CS2040S Data Structures and Algorithms

Lecture Note #3

Analysis of Algorithms

Objectives

1

- To introduce the theoretical basis for measuring the efficiency of algorithms

2

- To learn how to use such measure to compare the efficiency of different algorithms

Outline

1. What is an Algorithm?
2. What do we mean by Analysis of Algorithms?
3. Big-O notation – Upper Bound
4. How to find the complexity of a program?

1 What is an algorithm?

1 Algorithm

- A step-by-step procedure for solving a problem.
- Properties of an algorithm:
  - Each step of an algorithm must be exact.
  - An algorithm must terminate.
  - An algorithm must be effective.
  - *An algorithm should be general.

2 What do we mean by Analysis of Algorithms?

2.1 What is Analysis of Algorithms?

- Analysis of algorithms
  - Provides tools for comparing the **efficiency** of different methods of solution (rather than programs)
  - **Efficiency** = Complexity of algorithms
- A comparison of algorithms
  - Should focus on significant differences in the efficiency of the algorithms
  - Should not consider reductions in computing costs due to clever coding tricks. Tricks may reduce the readability of an algorithm.

2.2 Determining the Efficiency/Complexity of Algorithms

- To evaluate rigorously the resources (time and space) needed by an algorithm and represent the result of the analysis with a formula
- We will emphasize more on the time requirement rather than space requirement here
- The time requirement of an algorithm is also called its time complexity

2.3 By measuring the run time?

TimeTest.java

```
public class TimeTest {
  public static void main(String[] args) {
    long startTime = System.currentTimeMillis();
    long total = 0;
    for (int i = 0; i < 10000000; i++) {
      total += i;
    }
    long stopTime = System.currentTimeMillis();
    long elapsedTime = stopTime - startTime;
    System.out.println(elapsedTime);
  }
}
```

Note: The run time depends on the compiler, the computer used, and the current work load of the computer.

2.4 Exact run time is not always needed

- Using exact run time is not meaningful when we want to compare two algorithms
  - coded in different languages,
  - running on different computers or
  - using different data sets

2.5 Determining the Efficiency of Algorithms

- Algorithm analysis should be independent of
  - Specific implementations
  - Compilers and their optimizers
  - Computers
  - Data

2.6 Execution Time of Algorithms

- Instead of working out the exact timing, we count the number of some or all of the primitive operations (e.g. +, -, *, /, assignment, …) needed.
- Counting an algorithm's operations is a way to assess its efficiency
  - An algorithm’s execution time is related to the number of operations it requires.

2.7 Counting the number of statements

- To simplify the counting further, we can ignore
  - the different types of operations, and
  - different number of operations in a statement,

  and simply count the number of statements executed.

2.8 Computation cost of an algorithm

- How many operations are required?

```
for (int i=1; i<=n; i++) {
  perform 100 operations;          // A
  for (int j=1; j<=n; j++) {
     perform 2 operations;         // B
  }
}
```

Total Ops = A + B $= \sum_{i=1}^{n} 100 + \sum_{i=1}^{n} (\sum_{j=1}^{n} 2)$

$$= 100n + \sum_{i=1}^{n} 2n \;= 100n + 2n^2 \;= 2n^2 + 100n$$

2.9 Approximation of analysis results

- Very often, we are interested only in using a simple term to indicate how efficient an algorithm is. The exact formula of an algorithm's performance is not really needed.
- Example:

  Given the formula: $2n^2 + 100n$

  - the dominating term $2n^2$ can tell us approximately how the algorithm performs by providing us with a measure of the growth rate (how the number of operations executed grows as n increases in size) of the algorithm
- This is called asymptotic analysis of the algorithm

2.10 Asymptotic analysis

- Asymptotic analysis is an analysis of algorithms that focuses on
  - analyzing the problems of large input size,
  - considering only the leading term of the formula, and
  - ignoring the coefficient of the leading term
- Some notations are needed in asymptotic analysis

2.11 Algorithm Growth Rates (1/2)

- An algorithm's time requirement can be measured as a function of the problem size, say n
- An algorithm's growth rate
  - Enables the comparison of one algorithm with another
  - Examples
    - Algorithm A requires time proportional to n^2
    - Algorithm B requires time proportional to n

