PG .82

33. Let $\boldsymbol{A}$ be the 4×4 matrix obtained from $\boldsymbol{I}$ by the following sequence of elementary row operations:

$$\boldsymbol{I} \xrightarrow{\frac{1}{2}R_2} \xrightarrow{R_1 - R_2} \xrightarrow{R_2 \leftrightarrow R_4} \xrightarrow{R_3 + 3R_1} \boldsymbol{A}$$

(a) Write $\boldsymbol{A}$ as a product of four elementary matrices.

(b) Write $\boldsymbol{A}^{-1}$ as a product of four elementary matrices.

a)

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_2 \leftrightarrow R_4 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$R_3 + 3R_1$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

b)

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

39. A manufacturer makes three types of chairs A, B, C. The company has available 260 units of wood, 60 units of upholstery and 240 units of labor. The manufacturer wants a production schedule that uses all of these resources. The various products require the following amounts of resources.

	A	B	C
Wood	4	4	3
Upholstery	0	1	2
Labor	2	4	5

(a) Find the inverse of the data matrix above and hence determine how many pieces of each product should be manufactured.

(b) If the amount of wood is increased by 10 units, how will this change the number of type C chairs produced?

a) $\begin{cases} 4x_1 + 4x_2 + 3x_3 = 260 \\ x_2 + 2x_3 = 60 \\ 2x_1 + 4x_2 + 5x_3 = 240 \end{cases}$

$$\begin{pmatrix} 4 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 260 \\ 60 \\ 240 \end{pmatrix}$$

Also Can Use Cramer's Rule

Let $A = \begin{pmatrix} 4 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 5 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

A into reduced-row-echelon form $\longrightarrow$ Also can use a determinant formula

$2R_3 - R_1$ $\begin{pmatrix} 4 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & 4 & 7 \end{pmatrix}$ $R_3 - 4R_2$ $\begin{pmatrix} 4 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$ $\frac{1}{4}R_1$, $-1R_3$ $\begin{pmatrix} 1 & 1 & \frac{3}{4} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

$R_1 - \frac{3}{4}R_3$, $R_2 - 2R_3$ $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$R_1 - R_2$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Applying the same step to $I_{3\times3}$

$2R_3 - R_1$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$

$R_3 - 4R_2$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -4 & 2 \end{pmatrix}$

$\frac{1}{4}R_1$, $-1R_3$ $\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 4 & -2 \end{pmatrix}$

$R_1 - \frac{3}{4}R_3$
$R_2 - 2R_3$

$$\begin{pmatrix} -\frac{1}{2} & -3 & \frac{3}{2} \\ -2 & -7 & 4 \\ 1 & 4 & -2 \end{pmatrix}$$

$R_1 - R_2$

$$\begin{pmatrix} \frac{3}{2} & 4 & -\frac{5}{2} \\ -2 & -7 & 4 \\ 1 & 4 & -2 \end{pmatrix}$$

Using the inverse to find the solution

$$\begin{pmatrix} \frac{3}{2} & 4 & -\frac{5}{2} \\ -2 & -7 & 4 \\ 1 & 4 & -2 \end{pmatrix}\begin{pmatrix} 260 \\ 60 \\ 240 \end{pmatrix} = \begin{pmatrix} 130\times3 + 240 - 120\times5 \\ -520 - 420 + 960 \\ 260 + 240 - 480 \end{pmatrix}$$

$x_1 = 30$ wood
$x_2 = 20$ upholstery
$x_3 = 20$ labor

b) Since $10 \times [\text{the } (3,1)\text{-entry of } A^{-1}] = 10$
the number of chairs of type C is increased by 10.

42. Prove Theorem 2.4.14:

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two square matrices of the same order. Prove that if $\boldsymbol{A}$ is singular, then $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are singular. (Since we use Theorem 2.4.14 to prove Theorem 2.5.22.2, we cannot use determinants to do this question. Work out the proof using the definition of inverses together with Theorem 2.4.12.)

Contradiction:
Assume AB is invertible. Let C be the inverse of AB.
Then $(AB)C = I$ and hence $A(BC) = I$, A is invertable which contradicts that A is singular. Hence AB must be singular. Same for BA can be done.

43. Let $\boldsymbol{A}$ be an $m \times n$ matrix which is row equivalent to the following matrix:

$$\begin{pmatrix} \boldsymbol{R} \\ 0 \ \cdots \ 0 \end{pmatrix}$$

where the last row is a zero row and $\boldsymbol{R}$ is an $(m-1) \times n$ matrix. Show that there exists an $m \times 1$ matrix $\boldsymbol{b}$ such that the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent.

(Hint: If $\boldsymbol{A}$ is row equivalent to a matrix $\boldsymbol{C}$, then $\boldsymbol{A} = \boldsymbol{E_k} \cdots \boldsymbol{E_1 C}$ for some elementary matrices $\boldsymbol{E_1}, \ldots, \boldsymbol{E_k}$.)

Suppose $A = E_k \cdots E_1 \begin{pmatrix} R \\ 0 \cdots 0 \end{pmatrix}$ for some elementary matrices $E_1 \ldots E_k$.

Let $b = E_k \cdots E_1 \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$ which is one of many possible choices of b.

Then $Ax = b \iff \begin{pmatrix} R \\ 0 \cdots 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$ which is inconsistent.

53. Let $\boldsymbol{A}$ be a 4×4 matrix such that $\det(\boldsymbol{A}) = 9$. Find

(a) $\det(3\boldsymbol{A})$, (b) $\det(\boldsymbol{A}^{-1})$, (c) $\det(3\boldsymbol{A}^{-1})$, (d) $\det((3\boldsymbol{A})^{-1})$.

a) $\det(3A) = 3^4 \det(A) = 3^4 \times 9 = 729$

b) $\det(A^{-1}) = \det(A)^{-1} = \frac{1}{9}$

c) $\det(3A^{-1}) = 3^4 \det(A^{-1}) = 3^4 \times \frac{1}{9} = 9$

d) $\det((3A)^{-1}) = \det(3A)^{-1} = \frac{1}{729}$

61. Determine which of the following statements are true. Justify your answer.

(a) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices of the same size, then $\det(\boldsymbol{A}+\boldsymbol{B}) = \det(\boldsymbol{A}) + \det(\boldsymbol{B})$.

(b) If $\boldsymbol{A}$ is a square matrix, then $\det(\boldsymbol{A}+\boldsymbol{I}) = \det(\boldsymbol{A}^T + \boldsymbol{I})$.

(c) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices of the same size such that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^{-1}$ for some invertible matrix $\boldsymbol{P}$, then $\det(\boldsymbol{A}) = \det(\boldsymbol{B})$.

(d) If $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are square matrices of the same size such that $\det(\boldsymbol{A}) = \det(\boldsymbol{B})$, then $\det(\boldsymbol{A}+\boldsymbol{C}) = \det(\boldsymbol{B}+\boldsymbol{C})$.

b) True $\det(A+I) = \det((A+I)^T) = \det(A^T + I)$

c) True $\det(A) = \det(PBP^{-1}) = \det(P) \cdot \det(B) \cdot \det(P^{-1})$

$= \det(P) \cdot \det(P^{-1}) \cdot \det(B)$

$= \det(P) \cdot \det(P)^{-1} \cdot \det(B)$

$= \det(B)$