CS1231S: Discrete Structures

Tutorial #1: Propositional Logic and Proofs

Tutorials are meant to reinforce topics taught in lectures. Please try these questions on your own before coming to tutorial. In doing so, you may discover gaps in your understanding. Usually, a tutorial has a mix of easy, moderate and slightly challenging questions. It is perfectly fine if you cannot do some of the questions, but attempt them nonetheless, to at least get some partial solution.

You will be asked to present your answers. Your tutor's job is to guide you, not to provide the answers for you. Also, please keep in mind that the goal of tutorials is not to answer every question here, but to clarify doubts and reinforce concepts. Solutions to all tutorial questions will be given in the following week, but please treat them as a guide, for there are usually alternative ways of solving a problem.

You are also encouraged to raise your doubts or questions on **Canvas** or the **QnA** site

Tutorials are important so attendance is taken and it contributes 5% of your final grade. If you miss a tutorial with valid reason (eg: due to illness), please submit your document (eg: medical certificate) to your respective tutor (softcopy or hardcopy) in advance or within three days after your absence and you will not be penalized for your absence. You are to stick with your officially assigned tutorial group, or your attendance will not be taken. If you need to join a different group for just once with a valid reason, please email Aaron (tantc@comp.nus.edu.sg) in advance citing your reason.

Please take note of the above as it will not be repeated in subsequent tutorials.

Discussion Questions

The questions in this section will not be discussed in tutorial. You may discuss them or post your answer on Canvas.

D1. We discussed in class that English is an ambiguous language. Often, it also carries notions of perception, past experiences, cause and effect, etc. For example, in saying "today is rainy but hot", we use "but" because we often associate a rainy day as a cool day. In logic, there is no "but", "whenever", "unless", etc.; we need to find the appropriate logic connective. We would use "and" for "but" for the above example and write "today is rainy" ∧ "today is hot".

Restate the following symbolically. You may introduce your own statement variables.

(a) $P(B) \leq P(A)$ whenever $B \subseteq A$. (What is the meaning of "whenever"? Which logical connective should we use for it? 'And', 'or', 'if', 'only if', or others?)

(b) If $A - B$ is countable, then A is countable or B is uncountable.

(c) An undirected graph (V, E) that is connected and acyclic must have $|E| = |V| - 1$.

D2. Draw a truth table to show that the following statement is a tautology:

$$\big((a \to x) \land (b \to y)\big) \to \big((a \land b) \to (x \land y)\big).$$

D3. As mentioned above, we often rely on our past experiences and assumption in interpreting English sentences. For example, if you hear a mother utter this to her son:

"If you behave, you get ice-cream."

Suppose the above statement is true. Which of the following statements is/are true?

(a) If the child behaves, he gets ice-cream.

(b) If the child does not behave, he does not get ice-cream.

(c) If the child gets ice-cream, he behaves.

(d) If the child does not get ice-cream, he does not behave.

Let p represents "the child behaves" and q represents "the child gets ice-cream". Write out the above four statements in propositional logic, and relate them to what was discussed in lecture.

D4. Use the laws given in **Theorem 2.1.1 (Epp)** and the **implication law** to prove that the following are tautologies.

(a) $(p \wedge q) \rightarrow p$

(b) $\big((p \vee q) \wedge \sim p\big) \rightarrow q$

(c) $\big((p \rightarrow q) \wedge p\big) \rightarrow q$

(d) $\big(\sim p \rightarrow (q \wedge \sim q)\big) \rightarrow p$

Mathematical arguments are often constructed by using one implication (conditional statement) after another. Logically speaking, such an argument is constructed by using implications that are tautologies, like the ones above. For example, (c) is *modus ponens* and (d) is proof by contradiction. Part (b) is used in problems like Knights and Knaves in Q8 below.

D5. Mala has hidden her treasure somewhere on her property. She left a note in which she listed five statements (a-e below) and challenged the reader to use them to figure out the location of the treasure.

(a) If this house is next to a lake, then the treasure is not in the kitchen.

(b) If the tree in the front yard is an elm, then the treasure is in the kitchen.

(c) If the tree in the back yard is an oak, then the treasure is in the garage.

(d) The tree in the front yard us an elm or the treasure is buried under the flagpole.

(e) The house is next to a lake.

Where has Mala hidden her treasure?

II. Additional Notes

Given an argument:

$$\begin{array}{l} \quad p_1 \\ \quad p_2 \\ \quad \vdots \\ \quad p_k \\ \therefore q \end{array}$$

where $p_1, p_2, \ldots, p_k$ are the k premises and q the conclusion, we can say that "the argument is valid if and only if $(p_1 \wedge p_k \wedge \cdots \wedge p_k) \longrightarrow q$ is a tautology".

