Solutions to Tutorial 4

MA1521 Calculus for Computing

1. (a)

$$\int_1^{\sqrt{2}} \frac{s^2+\sqrt{s}}{s^2}\,ds = \int_1^{\sqrt{2}} (1+s^{-3/2})\,ds = (\sqrt{2}-1) - 2s^{-1/2}\Big|_1^{\sqrt{2}}$$
$$= (\sqrt{2}-1) - \frac{2}{\sqrt[4]{2}} + 2 = 1+\sqrt{2}-2^{3/4}.$$

(b)

$$\int_{-4}^{4} |x|\,dx = \int_0^4 x\,dx + \int_{-4}^{0}(-x)\,dx = \frac{1}{2}4^2 + \frac{1}{2}4^2 = 16.$$

(c)

$$\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|)\,dx = \int_0^{\pi/2} \frac{1}{2}(\cos x + |\cos x|)\,dx + \int_{\pi/2}^{\pi} \frac{1}{2}(\cos x + |\cos x|)\,dx$$
$$= \int_0^{\pi/2} \cos x\,dx + 0 = \sin x\Big|_0^{\pi/2} = 1.$$

(d)

$$\int_0^{\pi} \sin^2\Big(1+\frac{\theta}{2}\Big)\,d\theta = \int_0^{\pi} \frac{1}{2}\Big[1-\cos(2+\theta)\Big]\,d\theta = \frac{1}{2}\pi - \frac{1}{2}\sin(2+\theta)\Big|_0^{\pi}$$
$$= \frac{1}{2}\pi - \frac{1}{2}\Big[\sin(2+\pi)-\sin 2\Big] = \frac{1}{2}\pi + \sin 2.$$

2. The Fundamental Theorem of Calculus (I) says that

$$\frac{d}{du}\int_a^u f(t)\,dt = f(u)$$

for a continuous function f. Here a is a fixed number. It is a sort of *chain rule* to find

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt.$$

To see this, let

$$F(u) = \int_a^u f(t)\,dt \quad \text{and} \quad u = g(x).$$

It follows that

$$\frac{dF}{du} = \frac{d}{du}\int_a^u f(t)\,dt = f(u).$$

Furthermore,

$$F \circ g(x) = F(g(x)) = \int_a^{g(x)} f(t)\,dt.$$

By the chain rule, we have

$$\frac{dF(g(x))}{dx} = \frac{dF}{du}\frac{dg(x)}{dx} = f(u)\,g'(x) = f(g(x))\,g'(x).$$

(a) $y = \displaystyle\int_0^{\sqrt{x}} \cos t\,dt; \quad \cos\sqrt{x}\cdot\frac{d}{dx}\sqrt{x} = \frac{\cos\sqrt{x}}{2\sqrt{x}}.$

(b) $y = \displaystyle\int_0^{x^2} \cos\sqrt{t}\,dt; \quad \cos\sqrt{x^2}\cdot 2x = 2x\cos|x| = 2x\cos x.$

(c) $y = \displaystyle\int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}, \quad |x| < \frac{\pi}{2}. \quad \frac{1}{\sqrt{1-\sin^2 x}}\cdot\frac{d}{dx}\sin x = \frac{1}{\cos x}\cos x = 1.$

3. (a) $\displaystyle\int x^{1/2}\sin(x^{3/2}+1)\,dx = \int \sin(x^{3/2}+1)\cdot\frac{2}{3}\,d(x^{3/2}+1) = -\frac{2}{3}\cos(x^{3/2}+1) + C.$

Alternatively, let $u = x^{3/2}+1$. Then $du = \frac{3}{2}x^{\frac{1}{2}}\,dx$. Thus $\displaystyle\int x^{1/2}\sin(x^{3/2}+1)\,dx = \int \frac{2}{3}\sin u\,du$ $= -\dfrac{2}{3}\cos u + C = -\dfrac{2}{3}\cos(x^{3/2}+1) + C.$

