Solutions to Tutorial 7

MA1521 Calculus for Computing

1. Let $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

   Then $\mathbf{u} = \text{proj}_{\mathbf{w}}\mathbf{a}$ is parallel to $\mathbf{w}$

   and $\mathbf{v} = \mathbf{a} - \text{proj}_{\mathbf{w}}\mathbf{a}$ is perpendicular to $\mathbf{w}$

   and $\mathbf{a} = \mathbf{u} + \mathbf{v}$.

   We compute
   $$\mathbf{u} = \frac{\mathbf{a}\cdot\mathbf{w}}{||\mathbf{w}||^2}\mathbf{w} = \frac{3+6+4}{1+9+16}(\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = \frac{1}{2}(\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$$
   and
   $$\mathbf{v} = \mathbf{a} - \mathbf{u} = (3\mathbf{i} + 2\mathbf{j} + \mathbf{k}) - \frac{1}{2}(\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = \frac{1}{2}(5\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$

2. The line ℓ is perpendicular to lines ℓ_1 and ℓ_2. Therefore it is perpendicular to their direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Hence the cross product of $\mathbf{v}_1$ and $\mathbf{v}_2$ gives the direction vector of the line ℓ. Note that $\mathbf{v}_1 = \langle 3, -4, 4\rangle$ and $\mathbf{v}_2 = \langle 3, -1, 5\rangle$. So, a direction vector of ℓ is the cross product $\mathbf{v}_1 \times \mathbf{v}_2 = \langle -16, -3, 9\rangle$. Thus a parametric equation for ℓ through the point $P(1, -2, 3)$ and parallel to $\langle -16, -3, 9\rangle$ is given by: $(x, y, z) = (1, -2, 3) + t(-16, -3, 9)$.

3. Let $A(x, y, z)$ be the point of intersection of the two lines. Set $2t - 1 = 4s + 7$, $-3t + 2 = 2s - 2$ and $4t - 3 = -3s + 2$. Solving for t and s, we get $t = 2$ and $s = -1$. For ℓ_1, $t = 2$ and we get $A(x, y, z) = (3, -4, 5)$.

   A normal vector of the plane is given by $\langle 2, 3, 4\rangle \times \langle 4, 2, -3\rangle = \langle 1, 22, 16\rangle$. Therefore, the equation of the plane is $x + 22y + 16 = -5$.

4. (a) $\overrightarrow{AB} = (3\mathbf{i} + 0\mathbf{j} + \mathbf{k}) - (3\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}) = 0\mathbf{i} - 3\mathbf{j} + \mathbf{k}$,

   and $\overrightarrow{AC} = (0\mathbf{i} + 2\mathbf{j} + \mathbf{k}) - (3\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}) = -3\mathbf{i} - \mathbf{j} + 1\mathbf{k}$.

   A vector normal to the plane is $\overrightarrow{AB} \times \overrightarrow{AC} = -2\mathbf{i} - 3\mathbf{j} - 9\mathbf{k}$.

   An equation of the plane is given by
   $$\begin{aligned} -2x - 3y - 9z &= -2\cdot 0 - 3\cdot 2 - 9\cdot 1 \\ \Longrightarrow \quad 2x + 3y + 9z &= 15. \end{aligned}$$

   (b) The distance is given by
   $$\frac{|2\cdot 0 + 3\cdot 0 + 9\cdot 0 - 15|}{\sqrt{(2)^2 + (3)^2 + (9)^2}} = \frac{15}{\sqrt{94}}.$$

(c) $\overrightarrow{OD} = 4\mathbf{i}+2\mathbf{j}+\mathbf{k}$. Parametric equations of the line which contains the line segment OD are given by

$$(*) \qquad x = 4t, \quad y = 2t, \quad z = t.$$

Hence at the point of intersection (of the line and the plane) we have

$$2(4t)+3(2t)+9t = 15 \Rightarrow t = \frac{15}{23}.$$

By (*), the point of intersection is $\frac{15}{23}(4,\ 2,\ 1)$.

5. Note that two non-parallel planes will intersect (in a straight line), so that the shortest distance between them is 0. In this question, Π_1 and Π_2 are parallel because the vector $2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ (or the vector $4\mathbf{i}+4\mathbf{j}-2\mathbf{k}$) is perpendicular to them.

   Choose a point P on Π_1 and find the distance from P to Π_2.

