CS1231S: Discrete Structures

Tutorial #2: Logic of Quantified Statements (Predicate Logic)

Answers

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1. For each of the following statements, write its **converse**, **inverse** and **contrapositive**. Indicate which among the statement, its converse, its inverse, and its contrapositive are true and which are false. Give a counterexample for each that is false. Proof not required if it is true. The predicate $Even(n)$ means that n is an even integer.

   a. $\forall n \in \mathbb{Z}\ (6|n \rightarrow 2|n \wedge 3|n)$. (Note: "$d|n$" is as defined in D1.)

   b. $\forall x\ (x \in \mathbb{Q} \rightarrow x \in \mathbb{Z})$.

   c. $\forall p, q \in \mathbb{Z}\ \big(Even(p) \wedge Even(q) \rightarrow Even(p+q)\big)$.

   Answers:

   a. Statement: $\forall n \in \mathbb{Z}\ (6|n \rightarrow 2|n \wedge 3|n)$. (True)
      Converse: $\forall n \in \mathbb{Z}\ (2|n \wedge 3|n \rightarrow 6|n)$. (True)
      Inverse: $\forall n \in \mathbb{Z}\ (6 \nmid n \rightarrow 2 \nmid n \vee 3 \nmid n)$. (True)
      Contrapositive: $\forall n \in \mathbb{Z}\ (2 \nmid n \vee 3 \nmid n \rightarrow 6 \nmid n)$. (True)

   b. Statement: $\forall x\ (x \in \mathbb{Q} \rightarrow x \in \mathbb{Z})$. (False)
      Converse: $\forall x\ (x \in \mathbb{Z} \rightarrow x \in \mathbb{Q})$. (True)
      Inverse: $\forall x\ (x \notin \mathbb{Q} \rightarrow x \notin \mathbb{Z})$. (True)
      Contrapositive: $\forall x\ (x \notin \mathbb{Z} \rightarrow x \notin \mathbb{Q})$. (False)
      Counterexample for original statement and contrapositive: Let $x = 1/2$.

   c. Statement: $\forall p, q \in \mathbb{Z}\ \big(Even(p) \wedge Even(q) \rightarrow Even(p+q)\big)$. (True)
      Converse: $\forall p, q \in \mathbb{Z}\ \big(Even(p+q) \rightarrow Even(p) \wedge Even(q)\big)$. (False)
      Inverse: $\forall p, q \in \mathbb{Z}\ \big(\sim Even(p) \vee \sim Even(q) \rightarrow \sim Even(p+q)\big)$. (False)
      Contrapositive: $\forall p, q \in \mathbb{Z}\ \big(\sim Even(p+q) \rightarrow \sim Even(p) \vee \sim Even(q)\big)$. (True)
      Counterexample for converse and inverse: Let $p = q = 1$.

2. Given the predicate $Loves(x, y)$ which means "x loves y", translate the following English sentences into predicate logic statements. Do not start your answer with negation (~).

   For this question, you may leave out the domain of discourse in your statements (i.e. you do not need to specify what domain a variable belongs to, such as $\forall x \in D$; just write $\forall x$).

   a. "Everybody loves himself or herself."

      Comment on each of the following answers. Is the answer correct? If not, why? If it is correct, can it be simplified?

      (i) $\forall p, q\ Loves(p, q)$

      (ii) $\forall p, q\ \big((p = q) \wedge Loves(p, q)\big)$

      (iii) $\forall p, q\ \big((p = q) \rightarrow Loves(p, q)\big)$

   b. "Everybody loves somebody."

      Comment on each of the following answers:

      (i) $\exists p\ \forall q\ Loves(p, q)$

      (ii) $\exists p\ \forall q\ Loves(q, p)$

      Are the answers correct? If not, give the correct answer.

   c. "Everybody loves somebody else."

      Is this sentence the same as (b)? If not, give the answer.

   d. "Nobody except John loves Mary." (or: "Only John loves Mary; nobody else loves Mary.")