2.12 Algorithm Growth Rates (2/2)

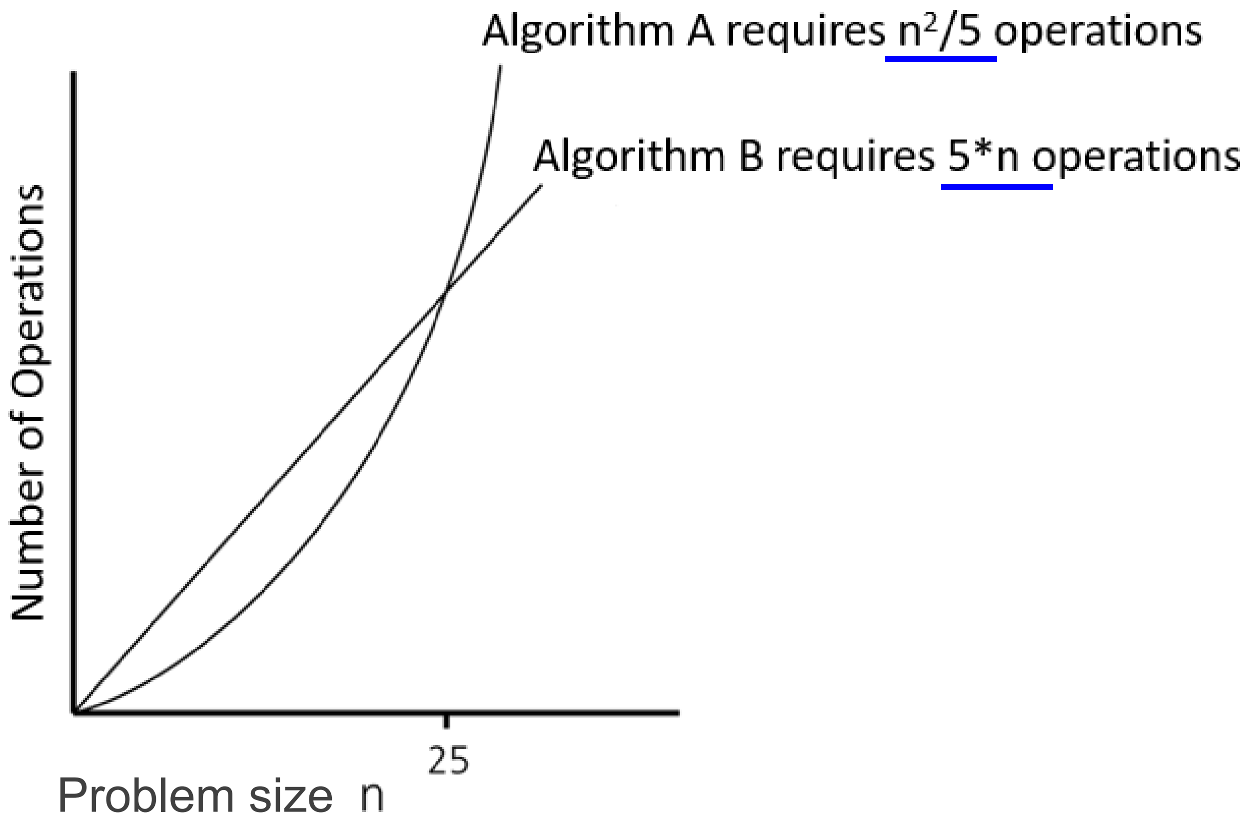


Figure - Time requirements as a function of the problem size n

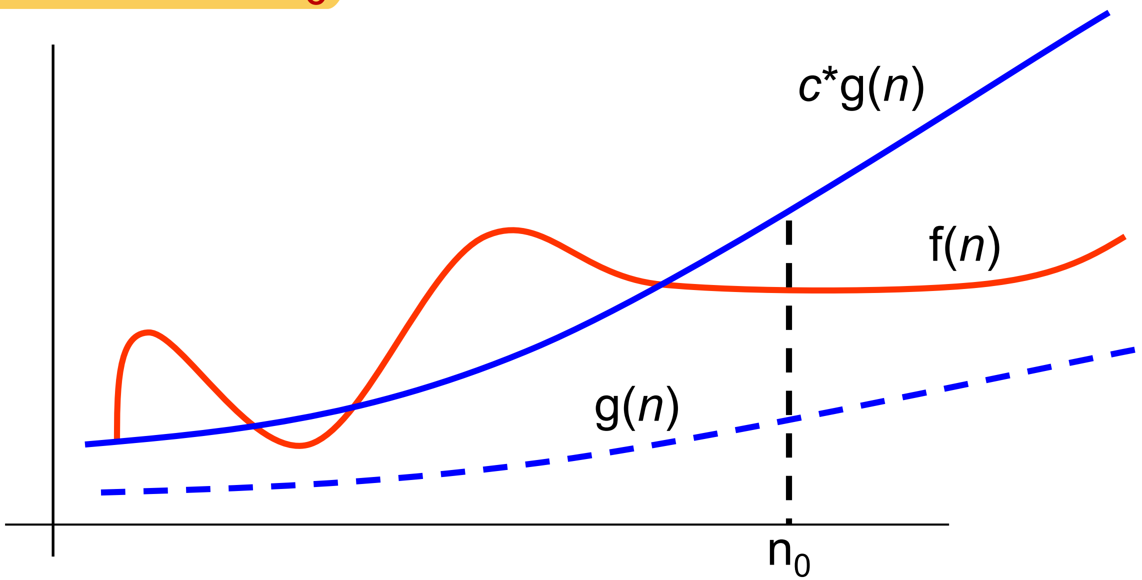
- Algorithm efficiency is typically a concern for large problems only. Why?

3 Big O notation

3.1 Definition – Big O notation

- Given a function f(n), we say g(n) is an (asymptotic) upper bound of f(n), denoted as f(n) = O(g(n)), if there exist a constant c > 0, and a positive integer n_0 such that f(n) $\leq$ c*g(n) for all $n \geq n_0$.

  - f(n) is said to be bounded from above by g(n).
  - O() is called the "big O" notation.



  - Another way of saying it is that O(g(n)) is the set of all functions f(n) where f(n) is asymptotically upper bounded by g(n)

3.2 Ignore the coefficients of all terms

- Based on the definition, $2n^2$ and $30n^2$ have the same upper bound n^2, i.e.,
  - $2n^2 = O(n^2)$

    f1(n) = $2n^2$; g(n) = n^2.
    Let c=3 and n_0=1, since $2n^2 \le cn^2 \ \forall\ n \ge n_0$
    Hence f1(n) = O(g(n))

    Why?

  - $30n^2 = O(n^2)$

    f2(n) = $30n^2$; g(n) = n^2.
    Let c=31 and n_0=1, since $30n^2 \le cn^2 \ \forall\ n \ge n_0$
    Hence f2(n) = O(g(n))

  They differ only in the choice of c.

  they are asymptoptically bounded by n^2 because we could satisfy the requirements of big O

