**(2022/23) Semester I  MA1521 Calculus for Computing  Midterm**

**Name:** ______________________  **SIS ID** ______________________

**Tutorial Group Number:** ___________

**Instruction: You must show your working and write down your answer in the box.**

(1) (6 marks) Evaluate $\lim_{x\to 0^+} \frac{\ln \sin 2x}{\ln \sin x}$.

Answer $\boxed{1}$

**Solution**. The limit is of the indeterminate form $\frac{\infty}{\infty}$. We apply L'Hôpital's rule.

$$\lim_{x\to 0^+} \frac{\ln \sin 2x}{\ln \sin x} \stackrel{L'H}{=} \lim_{x\to 0^+} \frac{\frac{2\cos 2x}{\sin 2x}}{\frac{\cos x}{\sin x}} = \lim_{x\to 0^+} \frac{2\sin x}{\sin 2x}\frac{\cos 2x}{\cos x} = \lim_{x\to 0^+} \frac{\sin x}{x}\frac{2x}{\sin 2x}\frac{\cos 2x}{\cos x} = 1.$$

(2) (8 marks) Find the maximal domain of the function $f(x) = \ln(4x - x^2)$; and $x$ at which the local extrema of the function $f$ is attained. (Note that it is possible that $f$ has no local minimum or maximum.)

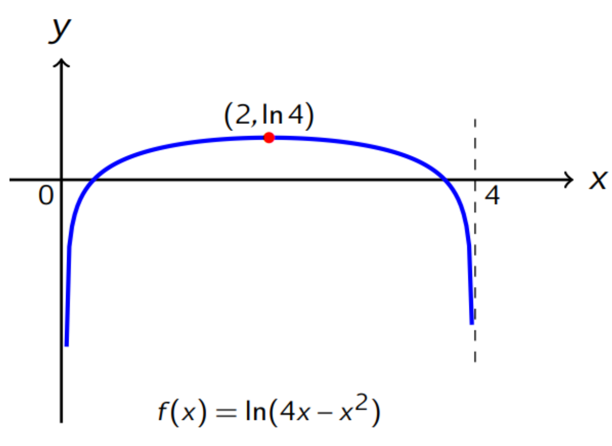
| Maximal domain | Local min | Local max |
|---|---|---|
| $(0, 4)$ | No local min | x=2 |

**Solution**. For $f$ to be defined, we need $4x - x^2 > 0$. That is $x(4-x) > 0 \Leftrightarrow x(x-4) < 0 \Leftrightarrow 0 < x < 4$ so that the maximal domain is the open interval $(0, 4)$.

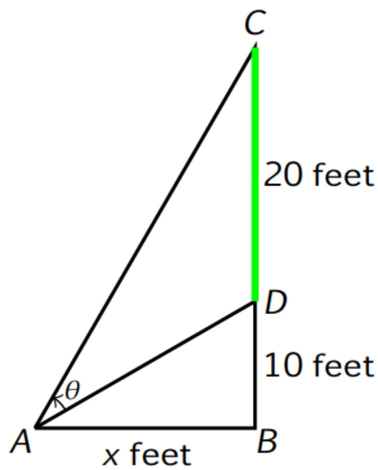
First we have $f'(x) = \frac{4-2x}{4x-x^2} = \frac{2(x-2)}{x(x-4)}$. Thus $f'(x) = 0 \Leftrightarrow x = 2$.

For $0 < x < 2$, $f'(x) > 0$. For $2 < x < 4$, $f'(x) < 0$. By the first derivative test, $f$ has a local maximum at $x = 2$.

Alternatively, $f''(x) = \frac{-2(x^2-4x+8)}{(4x-x^2)^2}$ and $f''(2) = -\frac{1}{2} < 0$. By the second derivative test, $f$ has a local maximum at $x = 2$.



