CS2040S
Data Structures and Algorithms

Augmented Trees!

This Week

On the importance of being balanced

- Height-balanced binary search trees
- AVL trees
- Rotations

Tries

- How to handle text?

Data structure design

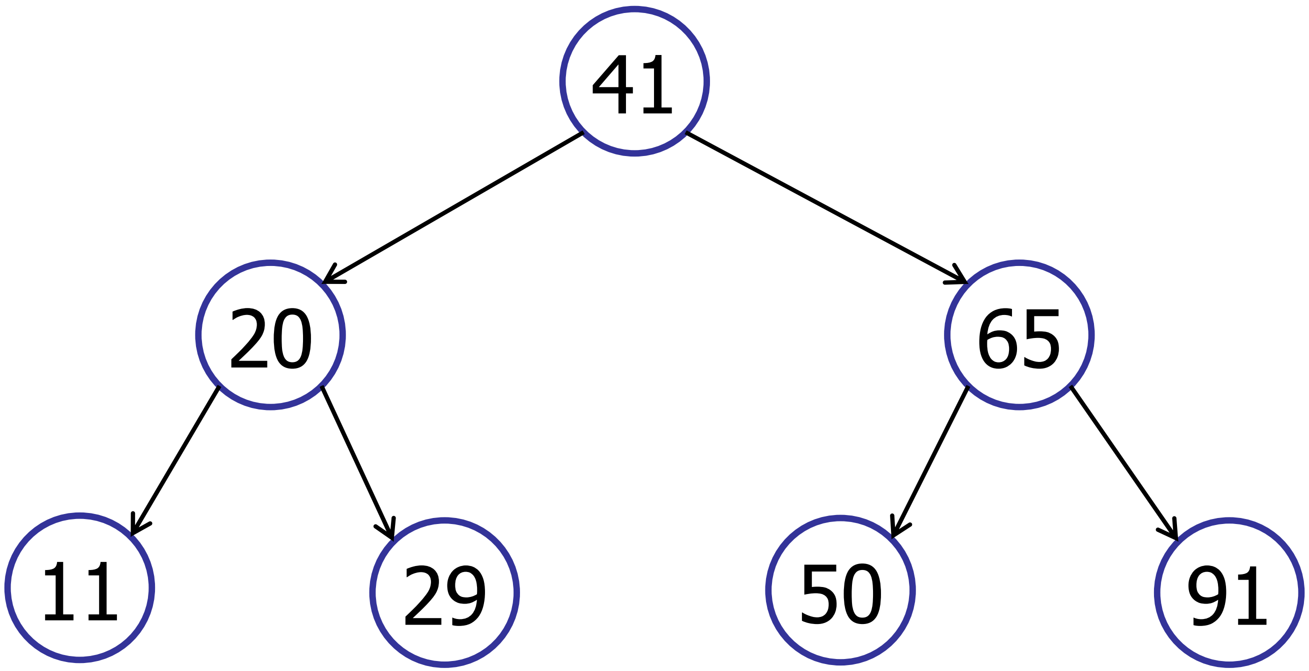
- How to build new structures on existing ideas?

Recap: Dictionary Interface

A collection of (key, value) pairs:

interface	**IDictionary**	
void	insert(Key k, Value v)	*insert (k,v) into table*
Value	search(Key k)	*get value paired with k*
Key	successor(Key k)	*find next key > k*
Key	predecessor(Key k)	*find next key < k*
void	delete(Key k)	*remove key k (and value)*
boolean	contains(Key k)	*is there a value for k?*
int	size()	*number of (k,v) pairs*

Recap: Binary Search Trees



- Two children: v.left, v.right
- Key: v.key
- **BST Property**: all in left sub-tree < key < all in right sub-right

The Importance of Being Balanced

How to get a balanced tree:

- Define a good property of a tree.
- Show that if the good property holds, then the tree is balanced.
- After every insert/delete, make sure the good property still holds. If not, fix it.

Invariant

AVL Trees [Adelson-Velskii & Landis 1962]

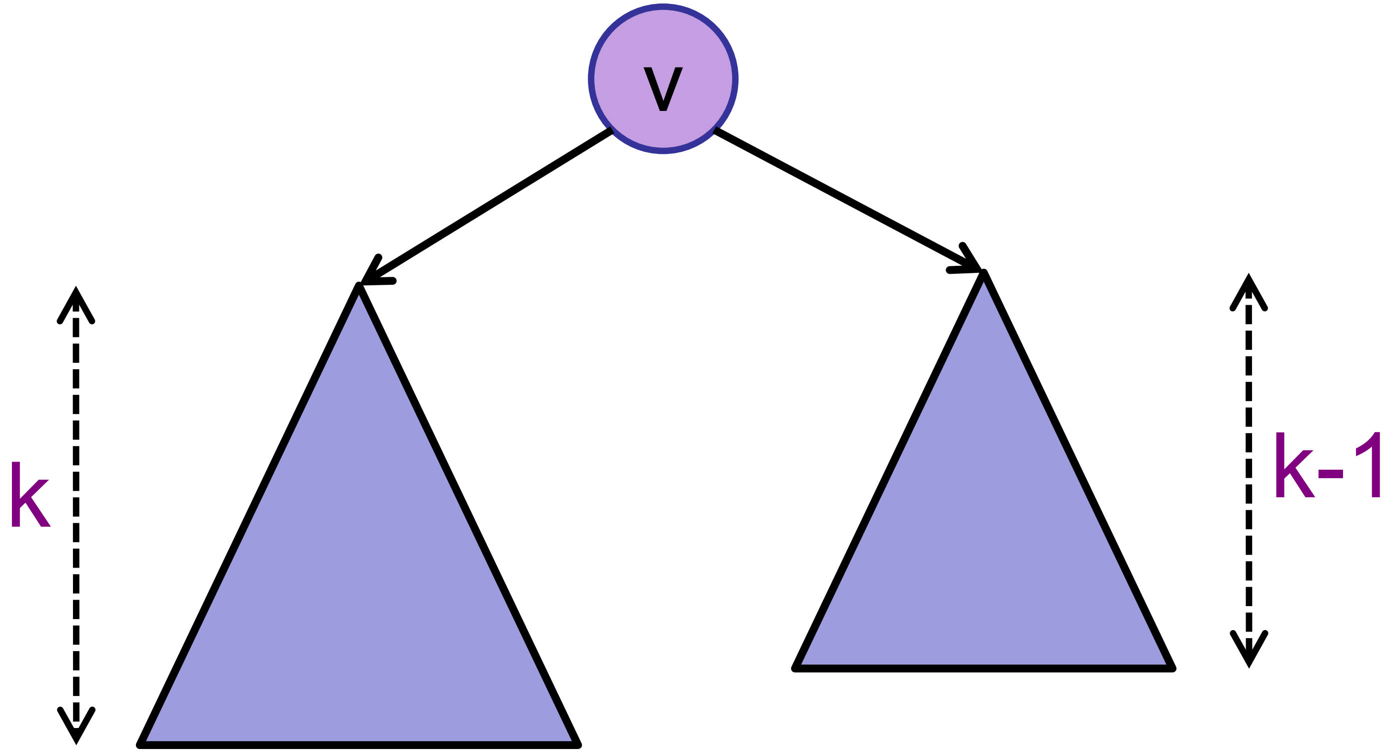
AVL Trees [Adelson-Velskii & Landis 1962]

Key definition

Step 1: Define Invariant

- A node v is **height-balanced** if:

$$|v.left.height - v.right.height| \leq 1$$



Height-Balanced Trees

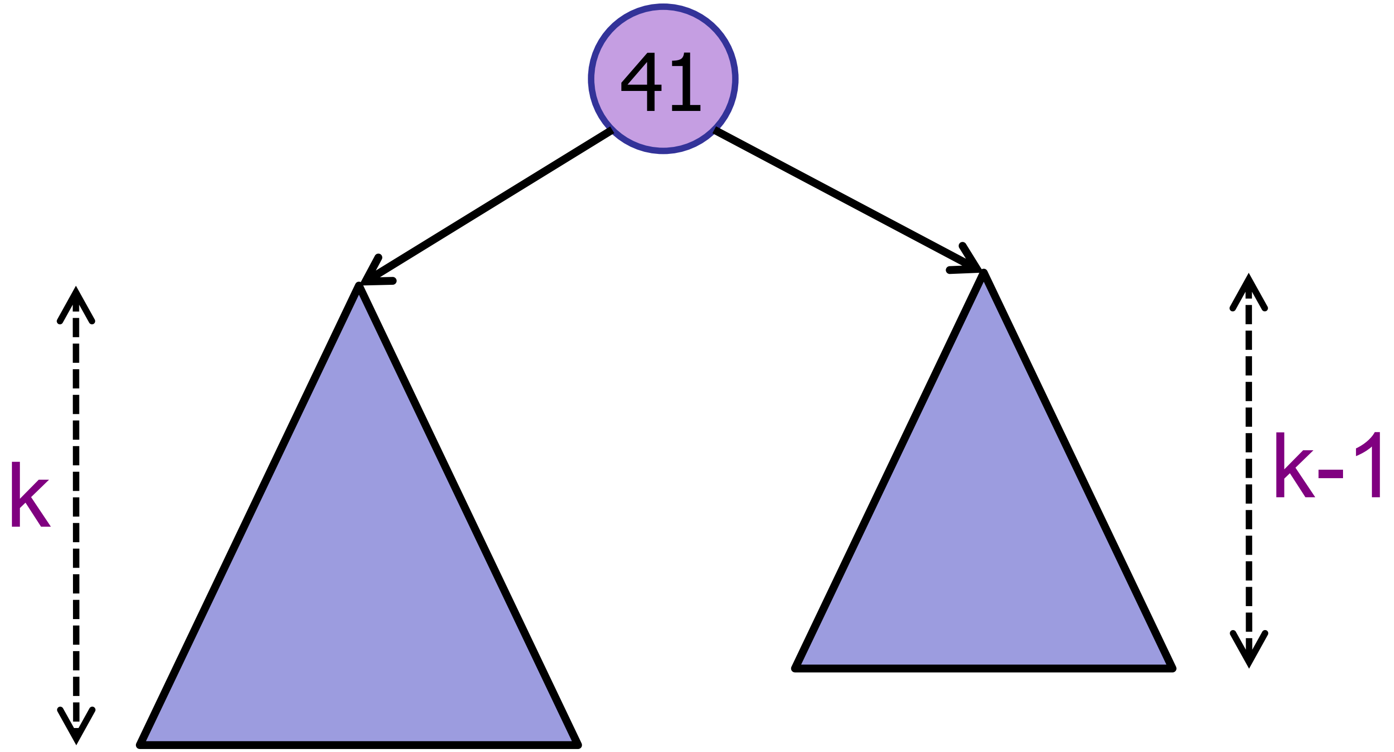
Theorem:

A height-balanced tree with n nodes has **at most** height $h < 2\log(n)$.

➔ A height-balanced tree is balanced.

AVL Trees [Adelson-Velskii & Landis 1962]

Step 2: Show how to maintain height-balance



Insert in AVL Tree

Summary:

- Insert key in BST.
- Walk up tree:
  - At every step, check for balance.
  - If out-of-balance, use rotations to rebalance and return.

Key observation:

- Only need to fix *lowest* out-of-balance node.
- Only need at most two rotations to fix.

Delete in AVL Tree

Summary:

- Delete key from BST.
- Walk up tree:
  - At every step, check for balance.
  - If out-of-balance, use rotations to rebalance.
  - Continue to root.

Key observation:

- It is *not* sufficient to only fix lowest out-of-balance node in tree.

Dynamic Data Structures

1. Maintain a set of items

2. Modify the set of items

3. Answer queries.

Big picture idea:

Trees are a good way to store, summarize, and search dynamic data.

Augmenting data structures

Basic methodology:

1. Choose underlying data structure

   (tree, hash table, linked list, stack, etc.)

2. Determine additional info needed.

3. Modify data structure to *maintain* additional info when the structure changes.

   (subject to insert/delete/etc.)

4. Develop new operations.

Plan

Two (or three?) examples of augmenting balanced BSTs

1. Order Statistics
2. Interval Queries
3. Orthogonal Range Searching

Order Statistics

Input

A set of integers.

Output: select(k)

The k^{th} item in the set.

52	**7**	**13**	**43**	**22**	**92**	**18**	**9**	**65**	**67**	**87**	**25**

select(4)

Dynamic Order Statistics

Implement a data structure that supports:

- insert(int key)
- delete(int key)

and also:

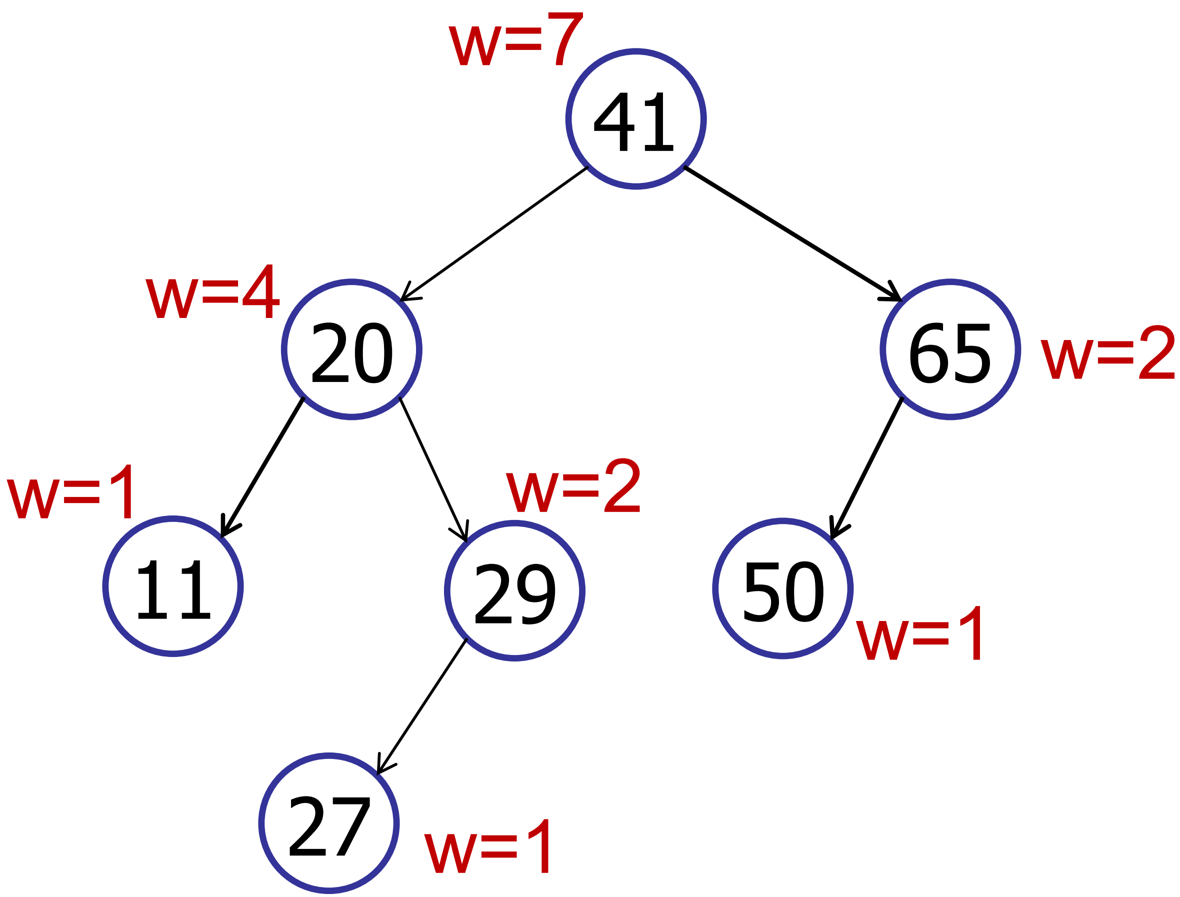
- select(int k)

7	**9**	**13**	**18**	**22**	**25**	**43**	**52**	**65**	**67**	**87**	**92**

select(4)

Dynamic Order Statistics

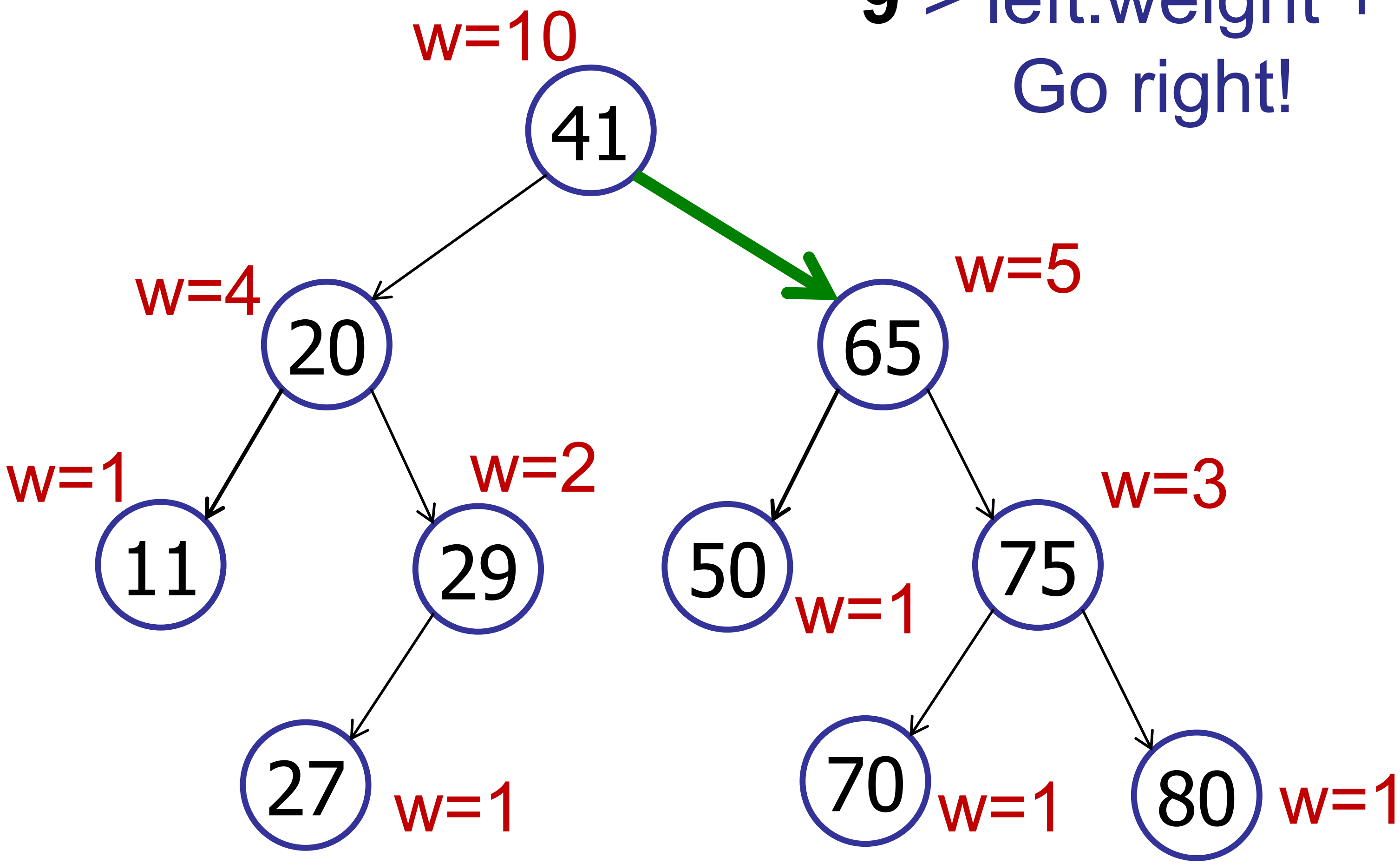
Idea: store *size* of sub-tree in every node



Dynamic Order Statistics

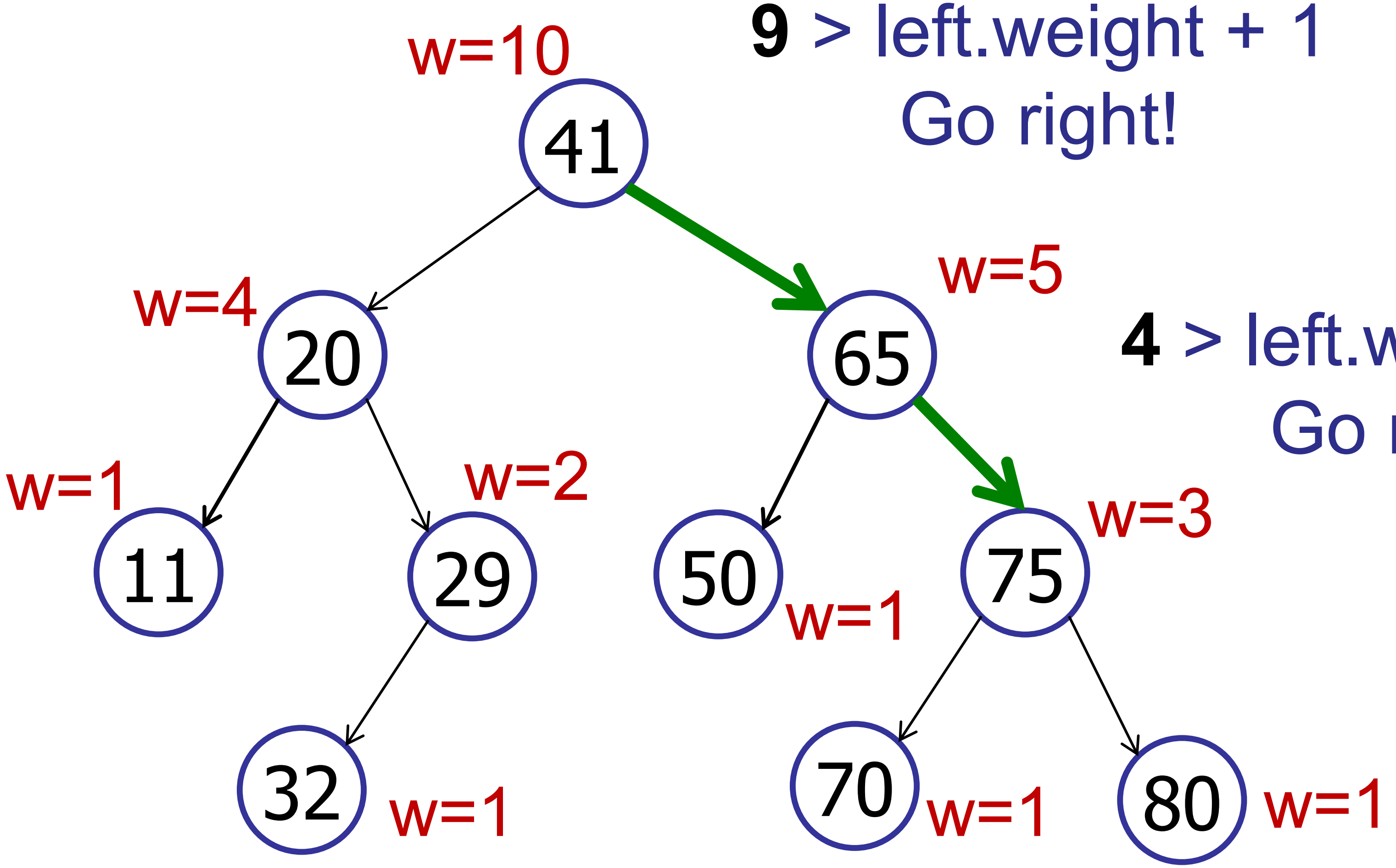
Example: select(9)

9 > left.weight + 1
Go right!



Dynamic Order Statistics

select(9)



9 > left.weight + 1

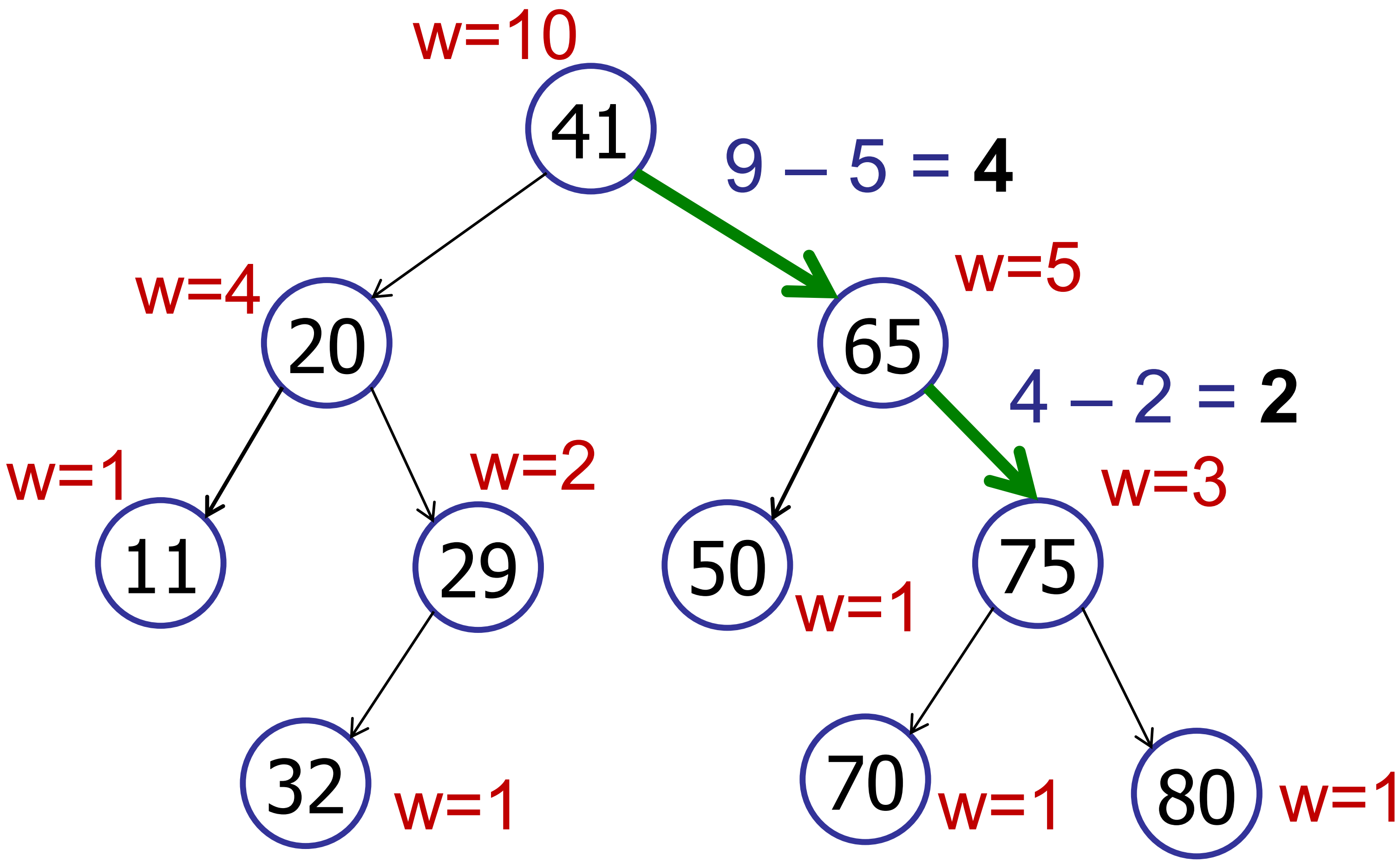
Go right!

4 > left.weight + 1

Go right!

Dynamic Order Statistics

select(9)



Dynamic Order Statistics

select(k)

```
rank = m_left.weight + 1;
if (k == rank) then
    return v;
else if (k < rank) then
    return m_left.select(k);
else if (k > rank) then
    return m_right.select(k–rank);
```

Dynamic Order Statistics

select(k) : finds the node with rank k

Example: find the 10th tallest student in the class.

