CS2040S
Data Structures and Algorithms

Dynamic Programming

Riddle of the Week: The Travelling SalesPeople

Three travelers show up at a hotel where a room costs $300. They each pay $100 and go to their room.

The manager realizes there is a special sale and the room only costs $250. He gives his assistant $50 to return to the travelers. The assistant only has tens for change, and so gives each traveler $10 in change, keeping $20 for himself.

So each traveler paid $90, and the assistant kept $20, leading to a total of 3*90+20 = 290 dollars. What happened to the remaining 10 dollars?

Semester Roadmap

Where are we?

- Searching
- Sorting
- Lists
- Trees
- Hash Tables
- Graphs
- **Dynamic Programming**

You are here

Roadmap

Today and Monday: Dynamic Programming

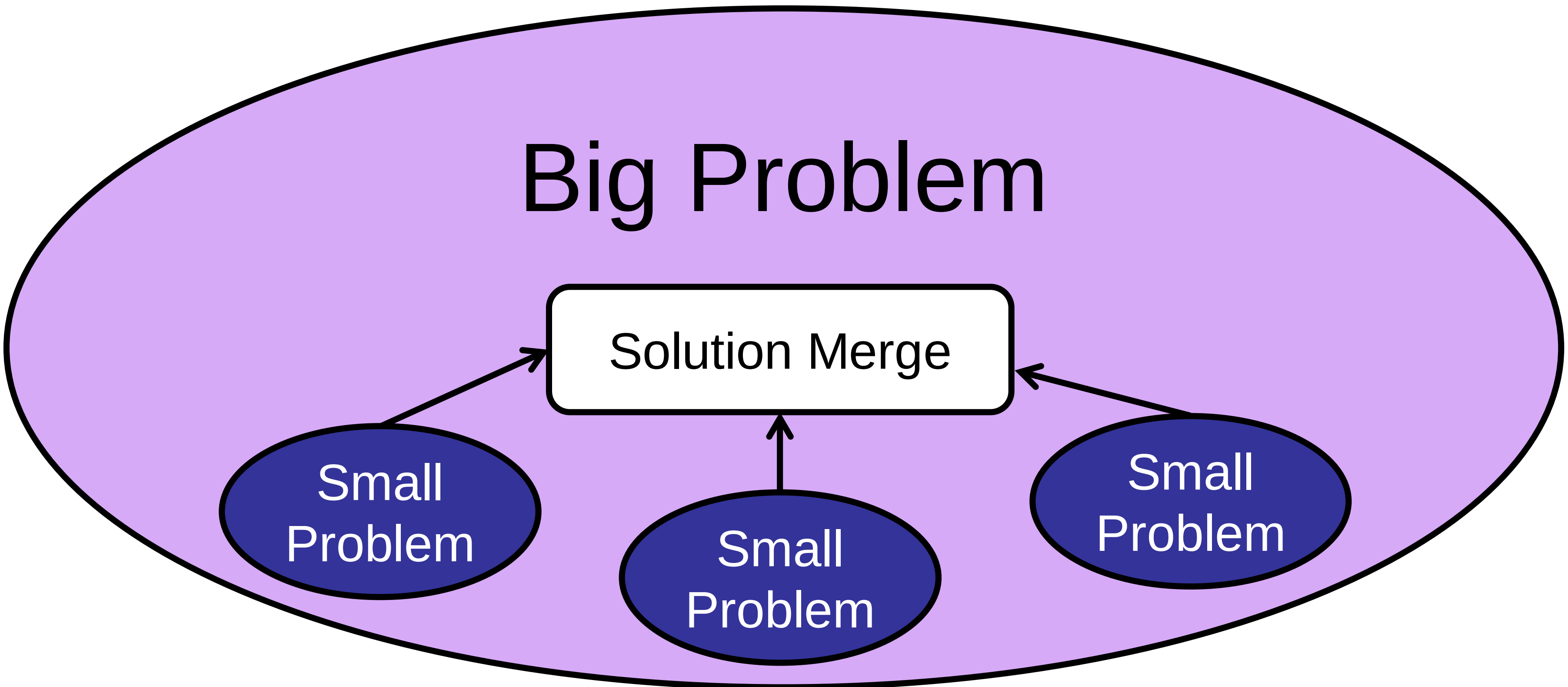
- Basics of DP
- Example: Longest Increasing Subsequence
- Example: Bounded Prize Collecting
- Example: Vertex Cover on a Tree
- Example: All-Pairs Shortest Paths

Dynamic Programming Basics

Dynamic Programming Basics

Optimal sub-structure:

Optimal solution can be constructed from optimal solutions to smaller sub-problems.



Which of these problems exhibit optimal sub-structure? (Choose all that apply.)

1. Sorting
2. Reversing a string
3. Merging two arrays
4. Shortest paths
5. Minimum spanning tree

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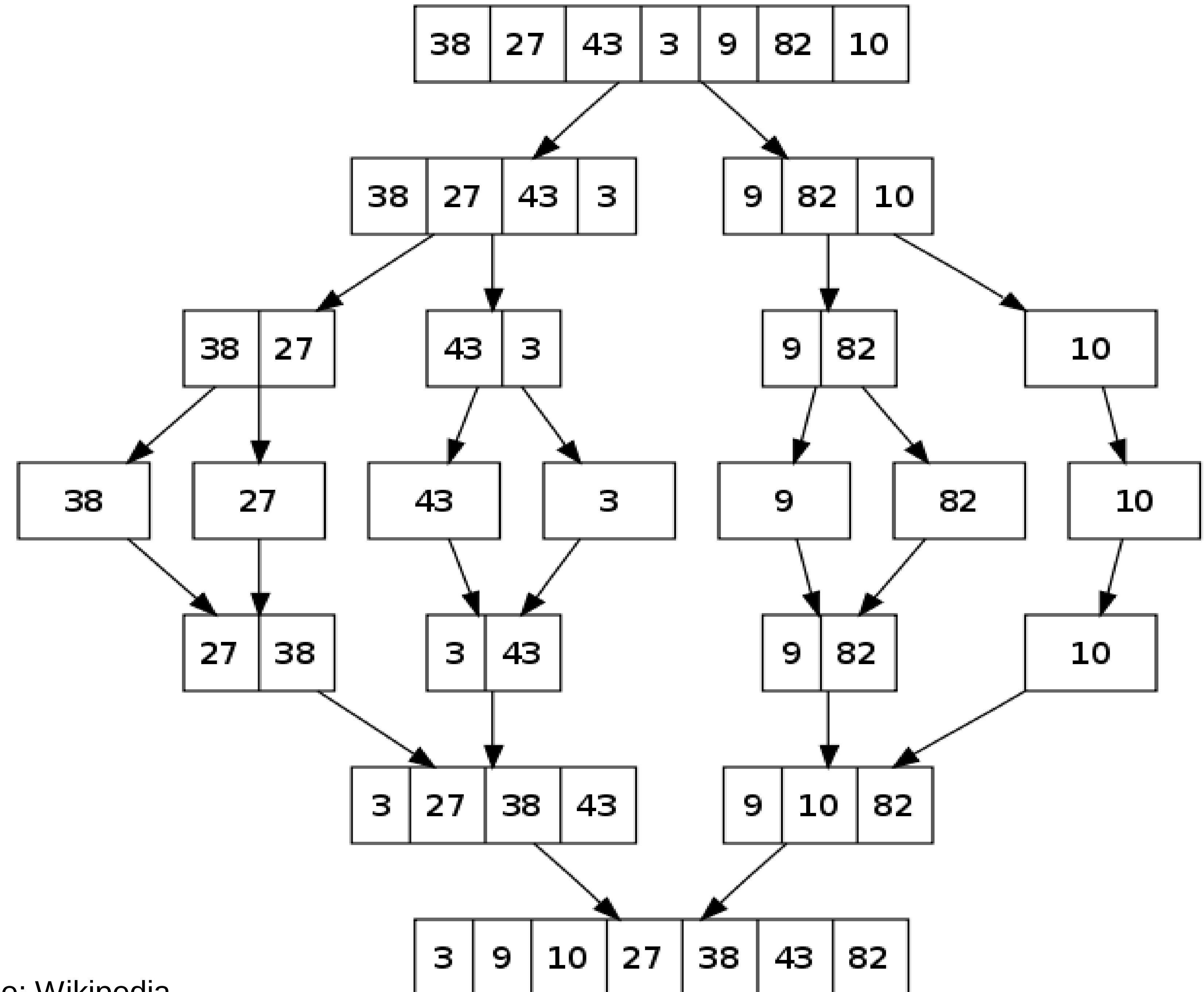
Optimal Sub-structure

Property of (nearly) every problem we study:

- Greedy algorithms
  - Dijkstra's Algorithm
  - Minimum Spanning Tree algorithms
- Divide-and-conquer algorithms
  - MergeSort
  - Fast Fourier Transform



Source: Wikipedia

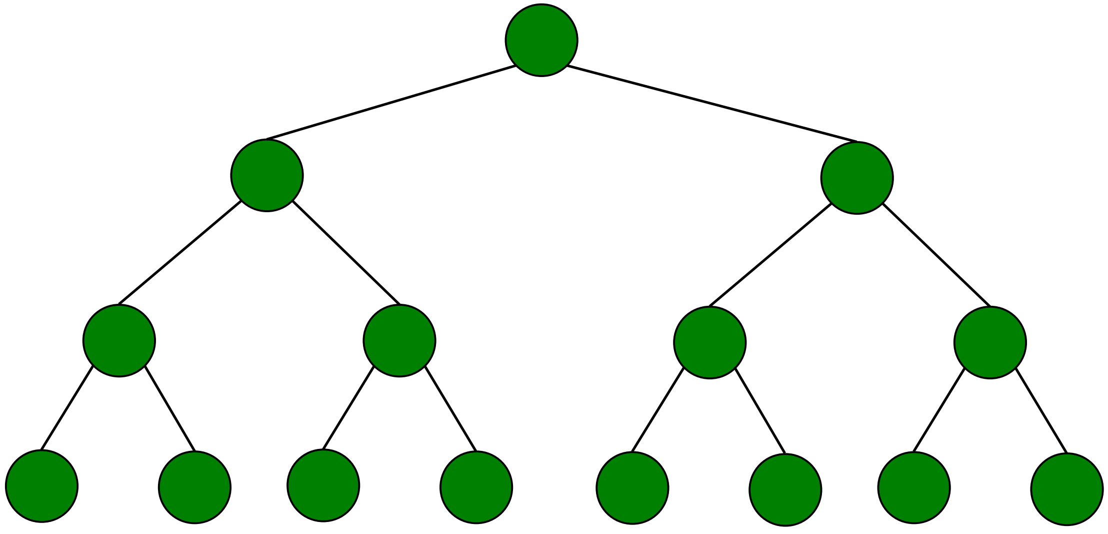
Optimal Sub-structure

Property of (nearly) every problem we study:

- Greedy algorithms
  - Dijkstra's Algorithm
  - Minimum Spanning Tree algorithms
- Divide-and-conquer algorithms
  - MergeSort
  - Fast Fourier Transform

Dynamic Programming

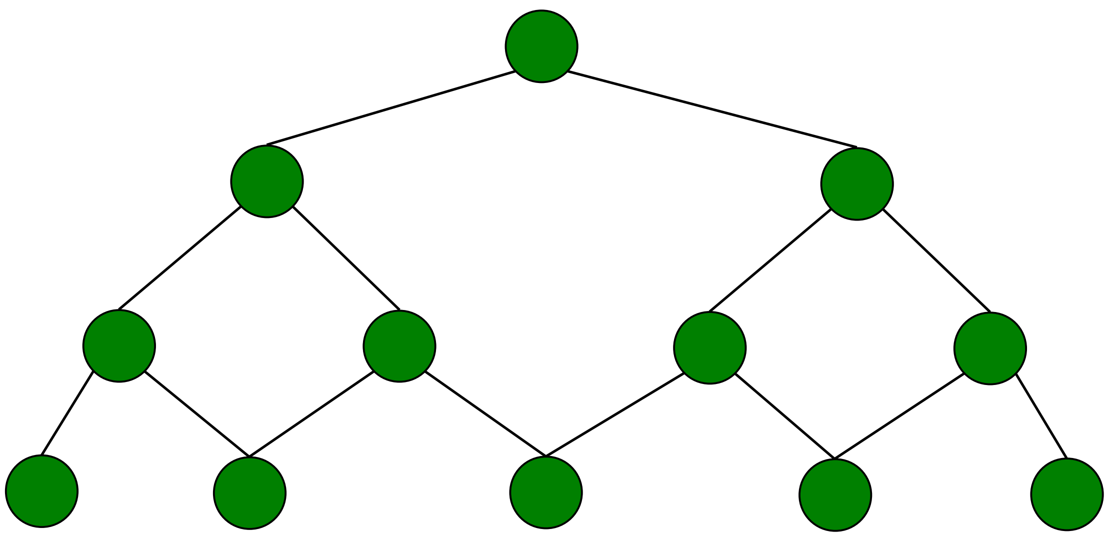
Optimal substructure (simple case):



Dynamic Programming

Optimal substructure (overlapping sub-problems):

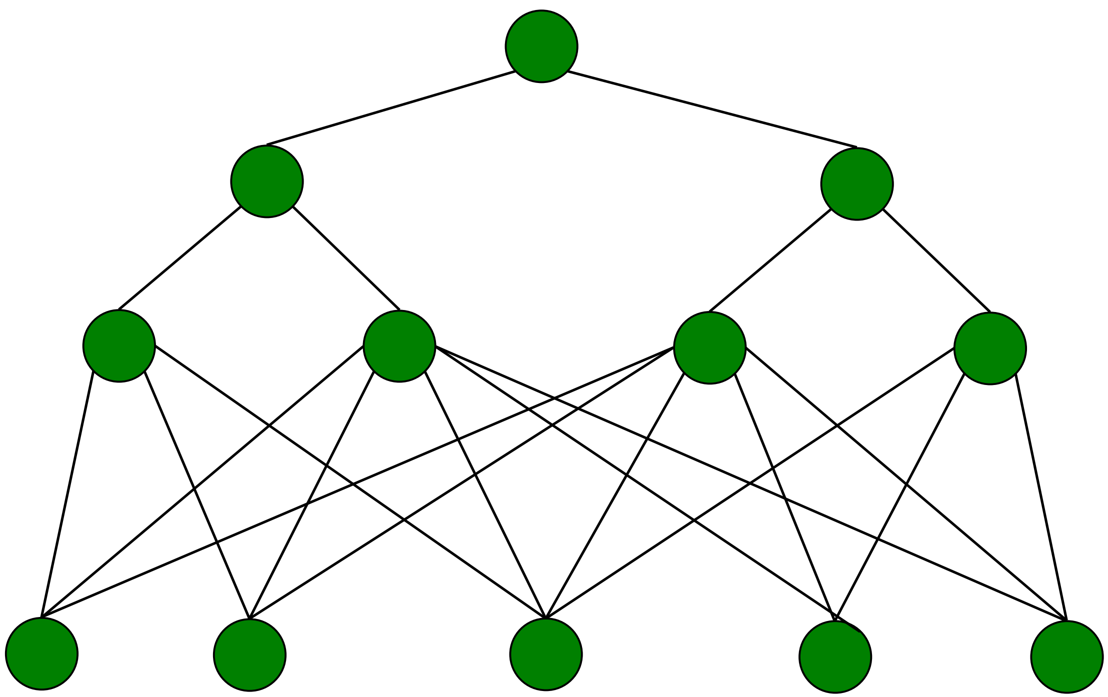
The same smaller problem is used to solve multiple different bigger problems.



Dynamic Programming

Overlapping sub-problems:

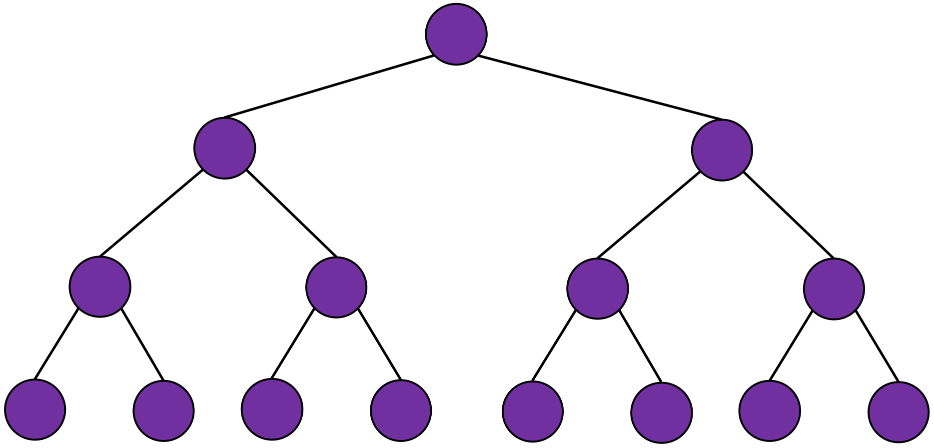
The same smaller problem is used to solve multiple different bigger problems.



