MA2001 LINEAR ALGEBRA

Linear Systems & Gaussian Elimination

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What will we learn in Linear Algebra I?

- Why **Linear Algebra**?
  - **Linear**:
    - Study *lines*, *planes*, and objects which are geometrically "*flat*".
    - The real world is too complicated. We may (have to) use "*flat*" objects to approximate the real world.
  - **Algebra**:
    - The objects of study are generalization of numbers.
    - The operations on the objects include addition, subtraction, multiplication and more.

What will we learn in Linear Algebra I?

- **Contents**:
  - Linear Equations & Gaussian Elimination.
    - Solve linear systems in systematical ways.
    - Determine the number of solutions of linear systems.
  - Matrices.
    - Definition and computations on matrices.
    - Determinant of square matrices.
  - Vector Spaces.
    - Euclidean spaces.
    - Subspaces.
    - Bases and dimensions.
    - Change of bases.

What will we learn in Linear Algebra I?

- **Contents**:
  - Vector Spaces Associated with Matrices.
    - Row spaces, column spaces and nullspaces.
  - Orthogonality.
    - Dot product.
    - Orthogonal and orthonormal Bases.
  - Diagonalization.
    - Eigenvalues and eigenvectors.
    - Diagonalization and orthogonal diagonalization.
    - Quadratic Forms and Conic Sections.
  - Linear Transformation.
    - Definition, ranges and kernels.
    - Geometric linear transformations.

Workload and Assessment

- Lessons are conducted physically in lecture theatres and seminar rooms.
  - Lecture Group 1:
    - Mondays and Wednesdays: 8:00–10:00 am.
  - Lecture Group 2:
    - Tuesdays and Fridays: 8:00–10:00 am.

    Recorded lectures will be uploaded to Canvas.
- Textbook:
  - Linear Algebra: Concepts & Techniques on Euclidean Spaces.
    - The E-version is available in NUS library.
  - The lecture notes is prepared based on the textbook.
  - Tutorial questions are taken from exercises of the textbook.
    - Refer to course outline for details.

Workload and Assessment

- Tutorials are conducted physically Week 3 – Week 11.
  - Tutorial questions are taken from exercises of the textbook.
- Laboratory.
  - Students learn MATLAB to solve linear algebra problems.
  - Download from `https://ntouch.nus.edu.sg`
  - Notes are prepared by lecturers.
- Homework Assignments (25%).
  - Four online homework assignments are submitted to Canvas.
- Mid-Term Test (25%).
  - The test is scheduled on Week 7 evening in MPSH (date TBC).
- Final Exam (50%).
  - The exam is scheduled on 25 April (Tuesday) 1:00–3:00 pm.

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Linear Systems & Their Solutions

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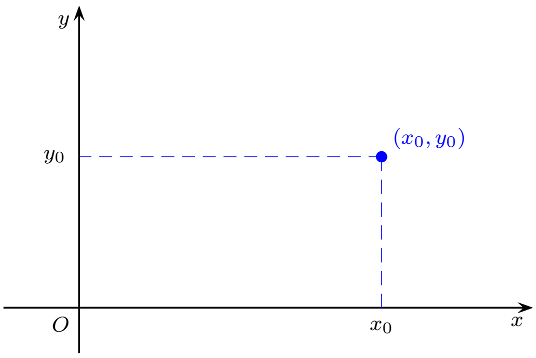
Lines on the plane

- Consider the xy-plane:



  - Every point on the xy-plane can be uniquely represented by a pair of real numbers (x_0, y_0).

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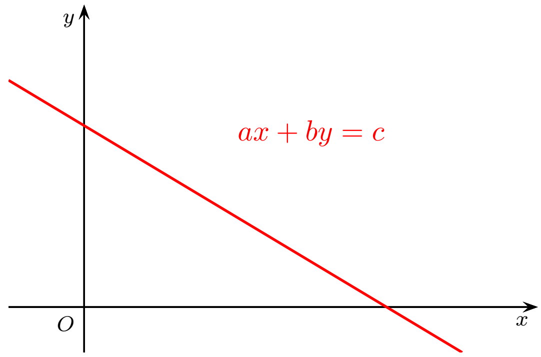
Lines on the plane

- Consider the xy-plane:



  - The points on a **straight line** are precisely all the points (x, y) on the xy-plane satisfying a linear equation
    - $\boxed{ax + by = c}$

    where a and b are not both zero.

Linear Equation

- A **linear equation** in n **variables** (**unknowns**) $x_1, x_2, \ldots, x_n$ is an equation in the form
  - $\boxed{a_1x_1 + a_2x_2 + \cdots + a_nx_n = b}$

  where $a_1, a_2, \ldots, a_n$ and b are real constants.
- **Note**: In a linear equation, $a_1, a_2, \ldots, a_n$ may be all zero.
  - If $a_1 = \cdots = a_n = 0$ but $b \neq 0$, it is **inconsistent**.
    - An equation that is not inconsistent is called **consistent**.
  - If $a_1 = \cdots = a_n = 0$ and $b = 0$, it is a **zero equation**.
    - An equation that is not zero is called a **zero equation**.

  For instance,
  - $0x_1 + 0x_2 = 1$ is an inconsistent equation;
  - $0x_1 + 0x_2 = 0$ is a zero equation;
  - $2x_1 - 3x_2 = 4$ is a nonzero and consistent equation.

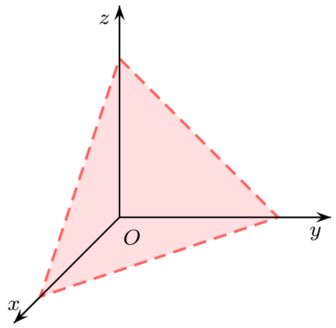
Examples

- The following equations are **linear equations**:
  - $x + 3y = 7$;
  - $x_1 + 2x_2 + 2x_3 + x_4 = x_5$;
    - $x_1 + 2x_2 + 2x_3 + x_4 - x_5 = 0$.
  - $y = x - \frac{1}{2}z + 4.5$;
    - $-x + y + \frac{1}{2}z = 4.5$.
- The following equations are NOT **linear equations**:
  - $xy = 2$;
  - $\sin\theta + \cos\phi = 0.2$;
  - $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$;
  - $x = e^y$.

Examples

- In the xyz-space, the linear equation
  - $\boxed{ax + by + cz = d}$,

  where a, b, c are not all zero, represents a **plane**.



For instance, $x + y + z = 1$ represents a plane in the xyz-space.

Solutions of a Linear Equation

- Let $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ be a linear equation in n variables $x_1, x_2, \ldots, x_n$.
  - For real numbers $s_1, s_2, \ldots, s_n$, if
    - $a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$,

    then $\boxed{x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n}$ is a **solution** to the given linear equation.
  - The set of all solutions is called the **solution set**.
    - The solution set of $ax + by = c$ (in x, y), where a, b are not all zero, represents a straight line in xy-plane.
    - The solution set of $ax + by + cz = d$ (in x, y, z), where a, b, c not all zero, represents a plane in xyz-space.
  - An expression that gives the entire solution set is a **general solution**.

Examples

- Linear equation $4x - 2y = 1$ in variables x and y.
  - x can take any arbitrary value, say t.
    - $x = t \Rightarrow y = 2t - \frac{1}{2}$.
    - General solution: $\begin{cases} x = t, \\ y = 2t - \frac{1}{2}, \end{cases}$ where t is a parameter.
  - y can take any arbitrary value, say s.
    - $y = s \Rightarrow x = \frac{1}{2}s + \frac{1}{4}$.
    - General solution: $\begin{cases} x = \frac{1}{2}s + \frac{1}{4}, \\ y = s, \end{cases}$ where s is a parameter.
- **Different representations** of the **same solution set**.
  - $\begin{cases} x = 1, \\ y = 1.5, \end{cases} \quad \begin{cases} x = 1.5, \\ y = 2.5, \end{cases} \quad \begin{cases} x = -1, \\ y = -2.5, \end{cases} \quad \ldots.$

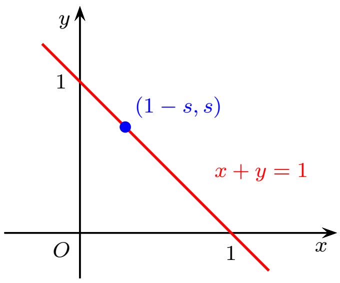
Examples

- $x_1 - 4x_2 + 7x_3 = 5$ in three variables x_1, x_2, x_3.
  - x_2 and x_3 can be chosen arbitrarily, say s and t.
    - $x_2 = s$ and $x_3 = t \Rightarrow x_1 = 5 + 4s - 7t$.
    - $\begin{cases} x_1 = 5 + 4s - 7t, \\ x_2 = s, \\ x_3 = t, \end{cases}$ where s, t are arbitrary parameters.
  - x_1 and x_2 can be chosen arbitrarily, say s and t.
    - $x_1 = s$ and $x_2 = t \Rightarrow x_3 = \frac{5}{7} - \frac{1}{7}s + \frac{4}{7}t$.
    - $\begin{cases} x_1 = s, \\ x_2 = t, \\ x_3 = \frac{5}{7} - \frac{1}{7}s + \frac{4}{7}t, \end{cases}$ where s, t are arbitrary parameters.