Answers:

a. (i) $\forall p, q\ Loves(p, q)$: Incorrect. This means "Everybody loves everybody."

   (ii) $\forall p, q\ \big((p = q) \wedge Loves(p, q)\big)$: Incorrect. This would only be correct if there is only one person. Otherwise, this statement is saying that all people are the same person.

   (iii) Correct.

   A simpler answer would be: $\forall p\ Loves(p, p)$.

b. (i) $\exists p\ \forall q\ Loves(p, q)$: Incorrect. This means "there is a person who loves everybody."

   (ii) $\exists p\ \forall q\ Loves(q, p)$: Incorrect. This means "there is a person whom everybody loves."

   The correct answer is: $\forall p\ \exists q\ Loves(p, q)$. ("Each person has someone he/she loves.")

c. The sentence is different from (b). Answer: $\forall p\ \exists q\ \big((p \neq q) \wedge Loves(p, q)\big)$.

d. $Loves(John, Mary) \wedge \forall x\big((x \neq John) \rightarrow {\sim}Loves(x, Mary)\big)$.

3. Recall the definition of rational numbers (Lecture 1 slide 37):

$$r \text{ is rational} \Leftrightarrow \exists a, b \in \mathbb{Z} \text{ s.t. } r = \frac{a}{b} \text{ and } b \neq 0.$$

Prove or disprove the following statements:

a. Integers are closed under division.

b. Rational numbers are closed under addition.

c. Rational number are closed under division.

Answers:

a. False. Counterexample: Let $a = 3, b = 2$, then $\frac{a}{b} = 1.5$ is not an integer.

b. 1. Let r and s be rational numbers.
   2. Then $\exists a, b, c, d \in \mathbb{Z}$ s.t. $r = \frac{a}{b}, s = \frac{c}{d}$ and $b \neq 0, d \neq 0$. (by defn of rational numbers)
   3. Hence, $r + s = \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ (by basic algebra)
   4. $ad + bd \in \mathbb{Z}$ and $bd \in \mathbb{Z}$. (closure of integers under $+$ and $\times$)
   5. Moreover, $bd \neq 0$ since $b \neq 0, d \neq 0$.
   6. Hence, $r + s$ is rational. (by definition of rational numbers)
   7. Therefore, rational numbers are closed under addition.

c. False. Counterexample: Let $r, s \in \mathbb{Q}$ and $s = 0$. Then r/s is undefined.

4. Given $A = \{1,3,5,7,11,13\}$ and $B = \{0,2,4,6\}$, for each of the statements (a) to (j), explain whether the statement is true or false.

a. $\forall x \in B\ \forall y \in B\ (x - y \in B)$

b. $\forall x \in A\ \forall y \in A\ \big((x < y) \wedge (y < 10) \rightarrow y - x \in B\big)$

c. $\forall x \in A\ \forall y \in B\ (x = y + 1)$

d. $\exists x \in A\ \exists y \in B\ (x = y + 1)$

e. $\exists y \in B\ \exists x \in A\ (x = y + 1)$

f. $\forall x \in A\ \forall y \in B\ (x \neq y + 1)$

g. $\exists x \in A\ \exists y \in B\ (x \neq y + 1)$

h. $\forall x \in A\ \exists y \in B\ (x > y)$

i. $\forall x \in A\ \exists y \in B\ (x \leq y)$

j. $\exists y \in B\ \forall x \in A\ (x > y)$

Answers:

a. $\forall x \in B\ \forall y \in B\ (x - y \in B)$

False. Counterexample: $x = 2, y = 4$.

b. $\forall x \in A\ \forall y \in A\ \big((x < y) \wedge (y < 10) \rightarrow y - x \in B\big)$

