NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 2, 2022/2023

MA2001 Linear Algebra **Homework Assignment 3**

ONLINE QUIZ

1. Find the rank and nullity of $\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 & 3 \\ 0 & -3 & 2 & -1 \\ -1 & 2 & -1 & -1 \end{pmatrix}$.

Apply Gaussian elimination to get a row-echelon form $\boldsymbol{R}$ of $\boldsymbol{A}$:

$$\boldsymbol{A} \xrightarrow{R_3+\frac{1}{2}R_1} \begin{pmatrix} 2 & -1 & 0 & 3 \\ 0 & -3 & 2 & -1 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} \end{pmatrix} \xrightarrow{R_3+\frac{1}{2}R_2} \begin{pmatrix} 2 & -1 & 0 & 3 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \boldsymbol{R}.$$

Since $\boldsymbol{R}$ has 2 pivot points,

$$\operatorname{rank}(\boldsymbol{A}) = 2 \quad \text{and} \quad \operatorname{nullity}(\boldsymbol{A}) = 4 - 2 = 2.$$

2. Consider the vectors

$$\boldsymbol{v}_1 = (2,2,-1,0,0), \quad \boldsymbol{v}_2 = (1,-2,1,2,2), \quad \boldsymbol{v}_3 = (2,0,0,-2,1).$$

View them as row vectors and apply Gaussian elimination:

$$\begin{pmatrix} 2 & 2 & -1 & 0 & 0 \\ 1 & -2 & 1 & 2 & 2 \\ 2 & 0 & 0 & -2 & 1 \end{pmatrix} \xrightarrow[R_3-R_1]{R_2-\frac{1}{2}R_1} \begin{pmatrix} 2 & 2 & -1 & 0 & 0 \\ 0 & -3 & \frac{3}{2} & 2 & 2 \\ 0 & -2 & 1 & -2 & 1 \end{pmatrix} \xrightarrow{R_3-\frac{2}{3}R_2} \begin{pmatrix} 2 & 2 & -1 & 0 & 0 \\ 0 & -3 & \frac{3}{2} & 2 & 2 \\ 0 & 0 & 0 & -\frac{10}{3} & -\frac{1}{3} \end{pmatrix}.$$

Note that the 3^{rd} and 5^{th} columns of the row-echelon form are non-pivot. Then

$$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{e}_3, \boldsymbol{e}_5\}$$

form a basis for $\mathbb{R}^5$.

3. A vector space (a subspace of $\mathbb{R}^n$) is of the form $V = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$. View each $\boldsymbol{v}_i$ as a row vector, and form the $k \times n$ matrix $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_k \end{pmatrix}$. Then V is the row space of $\boldsymbol{A}$.

4. There is some matrix whose row space equals it nullspace.

Assume that such a matrix exists. Let $\boldsymbol{r} = (c_1, \dots, c_n)$ be any row of the matrix. Then $\boldsymbol{r}^{\mathrm{T}}$ lies in its nullspace and thus

$$0 = \boldsymbol{r}\boldsymbol{r}^{\mathrm{T}} = c_1^2 + \cdots + c_n^2,$$

which implies that $c_1 = \cdots = c_n = 0$, i.e., $\boldsymbol{r} = \boldsymbol{0}$. Since $\boldsymbol{r}$ is an arbitrary row of $\boldsymbol{A}$, we have $\boldsymbol{A} = \boldsymbol{0}$. However, this would imply that the row space of $\boldsymbol{A}$ is $\boldsymbol{0}$, and the nullspace of $\boldsymbol{A}$ is $\mathbb{R}^n$.

5. There is some matrix whose column space equals its nullspace.

Let $\boldsymbol{A} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$. Then the column space and nullspace of $\boldsymbol{A}$ are both $\operatorname{span}\{(1,1)^{\mathrm{T}}\}$.

6. For any matrix $\boldsymbol{A}$, we have $\operatorname{nullity}(\boldsymbol{A}) = \operatorname{nullity}(\boldsymbol{A}^{\mathrm{T}})$.

Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then

$$\begin{aligned} \operatorname{nullity}(\boldsymbol{A}^{\mathrm{T}}) &= m - \operatorname{rank}(\boldsymbol{A}^{\mathrm{T}}) = m - \operatorname{rank}(\boldsymbol{A}), \\ \operatorname{nullity}(\boldsymbol{A}) &= n - \operatorname{rank}(\boldsymbol{A}). \end{aligned}$$

So $\operatorname{nullity}(\boldsymbol{A}) = \operatorname{nullity}(\boldsymbol{A}^{\mathrm{T}}) \Leftrightarrow m = n \Leftrightarrow \boldsymbol{A}$ is a square matrix.

7. If $\boldsymbol{A}$ has full rank, then $\operatorname{nullity}(\boldsymbol{A}) = 0$.

Let $\boldsymbol{A}$ be an $m \times n$ matrix with full rank. Then $\operatorname{rank}(\boldsymbol{A}) = \min\{m, n\}$. Hence,

$$\operatorname{nullity}(\boldsymbol{A}) = 0 \Leftrightarrow \operatorname{rank}(\boldsymbol{A}) = n \Leftrightarrow n = \min\{m, n\} \Leftrightarrow n \le m.$$

So if $n > m$, $\operatorname{nullity}(\boldsymbol{A}) \neq 0$.

8. If $\operatorname{nullity}(\boldsymbol{A}) = 0$, then $\boldsymbol{A}$ has full rank.

Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then

$$\operatorname{nullity}(\boldsymbol{A}) = 0 \Rightarrow \operatorname{rank}(\boldsymbol{A}) = n \ge \min\{m, n\} \ge \operatorname{rank}(\boldsymbol{A}).$$

So $\operatorname{rank}(\boldsymbol{A}) = \min\{m, n\} = n$; and thus $\boldsymbol{A}$ has full rank.

9. Suppose $\boldsymbol{A}$ is an $m \times n$ matrix, and $\boldsymbol{B}$ is an $n \times p$ matrix. If $\boldsymbol{A}$ and $\boldsymbol{B}$ both have full rank, then $\boldsymbol{AB}$ has full rank.

Let $\boldsymbol{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 1 \end{pmatrix}$. Then both $\boldsymbol{A}$ and $\boldsymbol{B}$ have full rank. However,

$$\boldsymbol{AB} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

has rank 1. So $\boldsymbol{AB}$ does not have full rank.

10. Suppose $\boldsymbol{A}$ is an $m \times n$ matrix and $\boldsymbol{B}$ is an $n \times p$ matrix. If $\boldsymbol{AB}$ has full rank, then $\boldsymbol{A}$ and $\boldsymbol{B}$ both have full rank.

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $\boldsymbol{AB} = \boldsymbol{A}$ has full rank but $\boldsymbol{B}$ does not have full rank.

11. If $\boldsymbol{A}$ has full rank, then $\boldsymbol{A}^{\mathrm{T}}$ has full rank.

Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then

$$\begin{aligned} \boldsymbol{A} \text{ has full rank} &\Leftrightarrow \operatorname{rank}(\boldsymbol{A}) = \min\{m, n\} \\ &\Leftrightarrow \operatorname{rank}(\boldsymbol{A}^{\mathrm{T}}) = \min\{m, n\} \\ &\Leftrightarrow \boldsymbol{A}^{\mathrm{T}} \text{ has full rank.} \end{aligned}$$

12. Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ be a basis for $\mathbb{R}^3$, where

$$\boldsymbol{v}_1 = (3, -1, 0), \quad \boldsymbol{v}_2 = (-2, 1, -1), \quad \boldsymbol{v}_3 = (1, 0, 1).$$

(i) Compute the transition matrix from the standard basis $\{e_1, e_2, e_3\}$ to S.

(ii) Find $(\boldsymbol{v})_S$ if $\boldsymbol{v} = (1, 2, 3)$.