Examples

- In xy-plane, $x + y = 1$ has a general solution
  - $(x, y) = (1 - s, s)$, where s is an arbitrary parameter.

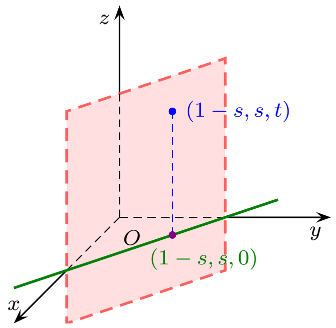
  These points form a straight line in the xy-plane:



Examples

- In xyz-space, $x + y = 1$ has a general solution
  - $(x, y, z) = (1 - s, s, t)$, where s, t are arbitrary parameters.

  These points form a plane in xyz-space:



  The projection of "the plane $x + y = 1$ in xyz-space" on the xy-plane is "the line $x + y = 1$ in xy-plane".

Examples

- The **zero equation** in n variables $x_1, x_2, \ldots, x_n$ is
  - $0x_1 + 0x_2 + \cdots + 0x_n = 0$ (or simply $0 = 0$).

  The equation is satisfied by any values of $x_1, x_2, \ldots, x_n$.
  - The general solution is given by
    - $(x_1, x_2, \ldots, x_n) = (t_1, t_2, \ldots, t_n)$,

    where $t_1, t_2, \ldots, t_n$ are arbitrary parameters.

- Let $b \neq 0$. An inconsistent equation in n variables $x_1, x_2, \ldots, x_n$:
  - $0x_1 + 0x_2 + \cdots + 0x_n = b$ (or simply $0 = b$).

  It is NOT satisfied by any values of $x_1, x_2, \ldots, x_n$.
  - An inconsistent equation has NO solution.

Linear System

- A **linear system** (**system of linear equations**) of m linear equations in n variables $x_1, x_2, \dots, x_n$ is
  - $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \quad\vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{cases}$$

  where a_{ij} and b_i are real constants.
  - a_{ij} is the **coefficient** of the variable x_j in the ith equation,
  - b_i is the **constant term** of the ith equation.
- If all a_{ij} and b_i are zero,
  - the linear system is called a **zero system**.

  If some a_{ij} or b_i is nonzero,
  - the linear system is called a **nonzero system**.

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Linear System

- A **linear system** (**system of linear equations**) of m linear equations in n variables $x_1, x_2, \dots, x_n$ is
  - $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \quad\vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{cases}$$

  where a_{ij} and b_i are real constants.
- If $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$ is a solution to **every equation** of the linear system, then it is called a **solution** to the system.
  - The **solution set** is the set of all solutions to the linear system.
  - A **general solution** is an expression which generates the solution set of the linear system.

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Example

- Linear system $\left\{ \begin{aligned} 4x_1 - x_2 + 3x_3 &= -1, \\ 3x_1 + x_2 + 9x_3 &= -4. \end{aligned} \right.$
  - $x_1 = 1, x_2 = 2, x_3 = -1$ is a solution to both equations, then it is a solution to the system.
  - $x_1 = 1, x_2 = 8, x_3 = 1$ is a solution to the first equation, but not a solution to the second equation; so it is not a solution to the system.

  Problem: How to find a general solution to the system?
  - For example, $\begin{cases} x_1 = 1 + 12t, \\ x_2 = 2 + 27t, \\ x_3 = -1 - 7t, \end{cases}$ where t is an arbitrary parameter.

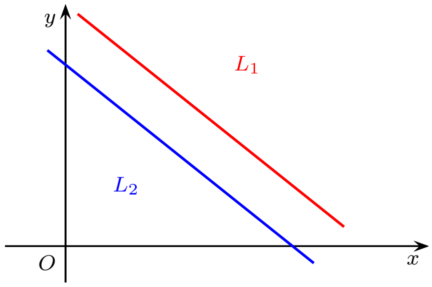
Consistency

- **Remark**. In a linear system, even if every equation has a solution, there may not be a solution to the entire system.
  - $\left\{ \begin{aligned} x + \ \ y &= 4, \\ 2x + 2y &= 6. \end{aligned} \right.$
    - $2x + 2y = 6 \Rightarrow x + y = 3.$
    - $x + y = 4$ & $x + y = 3 \Rightarrow 4 = 3$, impossible!
- **Definition**. A linear system is called
  - **consistent** if it has at least one solution;
  - **inconsistent** if it has no solution.
- **Remark**. A linear system has either
  - no solution, or
  - exactly one solution, or
  - infinitely many solutions. (To be proved in Chapter 2.)

Examples

- Linear system in variables x, y of two equations:
  - $\begin{cases} a_1x + b_1y = c_1, & (L_1) \\ a_2x + b_2y = c_2. & (L_2) \end{cases}$

  Assume a_1, b_1 are not both zero, a_2, b_2 are not both zero.
  - In xy-plane, each equation represents a straight line.



  - The system has no solution

    $\Leftrightarrow L_1$ and L_2 are parallel but distinct.

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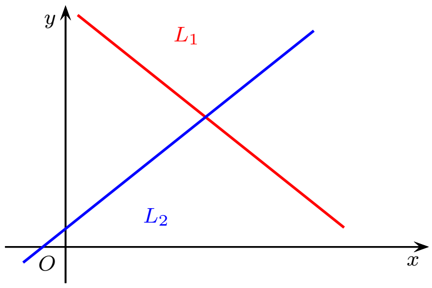
Examples

- Linear system in variables x, y of two equations:
  - $\begin{cases} a_1x + b_1y = c_1, & (L_1) \\ a_2x + b_2y = c_2. & (L_2) \end{cases}$

  Assume a_1, b_1 are not both zero, a_2, b_2 are not both zero.
  - In xy-plane, each equation represents a straight line.



  - The system has exactly one solution

    $\Leftrightarrow L_1$ and L_2 are not parallel.

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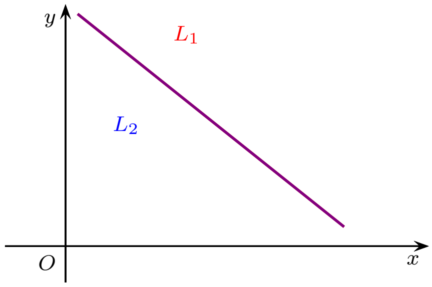
Examples

- Linear system in variables x, y of two equations:
  - $\begin{cases} a_1x + b_1y = c_1, & (L_1) \\ a_2x + b_2y = c_2. & (L_2) \end{cases}$

  Assume a_1, b_1 are not both zero, a_2, b_2 are not both zero.
  - In xy-plane, each equation represents a straight line.



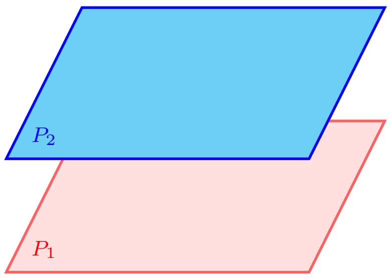
  - The system has infinitely many solutions

    $\Leftrightarrow$ L_1 and L_2 are the same line.

Examples

- Linear system in variables x, y, z of two equations:
  - $\begin{cases} a_1x + b_1y + c_1z = d_1, & (P_1) \\ a_2x + b_2y + c_2z = d_2. & (P_2) \end{cases}$

  Assume a_1, b_1, c_1 not all zero, a_2, b_2, c_2 not all zero.
  - Each equation represents a plane in xyz-space.



  - The system has no solution

    $\Leftrightarrow$ P_1 and P_2 are parallel but distinct.

Examples

- Linear system in variables x, y, z of two equations:
  - $\begin{cases} a_1x + b_1y + c_1z = d_1, & (P_1) \\ a_2x + b_2y + c_2z = d_2. & (P_2) \end{cases}$

  Assume a_1, b_1, c_1 not all zero, a_2, b_2, c_2 not all zero.
  - Each equation represents a plane in xyz-space.

  - The system has infinitely many solutions

    if P_1 and P_2 are the same plane.

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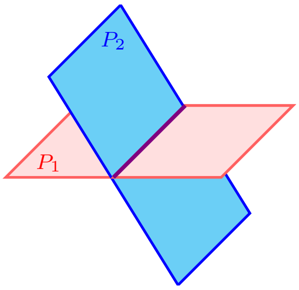
Examples

- Linear system in variables x, y, z of two equations:
  - $\begin{cases} a_1x + b_1y + c_1z = d_1, & (P_1) \\ a_2x + b_2y + c_2z = d_2. & (P_2) \end{cases}$

  Assume a_1, b_1, c_1 not all zero, a_2, b_2, c_2 not all zero.
  - Each equation represents a plane in xyz-space.



  - The system has infinitely many solutions

    if P_1 and P_2 intersect at a straight line.