- Therefore, in big O notation, we can omit the coefficients of all terms in a formula:
  - Example: $f(n) = 2n^2 + 100n = O(n^2) + O(n)$

3.3 Finding the constants c and n_0

- Given $f(n) = 2n^2 + 100n$, prove that $f(n) = O(n^2)$.

Observe that: $2n^2 + 100n \leq 2n^2 + n^2 = 3n^2$
whenever $n \geq 100$.

3n^2 > 2n^2 but 3n^2 = 2n^2 + n^2 so n^2 will only be bigger than 100n when n >= 100

→ Set the constants to be $c = 3$ and $n_0 = 100$.

By definition, we have $f(n) = O(n^2)$.

Notes:

1. $n^2 \leq 2n^2 + 100n$ for all n, i.e., $g(n) \leq f(n)$, and yet $g(n)$ is an asymptotic upper bound of $f(n)$
2. c and n_0 are not unique.
   For example, we can choose $c = 2 + 100 = 102$, and $n_0 = 1$ (because $f(n) \leq 102n^2\ \forall\ n \geq 1$)

yes because n^3 grows faster than n^2

Q: Can we write $f(n) = O(n^3)$? Yes

3.4 Is the bound tight?

- The complexity of an algorithm can be bounded by many functions.
- Example:
  - Let $f(n) = 2n^2 + 100n$.
  - $f(n)$ is bounded by n^2, n^3, n^4 and many others according to the definition of big O notation.
  - Hence, the following are all correct:
    - $f(n) = O(n^2)$; $f(n) = O(n^3)$; $f(n) = O(n^4)$
- However, we are more interested in the tightest bound which is n^2 for this case.

