Student Number: ____________________

NATIONAL UNIVERSITY OF SINGAPORE

MA1101R - Linear Algebra I

(Semester 1 : AY2014/2015)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number clearly in the space provided at the top of this page.** This booklet (and only this booklet) will be collected at the end of the examination.

2. Please write your matriculation/student number only. Do not write your name.

3. This examination paper contains **SIX** questions and comprises **NINETEEN** printed pages.

4. Answer **ALL** questions.

5. This is a CLOSED BOOK (with helpsheet) examination.

6. You are allowed to use two A4 size helpsheets.

7. You may use scientific calculators. However, you should lay out systematically the various steps in the calculations)

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
5	
6	
Total	

Question 1 [15 marks]

(a) [6 marks]

Let $\boldsymbol{A}$ be a 4×5 matrix such that its row echelon form is

$$\boldsymbol{R} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) Write down a basis for the row space of $\boldsymbol{A}$.

(ii) Extend the basis found in (i) to a basis for $\mathbb{R}^5$. (Just write down the additional vectors.)

(iii) Find a basis for the nullspace of $\boldsymbol{A}$. Show your working.

Show your working below.

Question 1

(b) [6 marks]

Let

$$\boldsymbol{B} = \begin{pmatrix} x & x(x-1) & 0 \\ 0 & x-1 & (x-1)(x+1) \\ 0 & 0 & x+1 \end{pmatrix}.$$

Find all values of x such that:
(i) rank($\boldsymbol{B}$)=1; (ii) rank($\boldsymbol{B}$)=2; (iii) rank($\boldsymbol{B}$)=3.

Justify your answer.

Show your working below.

Question 1

(c) [3 marks]

Give an example of a 3×4 matrix $\boldsymbol{C}$ with <u>no identical columns</u> such that the column space of $\boldsymbol{C}$ is span $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

(You are not required to justify your answer.)

What is the nullity of $\boldsymbol{C}$?

Show your working below.

Question 2 [15 marks]

(a) [6 marks]
Let $V = \{(x, y, z, w) \mid x = y + z, w = 2y\}$ be a subspace of $\mathbb{R}^4$.

(i) Write down an explicit form of a general vector in V.

(ii) Express V in linear span form.

(iii) Write down a basis for V and $\dim V$.

Show your working below.

Question 2

(b) [6 marks]

Let $S = \{\boldsymbol{u}, \boldsymbol{v}\}$ and $T = \{\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} + 2\boldsymbol{v}\}$ be two bases for a vector space U.

(i) Find the transition matrix from T to S.

(ii) Find the transition matrix from S to T.

(iii) Given the coordinate vector of $\boldsymbol{w} \in U$ with respect to T is $[\boldsymbol{w}]_T = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find $[\boldsymbol{w}]_S$.

Show your working below.

Question 2

(c) [3 marks]

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ be a basis for $\mathbb{R}^4$.

Suppose $U_1 = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $U_2 = \text{span}\{\boldsymbol{u}_3, \boldsymbol{u}_4\}$. Is it possible that

$$U_1 \cup U_2 = \mathbb{R}^4?$$

Justify your answer.

Show your working below.

Question 3 [15 marks]

(a) [6 marks]

Find the least squares solutions of the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

and hence find the smallest possible value of $\|\boldsymbol{Ax} - \boldsymbol{b}\|$ from among all $\boldsymbol{x} \in \mathbb{R}^2$.

Show your working below.

Question 3

(b) [4 marks]

Let $\boldsymbol{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

(i) Find an orthogonal basis $\{\boldsymbol{u}', \boldsymbol{v}'\}$ for $V = \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ such that $\boldsymbol{u} = \boldsymbol{u}'$.

(ii) Find the projection of $\boldsymbol{w}$ onto the subspace V.

Show your working below.

Question 3

(c) [5 marks]

Suppose the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

Show that the set $\left\{\frac{1}{\sqrt{2}}\boldsymbol{v}_1 - \frac{1}{\sqrt{2}}\boldsymbol{v}_2, \frac{1}{\sqrt{2}}\boldsymbol{v}_1 + \frac{1}{\sqrt{2}}\boldsymbol{v}_2, \boldsymbol{v}_3\right\}$ is also an orthonormal basis for $\mathbb{R}^3$.

Show your working below.

Question 4 [15 marks]

(a) [6 marks]

Let $\boldsymbol{A}$ be the matrix $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(i) Find all the eigenvalues of $\boldsymbol{A}$. Explain how you get your answer.

(ii) Find a basis for the eigenspace of $\boldsymbol{A}$ associated with each of the eigenvalues. Show your working.

Show your working below.

Question 4

(b) [4 marks]

Suppose $\boldsymbol{B}$ is a 2×2 matrix such that

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \boldsymbol{B} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}.$$

Find a matrix $\boldsymbol{C}$ such that $\boldsymbol{C}^2 = \boldsymbol{B}$.

Explain how you obtain your answer.

Show your working below.

Question 4

(c) [5 marks]

Let $\boldsymbol{M}$ be a non-invertible 3×3 symmetric matrix such that

$$\boldsymbol{M}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\2\\0\end{pmatrix} \quad \text{and} \quad \boldsymbol{M}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\\-1\end{pmatrix}.$$

What are the eigenvalues of $\boldsymbol{M}$?

Write down a basis for $\mathbb{R}^3$ consisting entirely of eigenvectors of $\boldsymbol{M}$.

Justify your answers.

Show your working below.

Question 5 [15 marks]

(a) [6 marks]

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ x+y \\ x-y \end{pmatrix}.$$

(i) Write down the standard matrix for T.

(ii) Find the kernel of T. Show your working.

(iii) Suppose $S : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation with standard matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$. Write down the formula for the composition $S \circ T$.

Show your working below.

Question 5

(b) [4 marks]

Given that $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation, P is a plane in $\mathbb{R}^3$ given by the equation $x + y + z = 0$, and ℓ is a line in $\mathbb{R}^3$ given by the set $\{(t, t, t) \mid t \in \mathbb{R}\}$.

Suppose F maps the plane P onto the line ℓ and maps the line ℓ to the origin.

Show that the linear transformation F^2 (i.e. $F \circ F$) is the zero transformation.

Show your working below.

Question 5

(c) [5 marks]

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation.

Suppose $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a basis for $\mathbb{R}^3$, and $\{T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), T(\boldsymbol{u}_3)\}$ spans $\mathbb{R}^2$.

Show that the standard matrix of T is of full rank.

Show your working below.

Question 6 [15 marks]

Determine whether each of the following parts is true or false. Justify your answer.

(a) [3 marks]

If $\boldsymbol{A}$ is a square matrix, then its row space is equal to its column space.

Show your working below.

(b) [3 marks]

The set $W = \{(a, b, c, abc) \mid a, b, c \in \mathbb{R}\}$ is not a subspace of $\mathbb{R}^4$.

Show your working below.

Question 6

(c) [3 marks]

Let $\boldsymbol{u}, \boldsymbol{v}$ be non-zero vectors in some vector space.
If $\text{span}\{\boldsymbol{u}\} \cap \text{span}\{\boldsymbol{v}\} = \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$, then $\text{span}\{\boldsymbol{u}\} = \text{span}\{\boldsymbol{v}\}$.

Show your working below.

(d) [3 marks]

There is no 3×3 matrix of rank 2 with only 1 eigenvalue.

Show your working below.

Question 6

(e) [3 marks]

Let $\boldsymbol{A}$ be a 3×2 matrix with two columns $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$, and $\boldsymbol{b}$ is a non-zero 3×1 column vector orthogonal to $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$. Then the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent.

Show your working below.