(iii) It is given that the transition matrix from S to another basis T for $\mathbb{R}^3$ is $\begin{pmatrix} 2 & 1 & 4 \\ 3 & 4 & 4 \\ 1 & -1 & 3 \end{pmatrix}$. Find the vectors in T.

(iv) If $(\boldsymbol{w})_S = (1, -2, -1)$, find $(\boldsymbol{w})_T$.

(v) Determine whether $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 0 \end{pmatrix}$ is the transition matrix from S to some other basis for $\mathbb{R}^3$.

Solution. (i) View all vectors as column vectors and set $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$.

$$(\boldsymbol{A} \mid \boldsymbol{I}) = \left(\begin{array}{rrr|rrr} 3 & -2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 + \frac{1}{3}R_1} \left(\begin{array}{rrr|rrr} 3 & -2 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 + 3R_2} \left(\begin{array}{rrr|rrr} 3 & -2 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & 0 \\ 0 & 0 & 2 & 1 & 3 & 1 \end{array}\right) \xrightarrow[\frac{1}{2}R_3]{\frac{1}{3}R_1 \\ 3R_2} \left(\begin{array}{rrr|rrr} 1 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right)$$

$$\xrightarrow[R_2 - R_3]{R_1 - \frac{1}{3}R_3} \left(\begin{array}{rrr|rrr} 1 & -\frac{2}{3} & 0 & \frac{1}{6} & -\frac{1}{2} & -\frac{1}{6} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right) \xrightarrow{R_1 + \frac{2}{3}R_2} \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right).$$

Since $\boldsymbol{A}$ is the transition matrix from S to $B = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, the transition matrix from B to S is given by:

$$\boldsymbol{A}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

(ii) Recall that $\boldsymbol{A}[\boldsymbol{v}]_S = \boldsymbol{v}$. Then

$$[\boldsymbol{v}]_S = \boldsymbol{A}^{-1}\boldsymbol{v} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}.$$

So $(\boldsymbol{v})_S = (0, 2, 5)$.

(iii) The transition matrix from S to T is given by $\boldsymbol{P} = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 4 & 4 \\ 1 & -1 & 3 \end{pmatrix}$. Then

$$(\boldsymbol{P} \mid \boldsymbol{I}) = \left(\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 3 & 4 & 4 & 0 & 1 & 0 \\ 1 & -1 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - \frac{1}{2}R_1]{R_2 - \frac{3}{2}R_1} \left(\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & \frac{5}{2} & -2 & -\frac{3}{2} & 1 & 0 \\ 0 & -\frac{3}{2} & 1 & -\frac{1}{2} & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 + \frac{3}{5}R_2} \left(\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & \frac{5}{2} & -2 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & -\frac{1}{5} & -\frac{7}{5} & \frac{3}{5} & 1 \end{array}\right) \xrightarrow[-5R_3]{\substack{\frac{1}{2}R_1 \\ \frac{2}{5}R)2}} \left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{4}{5} & -\frac{3}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 1 & 7 & -3 & -5 \end{array}\right)$$

$$\xrightarrow[R_2 + \frac{4}{5}R_3]{R_1 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & -\frac{27}{2} & 6 & 10 \\ 0 & 1 & 0 & 5 & -2 & -4 \\ 0 & 0 & 1 & 7 & -3 & -5 \end{array}\right) \xrightarrow{R_1 - \frac{1}{2}R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -16 & 7 & 12 \\ 0 & 1 & 0 & 5 & -2 & -4 \\ 0 & 0 & 1 & 7 & -3 & -5 \end{array}\right).$$

Since $\boldsymbol{P}$ is the transition matrix from S to $T = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, the transition matrix from T to S is

$$\boldsymbol{P}^{-1} = \begin{pmatrix} -16 & 7 & 12 \\ 5 & -2 & -4 \\ 7 & -3 & -5 \end{pmatrix} = \begin{pmatrix} [\boldsymbol{u}_1]_S & [\boldsymbol{u}_2]_S & [\boldsymbol{u}_3]_S \end{pmatrix}.$$