   In Π_1 : $2x+2y-z=1$, let $x=1$, $y=0$ to obtain $z=1$. Thus, $P(1,0,1)$ lies on Π_1. The distance from the point (x_0,y_0,z_0) to the plane $ax+by+cz=d$ is

$$\frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}.$$

   Thus,

$$\begin{aligned}\text{distance between } \Pi_1 \text{ and } \Pi_2 &= \text{distance from } P \text{ to } \Pi_2\\ &= \frac{|(4)(1)+(4)(0)+(-2)(1)-5|}{\sqrt{4^2+4^2+(-2)^2}}\\ &= \frac{1}{2}.\end{aligned}$$

6. For particles to collide, we equate the two vector functions using the *same* parameter t:

$$\mathbf{r}_1(t) = \mathbf{r}_2(t).$$

   Equating the three components, we get

$$t = 1+2t, \quad t^2 = 1+6t, \quad t^3 = 1+14t.$$

   This system does not have solutions. For example, from the first equation, we get $t=-1$. But $t=-1$ does not satisfy the other two equations.

   So, we conclude that the 2 particles do not collide.

For the path to intersect, we equate the two vector functions using *different* parameters s and t:

$$\mathbf{r}_1(t) = \mathbf{r}_2(s).$$

Equating the components, we get

$$t = 1 + 2s, \quad t^2 = 1 + 6s, \quad t^3 = 1 + 14s.$$

This system has a solution $t = 2, s = 1/2$.

So, we conclude that the 2 paths intersect.

7. $\lim\limits_{x\to 0} \frac{\|\boldsymbol{A}+x\boldsymbol{B}\|-\|\boldsymbol{A}\|}{x} = \lim\limits_{x\to 0} \frac{(\|\boldsymbol{A}+x\boldsymbol{B}\|-\|\boldsymbol{A}\|)(\|\boldsymbol{A}+x\boldsymbol{B}\|+\|\boldsymbol{A}\|)}{x(\|\boldsymbol{A}+x\boldsymbol{B}\|+\|\boldsymbol{A}\|)}$
$= \lim\limits_{x\to 0} \frac{\|\boldsymbol{A}+x\boldsymbol{B}\|^2-\|\boldsymbol{A}\|^2}{x(\|\boldsymbol{A}+x\boldsymbol{B}\|+\|\boldsymbol{A}\|)} = \lim\limits_{x\to 0} \frac{\boldsymbol{A}\cdot\boldsymbol{A}+2x\boldsymbol{A}\cdot\boldsymbol{B}+x^2\boldsymbol{B}\cdot\boldsymbol{B}-\boldsymbol{A}\cdot\boldsymbol{A}}{x(\|\boldsymbol{A}+x\boldsymbol{B}\|+\|\boldsymbol{A}\|)}$
$= \lim\limits_{x\to 0} \frac{2\boldsymbol{A}\cdot\boldsymbol{B}+x\boldsymbol{B}\cdot\boldsymbol{B}}{(\|\boldsymbol{A}+x\boldsymbol{B}\|+\|\boldsymbol{A}\|)} = \frac{2\boldsymbol{A}\cdot\boldsymbol{B}}{2\|\boldsymbol{A}\|} = \|\boldsymbol{B}\|\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}.$

   (2nd Solution). By the definition of derivative, $\frac{d}{dx}\|\boldsymbol{A}+x\boldsymbol{B}\|\Big|_{x=0} = \lim\limits_{x\to 0}\frac{\|\boldsymbol{A}+x\boldsymbol{B}\|-\|\boldsymbol{B}\|}{x}$. So we consider the function $\|\boldsymbol{A}+x\boldsymbol{B}\|$ and proceed to find its derivative at $x = 0$.

   Let $f(x) = \|\boldsymbol{A}+x\boldsymbol{B}\| = ((\boldsymbol{A}+x\boldsymbol{B})\cdot(\boldsymbol{A}+x\boldsymbol{B}))^{\frac{1}{2}} = (\boldsymbol{A}\cdot\boldsymbol{A}+2x\boldsymbol{A}\cdot\boldsymbol{B}+x^2\boldsymbol{A}\cdot\boldsymbol{A})^{\frac{1}{2}} = (\|\boldsymbol{A}\|^2+2x\boldsymbol{A}\cdot\boldsymbol{B}+x^2\|\boldsymbol{A}\|^2)^{\frac{1}{2}}$.

   Therefore, $f'(x) = \frac{2\boldsymbol{A}\cdot\boldsymbol{B}+2x\|\boldsymbol{A}\|^2}{2(\|\boldsymbol{A}\|^2+2x\boldsymbol{A}\cdot\boldsymbol{B}+x^2\|\boldsymbol{A}\|^2)^{\frac{1}{2}}}$. Thus $f'(0) = \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\|\boldsymbol{A}\|} = \|\boldsymbol{B}\|\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

Solutions to Further Exercises

1. A vector along the line is given by $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. Hence the parametric equations of the line are $x = 3t, y = 1 - t, z = 2 - 2t$.