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Examples

- Linear system in variables x, y, z of two equations:
  - $$\begin{cases} a_1x + b_1y + c_1z = d_1, & (P_1) \\ a_2x + b_2y + c_2z = d_2. & (P_2) \end{cases}$$

  Assume a_1, b_1, c_1 not all zero, a_2, b_2, c_2 not all zero.
  - Each equation represents a plane in xyz-space.

  1. P_1 and P_2 represent the same plane

     $\Leftrightarrow a_1 : a_2 = b_1 : b_2 = c_1 : c_2 = d_1 : d_2$.
  2. P_1 and P_2 are parallel but distinct planes

     $\Leftrightarrow a_1 : a_2 = b_1 : b_2 = c_1 : c_2 \neq d_1 : d_2$.
  3. P_1 and P_2 intersect at a line

     $\Leftrightarrow a_1 : a_2$, $b_1 : b_2$, $c_1 : c_2$ are not all the same.

Elementary Row Operations

Augmented Matrix

- A linear system in variables $x_1, x_2, \ldots, x_n$:
  - $$\begin{cases} a_{11}x_1 + \ a_{12}x_2 + \cdots + \ a_{1n}x_n = b_1, \\ a_{21}x_1 + \ a_{22}x_2 + \cdots + \ a_{2n}x_n = b_2, \\ \qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{cases}$$
  - The rectangular array of constants
    - $$\boxed{\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)}$$

    is called the **augmented matrix** of the linear system.
- A linear system in $y_1, y_2, \ldots, y_n$ with the same coefficients & constant terms has the same augmented matrix.

Example

- Linear system $\begin{cases} x_1 + x_2 + 2x_3 = 9, \\ 2x_1 + 4x_2 - 3x_3 = 1, \\ 3x_1 + 6x_2 - 5x_3 = 0. \end{cases}$
  - Augmented matrix: $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right)$

    This is also the augmented matrix for:
    - $\begin{cases} y_1 + y_2 + 2y_3 = 9, \\ 2y_1 + 4y_2 - 3y_3 = 1, \\ 3y_1 + 6y_2 - 5y_3 = 0. \end{cases}$
    - $\begin{cases} \spadesuit + \heartsuit + 2\clubsuit = 9, \\ 2\spadesuit + 4\heartsuit - 3\clubsuit = 1, \\ 3\spadesuit + 6\heartsuit - 5\clubsuit = 0. \end{cases}$

Elementary Row Operations

- To solve a linear system, we perform operations:
  - Multiply an equation by a nonzero constant.
  - Interchange two equations.
  - Add a constant multiple of an equation to another.
    - $E_1 \mapsto E_1 + cE_2 = E_3$.
    - $E_3 \mapsto E_3 + (-c)E_2 = E_1$.
- In terms of augmented matrix, they correspond to operations on the **rows** of the augmented matrix:
  - Multiply a row by a nonzero constant.
  - Interchange two rows.
  - Add a constant multiple of a row to another row.
    - $R_1 \mapsto R_1 + cR_2 = R_3$.
    - $R_3 \mapsto R_3 + (-c)R_2 = R_1$.

Elementary Row Operations

- The operations on rows of an augmented matrix:
    - Multiply a row by a nonzero constant;
    - Interchange two rows;
    - Add a constant multiple of a row to another row;

  are called the **elementary row operations**.
- **Remark.** Interchanging two rows can be obtained by using the other two operations.

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \xrightarrow{\text{add 2nd row to 1st row}} \begin{pmatrix} R_1 + R_2 \\ R_2 \end{pmatrix}$$
$$\xrightarrow{\text{add } (-1) \text{ times 1st row to 2nd row}} \begin{pmatrix} R_1 + R_2 \\ -R_1 \end{pmatrix}$$
$$\xrightarrow{\text{multiply 2nd row by } (-1)} \begin{pmatrix} R_1 + R_2 \\ R_1 \end{pmatrix}$$
$$\xrightarrow{\text{add } (-1) \text{ times 2nd row to 1st row}} \begin{pmatrix} R_2 \\ R_1 \end{pmatrix}$$

Example

- Compare operations of equations in a linear system and corresponding row operations of augmented matrix.
    - $\begin{cases} x + y + 3z = 0 & (1) \\ 2x - 2y + 2z = 4 & (2) \\ 3x + 9y = 3 & (3) \end{cases} \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$
        - Add (-2) times of (1) to (2) to obtain (4).
        - Add (-2) times of first row to second row.
    - $\begin{cases} x + y + 3z = 0 & (1) \\ - 4y - 4z = 4 & (4) \\ 3x + 9y = 3 & (3) \end{cases} \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$
        - Add (-3) times of (1) to (3) to obtain (5).
        - Add (-3) times of first row to third row.
    - $\begin{cases} x + y + 3z = 0 & (1) \\ - 4y - 4z = 4 & (4) \\ 6y - 9z = 3 & (5) \end{cases} \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right)$

Example

- Compare operations of equations in a linear system and corresponding row operations of augmented matrix.
  - $\begin{cases} x + \ \ y + 3z = 0 & (1) \\ \quad - 4y - 4z = 4 & (4) \\ \qquad 6y - 9z = 3 & (5) \end{cases}$ $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right)$
    - Add $(6/4)$ times of (4) to (5) to obtain (6).
    - Add $(6/4)$ times of second row to third row.
  - $\begin{cases} x + \ \ y + \ \ 3z = 0 & (1) \\ \quad - 4y - \ \ 4z = 4 & (4) \\ \qquad\quad - 15z = 9 & (6) \end{cases}$ $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$
    - (6) $\Rightarrow z = -3/5$.
    - Substitute $z = -3/5$ into (4):
      - $-4y - 4(-3/5) = 4 \Rightarrow y = -2/5$.
    - Substitute $y = -2/5$ and $z = -3/5$ into (1):
      - $x + (-2/5) + 3(-3/5) = 0 \Rightarrow x = 11/5$.

Example

- Compare operations of equations in a linear system and corresponding row operations of augmented matrix.
  - $\begin{cases} x + \ \ y + 3z = 0 & (1) \\ \quad - 4y - 4z = 4 & (4) \\ \qquad 6y - 9z = 3 & (5) \end{cases}$ $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right)$
    - Add $(6/4)$ times of (4) to (5) to obtain (6).
    - Add $(6/4)$ times of second row to third row.
  - $\begin{cases} x + \ \ y + \ \ 3z = 0 & (1) \\ \quad - 4y - \ \ 4z = 4 & (4) \\ \qquad\quad - 15z = 9 & (6) \end{cases}$ $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$

  The given linear system has exactly one solution:
  - $x = 11/5$, $y = -2/5$, $z = -3/5$.

  Note that this is the solution of every linear system in the procedure of solving the given linear system.

Row Equivalent Matrices

- **Definition**. Two **augmented matrices** are said to be **row equivalent** if one can be obtained from the other by a **series** of **elementary row operations**.
  - $\boldsymbol{A} \xrightarrow{\text{multiply a row by nonzero } c} \boldsymbol{B}$.
    - $\boldsymbol{B} \xrightarrow{\text{multiply the same row by } 1/c} \boldsymbol{A}$.
  - $\boldsymbol{A} \xrightarrow{\text{interchange two rows}} \boldsymbol{B}$.
    - $\boldsymbol{B} \xrightarrow{\text{interchange the two rows again}} \boldsymbol{A}$.
  - $\boldsymbol{A} \xrightarrow{\text{add } c \text{ times of row } i \text{ to row } j} \boldsymbol{B}$.
    - $\boldsymbol{B} \xrightarrow{\text{add } (-c) \text{ times of row } i \text{ to row } j} \boldsymbol{A}$.

  $\boldsymbol{A}$ is row equivalent to $\boldsymbol{B} \Leftrightarrow \boldsymbol{B}$ is row equivalent to $\boldsymbol{A}$.
  - $\boldsymbol{A} = \boldsymbol{A}_0 \to \boldsymbol{A}_1 \to \cdots \to \boldsymbol{A}_{k-1} \to \boldsymbol{A}_k = \boldsymbol{B}$.
  - $\boldsymbol{B} = \boldsymbol{A}_k \to \boldsymbol{A}_{k-1} \to \cdots \to \boldsymbol{A}_1 \to \boldsymbol{A}_0 = \boldsymbol{A}$.

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Row Equivalent Matrices

- **Theorem (and Exercise)**. Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be augmented matrices.
  - $\boldsymbol{A}$ is row equivalent to $\boldsymbol{A}$.
  - $\boldsymbol{A}$ is row equivalent to $\boldsymbol{B}$

    $\Rightarrow \boldsymbol{B}$ is row equivalent to $\boldsymbol{A}$.
  - $\boldsymbol{A}$ is row equivalent to $\boldsymbol{B}$ & $\boldsymbol{B}$ is row equivalent to $\boldsymbol{C}$

    $\Rightarrow \boldsymbol{A}$ is row equivalent to $\boldsymbol{C}$.
- **Theorem**. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be augmented matrices of two linear systems. Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent.
  - Then the corresponding linear systems have the same set of solutions.
- **Question**. Given an augmented matrix $\boldsymbol{A}$, how to find an row equivalent augmented matrix $\boldsymbol{B}$ which is of a **simple** (or the **simplest**) form?