Dynamic Programming

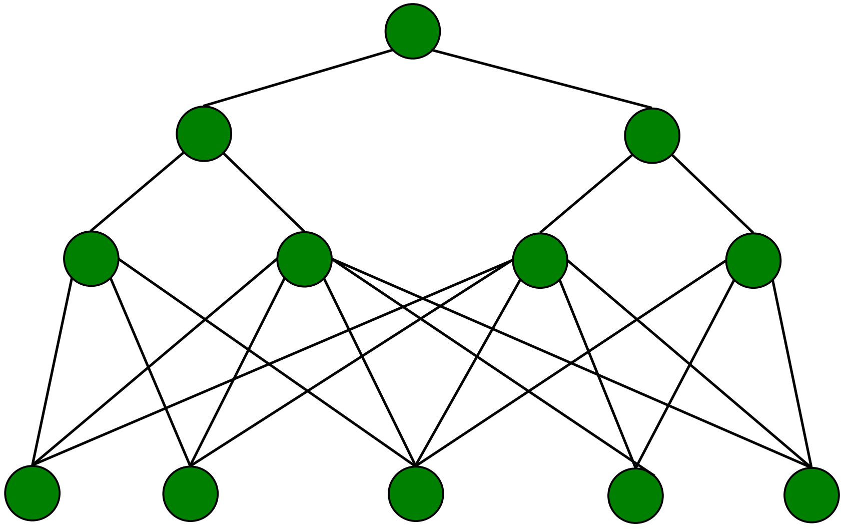
Contrast: Both have optimal substructure

No overlapping subproblems



Divide-and-Conquer

Overlapping subproblems



Dynamic Programming

Dynamic Programming

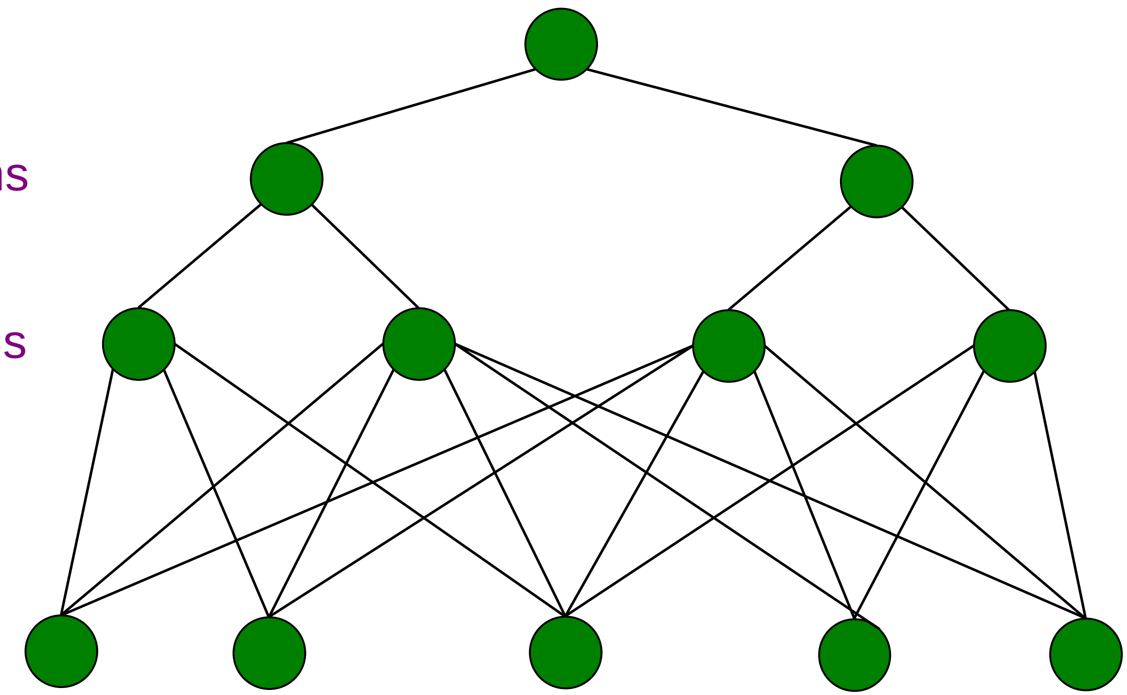
Basic strategy: *(bottom up dynamic programming)*

Step 4: solve root problem

Step 3: combine smaller problems

Step 2: combine smaller problems

Step 1: solve smallest problems

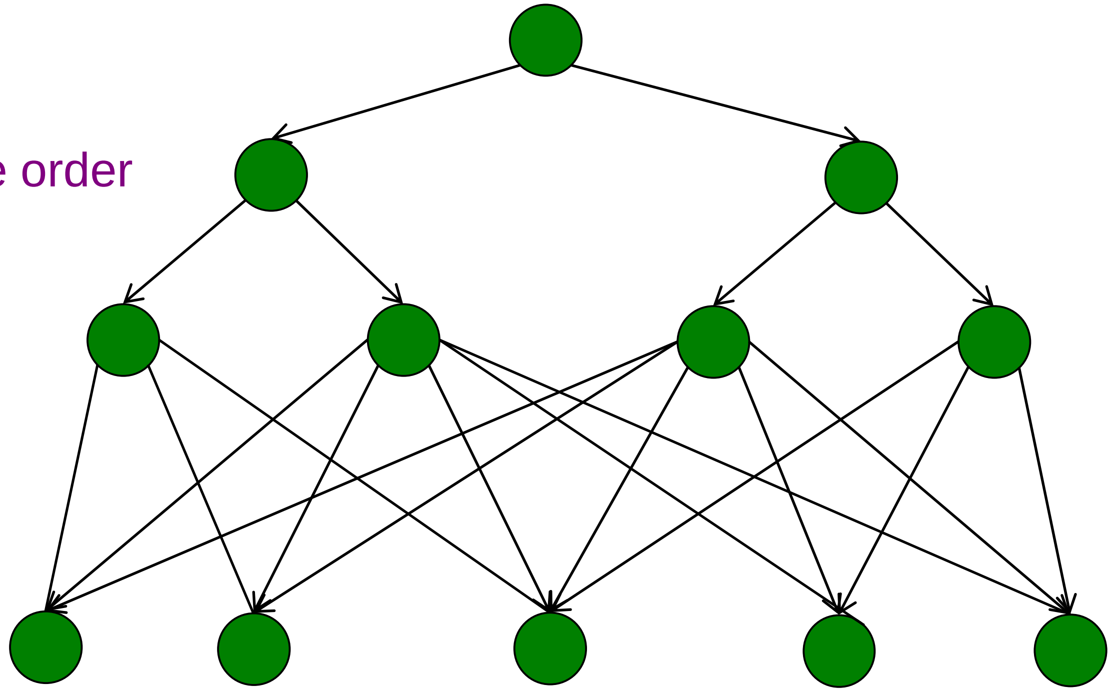


Dynamic Programming

Basic strategy: *(DAG + topological sort)*

Step 1: Topologically sort DAG

Step 2: Solve problems in reverse order



Dynamic Programming

Basic strategy:

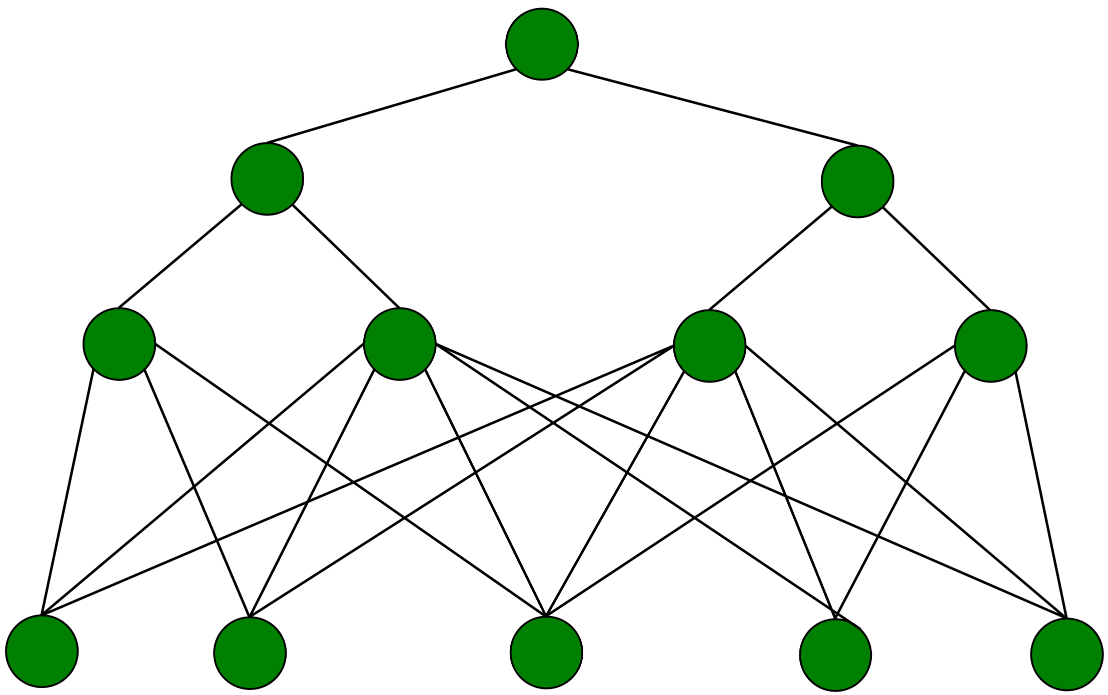
(top down dynamic programming)

Step 1: Start at root and recurse.

Step 2: Recurse.

Step 3: Recurse.

Step 4: Solve and memoize.
Only compute each
solution once.



Dynamic Programming

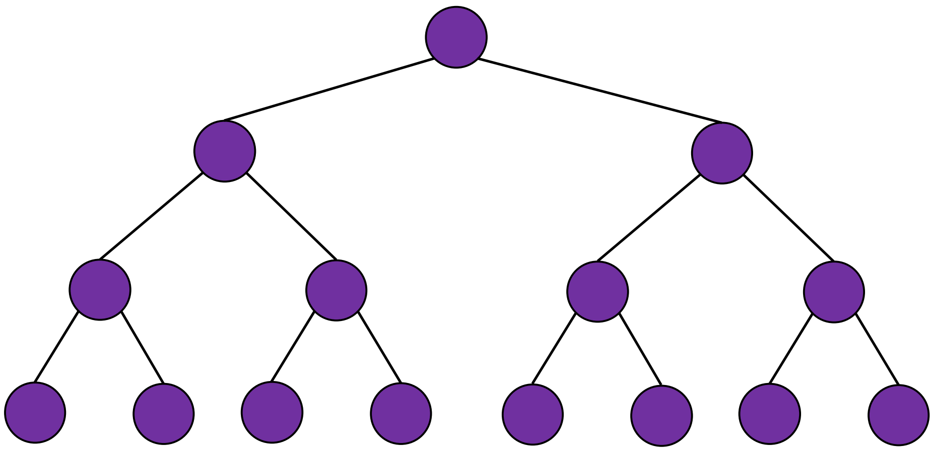
Table view:

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
1	17	22	14	19	8	4	9	12	15	7	5	9	13	14	18	4
2	15	12	13	13	7											
3																
4																
5																
6																
7																
8																
9																
10																
11																

Dynamic Programming

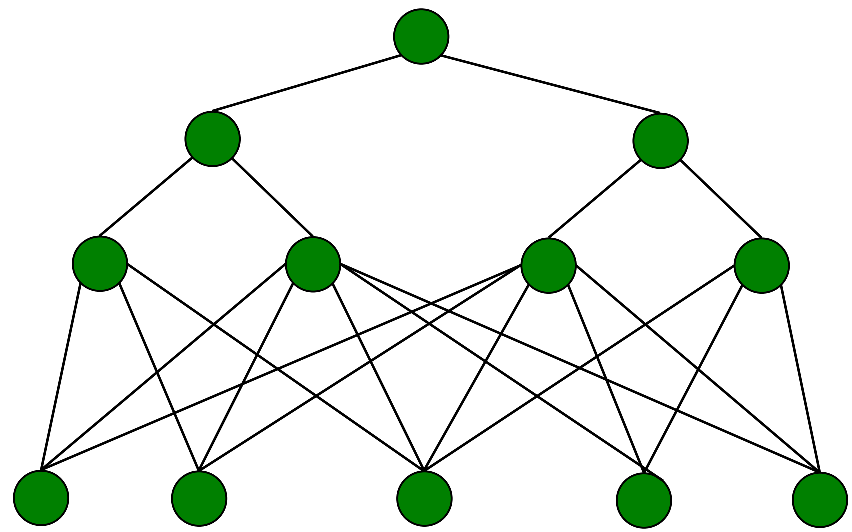
Contrast: Both have optimal substructure

No overlapping subproblems



Divide-and-Conquer

Overlapping subproblems



Dynamic Programming

Roadmap

Today and Monday: Dynamic Programming

- Basics of DP
- Example: Longest Increasing Subsequence
- Example: Bounded Prize Collecting
- Example: Vertex Cover on a Tree
- Example: All-Pairs Shortest Paths

Longest Increasing Subsequence

Input: Sequence of integers

- Example: {8, 3, 6, 4, 5, 7, 7}

Output: Increasing subsequence

- Example: {8, 3, 6, 4, 5, 7, 7}

Goal: Output sequence of maximum length

- Example: {8, 3, 6, 4, 5, 7, 7}

Longest Increasing Subsequence

Input: Sequence of integers

- Example: {8, 3, 6, 4, 5, 7, 7}

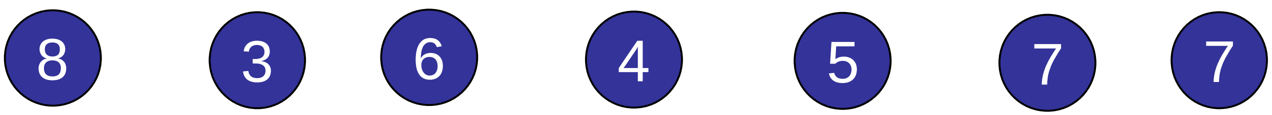
Output: <u>Length</u> of increasing subsequence

- Example: 3 → {8, 3, 6, 4, 5, 7, 7}

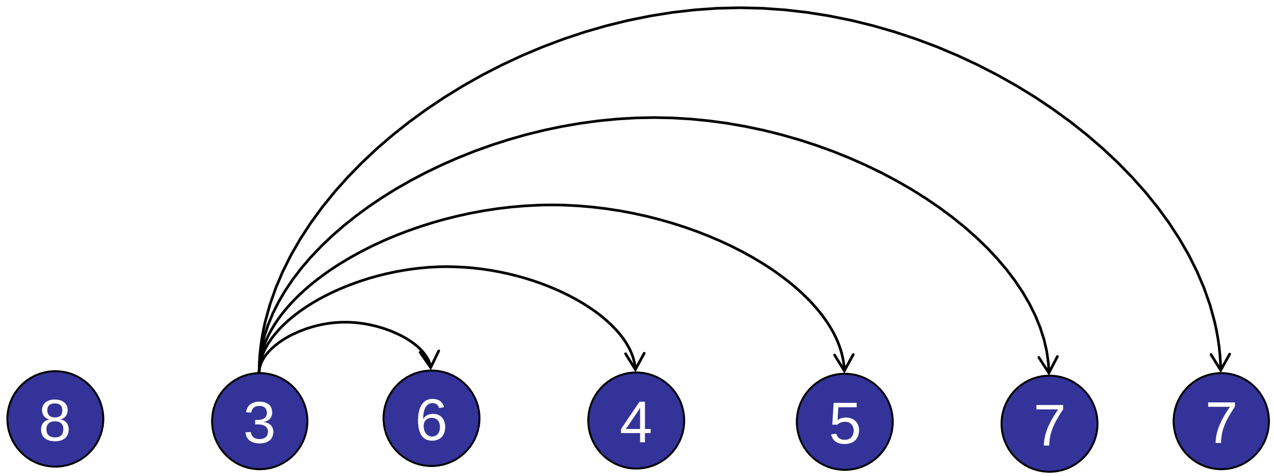
Goal: Output ~~sequence of~~ maximum length