So

$$\begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} [\boldsymbol{u}_1]_S & [\boldsymbol{u}_2]_S & [\boldsymbol{u}_3]_S \end{pmatrix} = \begin{pmatrix} -51 & 22 & 39 \\ 21 & -9 & -16 \\ 2 & -1 & -1 \end{pmatrix}.$$

Hence,

$$\boldsymbol{u}_1 = (-51, 21, 2), \quad \boldsymbol{u}_2 = (22, -9, -1), \quad \boldsymbol{u}_3 = (39, -16, -1).$$

If $(\boldsymbol{w})_S = (1, -2, -1)$, then

$$[\boldsymbol{w}]_T = \boldsymbol{P}[\boldsymbol{w}]_S = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 4 & 4 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -9 \\ 0 \end{pmatrix},$$

or $(\boldsymbol{w})_T = (-4, -9, 0)$.

Let $\boldsymbol{B} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 0 \end{pmatrix}$. Then

$$\det(\boldsymbol{B}) \xlongequal[R_3+R_1]{R_2-2R_1} \begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{vmatrix} \xlongequal{R_3+R_2} \begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

Since $\boldsymbol{B}$ is singular, it cannot be a transition matrix (of bases).

Written Assignment

1. Compute a basis for the nullspace of $\boldsymbol{A} = \begin{pmatrix} 2 & 2 & 0 & -2 & 3 \\ -2 & 3 & -3 & 3 & -2 \\ 0 & 4 & -3 & -2 & 3 \end{pmatrix}$.

Solution. Apply Gauss-Jordan elimination to $\boldsymbol{A}$:

$$\boldsymbol{A} \xrightarrow{R_2+R_1} \begin{pmatrix} 2 & 2 & 0 & -2 & 3 \\ 0 & 5 & -3 & 1 & 1 \\ 0 & 4 & -3 & -2 & 3 \end{pmatrix} \xrightarrow{R_3-\frac{4}{5}R_2} \begin{pmatrix} 2 & 2 & 0 & -2 & 3 \\ 0 & 5 & -3 & 1 & 1 \\ 0 & 0 & -\frac{3}{5} & -\frac{14}{5} & \frac{11}{5} \end{pmatrix}$$

$$\xrightarrow[-\frac{5}{3}R_3]{\substack{\frac{1}{2}R_1 \\ \frac{1}{5}R_2}} \begin{pmatrix} 1 & 1 & 0 & -1 & \frac{3}{2} \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{14}{3} & -\frac{11}{3} \end{pmatrix} \xrightarrow{R_2+\frac{3}{5}R_3} \begin{pmatrix} 1 & 1 & 0 & -1 & \frac{3}{2} \\ 0 & 1 & 0 & 3 & -2 \\ 0 & 0 & 1 & \frac{14}{3} & -\frac{11}{3} \end{pmatrix} \xrightarrow{R_1-R_2} \begin{pmatrix} 1 & 0 & 0 & -4 & \frac{7}{2} \\ 0 & 1 & 0 & 3 & -2 \\ 0 & 0 & 1 & \frac{14}{3} & -\frac{11}{3} \end{pmatrix}.$$

Set $x_4 = s$ and $x_5 = t$ as arbitrary parameters. Then

$$x_1 = 4s - \tfrac{7}{2}t, \quad x_2 = -3s + 2t, \quad x_3 = -\tfrac{14}{3}s + \tfrac{11}{3}t.$$

So

$$\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5) = s\left(4, -3, -\tfrac{14}{3}, 1, 0\right) + t\left(-\tfrac{7}{2}, 2, \tfrac{11}{3}, 0, 1\right).$$

Then the nullspace of $\boldsymbol{A}$ has a basis $\left\{\left(4, -3, -\frac{14}{3}, 1, 0\right),\ \left(-\frac{7}{2}, 2, \frac{11}{3}, 0, 1\right)\right\}$

2. Compute a basis for the vector space

$$\{(s-2t, -s-2u, 3s+t+5u, 3s-2t+4u) \in \mathbb{R}^4 \mid s, t, u \in \mathbb{R}\}.$$

Proof. The vector space consists all vectors of the form

$$(s-2t, -s-2u, 3s+t+5u, 3s-2t+4u) = s(1,-1,3,3) + t(-2,0,1,-2) + u(0,-2,5,4).$$

Hence, the vector space $V = \operatorname{span}\{(1,-1,3,3), (-2,0,1,-2), (0,-2,5,4)\}$.

