2}z^2 = 3x + C$. Therefore, $(y + 2x) + \frac{1}{2}(y + 2x)^2 = 3x + C$, or equivalently, $4x^2 + 4xy + y^2 - 2x + 2y = C$.

(b) To solve $y' = \left(\frac{x+y+1}{x+y+3}\right)^2$, we let $z = x + y + 1$. Then $y' = z' - 1$. Thus the DE can be written as $z' - 1 = \frac{z^2}{(z+2)^2}$, which can be simplified to $z' = \frac{2z^2+4z+4}{z^2+4z+4}$. Separating the variables, we have $\int \frac{z^2+4z+4}{2z^2+4z+4} dz = \int dx$. That is $x+C = \int \frac{1}{2} + \frac{z+1}{z^2+2z+2}\, dz = \frac{1}{2}z + \frac{1}{2}\ln|z^2+2z+2|$. In terms of x and y, we have $C = (-x+y+1)+\ln|(x+y+1)^2+2(x+y+1)+2|$, or equivalently $-x+y+\ln|(x+y+1)^2+2(x+y+1)+2| = C$.

(c) First $h=1, k=-2$ is the solution to the system of linear equations $3h-k-5=0$ and $-h+3k+7=0$. Thus by letting $x=z+1$, $y=w-2$, the DE is reduced to $\frac{dw}{dz} = \frac{3z-w}{-z+3w}$. Now let $w=vz$. Then $\frac{dw}{dz} = z\frac{dv}{dz}+v$, and the DE becomes $z\frac{dv}{dz}+v = \frac{3-v}{-1+3v}$. That is $\int \frac{3v-1}{3-3v^2}\, dv = \int \frac{dz}{z}$. Upon integration, we have $-\frac{1}{2}\ln|1-v^2| - \frac{1}{6}\ln\left|\frac{1+v}{1-v}\right| = \ln|z|+C$. That is $(1+v)^2(1-v)z^3 = C$, or $(z+w)^2(z-w) = C$. In terms of x and y, the general solution is $(x+y+1)^2(y-x+3) = C$.

Note that both $x+y+1=0$ and $y-x+3=0$ are solutions if we choose $C=0$.

3. The constant K measures the rapidity with which the rumour will spread. It depends on how interesting the rumour is, how much the people like to gossip, etc. The right hand side of the equation is designed to be small both near $R=1$ and near $R=1300$, when indeed the rumour can be expected to spread slowly either because not enough or too many people have heard it.

We have
$$\frac{dR}{dt} - 1300KR = -KR^2.$$
This is a Bernoulli equation as discussed in the notes. We solve it by letting $Z = 1/R$, which transforms the equation into a linear one:
$$\frac{dZ}{dt} + 1300KZ = K,$$
with solution
$$\frac{1}{R} = \frac{1}{1300} + Ce^{-1300Kt}.$$
The problem says that this highly interesting rumour was started by one person, so $R(0)=1$. Thus $C = 1299/1300$. Hence
$$\frac{1}{R} = \frac{1}{1300} + \frac{1299}{1300}e^{-1300Kt}.$$
From this, we know that as t tends to infinity, R tends to 1300.

Note that the equation $\frac{dR}{dt} - 1300KR = -KR^2$ is actually separable. It can be solved by direct integration.

4. Let U be the number of Uranium atoms and T the number of Thorium atoms in the coral sample at time t years after the coral died. From the lecture notes we have
$$\frac{T}{U} = \frac{k_U}{k_T - k_U}(1 - e^{(k_U-k_T)t}).$$

Since the half-life of Uranium is 245000 years, we get $k_U = \frac{\ln 2}{245000}$. Similarly, $k_T = \frac{\ln 2}{75000}$. Given T/U is 0.1, we have

$$0.1 = \frac{\frac{\ln 2}{245000}}{\frac{\ln 2}{75000} - \frac{\ln 2}{245000}}(1 - e^{(\frac{\ln 2}{245000} - \frac{\ln 2}{75000})t}).$$

Solving for t, we have $t = 40083.2$. Thus the answer is approximately 40 thousand years.

