32. Determine which of the following sets are bases for $\mathbb{R}^3$.

(a) $S_1 = \{(1,0,-1), (-1,2,3)\}$.

(b) $S_2 = \{(1,0,-1), (-1,2,3), (0,3,0)\}$.

(c) $S_3 = \{(1,0,-1), (-1,2,3), (0,3,3)\}$.

(d) $S_4 = \{(1,0,-1), (-1,2,3), (0,3,0), (1,-1,1)\}$.

Definition: Let $S = \{v_1 \dots v_k\}$ be a subset of a Vector $V$.
The $S$ is called a basis if $S$ is linearly independent and $\text{Span}(S) = V$

a) No $\text{Span}(S_1) \neq \mathbb{R}^3$. There are too little Vectors.

b) $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ -1 & 3 & 0 \end{pmatrix} \xrightarrow[R_3+R_1]{} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow[R_3-R_2]{} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{pmatrix}$ The System is consistent

$\therefore \text{Span}(S_2) = \mathbb{R}^3$
The System has only the trivial solution. Hence it is linearly independent.
So $S_2$ is a basis for $\mathbb{R}^3$

c) $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ -1 & 3 & 3 \end{pmatrix} \xrightarrow[R_3+R_1]{} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{pmatrix} \xrightarrow[R_3-R_2]{} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ linearly dependent.

$S_3$ is not a basis for $\mathbb{R}^3$

d) Too many vectors

33. Find a basis for the solution space of each of the following homogeneous systems.

(a) $x_1 + 3x_2 - x_3 + 2x_4 = 0$.

(b) $\begin{cases} x_1 + 3x_2 - x_3 + 2x_4 = 0 \\ -3x_2 + x_3 = 0. \end{cases}$

(c) $\begin{cases} x_1 + 3x_2 - x_3 + 2x_4 = 0 \\ -3x_2 + x_3 = 0 \\ x_1 - x_4 = 0. \end{cases}$

$$\begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & -3 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & -3 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \xrightarrow{R_3-R_1} \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & -3 & 1 & 0 \\ 0 & -3 & 1 & -3 \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$x_3 = t$, $x_4 = 0$, $x_2 = \frac{t}{3}$, $x_1 = t - t = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} 0 \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} \quad t \in \mathbb{R}$$

$\therefore \{(0, \frac{1}{3}, 1, 0)\}$ is a basis for the Solution Space.
is a set of one vector. $\therefore$ Linearly independent.

40. Let $\boldsymbol{u_1} = (1,0,1,1)$, $\boldsymbol{u_2} = (-3,3,7,1)$, $\boldsymbol{u_3} = (-1,3,9,3)$, $\boldsymbol{u_4} = (-5,3,5,-1)$ and let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ and $V = \text{span}(S)$.

(a) Find a non-trivial solution to the equation

$$a\boldsymbol{u_1} + b\boldsymbol{u_2} + c\boldsymbol{u_3} + d\boldsymbol{u_4} = \boldsymbol{0}.$$

(b) Express $\boldsymbol{u_3}$ and $\boldsymbol{u_4}$ (separately) as linear combinations of $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$.

(c) Find a basis for and determine the dimension of $V$.

(d) Find a subspace $W$ of $\mathbb{R}^4$ such that $\dim(W) = 3$ and $\dim(W \cap V) = 2$. Justify your answer.

a) $\begin{pmatrix} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 1 & 7 & 9 & 5 \\ 1 & 1 & 3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

let $c = s$, $d = t$

$b = -s-t$

$a = 2t-2s$

$(2t-2s, -s-t, s, t)$

let $s=1$, $t=0$

$a = -2$, $b=-1$, $c=1$, $d=0$

b) $\left(\begin{array}{cc|c|c} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 1 & 7 & 9 & 5 \\ 1 & 1 & 3 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c|c} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 4 & 4 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c|c} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$a = -1+3 = 2$ | $a = -5+3 = -2$

$b = 1$ | $b = 1$

$u_3 = 2u_1 + u_2$

$u_4 = -2u_1 + u_2$

c) View as column vectors

$\begin{pmatrix} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 1 & 7 & 9 & 5 \\ 1 & 1 & 3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

The corresponding pivots are $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 3 \\ 7 \\ 1 \end{pmatrix}$

which are $u_1$ and $u_2$.

$\dim(V) = 2$

d) Let $W = \text{span}\{u_1, u_2, (0,0,0,1)\}$. Then $\dim(W) = 3$. Since $W \cap V = V$, $\dim(W \cap V) = \dim(V) = 2$

44. Let $U = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $V = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ be subspaces of $\mathbb{R}^5$ such that $\dim(U \cap V) = 2$. Suppose $W$ is the smallest subspace of $\mathbb{R}^5$ that contains both $U$ and $V$. Determine all possible dimensions of $W$. Justify your answers.

