CS1231S: Discrete Structures

Tutorial #10: Counting and Probability II and Graphs I

(Week 12: 31 October – 4 November 2022)

I. Discussion Questions

You are strongly encouraged to discuss D1 – D3 on LumiNUS forum. No answers will be provided.

D1. Suppose a random sample of 2 lightbulbs is selected from a group of 8 bulbs in which 3 are defective, what is the expected value of the number of defective bulbs in the sample? Let X represent of the number of defective bulbs that occur on a given trial, where $X = 0,1,2$. Find $E[X]$.

D2. How many **injective functions** are there from a set A with m elements to a set B with n elements, where $m \leq n$?

D3. How many **surjective functions** are there from a 5-element set A to a 3-element set B?

II. Tutorial Questions

1. Find the term independent of x in the expansion of

$$\left(2x^2 + \frac{1}{x}\right)^9$$

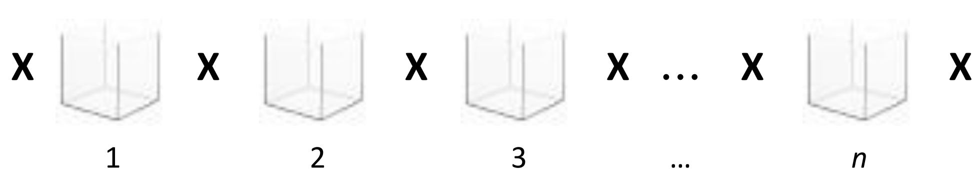
2. Let's revisit Tutorial #9 Question 5:

   Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

   Last week, the answer given was

   For k $(1 \ \leq k \leq n)$ consecutively numbered boxes that contain white balls, there are $n - k + 1$ ways. Therefore, total number of ways is $\sum_{k=1}^{n}(n - k + 1) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

   Now, let's use another approach to solve this problem. Draw crosses on the side of the boxes as shown below. How do you use these crosses?



3. [AY2020/21 Semester 2 Exam Question]
   On a die there are 6 numbers. We call 4, 5, 6 the big numbers and 1, 2, 3 the small numbers. Given a loaded die in which the probability of rolling any fixed big number is twice the probability of rolling any fixed small number, answer the following questions.
   (i) What is the probability of rolling a 6? Write your answer as a single fraction.
   (ii) If two such loaded dice are rolled, what is the expected value of the maximum of the two dice? Write your answer as a single fraction.

4. An urn contains five balls numbered 1, 2, 2, 8 and 8. If a person selects a set of two balls at random, what is the expected value of the sum of the numbers on the balls?

5. [AY2021/22 Semester 2 Exam Question]
   A rare disease broke out in a city with a prevalence of 0.1%, that is, it affects 1 out of every 1000 persons. A quick test kit has been developed that has a sensitivity of 85%, which is the probability that a person with the rare disease is tested positive. Among those who took the test, 10% of the time it came out positive. Write your answers correct to <u>3 significant figures</u>.
   (a) Divoc has shown symptoms of the disease. Should he be tested positive, what is the probability that he actually has the disease?
   (b) What is the probability of a false positive result, that is, a person does not have the disease but is tested positive?

6. [AY2015/16 Semester 1 Exam Question]
   Let $A = \{1, 2, 3, 4\}$. Since each element of $P(A \times A)$ is a subset of $A \times A$, it is a binary relation on A. ($P(S)$ denotes the powerset of S.)

   Assuming each relation in $P(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is (a) reflexive? (b) symmetric?

   Can you generalize your answer to any set A with n elements?

7. Let's revisit Tutorial #9 Question 1:

   > In a certain tournament, the first team to win four games wins the tournament. Suppose there are two teams A and B, and team A wins the first two games. How many ways can the tournament be completed?

   The solution given last week uses a possibility tree to depict the 15 ways. Now, let's approach this problem using combination.

   Let us define a function $W(a, b)$ to be the number of ways the tournament can be completed if team A has to win a more games to win, while team B has to win b more games to win. Hence,

$$W(a,b) = \begin{cases} 1, & \text{if } a = 0 \text{ or } b = 0; \\ W(a, b-1) + W(a-1, b), & \text{if a} > 0 \text{ and } b > 0. \end{cases}$$

We may express $W(a,b)$ as a simple combination formula as follows:

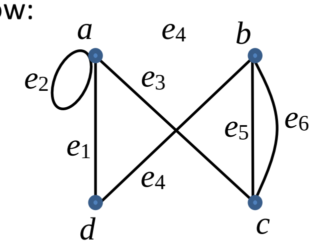
$$W(a,b) = \binom{a+b}{a}.$$

Verify the above.

Now, we denote the function $T(n,k)$ to be the number of ways the tournament can be completed, given that the first team to win n games wins the tournament, and team A wins the first k ($k \leq n$) games.

