CS1231S: Discrete Structures
Tutorial #8: Cardinality
Answers

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1. In lecture example #3, we showed that $\mathbb{Z}$ is countable by defining a bijection $f: \mathbb{Z}^+ \to \mathbb{Z}$ as follows:

$$f(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even;} \\ -\dfrac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

The above is based on the definition $\aleph_0 = |\mathbb{Z}^+|$. Suppose we adopt the definition $\aleph_0 = |\mathbb{N}|$ instead, define a bijection $g: \mathbb{N} \to \mathbb{Z}$ using a <u>single-line formula</u> to show that $\mathbb{Z}$ is countable.

Answer:

One such bijection (there could be others) $g: \mathbb{N} \to \mathbb{Z}$ is:

$$g(n) = (-1)^n \left\lfloor \frac{n+1}{2} \right\rfloor$$

Proof:

Note that (-1) to an even power is 1, and (-1) to an odd power is -1.

1. (Injectivity)
   - 1.1. Let $g(a), g(b) \in \mathbb{Z}$ and $g(a) = g(b)$.
   - 1.2. Then $g(a)$ and $g(b)$ must both be non-negative or both negative.
   - 1.3. Case 1: $g(a)$ and $g(b)$ are both non-negative.
     - 1.3.1. Then a and b must be even.
     - 1.3.2. Then $(-1)^a \left\lfloor \frac{a+1}{2} \right\rfloor = (-1)^b \left\lfloor \frac{b+1}{2} \right\rfloor \Rightarrow \left\lfloor \frac{a+1}{2} \right\rfloor = \left\lfloor \frac{b+1}{2} \right\rfloor \Rightarrow \frac{a}{2} = \frac{b}{2} \Rightarrow a = b$.
   - 1.4. Case 2: $g(a)$ and $g(b)$ are both negative.
     - 1.4.1. Then a and b must be odd.
     - 1.4.2. Then $(-1)^a \left\lfloor \frac{a+1}{2} \right\rfloor = (-1)^b \left\lfloor \frac{b+1}{2} \right\rfloor \Rightarrow -\left\lfloor \frac{a+1}{2} \right\rfloor = -\left\lfloor \frac{b+1}{2} \right\rfloor \Rightarrow \frac{a+1}{2} = \frac{b+1}{2} \Rightarrow a = b$.
   - 1.5. In all cases, $a = b$.
   - 1.6. Therefore g is injective.

2. (Surjectivity)
   - 2.1. Let $m \in \mathbb{Z}$. Then m is non-negative or negative.
   - 2.2. Case 1: m is non-negative.
     - 2.2.1. Let $n = 2m$.
     - 2.2.2. Then $n \in \mathbb{N}$ and $g(n) = (-1)^{2m} \left\lfloor \frac{2m+1}{2} \right\rfloor = \frac{2m}{2} = m$.
   - 2.3. Case 1: m is negative.
     - 2.3.1. Let $n = -2m - 1$.
     - 2.3.2. Then $n \in \mathbb{N}$ and $g(n) = (-1)^{-2m-1} \left\lfloor \frac{(-2m-1)+1}{2} \right\rfloor = -\frac{-2m}{2} = m$.
   - 2.4. In all cases, there exists $n \in \mathbb{N}$ such that $g(n) = m$.
   - 2.5. Therefore g is surjective.

3. Therefore g is a bijection from $\mathbb{N}$ to $\mathbb{Z}$.

2. Let B be a countably infinite set and C a finite set. Show that $B \cup C$ is countable

 (a) by using the sequence argument;

 (b) by defining a bijection $g: \mathbb{N} \to B \cup C$.

> **Lemma 9.2**
> An infinite set B is countable if and only if there is a sequence $b_0, b_1, b_2, \cdots$ in which every element of B appears.