(3) (10 marks) A movie screen on a wall is 20 feet high and 10 feet above your eye level. At what distance $x$ feet from the front of the room should you position yourself so that the viewing angle $\theta$ of the movie screen is as large as possible ?




Answer $\boxed{x = 10\sqrt{3}}$

**Solution**. We have $\theta = \tan^{-1}(\frac{30}{x}) - \tan^{-1}(\frac{10}{x})$, where $x > 0$. Thus

$$\frac{d\theta}{dx} = \frac{-\frac{30}{x^2}}{1 + (\frac{30}{x})^2} - \frac{-\frac{10}{x^2}}{1 + (\frac{10}{x})^2} = \frac{-30}{x^2 + 900} + \frac{10}{x^2 + 100} = \frac{20(300 - x^2)}{(x^2 + 100)(x^2 + 900)}.$$

Hence $\frac{d\theta}{dx} = 0 \Leftrightarrow x = \pm 10\sqrt{3}$. The negative root $x = -10\sqrt{3}$ is rejected as $x > 0$. Therefore there is only one critical point for $\theta$ at $x = 10\sqrt{3}$. For $0 < x < 10\sqrt{3}$, we have $\frac{d\theta}{dx} > 0$; and for $10\sqrt{3} < x$, we have $\frac{d\theta}{dx} < 0$. Thus by the first derivative test, $\theta(x)$ has the absolute maximum attained at $x = 10\sqrt{3}$. The maximum value of $\theta$ is $\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}} = 60° - 30° = 30°$.

(4) (8 marks) Find $\displaystyle\int \frac{1}{x[(\ln x)^2 + \ln x - 6]} dx.$

[Hint: Substitute $u = \ln x$.]

Answer $\boxed{\frac{1}{5} \ln \left| \frac{\ln x - 2}{\ln x + 3} \right| + C}$

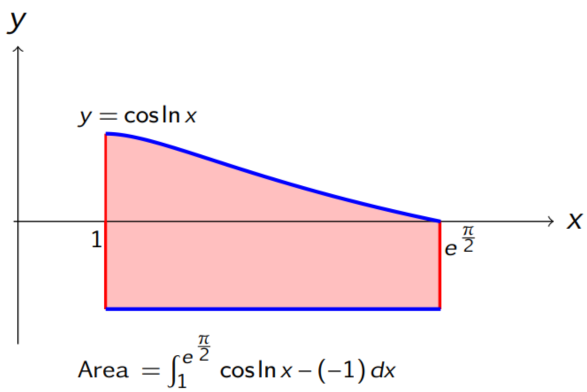
**Solution**. Let $u = \ln x$. Then $du = \frac{dx}{x}$. Thus

$$\int \frac{1}{x[(\ln x)^2 + \ln x - 6]} dx = \int \frac{1}{u^2 + u - 6} du = \int \frac{1}{(u+3)(u-2)} du$$
$$= \int \frac{1}{5} \left( \frac{-1}{u+3} + \frac{1}{u-2} \right) du = \frac{1}{5}(-\ln|u+3| + \ln|u-2|) + C$$
$$= \frac{1}{5} \ln \left| \frac{u-2}{u+3} \right| + C = \frac{1}{5} \ln \left| \frac{\ln x - 2}{\ln x + 3} \right| + C.$$

(5) (8 marks) Find the area of the region bounded by the curve $y = \cos\ln x$ and the lines $x = 1, x = e^{\pi/2}$ and $y = -1$.