True.
Case $x = 1, y = 3$: $y - x = 2 \in B$.
Case $x = 1, y = 5$: $y - x = 4 \in B$.
Case $x = 1, y = 7$: $y - x = 6 \in B$.
Case $x = 3, y = 5$: $y - x = 2 \in B$.
Case $x = 3, y = 7$: $y - x = 4 \in B$.
Case $x = 5, y = 7$: $y - x = 2 \in B$.

c. $\forall x \in A\ \forall y \in B\ (x = y + 1)$

False. Counterexample: $x = 1, y = 2$.

d. $\exists x \in A\ \exists y \in B\ (x = y + 1)$

True. Example: $x = 3, y = 2$.

e. $\exists y \in B\ \exists x \in A\ (x = y + 1)$

True. [(e) is equivalent to (d).]

f. $\forall x \in A\ \forall y \in B\ (x \neq y + 1)$

False. Counterexample: $x = 3, y = 2$. [(f) is the negation of (d).]

g. $\exists x \in A\ \exists y \in B\ (x \neq y + 1)$

True. Example: $x = 1, y = 2$. [(g) is the negation of (c).]

h. $\forall x \in A\ \exists y \in B\ (x > y)$

True. We can pick $y = 0$, so every element in A is larger than 0.

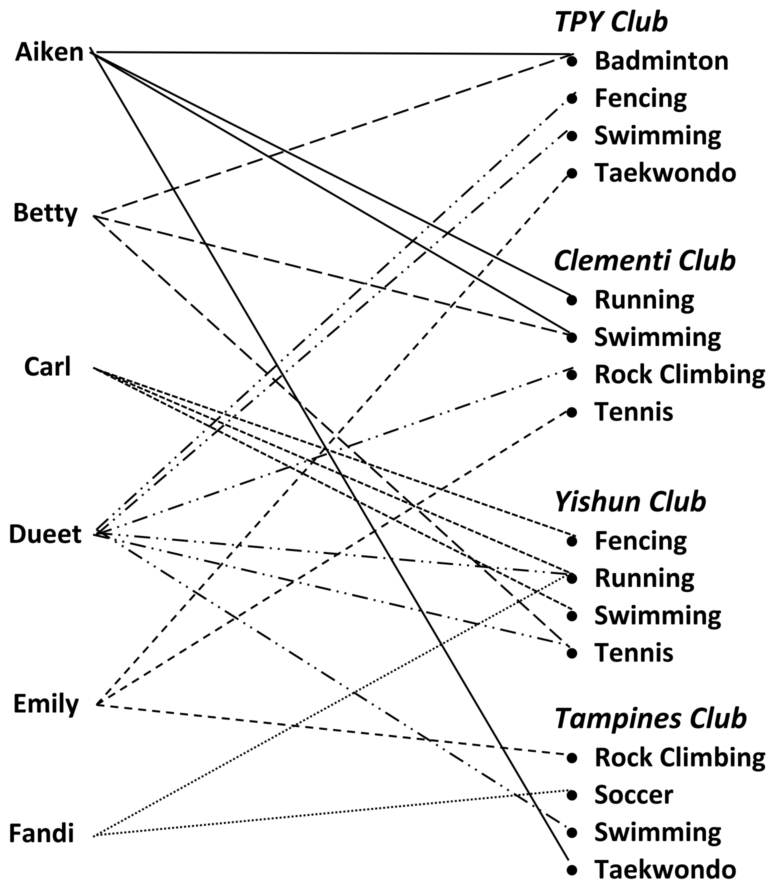
i. $\forall x \in A\ \exists y \in B\ (x \leq y)$

False. Counterexample: $x = 11$.

j. $\exists y \in B\ \forall x \in A\ (x > y)$

True. Example: $y = 0$.

5. (Midterm test AY2017/18 semester 1)
The diagram below shows the activities offered by four community clubs and the members who have subscribed to the activities at the clubs. A member can sign up for any activity at any of the clubs and a member may go for the same activity at different clubs.