Derive a simple combination formula for $T(n,k)$ (hint: relate function T to function W), and hence solve $T(4,2)$ which is the problem in Tutorial #9 Question 1.

8. Given the graph shown below:



   (a) Give the adjacency matrix A for the graph, with vertices in the order a, b, c, d.
   (b) Compute A^0, A^2 and A^3.
   (c) How many walks of length 2 are there from a to b? From c to itself? List out all the walks.
   (d) How many walks of length 3 are there from a to c? List out all the walks.

9. A lady hosted a party of n ($n \geq 2$) people (including herself). At the party, various friends met and some of them shook hands with each other. The thoughtful host made sure that she shook hands with everyone in the party.

   Prove that there are at least two people who have shaken hands the same number of times.

10. Let G be any simple graph with 6 vertices. Prove that G or its complementary graph $\bar{G}$ contains a triangle (a cycle of length 3).

   (Note: This problem is equivalent to the following: Show that in any group of 6 people, there must be either 3 mutual friends or 3 mutual strangers. We assume that for every pair of people, they are either friends or strangers. **BUT for your answers, please use the graph formulation.**)

The following definitions are used in Question 11.

A relation $\prec$ on a set A is said to be ***irreflexive,*** if and only if, $\forall a \in A, (a \nprec a)$.
(**Note:** "*irreflexive*" is not that same as "*not reflexive*".)

Alternative definition of anti-symmetry: $\forall x, y\ (x \neq y) \Rightarrow ((x, y) \in R) \Rightarrow ((y, x) \notin R)$.

A relation is a **strict partial order** if and only if it is irreflexive, antisymmetric and transitive.

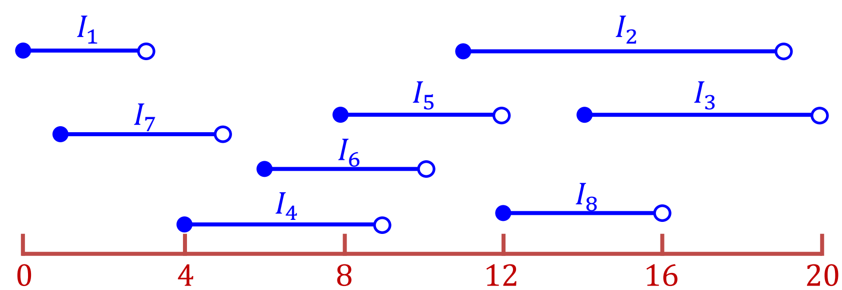
Let $\prec$ be a strict partial order on a set A. A subset C of A is called a **chain** if and only if each pair of *distinct* elements in C is comparable, that is, $\forall a, b \in C\ (a \neq b) \Rightarrow (a \prec b \vee b \prec a)$.

A **maximal chain** is a chain M such that $t \notin M \Rightarrow M \cup \{t\}$ is not a chain.

11. You are given a set of n jobs $J = \{J_1, J_2, J_3, \dots, J_n\}$. Each job J_k is represented by the interval $I_k = [s_k, e_k)$ where s_k is the *start time* and e_k (where $s_k < e_k$) is the *end time* of the job, for $k = 1, 2, \dots, n$. An instance of this problem with $n = 8$ is shown below.

Instance: $n = 8$, and $\{J_1, J_2, J_3, \dots, J_8\}$, and

$I_1 = [0, 3)$, $I_2 = [11, 19)$, $I_3 = [14, 20)$, $I_4 = [4, 9)$,
$I_5 = [8, 12)$, $I_6 = [6, 10)$, $I_7 = [1, 5)$, $I_8 = [12, 16)$,



Define a relation $\prec$ on J by: $(J_x \prec J_y) \Leftrightarrow (e_x \leq s_y)$

Namely, $J_x \prec J_y$ means that the job J_x is *to the left of* the job J_y.

(a) Show that the relation $\prec$ is a strict partial order.

(b) Draw the graph $G = (J, \prec)$ of the relation $\prec$ for this instance given above.

You want to assign all the jobs to workers so as to *minimize* the number of workers deployed. Each worker can only work on one job at a time. Each worker can do any number of jobs as long as the jobs do not overlap in time. (You can assume that a worker can (if necessary) start on a new job immediately after finishing a previous job. In the instance above, jobs T_5 and T_8 can be assigned to the same worker, if necessary.) This problem can be solved in parts (c) and (d) as follows:

(c) Give a maximal chain C in the graph G. Argue that the jobs in C can be assigned to a single worker.

(d) Now, partition the vertices in G into a *minimum* number of vertex-disjoint chains. For each chain, we assign a worker to do all the jobs in that chain. What is the minimum number of workers needed? Show the jobs assigned to each worker (W1, W2, etc.).