NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 2, 2022/2023

MA2001 Linear Algebra **Homework Assignment 4**

ONLINE QUIZ

1. Suppose $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

Note that $\boldsymbol{P}$ is invertible, then $\text{rank}(\boldsymbol{A}) = \text{rank}(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}) = 3$.

2. If $\boldsymbol{A}$ is orthogonal, then for any constant c,

$$(c\boldsymbol{A})^{\mathrm{T}}(c\boldsymbol{A}) = c^2\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = c^2\boldsymbol{I}.$$

So $c\boldsymbol{A}$ is orthogonal $\Leftrightarrow c^2 = 1 \Leftrightarrow c = \pm 1$.

3. If $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are orthogonal matrices of the same order, then

$$(\boldsymbol{ABC})^{\mathrm{T}}(\boldsymbol{ABC}) = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{ABC} = \boldsymbol{I}.$$

So $\boldsymbol{ABC}$ is orthogonal.

4. Let $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then 0 is the only eigenvalue of $\boldsymbol{A}$ but $\boldsymbol{A} \neq \boldsymbol{0}$.

5. If $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$, then $\boldsymbol{v} = -\boldsymbol{u}$ is also an eigenvector of $\boldsymbol{A}$, but $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{0}$ is not.

6. If $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$, then $\boldsymbol{v} = -\boldsymbol{u}$ is also an eigenvector of $\boldsymbol{A}$, and $\boldsymbol{u}, \boldsymbol{v}$ are linearly dependent.

7. Every symmetric matrix is orthogonally diagonalizable, in particular, it is diagonalizable.

8. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $\boldsymbol{A}$ is invertible but not diagonalizable.

9. Suppose $\boldsymbol{A}$ is diagonalized by $\boldsymbol{P}$. Let $\boldsymbol{Q} = (\boldsymbol{P}^{\mathrm{T}})^{-1} = (\boldsymbol{P}^{-1})^{\mathrm{T}}$. Then

$$\boldsymbol{Q}^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Q} = \boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}(P^{-1})^{\mathrm{T}} = (\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P})^{\mathrm{T}}$$

is a diagonal matrix. Hence, $\boldsymbol{A}^{\mathrm{T}}$ is diagonalizable.

10. Suppose $\boldsymbol{A}$ is diagonalized by $\boldsymbol{P}$. Then

$$\boldsymbol{P}^{-1}\boldsymbol{A}^2\boldsymbol{P} = (\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P})^2$$

is a diagonal matrix. Hence, $\boldsymbol{A}^2$ is diagonalizable.

11. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonally diagonalizable (symmetric), but $\boldsymbol{AB} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not.

12. The least squares solutions to $\boldsymbol{Ax} = \boldsymbol{b}$ are all solutions to $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Ax} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$. These solutions form a vector space $\Leftrightarrow$ $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{0}$.

13. The distance from (x_0, y_0, z_0) to the plane $ax + by + cz = 0$ is given by

$$\frac{|ax_0 + by_0 + cz_0|}{\|(a, b, c)\|}.$$

So the distance from $(3\sqrt{2}, \sqrt{3}, 0)$ to $x\sqrt{3} - y\sqrt{2} + z = 0$ is

$$d = \frac{|3\sqrt{2}\cdot\sqrt{3} + \sqrt{3}\cdot(-\sqrt{2}) + 0\cdot 1|}{\sqrt{3+2+1}} = 2.$$

14. The least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$ is the solution to $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Ax} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$:

$$(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} \mid \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}) = \left(\begin{array}{ccc|c} 10 & -2 & -4 & -1 \\ -2 & 7 & -2 & -12 \\ -4 & -2 & 4 & 6 \end{array}\right) \dashrightarrow \begin{pmatrix} 1 & 0 & 0 & -\frac{3}{10} \\ 0 & 1 & 0 & -\frac{17}{10} \\ 0 & 0 & 1 & \frac{7}{20} \end{pmatrix}.$$

So $\boldsymbol{u} = (-\frac{3}{10}, -\frac{17}{10}, \frac{7}{20})$.

15. $\|\boldsymbol{b} - \boldsymbol{Au}\|^2 = \|(0, 0, -\frac{8}{5}, -\frac{4}{5})\|^2 = 16/5$.