Dynamic Order Statistics

select(k) : finds the node with rank k

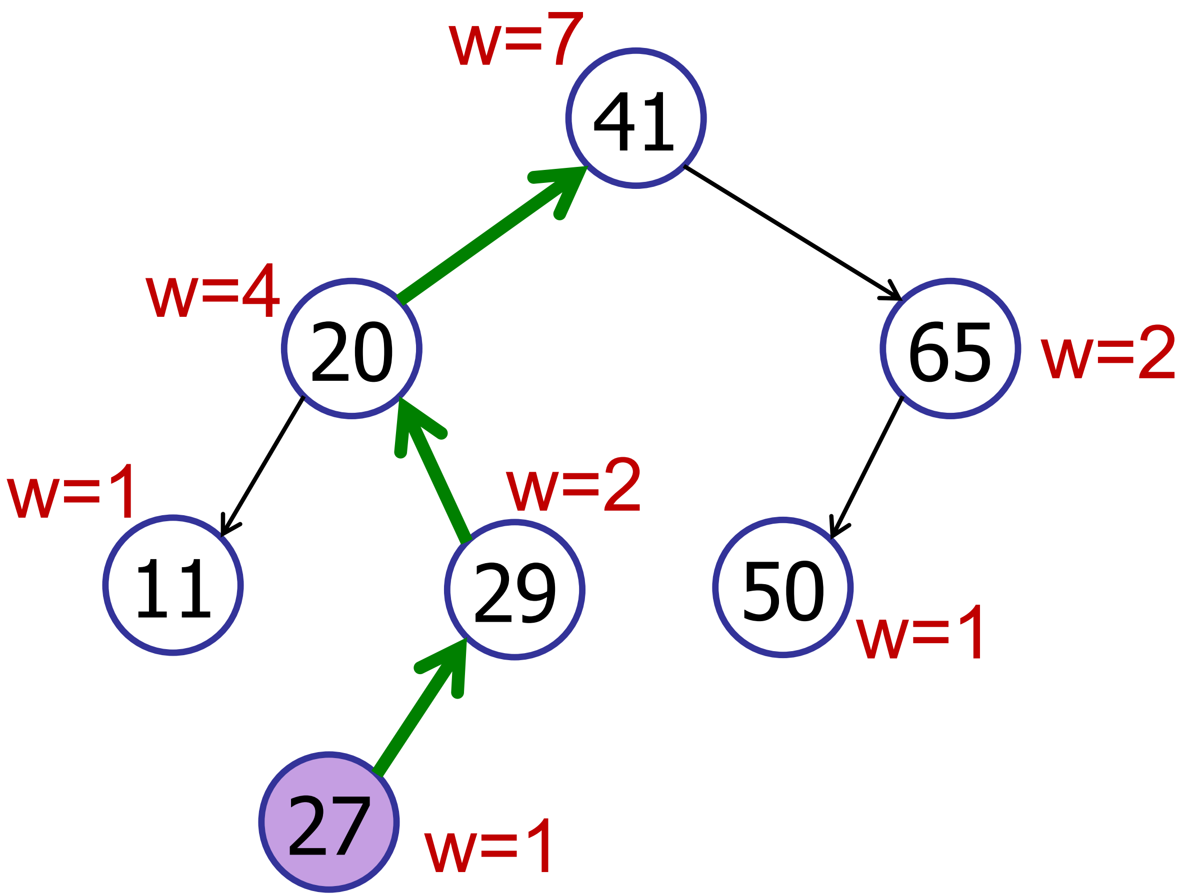
Example: find the 10th tallest student in the class.

rank(v) : computes the rank of a node v

Example: determine the percentile of Johnny's height.
Is Johnny in the 10^{th} percentile or the 90^{th} percentile?

Dynamic Order Statistics

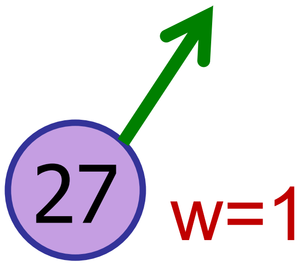
Example: rank(27)



rank = 1

Dynamic Order Statistics

Example: rank(27)



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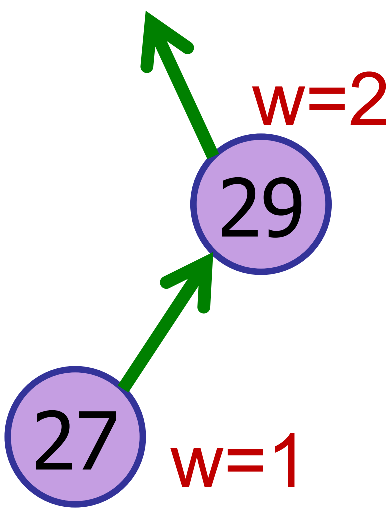
Dynamic Order Statistics

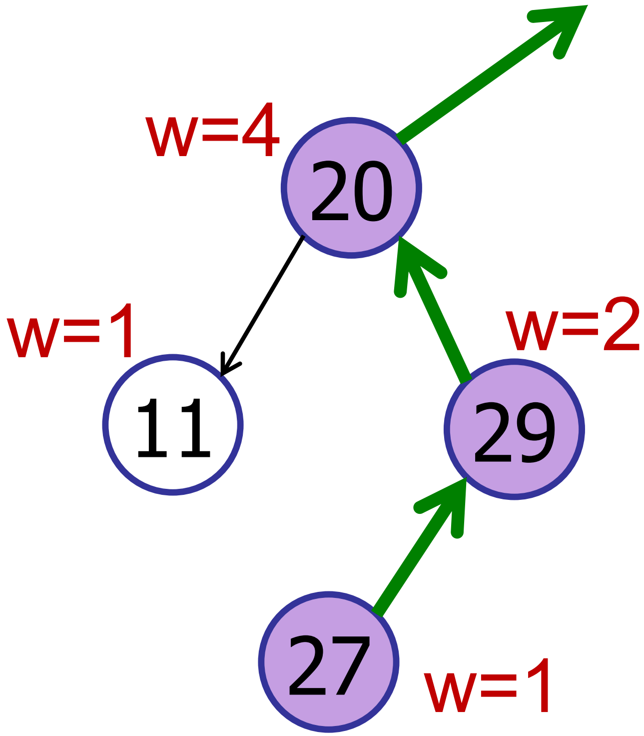
Example: rank(27)



rank = 1

Dynamic Order Statistics

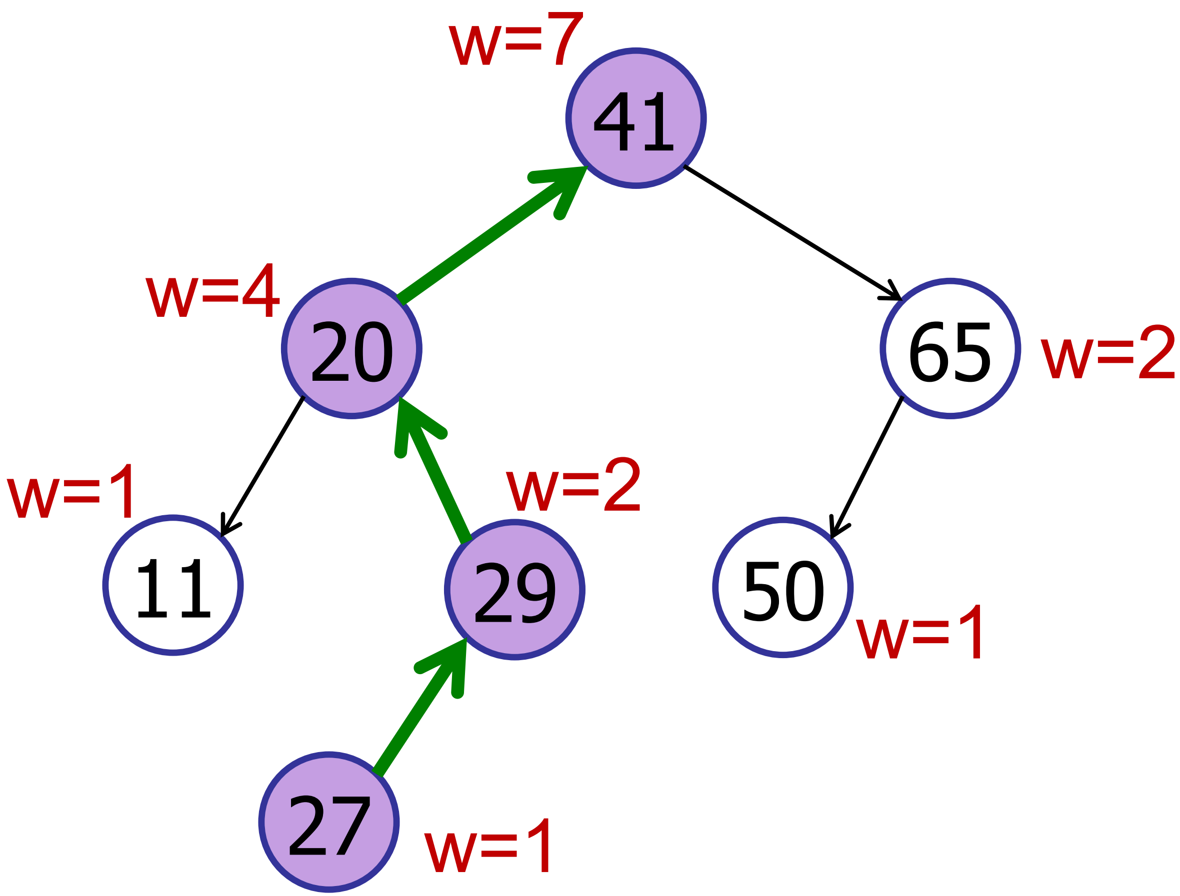
Example: rank(27)



rank = 1 + 2

Dynamic Order Statistics

Example: rank(27)



rank = 1 + 2 = 3

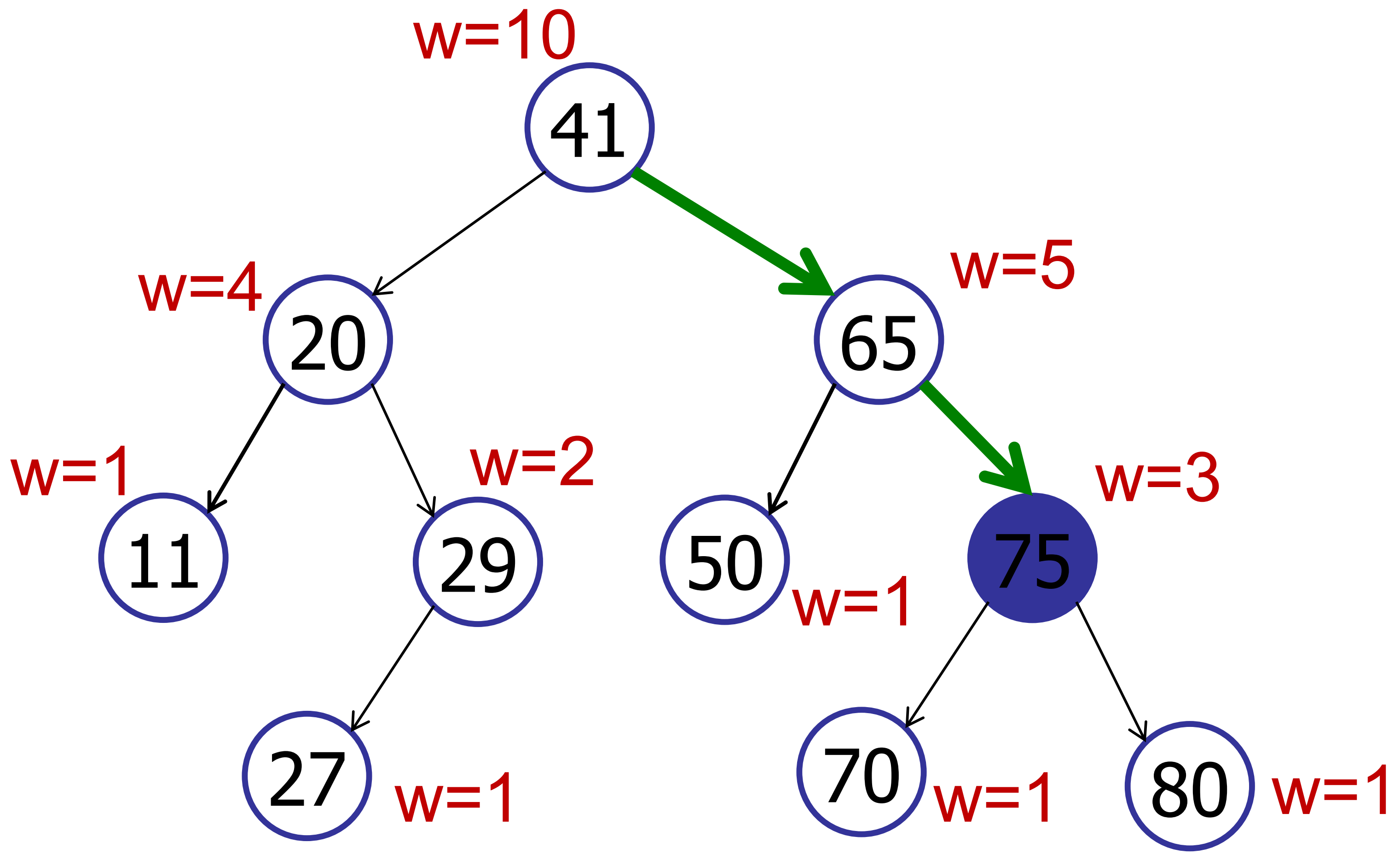
Dynamic Order Statistics

Rank(v) : computes the rank of a node v

```
rank(node)
    rank = node.left.weight + 1;
    while (node != null) do
        if node is left child then
            do nothing
        else if node is right child then
            rank += node.parent.left.weight + 1;
        node = node.parent;
    return rank;
```

Dynamic Order Statistics

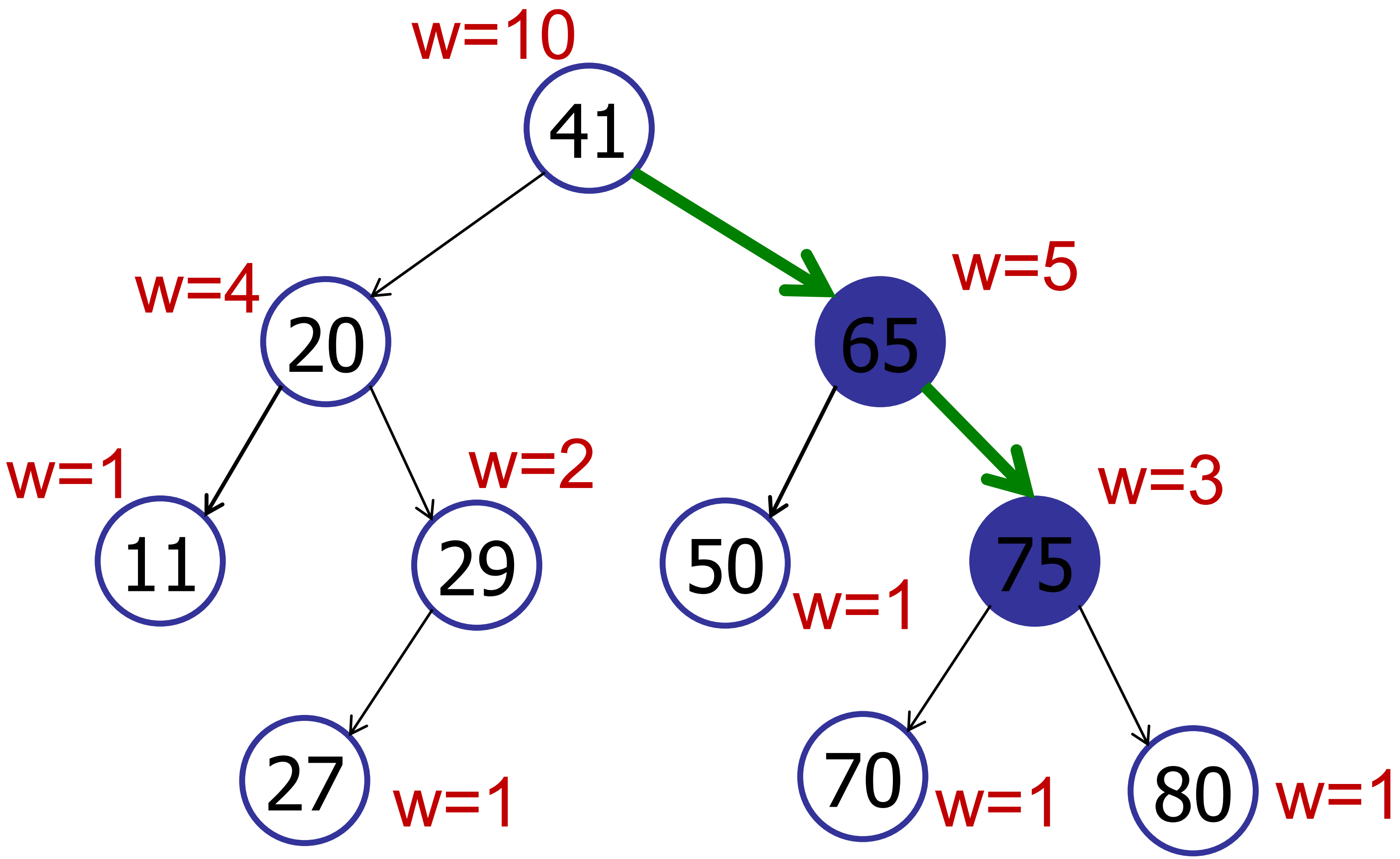
rank(75)



rank = 2

Dynamic Order Statistics

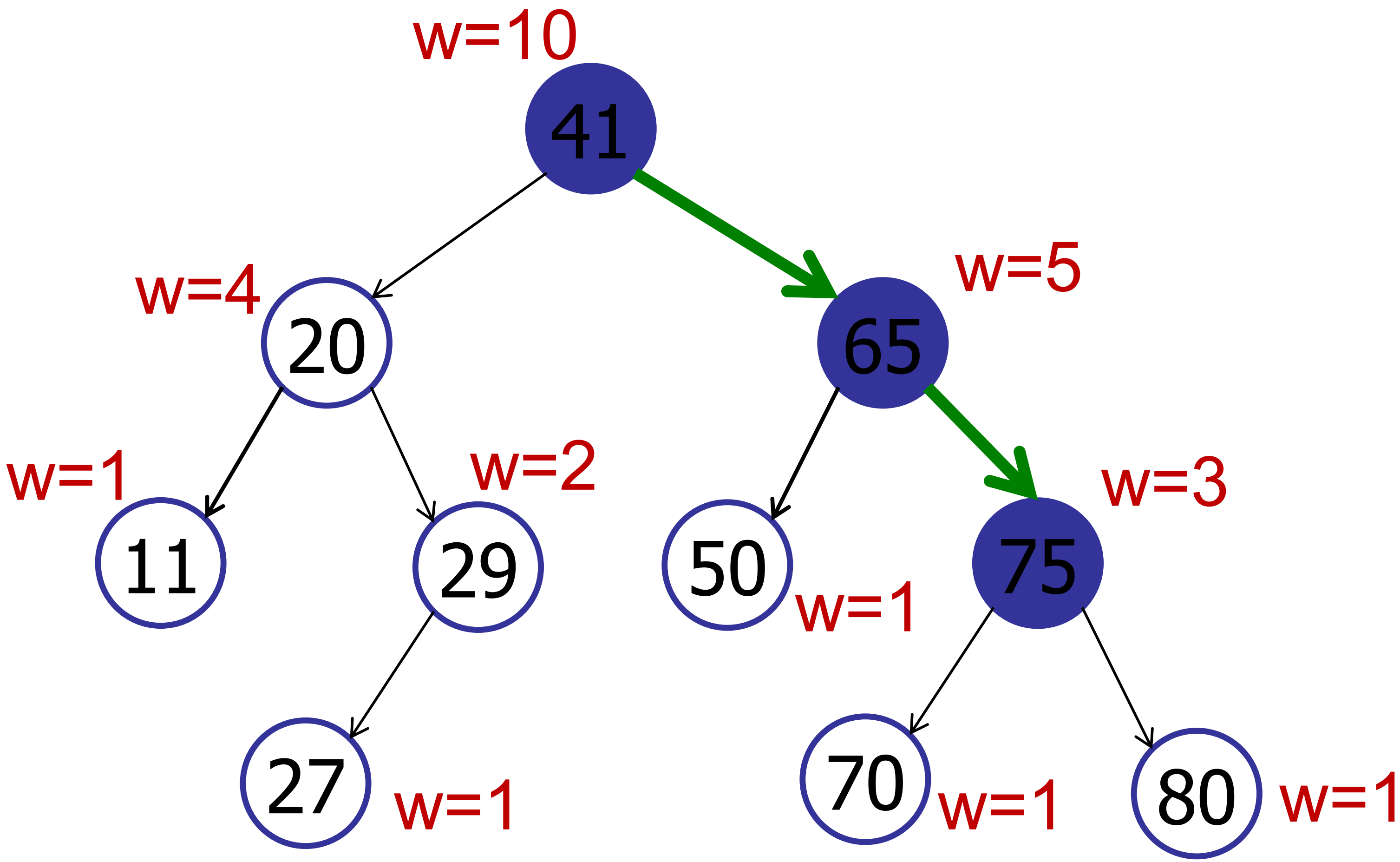
rank(75)



rank = 2 + 2

Dynamic Order Statistics

rank(75)



rank = 2 + 2 + 5 = 9

Dynamic Order Statistics

Rank(v) : computes the rank of a node v

```
rank(node)
    rank = node.left.weight + 1;
    while (node != null) do
        if node is left child then
            do nothing
        else if node is right child then
            rank += node.parent.left.weight + 1;
        node = node.parent;
    return rank;
```

Augmenting data structures

Basic methodology:

1. Choose underlying data structure:

    AVL tree

2. Determine additional info needed:

    Weight of each node

3. Maintained info as data structure is modified.

    Update weights as needed

4. Develop new operations using the new info.

    Select and Rank

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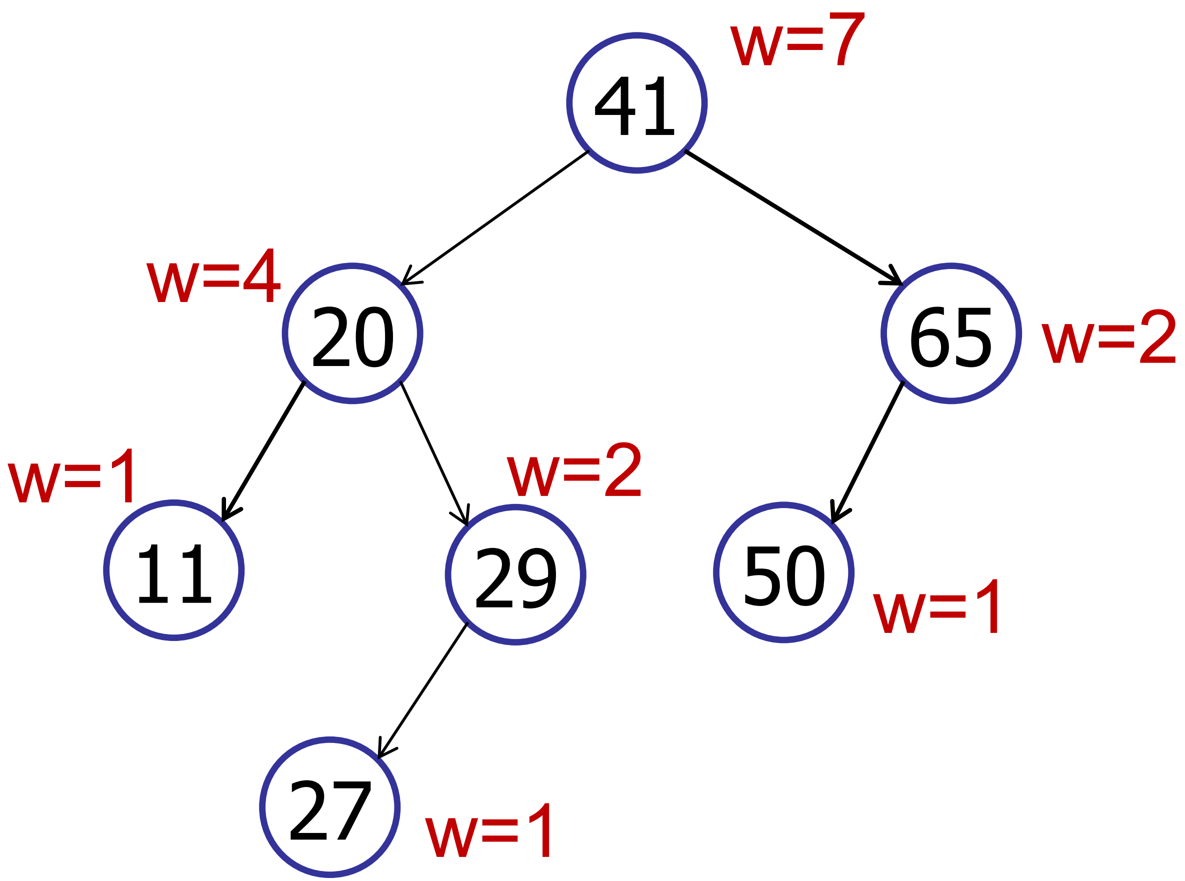
Augmented Trees

Maintain weight during insertions:



Augmented Trees

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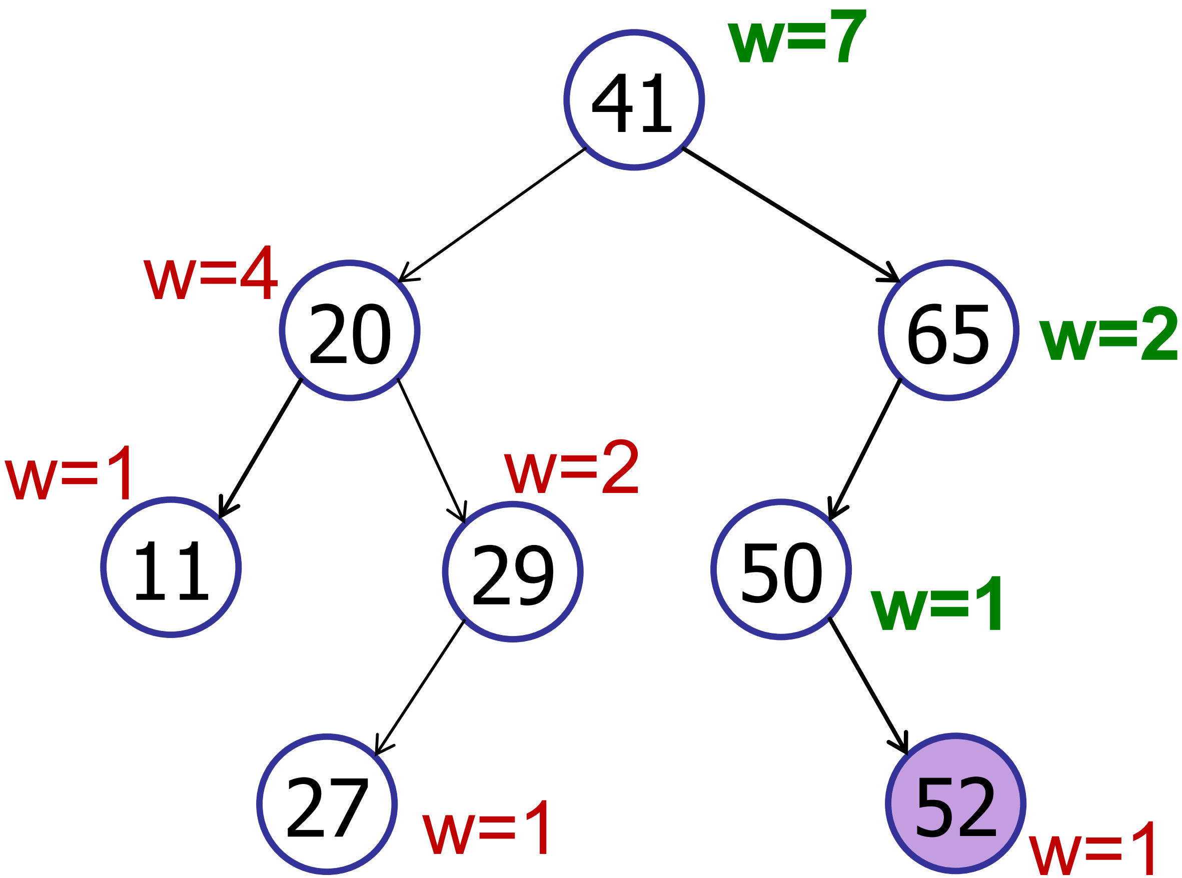
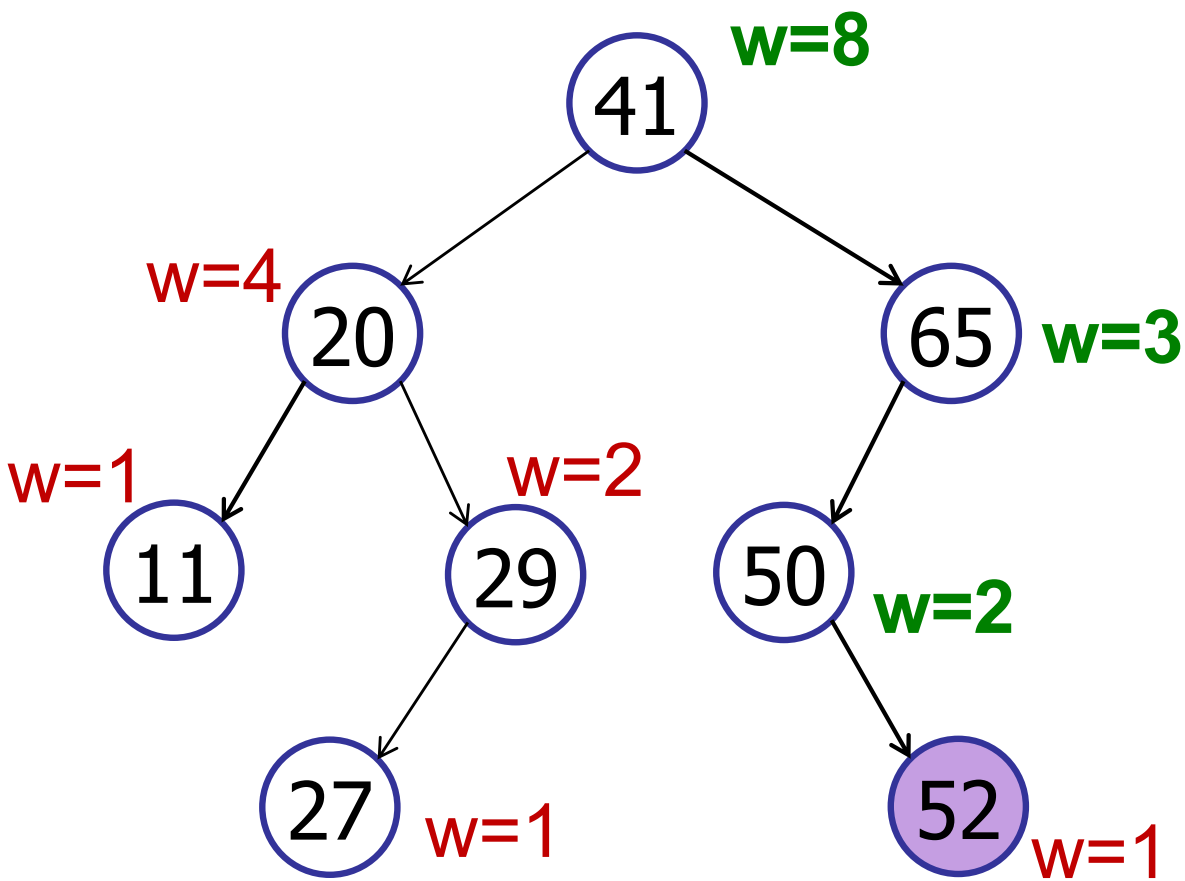