- Example: 4 →{8, 3, 6, 4, 5, 7, 7}

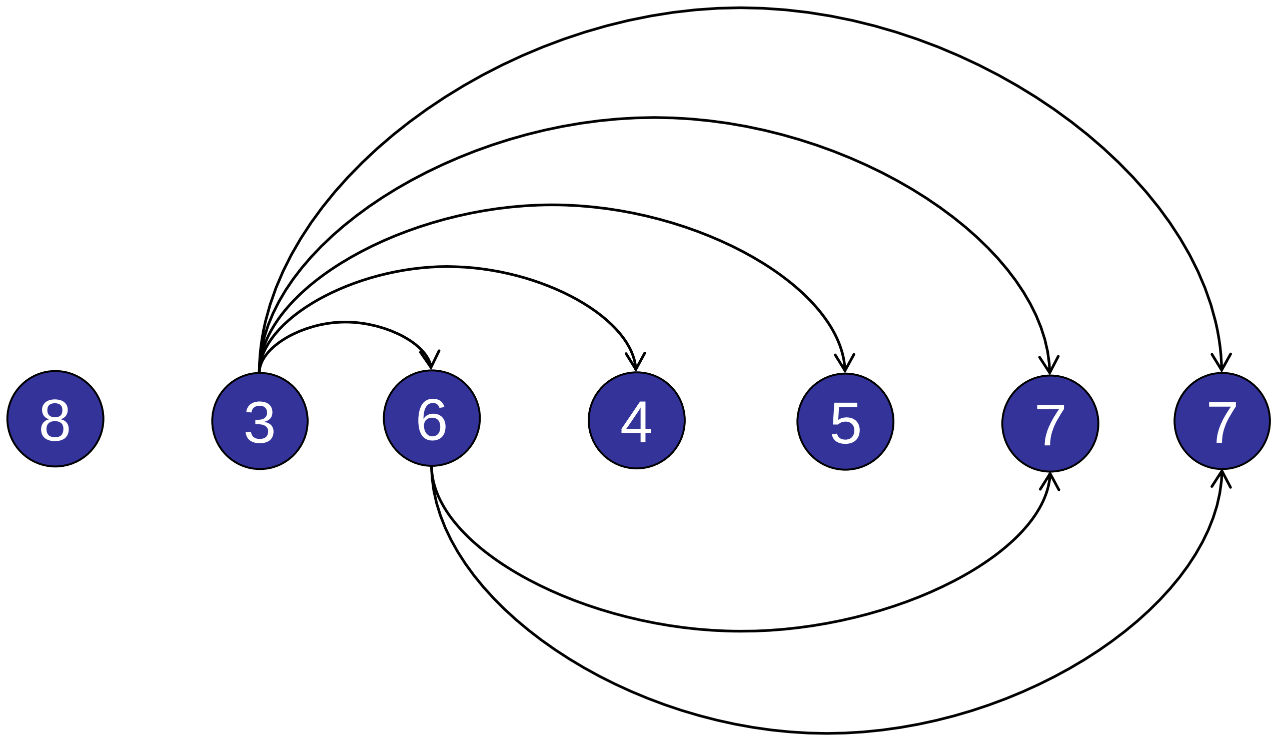
DAG Solution



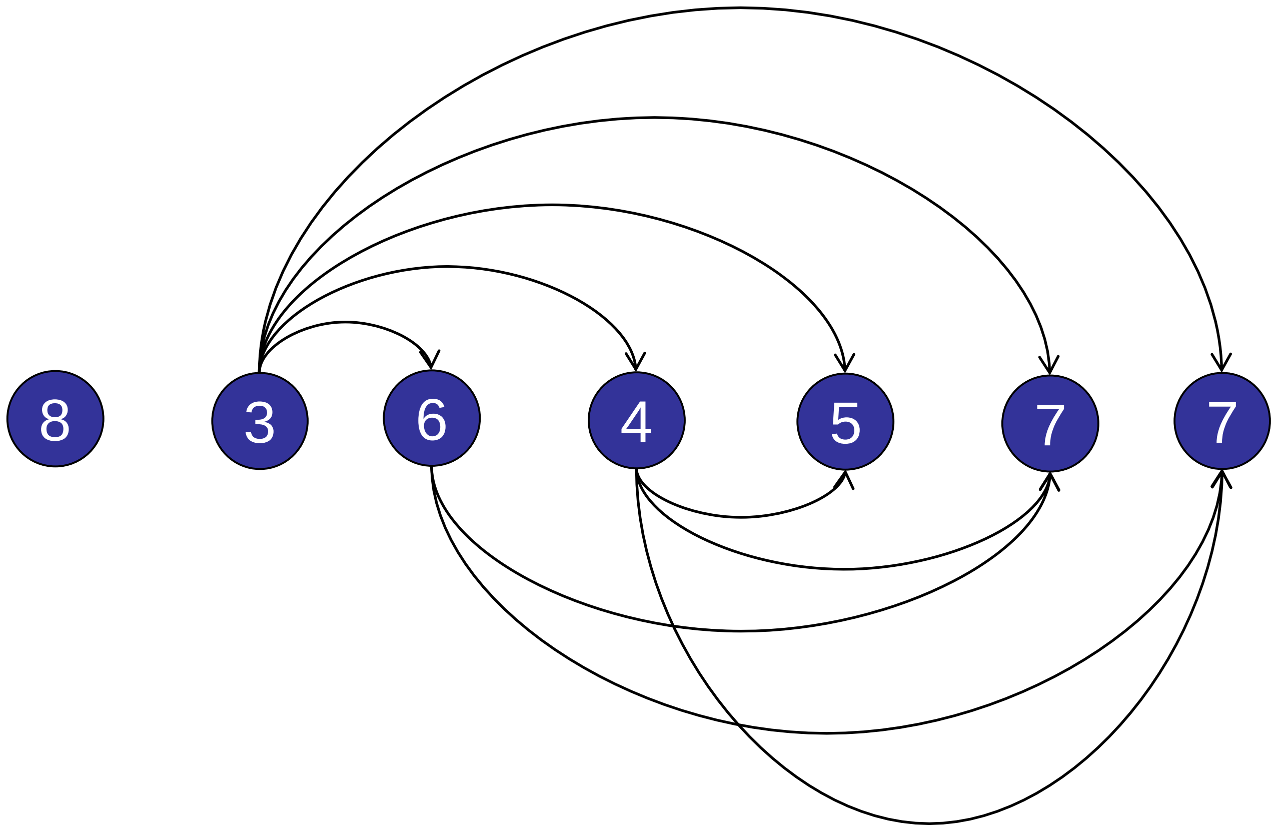
DAG Solution



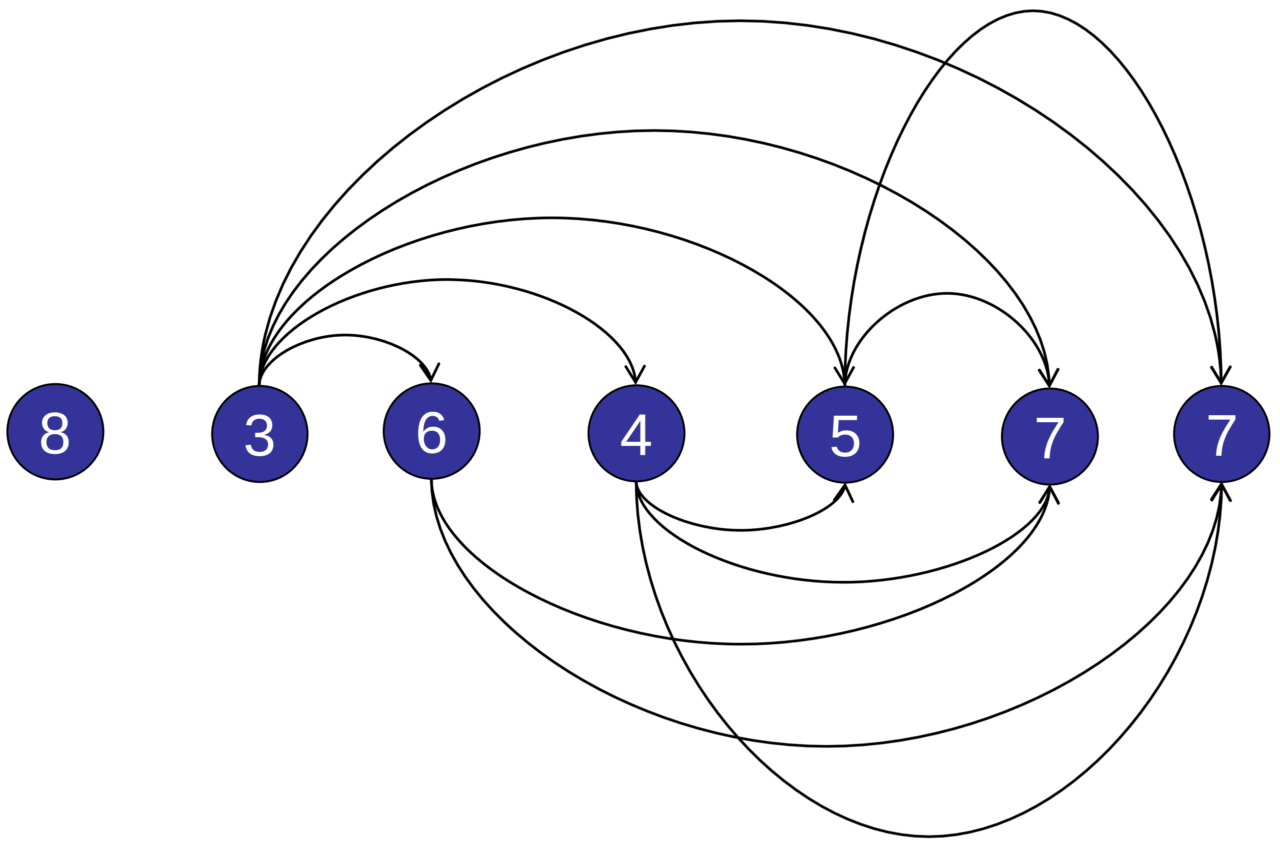
DAG Solution



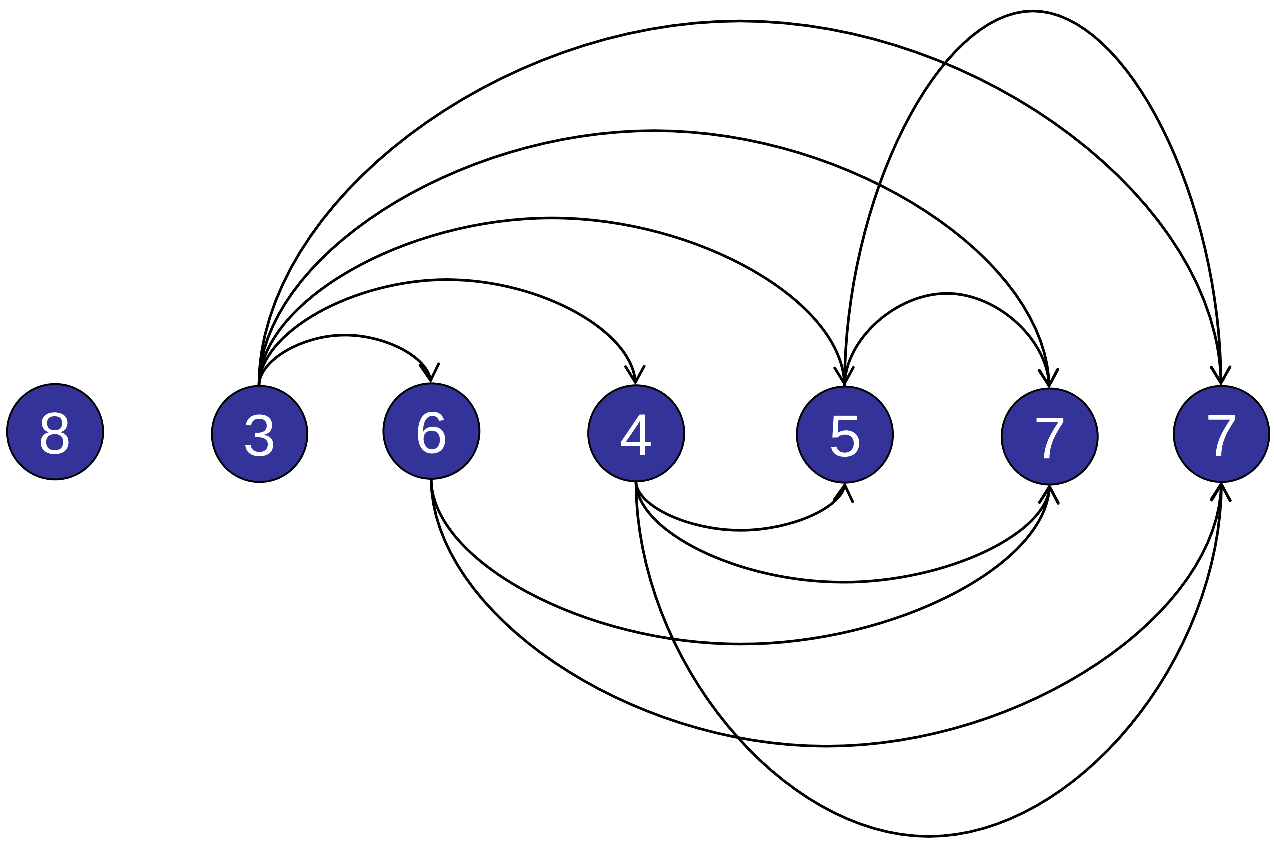
DAG Solution



DAG Solution

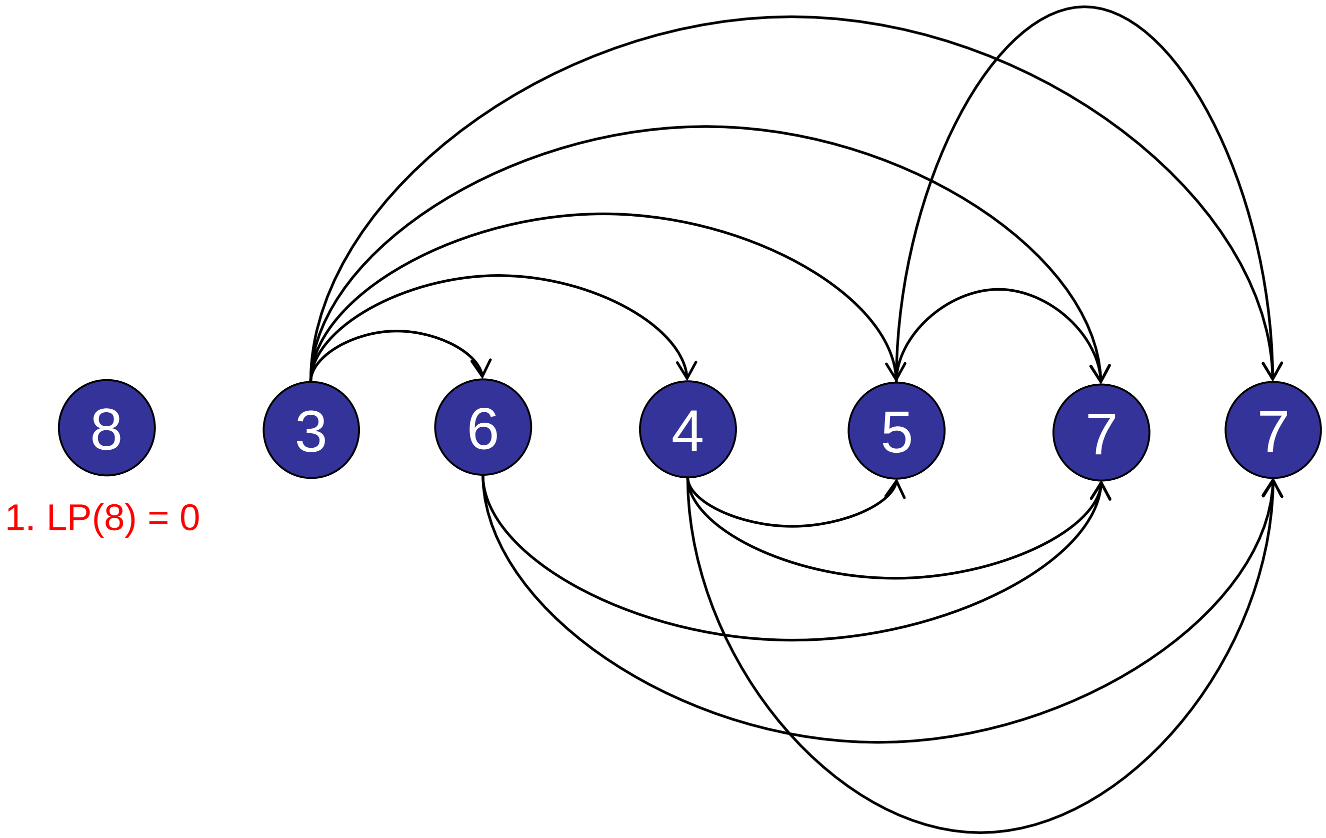


DAG Solution



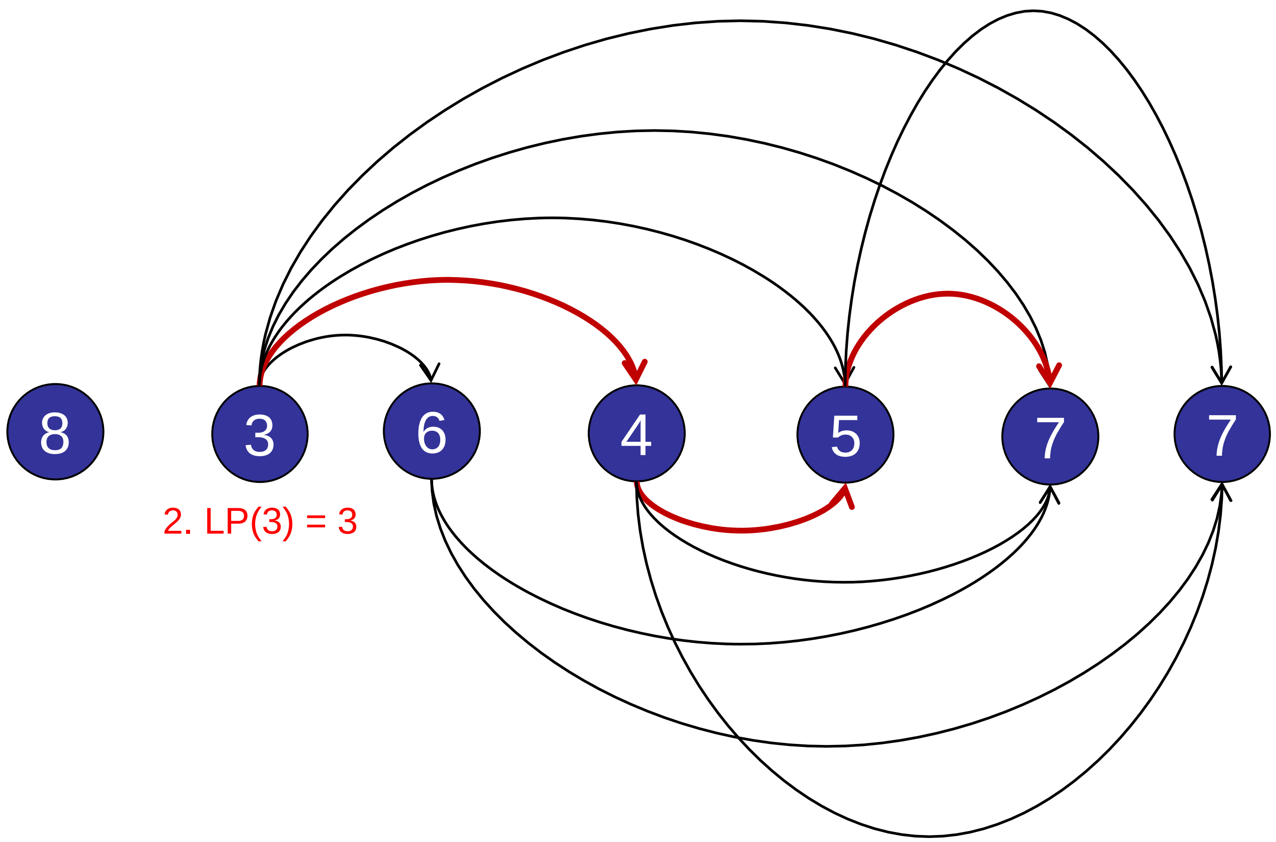
Step 1: Topological sort. (Oops, nothing to do.)

DAG Solution



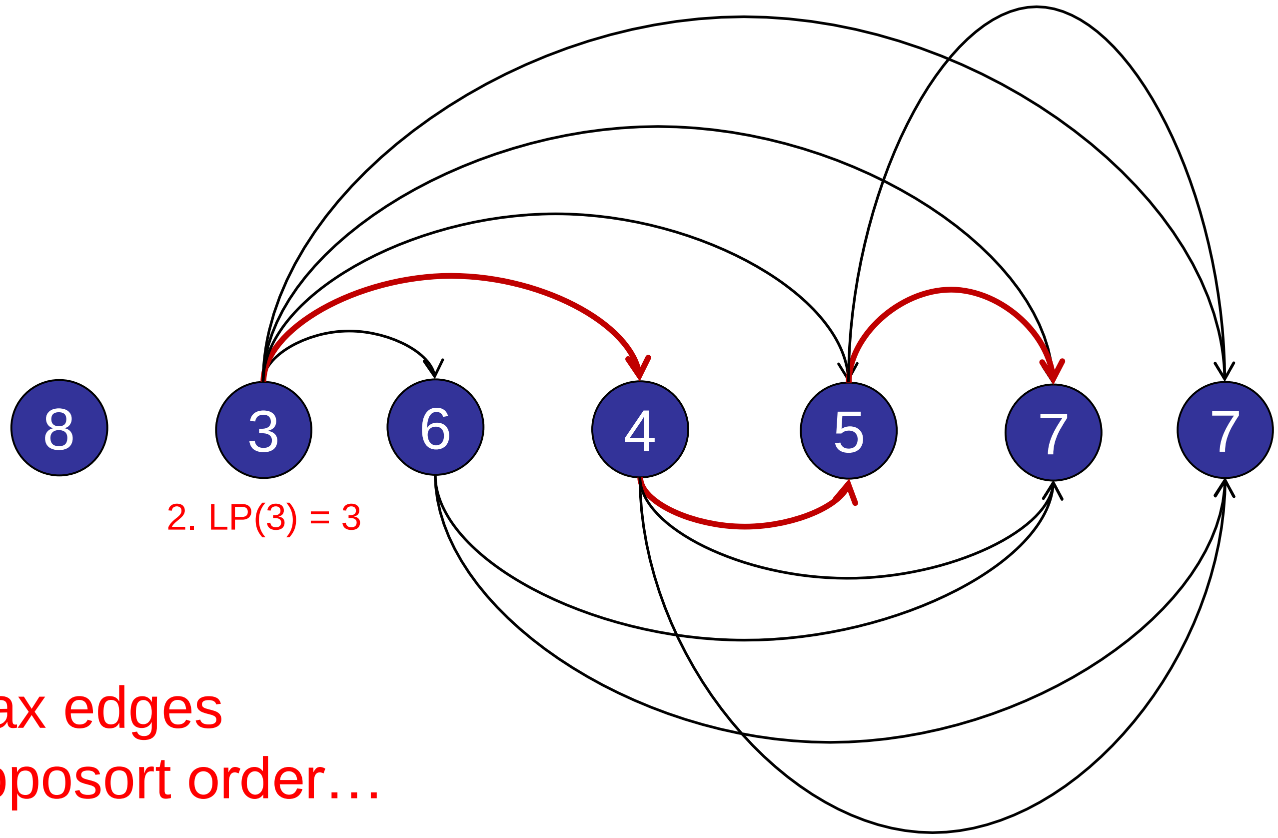
Step 2: Calculate longest paths.

DAG Solution



Step 2: Calculate longest paths: DAG_SSSP.

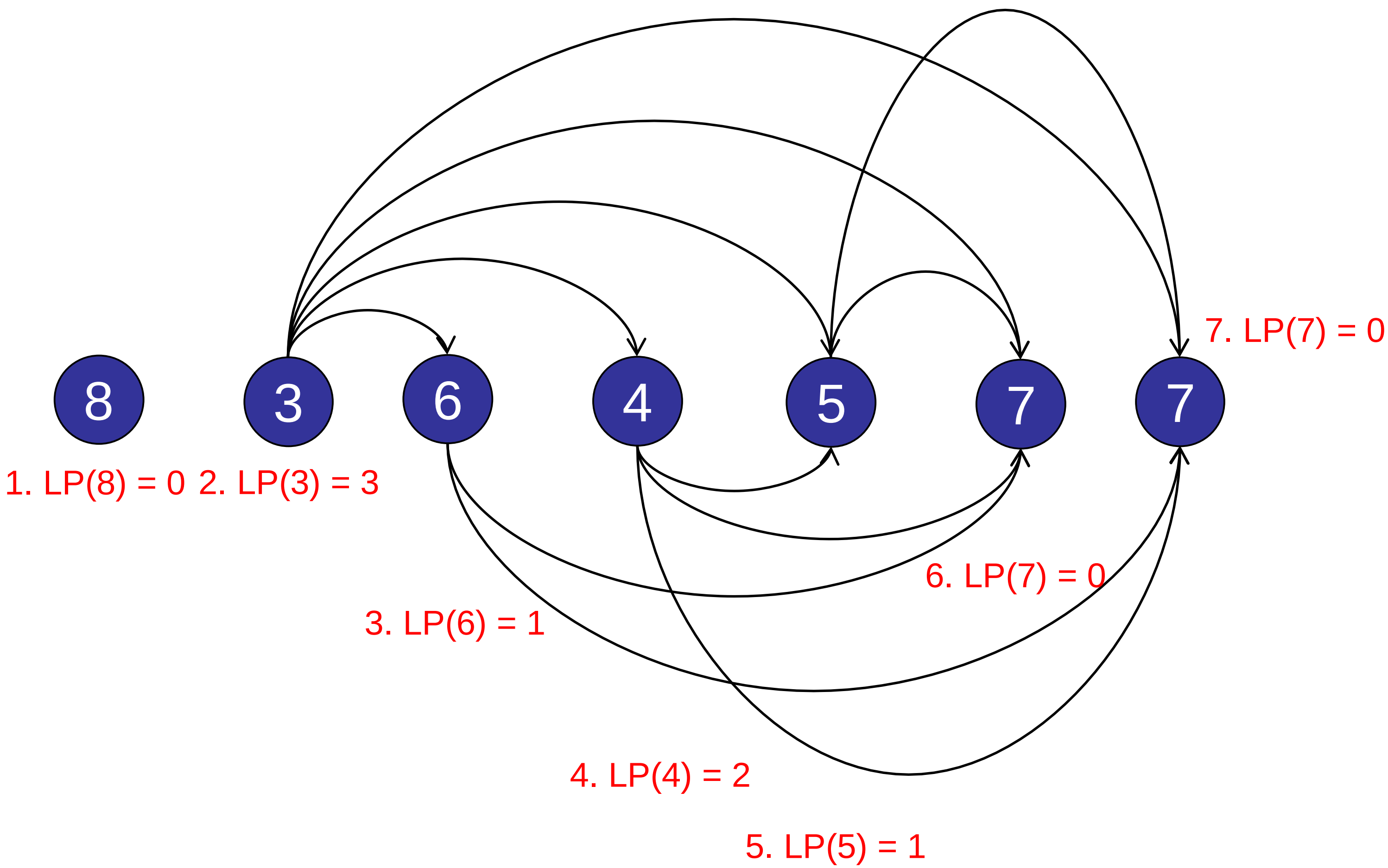
DAG Solution



Relax edges
in toposort order…

Step 2: Calculate longest paths: DAG_SSSP.