Written Quiz

1. Suppose $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is an orthonormal set of vectors in$\mathbb{R}^4$. Define

$$\begin{aligned} \boldsymbol{v}_1 &= \tfrac{2}{3}\boldsymbol{u}_1 + \tfrac{2}{3}\boldsymbol{u}_2 + \tfrac{1}{3}\boldsymbol{u}_3 \\ \boldsymbol{v}_2 &= -\tfrac{1}{\sqrt{2}}\boldsymbol{u}_1 + \tfrac{1}{\sqrt{2}}\boldsymbol{u}_2 \\ \boldsymbol{v}_3 &= -\tfrac{\sqrt{2}}{6}\boldsymbol{u}_1 - \tfrac{\sqrt{2}}{6}\boldsymbol{u}_2 + +\tfrac{2\sqrt{2}}{3}\boldsymbol{u}_3 \end{aligned}$$

(i) Prove that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is an orthonormal set of vectors.

(ii) Prove that $\operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

Proof. (i) View all vectors as column vectors and let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{pmatrix}$. Then $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{I}_3$. We can write

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \\ \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \end{pmatrix}.$$

Then

$$\begin{aligned} \boldsymbol{B}^{\mathrm{T}}\boldsymbol{B} &= \begin{pmatrix} \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \\ \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \end{pmatrix}^{\mathrm{T}} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} \begin{pmatrix} \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \\ \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \\ \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \\ \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \end{pmatrix} = \boldsymbol{I}_3. \end{aligned}$$

It follows that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is an orthonormal set of vectors.

(ii) Each $\boldsymbol{v}_i$ is a linear combination of $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$; so $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \subseteq \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$.

Since $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ are orthonormal sets, they are linearly independent. Then

$$\dim(\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}) = \dim(\operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}) = 3.$$

Hence, $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} = \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$.

2. Suppose V is a subspace of $\mathbb{R}^n$ with $\dim(V) = k$.

(i) Prove that there is a $k \times n$ matrix $\boldsymbol{A}$ such that $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{I}_k$ and for each $\boldsymbol{w} \in \mathbb{R}^n$, the projection of $\boldsymbol{w}$ onto V is $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{w}$.

(ii) Prove that $(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})^2 = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$.

Proof. (i) Let $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ be an orthonormal basis for V. View them as row vectors and form $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_k \end{pmatrix}$. So $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{I}_k$.

Note that V is the column space of $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$. For any $\boldsymbol{w} \in \mathbb{R}^n$, its projection onto V is $\boldsymbol{p} = \boldsymbol{B}\boldsymbol{x}$, where $\boldsymbol{x} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{x} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{w}$. Hence,

$$\boldsymbol{p} = \boldsymbol{B}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{w} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{w}.$$

Alternatively, we have

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{w} = \boldsymbol{A}^{\mathrm{T}} \begin{pmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_k \end{pmatrix} \boldsymbol{w} \quad = \boldsymbol{A}^{\mathrm{T}} \begin{pmatrix} \boldsymbol{v}_1 \cdot \boldsymbol{w} \\ \vdots \\ \boldsymbol{v}_k \cdot \boldsymbol{w} \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}_1 & \dots & \boldsymbol{v}_k \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1 \cdot \boldsymbol{w} \\ \vdots \\ \boldsymbol{v}_k \cdot \boldsymbol{w} \end{pmatrix} \quad = (\boldsymbol{v}_1 \cdot \boldsymbol{w})\boldsymbol{v}_1 + \cdots + (\boldsymbol{v}_k \cdot \boldsymbol{w})\boldsymbol{v}_k$$

which is the projection of $\boldsymbol{w}$ onto $V = \operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$. (Let us emphasize that this formula for the projection assumes $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is orthonormal, which is true by our choice.)

(ii) $(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})^2 = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$.

3. Let $\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 0 \\ 1 & -1 & 0 \\ -2 & -1 & -2 \end{pmatrix}$.

(i) Find the characteristic polynomial of $\boldsymbol{A}$.

(ii) Prove that $\boldsymbol{A}$ is not diagonalizable.

Proof. (i) The characteristic polynomial

$$\det(\lambda\boldsymbol{I}-\boldsymbol{A}) = \begin{vmatrix} \lambda-1 & 2 & 0 \\ -1 & \lambda+1 & 0 \\ 2 & 1 & \lambda+2 \end{vmatrix} = (\lambda+2)\begin{vmatrix} \lambda-1 & 2 \\ -1 & \lambda+1 \end{vmatrix}$$
$$= (\lambda+2)[(\lambda-1)(\lambda+1)-2(-1)] = (\lambda+2)(\lambda^2+1).$$

(ii) Since the characteristic polynomial of $\boldsymbol{A}$ cannot be completely factorized in $\mathbb{R}$, $\boldsymbol{A}$ is not diagonalizable.