2. Consider the equations $x = y$ and $x + 1 = y/2$ for the first and the second lines respectively. The solution to $x = y$ and $x + 1 = y/2$ is $x = y = -2$, Hence by the equation of the first line, $z = -2$, but this does not satisfy the equation $y/2 = z/3$ for the second line. Therefore the two lines do not intersect. Clearly they are not parallel as their direction numbers are not proportional. That is they form a pair of skew lines.

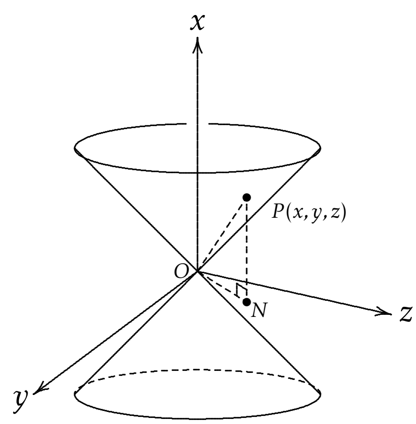
   A vector perpendicular to both of the lines is $\mathbf{n} = \langle 1,1,1\rangle \times \langle 1,2,3\rangle = \langle 1,-2,1\rangle$. Pick any point on each of the lines, say $P_1(-2,-2,-2)$ and $P_2(-2,-2,-3)$, and form the vector $\mathbf{b} = \boldsymbol{P_1P_2} = \langle 0,0,-1\rangle$. The distance between the two skew lines is given by the absolute value of the scalar projection of $\boldsymbol{P_1P_2}$ along $\mathbf{n}$, that is,

$$\frac{|\mathbf{n}\cdot\mathbf{b}|}{|\mathbf{n}|} = \frac{|1\cdot 0 - 2\cdot 0 - 1\cdot 1|}{\sqrt{1+4+1}} = \frac{1}{\sqrt{6}}.$$

Alternatively, $x-2y+z=0$ and $x-2y+z+1=0$ are the equations of the two parallel planes containing the lines $x=t, y=t, z=t$ and $x=s-1, y=2s, z=3s$ respectively. Then just find the distance from a point on one of the planes to the other plane. For example the distance from $(0,0,0)$ to the plane $x-2y+z+1=0$ is $\frac{1}{\sqrt{6}}$.

We may also find the point Q_1 on the line $\ell_1: x=t, y=t, z=t$ and the point Q_2 on the line $\ell_2: x=s-1, y=2s, z=3s$ such that the distance between Q_1 and Q_2 is $\frac{1}{\sqrt{6}}$. Let $Q_1=(t,t,t)$ and $Q_2=(-1+s,2s,3s)$. Then $\mathbf{Q_2Q_1}=\langle t+1-s, t-2s, t-3s\rangle$. We have $\langle t+1-s, t-2s, t-3s\rangle\cdot\langle 1,1,1\rangle=0$ and $\langle t+1-s, t-2s, t-3s\rangle\cdot\langle 1,2,3\rangle=0$. That is $3t-6s+1=0, 6t-14s+1=0$. Solving for t and s, we obtain $s=-\frac{1}{2}, t=-\frac{4}{3}$. Thus $Q_1=(-\frac{4}{3},-\frac{4}{3},-\frac{4}{3}), Q_2=(-\frac{3}{2},-1,-\frac{3}{2})$. Then $Q_1Q_2=\sqrt{(-\frac{4}{3}+\frac{3}{2})^2+(-\frac{4}{3}+1)^2+(-\frac{4}{3}+\frac{3}{2})^2}=\sqrt{\frac{1}{36}+\frac{1}{9}+\frac{1}{36}}=\frac{1}{\sqrt{6}}$.

3. The surface so generated is a double cone. Let $P(x,y,z)$ be a point on the double cone. Let N be the foot of perpendicular from P onto the y-z plane. Then PN/ON is the slope 3 of the line $3y=x$. Hence, $\frac{|x|}{\sqrt{y^2+z^2}}=3$. That is $x^2-9y^2-9z^2=0$.



A double cone

There is a general formula for the equation of surface of revolution as follow: Let $y=f(x)$ be a continuous function. We rotate the graph of $f(x)$ through 360° about the x-axis to get a surface. The equation of that surface is:

$$\sqrt{y^2+z^2}=|f(x)| \text{ or } y^2+z^2=f(x)^2.$$

In our case, $f(x)=x/3$. Thus we get the equation $y^2+z^2=x^2/9$.