(b) $\displaystyle\int \csc^2 2t\,\cot 2t\,dt = \int \cot 2t\cdot\left(-\frac{1}{2}\right)d(\cot 2t) = -\frac{1}{4}\cot^2 2t + C.$

(c) $\displaystyle\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta = -\int \sin\frac{1}{\theta}\,d\left(\sin\frac{1}{\theta}\right) = -\frac{1}{2}\sin^2\frac{1}{\theta} + C.$

(d) $\displaystyle\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)}\,dx = \int \frac{6\,d(\tan^3 x+2)}{(2+\tan^3 x)} = 6\ln|\tan^3 x + 2| + C.$

(e) $\displaystyle\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\cos^3\sqrt{\theta}}\,d\theta = -2\int (\cos\sqrt{\theta})^{-3}\,d(\cos\sqrt{\theta}) = (\cos\sqrt{\theta})^{-2} + C = \sec^2\sqrt{\theta} + C.$

4. (a)

$$\begin{aligned}\int x\sin\left(\frac{x}{2}\right)dx &= -2\int x\,d\left[\cos\left(\frac{x}{2}\right)\right] = -2\left[x\cos\left(\frac{x}{2}\right) - \int \cos\left(\frac{x}{2}\right)dx\right] + C\\ &= -2\left[x\cos\left(\frac{x}{2}\right) - 2\int \cos\left(\frac{x}{2}\right)d\left(\frac{x}{2}\right)\right] + C\\ &= -2\left[x\cos\left(\frac{x}{2}\right) - 2\sin\left(\frac{x}{2}\right)\right] + C.\end{aligned}$$

(b)

$$\int t^2 e^{4t}\,dt = \frac{1}{4}\int t^2\,d(e^{4t}) = \frac{1}{4}\left[t^2 e^{4t} - 2\int t\,e^{4t}\,dt\right] + C = \frac{1}{4}\left[t^2 e^{4t} - \frac{1}{2}\int t\,d(e^{4t})\right] + C$$

$$= \frac{1}{4}\left[t^2 e^{4t} - \frac{1}{2}\left(t\,e^{4t} - \int e^{4t}\,dt\right)\right] + C$$

$$= \frac{1}{4}\left[t^2 e^{4t} - \frac{1}{2}\left(t\,e^{4t} - \frac{e^{4t}}{4}\right)\right] + C \quad \text{(continue to simplify)}.$$

(c)

$$\int e^{-y}\cos y\,dy = \int e^{-y}\,d(\sin y) = e^{-y}\sin y + \int e^{-y}\sin y\,dy + C$$

$$= e^{-y}\sin y - \int e^{-y}\,d(\cos y) + C = e^{-y}\sin y - e^{-y}\cos y - \int e^{-y}\cos y\,dy$$

$$\Rightarrow \quad \int e^{-y}\cos y\,dy = \frac{e^{-y}}{2}(\sin y - \cos y) + C.$$

(There is no harm to rename $C/2$ as C.)

(d)

$$\int \theta^2 \sin(2\theta)\,d\theta = -\frac{1}{2}\int \theta^2 d[\cos(2\theta)] = -\frac{1}{2}\left[\theta^2\cos(2\theta) - 2\int \theta\cos(2\theta)\,d\theta\right] + C$$

$$= -\frac{1}{2}\left[\theta^2\cos(2\theta) - \int \theta\,d[\sin(2\theta)]\right] + C$$

$$= -\frac{1}{2}\left[\theta^2\cos(2\theta) - \theta\sin(2\theta) + \int \sin(2\theta)\,d\theta\right] + C$$

$$= -\frac{1}{2}\left[\theta^2\cos(2\theta) - \theta\sin(2\theta) - \frac{1}{2}\cos(2\theta)\right] + C.$$