DAG Solution



Step 2: Calculate longest paths. LIS = max(LP)+1

What is the running time of the DAG alg for a sequence of n numbers?

1. $O(n)$
2. $O(n \log n)$
3. $O(n^2)$
4. $O(n^2 \log n)$
5. $O(n^3)$
6. None of the above.

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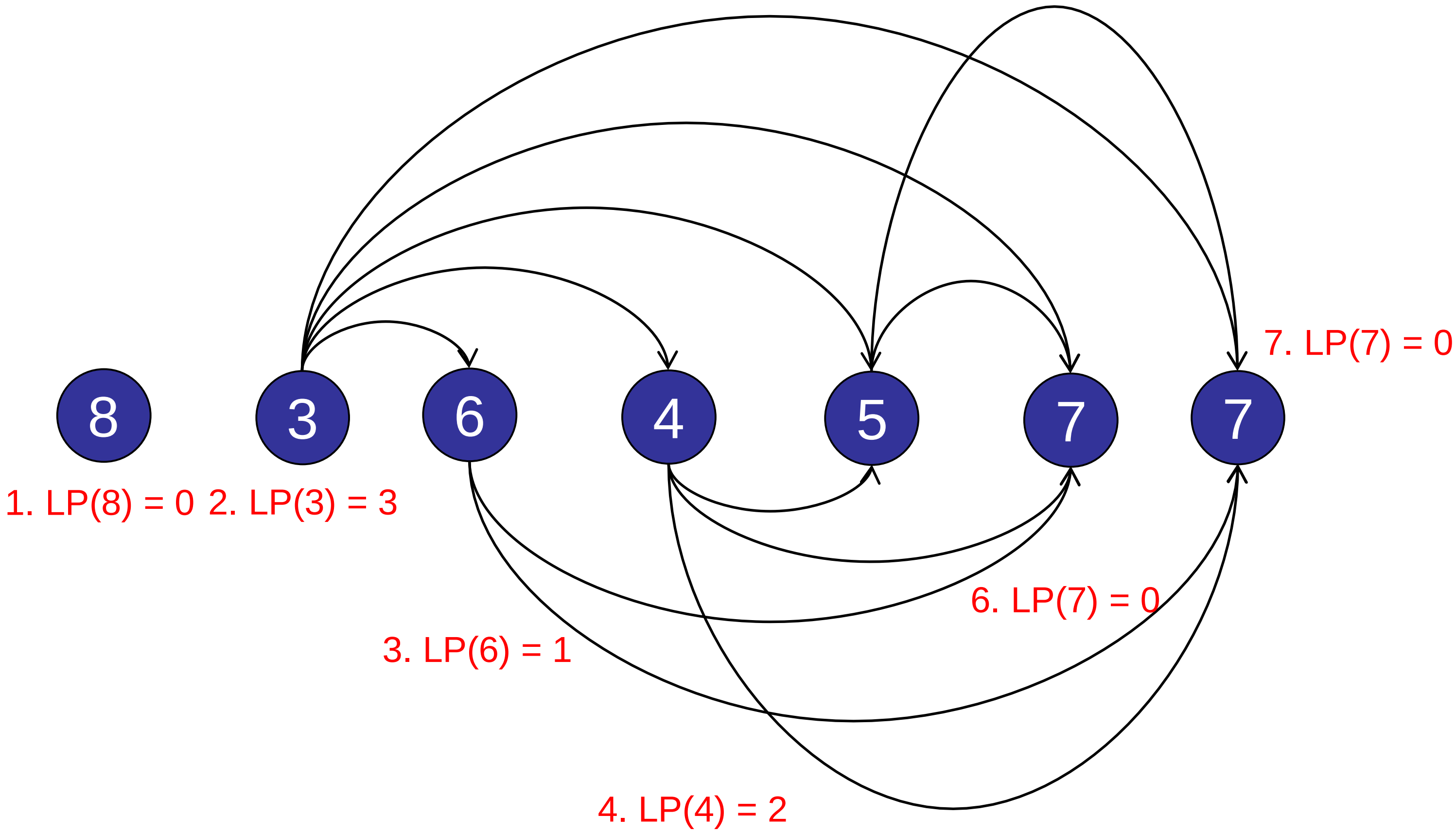
DAG Solution

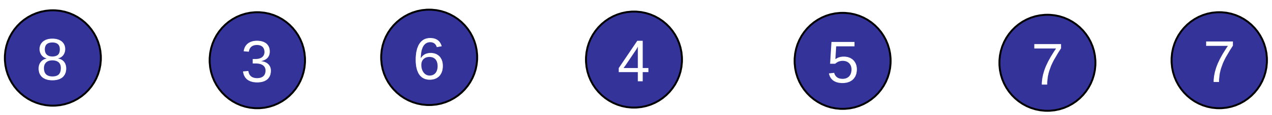
V = list of numbers
|V| = n
|E| = (n + n-1 + n-2 + …)



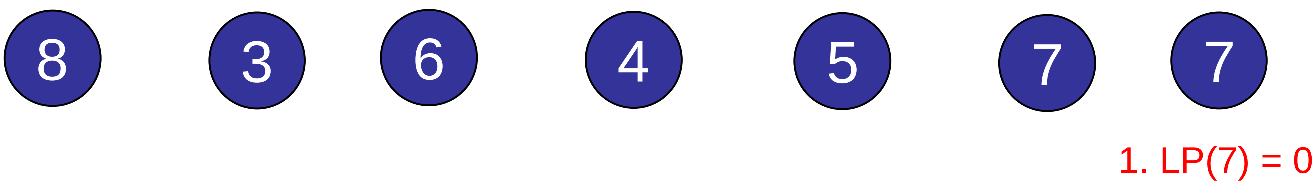
Longest path: O(V + E) = $O(n^2)$

Run longest path n times = $O(n^3)$

Overlapping Subproblems

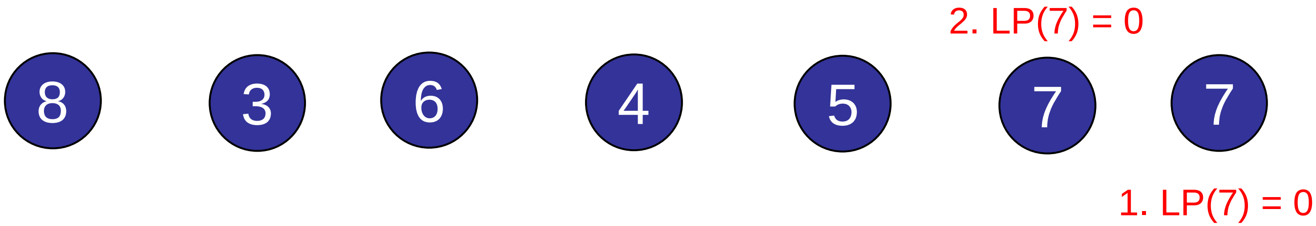


Overlapping Subproblems



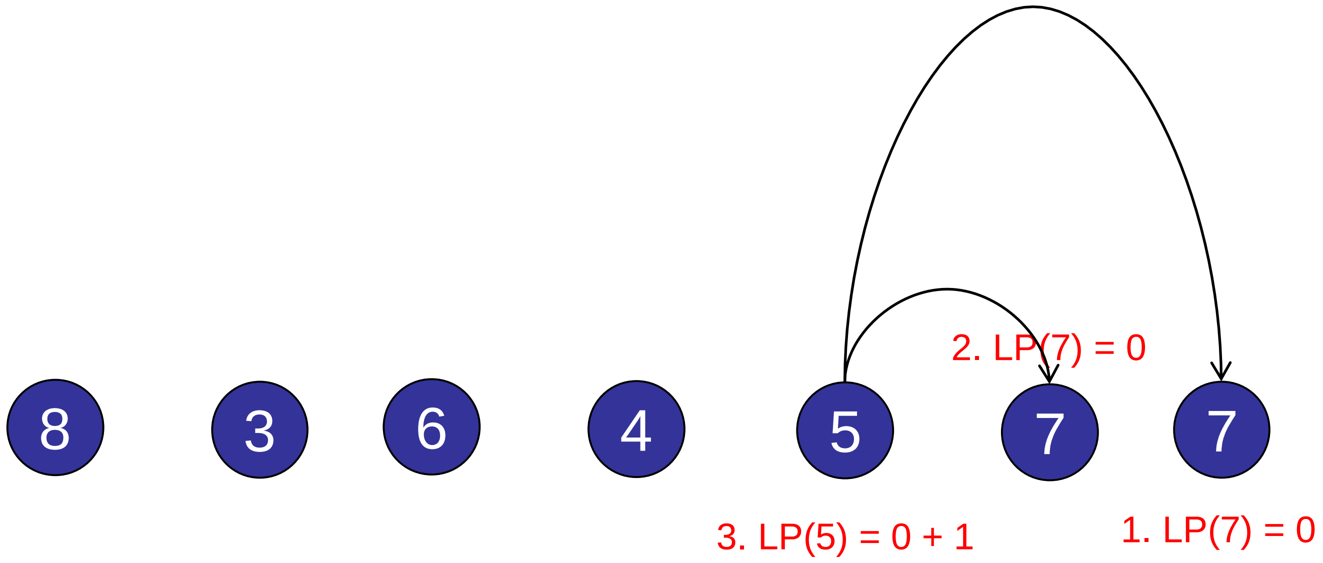
Start with the smallest sub-problem: LP(7)

Overlapping Subproblems



Start with the smallest sub-problem: LP(7)

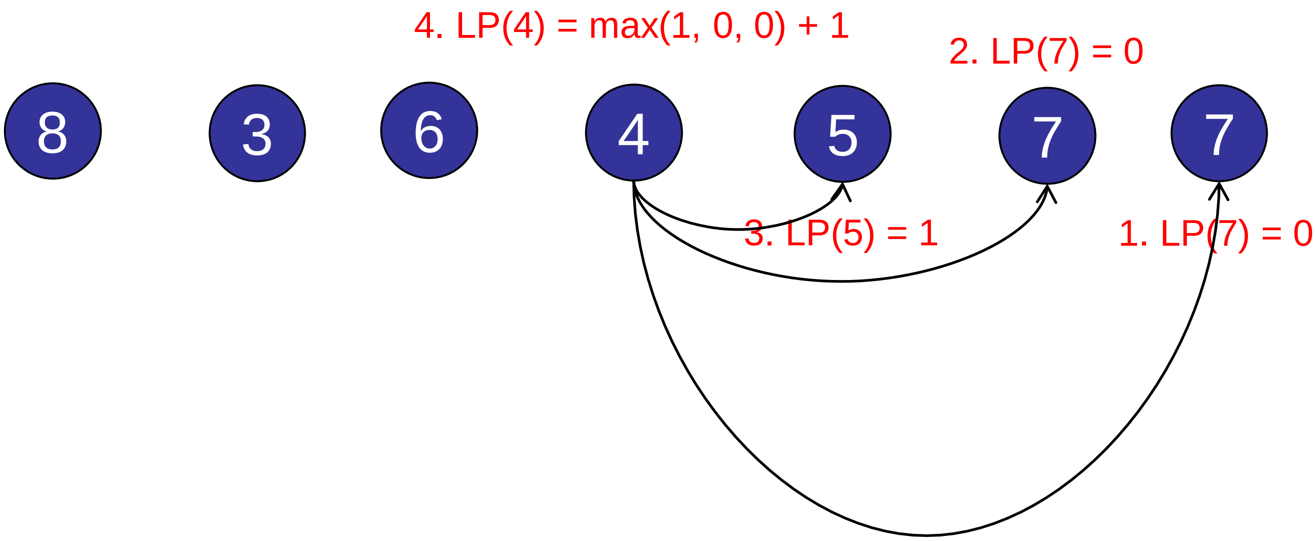
Overlapping Subproblems



Calculate LP(5):

- Examine each outgoing edge.
- Find the maximum.
- Add 1.

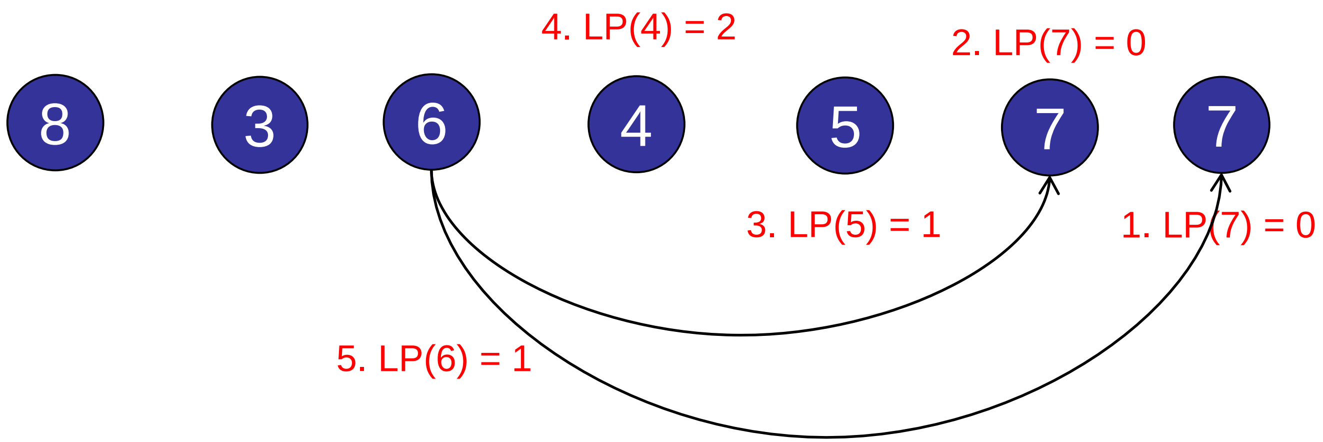
Overlapping Subproblems



Calculate LP(4):

- Examine each outgoing edge.
- Find the maximum.
- Add 1.

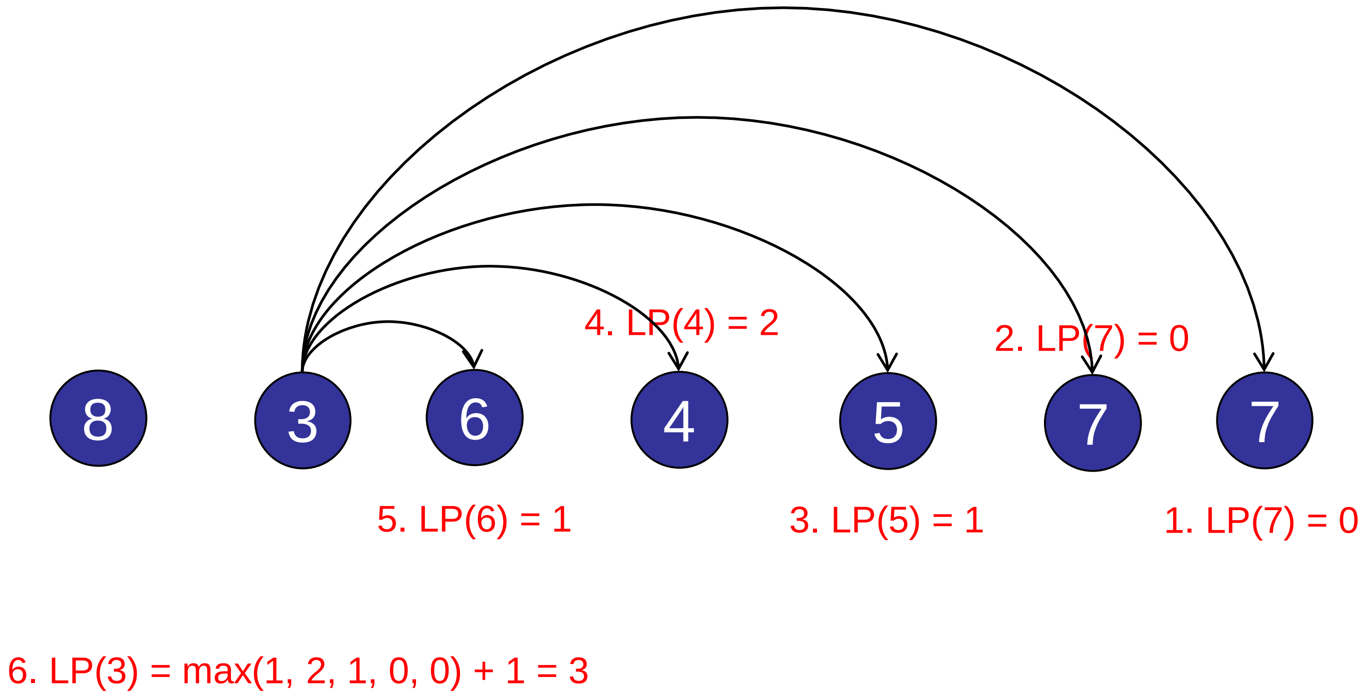
Overlapping Subproblems



Calculate LP(6):

- Examine each outgoing edge.
- Find the maximum.
- Add 1.

Overlapping Subproblems



Calculate LP(3):

- Examine each outgoing edge.
- Find the maximum.
- Add 1.

