CS1231S: Discrete Structures

Tutorial #3: Sets

(Week 5: 5 – 9 September 2022)

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1. Discussion Questions

These are meant for discussion on Canvas. No answers will be provided.

D1 Which of the following are true? Which are false?

(a) $\emptyset \in \emptyset$.

(b) $\emptyset \subseteq \emptyset$.

(c) $\emptyset \in \{\emptyset\}$.

(d) $\emptyset \subseteq \{\emptyset\}$.

(e) $\{\emptyset, 1\} = \{1\}$.

(f) $1 \in \{\{1,2\}, \{2,3\}, 4\}$.

(g) $\{1,2\} \subseteq \{3,2,1\}$.

(h) $\{3,3,2\} \subsetneq \{3,2,1\}$.

D2. Let $A = \{1, \{1,2\}, 2, \{2,1,1\}\}$. Find $|A|$.

D3. Let $A = \{0,1,4,5,6,9\}$ and $B = \{0,2,4,6,8\}$. Find $|A \cap B|$ and $|A \cup B|$.

2. Tutorial Questions

Note that the sets here are finite sets, unless otherwise stated.

1. Let $\mathcal{P}(A)$ denotes the power set of A. Find the following:

   a. $\mathcal{P}(\{a, b, c\})$;

   b. $\mathcal{P}\left(\mathcal{P}\left(\mathcal{P}(\emptyset)\right)\right)$.

2. Let $A = \{5,6,7, \ldots, 12\}$. Find the following:

   a. $\{n \in A : n \text{ is even}\}$;

   b. $\{n \in A : n = m^2 \text{ for some } m \in \mathbb{Z}\}$;

   c. $\{-5, -4, -3, \ldots, 5\} \setminus \{1,2,3, \ldots, 10\}$;

   d. $\overline{\{5,7,9\} \cup \{9,11\}}$, where A is considered the universal set;

   e. $\{(x, y) \in \{1,3,5\} \times \{2,4\} : x + y \geq 6\}$.

3. (Past year's exam question.)
   Denote by $|n|$ the absolute value of the integer n, i.e.,

$$|n| = \begin{cases} n, & \text{if } n \geq 0; \\ -n, & \text{if } n < 0. \end{cases}$$

   Given the set $S = \{-9, -6, 1, 3, 5, 8\}$, for each of the following statements, state whether it is true or false, with explanation.

   a. $\exists z \in S\ \forall x, y \in S\ z > |x - y|$.
   b. $\exists z \in S\ \forall x, y \in S\ z < |x - y|$.

4. Let $A = \{2n + 1 : n \in \mathbb{Z}\}$ and $B = \{2n - 5 : n \in \mathbb{Z}\}$. Is $A = B$? Prove that your answer is correct. What does this tell us about how odd numbers may be defined?

5. Using definitions of set operations (also called the **element method**), prove that for all sets A, B, C,

$$A \cap (B \setminus C) = (A \cap B) \setminus C.$$

6. (Past year's midterm test question.)
   Using **set identities** (Theorem 6.2.2), prove that for all sets A, B and C,

$$A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C).$$

7. For sets A and B, define $A \oplus B = (A \setminus B) \cup (B \setminus A)$.

   a. Let $A = \{1,4,9,16\}$ and $B = \{2,4,6,8,10,12,14,16\}$. Find $A \oplus B$.
   b. Using set identities (Theorem 6.2.2), prove that for all sets A and B,

$$A \oplus B = (A \cup B) \setminus (A \cap B).$$

8. Let A and B be set. Show that $A \subseteq B$ if and only if $A \cup B = B$.

9. Hogwarts School of Witchcraft and Wizardry where Harry Potter attended was divided into 4 houses: Gryffindor, Hufflepuff, Ravenclaw and Slytherin.

   Let $HSWW$ be the set of students in the Hogwarts School of Witchcraft and Wizardry, and G, H, R and S be the sets of students in the 4 houses.

   What are the necessary conditions for $\{G, H, R, S\}$ to be a partition of $HSWW$? Explain in English and the write logical statements.

For questions 10 to 12, for sets $A_m, A_{m+1}, \dots, A_n$, we define the following:

$$\bigcup_{i=m}^{n} A_i = A_m \cup A_{m+1} \cup \dots \cup A_n$$

and

$$\bigcap_{i=m}^{n} A_i = A_m \cap A_{m+1} \cap \dots \cap A_n$$

10. Let $A_i = \{x \in \mathbb{Z} : x \geq i\}$ for all integers i. Write down $\bigcup_{i=2}^{5} A_i$ and $\bigcap_{i=2}^{5} A_i$ in **roster notation**.

11. Let $V_i = \left\{x \in \mathbb{R} : -\frac{1}{i} \leq x \leq \frac{1}{i}\right\} = \left[-\frac{1}{i}, \frac{1}{i}\right]$ for all positive integers i.
    a. What is $\bigcup_{i=1}^{4} V_i$?
    b. What is $\bigcap_{i=1}^{4} V_i$?
    c. What is $\bigcup_{i=1}^{n} V_i$, where n is a positive integer?
    d. What is $\bigcap_{i=1}^{n} V_i$, where n is a positive integer?
    e. Are $V_1, V_2, V_3, \dots$ mutually disjoint?

12. Let $B_1, B_2, B_3, \dots, B_k$ and $C_1, C_2, C_3, \dots, C_l$ be sets such that

$$\bigcup_{i=1}^{k} B_i \subseteq \bigcap_{j=1}^{l} C_j$$

    Show that $B_i \subseteq C_j$ for any $i \in \{1,2, \dots, k\}$ and any $j \in \{1,2, \dots, l\}$.