CS1231S: Discrete Structures

Tutorial #7: Mathematical Induction and Recursion

(Week 9: 10 – 14 October 2022)

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1. Discussion Questions

These are meant for you to discuss on Canvas. No answers will be provided.

> **Definition:**
> An integer d is said to be a **linear combination** of integers a and b, if and only if, there exist integers s and t such that $as + bt = d$.

D1. Prove the following proposition:

$$\forall a, b, c \in \mathbb{Z}, \text{ if } a \mid b \text{ and } a \mid c, \text{ then } \forall x, y \in \mathbb{Z}\ (a \mid bx + cy).$$

The proposition states that if a divides both b and c, then a divides their linear combination.

D2. Aiken attempts to prove the following by Mathematical Induction:

$$\forall n \in \mathbb{N}\ \big(3 \mid (n^3 + 44n)\big).$$

However, his proof below is incorrect. Point out the mistakes.

Proof:

1. Note that $1^3 + 44(1) = 45$ which is divisible by 3.
2. So the statement is true for $n = 1$.
3. Now suppose the statement is true for some natural number k.
4. Then $k^3 + 44k$ is divisible by 3.
5. Therefore $(k+1)^3 + 44(k+1)$ is divisible by 3.
6. So by Mathematical Induction, the statement is true for all numbers.

D3. Dueet attempts to prove the following by Mathematical Induction.

Consider a group of n people, each of whom shakes hands exactly once with everybody else in the group. No one shakes his/her own hand. Let $S(n)$ be the total number of handshakes in any group of n people. Prove that

$$\forall n \in \mathbb{Z}^+ \left(S(n) = \frac{n(n-1)}{2} \right).$$

However, her proof below is incorrect. Point out the mistake.

Proof:

1. Let $P(n) \equiv (S(n) = \frac{n(n-1)}{2})$, for any $n \in \mathbb{Z}^+$.
2. Basis step: $n = 1$
   2.1. $S(1) = 0$ because nobody shakes his/her own hand.
   2.2. Also, $\frac{1(1-1)}{2} = 0 = S(1)$.
   2.3. Thus $P(1)$ is true.
3. Inductive step: Assume $P(k)$, i.e. $S(k) = \frac{k(k-1)}{2}$.
   3.1. For any $k \in \mathbb{Z}^+$, consider any group of k people.
   3.2. This group makes $S(k)$ handshakes, by the induction hypothesis.
   3.3. Now consider one new person joining the group. Since all the original k people have already shaken hands, they just need to shake the newcomer's hand, giving k additional handshakes in total.
   3.4. Thus $S(k+1) = S(k) + k = \frac{k(k-1)}{2} + k = \frac{(k+1)k}{2}$ by basic algebra.
   3.5. Hence $P(k+1)$ is true.
4. Therefore $P(n)$ is true for any $n \in \mathbb{Z}^+$, by mathematical induction.

2. Tutorial Questions

In writing Mathematical Induction proofs, please follow the format shown in class.

1. Prove by induction that for all $n \in \mathbb{Z}^+$,
$$\mathbf{1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).}$$

2. Let $x \in \mathbb{R}_{\geq -1}$. Prove by induction that $\mathbf{1 + nx \leq (1+x)^n}$ for all $n \in \mathbb{Z}^+$.

3. In Lecture #5, we claim that given any set A, $|\boldsymbol{\mathcal{P}(A)}| = \mathbf{2^n}$, where $\mathcal{P}(A)$ denotes the power set of A and $|A| = n$. Prove by induction on n that this claim is true by using the argument in Lecture #5.

4. Let a be an odd integer. Prove by induction that $\mathbf{2^{n+2} \mid a^{2^n} - 1}$ for all $n \in \mathbb{Z}^+$.
   Here you may use without proof the fact that the product of any two consecutive integers is even.
   (Note that $a^{b^c} = a^{(b^c)}$ by convention.)

5. Prove by induction that
$$\forall \boldsymbol{n} \in \mathbb{Z}_{\geq 8}\ \exists \boldsymbol{x}, \boldsymbol{y} \in \mathbb{N}\ (\boldsymbol{n} = \mathbf{3}\boldsymbol{x} + \mathbf{5}\boldsymbol{y}).$$
   (In other words, any integer-valued transaction of at least \$8 can be carried out using only \$3 and \$5 notes.)

6. Prove by induction that every positive integer can be written as a sum of *distinct* non-negative integer powers of 2, i.e.,

$$\forall \boldsymbol{n} \in \mathbb{Z}^+\ \exists \boldsymbol{l} \in \mathbb{Z}^+\ \exists \boldsymbol{i_1}, \boldsymbol{i_2}, \cdots, \boldsymbol{i_l} \in \mathbb{N}\ (\boldsymbol{i_1} < \boldsymbol{i_2} < \cdots < \boldsymbol{i_l}\ \wedge \boldsymbol{n} = \boldsymbol{2^{i_1}} + \boldsymbol{2^{i_2}} + \cdots + \boldsymbol{2^{i_l}}).$$

7. Let $F_0, F_1, F_2, \cdots$ be the Fibonacci sequence. Show that $\boldsymbol{F_{n+4}} = \boldsymbol{3F_{n+2}} - \boldsymbol{F_n}$ for all $n \in \mathbb{N}$.

8. Let $F_0, F_1, F_2, \cdots$ be the Fibonacci sequence. Show by induction that for for all $n \in \mathbb{N}$,

$$\boldsymbol{F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n.}$$

9. Let $a_0, a_1, a_2 \cdots$ be the sequence satisfying

$$a_0 = 0, \quad a_1 = 2, \quad a_2 = 7, \quad \text{and} \quad a_{n+3} = a_{n+2} + a_{n+1} + a_n$$

for all $n \in \mathbb{N}$. Prove by induction that $\boldsymbol{a_n} < \boldsymbol{3^n}$ for all $n \in \mathbb{N}$.

10. Define a set S recursively as follows.
    (1) $2 \in S$. (base clause)
    (2) If $x \in S$, then $3x \in S$ and $x^2 \in S$. (recursion clause)
    (3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

    Which of the numbers $0, 6, 15, 16, 36$ are in S? Which are not?

11. Let $A = \{1,2,3,4,5\}$and $B = \{1,3,5,7,9\}$. Define a set S recursively as follows.
    (1) $A, B \in S$. (base clause)
    (2) If $X, Y \in S$, then $X \cap Y \in S$ and $X \cup Y \in S$ and $X \setminus Y \in S$ (recursion clause)
    (3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

    For each of the following sets, determine whether it is in S, and use one sentence to explain your answer.

    (a) $C = \{2,4,7,9\}$.
    (b) $D = \{2,3,4,5\}$.