For each of the following, indicate whether the statement is true or false and explain why. (You are not required to write the quantified statements.)

a. None of the members goes to a single club for all his/her activities.
b. There is a unique activity which has the most number of members participating in it.
c. There are two members who do not share any common activity.
d. If you list out all the members attending each club, there are two clubs that are attended by the same group of members.
e. All the clubs are attended by the same number of members.

Answers:

a. **False.** Counter-example: Carl goes to Yishun Club for all his activities. (To prove $\forall x \sim P(x)$, we just need to show $\exists x\ P(x)$.)

b. **False.** Both swimming and running have four members participating.

c. **True.** Betty (Badminton, Swimming, Tennis) and Fandi (Running, Soccer) do not share any common activity. So are Emily (Taekwondo, Tennis, Rock Climbing) and Fandi.

d. **True.** TPY Club and Clementi Club are attended by Aiken, Betty, Dueet and Emily.

e. **True.** Every club is attended by four members.

6. This question illustrates that one can "prove" anything, including nonsense, using bad logic.

   a. The following is a proof for: $\forall x \in \mathbb{R}\ (x^2 \geq 0)$. What is wrong with this "proof"?

      "There are 3 cases to consider: $x < 0, x = 0$ and $x > 0$. If $x < 0$, for example, $x = -3$, then $x^2 = 9 \geq 0$; if $x = 0$, then $x^2 = 0$; if $x > 0$, say $x = 4$, then $x^2 = 16 \geq 0$. Therefore, in all cases, $x^2 \geq 0$."

   b. Use the same logic in (a) to prove: $\forall x \in \mathbb{R}\ (x^3 = x)$.

   c. The following is another proof for: $\forall x \in \mathbb{R}\ (x^2 \geq 0)$. What is wrong with this "proof"?

      "Prove by contradiction. Suppose $x^2 < 0$ for all real numbers x. Let $x = 3$, then $x^2 = 9 > 0$ which is a contradiction. Therefore, $\forall x \in \mathbb{R}\ (x^2 \geq 0)$."

   d. Use the same logic in (c) to prove: $\forall x \in \mathbb{R}\ (x^3 = x)$.

Answers:

a. This looks like proof by division into cases, but it is really a "proof by examples", which is invalid for a universal statement. Error: it does not consider arbitrary real number x.

b. Claim: $\forall x \in \mathbb{R}\ (x^3 = x)$.

   "There are 3 cases to consider: $x < 0, x = 0$ and $x > 0$. If $x < 0$, for example, $x = -1$, then $x^3 = -1 = x$; if $x = 0$, then $x^3 = 0 = x$; if $x > 0$, say $x = 1$, then $x^3 = 1 = x$. Therefore, in all cases, $x^3 = x$."

c. To prove by contradiction, one must consider $\sim\big(\forall x \in \mathbb{R}\ (x^2 \geq 0)\big)$, which is $\exists x \in \mathbb{R} \sim(x^2 \geq 0)$. What the proof considers is $\forall x \in \mathbb{R} \sim(x^2 \geq 0)$.

d. Claim: $\forall x \in \mathbb{R}\ (x^3 = x)$.

   "Prove by contradiction. Suppose $x^3 \neq x$ for all real numbers x. Let $x = 0$, then $x^2 = 0 = x$, which is a contradiction. Therefore, $\forall x \in \mathbb{R}\ (x^3 = x)$."

7. The following is a partial proof of the claim:

$$\forall x \in \mathbb{R}\left((x^2 > x) \rightarrow (x < 0) \vee (x > 1)\right).$$

> 1. Let r be an arbitrarily chosen real number.
> 2. Suppose $r^2 > r$.
>   2.1. Then $r^2 - r > 0$, or $r(r-1) > 0$. (by basic algebra)
>   2.2. So, both r and $r-1$ are positive, or both are negative. (by Appendix A, T25)
>   2.3. …
> 3. Therefore, $\forall x \in \mathbb{R}\left((x^2 > x) \rightarrow (x < 0) \vee (x > 1)\right)$.

