CS1231S: Discrete Structures

Tutorial #4: Relations & Equivalence Relations

(Week 6: 12 – 16 September 2022)

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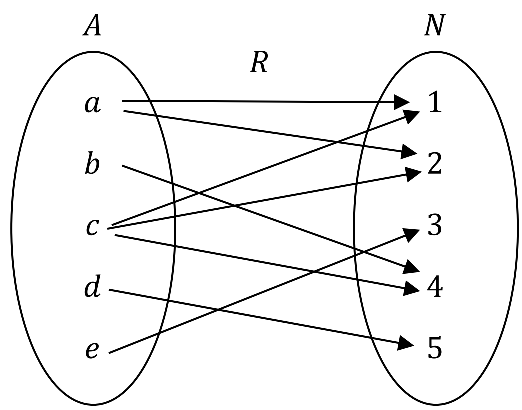
1. Discussion Questions

These are meant for you to discuss on Canvass. No answers will be provided.

D1. Let $A = \{0,1\}, B = \{a, b, c\}$ and $C = \{01,10\}$. Determine the following:

(a) $B \times C$ (b) $A \times B \times C$ (c) $\emptyset \times A$ (d) $\wp(\{\emptyset\}) \times A$

D2. Let $A = \{a, b, c, d, e\}$ and $N = \{1,2,3,4,5\}$. A relation R from A to N is shown in the arrow diagram below.



(a) Determine R^{-1}.

(b) Determine $R^{-1} \circ R$.

2. Tutorial Questions

1. Let $A = \{1,2, \dots ,10\}$ and $B = \{2,4,6,8,10,12,14\}$. Define a relation R from A to B by setting

$$x \, R \, y \ \Leftrightarrow x \text{ is prime and } x \mid y$$

for each $x \in A$ and each $y \in B$. Write down the sets R and R^{-1} in **roster notation**. Do not use ellipses (…) in your answers.

2. Let R be a relation on a set A. Show that the following are logically equivalent by using this strategy: (i) implies (ii), (ii) implies (iii), and (iii) implies (i).

(i) R is symmetric, i.e. $\forall x, y \in A \ (x \, R \, y \ \Rightarrow y \, R \, x)$.

(ii) $\forall x, y \in A \ (x \, R \, y \ \Leftrightarrow y \, R \, x)$.

(iii) $R = R^{-1}$.

3. For each of the relations defined below, determine whether it is (i) reflexive, (ii) symmetric, (iii) transitive, and (iv) an equivalence relation. If a property is false for the relation, give a counter-example.

   (a) Let $A = \{1,2,3\}$, $Q = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$, where Q is a relation on A.

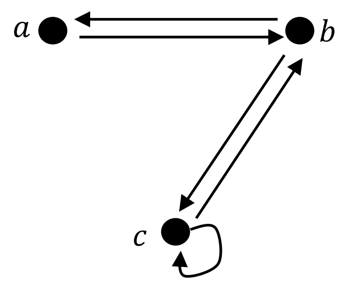
   (b) Define the relation E on $\mathbb{Q}$ by setting, for all $x, y \in \mathbb{Q}, x\ E\ y \ \Leftrightarrow x = y$.

   (c) Define the relation R on $\mathbb{Q}$ by setting, for all $x, y \in \mathbb{Q}, x\ R\ y \ \Leftrightarrow xy \geq 0$.

   (d) Define the relation S on $\mathbb{Q}$ by setting, for all $x, y \in \mathbb{Q}, x\ S\ y \ \Leftrightarrow xy > 0$.

   (e) Define the relation T on $\mathbb{Z}$ by setting, for all $x, y \in \mathbb{Z}, x\ T\ y \ \Leftrightarrow -2 \leq x - y \leq 2$.

4. The directed graph of a binary relation R on a set $A = \{a, b, c\}$ is shown below.



   Draw the directed graph for each of the following and determine if it is transitive or not. If it is not transitive, explain.

   (a) $R \circ R$ (b) $R \circ R \circ R$ (c) $(R \circ R) \cup R$

5. Consider the relation $S = \{(m, n) \in \mathbb{Z}^2 : m^3 + n^3 \text{ is even}\}$. (Recall that $\mathbb{Z}^2$ means $\mathbb{Z} \times \mathbb{Z}$.) Determine (a) S^{-1}, (b) $S \circ S$ and (c) $S \circ S^{-1}$.

   You may use theorems involving the sum of even and odd integers without quoting them (for example: the sum of two even integers is even; the sum of an even integer and odd integer is odd; etc.).

6. Let A, B, C, D be sets and $R \subseteq A \times B, S \subseteq B \times C$, and $T \subseteq C \times D$. Prove that

$$T \circ (S \circ R) = (T \circ S) \circ R.$$

   That is, composition of relations is associative.

7. (AY2020/21 Semester 1 exam question)
   Define an equivalence relation $\sim$ on $\mathbb{Z}^+ \times \mathbb{Z}^+$ by setting, for all $a, b, c, d \in \mathbb{Z}^+$,

$$(a, b) \sim (c, d) \Leftrightarrow ab = cd.$$

   Write down the equivalence classes $[(1,1)]$ and $[(4,3)]$ in **roster notation**.

8. Define a relation $\sim$ on $\mathbb{Z} \setminus \{0\}$ as follows: $\forall a, b \in \mathbb{Z} \setminus \{0\}\ (a \sim b \Leftrightarrow ab > 0)$.
   (a) Prove that $\sim$ is an equivalence relation. You may adopt the appropriate **order axioms** and **theorems** in *Appendix A: Properties of the Real Numbers* for the integers. (Appendix A is available on Canvas > Files as well as the CS1231S webpage at https://www.comp.nus.edu.sg/~cs1231s/2_resources/lectures.html.)
   (b) Determine all the distinct equivalence classes formed by this relation $\sim$.

9. Let $\mathscr{C}$ be a partition of a set A. Denote by $\sim$ the same-component relation with respect to $\mathscr{C}$, i.e. for all $x, y \in A$,

$$x \sim y \Leftrightarrow x \text{ is in the same component of } \mathscr{C} \text{ as } y.$$
$$\Leftrightarrow x, y \in S \text{ for some } S \in \mathscr{C}.$$

   (a) Prove that if $x \in S \in \mathscr{C}$, then $[x] = S$.
   (b) Prove that $A/{\sim} = \mathscr{C}$. (Recall that $A/{\sim}$ denotes the set of all equivalence classes w.r.t. $\sim$)