Longest Increasing Subsequence

Input:

- Array A[1..n]

Define sub-problems:

- S[i] = LIS(A[i..n]) starting at A[i]

Example: {8, 3, 6, 4, 5, 7, 7}

- S[5] = 2 → {8, 3, 6, 4, 5, 7, 7}
- S[2] = 4 → {8, 3, 6, 4, 5, 7, 7}

Dynamic Programming

Table view:

Entry	Longest path that starts at entry X
7	0
7	0
5	…
4	
6	
3	
8	

Longest Increasing Subsequence

Input:

- Array A[1..n]

Define sub-problems:

- S[i] = LIS(A[i..n]) starting at A[i]

Solve using sub-problems:

- S[n] = 0
- $S[i] = (\max_{(i,j) \in E} S[j]) + 1$

Dynamic Programming Recipe

Step 1: Identify optimal substructure

E.g., LIS can be built from suffix LIS

Step 2: Define sub-problems

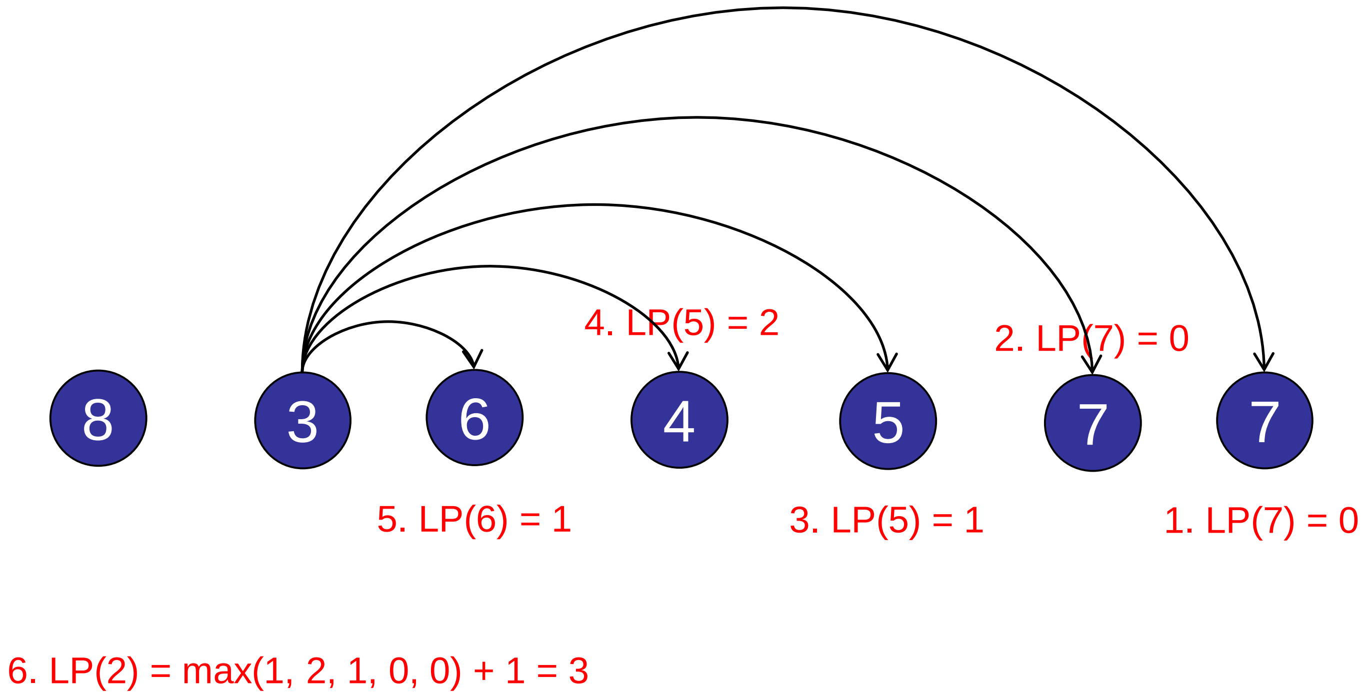
E.g., S[i] = LIS(A[i..n]) starting at A[i]

Step 3: Solve problem using sub-problems

E.g., $S[i] = (\max_{(i,j) \in E} S[j]) + 1$

Step 4: Write (pseudo)code.

Overlapping Subproblems



Calculate LP(2):

- Examine each outgoing edge.
- Find the maximum.
- Add 1.

Longest Increasing Subsequence

LIS(V): // Assume graph is already topo-sorted

```
int[] S = new int[V.length];  // Create memo array
for (i=0; i<V.length; i++) S[i] = 0; // Initialize array to zero
S[n-1] = 1; // Base case: node V[n-1]
for (int v = A.length-2; v>=0; v--) {
    int max = 0; // Find maximum S for any outgoing edge
    for (Node w : v.nbrList()) { // Examine each outgoing edge
            if (S[w] > max) max = S[w]; // Check S[w], which we already
                                        // calculated earlier.
    }
    S[v] = max + 1; // Calculate S[v] from max of outgoing edges.
}
```

Longest Increasing Subsequence

Let's stop thinking about this as a graph…

Input:

- Array A[1..n]

Alternate definition:

- S[i] = LIS(A[1..i]) **ending** at A[i]

Example: {8, 3, 6, 4, 5, 7, 7}

- S[4] = 2 → {8, 3, 6, 4, 5, 7, 7}
- S[5] = 3 → {8, 3, 6, 4, 5, 7, 7}

Longest Increasing Subsequence

Let's stop thinking about this as a graph…

Input:

- Array A[1..n]

Alternate definition:

- S[i] = LIS(A[1..i]) **ending** at A[i]

Solve using sub-problems:

- S[1] = 0
- $S[i] = (\max_{(j < i,\ A[j] < A[i])} S[j]) + 1$

Longest Increasing Subsequence

LIS(A):

```
int[] S = new int[A.length];  // Create memo array
for (i=0; i<A.length; i++) S[i] = 0; // Initialize array to zero
S[0] = 1; // Base case: length 1
for (int i = 0; i<A.length; i++) {
    int max = 0; // Find maximum S for any preceding node
    for (int j=0; j<i; j++) { // Examine each preceding element in the sequence
            if (A[j] < A[i]) // If A[i] is bigger than A[j]
                    if (S[j] > max)
                            max = S[j]; // If S[j] is longer sequence
    }
    S[i] = max + 1; // Calculate S[i] from max of preceding elements.
}
```

What is the running time of the LP-LIS alg for a sequence of n numbers?

1. $O(n)$
2. $O(n \log n)$
3. $O(n^2)$
4. $O(n^2 \log n)$
5. $O(n^3)$
6. None of the above.

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Longest Increasing Subsequence

LIS(A):

```
int[] S = new int[A.length];  // Create memo array
for (i=0; i<A.length; i++) S[i] = 0; // Initialize array to zero
S[0] = 1; // Base case: length 1
for (int i = 0; i<A.length; i++) {
    int max = 0; // Find maximum S for any preceding node
    for (int j=0; j<i; j++) { // Examine each preceding element in the sequence
            if (A[j] < A[i]) // If A[i] is bigger than A[j]
                    if (S[j] > max)
                            max = S[j]; // If S[j] is longer sequence
    }
    S[i] = max + 1; // Calculate S[i] from max of preceding elements.
}
```

Longest Increasing Subsequence

Summary:

Greedy subproblems: S[i] = LIS(A[1..i])

- n subproblems
- Subproblem i takes takes time O(i)

Total time: $O(n^2)$

Challenge of the Day:

How do you solve LIS in time O(n log n)?

Hint: use binary search to solve subproblems faster.

Roadmap

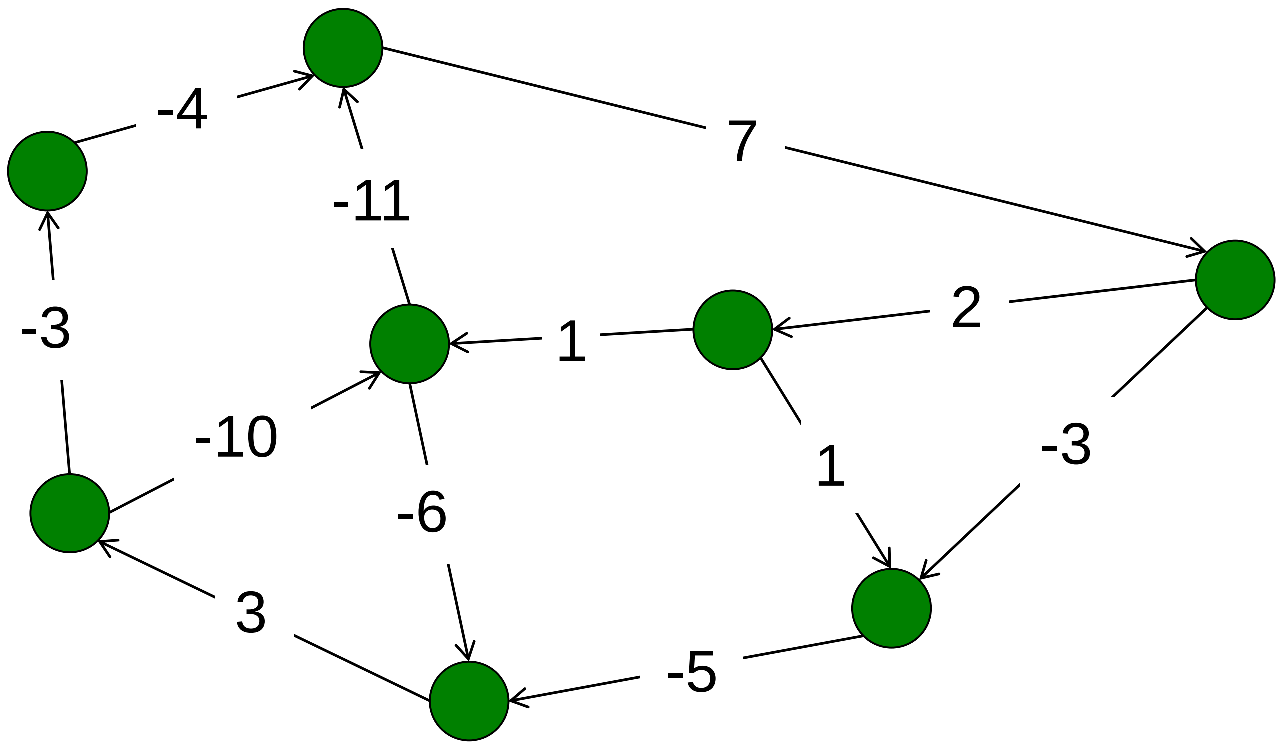
Today and Monday: Dynamic Programming

- DP Basics
- Longest Increasing Subsequence
- Prize Collecting
- Vertex Cover on a Tree
- All-Pairs-Shortest-Paths

Prize Collecting

Input:

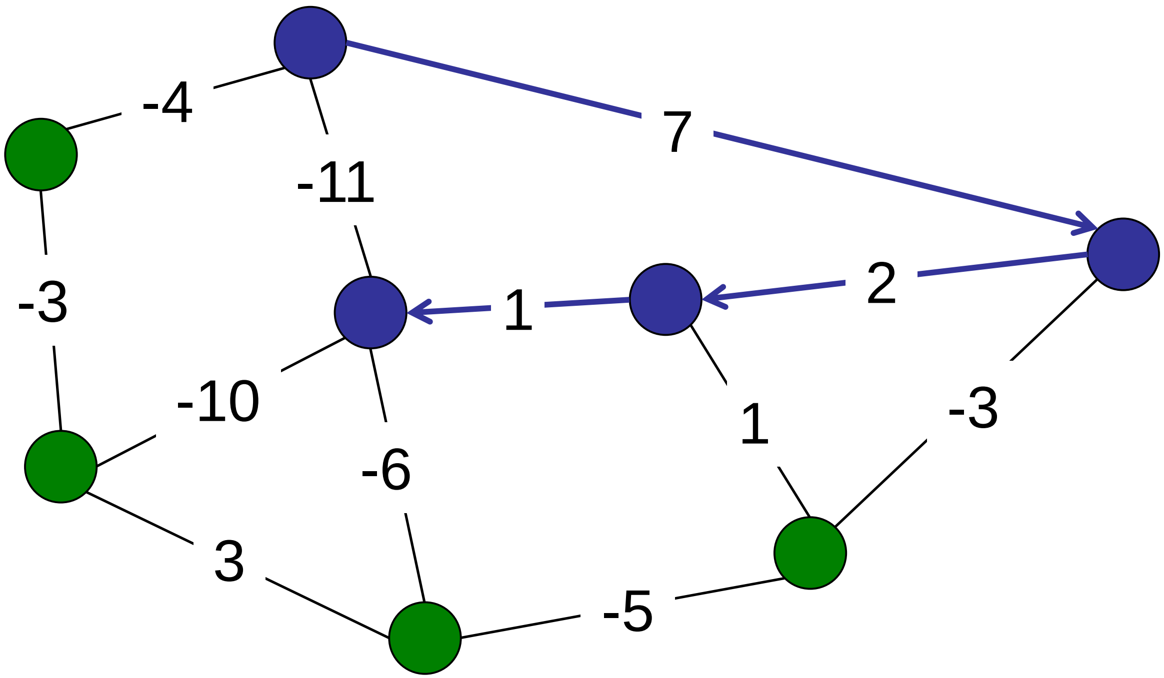
- Directed Graph G = (V,E)
- Edge weights **w** = prizes on each edge



Prize Collecting

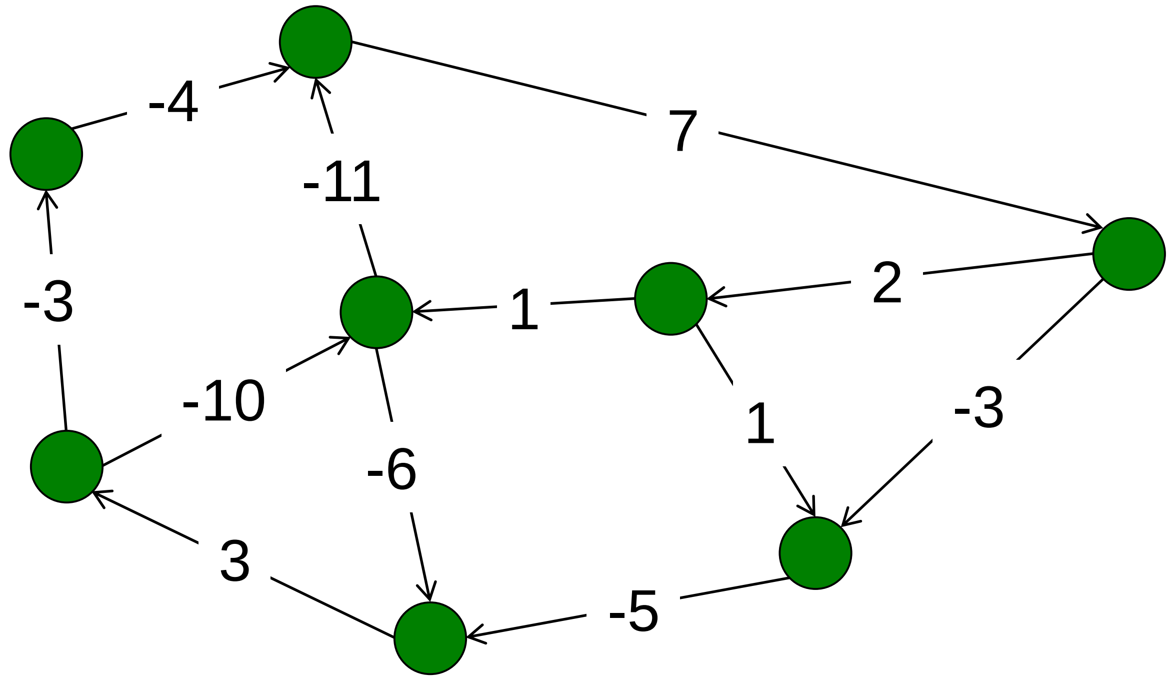
Output:

- Prize collecting path
- Example: 7 + 2 + 1 = 10



What is the maximum prize?

1. 1
2. 3
3. 10
4. 15
5. 17
6. Infinite



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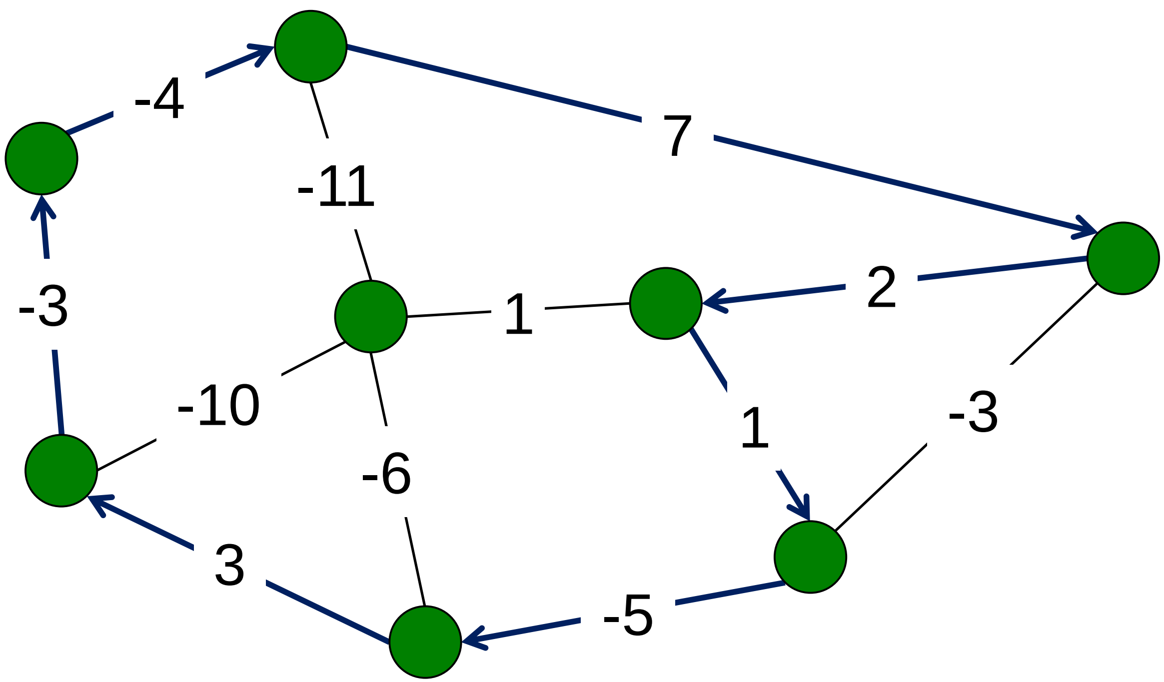
Prize Collecting

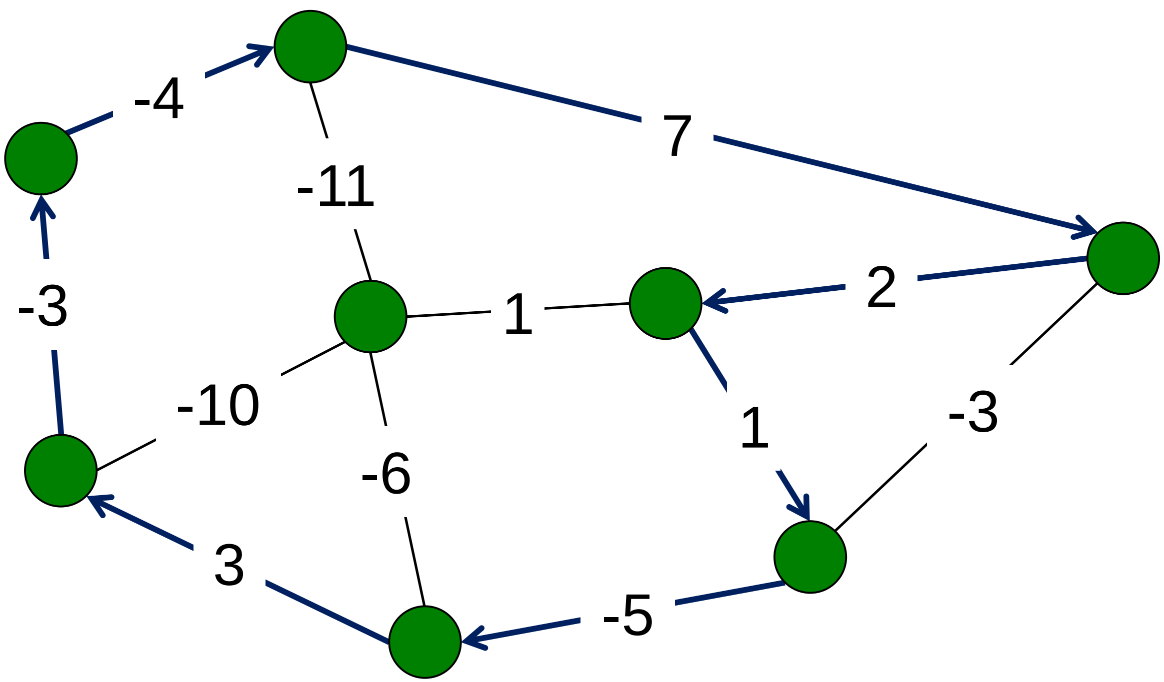
Output:

- Prize collecting path: 7 + 2 + 1 - 5 + 3 - 3 - 4 = 1
- Positive weight cycle → infinite prizes!