This serves as an alternative way to check whether an argument is valid, besides the critical row method shown in lecture. Go through the examples in the lecture yourself to verify the above.

Tutorial Questions

1. One of the most confusing concepts many students find is the difference between "if" and "only if", and the relationship among "if", "only if", "necessary condition" and "sufficient condition".
   a. Given these two statements: "I use the umbrella if it rains" and "I use the umbrella only if it rains". They may sound the same but in logic they are worlds apart! Now, rewrite them into propositional statements by using variable p for "I use the umbrella", variable q for "it rains" and the logical connective $\rightarrow$.
   b. "I use the umbrella if it rains": Is "I use the umbrella" a necessary condition for "it rains"? Or is "I use the umbrella" a sufficient condition for "it rains"? Is "it rains" a necessary condition for "I use the umbrella"? Or is "it rains" a sufficient condition for "I use the umbrella"?
   c. What if we say "I use the umbrella if and only if it rains". How would you write a logic statement using variables p and q, the imply connective $\rightarrow$, and either $\wedge$ or $\vee$? Is there a shorter way to write the logic statement using some other logical connective?
   d. "I use the umbrella if and only if it rains". What kind of condition is "I use the umbrella" for "it rains"?

2. Simplify the propositions below using the laws given in **Theorem 2.1.1 (Epp)** and the **implication law** (if necessary) with only negation ($\sim$), conjunction ($\wedge$) and disjunction ($\vee$) in your final answers. Supply a justification for every step.

   (For now, we want students to cite justification for every step. This is to ensure that you do not arrive at the answer by coincidence. Only after you have gained sufficient experience then would we relax this and allow you to skip obvious steps, or combine multiple steps in a line.)

   a. $\sim a \wedge \big(\sim a \rightarrow (b \wedge a)\big)$

      Aiken worked out his answer as shown below. However, he skipped some steps and hence his answer will not be awarded full credit. Can you point out the omissions? (Note: To show that two logical statements are equivalent, we use $\equiv$, not $=$.)

      $\sim a \wedge \big(\sim a \rightarrow (b \wedge a)\big)$
      $\equiv \sim a \ \wedge \big(a \ \vee (b \wedge a)\big)$ by the implication law
      $\equiv \sim a \ \wedge a$ by the absorption law
      $\equiv$ **false** by the negation law

      Reminder: We will use **true** and **false** instead of **t** and **c** (used in Susanna Epp's book) for tautology and contradiction respectively.

   b. $(p \vee \sim q) \rightarrow q$
   c. $\sim(p \vee \sim q) \vee (\sim p \wedge \sim q)$
   d. $(p \rightarrow q) \rightarrow r$

3. Prove, or disprove, that $(p \rightarrow q) \rightarrow r$ is logically equivalent to $p \rightarrow (q \rightarrow r)$.

4. Given the conditional statement "If $12x - 7 = 29$, then $x = 3$", write the **negation**, **contrapositive**, **converse** and **inverse** of the statement.

   Is the given conditional statement true? If it true, prove it; otherwise, give a counter-example.

   Is its converse true? If it is true, prove it; otherwise, give a counter-example.

   In general, is it possible for the converse of a conditional statement to be true while the inverse of the same statement is false? Why?

5. The conditional statement $p \rightarrow q$ is an important logical statement. Recall that it is defined by the following truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Oftentimes, students are perplexed by this definition. The first two rows look reasonable, but the last two rows seem strange. However, this way of defining $p \rightarrow q$ actually gives us the nice intuitive property of the following statement:

$$\big((p \rightarrow q) \land (q \rightarrow r)\big) \rightarrow (p \rightarrow r)$$

which is the **transitive rule of inference** we studied in lecture (Lecture #2, slide 65):

$$\begin{array}{l} \quad p \rightarrow q \\ \quad q \rightarrow r \\ \therefore p \rightarrow r \end{array}$$

For example, given premises "if x is a square then x is a rectangle" and "if x is a rectangle than x is a quadrilateral", the conclusion is "if x is a square then x is a quadrilateral". We use such intuitive reasoning very often in our life.

Show that if we define the conditional statement alternatively as follows, then the transitive rule of inference would no longer hold.