4. Let $\boldsymbol{A} = \begin{pmatrix} -3 & 2 & -2 \\ 2 & -3 & 4 \\ 4 & -5 & 6 \end{pmatrix}$.

(i) Compute all eigenvalues of $\boldsymbol{A}$.

(ii) For each eigenvalue λ of $\boldsymbol{A}$, compute a basis for the eigenspace E_λ.

(iii) Prove that $\boldsymbol{A}$ is not diagonalizable.

Proof. (i) The characteristic polynomial

$$\det(\lambda\boldsymbol{I}-\boldsymbol{A}) = \begin{vmatrix} \lambda+3 & -2 & 2 \\ -2 & \lambda+3 & -4 \\ -4 & 5 & \lambda-6 \end{vmatrix} \overset{R_3-2R_2}{=\!=\!=} \begin{vmatrix} \lambda+3 & -2 & 2 \\ -2 & \lambda+3 & -4 \\ 0 & -2\lambda-1 & \lambda+2 \end{vmatrix}$$
$$\overset{C_3+C_2}{=\!=\!=} \begin{vmatrix} \lambda+3 & -2 & 0 \\ -2 & \lambda+3 & \lambda-1 \\ 0 & -2\lambda-1 & -\lambda+1 \end{vmatrix} \overset{R_3+R_2}{=\!=\!=} \begin{vmatrix} \lambda+3 & -2 & 0 \\ -2 & \lambda+3 & \lambda-1 \\ -2 & -\lambda+2 & 0 \end{vmatrix}$$
$$= -(\lambda-1)\begin{vmatrix} \lambda+3 & -2 \\ -2 & -\lambda+2 \end{vmatrix} = -(\lambda-1)[(\lambda+3)(-\lambda+2)-4] = (\lambda+2)(\lambda-1)^2.$$

Then $\det(\boldsymbol{I}-\lambda)=0 \Leftrightarrow \lambda=-2$ or $\lambda=1$.

(ii) Let $\lambda=1$. Then

$$\lambda\boldsymbol{I}-\boldsymbol{A} = \begin{pmatrix} 4 & -2 & 2 \\ -2 & 4 & -4 \\ -4 & 5 & -5 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2+\frac{1}{2}R_1} \begin{pmatrix} 4 & -2 & 2 \\ 0 & 3 & -3 \\ 0 & 3 & -3 \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} 4 & -2 & 2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\xrightarrow[\frac{1}{3}R_2]{\frac{1}{4}R_1} \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1+\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $y=t$. Then $x=0$ and $y=t$. So $\boldsymbol{x}=(0,t,t)=t(0,1,1)$ and E_1 has a basis $\{(0,1,1)\}$.

Let $\lambda = -2$. Then

$$-2\boldsymbol{I}-\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -4 \\ -4 & 5 & -8 \end{pmatrix} \xrightarrow[R_3+4R_1]{R_2+2R_1} \begin{pmatrix} 1 & -2 & 2 \\ 0 & -3 & 0 \\ 0 & -3 & 0 \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} 1 & -2 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_2} \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1+2R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $z = t$. Then $x = -2t$ and $y = 0$. So $\boldsymbol{x} = (-2t, 0, t) = t(-2, 0, 1)$ and E_{-2} has a basis $\{(-2, 0, 1)\}$.

(iii) Since $\boldsymbol{A}$ has only 2 linearly independent eigenvectors, it is not diagonalizable.

5. Define a sequence by real numbers $\{a_n\}_{n=0}^{\infty}$ by

$$a_0 = 0,\ a_0 = 1,\ a_2 = 1,\ a_{n+3} = -a_{n+2} + 4a_{n+1} + 4a_n \quad \text{for all } n.$$

(i) Write down a 3×3 matrix $\boldsymbol{A}$ such that $\begin{pmatrix} a_{n+1} \\ a_{n+2} \\ a_{n+3} \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{pmatrix}$ for all n.

(ii) Without computing any eigenvectors, explain why $\boldsymbol{A}$ is diagonalizable.

(iii) Diagonalize $\boldsymbol{A}$.

(iv) Use the previous parts to derive a general formula for a_n, i.e., express a_n in terms of n.

(v) Is $\boldsymbol{A}$ orthogonally diagonalizable?

Solution. (i) $\begin{pmatrix} a_{n+1} \\ a_{n+2} \\ a_{n+3} \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ a_{n+2} \\ -a_{n+2}+4a_{n+1}+4a_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 4 & -1 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{pmatrix}$. So $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 4 & -1 \end{pmatrix}$.