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Row-Echelon Form

- **Definition.** An augmented matrix is said to be in **row-echelon form** if the following properties are satisfied.
  1. The **zero rows** are grouped together at the bottom.

nonzero row
······
nonzero row
zero row
······
zero row

  2. For any two successive nonzero rows, the first nonzero number (**leading entry**) in the lower row appears to the right of the first nonzero number in the higher row.

$$\begin{array}{cccccccccccc} \hline 0 & \cdots & 0 & \otimes & * & \cdots & * & * & * & \cdots \\ \hline 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \otimes & * & \cdots \\ \hline\hline \end{array}, \ \otimes \text{ nonzero.}$$

Row-Echelon Form

- **Definition.** Suppose an augmented matrix is in **row-echelon form**.
  - The **leading entry** of a nonzero row is a **pivot point**.
  - A column of the augmented matrix is called a
    - **pivot column** if it contains a pivot point;
    - **non-pivot column** if it contains no pivot point.

$$\begin{array}{ccccccccccccc} \hline 0 & \cdots & 0 & * & \cdots & * & * & \cdots & * & * & \cdots \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & * & \cdots & * & * & \cdots \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & * & \cdots \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \hline \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \hline\hline & & & \uparrow & & & \uparrow & & & \uparrow & \end{array}$$

  - A pivot column contains exactly one pivot point.

Examples

- The following augmented matrices are in row-echelon form:
  - $\left(\begin{array}{cc|c} 3 & 2 & 1 \end{array}\right)$
  - $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right)$
  - $\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$
  - $\left(\begin{array}{ccc|c} -1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 3 \end{array}\right)$
  - $\left(\begin{array}{cccc|c} 0 & 1 & 2 & 8 & 1 \\ 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

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Examples

- These augmented matrices are NOT in row-echelon form:
  - $\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right)$
  - $\left(\begin{array}{cc|c} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right)$
  - $\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right)$
  - $\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

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Reduced Row-Echelon Form

- **Definition.** Suppose an augmented matrix is in **row-echelon form**. It is in **reduced row-echelon form** if

3. The leading entry of every nonzero row is 1;
   - Equivalently, every pivot point is 1.
4. In each pivot column, except the pivot point, all other entries are 0.

$$
\begin{array}{cccccccccccc}
\hline\hline
0 & \cdots & 0 & 1 & \cdots & * & 0 & \cdots & * & 0 & \cdots \\
\hline
0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & * & 0 & \cdots \\
\hline
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots \\
\hline
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\
\hline
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots \\
\hline
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots \\
\hline\hline
 & & & \uparrow & & & \uparrow & & & \uparrow &
\end{array}
$$

Examples

- These are in reduced row-echelon form:
  - $\left(\begin{array}{cc|c} 1 & 2 & 3 \end{array}\right)$
  - $\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$
  - $\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$
  - $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$
  - $\left(\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Examples

- These row-echelon forms are NOT reduced:
  - $\left(\begin{array}{cc|c} 3 & 2 & 1 \end{array}\right)$
  - $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right)$
  - $\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$
  - $\left(\begin{array}{ccc|c} -1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 3 \end{array}\right)$
  - $\left(\begin{array}{cccc|c} 0 & 1 & 2 & 8 & 1 \\ 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Solve Linear System

- Suppose that the augmented matrix of a linear system is in **(reduced) row-echelon form**.
  - Is it convenient to find a solution to the linear system?
- **Example**.
  - Augmented matrix $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$.
  - Linear system $\begin{cases} 1x_1 + 0x_2 + 0x_3 = 1 \\ 0x_1 + 1x_2 + 0x_3 = 2 \\ 0x_1 + 0x_2 + 1x_3 = 3. \end{cases}$
    - Equivalently $\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3. \end{cases}$
  - The system has one solution $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

Solve Linear System

- Suppose that the augmented matrix of a linear system is in **(reduced) row-echelon form**.
  - Is it convenient to find a solution to the linear system?
- **Example**.
  - Augmented matrix $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$.
  - Linear system $\begin{cases} 0x_1 + 0x_2 + 0x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0. \end{cases}$
  - This is a zero system in three variables. It has infinitely many solutions
    - $x_1 = r$, $x_2 = s$, $x_3 = t$, where r, s, t are arbitrary parameters.

Solve Linear System

- Suppose that the augmented matrix of a linear system is in **(reduced) row-echelon form**.
  - Is it convenient to find a solution to the linear system?
- **Example**.
  - Augmented matrix $\left(\begin{array}{cc|c} 3 & 1 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{array}\right)$.
  - Linear system $\begin{cases} 3x_1 + 1x_2 = 4 \\ 0x_1 + 2x_2 = 1 \\ 0x_1 + 0x_2 = 1 \end{cases}$
  - The last equation is inconsistent; so the system is inconsistent.

Examples

- Augmented matrix $\left(\begin{array}{cccc|c} 1 & -1 & 0 & 3 & -2 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$.
  - $\begin{cases} 1x_1 - 1x_2 + 0x_3 + 3x_4 = -2 \\ 0x_1 + 0x_2 + 1x_3 + 2x_4 = \ \ 5 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 = \ \ 0 \end{cases}$

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Examples

- Augmented matrix $\left(\begin{array}{cccc|c} 1 & -1 & 0 & 3 & -2 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$.
  - $\begin{cases} x_1 - x_2 \qquad + 3x_4 = -2 \\ \qquad\qquad x_3 + 2x_4 = \ \ 5 \end{cases}$

  1. Let $x_4 = t$ and substitute into the second equation.
     - $x_3 + 2t = 5 \Rightarrow x_3 = 5 - 2t$.
  2. Substitute $x_4 = t$ into the first equation.
     - $x_1 - x_2 + 3t = -2$.
     - Let $x_2 = s$. Then $x_1 = -2 + s - 3t$.

  Infinitely many solutions (s and t are arbitrary parameters)

  - $x_1 = -2 + s - 3t$, $x_2 = s$, $x_3 = 5 - 2t$, $x_4 = t$.

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Examples

- Augmented matrix $\left(\begin{array}{ccccc|c} 0 & 2 & 2 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{array}\right)$.
  - $\begin{cases} 0x_1 + 2x_2 + 2x_3 + 1x_4 - 2x_5 = 2 \\ 0x_1 + 0x_2 + 1x_3 + 1x_4 + 1x_5 = 3 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 2x_5 = 4. \end{cases}$

Examples

- Augmented matrix $\left(\begin{array}{ccccc|c} 0 & 2 & 2 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{array}\right)$.
  - $\begin{cases} 2x_2 + 2x_3 + x_4 - 2x_5 = 2 \\ \qquad\quad x_3 + x_4 + \ \ x_5 = 3 \\ \qquad\qquad\qquad\quad 2x_5 = 4. \end{cases}$

  1. By the third equation, $2x_5 = 4 \Rightarrow x_5 = 2$.
  2. Substitute $x_5 = 2$ into the second equation:
     - $x_3 + x_4 + 2 = 3$, i.e., $x_3 + x_4 = 1$.
     - Let $x_4 = t$. Then $x_3 = 1 - t$.
  3. Substitute $x_5 = 2$, $x_3 = 1 - t$, $x_4 = t$ into the first:
     - $2x_2 + 2(1 - t) + t - 2 \cdot 2 = 2$. So $x_2 = 2 + \frac{1}{2}t$.

Examples

- Augmented matrix $\left(\begin{array}{ccccc|c} 0 & 2 & 2 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{array}\right)$.
  - $\begin{cases} 2x_2 + 2x_3 + x_4 - 2x_5 = 2 \\ \qquad\quad x_3 + x_4 + \ x_5 = 3 \\ \qquad\qquad\qquad\quad 2x_5 = 4. \end{cases}$

  The system has infinitely many solutions

  - $\begin{cases} x_1 = s \\ x_2 = 2 + \frac{1}{2}t \\ x_3 = 1 - t \\ x_4 = t \\ x_5 = 2, \end{cases}$

  where s and t are arbitrary parameters.

Algorithm

- Suppose that the augmented matrix corresponding to a linear system is in **row-echelon form**.
  1. **Set** the variables corresponding to **non-pivot columns** to be arbitrary parameters.
  2. **Solve** the variables corresponding to **pivot columns** by **back substitution** (from last equation to first.)

  Example. $\begin{cases} 0x_1 + 2x_2 + 2x_3 + x_4 - 2x_5 = 2 \\ \qquad\qquad\qquad x_3 + x_4 + \ x_5 = 3 \\ \qquad\qquad\qquad\qquad\qquad 2x_5 = 4. \end{cases}$

  - Variables corresponding to pivot columns: x_2, x_3, x_5.
  - Variables corresponding to non-pivot columns: x_1, x_4.
    - Set $x_1 = s$ and $x_4 = t$ as arbitrary parameters.
    - Solve $x_5 = 2$, $x_3 = 1 - t$ and $x_2 = 2 + \frac{1}{2}t$.