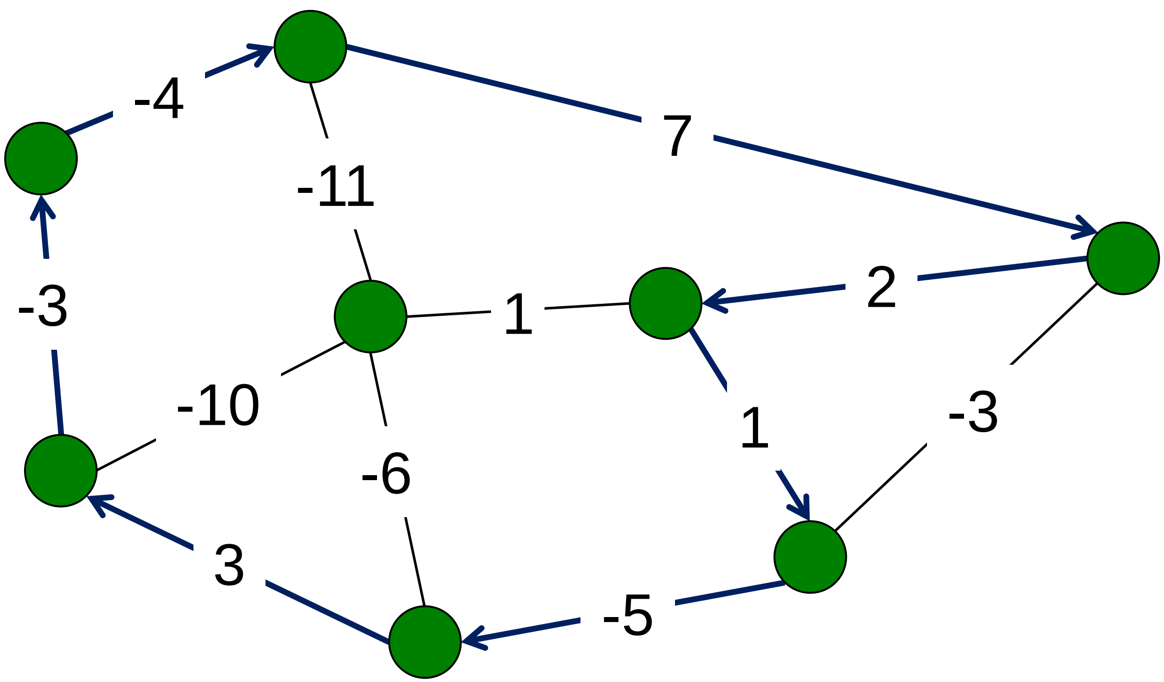
Prize Collecting

Aside: How could we determine if there is a positive weight cycle in a graph?



Prize Collecting

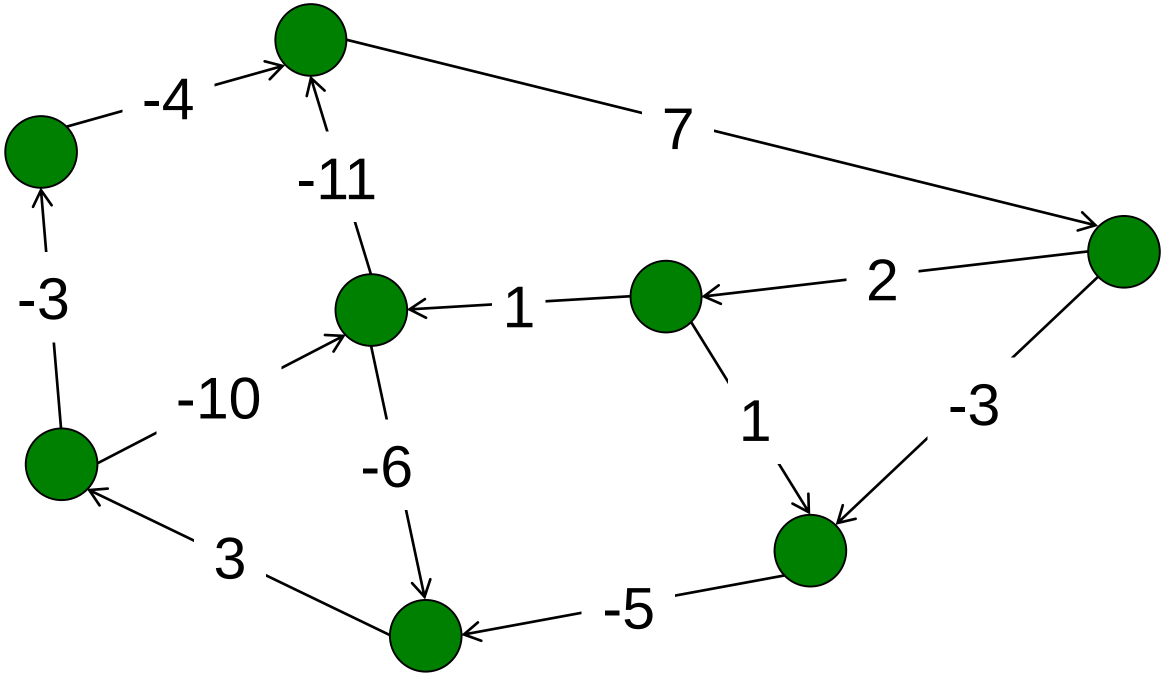
1. Check for positive weight cycles.
2. Negate the edges, run BF.



Lazy Prize Collecting

Input:

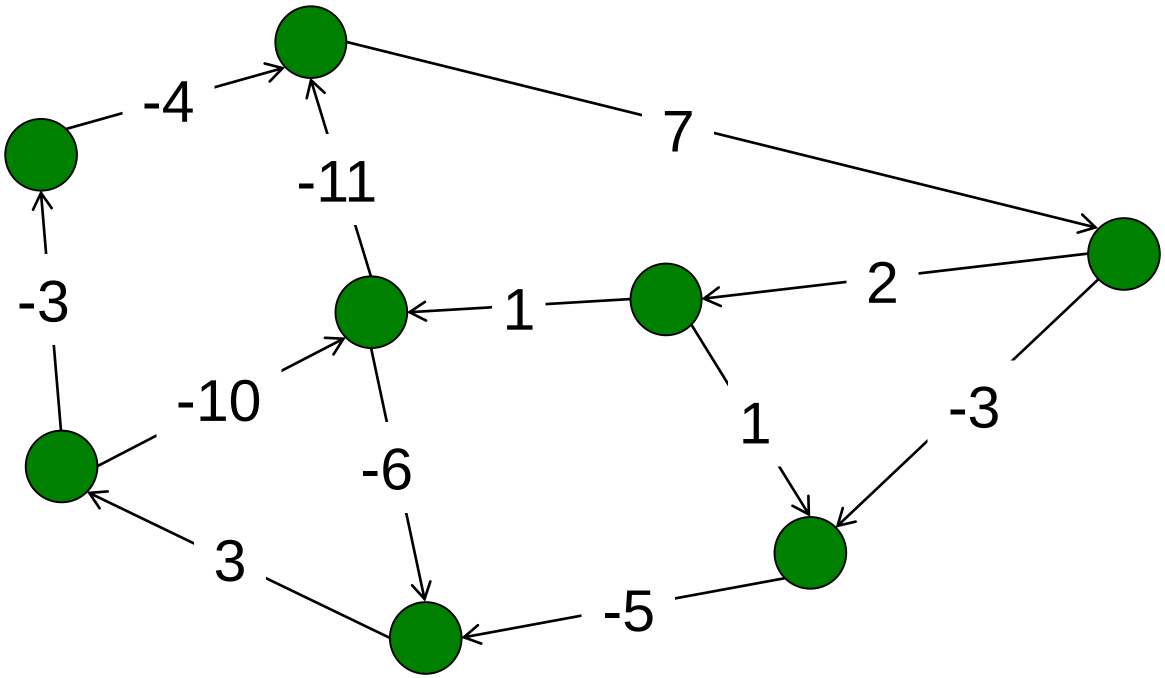
- Graph G = (V,E)
- Edge weights w = prizes on each edge
- Limit k: only cross at most k edges



Lazy Prize Collecting

Example:

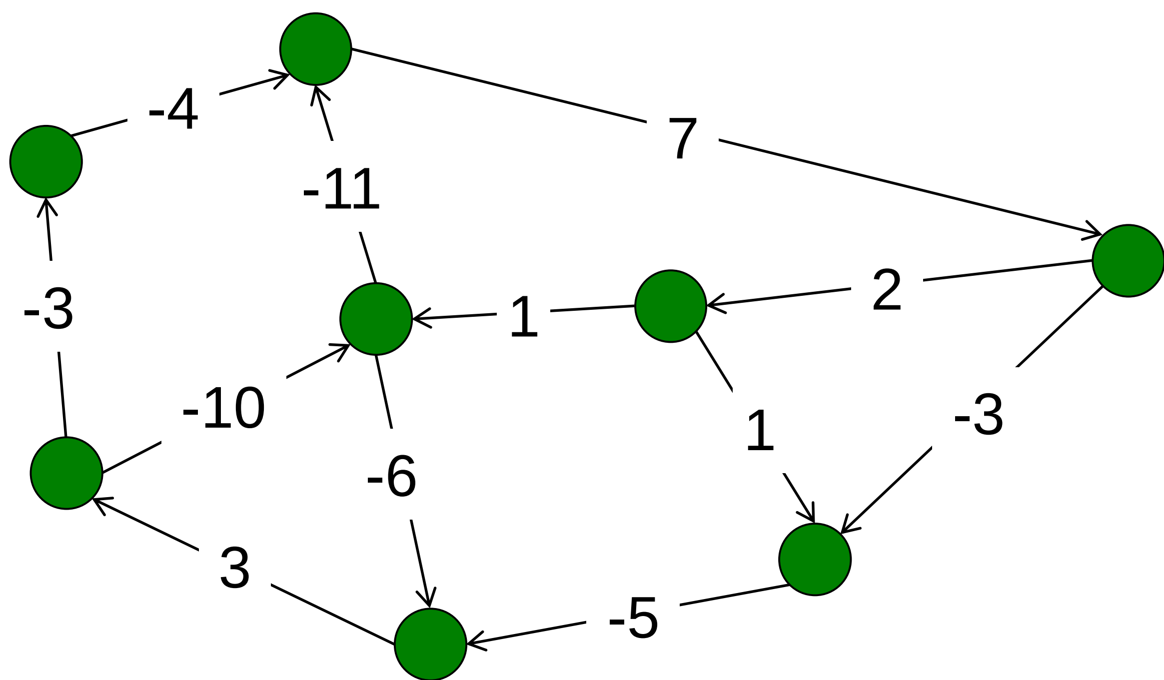
- k = 1 → 7
- k = 2 → 9
- k = 3 → 10
- …
- k = 71 → 17



Lazy Prize Collecting

Note: Not a shortest path problem

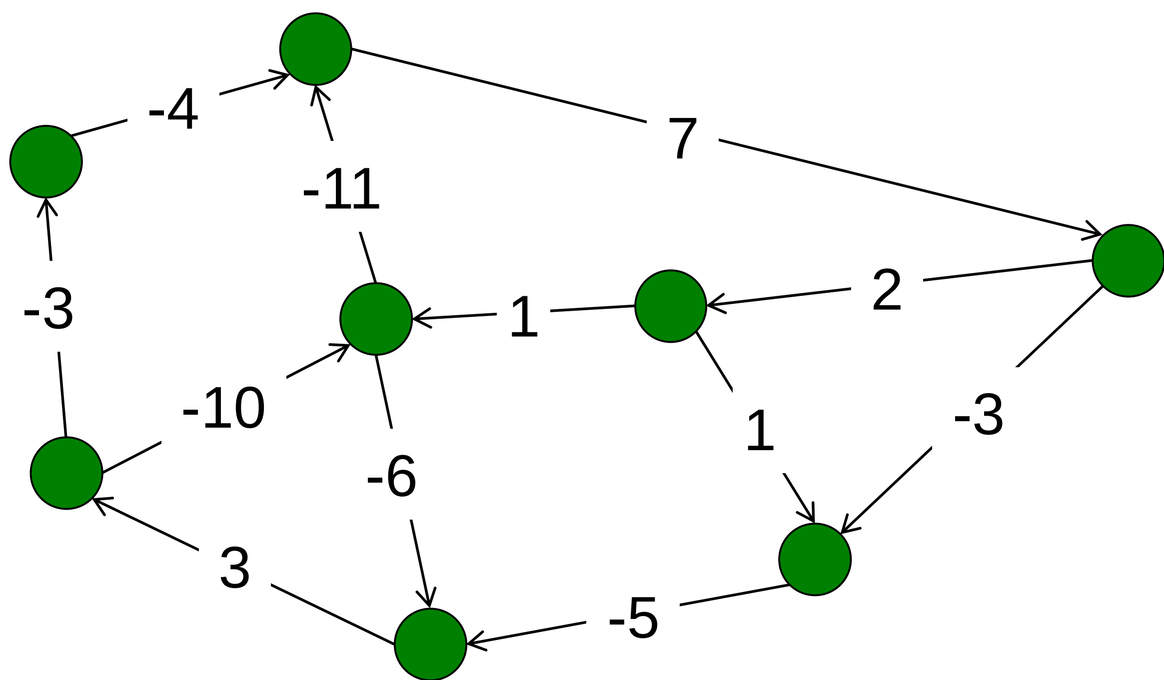
- Not a shortest path problem! Longest path…
- Negative weight cycles.
- Positive weight cycles.



Lazy Prize Collecting

Idea 1:

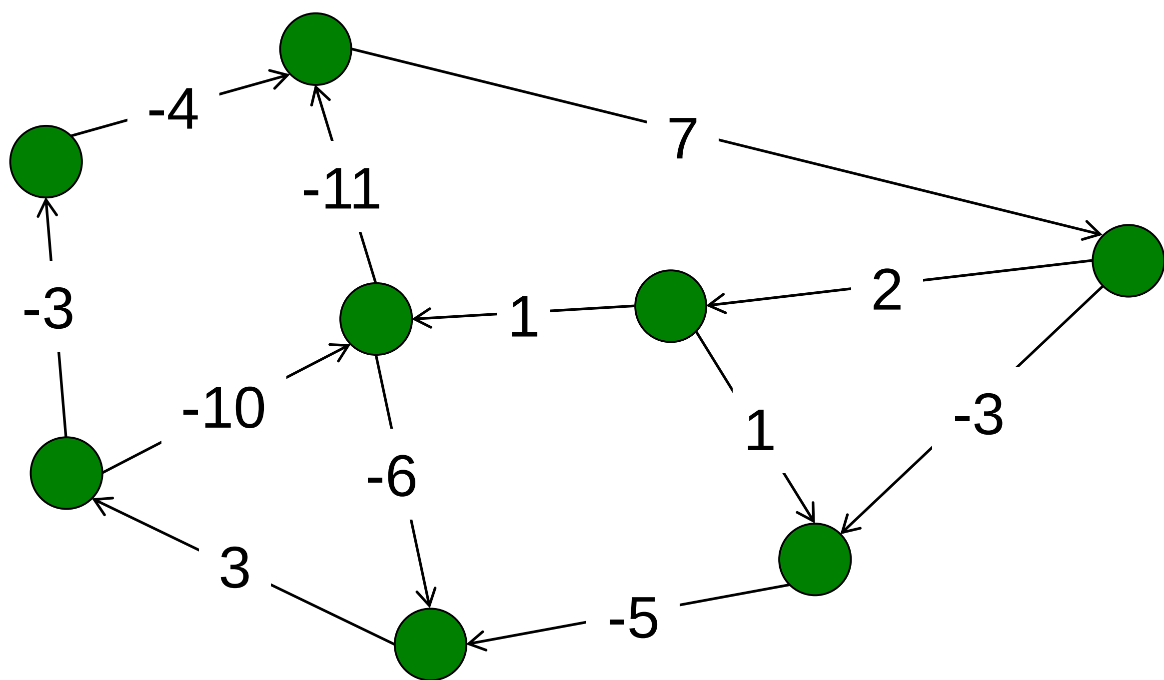
- Transform G into a DAG



Lazy Prize Collecting

Idea 1:

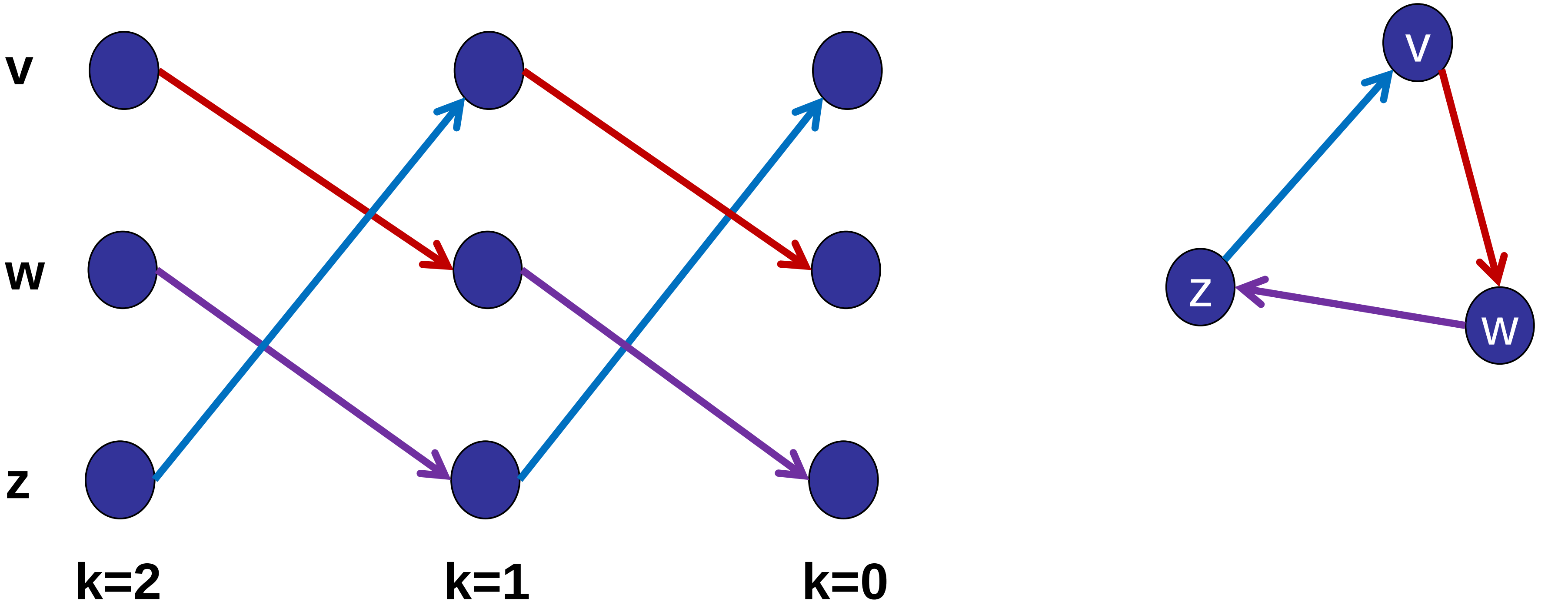
- Transform G into a DAG
- Make **k** copies of every node: (v,1), (v,2), (v,3), ...