Alternative 1: $\rightarrow_a$

p	q	$p \rightarrow_a q$
T	T	T
T	F	F
F	T	F
F	F	F

Alternative 2: $\rightarrow_b$

p	q	$p \rightarrow_b q$
T	T	T
T	F	F
F	T	T
F	F	F

Alternative 3: $\rightarrow_c$

p	q	$p \rightarrow_c q$
T	T	T
T	F	F
F	T	F
F	F	T

6. Some of the arguments below are valid, whereas others exhibit the converse or inverse error. Use symbols to write the logical form of each argument. If the argument is valid, identify the rule of inference that guarantees its validity. Otherwise, state whether the converse or the inverse error is made.

   a. Sandra knows Java and Sandra knows C++.
   ∴ Sandra knows C++.

   b. If at least one of these two numbers is divisible by 6, then the product of these two numbers is divisible by 6.
   Neither of these two numbers is divisible by 6.
   ∴ The product of these two numbers is not divisible by 6.

   c. If there are as many rational numbers as there are irrational numbers, then the set of all irrational numbers is infinite.
   The set of all irrational numbers is infinite.
   ∴ There are as many rational numbers as there are irrational numbers.

   d. If I get a Christmas bonus, I'll buy a stereo.
   If I sell my motorcycle, I'll buy a stereo.
   ∴ If I get a Christmas bonus or I sell my motorcycle, I'll buy a stereo.

7a. Given the following argument:

$$p \lor (q \land r)$$
$$\sim p$$
$$\therefore q \land r$$

<u>Without</u> actually drawing the truth table, determine the values of p, q and r in the critical row(s) of the truth table. Is the argument valid?

b. Give a counterexample to show that the following argument is invalid.

$$p \lor (q \land r)$$
$$\sim (p \land q)$$
$$\therefore r$$

c. Determine whether the following argument is valid or invalid. Use variables to represent the statements (for example: let p be "I go to the beach".)

   If I go to the beach, I will take my shades or my sunscreen.
   I am taking my shades but not my sunscreen.
   ∴ I will go to the beach.

d. Determine whether the following argument is valid or invalid. Use variables to represent the statements.

   I will buy a new goat or a used Yugo.
   If I buy both a new goat and a used Yugo, I will need a loan.
   I bought a used Yugo but I don't need a loan.
   ∴ I didn't buy a new goat.

8. The island of Wantuutrewan is inhabited by exactly two types of people: **knights** who always tell the truth and **knaves** who always lie. Every native is a knight or a knave, but not both. You visit the island and have the following encounters with the natives.

   a. Two natives *A* and *B* speak to you:

      A says: Both of us are knights.
      B says: *A* is a knave.

      What are *A* and *B*?

   b. Two natives *C* and *D* speak to you:

      C says: *D* is a knave.
      D says: *C* is a knave.

      How many knights and knaves are there?

Part (a) has been solved for you (see below). Study the solution, and use the same format in answering part (b).

Answer for part (a):

Proof (by contradiction).

1. If *A* is a knight, then:
   - 1.1 What *A* says is true. (by definition of knight)
   - 1.2 $\therefore$ *B* is a knight too. (that's what *A* says)
   - 1.3 $\therefore$ What *B* says is true. (by definition of knight)
   - 1.4 $\therefore$ *A* is a knave. (that's what *B* says)
   - 1.5 $\therefore$ *A* is not a knight. (since *A* is either a knight or a knave, but not both)
   - 1.6 $\therefore$ Contradiction to 1.
2. $\therefore$ *A* is not a knight.
3. $\therefore$ *A* is a knave. (since *A* is either a knight or a knave, but not both)
4. $\therefore$ What *B* says is true.
5. $\therefore$ *B* cannot be a knave. (as *B* has said something true)
6. $\therefore$ *B* is a knight. (one is a knight or a knave)
7. Conclusion: *A* is a knave and *B* is a knight.

Notes:
- It is tempting to say "Contradiction" right after line 1.4. However, this is not valid because contradiction requires $p \wedge \sim p$, but 'knave' is not the negation of 'knight'. Hence line 1.5 is required before we arrive at the contradiction in 1.6.

9. Recall the definitions of even and odd integers in Lecture #1 slide 27:

> If n is an integer, then
> n is even if and only if $\exists k \in \mathbb{Z}$ s.t. $n = 2k$;
> n is odd if and only if $\exists k \in \mathbb{Z}$ s.t. $n = 2k + 1$.

Prove the following:

The product of any two odd integers is an odd integer.

10. Your classmate Smart came across this claim:

Let n be an integer. Then n^2 is odd if and only if n is odd.

a. Smart attempts to prove the above claim as follows:

Proof (by contradiction).

1. Suppose n is an even integer.
2. Then $\exists k \in \mathbb{Z}$ s.t. $n = 2k$.
3. Squaring both sides, we get $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
4. Since k is an integer, so is $2k^2$.
5. Hence $n^2 = 2p$, with $p = 2k^2 \in \mathbb{Z}$.
6. Therefore, n^2 is even.
7. So, if n is even, then n^2 is even, which is the same as saying, if n^2 is odd, then n is odd.
8. Therefore, n^2 is odd if and only if n is odd.

Comment on Smart's proof.

b. Write your own proof.