Row Echelon Form

- **Definition.** Let $\boldsymbol{A}$ and $\boldsymbol{R}$ be augmented matrices.
  - Suppose that $\boldsymbol{A}$ is row equivalent to $\boldsymbol{R}$.
    - i.e., $\boldsymbol{R}$ can be obtained from $\boldsymbol{A}$ by a series of elementary row operations.
      $\boldsymbol{A} = \boldsymbol{A}_0 \to \boldsymbol{A}_1 \to \boldsymbol{A}_2 \to \cdots \to \boldsymbol{A}_k = \boldsymbol{R}$.

  1. If $\boldsymbol{R}$ is in row-echelon form,
     - $\boldsymbol{R}$ is called a **row-echelon form** of $\boldsymbol{A}$.
  2. If $\boldsymbol{R}$ is in reduced row-echelon form,
     - $\boldsymbol{R}$ is called a **reduced row-echelon form** of $\boldsymbol{A}$.
- Solve a linear system with augmented matrix $\boldsymbol{A}$

  $\Leftrightarrow$ solve a linear system with augmented matrix $\boldsymbol{R}$.

Gaussian Elimination

- Given an augmented matrix, we need an **algorithm** to find its (reduced) row-echelon form of $\boldsymbol{A}$.
- **Example.** $\left(\begin{array}{cccccc|c} 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right)$

  1. Find the **leftmost column** which is not entirely zero. $\left(\begin{array}{cccccc|c} 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right)$
  2. Check the **top entry** of such column. If it is 0,
     - replace it by a nonzero number by interchanging the top row with another row below.

$$\left(\begin{array}{cccccc|c} 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right)$$

Gaussian Elimination

- **Example.** $\left(\begin{array}{cccccc|c} 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right)$

1. Find the **leftmost column** which is not entirely zero.
2. If the **top entry** of such column is 0,
   - then replace it by a nonzero number by interchanging the top row with another row below.
3. For **each row below** the top row,
   - add a suitable multiple of the **top row** to it so that its **leading entry** becomes 0.

   Add 2 times the first row to the third row:

   - $\left(\begin{array}{cccccc|c} 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 3 & 6 & 9 & -12 \end{array}\right)$

Gaussian Elimination

- **Example.** $\left(\begin{array}{cccccc|c} 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 3 & 6 & 9 & -12 \end{array}\right)$

4. Cover the top row and repeat the procedure to the matrix remained.
1. The 4th column is the leftmost nonzero column.
   - $\left(\begin{array}{cccccc|c} 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ \hline 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 3 & 6 & 9 & -12 \end{array}\right)$
2. The top entry is nonzero. No action.
3. Add $-3/2$ times the 2nd row to the 3rd row.
   - $\left(\begin{array}{cccccc|c} 0 & 1 & 2 & 4 & 5 & 3 & -9 \\ \hline 0 & 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & 6 & -24 \end{array}\right)$
4. This is in row-echelon form. Done!

Gaussian Elimination

- **Gaussian Elimination**. Use elementary row operations to reduce an augmented matrix to row-echelon form.
  1. Find the leftmost column which is not entirely zero.
  2. If the top entry of such column is 0,
     - then replace it by a nonzero number by interchanging the top row with another row.
  3. For each row below the top row,
     - add a suitable multiple of the top row to it so that its leading entry becomes 0.
  4. Cover the top row and repeat the procedure to the remained matrix.
     - Continue this way until the entire matrix is in row-echelon form.

Example

- $$\begin{cases} \phantom{x_1 + 2x_2 +{}} 2x_3 + 4x_4 + 2x_5 = 8 \\ x_1 + 2x_2 + 4x_3 + 5x_4 + 3x_5 = -9 \\ -2x_1 - 4x_2 - 5x_3 - 4x_4 + 3x_5 = 6 \end{cases}$$
  - Augmented matrix: $\left(\begin{array}{ccccc|c} 0 & 0 & 2 & 4 & 2 & 8 \\ 1 & 2 & 4 & 5 & 3 & -9 \\ -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right)$

  We have found a row-echelon form
  - $\left(\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 0 & 6 & -24 \end{array}\right)$

  It corresponds to the linear system
  - $$\begin{cases} x_1 + 2x_2 + 4x_3 + 5x_4 + 3x_5 = -9 \\ 2x_3 + 4x_4 + 2x_5 = 8 \\ 6x_5 = -24 \end{cases}$$

Example

- The given linear system has the same solution set as
  - $$\begin{cases} x_1 + 2x_2 + 4x_3 + 5x_4 + 3x_5 = -9 \\ \qquad\qquad 2x_3 + 4x_4 + 2x_5 = 8 \\ \qquad\qquad\qquad\qquad 6x_5 = -24 \end{cases}$$

1. Set the variables corresponding to non-pivot columns as arbitrary parameters.
   - $x_2 = s$ and $x_4 = t$.
2. Solve the variables corresponding to pivot columns.
   - $6x_5 = -24 \Rightarrow x_5 = -4$.
   - $2x_3 + 4 \cdot t + 2(-4) = 8 \Rightarrow x_3 = 8 - 2t$.
   - $x_1 + 2 \cdot s + 4(8 - 2t) + 5 \cdot t + 3(-4) = -9$

     $\Rightarrow x_1 = -29 - 2s + 3t$.

Example

- $$\begin{cases} \qquad\qquad 2x_3 + 4x_4 + 2x_5 = 8 \\ x_1 + 2x_2 + 4x_3 + 5x_4 + 3x_5 = -9 \\ -2x_1 - 4x_2 - 5x_3 - 4x_4 + 3x_5 = 6 \end{cases}$$

  This system has general solution
  - $$\begin{cases} x_1 = -29 - 2s + 3t \\ x_2 = s \\ x_3 = 8 - 2t \\ x_4 = t \\ x_5 = -4 \end{cases}$$

  where s and t are arbitrary parameters.

Gauss-Jordan Elimination

- Suppose an augmented matrix is in row-echelon form. Is there an algorithm to get its **reduced** row-echelon form?
- **Example.** $\left(\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 0 & 6 & -24 \end{array}\right)$.

1. All the pivot points must be 1.
   - Multiply $1/2$ to 2nd row, multiply $1/6$ to 3rd row.
   - $\left(\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right)$.

Gauss-Jordan Elimination

- **Example.** $\left(\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right)$.

2. In each pivot column, all entries other than the pivot point must be 0.
   - Add (-3) times 3rd row to 1st row, and add (-1) times 3rd row to 2nd row.

     $\left(\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right)$.
   - Add (-4) times 2nd row to 1st row.

     $\left(\begin{array}{ccccc|c} 1 & 2 & 0 & -3 & 0 & -29 \\ 0 & 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right)$.

Gauss-Jordan Elimination

- **Gauss-Jordan Elimination**. Use elementary row operations to reduce a matrix to reduced row-echelon form.

  1-4. Use Gaussian Elimination to get a row-echelon form.

  5. For each nonzero row, multiple a suitable constant so that the pivot point becomes 1.
  6. Begin with the last nonzero row, work backwards.
     - Add suitable multiple of each row to the rows above to introduce 0 above the pivot points.

- **Remarks.**
  - Every matrix has a unique reduced row-echelon form.
    - (Can you prove it? It is very challenging!)
  - Every nonzero matrix has infinitely many (non-reduced) row-echelon forms.

Example

- $$\begin{cases} 2x_3 + 4x_4 + 2x_5 = 8 \\ x_1 + 2x_2 + 4x_3 + 5x_4 + 3x_5 = -9 \\ -2x_1 - 4x_2 - 5x_3 - 4x_4 + 3x_5 = 6 \end{cases}$$
  - Augmented matrix: $\left(\begin{array}{ccccc|c} 0 & 0 & 2 & 4 & 2 & 8 \\ 1 & 2 & 4 & 5 & 3 & -9 \\ -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right)$

  We have found a reduced row-echelon form
  - $\left(\begin{array}{ccccc|c} 1 & 2 & 0 & -3 & 0 & -29 \\ 0 & 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right)$

  It corresponds to the linear system
  - $\begin{cases} x_1 + 2x_2 \quad - 3x_4 + \quad = -29 \\ x_3 + 2x_4 \quad = 8 \\ x_5 = -4 \end{cases}$

Example

- $$\begin{cases} \phantom{x_1 + 2x_2 + {}} 2x_3 + 4x_4 + 2x_5 = 8 \\ x_1 + 2x_2 + 4x_3 + 5x_4 + 3x_5 = -9 \\ -2x_1 - 4x_2 - 5x_3 - 4x_4 + 3x_5 = 6 \end{cases}$$

  It has the same solution set as the linear system

  - $$\begin{cases} x_1 + 2x_2 - 3x_4 + = -29 \\ x_3 + 2x_4 = 8 \\ x_5 = {-4} \end{cases}$$

  1. Set the variables corresponding to non-pivot columns as arbitrary parameters: $x_2 = s$ and $x_4 = t$.
  2. Solve other variables:
     - $x_1 + 2s - 3t = -29 \Rightarrow x_1 = -29 - 2s + 3t.$
     - $x_3 + 2t = 8 \Rightarrow x_3 = 8 - 2t.$
     - $x_5 = -4.$

Consistency

- Suppose that $\boldsymbol{A}$ is the augmented matrix of a linear system, and $\boldsymbol{R}$ is a row-echelon form of $\boldsymbol{A}$.
  - When the system has no solution (i.e., is inconsistent)?
  - When the system has exactly one solution?
  - When the system has infinitely many solutions?
- Recall the procedure of finding solution:
  1. Set the variables corresponding to non-pivot columns as arbitrary parameters.
  2. Solve variables corresponding to pivot columns.