$\dim(U) \leq 3$ , $\dim(V) \leq 3$

Since $\dim(U \cap V) = 2$, $\dim(U) \geq 2$ and $\dim(V) \geq 2$

Suppose $\dim(U) = 2$. Then $U \cap V = U$. As the smallest subspace that contains both U and V, we have $W = V$ and hence $\dim(W) = \dim(V) = 2$ or $3$

Suppose $\dim(V) = 2$. Then similarly, $W = U$ and hence $\dim(W) = \dim(U) = 2$ or $3$

Finally, if $\dim(U) = \dim(V) = 3$, then $\dim(W) = 3+3-2 = 4$ $(\dim(U+V) = \dim(U) + \dim(V) - d(U \cap V))$

Therefore the possible dimension of W are 2, 3 and 4.

46. (a) Let $\boldsymbol{u_1} = (1, 2, -1)$, $\boldsymbol{u_2} = (0, 2, 1)$, $\boldsymbol{u_3} = (0, -1, 3)$. Show that $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ forms a basis for $\mathbb{R}^3$.

(b) Suppose $\boldsymbol{w} = (1, 1, 1)$. Find the coordinate vector of $\boldsymbol{w}$ relative to $S$.

(c) Let $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ be another basis for $\mathbb{R}^3$ where $\boldsymbol{v_1} = (1, 5, 4)$, $\boldsymbol{v_2} = (-1, 3, 7)$, $\boldsymbol{v_3} = (2, 2, 4)$. Find the transition matrix from $T$ to $S$.

(d) Find the transition matrix from $S$ to $T$.

(e) Use the vector $\boldsymbol{w}$ in Part (b). Find the coordinate vector of $\boldsymbol{w}$ relative to $T$.

a) $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix}$

all colums are pivot colums. The system has a unique solution.
The rows form a basis for $\mathbb{R}^3$.

b) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 2 & -1 & 1 \\ -1 & 1 & 3 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & -\frac{1}{2} & \frac{1}{2} \\ -1 & 1 & 3 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 3 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{7}{2} & \frac{5}{2} \end{array}\right)$

$c = \frac{5}{2} \times \frac{2}{7} = \frac{5}{7}$, $b = -\frac{1}{7}$

$a = 1$

$(w)_S = (1, -\frac{1}{7}, \frac{5}{7})$

c) $\left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & -1 & 2 \\ 2 & 2 & -1 & 5 & 3 & 2 \\ -1 & 1 & 3 & 4 & 7 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right)$

$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

d) $\frac{1}{8}\begin{pmatrix} 6 & 4 & -6 \\ -4 & 0 & 4 \\ -1 & -2 & 5 \end{pmatrix}$ Can be found using previous method. Or the inverse of the previous matrix.

e) $(w)_S = (1, -\frac{1}{7}, \frac{5}{7})$

$$\frac{1}{8}\begin{pmatrix} 6 & 4 & -6 \\ -4 & 0 & 4 \\ -1 & -2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -\frac{1}{7} \\ \frac{5}{7} \end{pmatrix} = \frac{1}{8}\begin{bmatrix} 6 - \frac{4}{7} - \frac{30}{7} \\ -4 - \frac{4}{7} \\ -1 + \frac{2}{7} + \frac{25}{7} \end{bmatrix} = \begin{pmatrix} \frac{1}{7} \\ -\frac{1}{7} \\ \frac{5}{14} \end{pmatrix}$$

$(w)_T = (\frac{1}{7}, -\frac{1}{7}, \frac{5}{14})$

49. Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ be a basis for $\mathbb{R}^3$ and $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ where

$$\boldsymbol{v_1} = \boldsymbol{u_1} + \boldsymbol{u_2} + \boldsymbol{u_3}, \quad \boldsymbol{v_2} = \boldsymbol{u_2} + \boldsymbol{u_3} \text{ and } \boldsymbol{v_3} = \boldsymbol{u_2} - \boldsymbol{u_3}.$$

(a) Show that $T$ is a basis for $\mathbb{R}^3$.

(b) Find the transition matrix from $S$ to $T$.

a) $c_1v_1 + c_2v_2 + c_3v_3 = 0 \iff c_1u_1 + (c_1 + c_2 + c_3)u_2 + (c_1 + c_2 - c_3)u_3 = 0$

Since $u_1, u_2, u_3$ are linearly independent,

$$\begin{cases} c_1 = 0 \\ c_1 + c_2 + c_3 = 0 \\ c_1 + c_2 - c_3 = 0 \end{cases}$$

The system has only the trivial solution. So $T$ is linearly independent.
Since $\dim(T) = 3$, $T$ is a basis for $\mathbb{R}^3$.

b) $[v_1]_S = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $[v_2]_S = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $[v_3]_S = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

The transition matrix from $T$ to $S$ is $P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

and the transition matrix from $S$ to $T$ is $P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$