Augmented Trees

Maintain weight during insertions:



Augmented Trees

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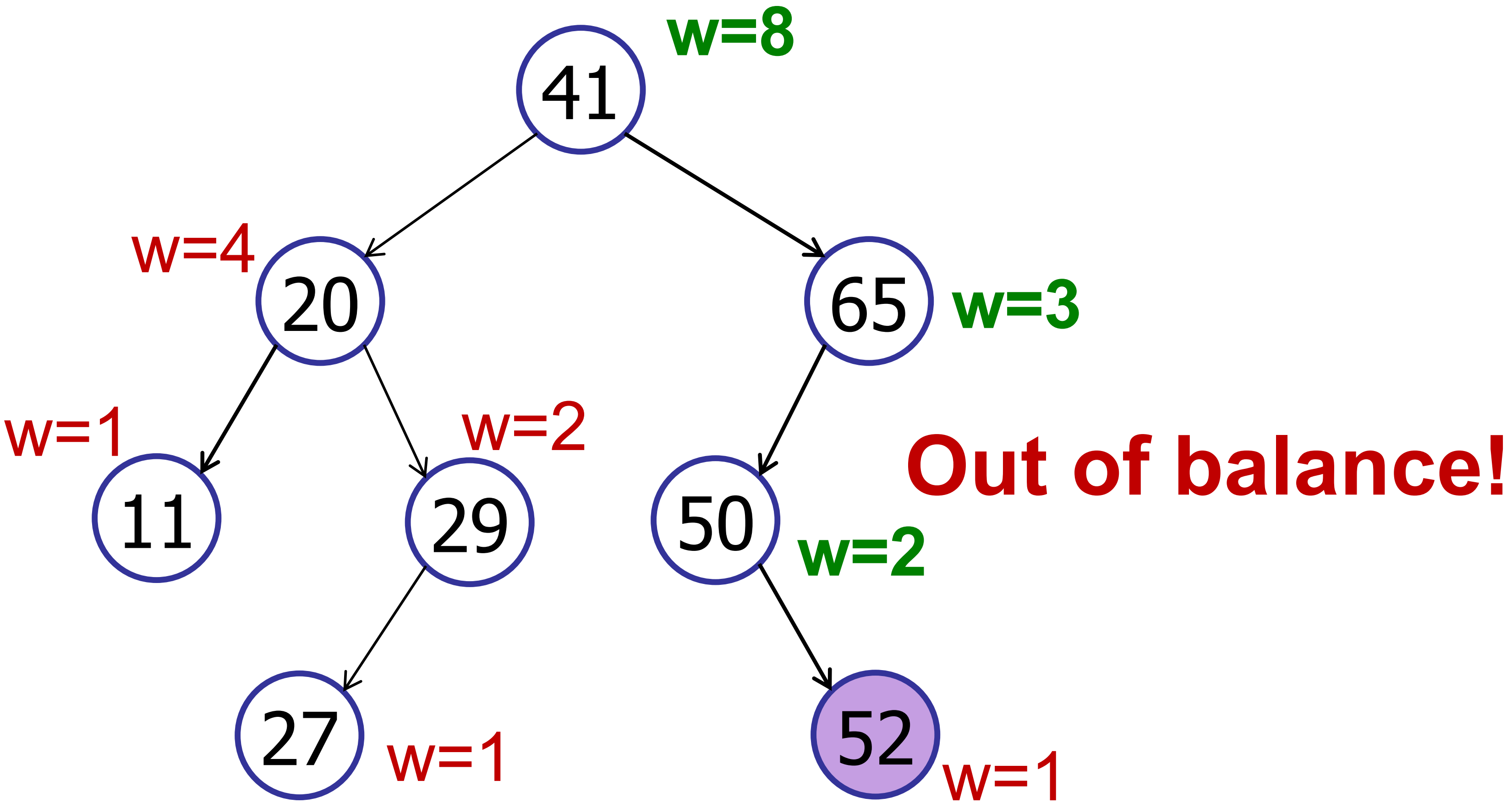
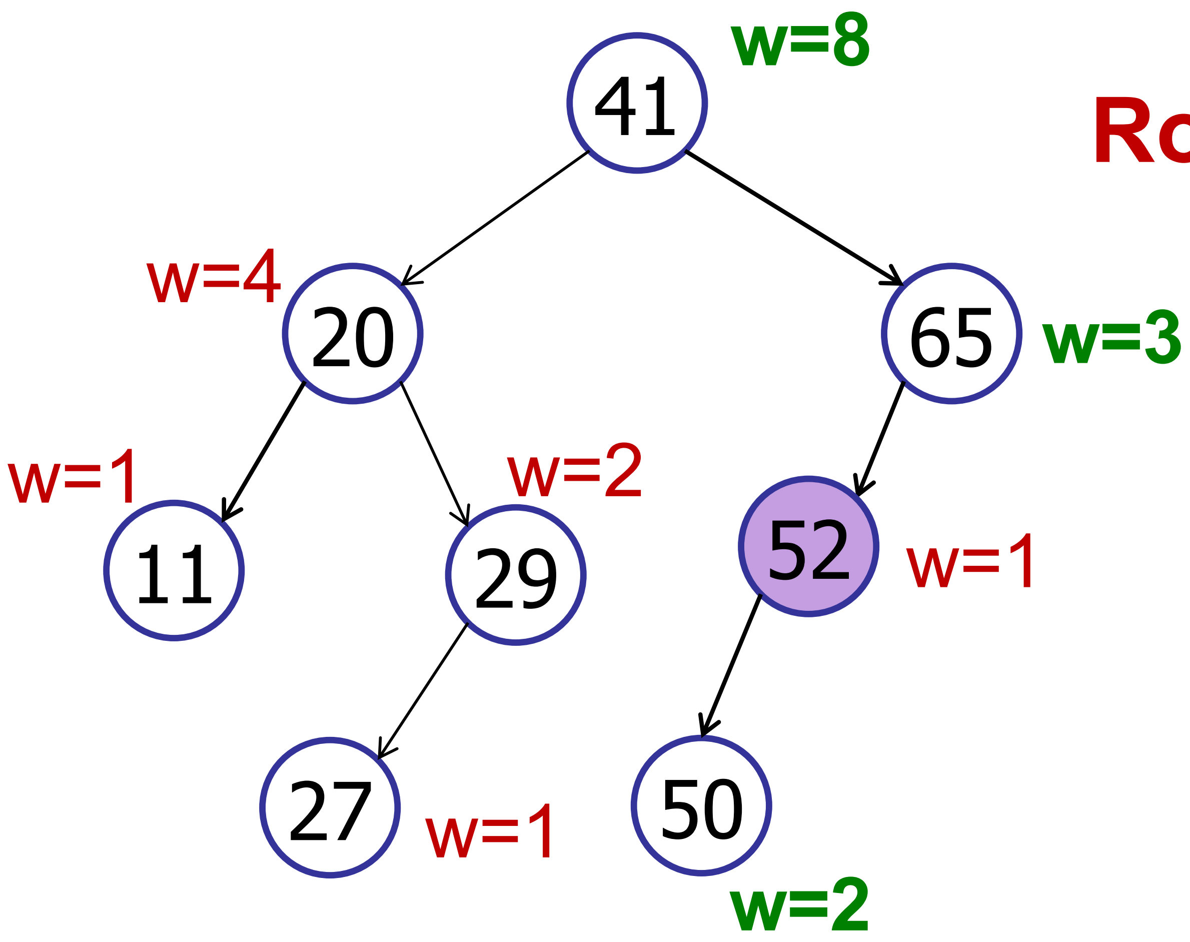


Augmented Trees

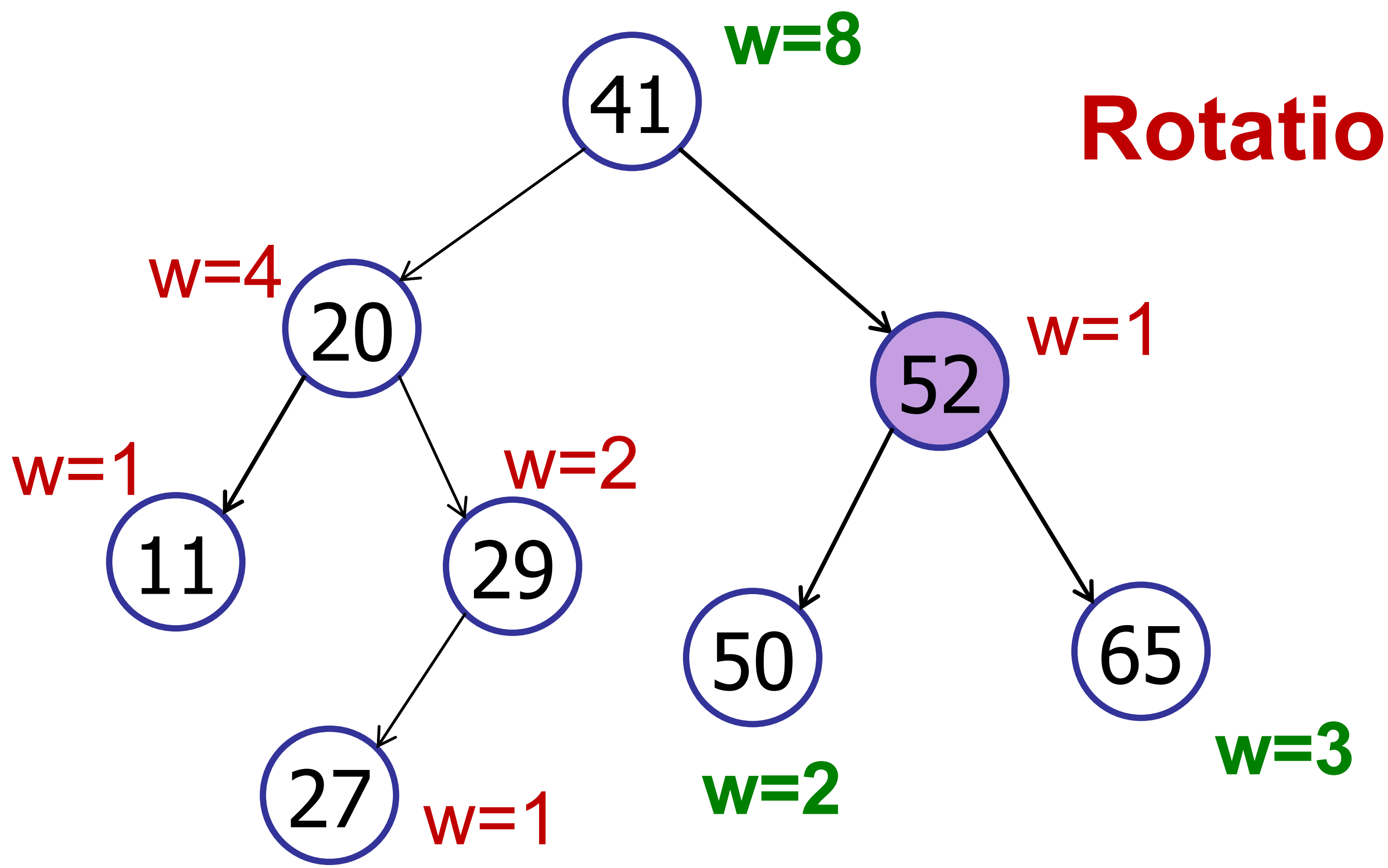
Maintain weight during insertions:



Rotation 1:

Augmented Trees

Maintain weight during insertions:

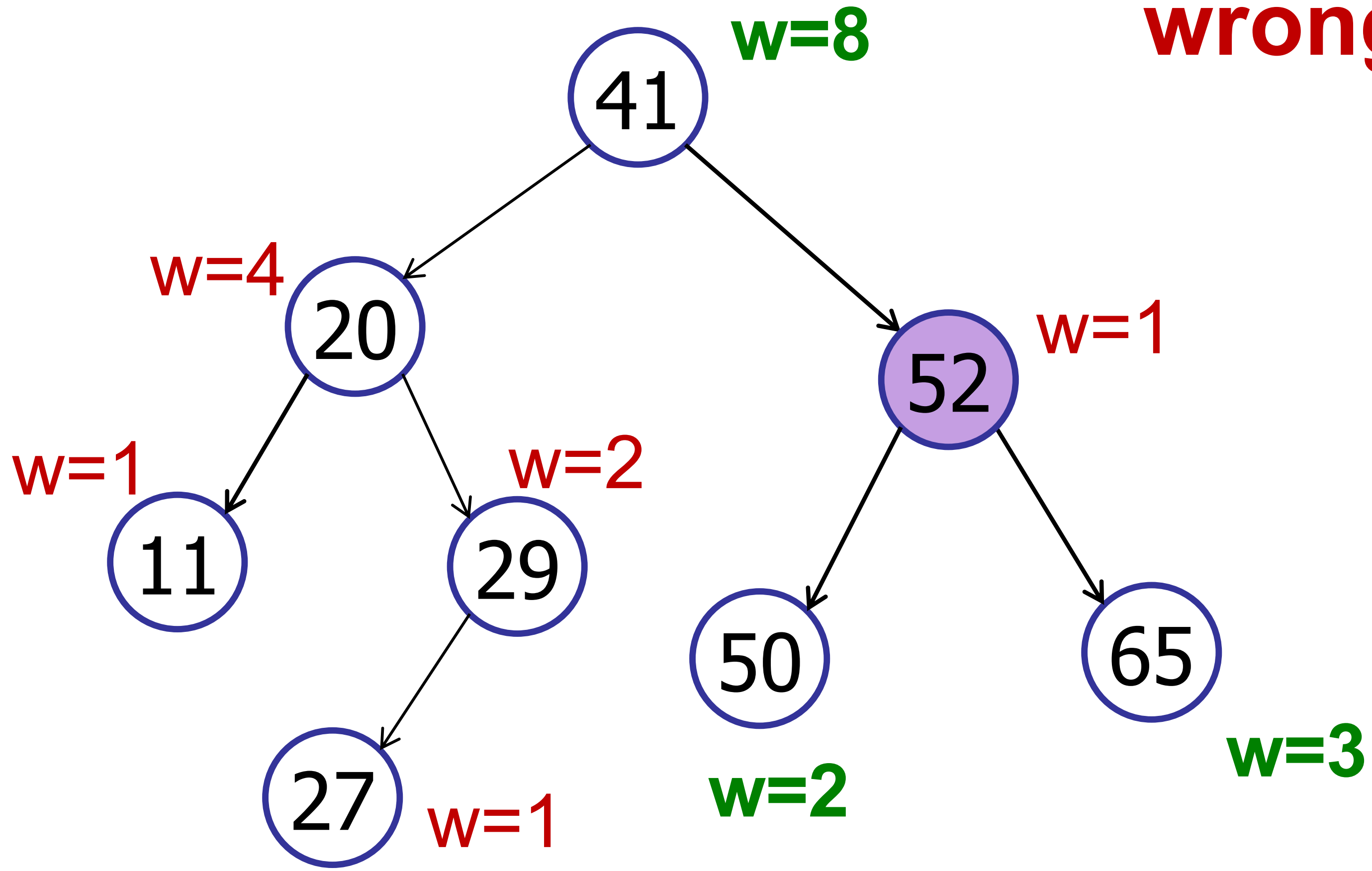


Rotation 2:

Augmented Trees

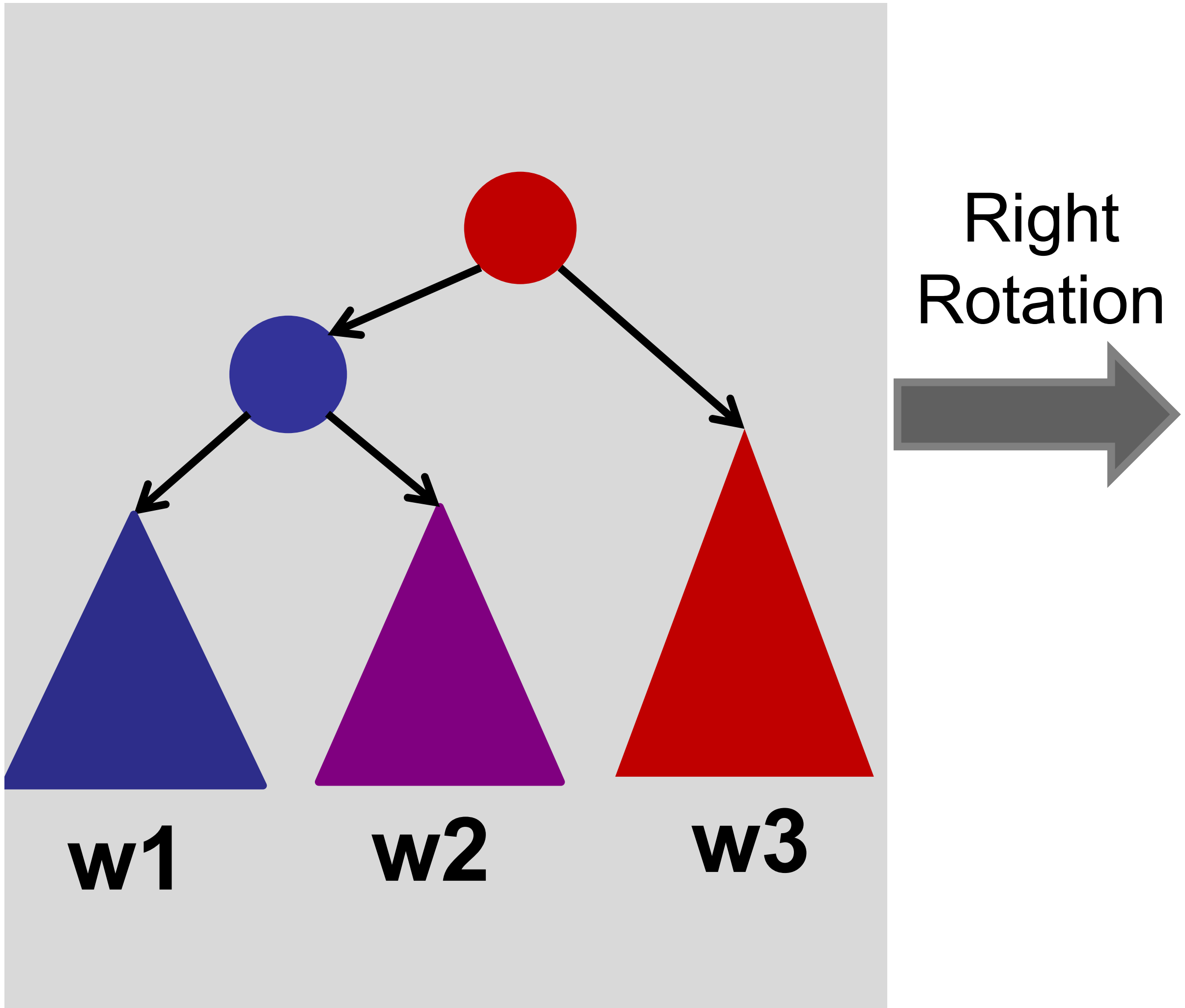
How to update weights on rotation?

Weights all wrong!



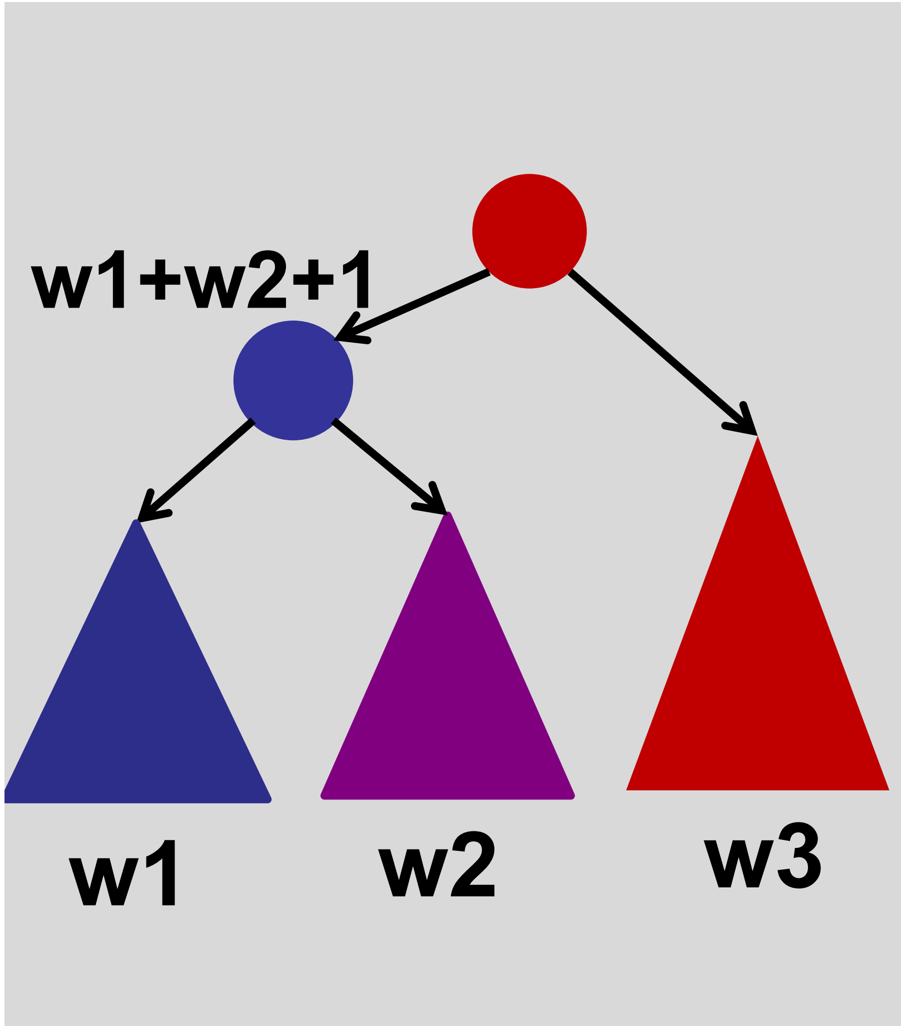
Augmented Trees

Maintain weight during rotations:



Augmented Trees

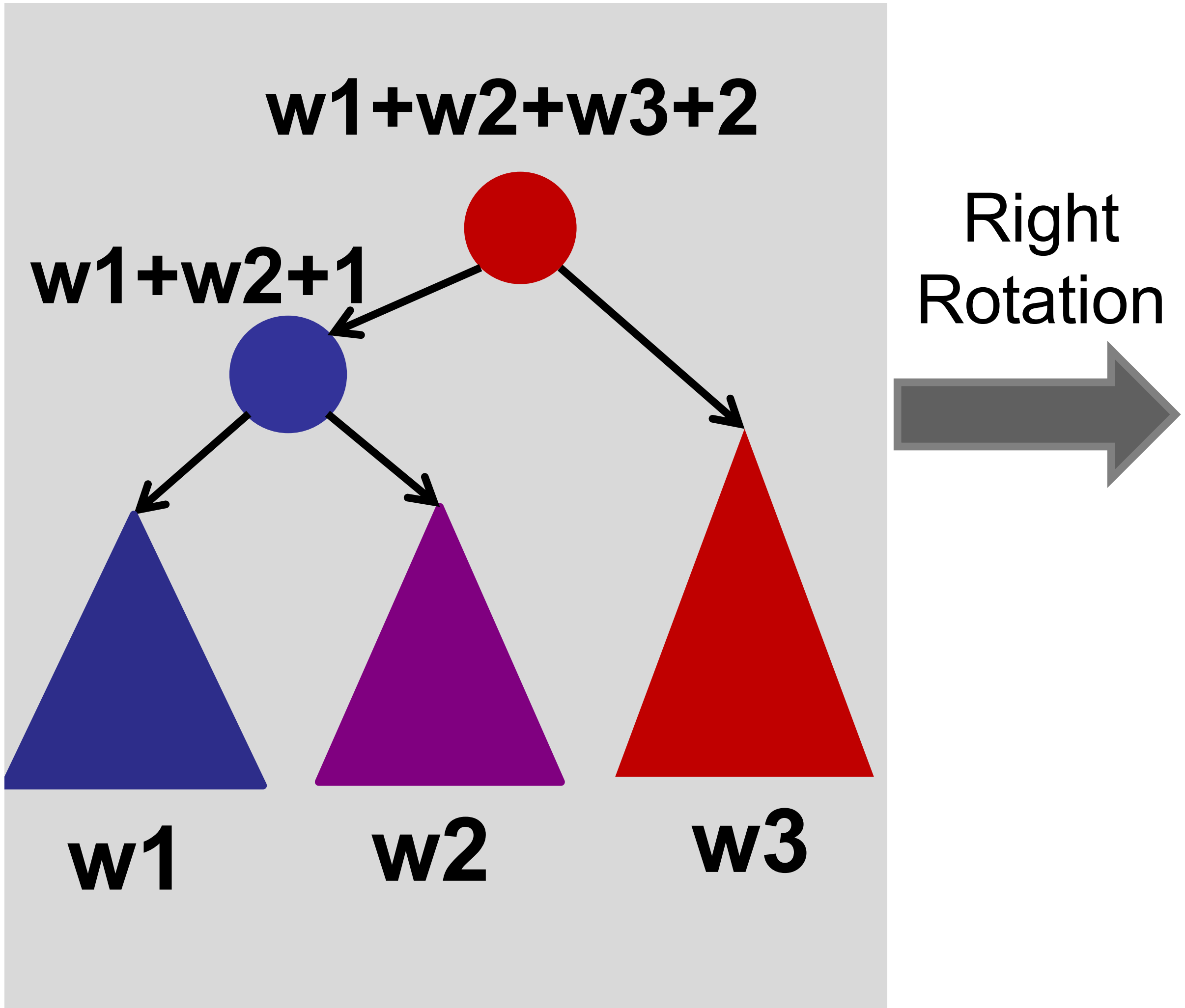
Maintain weight during rotations:



Right
Rotation

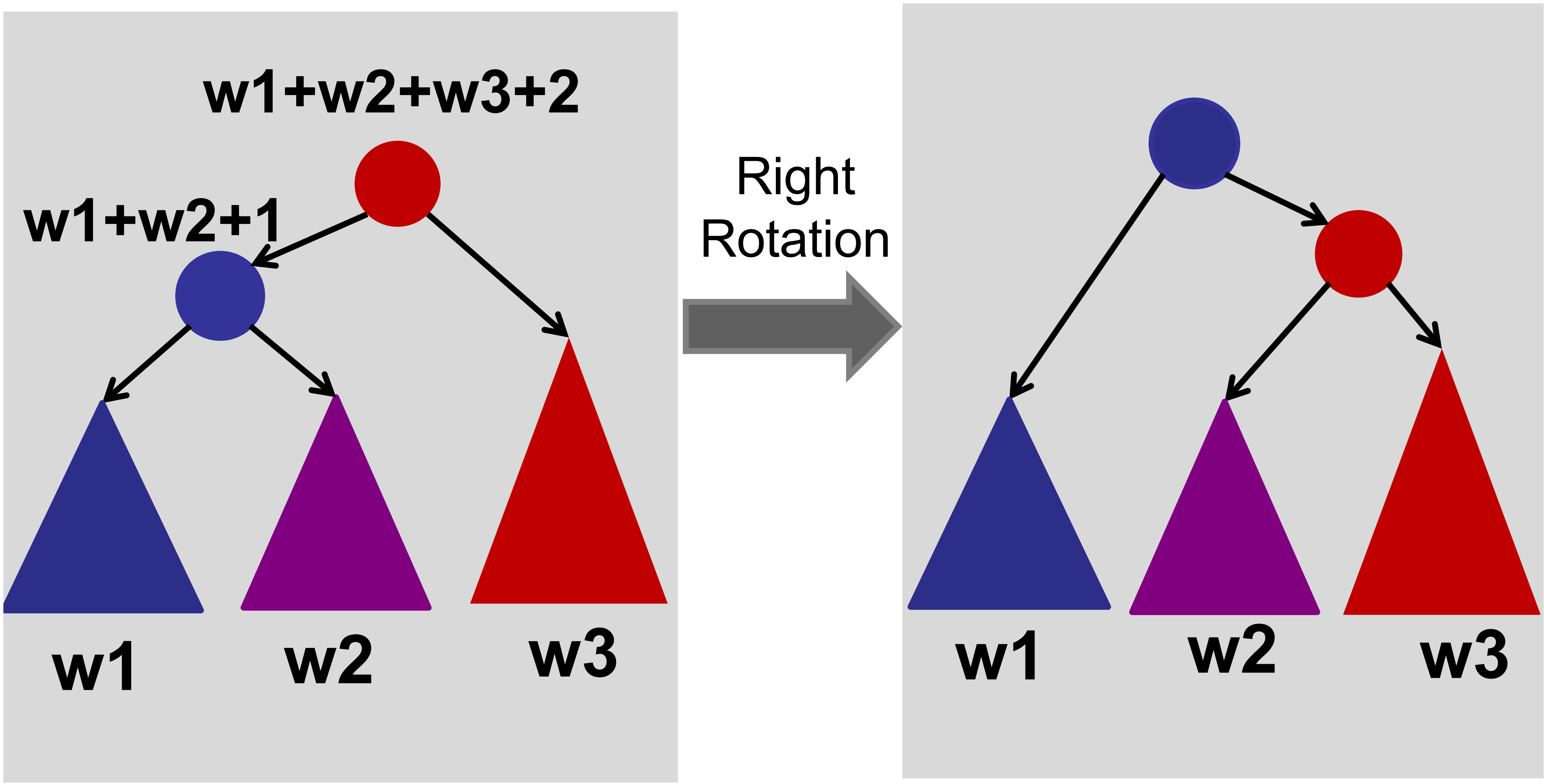
Augmented Trees

Maintain weight during rotations:



Augmented Trees

Maintain weight during rotations:



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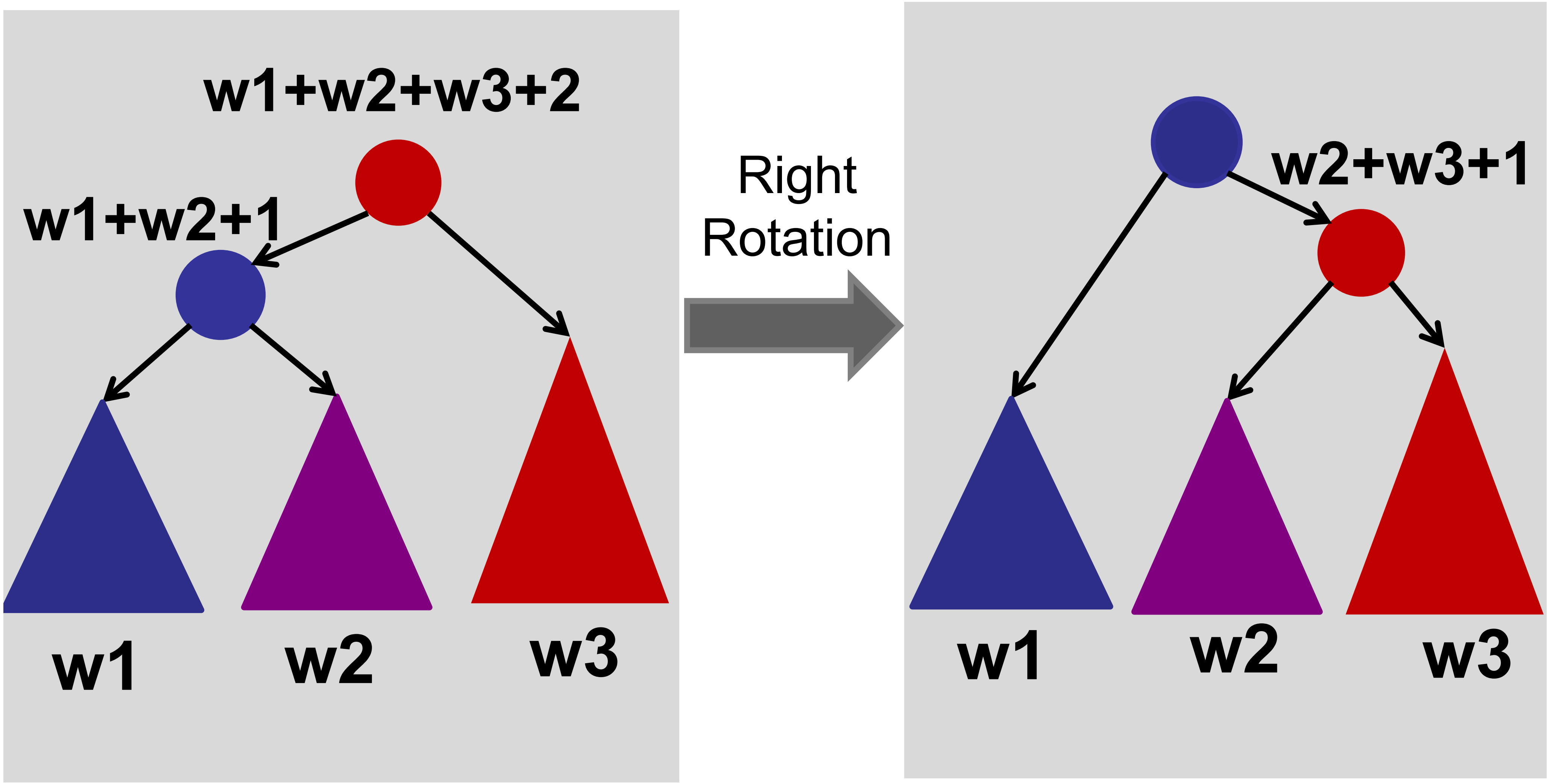

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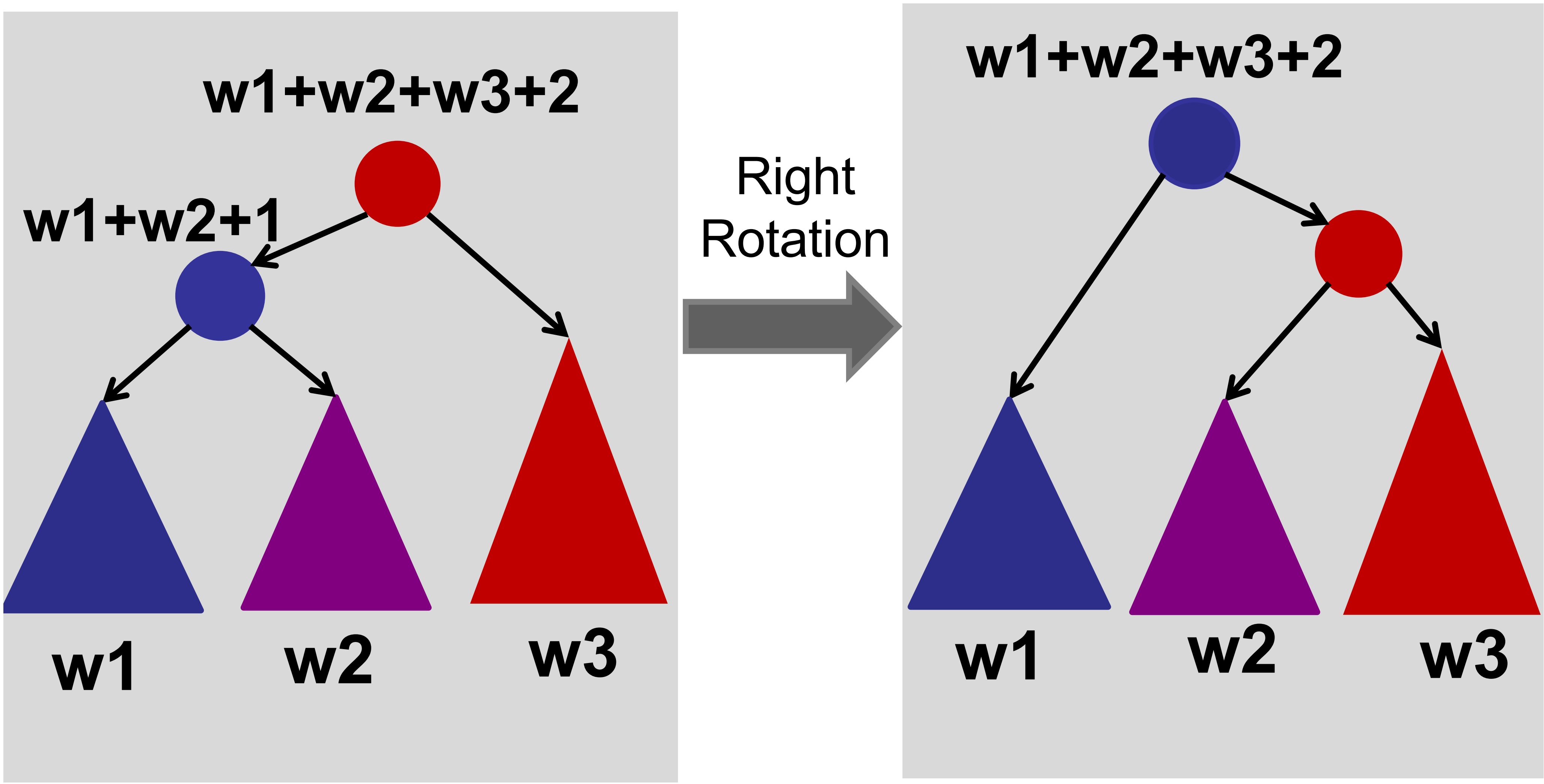
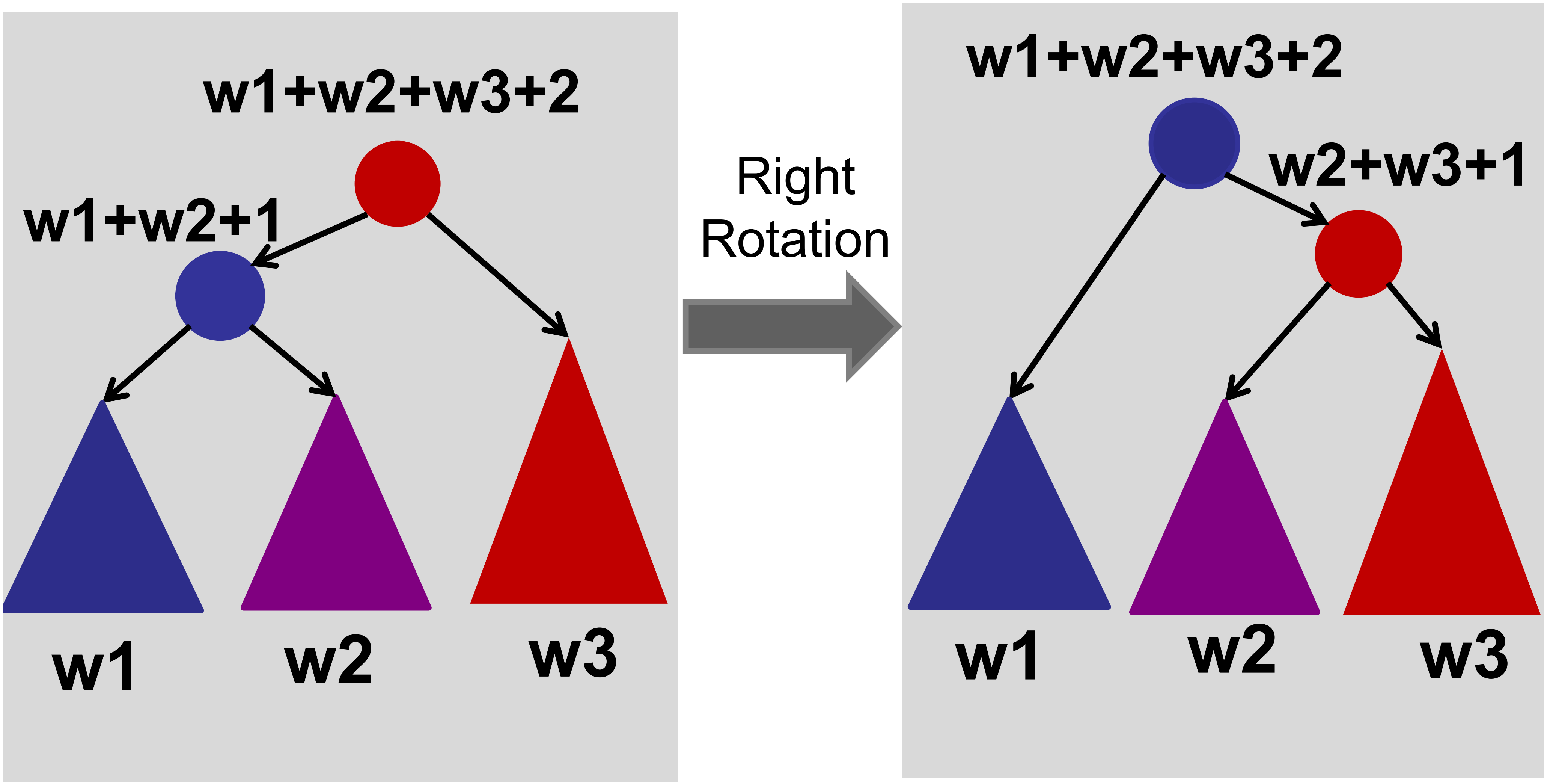


Augmented Trees

Maintain weight during rotations:



How long does it take to update the weights during a rotation?

1. O(1)
2. O(log n)
3. O(n)
4. O(n^2)
5. What is a rotation?

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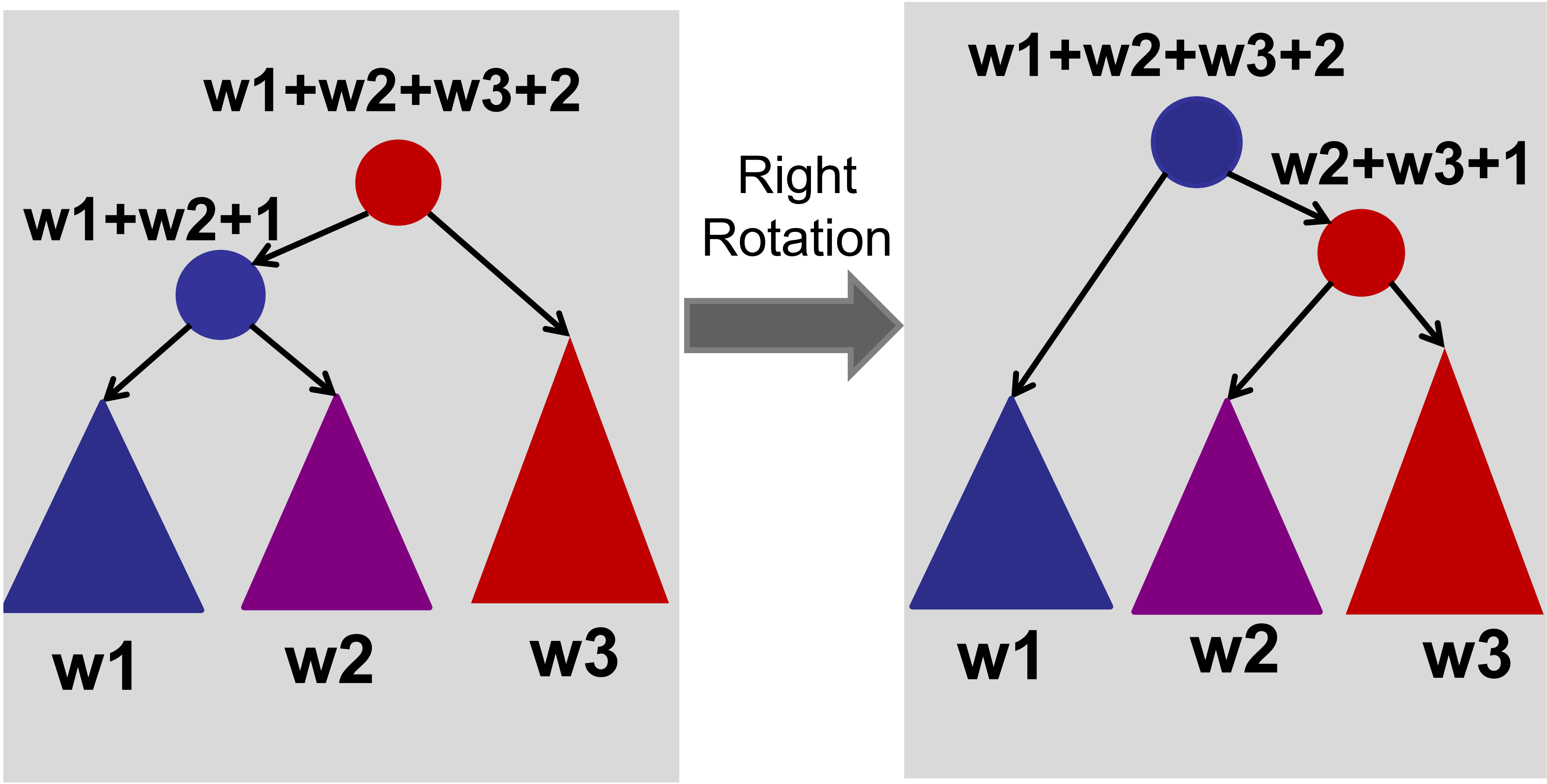
is open

Augmented Trees

Maintain weight during rotations:



Augmenting data structures

Basic methodology:

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   (tree, hash table, linked list, stack, etc.)

2. Determine additional info needed.

3. Verify that the additional info can be maintained as the data structure is modified.

   (subject to insert/delete/etc.)

4. Develop new operations using the new info.

Next few lectures…

Three examples of augmenting balanced BSTs

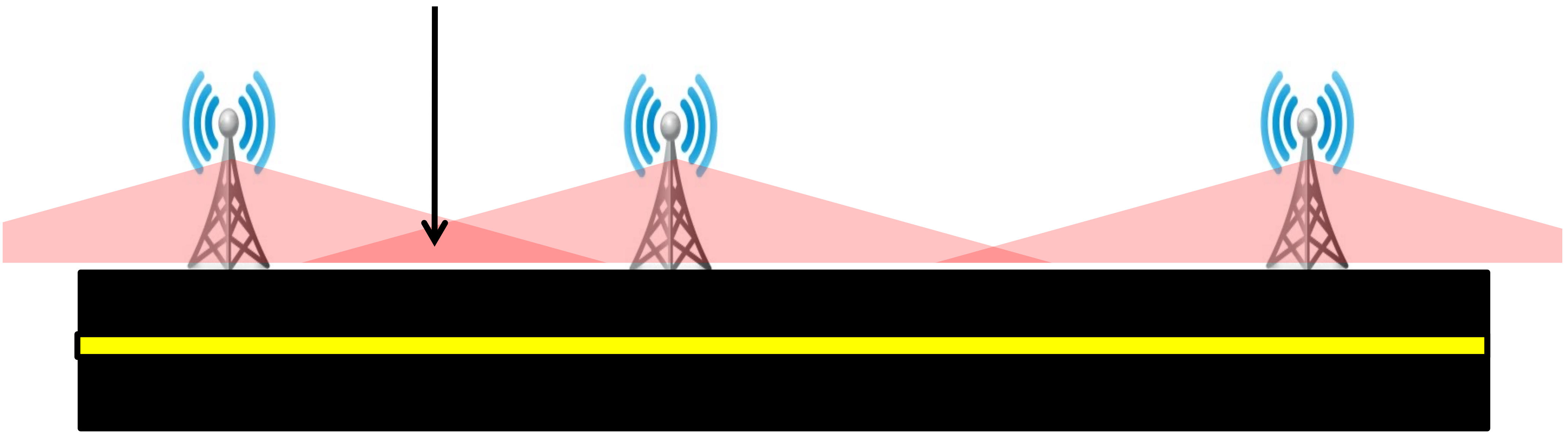
1. Order Statistics

2. Interval Queries

3. Orthogonal Range Searching

Cell Tower Coverage

Find a tower that covers my location.



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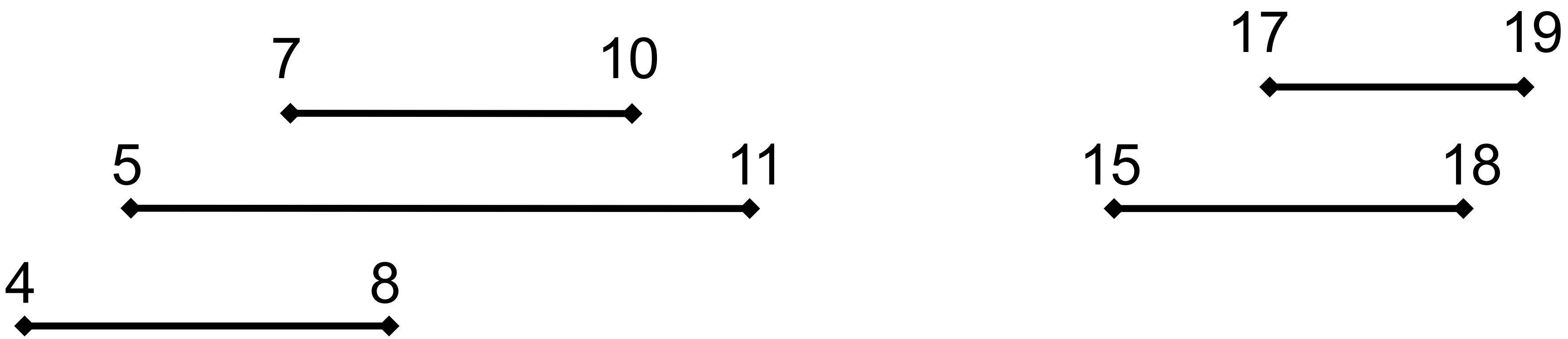
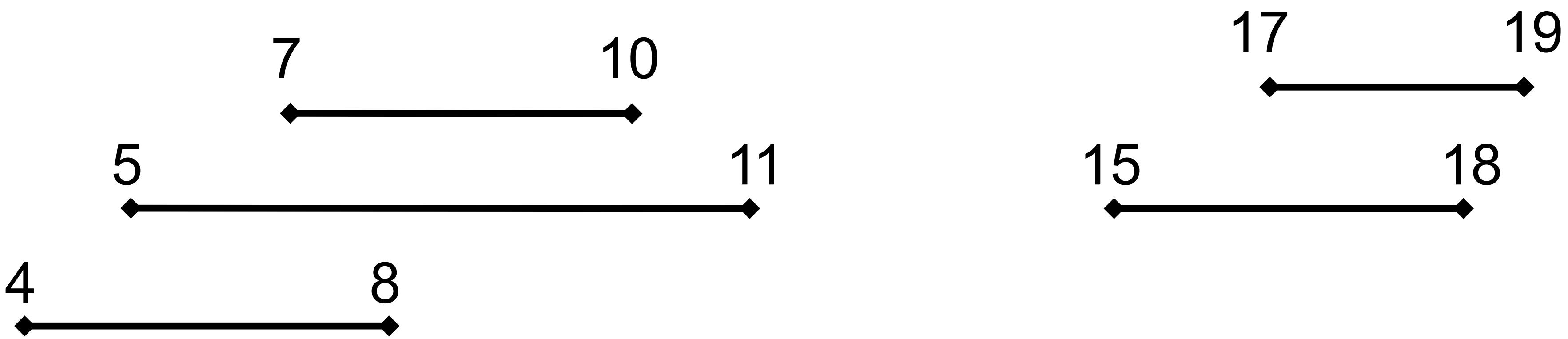


Cell Tower Coverage

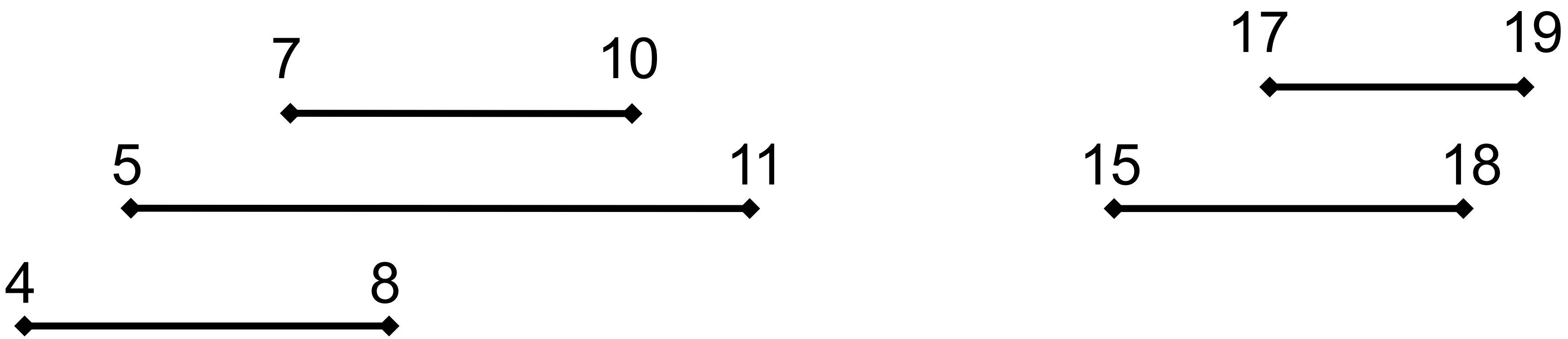
Dynamic data structure: supports new towers.



insert(begin, end)
delete(begin, end)

Cell Tower Coverage

Find a tower that covers my location.