  The procedure is valid as long as
  - Every row of $\boldsymbol{R}$ corresponds to a consistent equation.
  - i.e., no row corresponds to an inconsistent equation:
    - $0x_1 + 0x_2 + \cdots + 0x_n = \otimes \leftarrow$ nonzero.

Consistency

- Suppose that $\boldsymbol{A}$ is the augmented matrix of a linear system, and $\boldsymbol{R}$ is a row-echelon form of $\boldsymbol{A}$.
  - When the system has no solution (i.e., is inconsistent)?

  Answer: There is a row in $\boldsymbol{R}$ with the form
  - $\left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & \otimes \end{array}\right)$, where $\otimes$ is nonzero.

  Or equivalently, the last column is a pivot column.

  Note: Such a row must be the last nonzero row of $\boldsymbol{R}$.
- **Examples.**
  - $\left(\begin{array}{ccc|c} 3 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$, $\left(\begin{array}{ccc|c} 3 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$.

Consistency

- Suppose that $\boldsymbol{A}$ is the augmented matrix of a linear system, and $\boldsymbol{R}$ is a row-echelon form of $\boldsymbol{A}$.
  - When the system has exactly one solution?
- Recall the procedure of finding solution:
  1. Set the variables corresponding to non-pivot columns as arbitrary parameters.
  2. Solve variables corresponding to pivot columns.

  For consistency, the last column is non-pivot. We also need
  - No variables corresponding to non-pivot columns.

  Answer:
  - The last column is a non-pivot column, and
  - All other columns are pivot columns.

Consistency

- Suppose that $\boldsymbol{A}$ is the augmented matrix of a linear system, and $\boldsymbol{R}$ is a row-echelon form of $\boldsymbol{A}$.
  - When the system has exactly one solution?

  Answer:

  - The last column is a non-pivot column, and
  - All other columns are pivot columns.

  Example: (Here $\otimes$ are pivot points, which are nonzero.)

  - $$\left(\begin{array}{cccccc|c} \otimes & * & * & \cdots & * & * \\ 0 & \otimes & * & \cdots & * & * \\ 0 & 0 & \otimes & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & * & * \\ 0 & 0 & 0 & \cdots & \otimes & * \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right)$$

Consistency

- Suppose that $\boldsymbol{A}$ is the augmented matrix of a linear system, and $\boldsymbol{R}$ is a row-echelon form of $\boldsymbol{A}$.
  - When the system has infinitely many solutions?

  Answer:

  - The last column is a non-pivot column, and
  - Some other columns are non-pivot columns.

  Note: The number of arbitrary parameters is the same as the number of non-pivot columns (except the last column).

- **Examples**:
  - $$\left(\begin{array}{cccc|c} 5 & 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right), \left(\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Notations

- Notations for elementary row operations.
  - Multiply the ith row by (nonzero) constant k: kR_i.
  - Interchange the ith and the jth rows: $R_i \leftrightarrow R_j$.
  - Add k times the ith row to the jth row: $R_j + kR_i$.

  Note:

  - $R_1 + R_2$ means "add the 2nd row to the 1st row".
  - $R_2 + R_1$ means "add the 1st row to the 2nd row".
- **Example**.
  - $$\begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{R_1+R_2} \begin{pmatrix} a+b \\ b \end{pmatrix} \xrightarrow{R_2+(-1)R_1} \begin{pmatrix} a+b \\ -a \end{pmatrix}$$
    $$\xrightarrow{R_1+R_2} \begin{pmatrix} b \\ -a \end{pmatrix} \xrightarrow{(-1)R_2} \begin{pmatrix} b \\ a \end{pmatrix}.$$

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Example 1

- What is the condition so that the system is consistent?
  - $$\begin{cases} x + 2y - 3z = a \\ 2x + 6y - 11z = b \\ x - 2y + 7z = c. \end{cases}$$
- The augmented matrix is $\left(\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 6 & -11 & b \\ 1 & -2 & 7 & c \end{array}\right)$.

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 6 & -11 & b \\ 1 & -2 & 7 & c \end{array}\right) \xrightarrow{R_2+(-2)R_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 1 & -2 & 7 & c \end{array}\right)$$
$$\xrightarrow{R_3+(-1)R_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & -4 & 10 & c-a \end{array}\right)$$
$$\xrightarrow{R_3+2R_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & 0 & 0 & 2b+c-5a \end{array}\right)$$

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Example 1

- What is the condition so that the system is consistent?
  - $$\begin{cases} x + 2y - 3z = a \\ 2x + 6y - 11z = b \\ x - 2y + 7z = c. \end{cases}$$
- A row-echelon form of the augmented matrix is
  - $$\left(\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & 0 & 0 & 2b+c-5a \end{array}\right)$$
  - The system is consistent

    $\Leftrightarrow$ the last column is non-pivot

    $\Leftrightarrow 2b + c - 5a = 0.$
  - Moreover, suppose the system is consistent.
    - The 3rd column is non-pivot
    - Infinitely many solutions (one arbitrary parameter).

Example 2

- Find the number of solutions: $\begin{cases} x + 2y + z = 1 \\ 2x + by + 2z = 2 \\ 4x + 8y + b^2 z = 2b \end{cases}$

  Find a row-echelon form of augmented matrix.
  - $$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & b & 2 & 2 \\ 4 & 8 & b^2 & 2b \end{array}\right)$$

    $$\xrightarrow{R_3+(-4)R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & b & 2 & 2 \\ 0 & 0 & b^2-4 & 2b-4 \end{array}\right)$$

    $$\xrightarrow{R_2+(-2)R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & b-4 & 0 & 0 \\ 0 & 0 & b^2-4 & 2b-4 \end{array}\right)$$

Example 2

- Find the number of solutions: $\begin{cases} x + 2y + z = 1 \\ 2x + by + 2z = 2 \\ 4x + 8y + b^2 z = 2b \end{cases}$
- Find a row-echelon form of augmented matrix.
  - $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & b & 2 & 2 \\ 4 & 8 & b^2 & 2b \end{array}\right) \to \cdots \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & b-4 & 0 & 0 \\ 0 & 0 & b^2-4 & 2b-4 \end{array}\right)$
    - If $b = 4$, then we can continue

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & b-4 & 0 & 0 \\ 0 & 0 & b^2-4 & 2b-4 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 4 \end{array}\right)$$
$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 12 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

  - The second column and the last column are non-pivot. Infinitely many solutions (one parameter).

Example 2

- Find the number of solutions: $\begin{cases} x + 2y + z = 1 \\ 2x + by + 2z = 2 \\ 4x + 8y + b^2 z = 2b \end{cases}$

Let $b \neq 4$. Row-echelon form: $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & b-4 & 0 & 0 \\ 0 & 0 & b^2-4 & 2b-4 \end{array}\right)$

- No solution $\Leftrightarrow$ The last column is a pivot column.

The last column is pivot $\Leftrightarrow$ $2b-4$ is the pivot point

$$\Leftrightarrow \begin{cases} b^2 - 4 = 0 \\ 2b - 4 \neq 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} b = 2 \text{ or } -2 \\ b \neq 2 \end{cases}$$
$$\Leftrightarrow b = -2.$$

Example 2

- Find the number of solutions: $\begin{cases} x + 2y + z = 1 \\ 2x + by + 2z = 2 \\ 4x + 8y + b^2 z = 2b \end{cases}$
- Let $b \neq 4$. Row-echelon form: $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & b-4 & 0 & 0 \\ 0 & 0 & b^2-4 & 2b-4 \end{array}\right)$
  - Unique solution $\Leftrightarrow$ Only the last column is non-pivot.

Only the last column is non-pivot

$\Leftrightarrow$ the first three columns are pivot

$$\Leftrightarrow \begin{cases} 1 \neq 0 \\ b-4 \neq 0 \\ b^2-4 \neq 0 \end{cases}$$

$\Leftrightarrow b \neq 4,\ b \neq -2,\ b \neq 2.$

Example 2

- Find the number of solutions: $\begin{cases} x + 2y + z = 1 \\ 2x + by + 2z = 2 \\ 4x + 8y + b^2 z = 2b \end{cases}$
- Let $b \neq 4$. Row-echelon form: $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & b-4 & 0 & 0 \\ 0 & 0 & b^2-4 & 2b-4 \end{array}\right)$
  - Infinitely many solutions

$\Leftrightarrow$ The last and some other columns are non-pivot.

last column is non-pivot $\Leftrightarrow b \neq -2$

some other columns are non-pivot $\Leftrightarrow \begin{cases} 1 \neq 0 \\ b-4 \neq 0 \\ b^2-4 = 0 \end{cases}$

$\Leftrightarrow b = -2$ or $b = 2$.