3.5 Growth Terms: Order-of-Magnitude

- In asymptotic analysis, a formula can be simplified to a single term with coefficient 1
- Such a term is called a growth term (rate of growth, order of growth, order-of-magnitude)
- The most common growth terms can be ordered as follows: (note: many others are not shown)

$$O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(n^3) < O(2^n) < O(n!)$$

"fastest" *"slowest"*

Note:

- "log" = log base 2, or $\log_2$; "$\log_{10}$" = log base 10; "ln" = log base e. In big O, all these log functions are the same. (Why?) because you can multiply from one base to another by multiplying by a constant(change of base formula)

3.6 Some more properties of Big O

1. if $f1(n) = O(g1(n))$ and $f2(n) = O(g2(n))$
   then $f1(n)+f2(n) = O(g1(n)+g2(n))$

2. if $f1(n) = O(g1(n))$ and $f2(n) = O(g2(n))$
   then $f1(n)*f2(n) = O(g1(n)*g2(n))$

this 2 must be either <= 2*f1(n) or 2*f2(n) = 2O(g(n)) which is Og(n))

3. if $f1(n), f2(n) = O(g(n))$ then $f1(n)+f2(n) = O(g(n))$

4. if $g1(n) = O(g2(n))$ then $g1(n)+g2(n) = O(g2(n))$

3.7 Examples on Big O notation

- $f1(n) = \frac{1}{2}n + 4$
  $= O(n)$
- $f2(n) = 240n + 0.001n^2$
  $= O(n^2)$
- $f3(n) = n \log n + \log n + n \log (\log n)$
  $= O(n \log n)$

3.8 Order-of-Magnitude Analysis and Big O Notation (1/2)

(a)

Function	n = 10	100	1,000	10,000	100,000	1,000,000
1	1	1	1	1	1	1
$\log_2 n$	3	6	9	13	16	19
n	10	10^2	10^3	10^4	10^5	10^6
$n * \log_2 n$	30	664	9,965	10^5	10^6	10^7
n^2	10^2	10^4	10^6	10^8	10^{10}	10^{12}
n^3	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}
2^n	10^3	10^{30}	10^{301}	$10^{3,010}$	$10^{30,103}$	$10^{301,030}$

