Student Number: ______________________

NATIONAL UNIVERSITY OF SINGAPORE

MA1101R - Linear Algebra I

(Semester 2 : AY2014/2015)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number clearly in the space provided at the top of this page.** This booklet (and only this booklet) will be collected at the end of the examination.

2. Please write your matriculation/student number only. Do not write your name.

3. This examination paper contains **FOUR** questions and comprises **NINETEEN** printed pages.

4. Answer **ALL** questions.

5. This is a CLOSED BOOK (with helpsheet) examination.

6. You are allowed to use two A4 size helpsheets.

7. You may use scientific calculators. However, you should lay out systematically the various steps in the calculations)

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
Total	

Question 1 [25 marks]

(a) [15 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

(i) Is $\boldsymbol{A}$ invertible? Justify your answer.

(ii) Find elementary matrices $\boldsymbol{E_1}, \boldsymbol{E_2}, \cdots, \boldsymbol{E_k}$ such that $\boldsymbol{A} = \boldsymbol{E_k} \cdots \boldsymbol{E_2}\boldsymbol{E_1}\boldsymbol{R}$ where $\boldsymbol{R}$ is a matrix in row-echelon form.

(iii) Find a matrix $\boldsymbol{P}$ that orthogonally diagonalizes $\boldsymbol{A}$ and determine $\boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P}$. (You may assume that the characteristic polynomial for $\boldsymbol{A}$ is $(\lambda + 1)^2(\lambda - 2)$.)

Show your working below.

More working space for Question 1(a)

Question 1

(b) [10 marks]

Let $\boldsymbol{B} = \begin{pmatrix} -1 & k & 2 \\ -3 & 2 & 1 \\ k & 0 & 1 \end{pmatrix}$ where k is a real number.

(i) Compute $\det(\boldsymbol{B})$ in terms of k.

(ii) Find all values of k such that $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution. Justify your answer.

(iii) Find all values of k such that the solution space of $\boldsymbol{Bx} = \boldsymbol{0}$ has dimension at least 1. Justify your answer.

(iv) What is the smallest possible value of $\text{rank}(\boldsymbol{B})$? Justify your answer.

(v) Are there values of k such that the solution space of $\boldsymbol{B}^T\boldsymbol{x} = \boldsymbol{0}$ is a plane in $\mathbb{R}^3$ that contains the origin? Justify your answer.

Show your working below.

More working space for Question 1(b)

Question 2 [25 marks]

(a) [11 marks]

Let $V = \{(a, a, a, 0) \mid a \in \mathbb{R}\}$.

(i) Find a basis for V and determine $\dim(V)$.

(ii) Find a subspace W of $\mathbb{R}^4$ such that $\dim(W) = 3$ and $\dim(W \cap V) = 1$. Justify your answer.

(iii) Let $U = \{\boldsymbol{u} \in \mathbb{R}^4 \mid \boldsymbol{u} \cdot \boldsymbol{v} = 0 \text{ for all } \boldsymbol{v} \in V\}$. Find a basis for and determine the dimension of U.

Show your working below.

More working space for Question 2(a)

Question 2

(b) [14 marks]

Suppose $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ and $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ are two different bases for $\mathbb{R}^3$. Let

$$\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 2 \end{pmatrix}$$

be the transition matrix from S to T.

(i) Write down the coordinate vectors $[\boldsymbol{u_1}]_T$, $[\boldsymbol{u_2}]_T$ and $[\boldsymbol{u_3}]_T$.

(ii) Suppose

$$\boldsymbol{v_1} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \boldsymbol{v_3} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}, \quad \boldsymbol{u_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Find vectors $\boldsymbol{v_2}, \boldsymbol{u_1}, \boldsymbol{u_3}$.

(iii) Let $\boldsymbol{w} = (-2, 1, 1)$. You may assume that $[\boldsymbol{w}]_S = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. Compute $[\boldsymbol{w}]_T$.