Example 2

- Find the number of solutions: $\begin{cases} x + 2y + \ \ z = 1 \\ 2x + by + 2z = 2 \\ 4x + 8y + b^2 z = 2b \end{cases}$

- Let $b \neq 4$. Row-echelon form: $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & b-4 & 0 & 0 \\ 0 & 0 & b^2-4 & 2b-4 \end{array}\right)$
  - Infinitely many solutions:
    - $b = 4$ or $b = 2$.
  - No solution:
    - $b = -2$.
  - Exactly one solution:
    - $b \neq 4$, $b \neq -2$, $b \neq 2$.

Example 3

- Find the number of solutions: $\begin{cases} ax + y \qquad\ \ = a \\ \ \ x + y + \ \ z = 1 \\ \qquad\ \ y + az = b \end{cases}$

$$\left(\begin{array}{ccc|c} a & 1 & 0 & a \\ 1 & 1 & 1 & 1 \\ 0 & 1 & a & b \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ a & 1 & 0 & a \\ 0 & 1 & a & b \end{array}\right)$$

$$\xrightarrow{R_2 + (-a)R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1-a & -a & 0 \\ 0 & 1 & a & b \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & a & b \\ 0 & 1-a & -a & 0 \end{array}\right)$$

$$\xrightarrow{R_3 + (a-1)R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & a & b \\ 0 & 0 & a^2-2a & (a-1)b \end{array}\right)$$

Example 3

- Find the number of solutions: $\begin{cases} ax + y = a \\ x + y + z = 1 \\ \phantom{x +{}} y + az = b \end{cases}$
- Row-echelon form: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & a & b \\ 0 & 0 & a^2 - 2a & (a-1)b \end{array}\right)$

No solution $\Leftrightarrow$ last column is pivot
$\Leftrightarrow a^2 - 2a = 0$ and $(a-1)b \neq 0$
$\Leftrightarrow (a = 0$ or $a = 2)$ and $(a \neq 1$ and $b \neq 0)$
$\Leftrightarrow (a = 0$ or $a = 2)$ and $b \neq 0$.

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Example 3

- Find the number of solutions: $\begin{cases} ax + y = a \\ x + y + z = 1 \\ \phantom{x +{}} y + az = b \end{cases}$
- Row-echelon form: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & a & b \\ 0 & 0 & a^2 - 2a & (a-1)b \end{array}\right)$

Unique solution $\Leftrightarrow$ Only the last column is non-pivot
$\Leftrightarrow a^2 - 2a \neq 0$
$\Leftrightarrow a \neq 0$ and $a \neq 2$.

Infinite solutions $\Leftrightarrow$ last and some other columns are non-pivot
$\Leftrightarrow a^2 - 2a = 0$ and $(a-1)b = 0$
$\Leftrightarrow (a = 0$ or $a = 2)$ and $(a = 1$ or $b = 0)$
$\Leftrightarrow (a = 0$ or $a = 2)$ and $b = 0$.

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Example 4

- Find a cubic curve $y = a + bx + cx^2 + dx^3$ that contains points $(0, 10), (1, 7), (3, -11), (4, -14)$.
  - Substitute the (x, y)-coordinates into the cubic curve.
    - We obtain four equations in variables a, b, c, d:

$$\begin{cases} 10 = a + 0b + 0c + 0d \\ 7 = a + 1b + 1c + 1d \\ -11 = a + 3b + 9c + 27d \\ -14 = a + 4b + 16c + 64d \end{cases}$$

In the following, solve the linear system in a, b, c, d to complete the question.

- Augmented matrix: $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 1 & 1 & 1 & 1 & 7 \\ 1 & 3 & 9 & 27 & -11 \\ 1 & 4 & 16 & 64 & -14 \end{array}\right)$

Example 4

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 1 & 1 & 1 & 1 & 7 \\ 1 & 3 & 9 & 27 & -11 \\ 1 & 4 & 16 & 64 & -14 \end{array}\right) \xrightarrow[R_4+(-1)R_1]{\substack{R_2+(-1)R_1 \\ R_3+(-1)R_1}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 1 & 1 & -3 \\ 0 & 3 & 9 & 27 & -21 \\ 0 & 4 & 16 & 64 & -24 \end{array}\right)$$

$$\xrightarrow[R_4+(-4)R_2]{R_3+(-3)R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 1 & 1 & -3 \\ 0 & 0 & 6 & 24 & -12 \\ 0 & 0 & 12 & 60 & -12 \end{array}\right)$$

$$\xrightarrow{R_4+(-2)R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 1 & 1 & -3 \\ 0 & 0 & 6 & 24 & -12 \\ 0 & 0 & 0 & 12 & 12 \end{array}\right)$$

$$\xrightarrow[\frac{1}{12}R_4]{\frac{1}{6}R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 1 & 1 & -3 \\ 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

Example 4

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 1 & 1 & 1 & 1 & 7 \\ 1 & 3 & 9 & 27 & -11 \\ 1 & 4 & 16 & 64 & -14 \end{array}\right) \longrightarrow \cdots \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 1 & 1 & -3 \\ 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow[R_3+(-4)R_4]{R_2+(-1)R_4} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_2+(-1)R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

- Therefore, $a = 10, b = 2, c = -6$ and $d = 1$.
  - The cubic curve is $y = 10 + 2x - 6x^2 + x^3$.

Geometric Interpretation

- Linear system of three equations in three variables x, y, z:
  - $\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases}$

  Suppose that a_{i1}, a_{i2}, a_{i3} are not all zero, $i = 1, 2, 3$.
  - Each equation represents a plane in the xyz-space.

  What is the reduced row-echelon form of the augmented matrix? What is the geometric interpretation?
  - The reduced row-echelon form $\boldsymbol{R}$ has three rows and four columns.
    - The system may be consistent.
    - The system may be inconsistent.

Geometric Interpretation

- Assume that the system is consistent, i.e., the last column is of $\boldsymbol{R}$ is a non-pivot column.
  - Each nonzero row contains exactly one pivot point.
  - Each pivot column contains exactly one pivot point.

$$\begin{aligned}\text{no. of nonzero rows} &= \text{no. of pivot points}\\ &= \text{no. of pivot columns.}\end{aligned}$$

1. Suppose that $\boldsymbol{R}$ has three nonzero rows.
   - The first three columns are all pivot columns.
   - $\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$

   The system has a unique solution.

   The three planes meet at a common point.

Geometric Interpretation

- Assume that the system is consistent, i.e., the last column is of $\boldsymbol{R}$ is a non-pivot column.
  - Each nonzero row contains exactly one pivot point.
  - Each pivot column contains exactly one pivot point.

$$\begin{aligned}\text{no. of nonzero rows} &= \text{no. of pivot points}\\ &= \text{no. of pivot columns.}\end{aligned}$$

2. Suppose that $\boldsymbol{R}$ has two nonzero rows.
   - One of the first three columns is non-pivot.
   - $\left(\begin{array}{ccc|c} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc|c} 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right)$

   The system has infinitely many solutions with one arbitrary parameter.

   The three planes meet at a straight line.

Geometric Interpretation

- Assume that the system is consistent, i.e., the last column is of $\boldsymbol{R}$ is a non-pivot column.
  - Each nonzero row contains exactly one pivot point.
  - Each pivot column contains exactly one pivot point.

$$\begin{aligned}\text{no. of nonzero rows} &= \text{no. of pivot points}\\ &= \text{no. of pivot columns}.\end{aligned}$$

3. Suppose that $\boldsymbol{R}$ has one nonzero row.
   - Only one of the first three columns is pivot.
   - $\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc|c} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc|c} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

   The system has infinitely many solutions with two arbitrary parameters.

   The three planes coincide.

Examples

- $\begin{cases} x + y + 2z = 1 \\ x - y - \;\; z = 0 \\ x + y - \;\; z = 2 \end{cases}.$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & -1 & 2 \end{array}\right) \xrightarrow[R_3+(-1)R_1]{R_2+(-1)R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -2 & -3 & -1 \\ 0 & 0 & -3 & 1 \end{array}\right)$$

$$\xrightarrow[(-\frac{1}{3})R_3]{(-\frac{1}{2})R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right)$$

$$\xrightarrow[R_2+(-\frac{3}{2})R_3]{R_1+(-2)R_3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right)$$

$$\xrightarrow{R_1+(-1)R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right)$$

$x = 2/3$, $y = 1$, $z = -1/3$. The planes meet at point $(2/3, 1, -1/3)$.