Answer $\boxed{\frac{3}{2}(e^{\frac{\pi}{2}} - 1)}$

**Solution**.




Area $= \int_1^{e^{\frac{\pi}{2}}} \cos\ln x - (-1)\,dx = \int_1^{e^{\frac{\pi}{2}}} \cos\ln x\,dx + e^{\frac{\pi}{2}} - 1.$

Using integration by parts, we have $\int \cos\ln x\,dx = (\cos\ln x)x - \int(-\frac{1}{x}\sin\ln x)x\,dx$
$= x\cos\ln x + \int \sin\ln x\,dx = x\cos\ln x + x\sin\ln x - \int(\frac{1}{x}\cos\ln x)x\,dx$
$= x\cos\ln x + x\sin\ln x - \int\cos\ln x\,dx.$

Hence, $\int\cos\ln x\,dx = \frac{x}{2}(\cos\ln x + \sin\ln x) + C.$

Therefore, $\int_1^{e^{\frac{\pi}{2}}} \cos\ln x\,dx = \left[\frac{x}{2}(\cos\ln x + \sin\ln x)\right]_1^{e^{\frac{\pi}{2}}}$
$= \frac{e^{\frac{\pi}{2}}}{2}(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}) - \frac{1}{2}(\cos 0 - \sin 0)$
$= \frac{e^{\frac{\pi}{2}}}{2} - \frac{1}{2}.$

Consequently, the area is $\frac{e^{\frac{\pi}{2}}}{2} - \frac{1}{2} + e^{\frac{\pi}{2}} - 1 = \frac{3}{2}(e^{\frac{\pi}{2}} - 1).$

(6) (6 marks) Let $f$ be a function defined on $\mathbb{R}$ such that $f''$ is continuous. Suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, and $f'(4) = 3$. Evaluate $\displaystyle\int_1^4 xf''(x)\,dx$.

Answer $\boxed{2}$

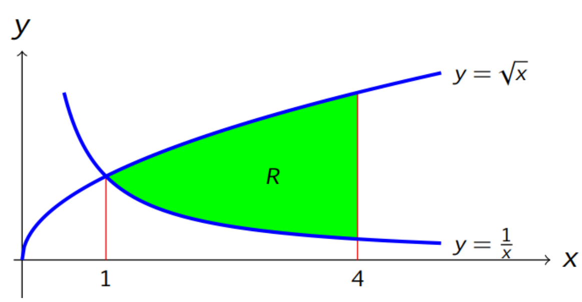
**Solution**. Using integration by parts, we have

$$\int_1^4 xf''(x)\,dx = [xf'(x)]_1^4 - \int_1^4 f'(x)\,dx = (4f'(4) - f'(1)) - (f(4) - f(1)) = 2.$$

(7) (8 marks) Let $R$ be the region bounded by the curves $y = \sqrt{x}$, $y = \frac{1}{x}$, the lines $x = 1$ and $x = 4$. Find the volume of the solid of revolution generated by revolving $R$ around the $y$-axis. [Hint: Use cylindrical shell method.]

Answer $\boxed{\dfrac{94\pi}{5}}$

**Solution**. The volume is $2\pi \int_1^4 x(\sqrt{x} - \frac{1}{x})\, dx = 2\pi \int_1^4 x^{\frac{3}{2}} - 1\, dx = 2\pi \left[\frac{2}{5}x^{\frac{5}{2}} - x\right]_1^4 = 2\pi((\frac{64}{5} - 4) - (\frac{2}{5} - 1)) = \frac{94\pi}{5}$.



(8) (6 marks) Find the length of the curve $y = 12 + \frac{1}{x} + \frac{x^3}{12}$ for $x = 1$ to $x = 4$.

Answer $\boxed{6}$

**Solution**. We have $y' = -\frac{1}{x^2} + \frac{x^2}{4} \Rightarrow \sqrt{1 + y'^2} = \sqrt{1 + (-\frac{1}{x^2} + \frac{x^2}{4})^2} = \frac{1}{x^2} + \frac{x^2}{4}$. Thus the length is $\int_1^4 \sqrt{1 + y'^2}\, dx = \int_1^4 \frac{1}{x^2} + \frac{x^2}{4}\, dx = \left[-\frac{1}{x} + \frac{x^3}{12}\right]_1^4 = (-\frac{1}{4} + \frac{16}{3}) - (-1 + \frac{1}{12}) = 6$.