Answers:

(a) 1. Apply Lemma 9.2 to obtain a sequence $b_0, b_1, b_2, \cdots$ in which every element B appears.
 2. Suppose $|C| = n \in \mathbb{N}$. We may write $C = \{c_0, c_1, c_2, \cdots, c_{n-1}\}$.
 3. Then $c_0, c_1, c_2, \cdots, c_{n-1}, b_0, b_1, b_2, \cdots$ is a sequence in which every element of $B \cup C$ appears.
 4. So $B \cup C$ is countable by Lemma 9.2.

(b) 1. As B is a countably infinite set, we have a bijection $f: \mathbb{N} \to B$.
 2. Remove all elements in C that are in B. After removal, $C = \{c_0, c_1, c_2, \cdots, c_{k-1}\}$.
 3. Define a function $g: \mathbb{N} \to B \cup C$ such that
 $$g(i) = \begin{cases} c_i & \text{if } i < k; \\ f(i-k) & \text{otherwise.} \end{cases}$$
 4. As the c_i's are distinct, $g(i) = g(j) \Rightarrow c_i = c_j \Rightarrow i = j$, and hence g is injective from $\{0,1,\cdots,k-1\}$ to C.
 5. For every c_i, there exists an i such that $g(i) = c_i$, hence g is surjective from $\{0,1,\cdots,k-1\}$ to C.
 6. Therefore g is a bijection between $\{0,1,\cdots,k-1\}$ and C.
 7. g is a bijection between $\{k, k+1, \cdots\}$ and B as f is bijective between $\{0,1,2,\cdots\}$ and B.
 8. Therefore g is a bijection between $\mathbb{N}$ and $B \cup C$.

Note: You can see that the sequence argument "shields off" a lot of details.

3. Recall the definition of $\bigcup_{i=m}^{n} A_i$ in Tutorial 3.

 (a) Consider this claim:

 "Suppose $A_1, A_2, \cdots$ are finite sets. Then $\bigcup_{i=1}^{n} A_i$ is finite for any $n \geq 2$."

 The above statement is true. However, consider the following "proof":

 "We will prove by induction on n. Since A_1 and A_2 are finite, then $A_1 \cup A_2$ is finite, so the claim is true for $n = 2$. Now suppose the claim is true for $n = k$, so $\bigcup_{i=1}^{k} A_i$ is finite. Let $A_{k+1} = \emptyset$. Then $\bigcup_{i=1}^{k+1} A_i = (\bigcup_{i=1}^{k} A_i) \cup A_{k+1} = \bigcup_{i=1}^{k} A_i$ which is finite by the induction hypothesis, so the claim is true for $n = k+1$. Therefore, the claim is true for all $n \geq 2$."

 What is wrong with this "proof"?

 (b) Prove the following is false: "Suppose $A_1, A_2, \cdots$ are finite sets. Then $\bigcup_{k=1}^{\infty} A_k$ is finite."
 [The point here is: induction takes you to any finite n, but not to infinity.]

Answers:

(a) There is an implicit universal quantification on $A_1, A_2, \cdots$, i.e. we have to prove the claim is true for all possible $A_1, A_2, \cdots$, so we cannot just consider the special case $A_{k+1} = \emptyset$.

(b) Let $A_i = \{i\}$ for all $i \geq 1$. Then $\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}^+$, which is infinite.

4. Suppose $A_1, A_2, A_3, \cdots$ are countable sets.

   (a) Prove, by induction, that $\bigcup_{i=1}^{n} A_i$ is countable for any $n \in \mathbb{Z}^+$.

   (b) Does (a) prove that $\bigcup_{i=1}^{\infty} A_i$ is countable?

   Answers:

   > **Lemma 9.4**
   > Let A and B be countably infinite sets. Then $A \cup B$ is countable.

   (a) $\bigcup_{i=1}^{n} A_i$ is countable.