Examples

- $\begin{cases} x + y + 2z = 1 \\ x - y - z = 0 \\ 2x + z = 1 \\ 3x - y = 1 \end{cases}$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & -1 & -1 & 0 \\ 2 & 0 & 1 & 1 \\ 3 & -1 & 0 & 1 \end{array}\right) \xrightarrow[R_4+(-3)R_1]{R_2+(-1)R_1,\ R_3+(-2)R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -2 & -3 & -1 \\ 0 & -2 & -3 & -1 \\ 0 & -4 & -6 & -2 \end{array}\right)$$

$$\xrightarrow[R_4+(-2)R_2]{R_3+(-1)R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -2 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{(-\frac{1}{2})R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Examples

- $\begin{cases} x + y + 2z = 1 \\ x - y - z = 0 \\ 2x + z = 1 \\ 3x - y = 1 \end{cases}$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & -1 & -1 & 0 \\ 2 & 0 & 1 & 1 \\ 3 & -1 & 0 & 1 \end{array}\right) - \cdots \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1+(-1)R_2} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

  - Let $z = t$ be an arbitrary parameter (non-pivot column).
    - $x + \frac{1}{2}t = \frac{1}{2} \Rightarrow x = \frac{1}{2} - \frac{1}{2}t$.
    - $y + \frac{3}{2}t = \frac{1}{2} \Rightarrow y = \frac{1}{2} - \frac{3}{2}t$.

Examples

- $\begin{cases} x + y + 2z = 1 \\ x - y - z = 0 \\ 2x + z = 1 \\ 3x - y = 1 \end{cases}$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & -1 & -1 & 0 \\ 2 & 0 & 1 & 1 \\ 3 & -1 & 0 & 1 \end{array}\right) \longrightarrow \cdots \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1+(-1)R_2} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

  - The four planes intersect at the straight line
    - $(\frac{1}{2} - \frac{1}{2}t, \frac{1}{2} - \frac{3}{2}t, t)$, where t is an arbitrary parameter.

Examples

- $\begin{cases} x + y + 2z = 1 \\ 3x + 3y + 6z = 3 \end{cases}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 3 & 3 & 6 & 3 \end{array}\right) \xrightarrow{R_2+(-3)R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

  - Let $y = s$ and $z = t$ be arbitrary parameters.
    - $x + s + 2t = 1 \Rightarrow x = 1 - s - 2t$.
  - The two planes are the same, parameterized by
    - $(1 - s - 2t, s, t)$, where s, t are arbitrary parameters.

Homogeneous Linear Equations & Systems

- **Definition.** A linear equation in variables $x_1, x_2, \ldots, x_n$ is called **homogeneous** if it is of the form
  - $\boxed{a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0}$
- A linear equation in $x_1, x_2, \ldots, x_n$ is homogeneous

  $\Leftrightarrow x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is a solution.
- **Definition.** A linear system is **homogeneous** if every linear equation of the system is homogeneous.
  - $$\begin{cases} a_{11}x_1 + \ a_{12}x_2 + \cdots + \ a_{1n}x_n = 0 \\ a_{21}x_1 + \ a_{22}x_2 + \cdots + \ a_{2n}x_n = 0 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$
- A linear system in $x_1, x_2, \ldots, x_n$ is homogeneous

  $\Leftrightarrow x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is a solution.

  This is the **trivial solution** of a homogeneous linear system.

  Other solutions are called **non-trivial solutions**.

Example

- Find the equation $ax^2 + by^2 + cz^2 = d$ in the xyz-space which contains points $(1, 1, -1), (1, 3, 3), (-2, 0, 2)$.
- Substitute $(x, y, z) = (1, 1, -1), (1, 3, 3), (-2, 0, 2)$ to get three equations in a, b, c, d.
  - $$\begin{cases} a + \ b + \ c = d \\ a + 9b + 9c = d \\ 4a \qquad + 4c = d \end{cases}$$

  This is a homogeneous system in a, b, c, d:
  - $$\begin{cases} a + \ b + \ c - d = 0 \\ a + 9b + 9c - d = 0 \\ 4a \qquad + 4c - d = 0 \end{cases}$$
  - Augmented matrix: $\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 1 & 9 & 9 & -1 & 0 \\ 4 & 0 & 4 & -1 & 0 \end{array}\right)$

Example

- Find the equation $ax^2 + by^2 + cz^2 = d$ in the xyz-space which contains points $(1, 1, -1), (1, 3, 3), (-2, 0, 2)$.
- $\begin{cases} a + b + c - d = 0 \\ a + 9b + 9c - d = 0 \\ 4a + 4c - d = 0 \end{cases}$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 1 & 9 & 9 & -1 & 0 \\ 4 & 0 & 4 & -1 & 0 \end{array}\right) \xrightarrow[R_3+(-4)R_1]{R_2+(-1)R_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 8 & 8 & 0 & 0 \\ 0 & -4 & 0 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{R_3+\frac{1}{2}R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 8 & 8 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \end{array}\right)$$

$$\xrightarrow[\frac{1}{4}R_3]{\frac{1}{8}R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{4} & 0 \end{array}\right)$$

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Example

- Find the equation $ax^2 + by^2 + cz^2 = d$ in the xyz-space which contains points $(1, 1, -1), (1, 3, 3), (-2, 0, 2)$.
- $\begin{cases} a + b + c - d = 0 \\ a + 9b + 9c - d = 0 \\ 4a + 4c - d = 0 \end{cases}$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 1 & 9 & 9 & -1 & 0 \\ 4 & 0 & 4 & -1 & 0 \end{array}\right) - \cdots \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{4} & 0 \end{array}\right)$$

$$\xrightarrow[R_1+(-1)R_3]{R_2+(-1)R_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & -\frac{7}{4} & 0 \\ 0 & 1 & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{3} & 0 \end{array}\right)$$

$$\xrightarrow{R_1+(-1)R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{4} & 0 \end{array}\right)$$

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Example

- Find the equation $ax^2+by^2+cz^2=d$ in the xyz-space which contains points $(1,1,-1),(1,3,3),(-2,0,2)$.
- $\begin{cases} a+\ \ b+\ \ c-d=0 \\ a+9b+9c-d=0 \\ 4a\qquad\ +4c-d=0 \end{cases}$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 1 & 9 & 9 & -1 & 0 \\ 4 & 0 & 4 & -1 & 0 \end{array}\right) - \cdots \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{4} & 0 \end{array}\right)$$

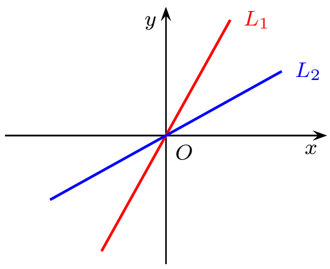
  - Set $d=t$ as an arbitrary parameter. Then
    - $a=t$, $b=\frac{3}{4}t$ and $c=-\frac{3}{4}t$.

  For $t\neq 0$, the equation is $tx^2+\frac{3}{4}ty^2-\frac{3}{4}tz^2=t$.
  - It is equivalent to $x^2+\frac{3}{4}y^2-\frac{3}{4}z^2=1$.

Geometric Interpretation

- In the xy-plane, the homogeneous system of two equations
  - $\begin{cases} a_1x+b_1y=0 & (L_1) \\ a_2x+b_2y=0 & (L_2) \end{cases}$

  where a_1,b_1 not all zero, a_2,b_2 not all zero, represent straight lines through the origin $O(0,0)$.



  - The system has only the trivial solution

    $\Leftrightarrow L_1$ and L_2 are different.

Geometric Interpretation

- In the xy-plane, the homogeneous system of two equations
  - $\begin{cases} a_1x + b_1y = 0 & (L_1) \\ a_2x + b_2y = 0 & (L_2) \end{cases}$

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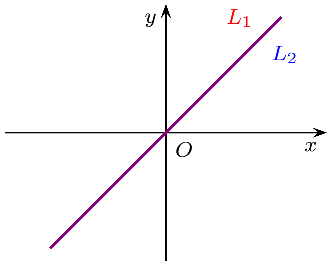


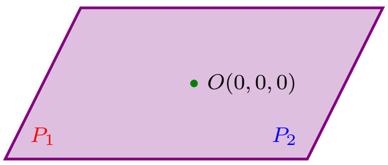
  - The system has non-trivial solutions

    $\Leftrightarrow L_1$ and L_2 are the same.

Geometric Interpretation

- In xyz-space, the homogeneous system of two equations
  - $\begin{cases} a_1x + b_1y + c_1z = 0 & (P_1) \\ a_2x + b_2y + c_2z = 0 & (P_2) \end{cases}$

  where a_1, b_1, c_1 not all zero, a_2, b_2, c_2 not all zero, represent planes containing the origin $O(0, 0, 0)$.
  - The system has (infinitely many) non-trivial solutions.

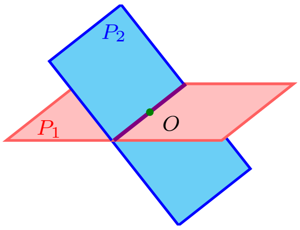


  - Case 1: The two planes are the same.

Geometric Interpretation

- In xyz-space, the homogeneous system of two equations
  - $\left\{\begin{array}{ll} a_1x + b_1y + c_1z = 0 & (P_1) \\ a_2x + b_2y + c_2z = 0 & (P_2) \end{array}\right.$

  where a_1, b_1, c_1 not all zero, a_2, b_2, c_2 not all zero, represent plans containing the origin $O(0,0,0)$.
  - The system has (infinitely many) non-trivial solutions.



  - Case 2: The two planes intersect at a straight line passing through $O(0,0,0)$.