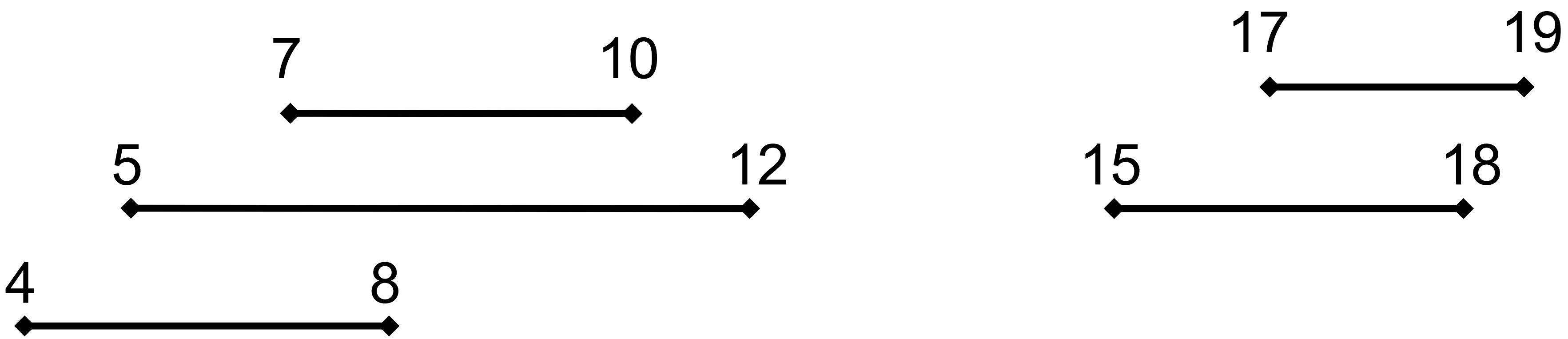


insert(begin, end)
delete(begin, end)

query(x): find an interval that overlaps x.

Cell Tower Coverage

Find a tower that covers my location.



Idea 1: Keep intervals in a list.

Sort by minimum value in interval.

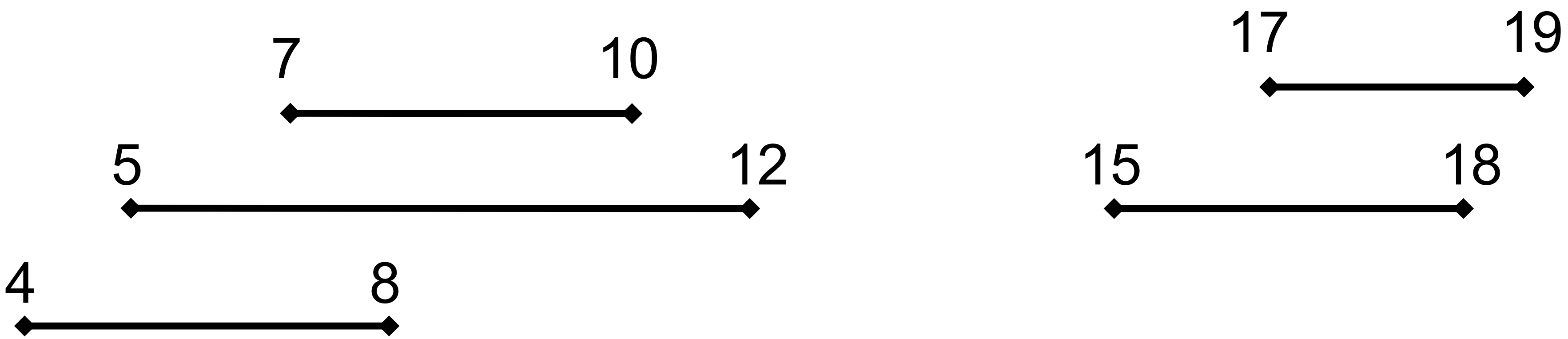
Query: scan entire list.

Does sorting help? Can we binary search?

Cell Tower Coverage

Find a tower that covers my location.

example: query(11)



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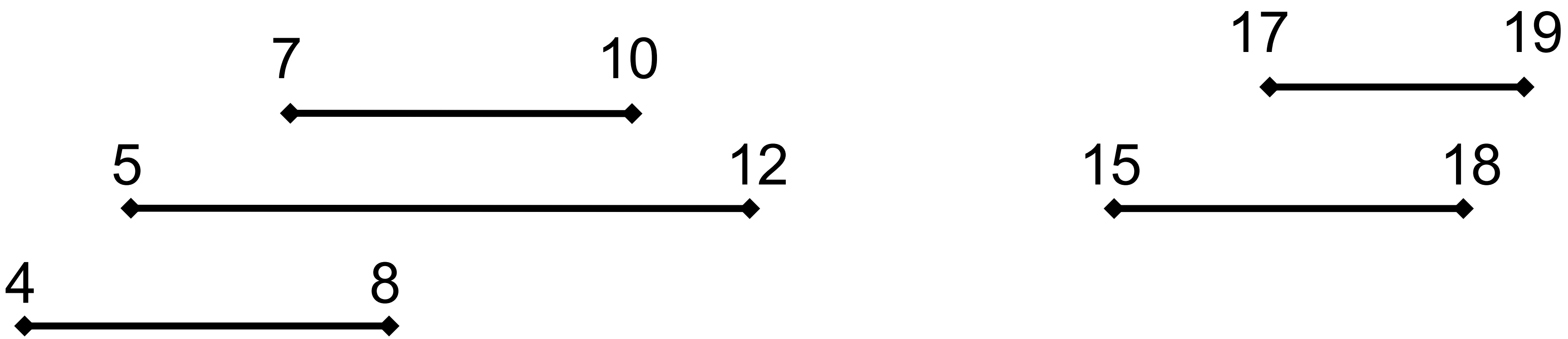
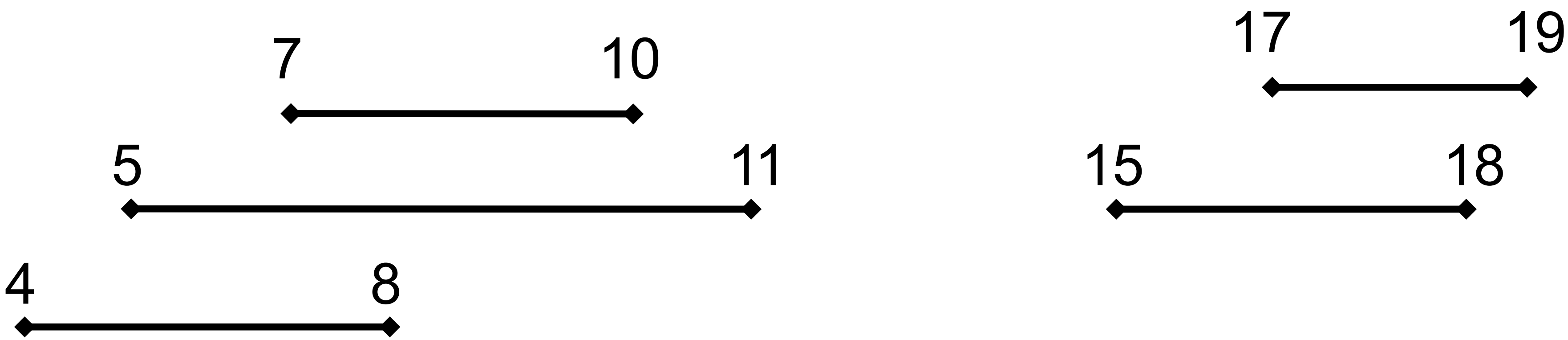


Idea 2: O(1) queries??

Cell Tower Coverage

Find a tower that covers my location.



Idea 2: O(1) queries

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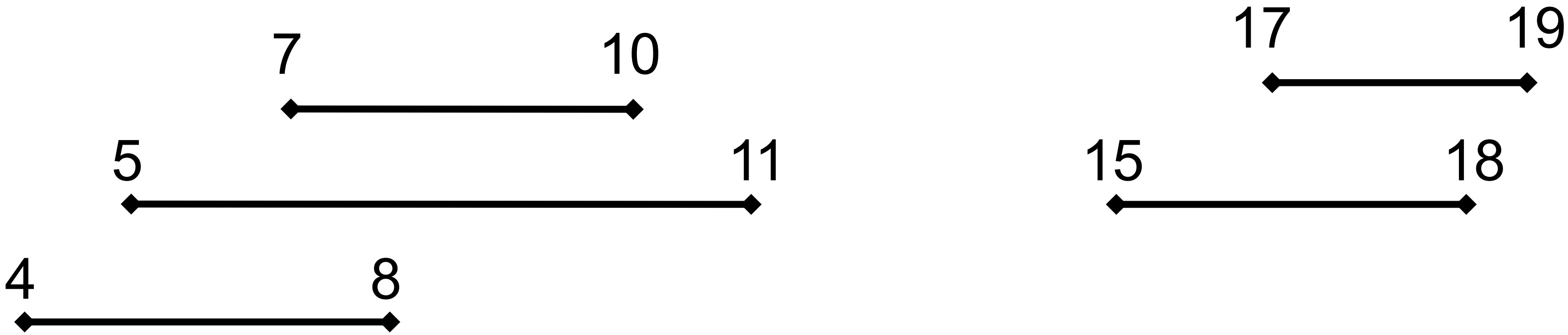
is open

			A	**A**	**A**	**A**	**A**	**B**	**B**	**C**				**D**	**D**	**D**	**D**	**E**	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Problems??

Cell Tower Coverage

Find a tower that covers my location.



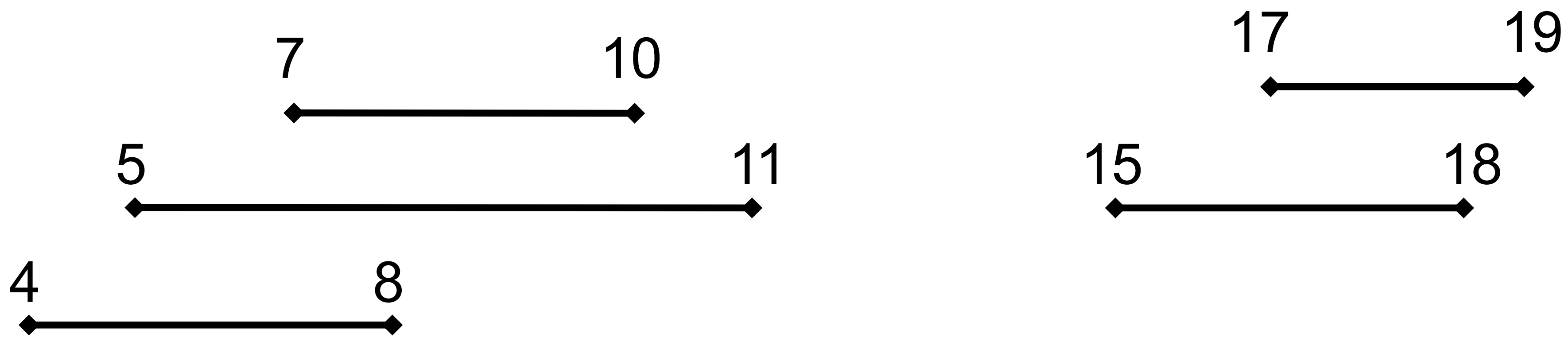
Idea 2: O(1) queries

			A	**A**	**A**	**A**	**A**	**B**	**B**	**C**				**D**	**D**	**D**	**D**	**E**	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Space usage, requires discrete integers, potentially expensive to update.

Cell Tower Coverage

Find a tower that covers my location.

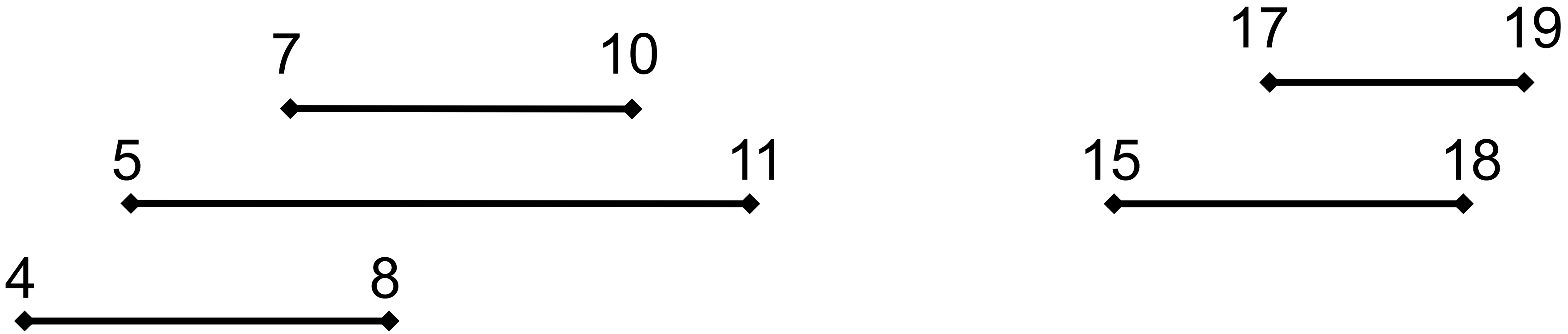


Not ideal solution:

- Space depends on the values stored.
- Time depends on the values stored.

Cell Tower Coverage

Find a tower that covers my location.



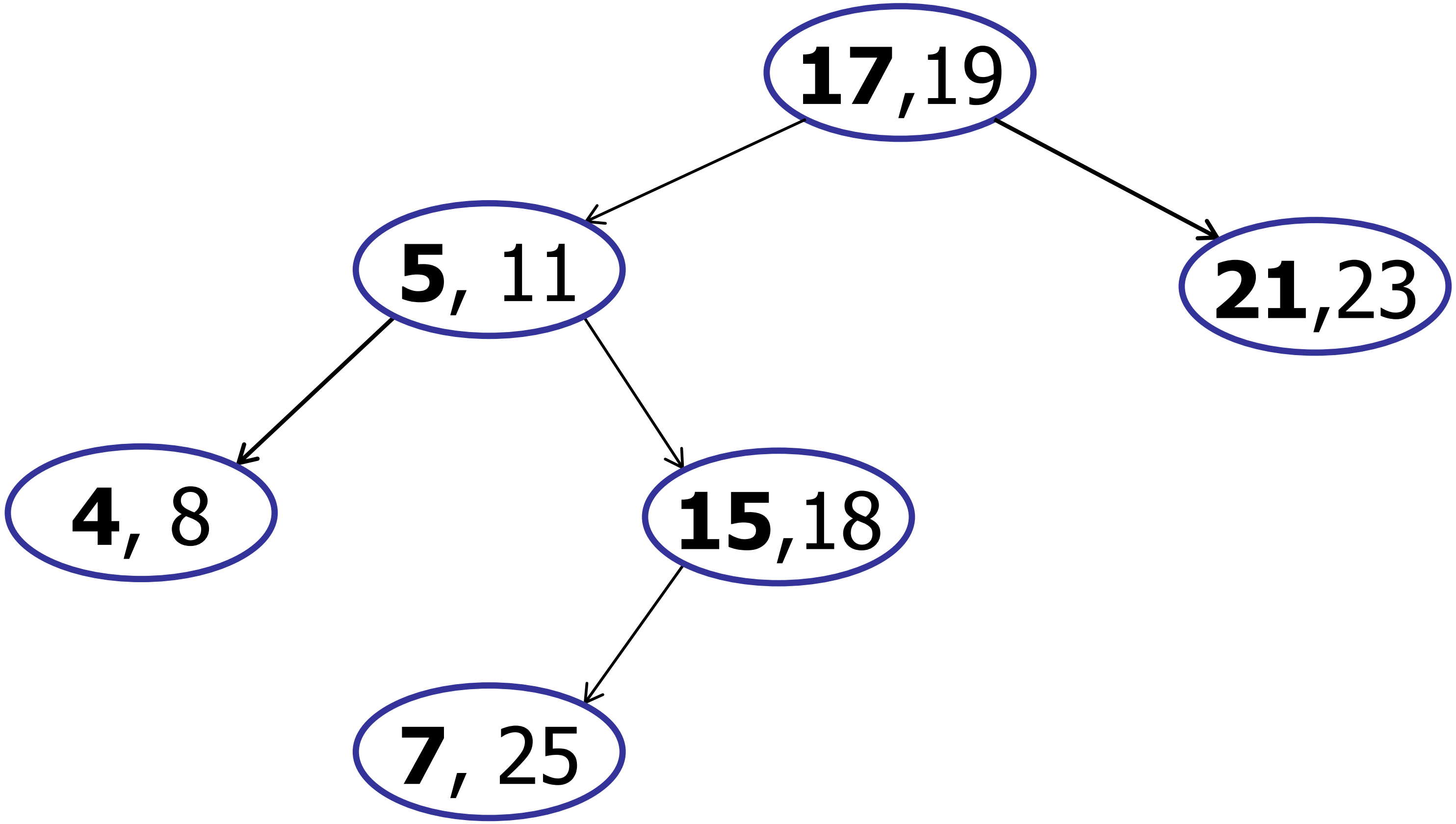
Goal:

- Solutions where space is linear (or near linear) in the number of things stored (i.e., intervals).
- Operations are logarithmic in # of things stored.

Idea 3: Interval Trees

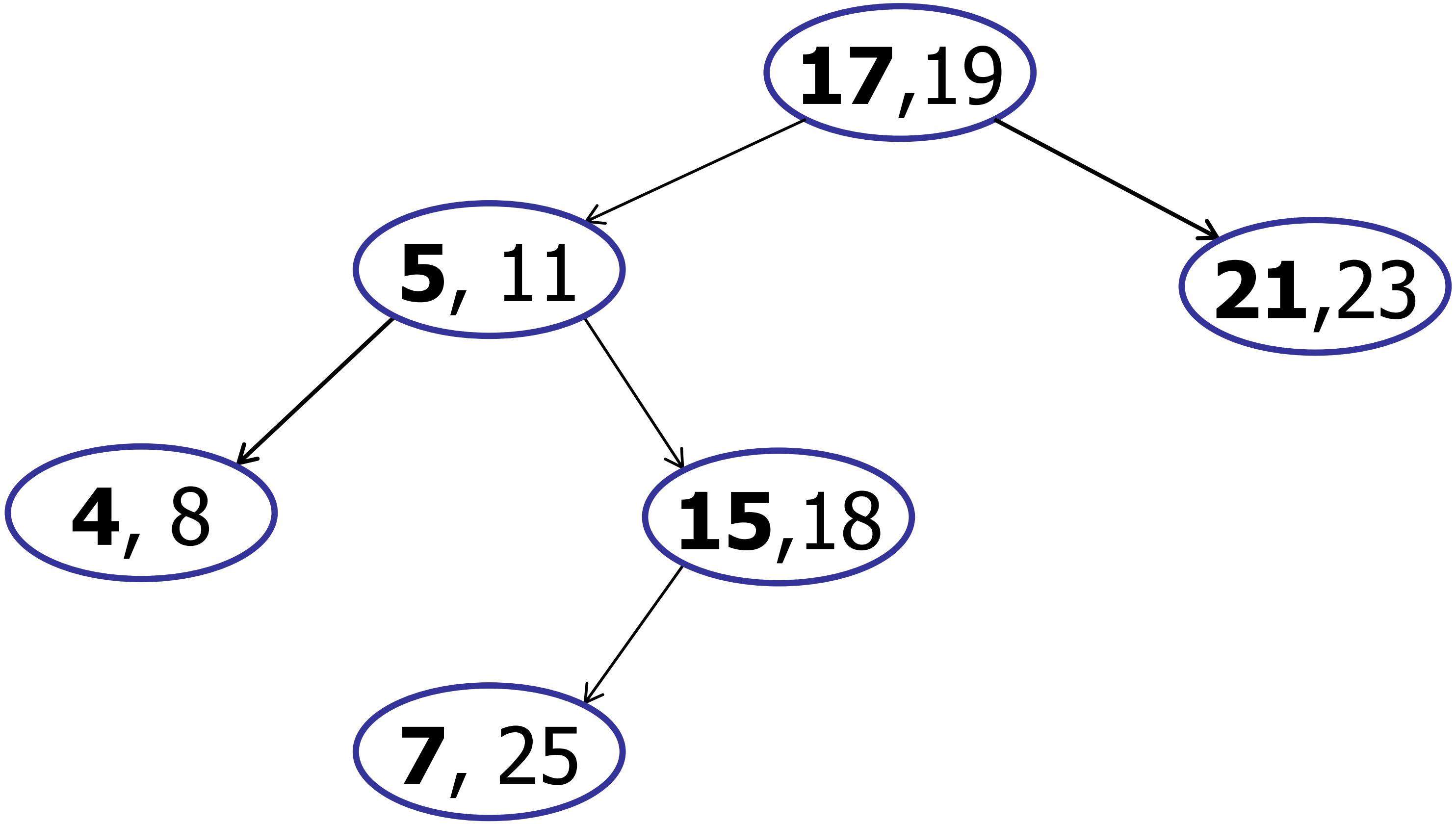
Interval Trees

Each node is an interval



Interval Trees

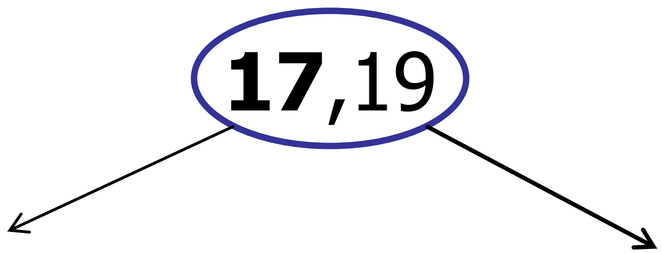
Sorted by left endpoint



Important: always specify what your tree is sorted by!

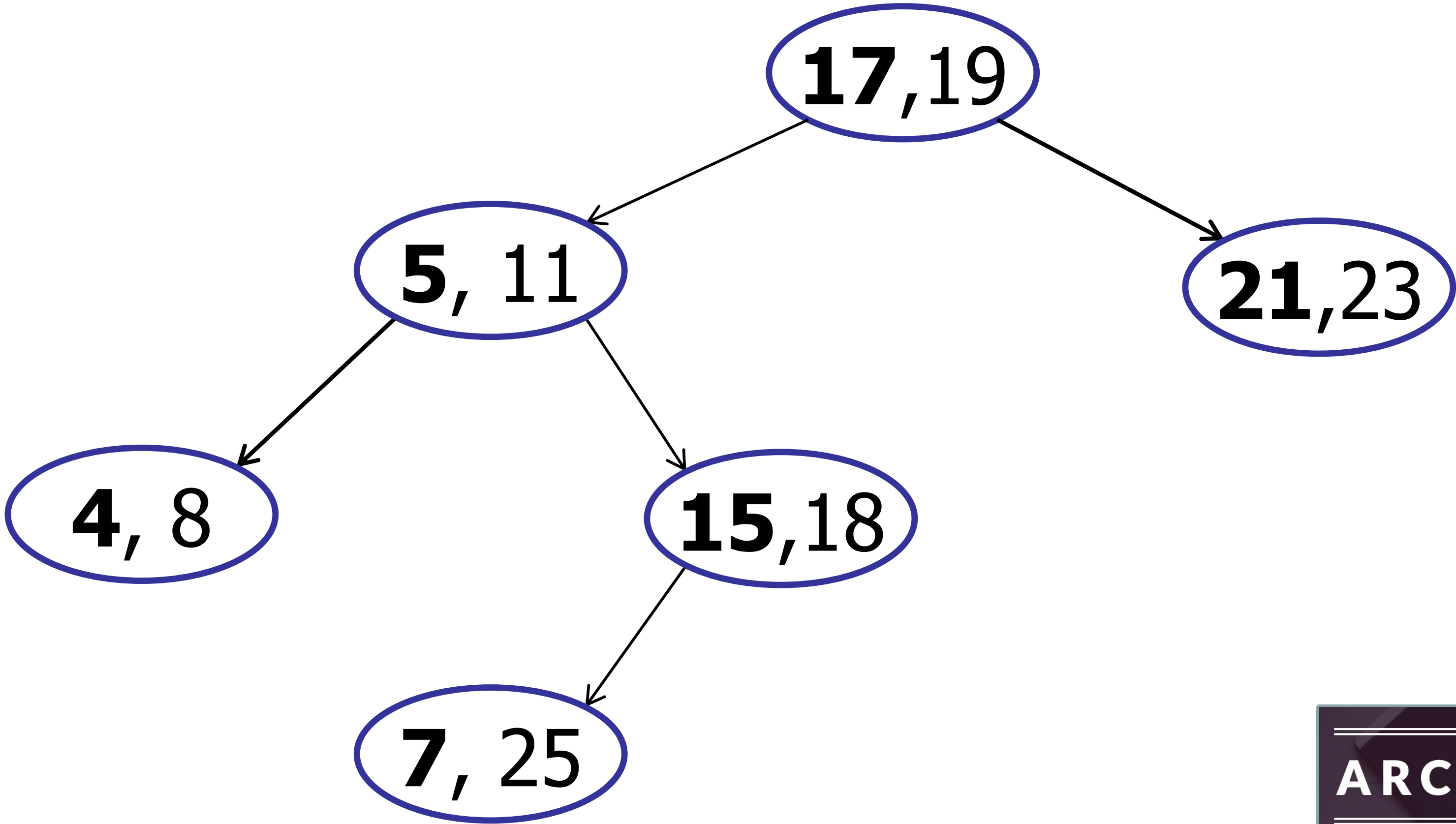
Interval Trees

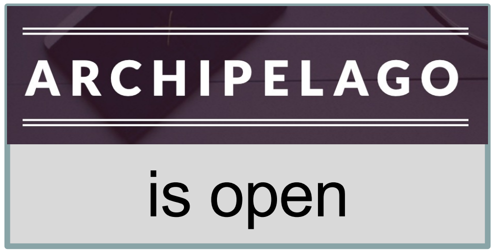
search-interval(25) = ?



Interval Trees

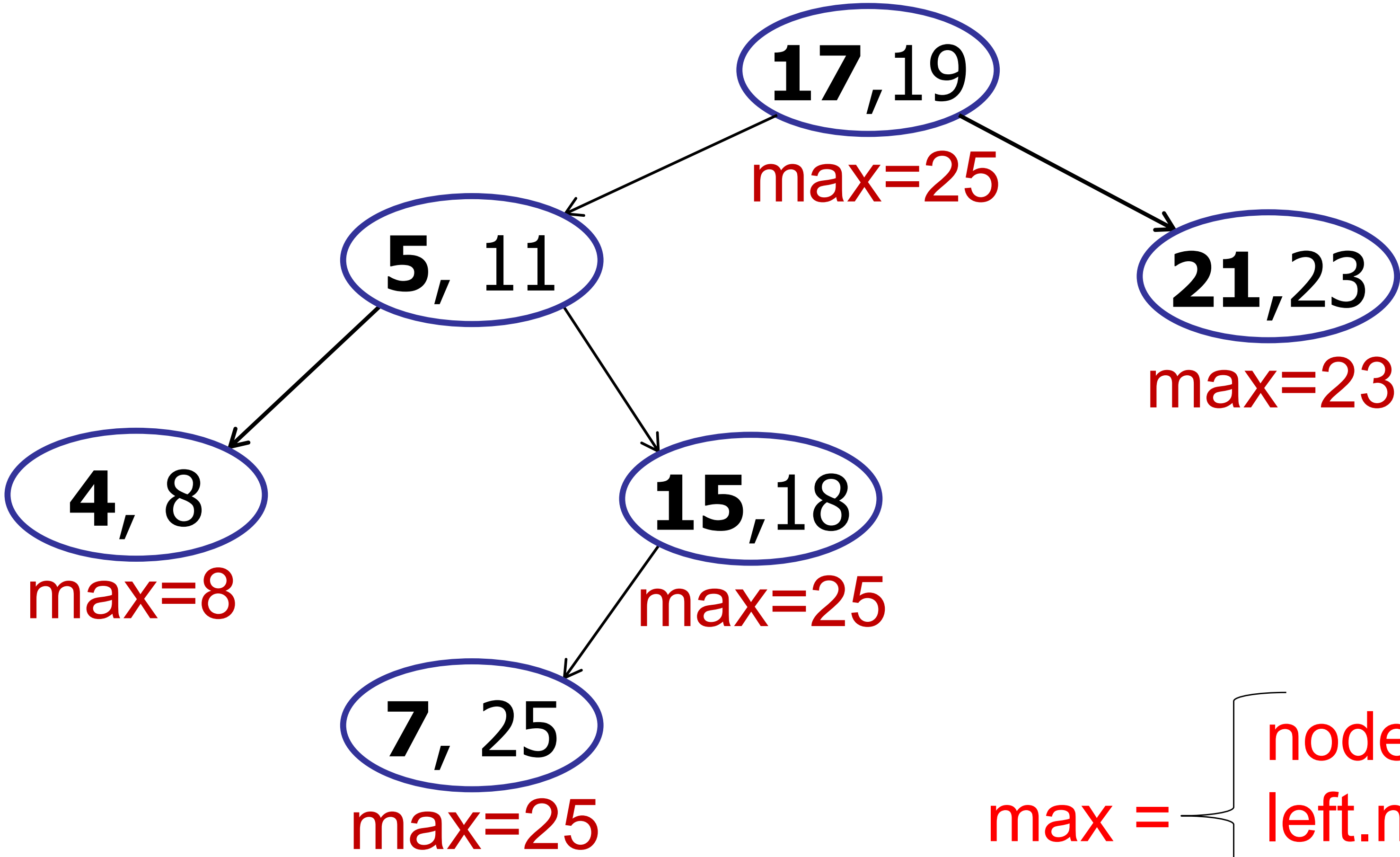
Augment: ??





