(2022/23) Semester I   **MA1521 Calculus for Computing**   **Tutorial 11**

1. Solve the following differential equations:

   (a) $xy' + (1+x)y = e^{-x}$, $x > 0$,

   (b) $y' - (1 + \frac{3}{x})y = x + 2$, $y(1) = e - 1$, $x > 0$,

   (c) $y' + y + \frac{x}{y} = 0$,

   (d) $2xyy' + (x-1)y^2 = x^2 e^x$, $x > 0$.

   Ans.   (a) $y = e^{-x}(1 + \frac{C}{x})$, (b) $y = -x + x^3 e^x$, (c) $y^2 = \frac{1}{2} - x + Ce^{-2x}$, (d) $y^2 = \frac{1}{2}xe^x + Cxe^{-x}$.

2. Solve the following equations:

   (a) $y' = \frac{1 - 2y - 4x}{1 + y + 2x}$

   (b) $y' = \left(\frac{x+y+1}{x+y+3}\right)^2$

   (c) $y' = \dfrac{3x - y - 5}{-x + 3y + 7}$

   [Hint: For (c), let $x = z + 1$, $y = w - 2$.]

   Ans.   (a) $4x^2 + 4xy + y^2 - 2x + 2y = C$, (b) $-x + y + \ln|(x+y+1)^2 + 2(x+y+1) + 2| = C$, (c) $(x+y+1)^2(y-x+3) = C$.

3. A certain person starts a rumour in a small town. The number of people who have heard the rumour, $R(t)$, is given by

$$\frac{dR}{dt} = KR(1300 - R),$$

   where K is a positive constant, and 1300 is the number of residents in that small town. What is the meaning of K? Is this equation reasonable? [Hint: surely the rumour will spread slowly both when hardly anyone has heard it yet, **but also** when nearly everyone has already heard it!] By regarding this equation as a Bernoulli equation, find $R(t)$.

4. The half-life of Thorium 230 is about 75000 years, while that of Uranium 234 is about 245000 years. A certain sample of ancient coral has a Thorium/Uranium ratio of 10 percent. How old is the coral?

5. The number of bacteria in a certain culture is 10000 initially. Two and a half hours later there are 11000 of them. Assuming a standard Malthus' model, how many bacteria will there be 10 hours after the start of the experiment? How long will it take for the number to reach 20000?

   Ans. about 14600; about 18.18 hours.

Further Exercises

1. Using the transformation $y = xv$, solve the differential equation
$$y' = \frac{x^2 + xy + y^2}{x^2}.$$
   Ans. $y = x \tan(\ln |x| + c)$.

2. You have 200 bugs in a bottle. Every day you supply them with food and count them. After two days you have 360 bugs. It is known that the birth rate for this kind of bug is 150% per day. Assuming that the population is given by the improved Malthus' model, find the number of bugs after 3 days. Predict how many bugs you will have eventually.

   Ans. about 372, about 376.

3. If a cable is held up at two ends at the same height, then it will sag in the middle, making a U-shaped curve called a **catenary**. This is the shape seen in electricity cables suspended between poles. It can be shown using simple physics that if the shape is given by a function $y(x)$, then this function satisfies
$$\frac{dy}{dx} = \frac{\mu}{T} \int_0^x \sqrt{\left(\frac{dy}{dt}\right)^2 + 1} \, dt,$$
   where $x = 0$ at the lowest point of the catenary and $y(0) = 0$, where μ is the weight per unit length of the cable, and where T is the horizontal component of its tension; this horizontal component is a constant along the cable. Find a formula for the shape of the cable. [Hint: Use the Fundamental Theorem of Calculus, and think of the resulting equation as a **first-order** ODE.]