   Proof:

   1. Let $P(n)$ means "$\bigcup_{i=1}^{n} A_i$ is countable".
   2. Basis step: $\bigcup_{i=1}^{1} A_i = A_1$ is countable, so $P(1)$ is true.
   3. Induction step: Suppose $P(k)$ is true.
      3.1. $\bigcup_{i=1}^{k+1} A_i = (\bigcup_{i=1}^{k} A_i) \cup A_{k+1}$.
      3.2. Since $\bigcup_{i=1}^{k} A_i$ is countable (by induction hypothesis) and A_{k+1} is countable, so $(\bigcup_{i=1}^{k} A_i) \cup A_{k+1}$ is countable (by Lemma 9.4).
      3.3. Hence $P(k+1)$ *is true*.
   4. Therefore $\bigcup_{i=1}^{n} A_i$ is countable for any $n \in \mathbb{Z}^+$ by MI.

   (b) No. Question 3(b) shows that a proof that $\bigcup_{i=1}^{n} A_i$ is finite for every $n \geq 2$ does not imply $\bigcup_{i=1}^{\infty} A_i$ is finite. Similarly here, a proof that $\bigcup_{i=1}^{n} A_i$ is countable for every $n \geq 1$ does not imply that $\bigcup_{i=1}^{\infty} A_i$ is countable.

   Note that $\bigcup_{i=1}^{\infty} A_i$ is indeed countable, just that it cannot be proved using the approach in part (a). We will prove it in the next question.

5. Let S_i be a countably infinite set for each $i \in \mathbb{Z}^+$. Prove that $\bigcup_{i \in \mathbb{Z}^+} S_i$ is countable.
   [Hint: Use this theorem covered in class: $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.]

   Answer:

   > **Lemma 9.2**
   > An infinite set B is countable if and only if there is a sequence $b_0, b_1, b_2, \cdots$ in which every element of B appears.

   1. Note that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.
   2. Hence there is a bijection $f: \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$.
   3. For each $i \in \mathbb{Z}^+$, since S_i is countable, apply Lemma 9.2 to find a sequence $b_{i,1}, b_{i,2}, b_{i,3}, \cdots$ in which every element of S_i appears.
   4. Define a sequence $c_1, c_2, c_3, \cdots$ by setting each $c_k = b_{i,j}$, where $(i,j) = f(k)$.
   5. In view of Lemma 9.2, it suffices to show that every element of $\bigcup_{i \in \mathbb{Z}^+} S_i$ appears in the sequence $c_1, c_2, c_3, \cdots$.
      5.1. Take $x \in \bigcup_{i \in \mathbb{Z}^+} S_i$.
      5.2. The definition of $\bigcup_{i \in \mathbb{Z}^+} S_i$ gives $i \in \mathbb{Z}^+$ such that $x \in S_i$.
      5.3. So line 3 tell us that x appears in the sequence $b_{i,1}, b_{i,2}, b_{i,3}, \cdots$.
      5.4. Let $j \in \mathbb{Z}^+$ such that $x = b_{i,j}$.
      5.5. From the surjectivity of f, we obtain $k \in \mathbb{Z}^+$ such that $f(k) = (i,j)$.
      5.6. Then $x = b_{i,j} = c_k$ by the definition of c_k.

6. Let B be a (not necessarily countable) infinite set and C be a finite set. Define a bijection $B \cup C \to B$.

> **Proposition 9.3**
> Every infinite set has a countably infinite subset.

Answer:

1. Use Proposition 9.3 to find a countably infinite subset $B_0 \subseteq B$.
2. Let $C_0 = C \setminus B$, so that C_0 is finite.
3. Then $B_0 \cup C_0$ is countable by question 2.
4. Hence $|B_0 \cup C_0| = |\mathbb{Z}^+| = |B_0|$ by the definition of countably infinite sets.
5. Hence there is a bijection $f: B_0 \cup C_0 \to B_0$.
6. Define $g: B \cup C \to B$ as follows: for each $x \in B \cup C$,

$$g(x) = \begin{cases} f(x), & \text{if } x \in B_0 \cup C_0; \\ x, & \text{otherwise.} \end{cases}$$

7. g is the required bijection.

