7. Let W be a subspace of $\mathbb{R}^n$. Define $W^{\perp} = \{\boldsymbol{u} \in \mathbb{R}^n \mid \boldsymbol{u} \text{ is orthogonal to } W\}$.

(a) Let $W = \text{span}\{(1,0,1,1),(1,-1,0,2),(1,2,3,-1)\}$. Find $W^{\perp}$.

(b) Show that $W^{\perp}$ is a subspace of $\mathbb{R}^n$. (*Hint*: Show that $W^{\perp}$ is a solution set of a homogeneous system of linear equations.)

a)
$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 2 \\ 1 & 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 2 \\ 1 & 2 & 3 & -1 \end{pmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 2 & 2 & -2 \end{pmatrix} \xrightarrow{R_3+2R_2} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$s \quad t$

$b = -s + t$
$a = -s - t$

$$s \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$W^{\perp} = \{ s(-1,-1,1,0) + t(-1,1,0,1) \mid s,t \in \mathbb{R} \}$

b) Let $\{w_1, \dots, w_k\}$ be a basis for W

$$u \in W^{\perp} \iff \begin{cases} w_1 \cdot u = 0 \\ \vdots \\ w_k \cdot u = 0 \end{cases} \iff \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} u^T = 0$$

So $W^{\perp}$ is a solution set of a homogeneous systems of linear equations.
Hence $W^{\perp}$ is a subspace of $\mathbb{R}^n$

12. Use Gram-Schmidt Process to transform each of the following bases for $\mathbb{R}^3$ to an orthonormal basis.

(a) $\{(1,0,1),(0,1,2),(2,1,0)\}$.

(b) $\{(1,1,1),(1,-1,1),(1,1,-1)\}$.

$V_1 = u_1 = (1,0,1)$

$V_2 = u_2 - \dfrac{u_2 \cdot V_1}{V_1 \cdot V_1} V_1 = (0,1,2) - \dfrac{2}{2}(1,0,1) = (-1,1,1)$

$$V_3 = u_3 - \frac{u_3 \cdot V_1}{V_1 \cdot V_1} V_1 - \frac{u_3 \cdot V_2}{V_2 \cdot V_2} V_2$$

$$= (2,1,0) - \frac{2}{2}(1,0,1) - \frac{-2+1}{3}(-1,1,1)$$

$$= (2,1,0) - (1,0,1) + \frac{1}{3}(-1,1,1)$$

$$= (1,1,-1) + \frac{1}{3}(-1,1,1)$$

$$= \left(1 - \frac{1}{3},\ 1 + \frac{1}{3},\ -1 + \frac{1}{3}\right) = \left(\frac{2}{3}, \frac{4}{3}, -\frac{2}{3}\right) = \frac{3}{2}(1,2,-1)$$

$$\left\{ \frac{1}{\sqrt{2}}(1,0,1),\ \frac{1}{\sqrt{3}}(-1,1,1),\ \frac{1}{\sqrt{6}}(1,2,-1) \right\}$$

13. Use Gram-Schmidt Process to transform the following basis for $\mathbb{R}^4$ to an orthonormal basis:

$$\{(2,1,0,0),(-1,0,0,1),(2,0,-1,1),(0,0,1,1)\}.$$

$v_1 = (2,1,0,0)$

$v_2 = (-1,0,0,1) - \dfrac{(-1,0,0,1)\cdot(2,1,0,0)}{4+1}(2,1,0,0)$

$= (-1,0,0,1) - -\frac{2}{5}(2,1,0,0)$

$= (-1,0,0,1) + \left(\frac{4}{5}, \frac{2}{5}, 0, 0\right)$

$= \left(-\frac{1}{5}, \frac{2}{5}, 0, 1\right)$

$v_2 = (-1, 2, 0, 5)$

$v_3 = (2,0,-1,1) - \dfrac{(2,0,-1,1)\cdot(2,1,0,0)}{5}(2,1,0,0) - \dfrac{(2,0,-1,1)\cdot(-1,2,0,5)}{1+4+25}(-1,2,0,5)$

$= (2,0,-1,1) - \frac{4}{5}(2,1,0,0) - \dfrac{-2+5}{30}(-1,2,0,5)$

$= (2,0,-1,1) - \left(\frac{8}{5}, \frac{4}{5}, 0, 0\right) - \left(-\frac{1}{10}, \frac{1}{5}, 0, \frac{1}{2}\right)$

$(20, 0, -10, 10) - (16, 8, 0, 0) - (-1, 2, 0, 5)$

$(5, -10, -10, 5)$

$v_3 = (1, -2, -2, 1)$

$v_4 = (-1, -2, 3, 1)$

$\left\{ \frac{1}{\sqrt{5}}(2,1,0,0), \frac{1}{\sqrt{30}}(-1,2,0,5), \frac{1}{\sqrt{10}}(1,-2,-2,1), \frac{1}{\sqrt{15}}(1,-2,3,1) \right\}$

19. (All vectors in this question are written as column vectors.) Let $\boldsymbol{A}$ be a square matrix of order n such that $\boldsymbol{A}^2 = \boldsymbol{A}$ and $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$.