Figure - Comparison of growth-rate functions in tabular form

3.8 Order-of-Magnitude Analysis and Big O Notation (2/2)

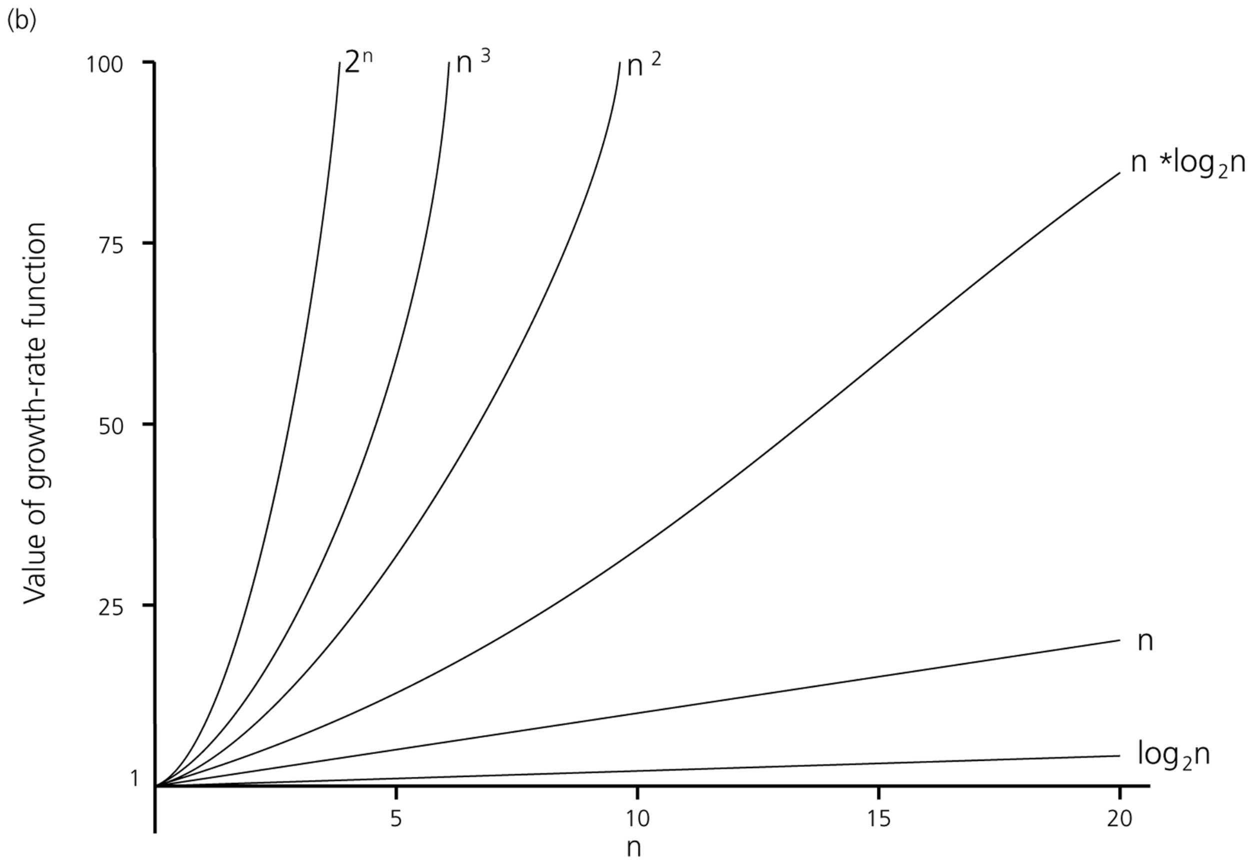


Figure - Comparison of growth-rate functions in graphical form

3.9 Summary: Order-of-Magnitude Analysis and Big O Notation

- Order of growth of some common functions:

  O(1) < O(log n) < O(n) < O(n log n) < O(n^2) < O(n^3) < O(2^n) <O(n!)

- Properties of growth-rate functions/Big O notation
  - You can ignore low-order terms
  - You can ignore a multiplicative constant in the high-order term
  - Other properties in slide 25

4 How to find the complexity of a program?

4.1 Examples on finding complexity (1/3)

- What is the complexity of the following code fragment?

```
int sum = 0;
for (int i=1; i<n; i=i*2) {
    sum++;
}
```

- It is clear that sum is incremented only when

  i = 1, 2, 4, 8, …, 2^k where $k = \lfloor \log_2 n \rfloor$

  There are k+1 iterations. So the complexity is O(k) or O(log n)

Note:

- In Computer Science, log n means $\log_2 n$.
- When 2 is replaced by 10 in the 'for' loop, the complexity is O($\log_{10} n$) which is the same as O($\log_2 n$). (Why?)
- $\log_{10} n = \log_2 n / \log_2 10$

4.1 Examples on finding complexity (2/3)

- What is the complexity of the following code fragment?

```
int sum = 0; O(1)
for (int i=1; i<n; i=i*2) {          O(log n)
  for (int j=n; j>1; j=j/2)          O(log n)
    sum++; O(1)
}
```

O(log n)*O(log n) = O((log n)^2)

4.1 Examples on finding complexity (3/3)

- What is the complexity of the following code fragment?
  (For simplicity, let's assume that *n* is some power of 3.)

```
int sum = 0;
for (int i=1; i<n; i=i*3) {
   for (j=1; j<=i; j++) {
      sum++;
   }
}
```

first time run inner loops runs 1 time, then since i*3 and j <= 1, second time it runs 3 times, third time it runs 9 times

- $T(n) = 1 + 3 + 9 + 27 + \ldots + 3^{(\log_3 n)}$

  $= 1 + 3 + \ldots + n/9 + n/3 + n$

  $= n + n/3 + n/9 + \ldots + 3 + 1$ (reversing the terms in previous step)

  $= n * (1 + 1/3 + 1/9 + \ldots)$

  $\leq n * (3/2)$

  $= 3n/2$

  $= O(n)$

Time complexity normally bounded by how many times the inner loop is executed

Why is $(1 + 1/3 + 1/9 + \ldots) \leq 3/2$?
This is sum of infinite geometric series see word file "analysis_of_algorithms_useful_equalities.pdf"

4.2 Non-recursive Binary Search Algorithm (1)

- Requires array to be sorted in ascending order
- Maintain subarray where x (the search key) might be located
- Repeatedly compare x with m, the middle element of current subarray
  - If x = m, found it!
  - If x > m, continue search in subarray after m
  - If x < m, continue search in subarray before m