Note: Some students drew diagrams (eg: graphs) and used them as proofs. In this module, do not use diagrams for proofs, unless otherwise instructed. If you use diagrams, you need to explain the diagrams.

a. In step 2, we explore the case $r^2 > r$. Do we need to include the case $r^2 \leq r$? Why?

b. Complete the proof.

c. Step 3 is an application of **universal generalization**. Explain what it means.

Answers:

a. A conditional statement $p \rightarrow q$ is always true when the hypotheses/antecedent p is false, therefore, there is no need to examine the case $r^2 \leq r$.

b.

1. Let r be an arbitrarily chosen real number.
2. Suppose $r^2 > r$.
   2.1. Then $r^2 - r > 0$, or $r(r-1) > 0$. (by basic algebra)
   2.2. So, both r and $r-1$ are positive, or both are negative. (by Appendix A, T25)
   2.3. Case 1: Both r and $r-1$ are positive.
        2.3.1. $(r > 0) \wedge (r - 1 > 0) \rightarrow (r > 1)$.
   2.4. Case 2: Both r and $r-1$ are negative.
        2.4.1 $(r < 0) \wedge (r - 1 < 0) \rightarrow (r < 0)$.
   2.5. From lines 2.3.1 and 2.4.1, we have $(r > 1) \vee (r < 0)$.
   2.6. Therefore, $(r^2 > r) \rightarrow (r > 1) \vee (r < 0)$.
3. Therefore, $\forall x \in \mathbb{R}\left((x^2 > x) \rightarrow (x < 0) \vee (x > 1)\right)$.

c. Let predicates $P(x)$ be "$x^2 > x$" and $Q(x)$ be "$(x < 0) \vee (x > 1)$". Since we have established that for an arbitrarily chosen real number r, $P(r) \rightarrow Q(r)$, we may generalize it to all real numbers, i.e., $\forall x \in \mathbb{R}\left(P(x) \rightarrow Q(x)\right)$.

8. Let V be the set of all visitors to Universal Studios Singapore on a certain day, $T(v)$ be "v took the Transformers ride", $G(v)$ be "v took the Battlestar Galactica ride", $E(v)$ be "v visited the Ancient Egypt", and $W(v)$ be "v watched the Water World show".

   Express each of the following statements using quantifiers, variables, and the predicates $T(v)$, $G(v)$, $E(v)$ and $W(v)$. The statements are not related to one another. Part (a) has been done for you.

   a. Every visitor watched the Water World show.

Answer for (a): $\forall v \in V\ (W(v))$.

   b. Every visitor who took the Battlestar Galactica ride also took the Transformers ride.

   c. There is a visitor who took both the Transformers ride and the Battlestar Galactica ride.

   d. No visitor who visited the Ancient Egypt watched the Water World show.

   e. Some visitors who took the Transformers ride also visited the Ancient Egypt but some (who took the Transformers ride) did not (visit the Ancient Egypt).

Answers

b. $\forall v \in V\ (G(v) \rightarrow T(v))$

c. $\exists v \in V\ (T(v) \wedge G(v))$

d. $\forall v \in V\ (E(v) \rightarrow \sim W(v))$
   Alternatively: $\forall v \in V\ (\sim E(v) \vee \sim W(v))$

e. $\big(\exists v \in V\ (T(v) \wedge E(v))\big) \wedge \big(\exists u \in V\ (T(u) \wedge \sim E(u))\big)$

   Note that the above is a conjunction (AND) of two existential clauses. Each existential variable v or u has a scope only within its own clause. Therefore, there is no ambiguity to use the same variable in both clauses, as shown below:

   $$\big(\exists v \in V\ (T(v) \wedge E(v))\big) \wedge \big(\exists v \in V\ (T(v) \wedge \sim E(v))\big)$$

   Nevertheless, you may use different variable names to avoid confusion.