View the vectors as row vectors to form a matrix

$$\begin{pmatrix} 1 & -1 & 3 & 3 \\ -2 & 0 & 1 & -2 \\ 0 & -2 & 5 & 4 \end{pmatrix} \xrightarrow{R_2+2R_1} \begin{pmatrix} 1 & -1 & 3 & 3 \\ 0 & -2 & 7 & 4 \\ 0 & -2 & 5 & 4 \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} 1 & -1 & 3 & 3 \\ 0 & -2 & 7 & 4 \\ 0 & 0 & -2 & 0 \end{pmatrix}.$$

Since all rows of the row-echelon form are nonzero, the matrix has rank 3. Then

$$\{(1,-1,3,3),\ (-2,0,1,-2),\ (0,-2,5,4)\} \quad \text{or} \quad \{(1,-1,3,3),\ (0,-2,7,4),\ (0,0,-2,0)\}$$

is a basis of the given vector space.

3. Suppose $\boldsymbol{A}$ is an $n \times n$ matrix and $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{A}^3\boldsymbol{x} = \boldsymbol{0}$ but $\boldsymbol{A}^2\boldsymbol{x} \neq \boldsymbol{0}$. Prove that $\{\boldsymbol{x}, \boldsymbol{A}\boldsymbol{x}, \boldsymbol{A}^2\boldsymbol{x}\}$ is linearly independent.

Proof. Let $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$\boldsymbol{0} = c_1\boldsymbol{x} + c_2\boldsymbol{A}\boldsymbol{x} + c_3\boldsymbol{A}^2\boldsymbol{x}.$$

Pre-multiplication of $\boldsymbol{A}^2$ yields

$$\boldsymbol{0} = c_1\boldsymbol{A}^2\boldsymbol{x} + c_2\boldsymbol{A}^3\boldsymbol{x} + c_3\boldsymbol{A}^4\boldsymbol{x} = c_1\boldsymbol{A}^2\boldsymbol{x}.$$

It is given that $\boldsymbol{A}^2\boldsymbol{x} \neq 0$. Then $c_1 = 0$, and the relation is

$$\boldsymbol{0} = c_2\boldsymbol{A}\boldsymbol{x} + c_3\boldsymbol{A}^2\boldsymbol{x}.$$

Pre-multiplication of $\boldsymbol{A}$ yields

$$\boldsymbol{0} = c_2\boldsymbol{A}^2\boldsymbol{x} + c_3\boldsymbol{A}^3\boldsymbol{x} = c_2\boldsymbol{A}^2\boldsymbol{x}.$$

Again, since $\boldsymbol{A}^2\boldsymbol{x} \neq 0$, we must have $c_2 = 0$, and the relation is

$$\boldsymbol{0} = c_3\boldsymbol{A}^2\boldsymbol{x}.$$

Using the condition $\boldsymbol{A}^2\boldsymbol{x} \neq \boldsymbol{0}$ again, we have $c_3 = 0$.

Since $c_1 = c_2 = c_3 = 0$, we conclude that $\boldsymbol{x}, \boldsymbol{A}\boldsymbol{x}, \boldsymbol{A}^2\boldsymbol{x}$ are linearly independent vectors.

4. Suppose $\boldsymbol{M}$ is the matrix $\left(\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \end{array}\right)$ with blocks $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$.

(a) For each of the following cases, write down 1×1 matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ such that the condition is fulfilled. Write down nullity($\boldsymbol{M}$) and nullity($\boldsymbol{A}$) for your choices.