4.2 Non-recursive Binary Search Algorithm (2)

- Data in the array a[] are sorted in ascending order

```
public static int binSearch(int[] a, int len, int x)
{
    int mid, low = 0;
    int high = len - 1;
    while (low <= high) {
        mid = (low + high) / 2;
        if (x == a[mid]) return mid;
        else if (x > a[mid]) low = mid + 1;
        else high = mid - 1;
    }
    return -1;
}
```

best case: O(1)
word case: O(log n)

4.2 Non-recursive Binary Search Algorithm (3)

- At any point during binary search, part of array is "*alive*" (might contain the point x)
- Each iteration of loop eliminates at least half of previously "*alive*" elements
- At the beginning, all n elements are "*alive*", and after
  - After 1 iteration, at most $n/2$ elements are left, or alive
  - After 2 iterations, at most $(n/2)/2 = n/4 = n/2^2$ are left
  - After 3 iterations, at most $(n/4)/2 = n/8 = n/2^3$ are left

    :
  - After i iterations, at most $n/2^i$ are left
  - At the final iteration, at most 1 element is left

4.2 Non-recursive Binary Search Algorithm (4)

In the worst case, we have to search all the way up to the last iteration k with only one element left.

We have:

$$n/2^k = 1$$

$$2^k = n$$

$$k = \log n$$

Hence, the binary search algorithm takes O(f(n)) , or O($\log n$) times

4.3 Time complexity of recursion: An example (1/2)

- What is the complexity of the following recursive function?

```
// Precond: n >= 0
public static int f(int n) {
  if (n == 0)
    return 1;
  else {
    int sum = 0;
    for (int i=0; i<n; i++)
      sum++; basically sum = n
    return f(n-1)+sum;
  }
}
```

Recurrence: write down how to express the algo mathematically

Recurrence for the recursive function f

$f(n) = 1$	for $n == 0$	base case
$= f(n-1) + n$	for $n > 0$	

4.3 Time complexity of recursion: An example (2/2)

- Function for the total operations of a recursive function is similar in form to the recursive case of the recurrence

# of operations to compute f(n) and f(n-1) respectively

# of operation executed by the for loop in the else clause

$T(n) = T(n\text{-}1) + c{*}n$ first order linear recurrence

expand T(n-1)

$= [T(n\text{-}2) + n\text{-}1] + n$

$= T(n\text{-}2) + (n\text{-}1) + n$

Iterative Method

$= T(n\text{-}3) + (n\text{-}2) + (n\text{-}1) + n$

…

$= T(0) + 1 + 2 + 3 + \ldots + n$ (must stop when n=0)

this arithmetic progression
= 1 + n(n+1)/2
= 1 + n^2/2 + n/2

$= 1 + 1 + 2 + 3 + \ldots + n = O(n^2)$

4.3 Time complexity of recursion: Fibonacci numbers

- Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, …
  - The first two Fibonacci numbers are both 1 (arbitrary numbers)
  - The rest are obtained by adding the previous two together.
- Calculating the n^{th} Fibonacci number recursively:

```
// Precond: n > 0
public static int fib(int n) {
  if (n <= 2)
    return 1;
  else
    return fib(n-1) + fib(n-2);
}
```

Recurrence for fib

$$
\begin{aligned}
fib(n) &= 1 && \text{for } n=1, 2 \\
&= fib(n-1) + fib(n-2) && \text{for } n > 2
\end{aligned}
$$

4.3 Time complexity of recursion: Fibonacci numbers

- Again the function for total number of operations is similar to the the recursive case of Fibonacci
  - $\boxed{T(n) = T(n-1)+T(n-2)+4}$ 4 operations that take constant time (1 comparison, 1 addition, 2 subtraction)

    2nd order linear recurrence → Hard !

    approx it to become a first order linear recurrence

$\Rightarrow$ $\boxed{T(n) < 2T(n-1)+c}$ → 1st order linear recurrence → Easier !