   Also, note that it is incorrect to write

   $$\exists v \in V\ \big((T(v) \wedge E(v)) \wedge (T(v) \wedge \sim E(v))\big)$$

   because the scope of v here covers both $(T(v) \wedge E(v))$ as well as $(T(v) \wedge \sim E(v))$. This means that if there exists a visitor who took the Transformers Ride and visited Ancient Egypt, the <u>same visitor</u> took the Transformers Ride but did not visit Ancient Egypt! The statement becomes false (by applying commutativity and associativity of conjunction).

9. Given the following argument:

> 1. If an object is above all the triangles, then it is above all the blue objects.
> 2. If an object is not above all the gray objects, then it is not a square.
> 3. Every black object is a square.
> 4. Every object that is above all the gray objects is above all the triangles.
>
> ∴ If an object is black, then it is above all the blue object.

a. Reorder the premises in the argument to show that the conclusion follows as a valid consequence from the premises, by applying universal transitivity (Lecture 3 slide 96). (Hint: It may be helpful to rewrite the statements in if-then form and replace some statements by their contrapositives.)

   You may use self-explanatory predicate names such as $Triangle(x)$, $Square(x)$, etc.

b. Rewrite your answer in part (a) using predicates and quantified statements.

Answers:

a.

3. If an object is black, then it is a square.
2. (Contrapositive form) If an object is a square, then it is above all the gray objects.
4. If an object is above all the gray objects, then it is above all the triangles.
1. If an object is above all the triangles, then it is above all the blue objects.

∴ If an object is black, then it is above all the blue objects.

b.

Let O, the domain, be the set of objects.

3. $\forall x \in O\ \big(Black(x) \to Square(x)\big)$.
2. (Contrapositive form) $\forall x \in O\ \Big(Square(x) \to \big(\ \forall y \in O(Gray(y) \to Above(x,y))\big)\Big)$.
4. $\forall x \in O\ \Big(\big(\forall y \in O\ (Gray(y) \to Above(x,y))\big) \to \big(\forall z \in O\ (Triangle(z) \to Above(x,z))\big)\Big)$.
1. $\forall x \in O\ \Big(\big(\forall z \in O\ (Triangle(z) \to Above(x,z))\big) \to \big(\forall w \in O\ (Blue(w) \to Above(x,w))\big)\Big)$.

∴ $\forall x \in O\ \Big(Black(x) \to \big(\ \forall w \in O\ (Blue(w) \to Above(x,w))\big)\Big)$.

10. [Past year's midterm test question]

Prove that if n is a product of two positive integers a and b, then $a \leq n^{1/2}$ or $b \leq n^{1/2}$.

Answers:

Proof by contraposition. (Recall: $p \rightarrow q \equiv \sim q \rightarrow \sim p$)

1. The contrapositive of the given statement is:

   If $a > n^{1/2}$ and $b > n^{1/2}$, then n is not a product of a and b. (by De Morgan's law)
2. Suppose $a > n^{1/2}$ and $b > n^{1/2}$, then $ab > n^{1/2} \cdot n^{1/2} = n$. (by Appendix A T27)
3. Since $ab \neq n$, the contrapositive statement is true.
4. Therefore, the original statement is true.

What if students choose to use proof by contradiction? (Recall: $\sim(p \rightarrow q) \equiv p \wedge \sim q$)

1. Suppose not (that is, taking the negation of the given statement), we have

   $n = ab$ and $a > n^{1/2}$ and $b > n^{1/2}$. (by De Morgan's law)
2. Since $a > n^{1/2}$ and $b > n^{1/2}$, we have $ab > n^{1/2} \cdot n^{1/2} = n$. (by Appendix A T27)
3. This contradicts $n = ab$.
4. Therefore, the original statement is true.

Note: Appendix A is given in Canvas as well as CS1231S "Lectures" page.

Appendix A T27.

$$\forall a, b, c \in \mathbb{R}, \text{ if } 0 < a < c \text{ and } 0 < b < d, \text{ then } 0 < ab < cd.$$