(a) For any two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, show that $(\boldsymbol{Au})\cdot\boldsymbol{v} = \boldsymbol{u}\cdot(\boldsymbol{Av})$.

(b) For any vector $\boldsymbol{w} \in \mathbb{R}^n$, show that $\boldsymbol{Aw}$ is the projection of $\boldsymbol{w}$ onto the subspace $V = \{\boldsymbol{u} \in \mathbb{R}^n \mid \boldsymbol{Au} = \boldsymbol{u}\}$ of $\mathbb{R}^n$.

a) $(Au)\cdot v = (Au)^T v = u^T A^T v = u^T A v = u\cdot(Av)$

b) Since $A(Aw) = A^2 w = Aw$, $Aw \in V$

Let $v = w - Aw$. For any $u \in V$

$u\cdot v = u\cdot w - u\cdot(Aw) = u\cdot w - (Au)\cdot w = u\cdot w - u\cdot w = 0$

So v is orthogonal to V.

Since we can write $w = Aw + v$ where $Aw \in V$ and v is orthogonal to V, Aw is the projection of w onto V.

23. A father wishes to distribute an amount of money among his three sons Jack, Jim and John.

(a) Show that it is not possible to have a distribution such that the following conditions are all satisfied.

(i) The amount Jack receives plus twice the amount Jim receives is \$300.

(ii) The amount Jim receives plus the amount John receives is \$300.

(iii) Jack receives \$300 more than twice of what John receives.

(b) Since there is no solution to the distribution problem above, find a least squares solution. (Make sure that your least squares solution is feasible. For example, one cannot give a negative amount of money to anybody.)

a) Jack = a , Jim = b, John = c

$a + 2b = 300$

$b + c = 300$

$a = 300 + 2c$

$$\begin{cases} a + 2b = 300 \\ b + c = 300 \\ a - 2c = 300 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 300 \\ 300 \\ 300 \end{pmatrix}$$

The determinant is $\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = -2 + 2 = 0$

hence the system is inconsistent

b) Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix}$, then $A^T = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 300 \\ 300 \\ 300 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 2 & -2 & 600 \\ 2 & 5 & 1 & 900 \\ -2 & 1 & 5 & -300 \end{array}\right)$$

$$\xrightarrow[R_2 - R_1]{R_3 + R_1} \left(\begin{array}{ccc|c} 2 & 2 & -2 & 600 \\ 0 & 3 & 3 & 300 \\ 0 & 3 & 3 & 300 \end{array}\right)$$

$$\xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 300 \\ 0 & 3 & 3 & 300 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

t

$c = t$

$3b = 300 - 3t \qquad b = 100 - t$

$a = 300 - 100 + t + t = 200 + 2t$

$\therefore$ $a = 200 + 2t$

$b = 100 - t$

$c = t$

27. (a) Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$.

(i) Solve the linear system $\boldsymbol{Ax} = \boldsymbol{b}$.

(ii) Find the least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

(b) Suppose a linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent. Show that the solution set of $\boldsymbol{Ax} = \boldsymbol{b}$ is equal to the solution set of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Ax} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$.

(*Hint*: You need Theorem 4.3.6 and the result of Question 4.25(a).)

a)
$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 0 & 2 \end{array}\right) \xrightarrow[R_3-R_1]{R_2-R_1} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{array}\right) \xrightarrow{R_3+R_2} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1-R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

$$x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

b) Let v be a solution of $Ax=b$, ie. $Av=b$. Since $A^TAv = A^Tb$, v is also the solution of $A^TAx = A^Tb$. Then

$$\begin{aligned} \text{the solution set of } (Ax=b) &= \{u+v \mid u \in \text{the nullspace of } A\} \\ &= \{u+v \mid u \in \text{the null space of } A^TA\} \\ &= \text{the solution set of } (A^TAx = A^Tb) \end{aligned}$$