7. Prove that a set B is infinite if and only if there is $A \subsetneq B$ such that $|A| = |B|$.

Answer:

1. ("Only if")
   1.1. Suppose B is infinite.
   1.2. Taken any $b \in B$.
   1.3. Define $A = B \setminus \{b\}$.
   1.4. Then $A \subsetneq B$.
   1.5. As B is infinite, we know A is infinite too.
   1.6. So $|B| = |A \cup \{b\}| = |A|$ by question 6.
2. ("If") We prove the contrapositive: If B is finite then for all $A \subsetneq B$, $|A| \neq |B|$.
   2.1. Suppose B is finite.
   2.2. Taken any $A \subsetneq B$.
   2.3. As B is finite, we know A is also finite.
   2.4. There are strictly few elements in A than in B.
   2.5. Hence there is no bijection $A \to B$.
   2.6. This means $|A| \neq |B|$ by Equality of Cardinality of Finite Sets Theorem.

> **Theorem: Equality of Cardinality of Finite Sets**
> Let A and B be any finite sets. $|A| = |B|$ iff there is a bijection $f: A \to B$.

8. Prove that $\mathbb{C}$ (the set of complex numbers) is uncountable.

> **Corollary 7.4.4**
> Any set with an uncountable subset is uncountable.

Answer:

1. $\mathbb{R} \subseteq \mathbb{C}$.
2. We know that $\mathbb{R}$ is uncountable from lecture #9 example #5.
3. Therefore $\mathbb{C}$ is uncountable by Corollary 7.4.4.

9. Let A be a countably infinite set. Prove that $\mathcal{P}(A)$ is uncountable.
(Note: $\mathcal{P}(A)$ is the power set of A.)

Answer:

Sketch: We prove by contradiction. Assuming that $\mathcal{P}(A)$ is countable, we provide a sequence of elements of $\mathcal{P}(A)$. Then we produce an element of $\mathcal{P}(A)$ that does not appear in the sequence that claims to contain all elements of $\wp(A)$.

1. Suppose not, that is, $\mathcal{P}(A)$ is countable.
2. $\mathcal{P}(A)$ is infinite as A is infinite and $\{a\} \in \mathcal{P}(A)$ for every $a \in A$.
3. By Proposition 9.1, there is a sequence $a_0, a_1, a_2, \cdots \in A$ in which every element of A appears exactly once.
4. By Proposition 9.1, there is a sequence $B_0, B_1, B_2, \cdots \in \mathcal{P}(A)$ in which every element of $\mathcal{P}(A)$ appears exactly once.
5. Now, define $B = \{a_i : a_i \notin B_i\}$.
6. Note that $B \in \mathcal{P}(A)$ since $a_0, a_1, a_2, \cdots \in A$.
7. To show $B \neq B_i$ for all $i \in \mathbb{N}$.
   - 7.1. Let $i \in \mathbb{N}$.
   - 7.2. Case 1: If $a_i \notin B_i$, then $a_i \in B$ by the definition of B.
   - 7.3. Case 2: If $a_i \in B_i$, then $a_i \notin B$ by the definition of B (as every element a_i of A appears exactly once in the sequence $a_0, a_1, a_2, \cdots$, so no $a_i = a_j$ if $i \neq j$.)
   - 7.4. In all cases, $B \neq B_i$.
8. Since B is not in the sequence $B_0, B_1, B_2, \cdots \in \mathcal{P}(A)$, this contradicts the claim that $\mathcal{P}(A)$ is countable.
9. Therefore, $\mathcal{P}(A)$ is uncountable.

Proposition 9.1
An infinite set B is countable if and only if there is a sequence $b_0, b_1, b_2, \cdots \in B$ in which every element of B appears exactly once.