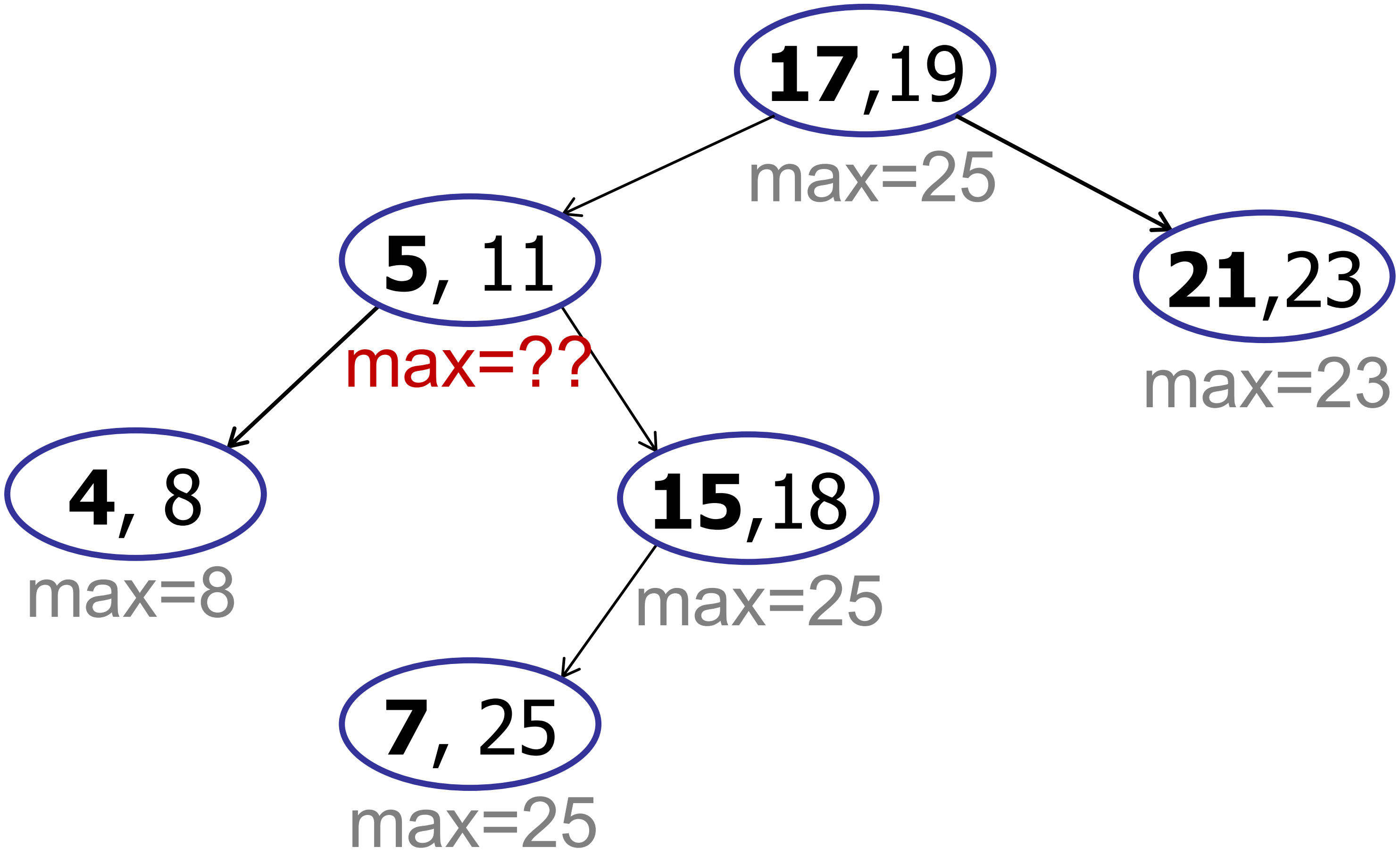
Interval Trees

Augment: maximum endpoint in subtree



$$\text{max} = \begin{cases} \text{node.right} \\ \text{left.max} \\ \text{right.max} \end{cases}$$

max=??

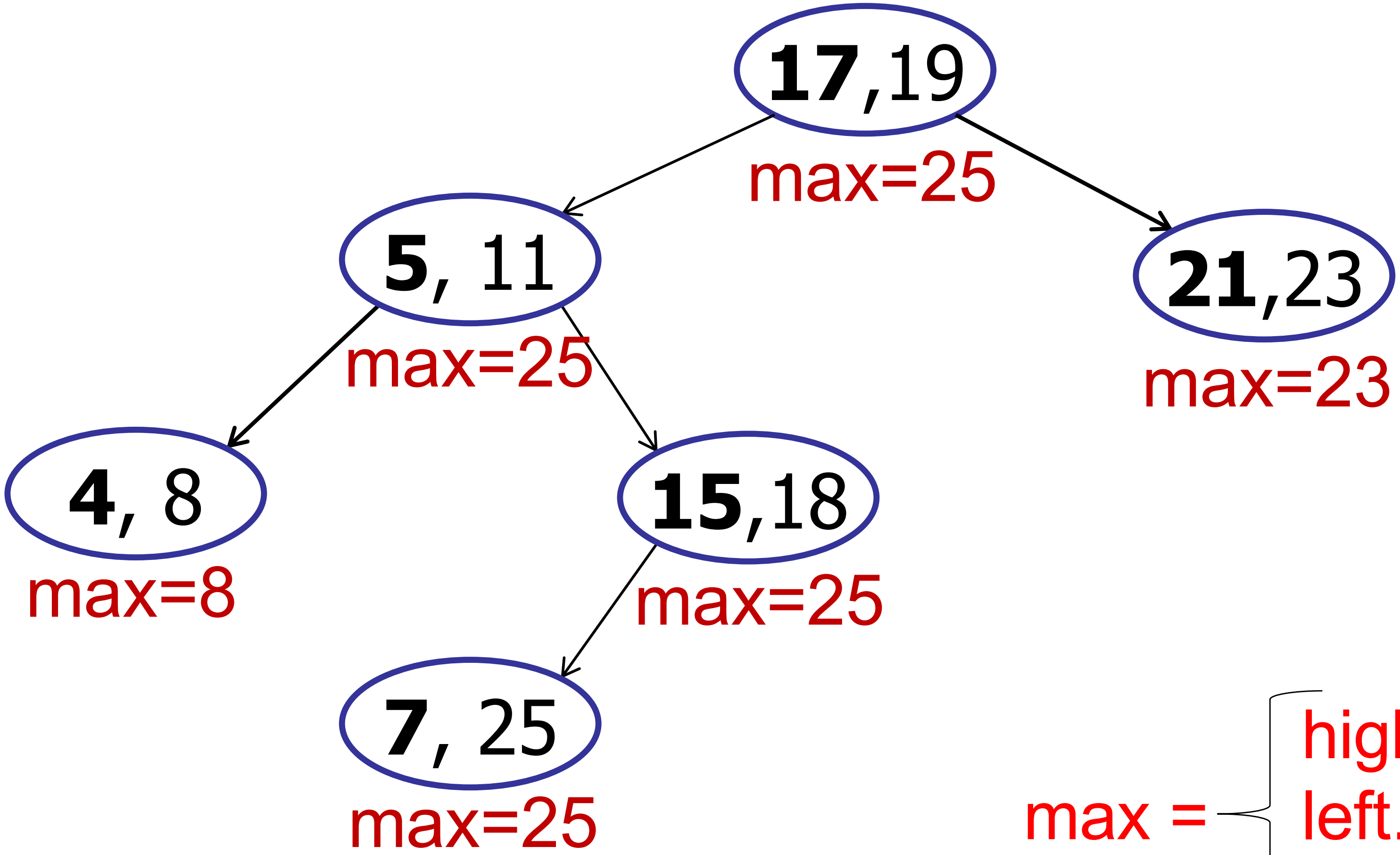


1. 5
2. 8
3. 11
4. 18
5. 25
6. 19

ARCHIPELAGO
is open

Interval Trees

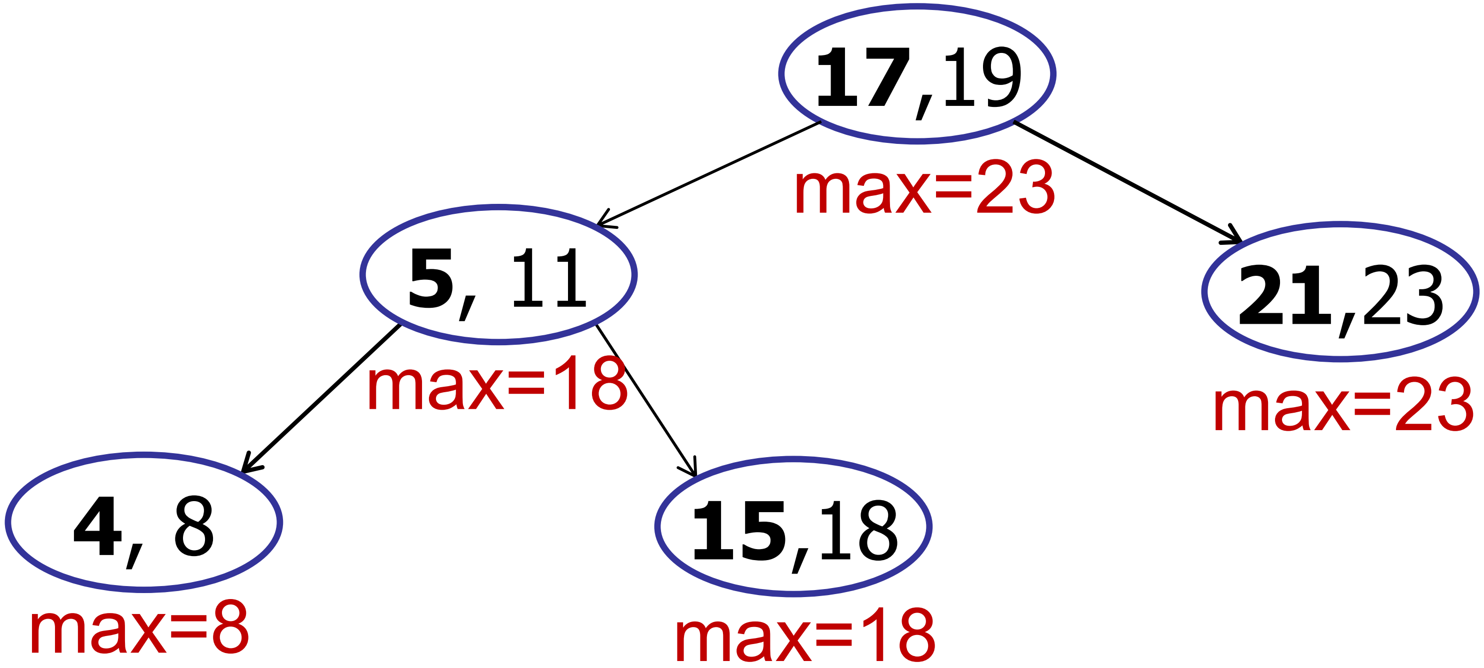
Augment: maximum endpoint in subtree



$$\text{max} = \begin{cases} \text{high} \\ \text{left.max} \\ \text{right.max} \end{cases}$$

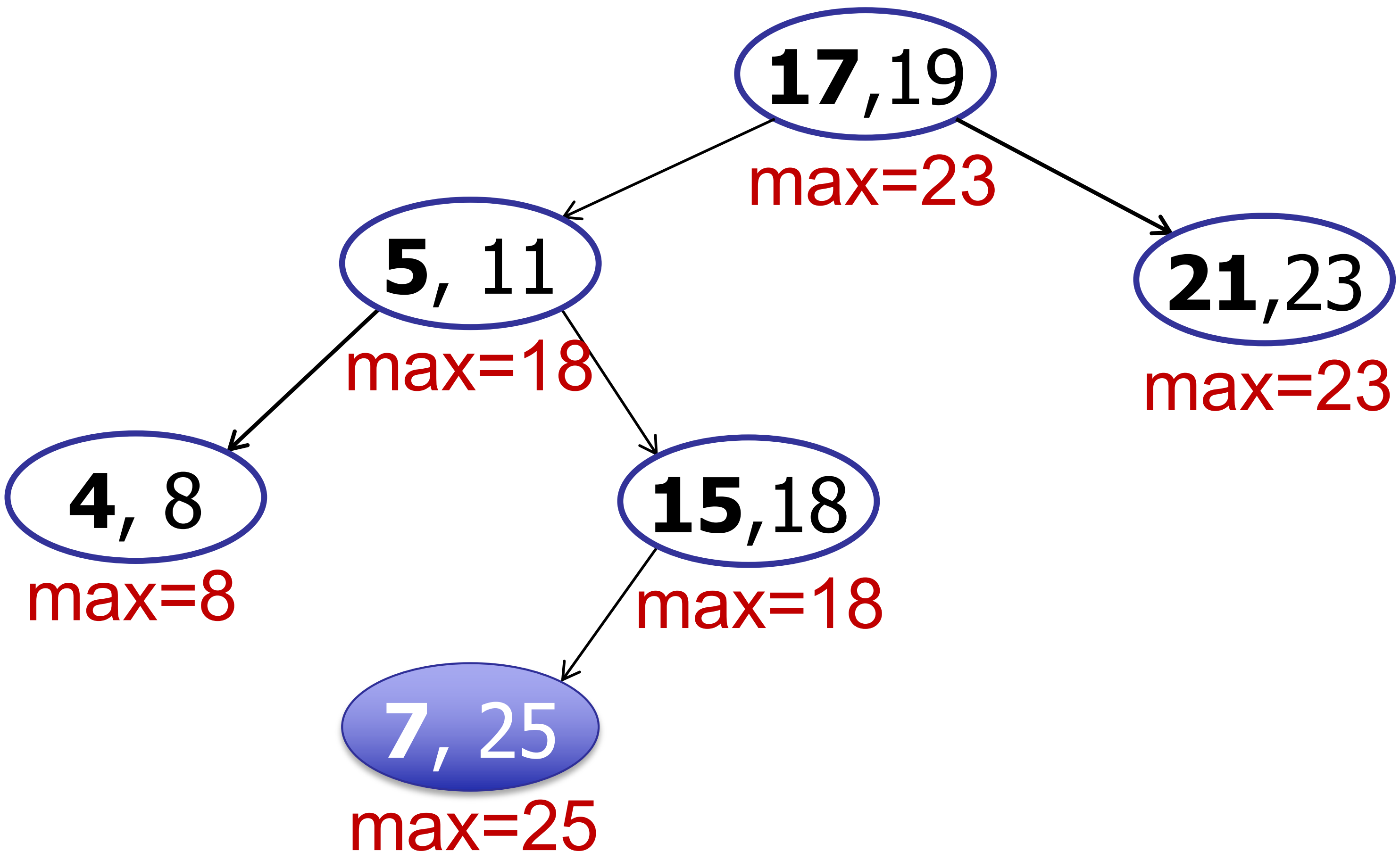
Interval Trees

Insertion: *example* – **insert(7, 25)**



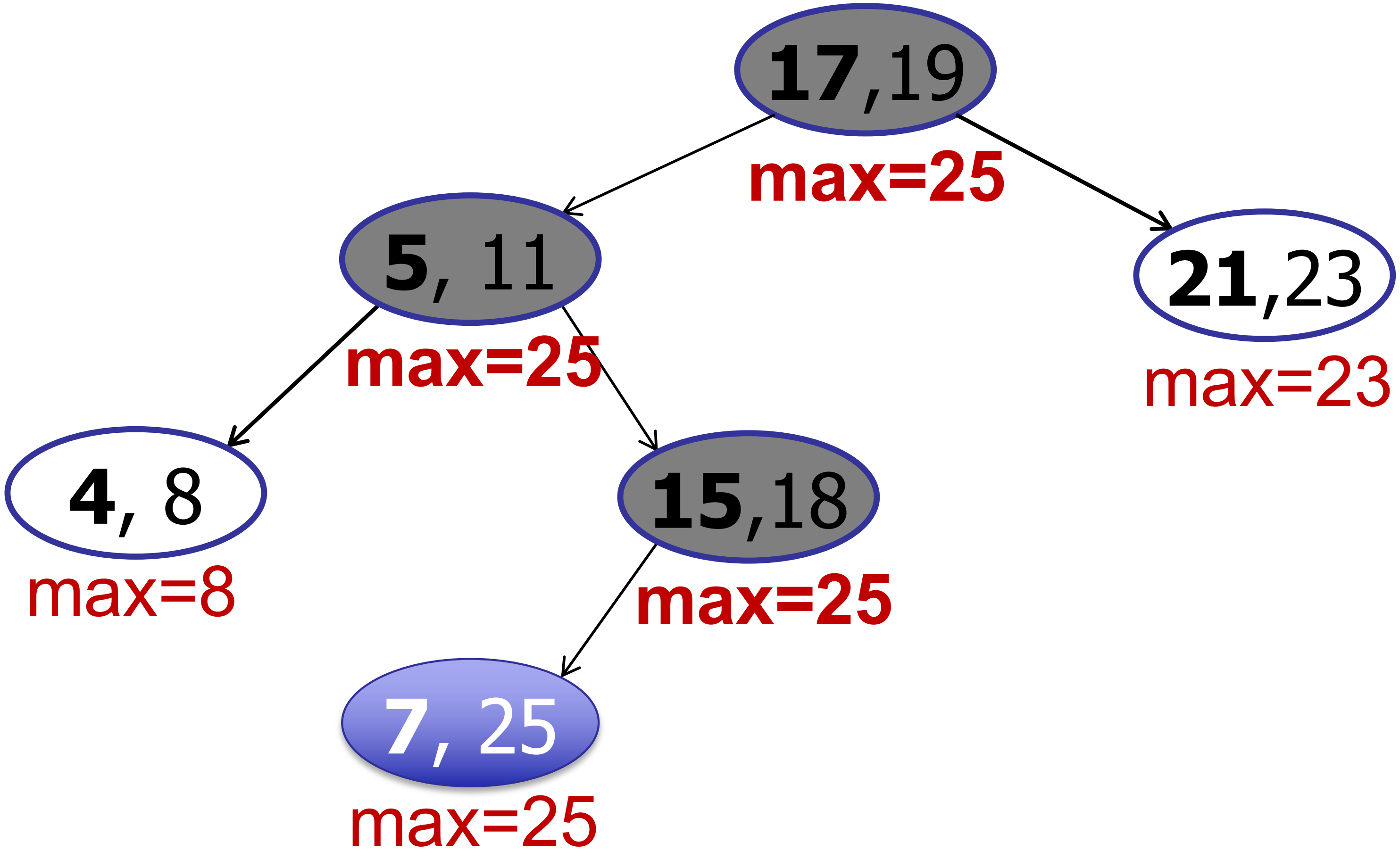
Interval Trees

Insertion: *example* – **insert(7, 25)**



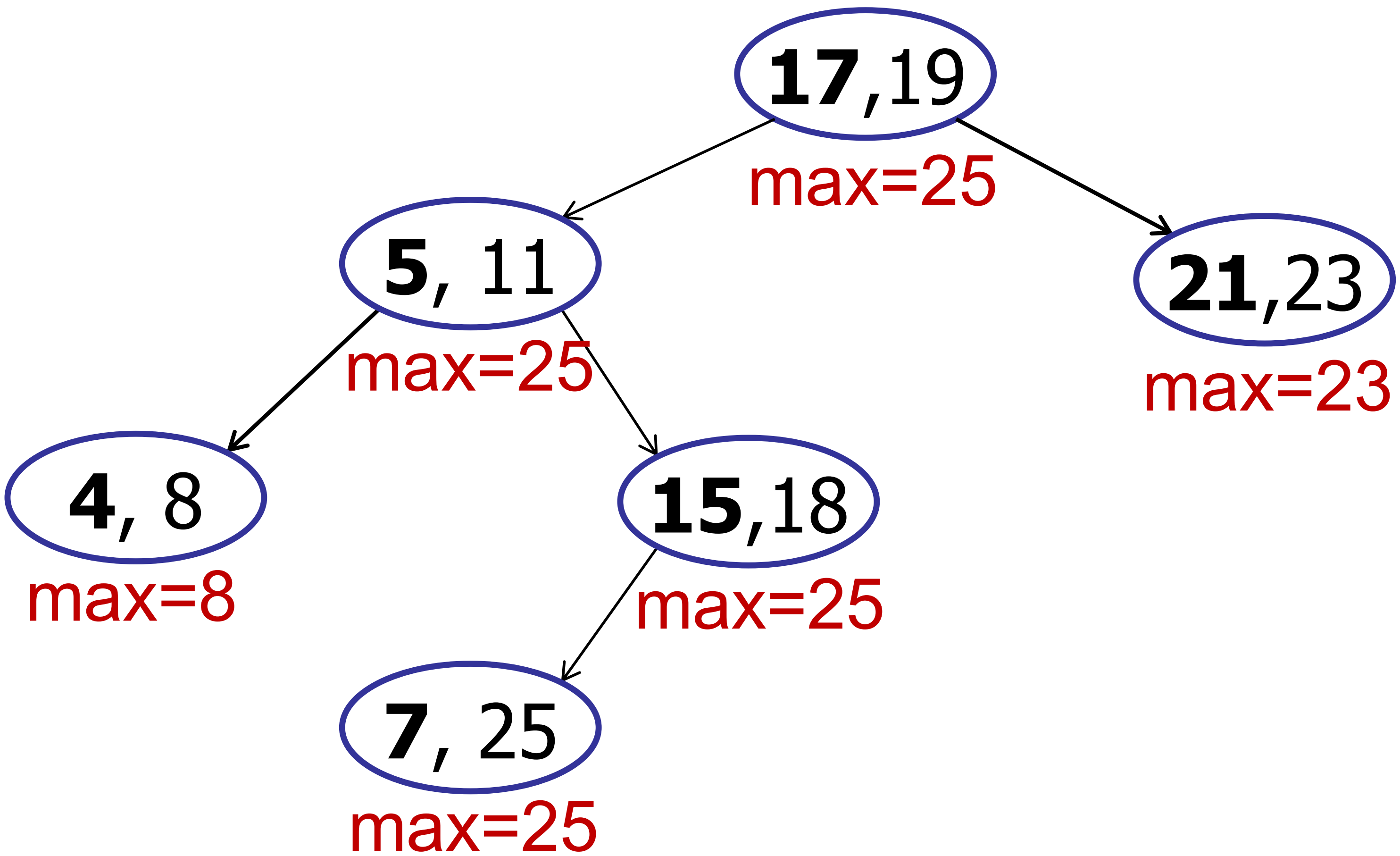
Interval Trees

Insertion: *example* – **insert(7, 25)**



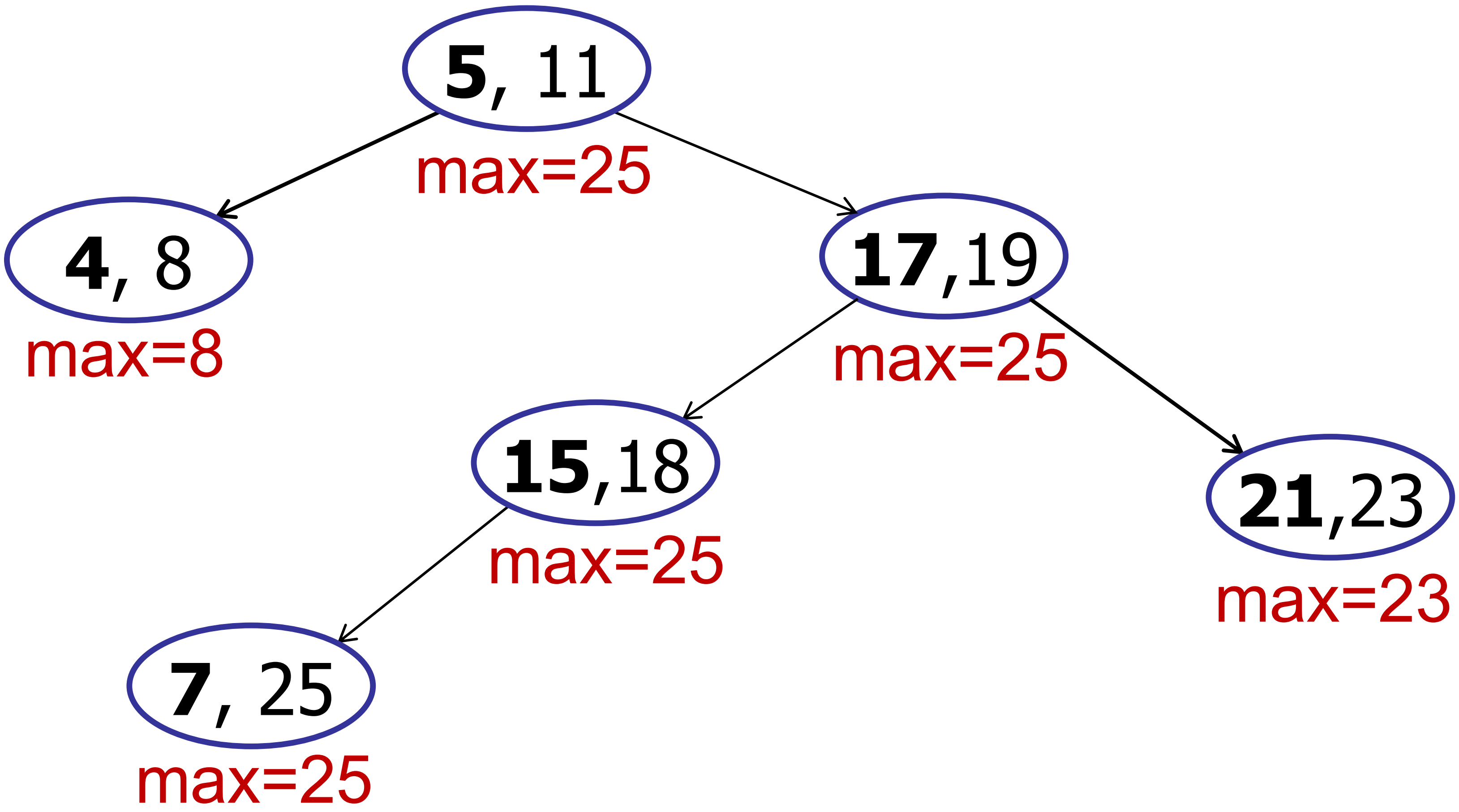
Interval Trees

Insertion: *out-of-balance*



Interval Trees

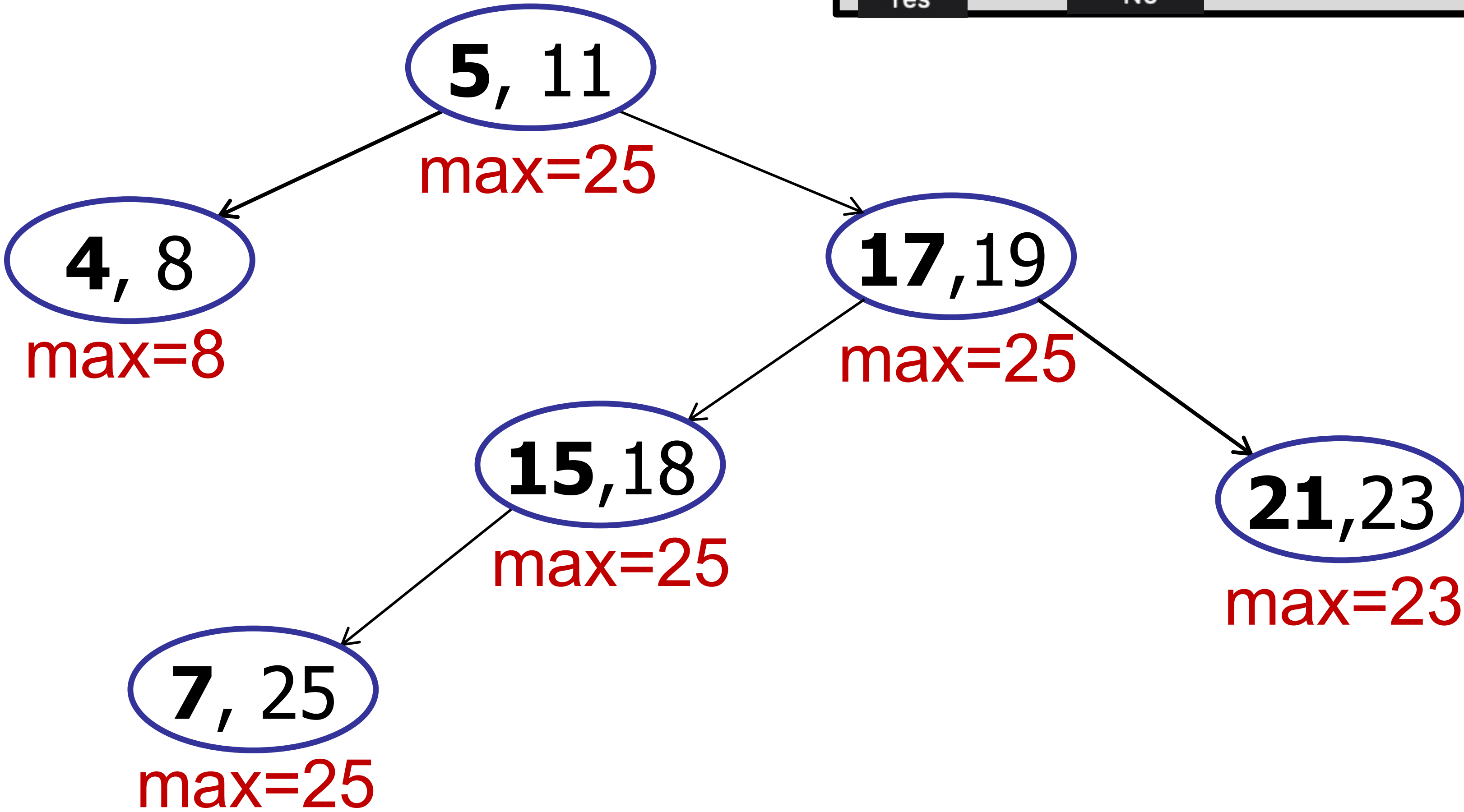
Insertion: right-rotate (17, 19)