(e)

$$\int z(\ln z)^2\,dz = \frac{1}{2}\int (\ln z)^2\,d(z^2) = \frac{1}{2}\left[z^2(\ln z)^2 - 2\int z(\ln z)\,dz\right] + C$$

$$= \frac{1}{2}\left[z^2(\ln z)^2 - \int (\ln z)\,d(z^2)\right] + C$$

$$= \frac{1}{2}\left[z^2(\ln z)^2 - z^2(\ln z) + \int z\,dz\right] + C$$

$$= \frac{1}{2}\left[z^2(\ln z)^2 - z^2(\ln z) + \frac{z^2}{2}\right] + C.$$

5. (a)

$$\int_0^1 \frac{1}{(x-1)^{\frac{4}{5}}}\,dx = \lim_{c\to 1^-}\int_0^c \frac{1}{(x-1)^{\frac{4}{5}}}\,dx = \lim_{c\to 1^-}\Big[5(x-1)^{\frac{1}{5}}\Big]_0^c = \lim_{c\to 1^-} 5(c-1)^{\frac{1}{5}}+5 = 5.$$

(b) Using integration by parts,

$$\int_1^b \frac{\ln x}{x^3}\,dx = \left[\frac{-\ln x}{2x^2}\right]_1^b + \int_1^b \frac{1}{2x^3}\,dx = -\frac{\ln b}{2b^2} + \left[-\frac{1}{4x^2}\right]_1^b = -\frac{\ln b}{2b^2} - \frac{1}{4b^2} + \frac{1}{4}.$$

Thus, $\displaystyle\int_1^\infty \frac{\ln x}{x^3}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^3}\,dx = \lim_{b\to\infty}\left(-\frac{\ln b}{2b^2} - \frac{1}{4b^2} + \frac{1}{4}\right) = \frac{1}{4},$

since by L'Hôpital's rule, $\displaystyle\lim_{b\to\infty}\frac{\ln b}{2b^2} = \lim_{b\to\infty}\frac{\frac{1}{b}}{4b} = \lim_{b\to\infty}\frac{1}{4b^2} = 0.$

Solutions to Further Exercises

1. Let $I = \displaystyle\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$. Apply the substitution $x = a\sin\theta$, we have

$$I = \int_0^{\frac{\pi}{2}} \frac{a\cos\theta}{a\sin\theta + a\cos\theta}\,d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta.$$

Now observe that $\cos\theta = \frac{1}{2}(\cos\theta+\sin\theta) + \frac{1}{2}(\cos\theta-\sin\theta)$. Therefore

$$I = \int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}(\cos\theta+\sin\theta)+\frac{1}{2}(\cos\theta-\sin\theta)}{\sin\theta+\cos\theta}\,d\theta = \frac{\pi}{4} + \left[\frac{1}{2}\ln|\sin\theta+\cos\theta|\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

2. First $y = \sec^{-1}x \Leftrightarrow x = \sec y$. Also $x=1 \Rightarrow y = 0$ and $x = 2 \Rightarrow y = \sec^{-1}2 = \frac{\pi}{3}$. Therefore,

$$\int_1^2 \sec^{-1}x\,dx = 2(\frac{\pi}{3}) - 0 - \int_0^{\frac{\pi}{3}} \sec x\,dx = \frac{2\pi}{3} - [\ln|\sec x + \tan x|]_0^{\frac{\pi}{3}} = \frac{2\pi}{3} - \ln(2+\sqrt{3}).$$

3. (a) $\displaystyle\int_0^\pi \frac{\sin x}{\sqrt{9-\cos^2 x}}\,dx = -\int_0^\pi \frac{d(\cos x)}{\sqrt{3^2-\cos^2 x}}\,dx = \left[-\sin^{-1}(\frac{\cos x}{3})\right]_0^\pi = 2\sin^{-1}\frac{1}{3}.$

(b) $\displaystyle\int_0^\pi \frac{x\sin x}{\sqrt{9-\cos^2 x}}\,dx = \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{\sqrt{9-\cos^2(\pi-x)}}\,dx = \int_0^\pi \frac{\pi\sin x - x\sin x}{\sqrt{9-\cos^2 x}}\,dx.$

This implies $\displaystyle\int_0^\pi \frac{x\sin x}{\sqrt{9-\cos^2 x}}\,dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{\sqrt{9-\cos^2 x}}\,dx = \pi\sin^{-1}\frac{1}{3}$ by (a).