(ii) The characteristic polynomial

$$\det(\lambda\boldsymbol{I}-\boldsymbol{A}) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & -4 & \lambda+1 \end{vmatrix} = \lambda^2(\lambda+1) - 4 - 4\lambda = \lambda^3 + \lambda^2 - 4\lambda - 4 = (\lambda+2)(\lambda+1)(\lambda-2).$$

Then the eigenvalues of $\boldsymbol{A}$ are $\lambda = -2$, $\lambda = -1$, $\lambda = 2$.

Since $\boldsymbol{A}$ has 3 distinct eigenvalues, it is diagonalizable.

(iii) Let $\lambda = -2$. The

$$-2\boldsymbol{I}-\boldsymbol{A} = \begin{pmatrix} -2 & -1 & 0 \\ 0 & -2 & -1 \\ -4 & -4 & -1 \end{pmatrix} \xrightarrow{R_3-2R_1} \begin{pmatrix} -2 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} -2 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $z = t$. Then $y = -\frac{1}{2}t$ and $z = \frac{1}{4}t$. So $\boldsymbol{x} = t(\frac{1}{4}, -\frac{1}{2}, 1)$ and E_{-2} has a basis $\{(\frac{1}{4}, -\frac{1}{2}, 1)\}$.

Let $\lambda = -1$. Then

$$-1\boldsymbol{I}-\boldsymbol{A} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ -4 & -4 & 0 \end{pmatrix} \xrightarrow{R_3-4R_1} \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $z = t$. Then $y = -t$ and $x = t$. So $\boldsymbol{x} = t(1,-1,1)$ and E_{-1} has a basis $\{(1,-1,1)\}$.

Let $\lambda = 2$. Then

$$2\boldsymbol{I}-\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -4 & -4 & 3 \end{pmatrix} \xrightarrow{R_3+2R_1} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -6 & 3 \end{pmatrix} \xrightarrow{R_3+3R_2} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $z = t$. Then $y = \frac{1}{2}t$ and $x = \frac{1}{4}t$. So $\boldsymbol{x} = t(\frac{1}{4},\frac{1}{2},1)$ and E_2 has a basis $\{(\frac{1}{4},\frac{1}{2},1)\}$.

Let $\boldsymbol{P} = \begin{pmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ -\frac{1}{2} & -1 & \frac{1}{2} \\ 1 & 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \boldsymbol{D}$.

(iv) Find inverse of $\boldsymbol{P}$:

$$(\boldsymbol{P} \mid \boldsymbol{I}) = \left(\begin{array}{ccc|ccc} \frac{1}{4} & 1 & \frac{1}{4} & 1 & 0 & 0 \\ -\frac{1}{2} & -1 & \frac{1}{2} & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3-4R_1]{R_2+2R_1} \left(\begin{array}{ccc|ccc} \frac{1}{4} & 1 & \frac{1}{4} & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & -3 & 0 & -4 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3+3R_2} \left(\begin{array}{ccc|ccc} \frac{1}{4} & 1 & \frac{1}{4} & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 3 & 1 \end{array}\right) \xrightarrow[\frac{1}{3}R_3]{4R_1} \left(\begin{array}{ccc|ccc} 1 & 4 & 1 & 4 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 1 & \frac{1}{3} \end{array}\right)$$

$$\xrightarrow[R_1-R_3]{R_2-R_3} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & \frac{10}{3} & -1 & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{4}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} & 1 & \frac{1}{3} \end{array}\right) \xrightarrow{R_1-4R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 1 \\ 0 & 1 & 0 & \frac{4}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} & 1 & \frac{1}{3} \end{array}\right) = (\boldsymbol{I}\boldsymbol{P}^{-1}).$$

Let $\boldsymbol{x}_0 = (a_0,a_1,a_2)^{\mathrm{T}} = (0,0,1)^{-1}$. Then

$$\boldsymbol{x}_n = \boldsymbol{A}^n\boldsymbol{x}_0 = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}\boldsymbol{x}_0 = \begin{pmatrix} \frac{1}{4}(-2)^n - \frac{1}{3}(-1)^n + \frac{1}{12}(2^n) \\ -\frac{1}{2}(-2)^n + \frac{1}{3}(-1)^n + \frac{1}{6}(2^n) \\ (-2)^n - \frac{1}{3}(-1)^n + \frac{1}{3}(2^n). \end{pmatrix}.$$

In particular, its first entry gives

$$a_n = \frac{1}{4}(-2)^n - \frac{1}{3}(-1)^n + \frac{1}{12}(2^n).$$

(v) $\boldsymbol{A}$ is not symmetric; so it is not orthogonally diagonalizable.