Interval Trees

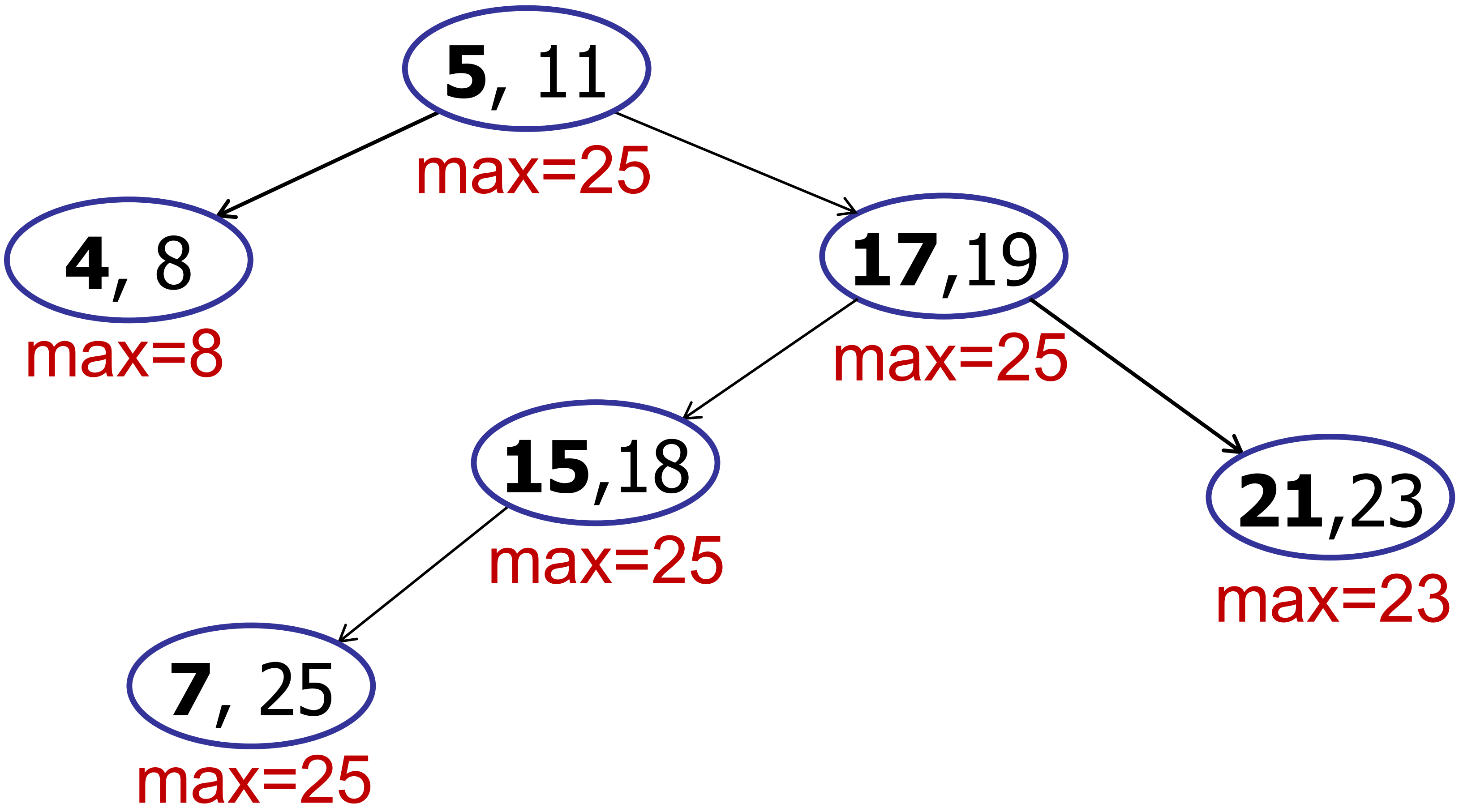
Insertion: right-rotate (17, 19)

Is the tree now balanced?



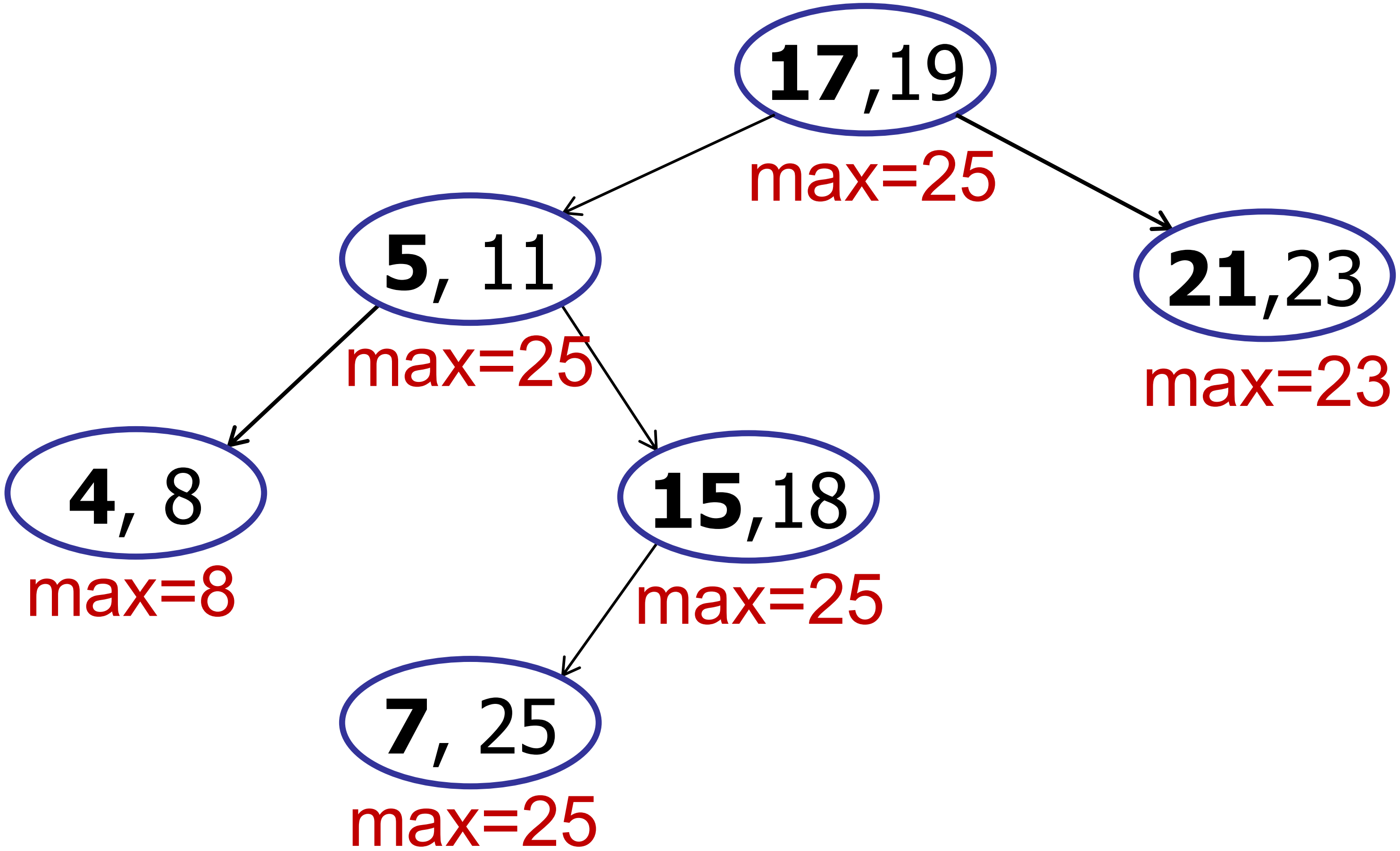
Interval Trees

Insertion: *right-rotate (17, 19),* OOPS!



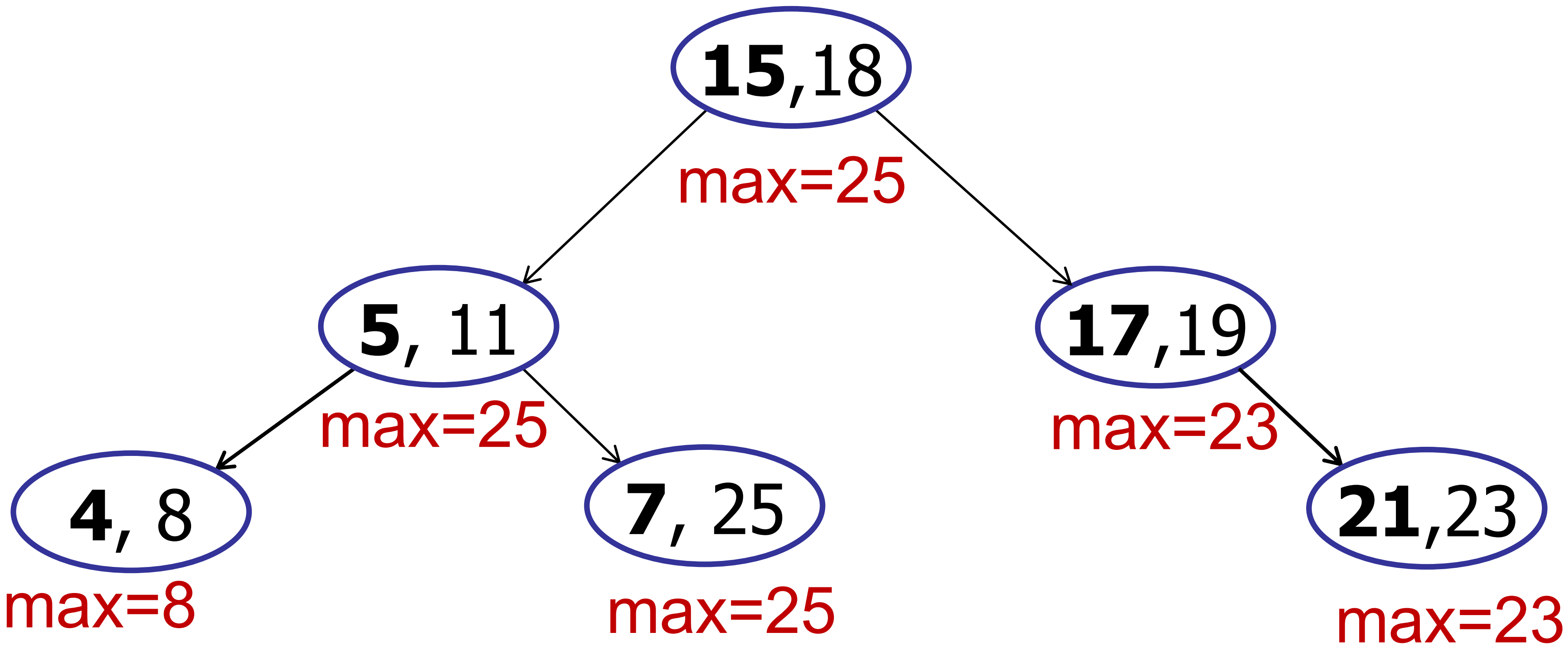
Interval Trees

Insertion: *out-of-balance*



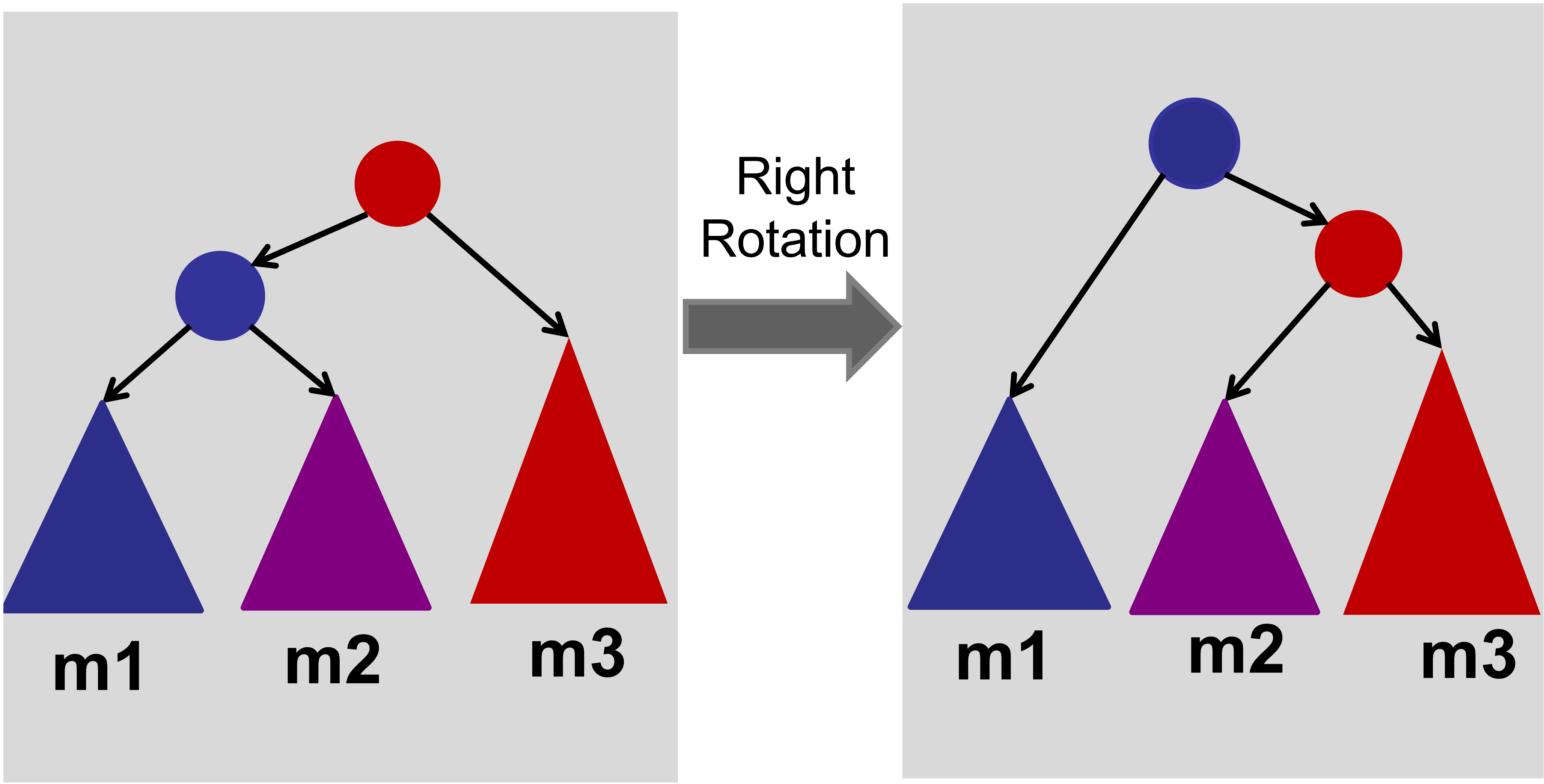
Interval Trees

Insertion: left-rotate, right-rotate



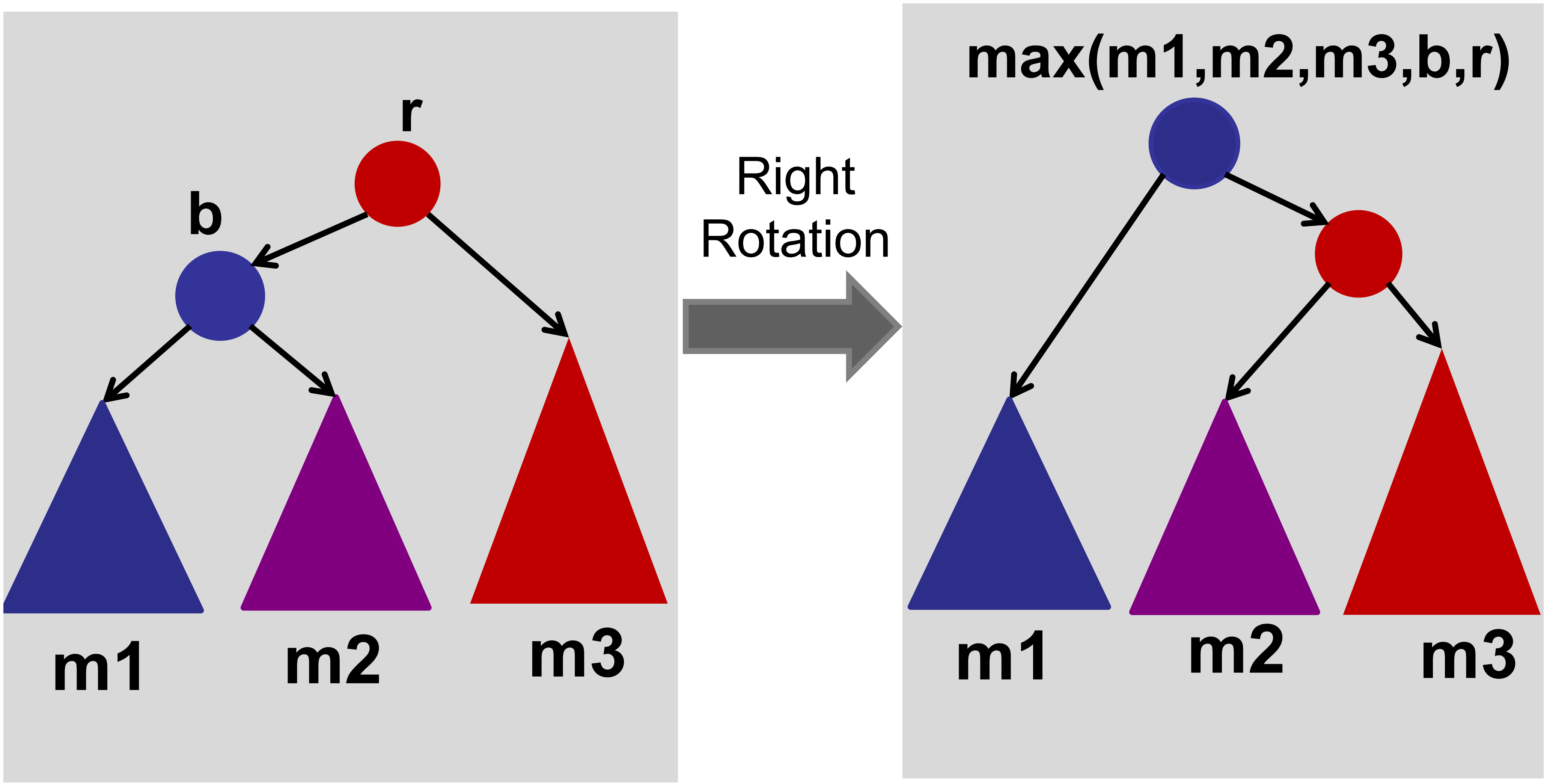
Interval Trees

Maintain MAX during rotations:



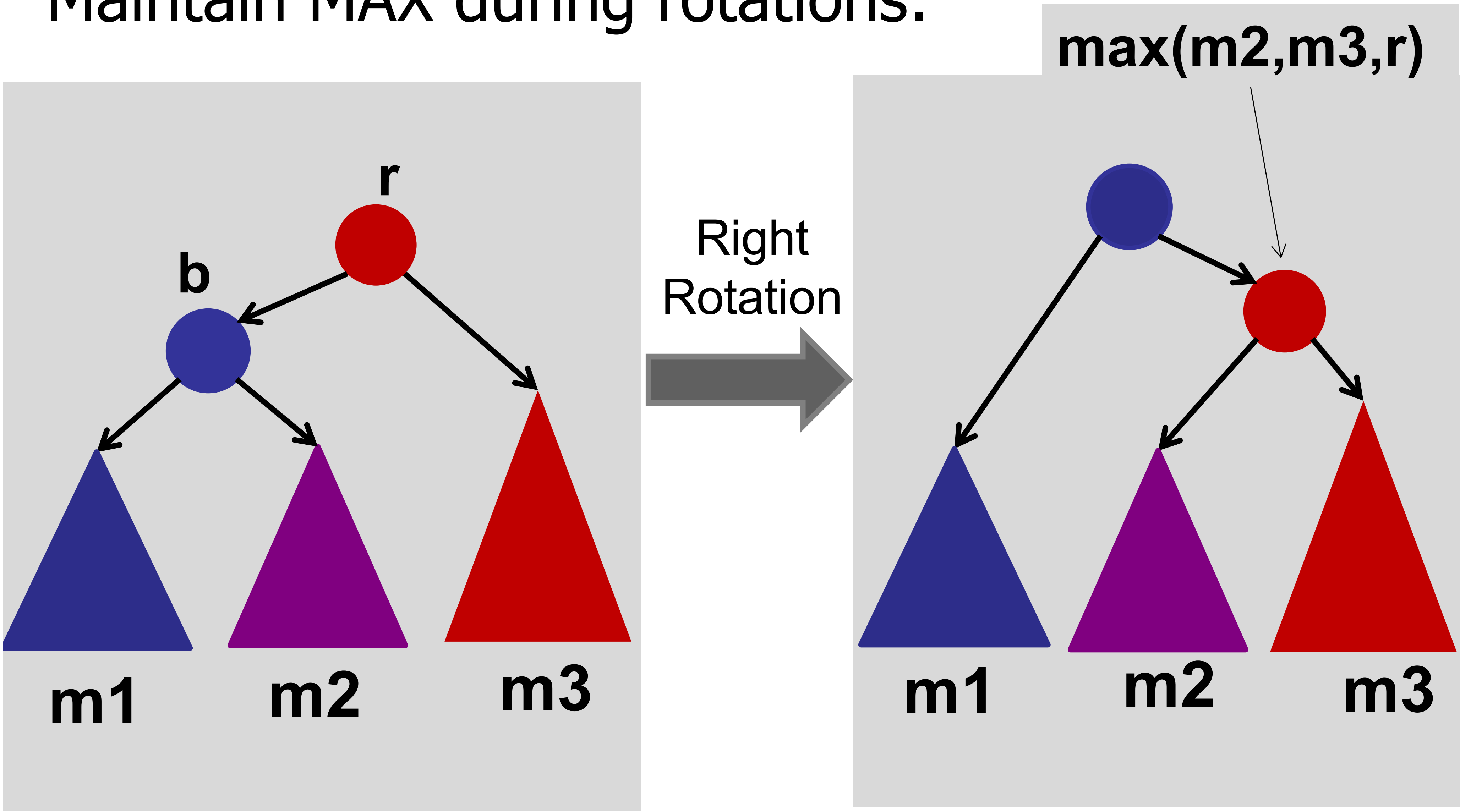
Interval Trees

Maintain MAX during rotations:



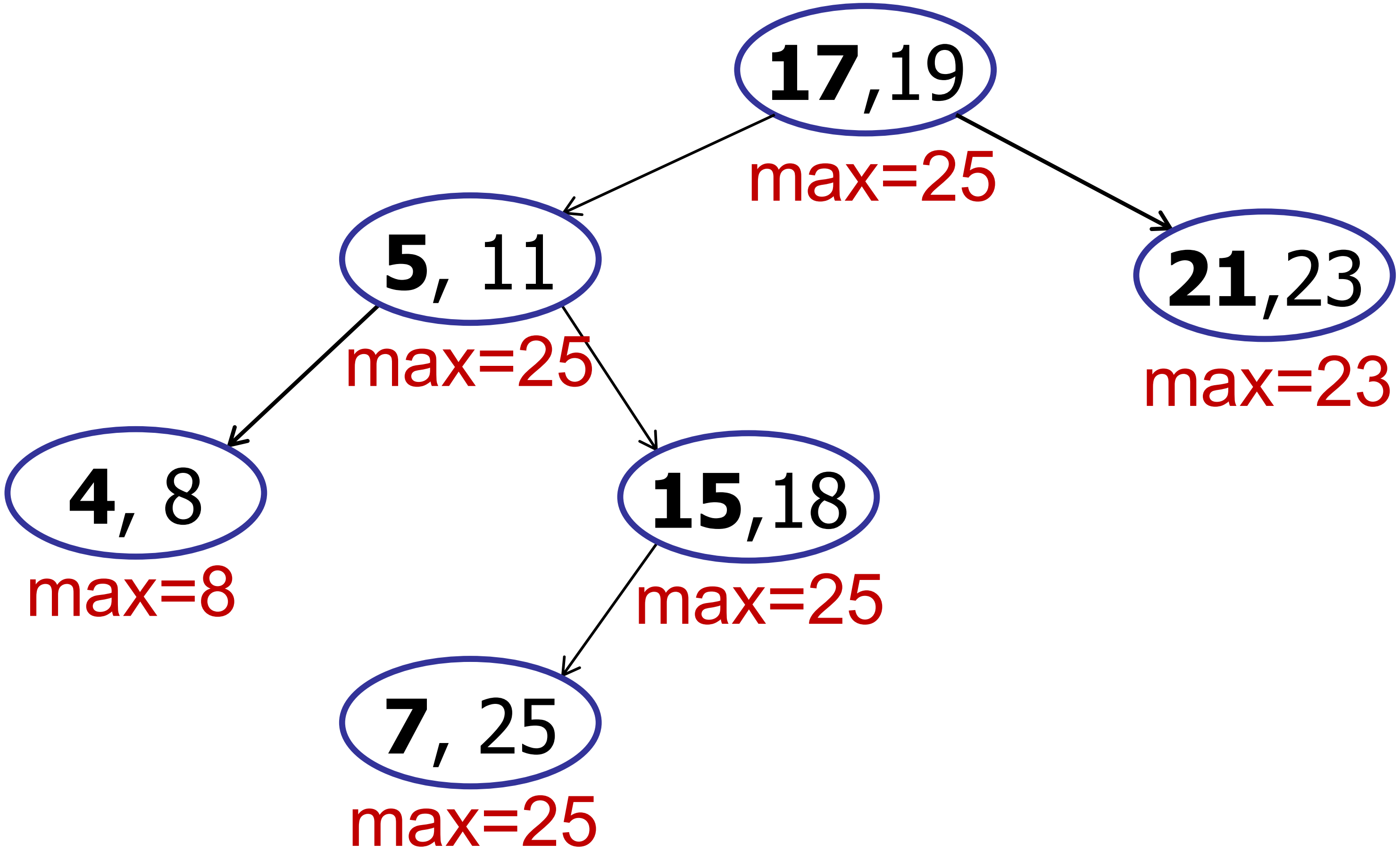
Interval Trees

Maintain MAX during rotations:



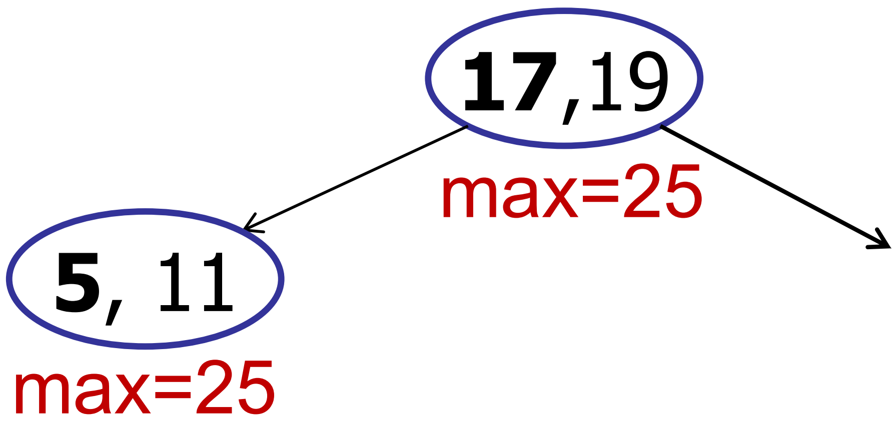
Interval Trees

Searching: interval-search(22)



Interval Trees

Searching: interval-search(22)



It is possible that 22 is covered in the left subtree.