Lazy Prize Collecting

Idea 1:

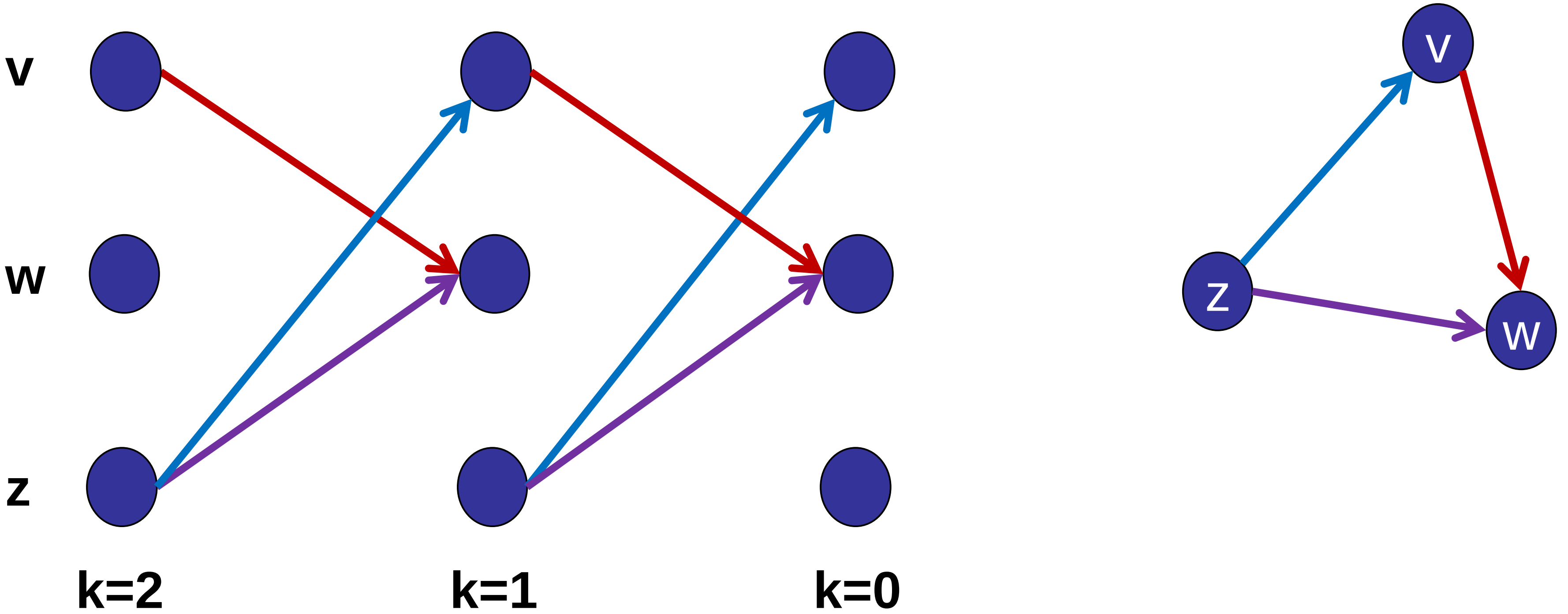
- Transform G into a DAG
- Make **k** copies of every node: (v,1), (v,2), (v,3), …



Lazy Prize Collecting

Idea 1:

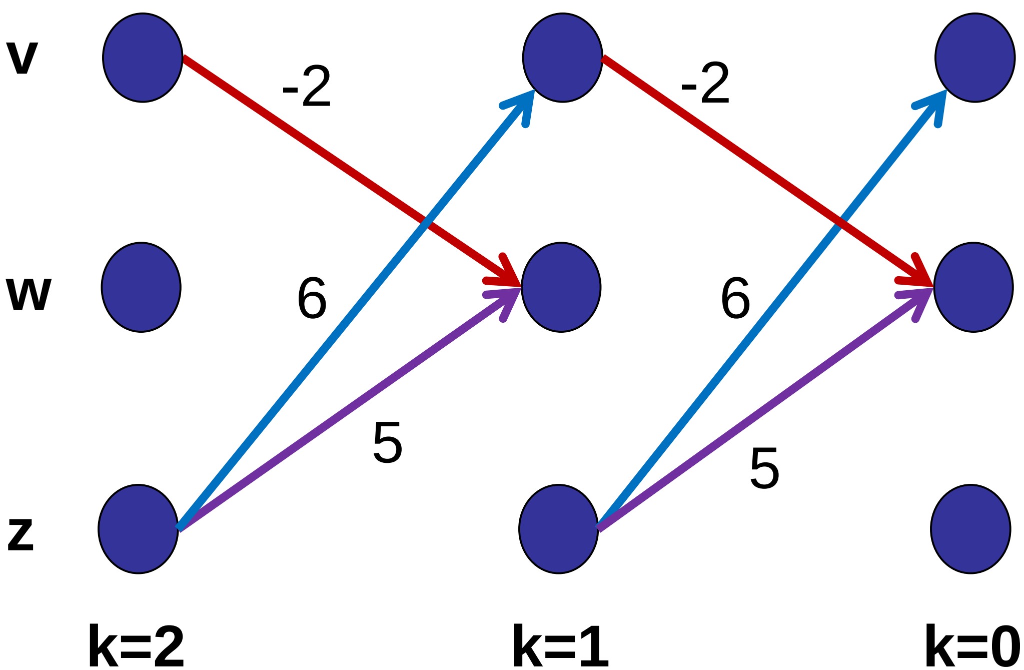
- Transform G into a DAG
- Make **k** copies of every node: (v,1), (v,2), (v,3), …

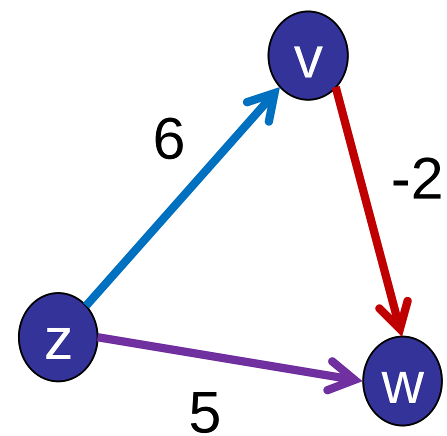


Lazy Prize Collecting

Idea 1:

- Transform G into a DAG
- Make **k** copies of every node: (v,1), (v,2), (v,3), …
- Solve prize collecting via DAG_SSSP (longest path)

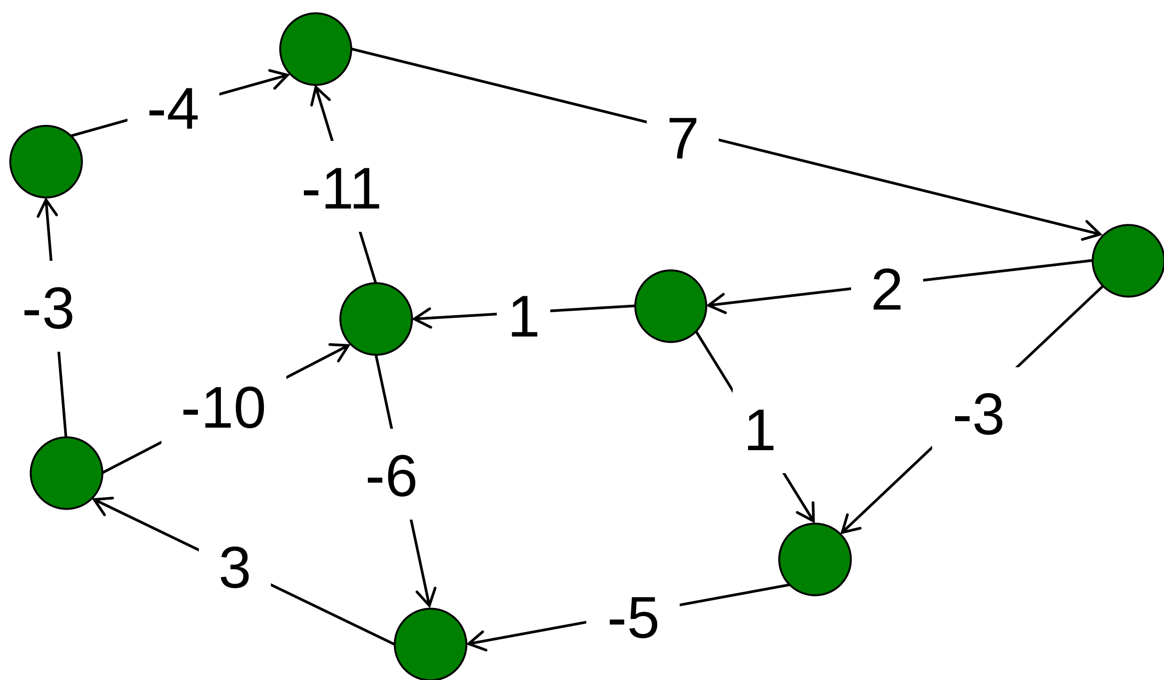




Lazy Prize Collecting

Idea 1:

- Transform G into a DAG
- Make **k** copies of every node: (v,1), (v,2), (v,3), …
- Solve longest-path problem for each source.



What is the running time of Idea 1?

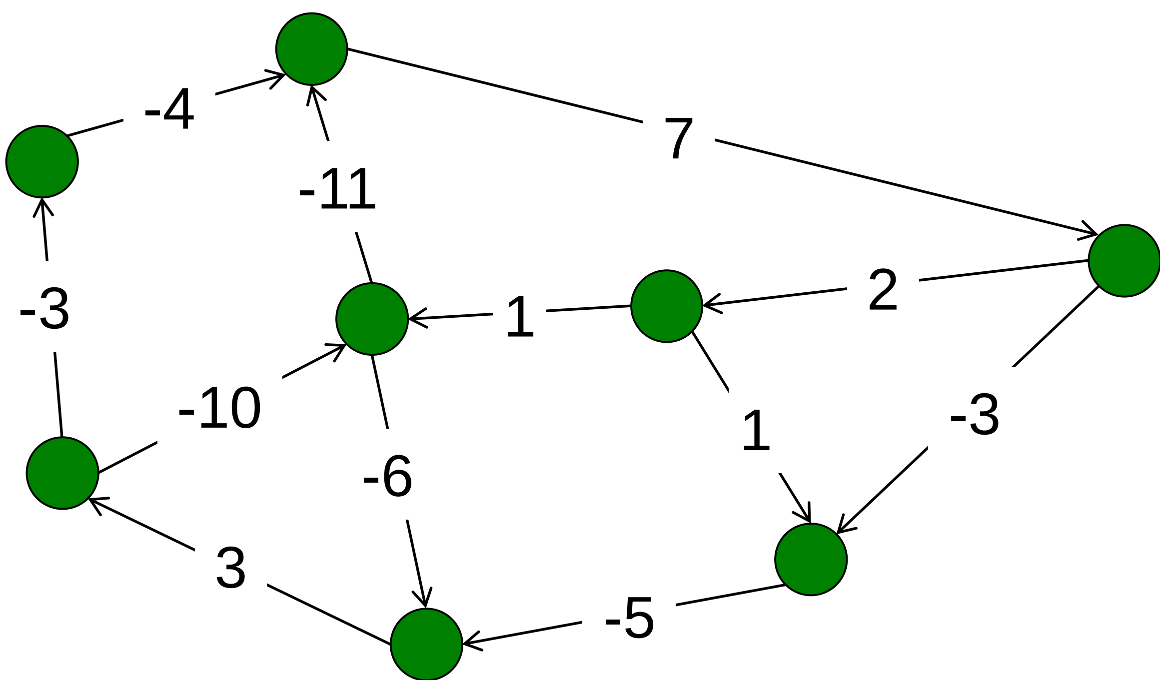
1. $O(E)$
2. $O(VE)$
3. $O(kE)$
4. $O(kVE)$
5. $O(kV^2E)$
6. None of the above

Lazy Prize Collecting

Running Time:

- Transformed graph: kV nodes, kE edges
- Topo-sort / Longest path: O(kV + kE)
- Once per source: repeat V times ➔ O(kVE)?

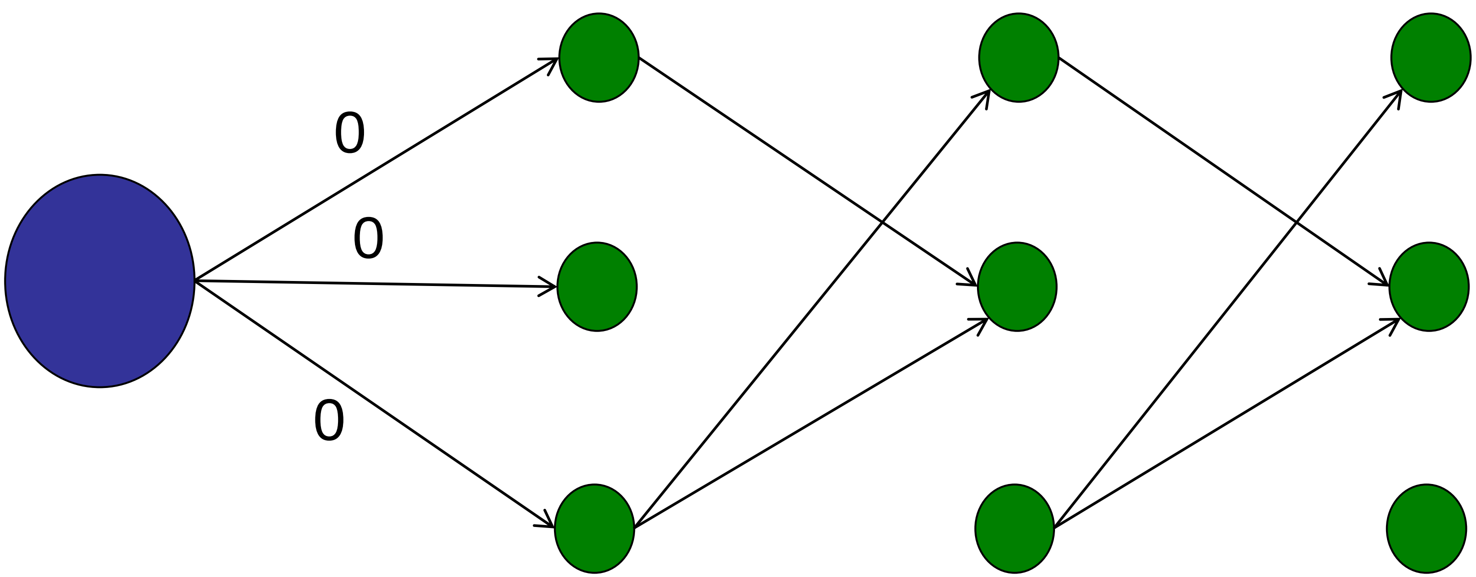
Whenever you transform a graph, do NOT forget to recompute the number of nodes and edges in the new graph.



Lazy Prize Collecting

Running Time:

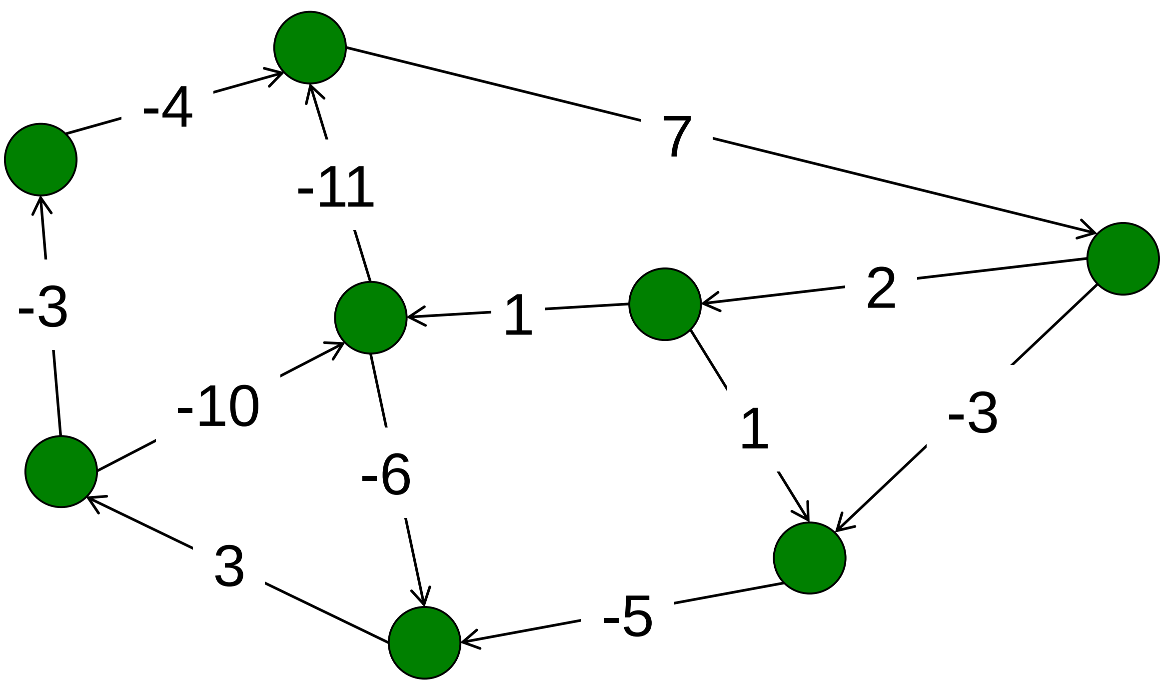
- Transformed graph: kV nodes, kE edges
- Topo-sort / Longest path: O(kV + kE)
- Create super-source….



Lazy Prize Collecting

Idea 2: Dynamic Programming

If you know the optimal solution for (k-1), then it is easy to computer optimal solution for k.



Dynamic Programming Recipe

Step 1: Identify optimal substructure

E.g., solution for (k-1) → solution for k

Step 2: Define sub-problems

Step 3: Solve problem using sub-problems

Step 4: Write (pseudo)code.

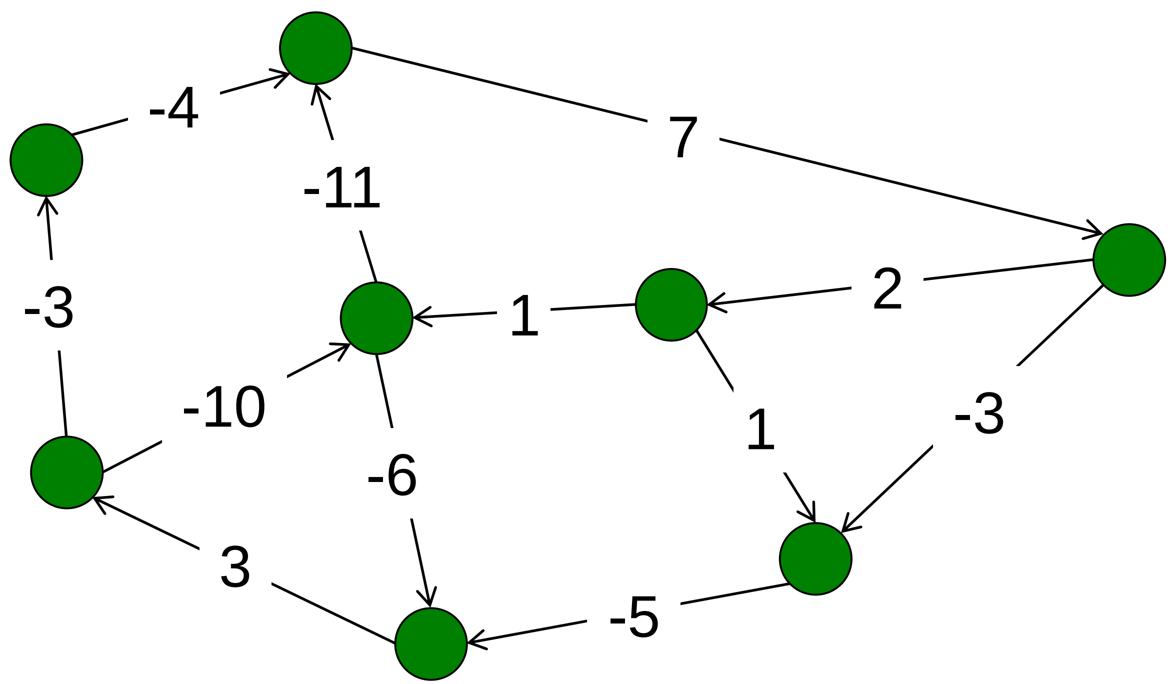
Lazy Prize Collecting

Idea 2: Dynamic Programming

P[v, k] = maximum prize that you can collect starting at v and taking *exactly* k steps.