(iv) Use your answer in (iii) to verify that your answer for $\boldsymbol{v_2}$ in (ii) is correct.

Show your working below.

More working space for Question 2(b)

Continue on last page if you need more writing space. $\cdots$ – 10–

Question 3 [25 marks]

(a) [10 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 & 1 \end{pmatrix}$.

(i) Find a basis for each of the row space and column space of $\boldsymbol{A}$ and state its rank.

(ii) Extend the basis for the row space of $\boldsymbol{A}$ in part (i) to a basis for $\mathbb{R}^5$.

(iii) Is it possible to find a full rank 5×3 matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \mathbf{0}$? Justify your answer.

Show your working below.

More working space for Question 3(a)

Question 3

(b) [9 marks]

Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ with $\boldsymbol{v}_1 = (1, 1, 0, 1), \boldsymbol{v}_2 = (0, 1, 1, -1), \boldsymbol{v}_3 = (1, -1, 1, 0)$.

(i) Show that S is an orthogonal set.

(ii) Let $\boldsymbol{w} = (5, -2, 2, 3)$. Find the projection of $\boldsymbol{w}$ onto the subspace $V = \text{span}(S)$. Does $\boldsymbol{w}$ belong to V?

(iii) Without performing Gaussian elimination, can you tell whether the system

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 2 \\ 3 \end{pmatrix}$$

has no solution, exactly one solution, or infinitely many solutions? Why?

Show your working below.

More working space for Question 3(b)

Question 3

(c) [6 marks]

(i) Let $\boldsymbol{A}$ be a 2×3 matrix and $\boldsymbol{b}$ a 2×1 column vector. Suppose

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 5 \end{pmatrix} \quad \text{and} \quad \boldsymbol{A}^T\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Find the least squares solutions of $\boldsymbol{Ax} = \boldsymbol{b}$.

(ii) True or false: Given any 2×3 matrix $\boldsymbol{M}$ and 2×1 column vector $\boldsymbol{c}$, the linear system $\boldsymbol{Mx} = \boldsymbol{c}$ always has infinitely many least squares solutions. Justify your answer.

Show your working below.

More working space for Question 3(c)

Question 4 [25 marks]

(a) [21 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & -3 \\ 3 & -3 & 1 \end{pmatrix}$ and $\boldsymbol{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

(i) Compute $\boldsymbol{A}\boldsymbol{v}_1, \boldsymbol{A}\boldsymbol{v}_2$ and $\boldsymbol{A}\boldsymbol{v}_3$.

(ii) Write down all the eigenvalues of $\boldsymbol{A}$.

(iii) For each eigenvalue of $\boldsymbol{A}$, write down a basis for the corresponding eigenspace.

(iv) Diagonalize the matrix $\boldsymbol{A}$.

(v) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation with standard matrix $\boldsymbol{A}$. Find the range $R(T)$ and kernel $\ker(T)$ of T. Justify your answers.

(vi) Write down the equation of a plane P in the xyz-space that is not transformed to a different plane* under the linear transformation T in part (v).
(* This means for any vector $\boldsymbol{v}$ on plane P, $T(\boldsymbol{v})$ would still be a vector on P.)

(vii) Find a linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$(S \circ T)(\boldsymbol{v}_1) = 4\boldsymbol{v}_1, \quad (S \circ T)(\boldsymbol{v}_2) = 4\boldsymbol{v}_2, \quad (S \circ T)(\boldsymbol{v}_3) = -4\boldsymbol{v}_3.$$

(You may give your answer in terms of the standard matrix of S.)

Show your working below.

More working space for Question 4(a)

More working space for Question 4(a)

Question 4

(b) [4 marks]

Prove the following:
Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be two $n \times n$ matrices. Suppose $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ is a set of linearly independent eigenvectors for both $\boldsymbol{M}$ and $\boldsymbol{N}$. Then $\boldsymbol{MN} = \boldsymbol{NM}$.

Show your working below.

END OF PAPER