5. Following the solution for the standard Malthus' Model, we have

   $N = \hat{N}e^{kt}, N(0) = 10000 = \hat{N}$. As $N(2.5) = 10000e^{2.5k} = 11000$, this implies $e^{2.5k} = 1.1$. Thus $k = \frac{1}{2.5}\ln(1.1), e^{2.5k} = 1.1$, so that $k = 0.0381$.

   Then $N(10) = 10000e^{10k} = 10000e^{10(0.0381)} \approx 14600$.

   Also $20000 = 10000e^{kt} \Rightarrow t = \frac{1}{k}\ln(2) \Rightarrow t = 18.18$ hours.

Solutions to Further Exercises

1. Let $y = xv$. Then $y' = v + xv'$. Thus the given differential equation is equivalent to $v + xv' = 1 + v + v^2$ which simplifies to $xv' = 1 + v^2$, or $v'/(1 + v^2) = 1/x$. Upon integration, we obtain $\tan^{-1} v = \ln|x| + c$, where c is an arbitrary constant. Consequently, $y = x\tan(\ln|x| + c)$.

2. Recall that the solution of the improved Malthus' model is

$$N = \frac{B}{s + \left(\frac{B}{\hat{N}} - s\right)e^{-Bt}} = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right)e^{-Bt}}.$$

   Here $\hat{N} = 200$, $B = 1.5$, so at $t = 2$ we have

$$360 = \frac{N_\infty}{1 + \left(\frac{N_\infty}{200} - 1\right)e^{-1.5\times 2}} \Rightarrow 360 + \frac{360}{200}e^{-3}N_\infty - 360e^{-3} = N_\infty.$$

   Thus $N_\infty = \frac{360(1-e^{-3})}{1-\frac{360}{200}e^{-3}} \approx 376$. Also $N(3) = \frac{N_\infty}{1+\left(\frac{N_\infty}{200}-1\right)e^{-4.5}} \approx 372$.

3. We are given $\frac{dy}{dx} = \frac{\mu}{T}\int_0^x \sqrt{\left(\frac{dy}{dt}\right)^2 + 1}\, dt$. Differentiating with respect to x and using the fundamental theorem of calculus, we obtain $\frac{d^2y}{dx^2} = \frac{\mu}{T}\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}$. Let $u = \frac{dy}{dx}$. Then the above equation can be written as $\frac{du}{dx} = \frac{\mu}{T}\sqrt{u^2 + 1}$. That is $\frac{\mu}{T}dx = \frac{du}{\sqrt{u^2+1}}$. Integrating, $\frac{\mu x}{T} + C_1 = \int \frac{du}{\sqrt{u^2+1}}$. That is $\frac{\mu x}{T} + C_1 = \ln|u + \sqrt{u^2 + 1}|$; or equivalently, $Ae^{\frac{\mu x}{T}} = u + \sqrt{u^2 + 1}$, where $A = \pm e^{C_1}$. When $x = 0$, it is at the lowest point of the catenary which is a minimum. This means $u(0) = y'(0) = 0$, giving $A = 1$. Therefore, $e^{\frac{\mu x}{T}} = u + \sqrt{u^2 + 1}$.

Thus $e^{-\frac{\mu x}{T}} = \frac{1}{u+\sqrt{u^2+1}} = \frac{1}{u+\sqrt{u^2+1}} \cdot \frac{u-\sqrt{u^2+1}}{u-\sqrt{u^2+1}} = \frac{u-\sqrt{u^2+1}}{-1} = -u + \sqrt{u^2+1}$. It follows that $u = \frac{1}{2}(e^{\frac{\mu x}{T}} - e^{\frac{-\mu x}{T}})$.

Then $y = \int u\,dx = \int \frac{1}{2}(e^{\frac{\mu x}{T}} - e^{\frac{-\mu x}{T}})\,dx$. That is $y = \frac{T}{2\mu}(e^{\frac{\mu x}{T}} + e^{\frac{-\mu x}{T}}) + C_2$. When $x = 0$, we have $y = 0$. Thus $C_2 = -\frac{T}{\mu}$. Therefore, $y = \frac{T}{\mu}\left(\frac{1}{2}(e^{\frac{\mu x}{T}} + e^{\frac{-\mu x}{T}}) - 1\right)$. That is $y = \frac{T}{\mu}(\cosh x - 1)$. This is the equation of the catenary.