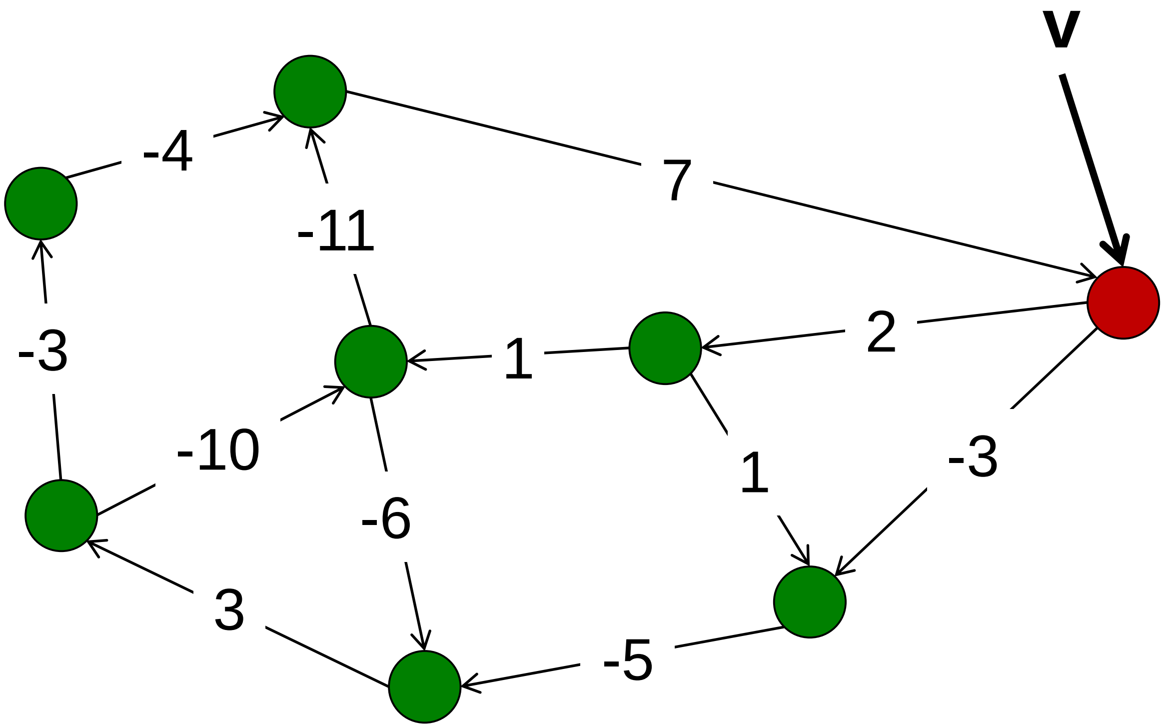
<u>Modified subproblem:</u>
Leads to better optimal substructure.

Often, useful to solve modified problem.



P(v, 0) = ??

1. 0
2. 2
3. -3
4. 4
5. 5

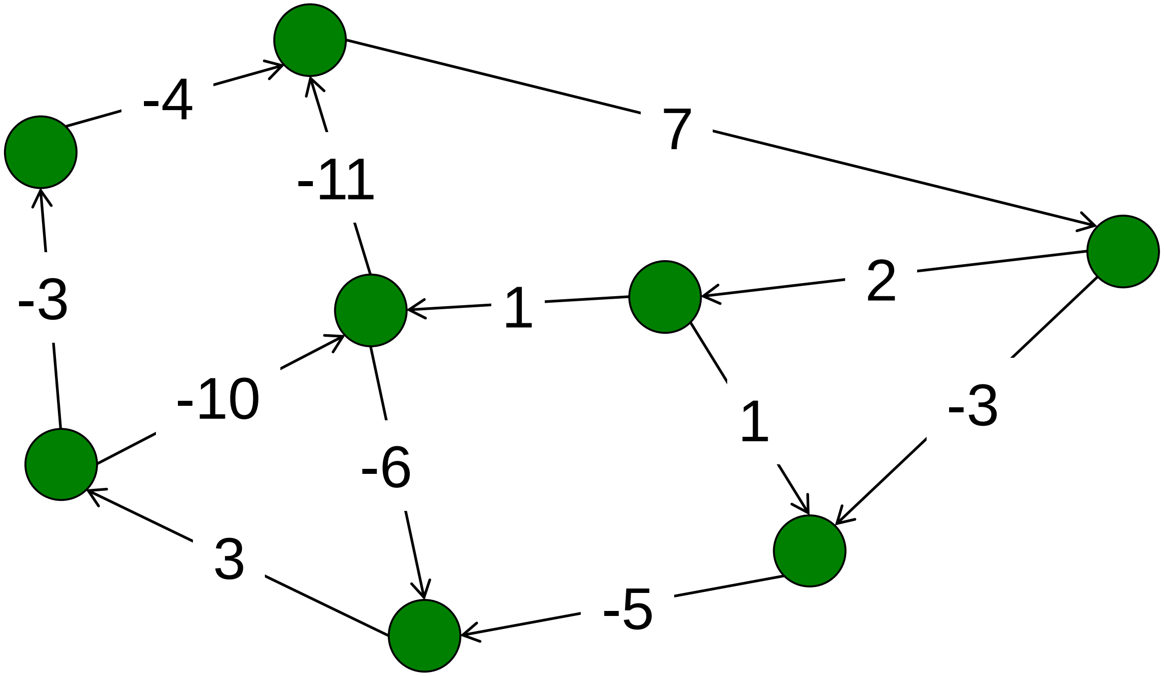


Lazy Prize Collecting

Idea 2: Dynamic Programming

P[v, k] = maximum prize that you can collect starting at v and taking exactly k steps.

P[v, 0] = 0



Lazy Prize Collecting

Idea 2: Dynamic Programming

P[v, k] = maximum prize that you can collect starting at v and taking *exactly* k steps.

<u>Solve P[v,k] using subproblems:</u>

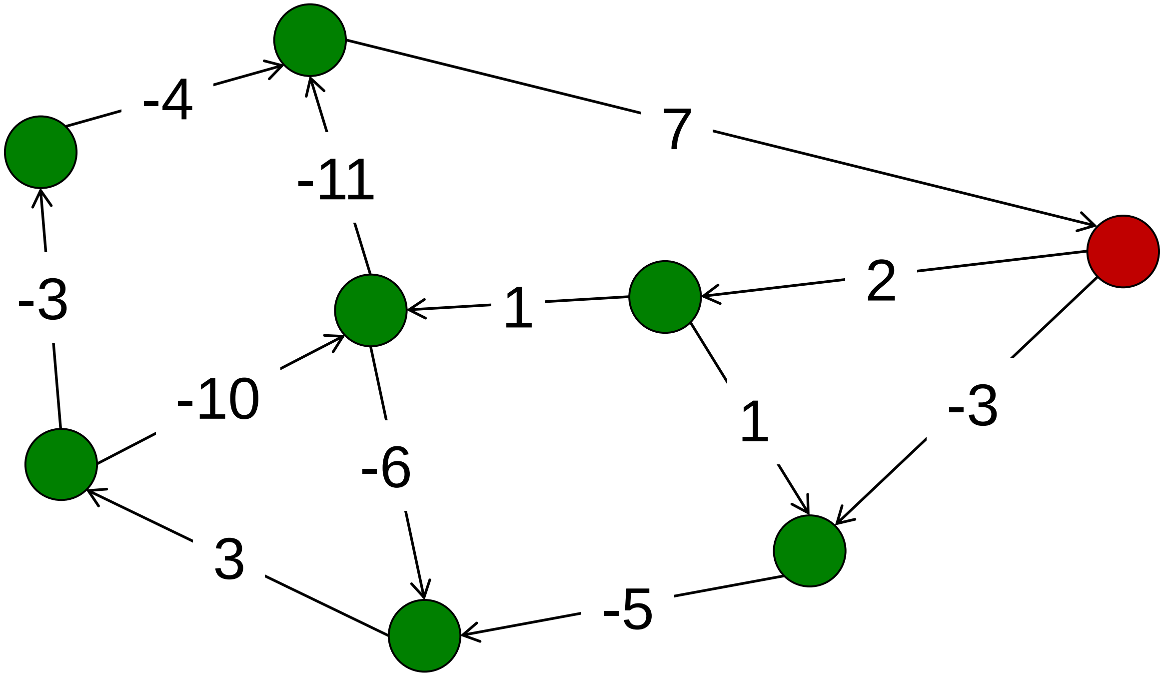
$$P[v, k] = \text{MAX} \{ \quad P[w_1, k-1] + w(v, w_1),$$
$$P[w_2, k-1] + w(v, w_2),$$
$$P[w_3, k-1] + w(v, w_3), \ldots \quad \}$$

where v.nbrList() = $\{w_1, w_2, w_3, \ldots\}$

Lazy Prize Collecting

Idea 2: Dynamic Programming

$P[v, 1] = \max(0+2, 0-3) = 2$

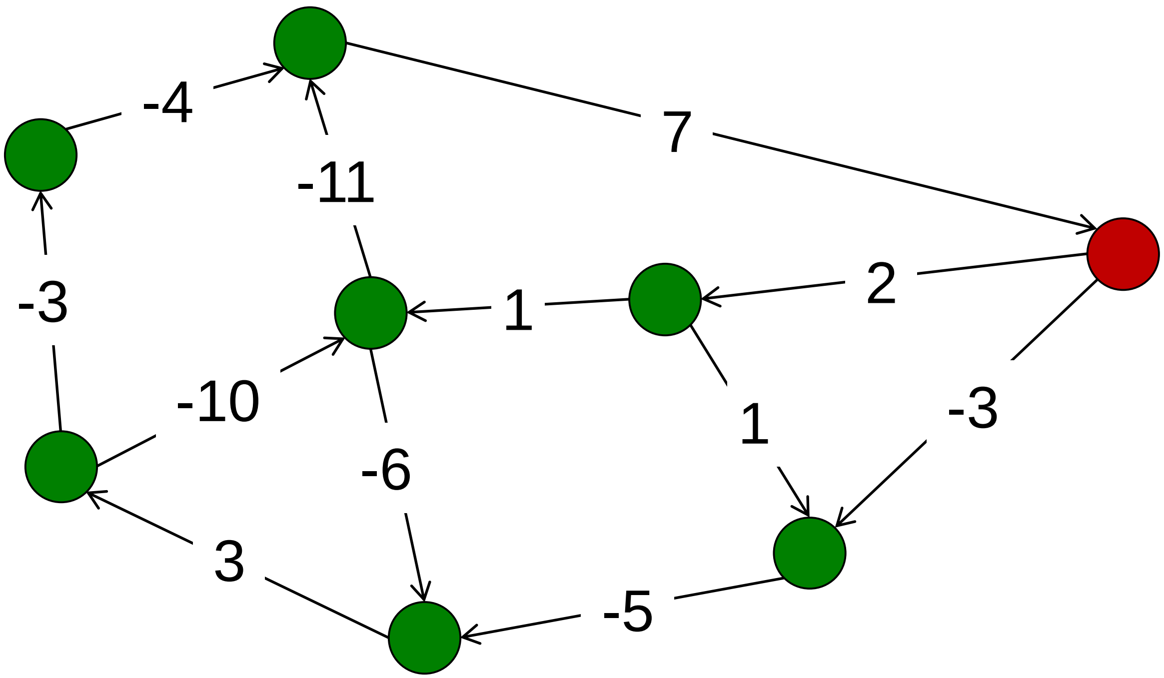


Lazy Prize Collecting

Idea 2: Dynamic Programming

$P[v, 1] = \max(0+2, 0-3) = 2$

$P[v, 2] = \max(1+2, -5-3) = 3$



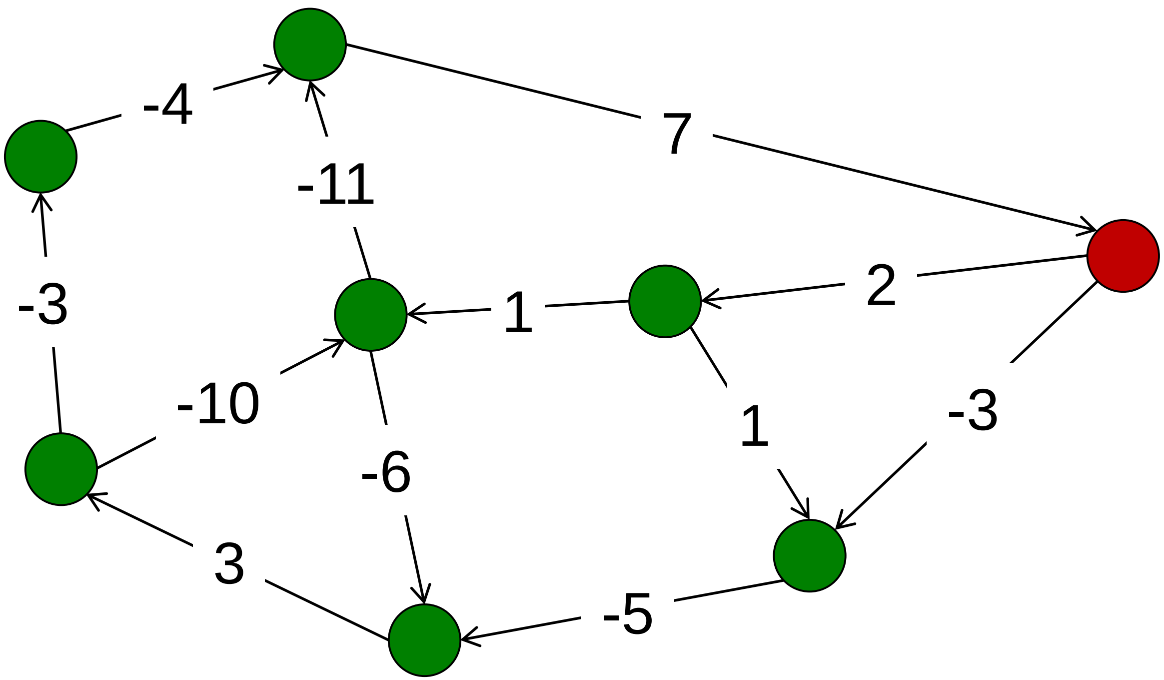
Lazy Prize Collecting

Idea 2: Dynamic Programming

$P[v, 1] = \max(0+2, 0-3) = 2$

$P[v, 2] = \max(1+2, -5-3) = 3$

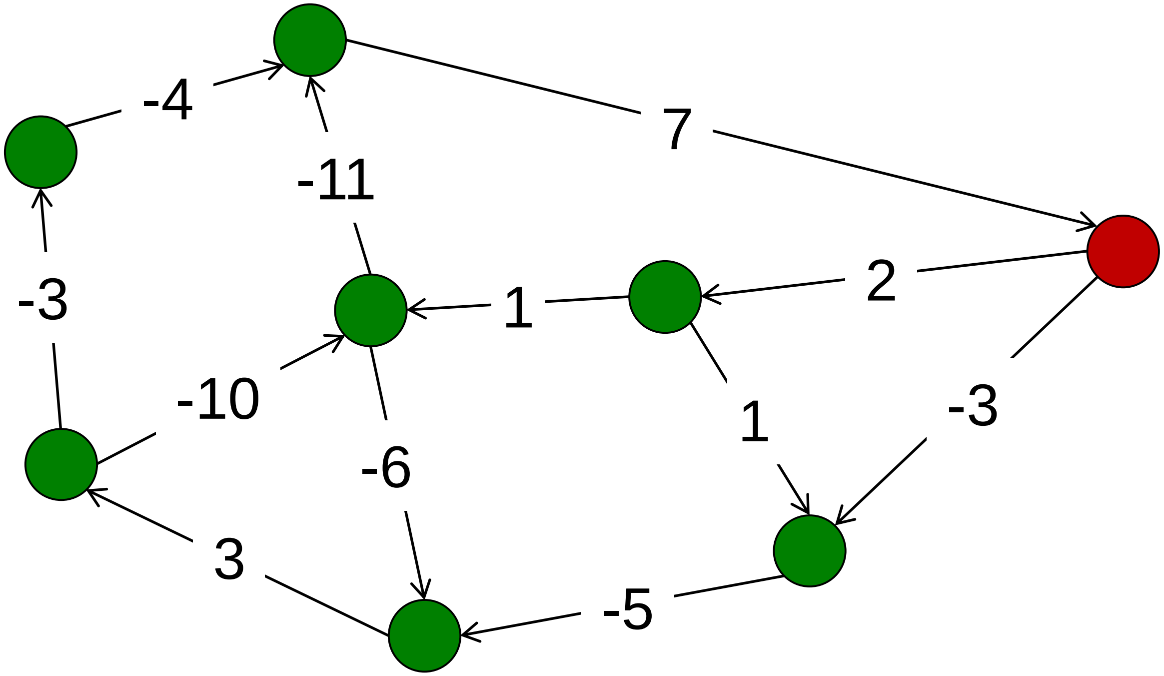
$P[v, 3] = \max(-4+2, -2-3) = -2$



Lazy Prize Collecting

Idea 2: Dynamic Programming

When is it worth crossing a negative edge?



Dynamic Programming

Table view: P[k, v]

k	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
1	17	22	14	19	8	4	9	12	15	7
2	15	12	13	13	7					
3										
4										
5										
6										
7										
8										
9										
10										
11										

```
int LazyPrizeCollecting(V, E, kMax) {

    int[][] P = new int[V.length][kMax+1]; // create memo table P
    for (int i=0; i<V.length; i++) // initialize P to zero
       for (int j=0; j<kMax+1; j++)
              P[i][j] = 0;

    for (int k=1; k<kMax+1; k++) { // Solve for every value of k
       for (int v = 0; v<V.length; v++) { // For every node…
              int max = -INFTY;
              // …find max prize in next step
              for (int w : V[v].nbrList()) {
                     if (P[w,k-1] + E[v,w] > max)
                            max = P[w,k-1] + E[v,w];
              }
              P[v, k] = max;
       }
    }
    return maxEntry(P); // returns largest entry in P
}
```

Lazy Prize Collecting

Idea 2: Dynamic Programming

P[v, k] = maximum prize that you can collect starting at v and taking exactly k steps.

Total Cost:

Two factors:

– Number of subproblems: kV

– Cost to solve each subproblem: |v.nbrList|

Total: $O(kV^2)$

Dynamic Programming

Table view: P[k, v]

k	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
1	17	22	14	19	8	4	9	12	15	7
2	15	12	13	13	7					
3										
4										
5										
6										
7										
8										
9										
10										
11										

Lazy Prize Collecting

Idea 2: Dynamic Programming

P[v, k] = maximum prize that you can collect starting at v and taking exactly k steps.

Total Cost:

Two factors:

– Number of rows: k

– Cost to solve all problems in a row: E

Total: O(kE)

Roadmap

Today and Monday: Dynamic Programming

- DP Basics
- Longest Increasing Subsequence
- Prize Collecting
- Vertex Cover on a Tree
- All-Pairs-Shortest-Paths