$\Rightarrow$ $< 2(2T(n-2)+c)+c$ (expand T(n-1) to be 2*T(n-2)+c)

$\Rightarrow$ $< 4T(n-2)+2c+c$

When we approximate it, we want to find the upper bound(something that will take more than or equal time). Since, T(n - 1) > T(n - 2), we can do 2T(n-1) as an upper bounded approximation s.t. T(n - 1) + T(n - 2) will confirm be within the bound

$\Rightarrow$ $< 8T(n-3)+4c+2c+c$ (cont. the expansion)

$\Rightarrow$ … (will stop when n is 1)

$\Rightarrow$ $< 2^{n-1}(T(1))+2^{n-2}c+ 2^{n-3}c+\ldots+c$ (T(1) is c)

$\Rightarrow$ $< 2^n*(1/2+1/4+1/8+\ldots)*c$

sum of g.p with r < 1 will converge into a constant so we ignore it

$\Rightarrow$ $= O(2^n)$

4.4 Some rules of thumb and examples

- Basically just count the number of statements executed.
- If there are only a small number of simple statements in a program – O(1)
- If there is a 'for' loop dictated by a loop index that goes up to n – O(n)
- If there is a nested 'for' loop with outer one controlled by n and the inner one controlled by m – O($n*m$)
- For a loop with a range of values n, and each iteration reduces the range by a fixed constant fraction (eg: ½) – O(log n)
- For a recursive method, each call is usually O(1). So
  - if n calls are made – O(n)
  - if n log n calls are made – O(n log n)

4.5 Analysis of Different Cases (1)

Worst-Case Analysis

- Interested in the worst-case behaviour.
- A determination of the maximum amount of time that an algorithm requires to solve problem/input of size *n*
- This is the one we will be mostly looking at for the rest of the course

Best-Case Analysis

- Interested in the best-case behaviour
- Not useful

Average-Case Analysis

- A determination of the amount of time that an algorithm requires to solve an “average” input of size *n*
- Have to know the probability distribution of the inputs to determine what is an “average input”
- Not covered in this module (except for some simple examples)

4.5 Analysis of Different Cases (2)

Expected-Case Analysis

- Analysis performed on randomized algorithms – i.e algorithms that employ randomness in their logic
- Often confused with average-case analysis (although they are related)
- Not covered in this module

Amortized Analysis

e.g. first run is n operations, subsequent is 1,1,1,1,1,
then last one is n operations again, we sum it all up
and divide by the total no. of runs

- Sometimes worst case behavior cannot be possible for every run of the algorithm, meaning that across several runs, only some can induce worst case behavior while others don't
- Amortized analysis determines the total time complexity required for a sequence of runs and thus the "amortized" cost per run
- Need more advanced techniques which will be covered in CS3230, so will only cover some very simple examples in CS2040S

4.6 The Efficiency of Searching Algorithms

- Example: Efficiency of Sequential Search (data not sorted)
  - Worst case: O(n)

    Which case? if in last place, or item not inside
  - Average case: O(n) probability distribution: assuming each slot has a 1/n(equal) chance of being the item, the total is still O(n)
  - Best case: O(1)

    Why? Which case?
  - Unsuccessful search?
- Q: What is the best case complexity of Binary Search (data sorted)?
  - Best case complexity is not interesting. Why?

4.7 Keeping Your Perspective

- If the problem size is always small, you can probably ignore an algorithm's efficiency
- Weigh the trade-offs between an algorithm's time requirements and its memory requirements
- Order-of-magnitude analysis focuses on large problems
- There are other measures, such as big Omega (Ω), big theta (Θ), little oh (o), and little omega (ω). These may be covered in more advanced module.

End of file

Question Time:

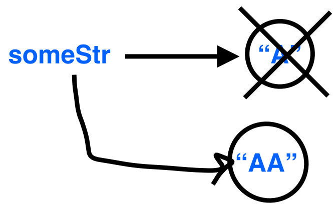
- What is the complexity of the following code fragment?

```
String someStr = "A";

for (int i=1; i<n; i++) {
    someStr = someStr + "A";
}
```

mutable string class: string builder which will give us O(N)

1. O(n)
2. O(nlogn)
3. O(logn)
4. O(n^2)



garbage collected, memory for this will be destroyed

primitive operation O(1) time: copying each character and adding to the back

1st: 1 just copy one char
2nd: 2, then two
3rd: 3, then three
….
nth: n

add it all together becomes AP = n(n+1)/2 = n^2

String being used is java string which is immutable, when the concatonation is done, it will make a new string object with each iteration