(i) nullity($\boldsymbol{M}$) > nullity($\boldsymbol{A}$);

(ii) nullity($\boldsymbol{M}$) = nullity($\boldsymbol{A}$);

(iii) nullity($\boldsymbol{M}$) < nullity($\boldsymbol{A}$).

(b) Prove that in general, rank($\boldsymbol{M}$) ≥ rank($\boldsymbol{A}$).

Solution. (a) Let $\boldsymbol{M} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then nullity($\boldsymbol{M}$) = 2 > 1 = nullity($\boldsymbol{A}$).

Let $\boldsymbol{M} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then nullity($\boldsymbol{M}$) = 1 = 1 = nullity($\boldsymbol{A}$).

Let $\boldsymbol{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then nullity($\boldsymbol{M}$) = 0 < 1 = nullity($\boldsymbol{A}$).

(b) Suppose $\boldsymbol{X} = \left(\begin{array}{c|c} \boldsymbol{X}_1 & \boldsymbol{X}_2 \end{array} \right)$. Apply Gaussian elimination to get a row-echelon form

$$\boldsymbol{R} = \left(\begin{array}{c|c} \boldsymbol{R}_1 & \boldsymbol{R}_2 \end{array} \right).$$

Note that $\boldsymbol{R}_1$ is a row-echelon form of $\boldsymbol{X}_1$. So

$$\begin{aligned} \operatorname{rank}(\boldsymbol{X}_1) &= \text{the number of pivot columns of } \boldsymbol{R}_1 \\ &\le \text{the number of pivot columns of } \boldsymbol{R} \\ &= \operatorname{rank}(\boldsymbol{X}). \end{aligned}$$

Suppose $\boldsymbol{Y} = \left(\begin{array}{c} \boldsymbol{Y}_1 \\ \hline \boldsymbol{Y}_2 \end{array} \right)$. Then $\boldsymbol{Y}^{\mathrm{T}} = \left(\begin{array}{c|c} \boldsymbol{Y}_1^{\mathrm{T}} & \boldsymbol{Y}_2^{\mathrm{T}} \end{array} \right)$. So

$$\operatorname{rank}(\boldsymbol{Y}_1) = \operatorname{rank}(\boldsymbol{Y}_1^{\mathrm{T}}) \le \operatorname{rank}(\boldsymbol{Y}^{\mathrm{T}}) = \operatorname{rank}(\boldsymbol{Y}).$$

Let $\boldsymbol{M} = \left(\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \end{array} \right)$. Then

$$\operatorname{rank}(\boldsymbol{A}) \le \operatorname{rank}(\boldsymbol{A} \mid \boldsymbol{B}) \le \operatorname{rank} \left(\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \end{array} \right) = \operatorname{rank}(\boldsymbol{M}).$$

5. Suppose $\boldsymbol{A}$ is an $m \times n$ matrix. Prove that if $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has a unique solution, then for every $\boldsymbol{b} \in \mathbb{R}^n$, the system $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{b}$ is consistent.

Proof. Suppose that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has a unique solution, i.e., the nullspace of $\boldsymbol{A}$ is $\{\boldsymbol{0}\} \subseteq \mathbb{R}^n$. Then nullity$(\boldsymbol{A}) = 0$. It follows that

$$\operatorname{rank}(\boldsymbol{A}^{\mathrm{T}}) = \operatorname{rank}(\boldsymbol{A}) = n - \operatorname{nullity}(\boldsymbol{A}) = n.$$

Note that the column space of $\boldsymbol{A}^{\mathrm{T}}$ is a subspace of $\mathbb{R}^n$ having dimension n. Then the column space of $\boldsymbol{A}^{\mathrm{T}}$ equals $\mathbb{R}^n$.

If $\boldsymbol{b} \in \mathbb{R}^n$, then $\boldsymbol{b}$ belongs to the column space of $\boldsymbol{A}^{\mathrm{T}}$, which means that $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{b}$ is consistent.