Do we know *for sure* that going left will work?

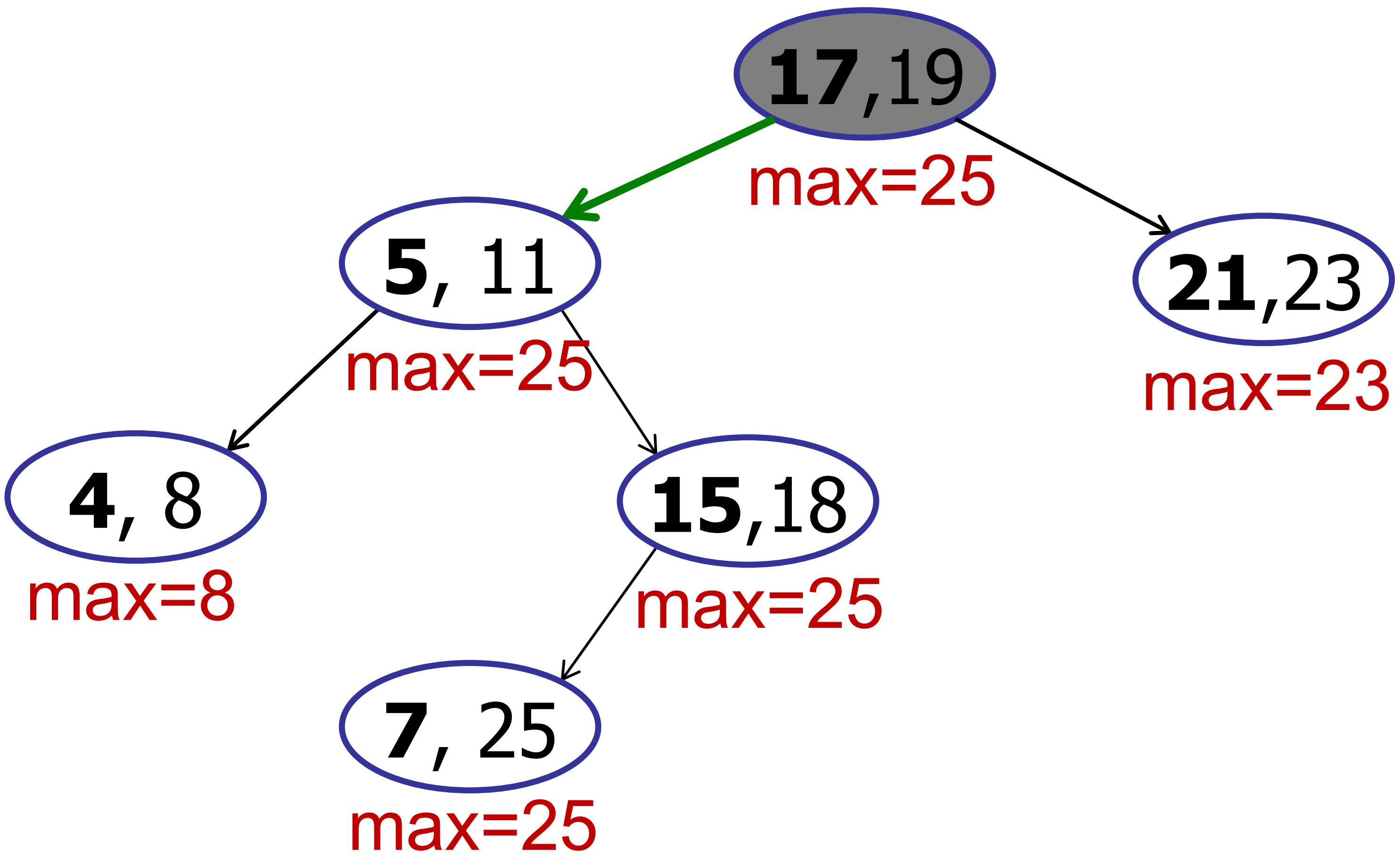
Interval Trees

interval-search(x) : find interval containing x

```
interval-search(x)
    c = root;
    while (c != null and x is not in c.interval) do
        if (c.left == null) then
            c = c.right;
        else if  (x > c.left.max)  then
            c = c.right;
        else c = c.left;
    return c.interval;
```

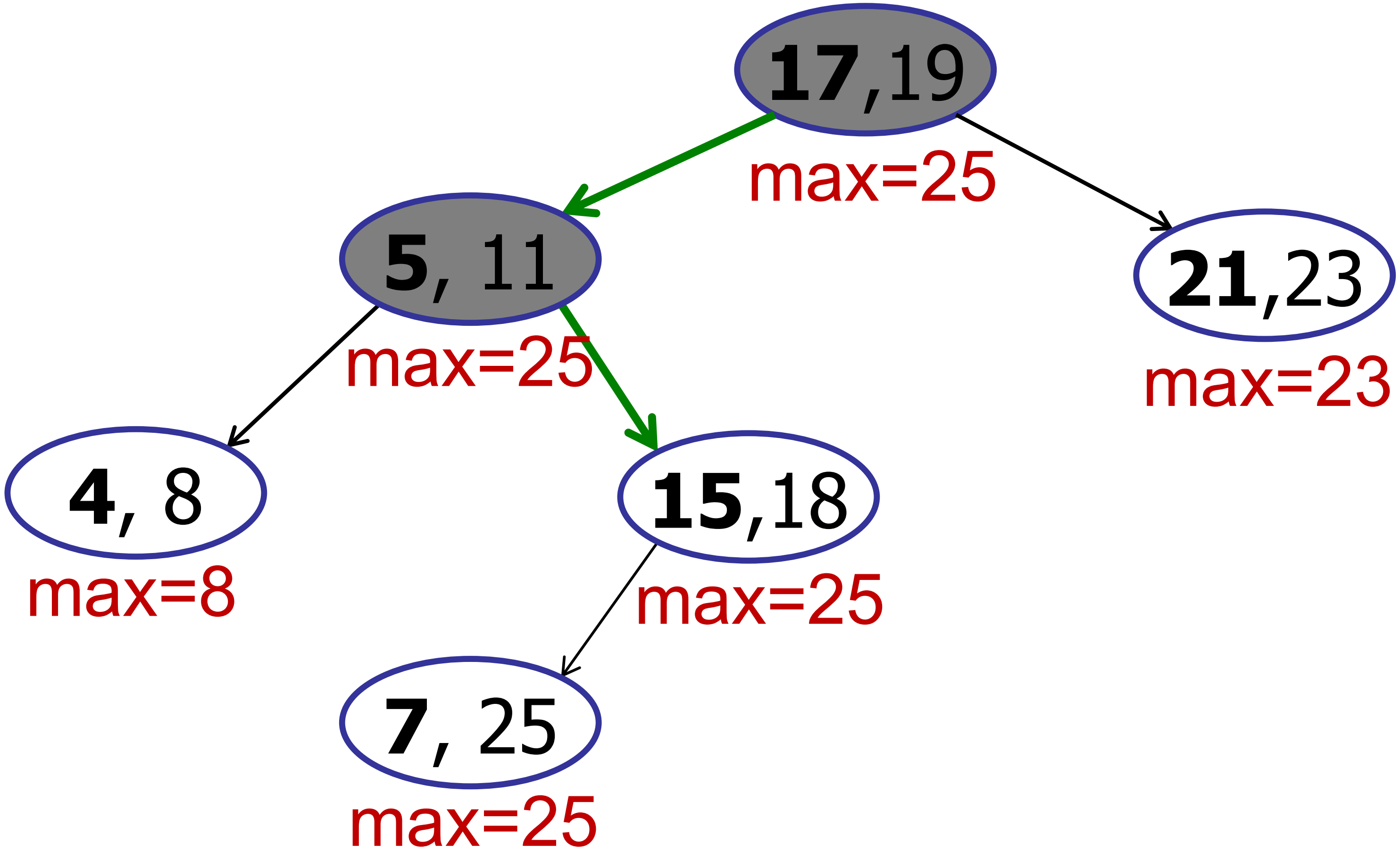
Interval Trees

Searching: interval-search(22)



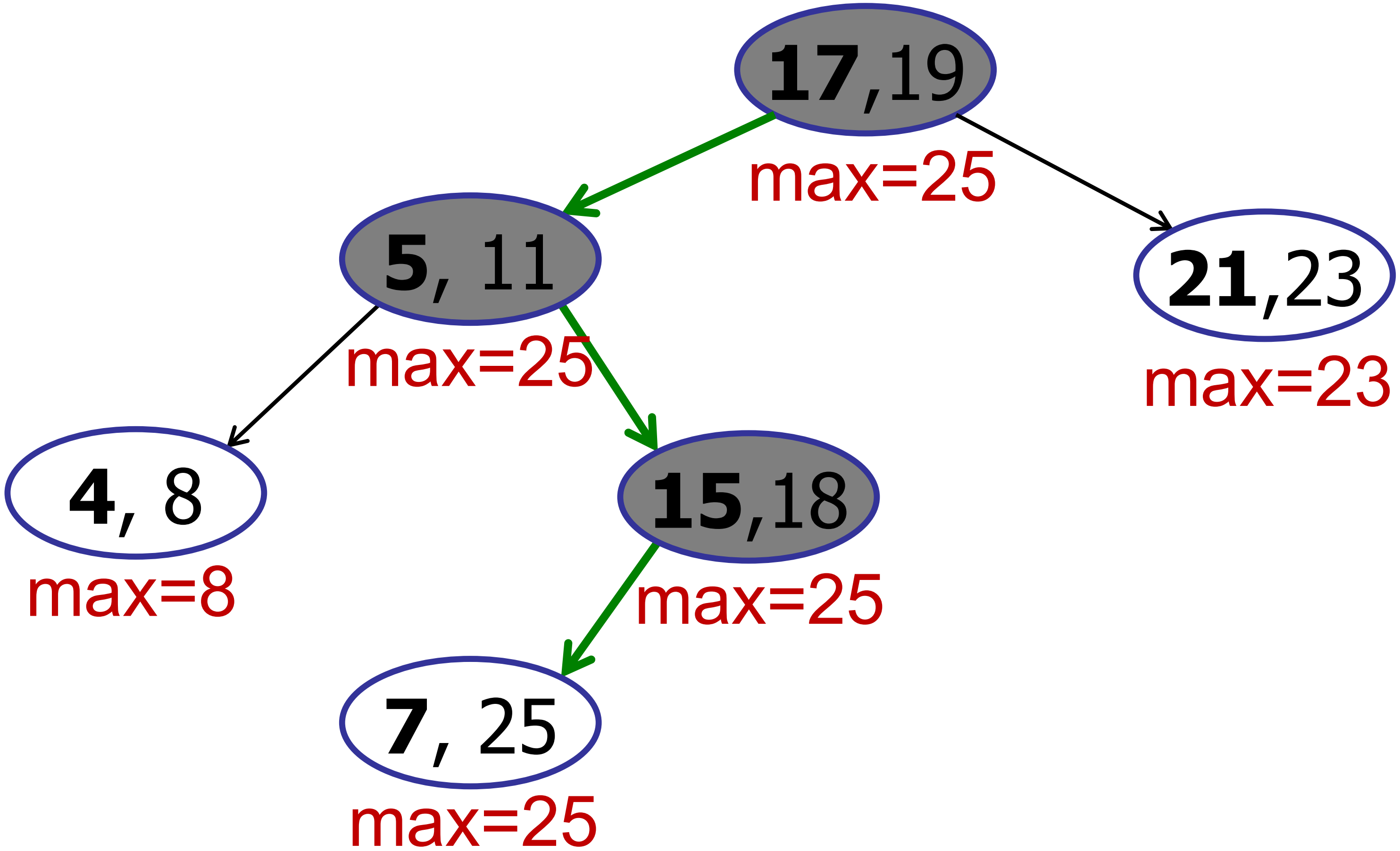
Interval Trees

Searching: interval-search(22)



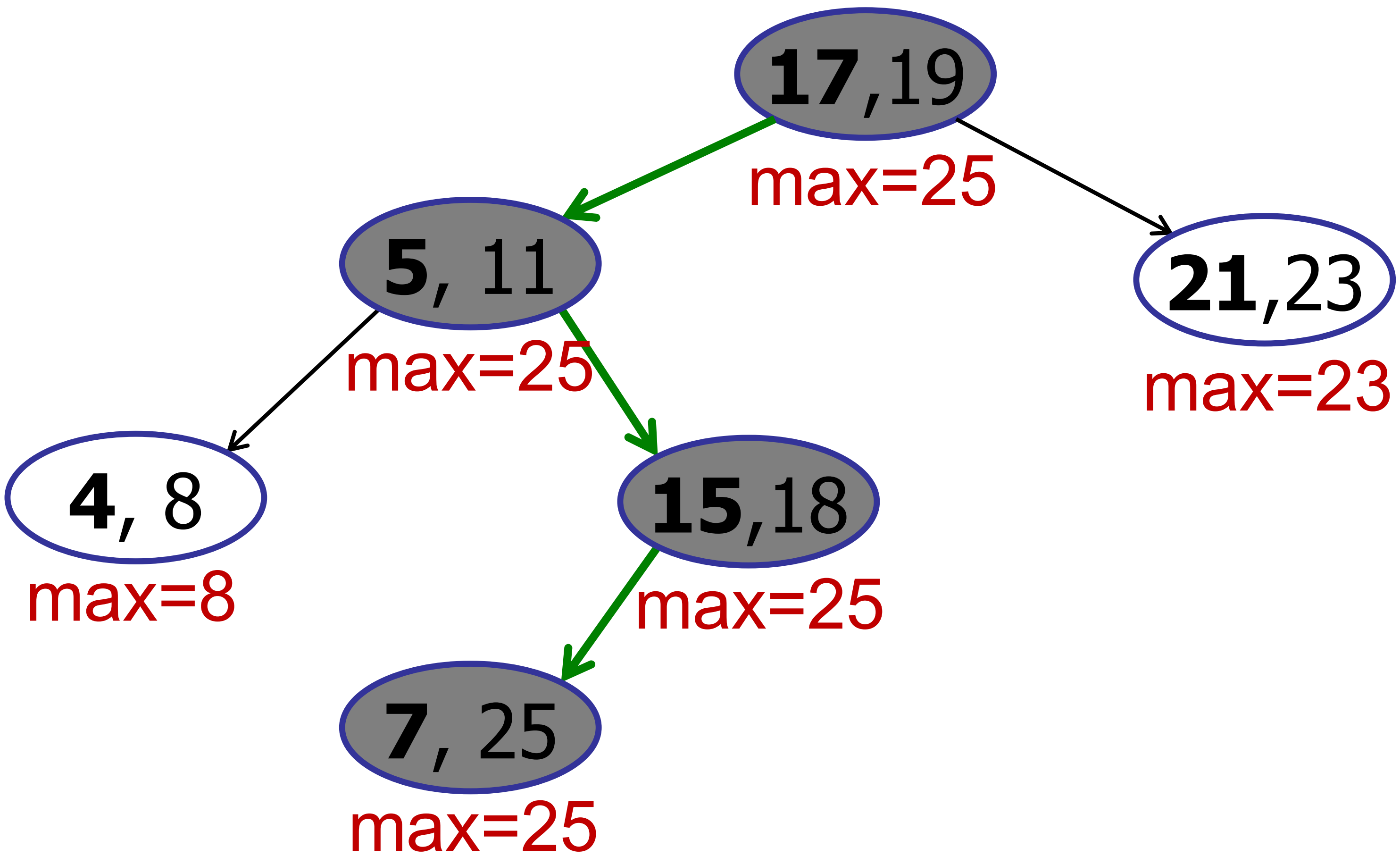
Interval Trees

Searching: interval-search(22)



Interval Trees

Searching: interval-search(22)



Interval Trees

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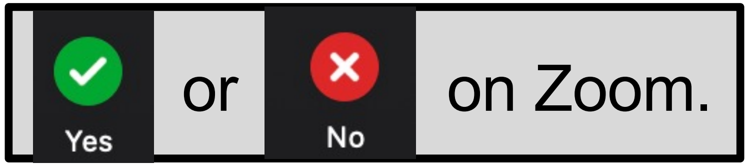
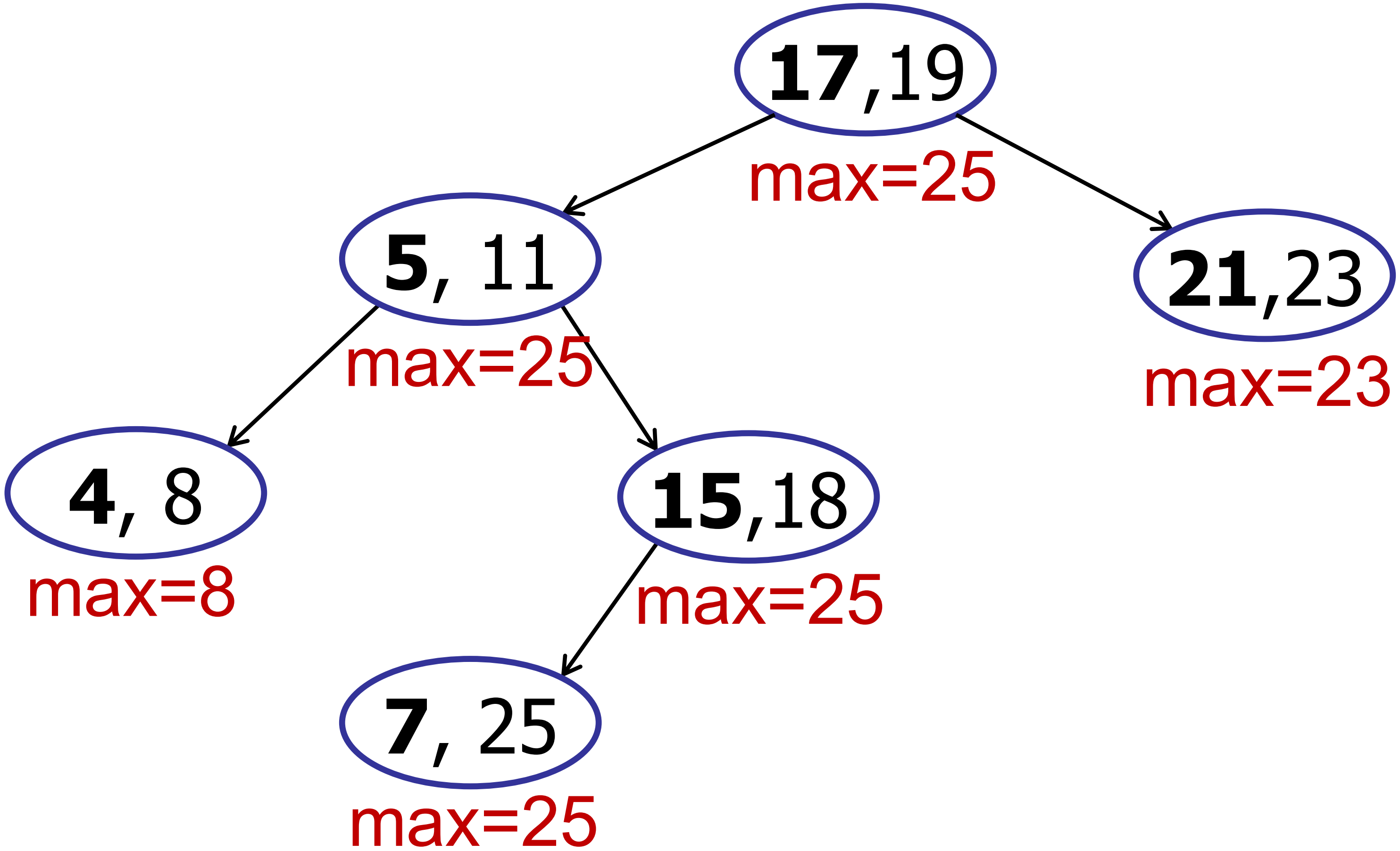
            c = c.right;
        else if  (x > c.left.max)  then
            c = c.right;
        else c = c.left;
    return c.interval;
```

Interval Trees

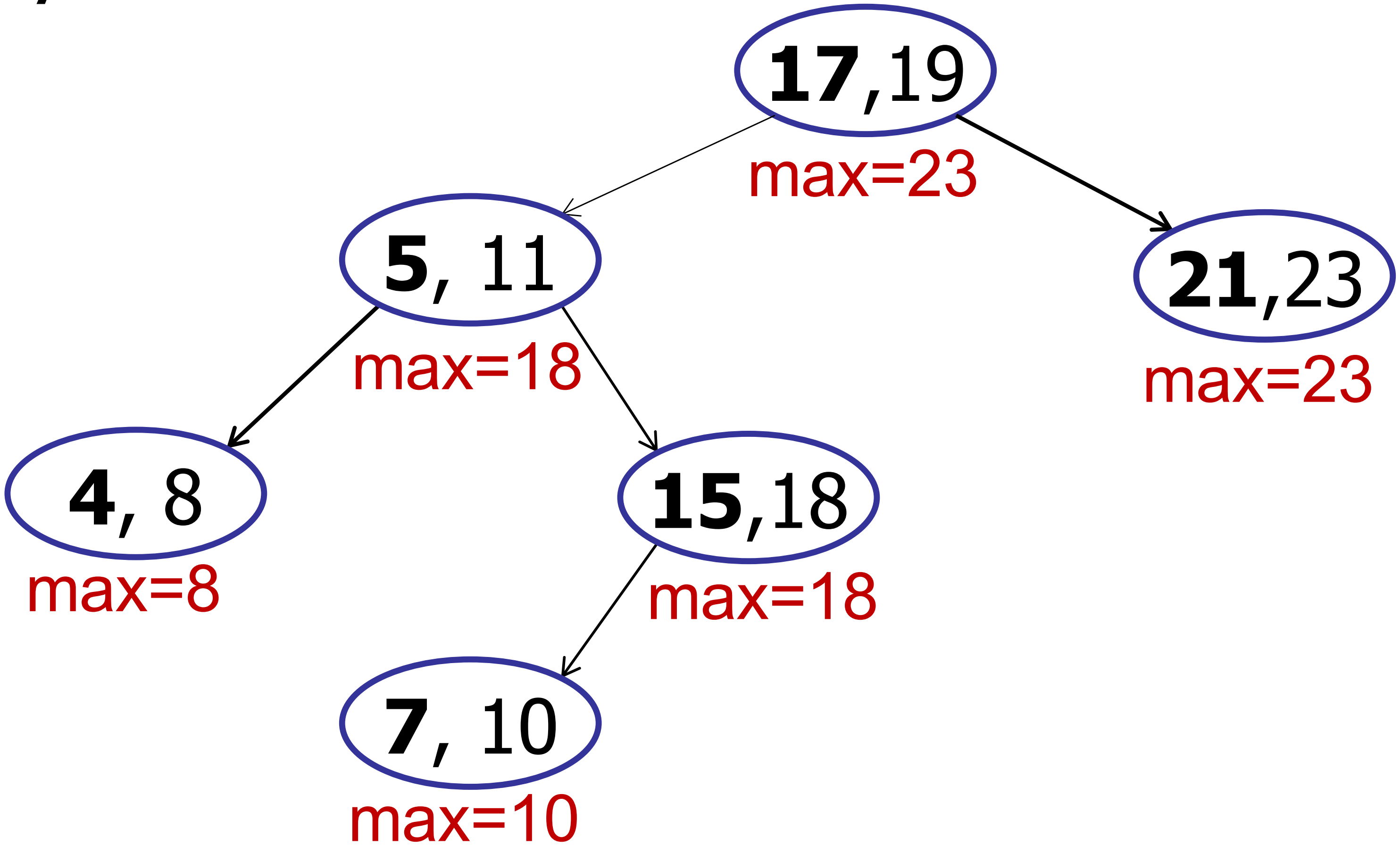
Will any search find (21, 23)?





Interval Trees

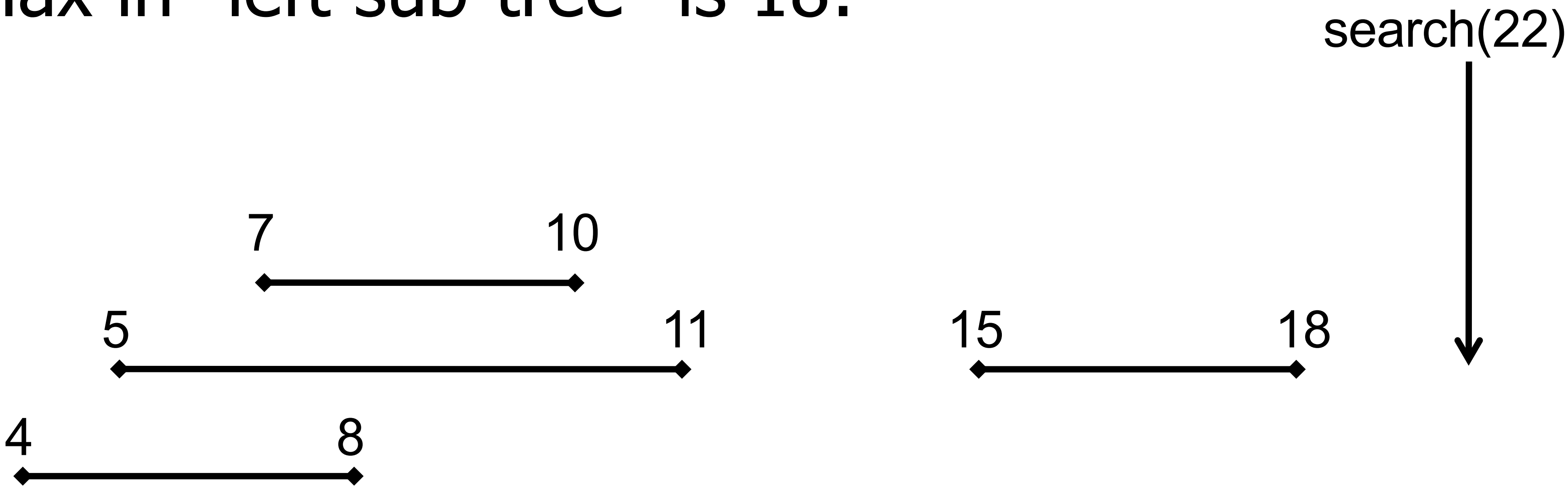
Why does it work?



Claim: If search goes right, then no overlap in left subtree.

Interval Trees

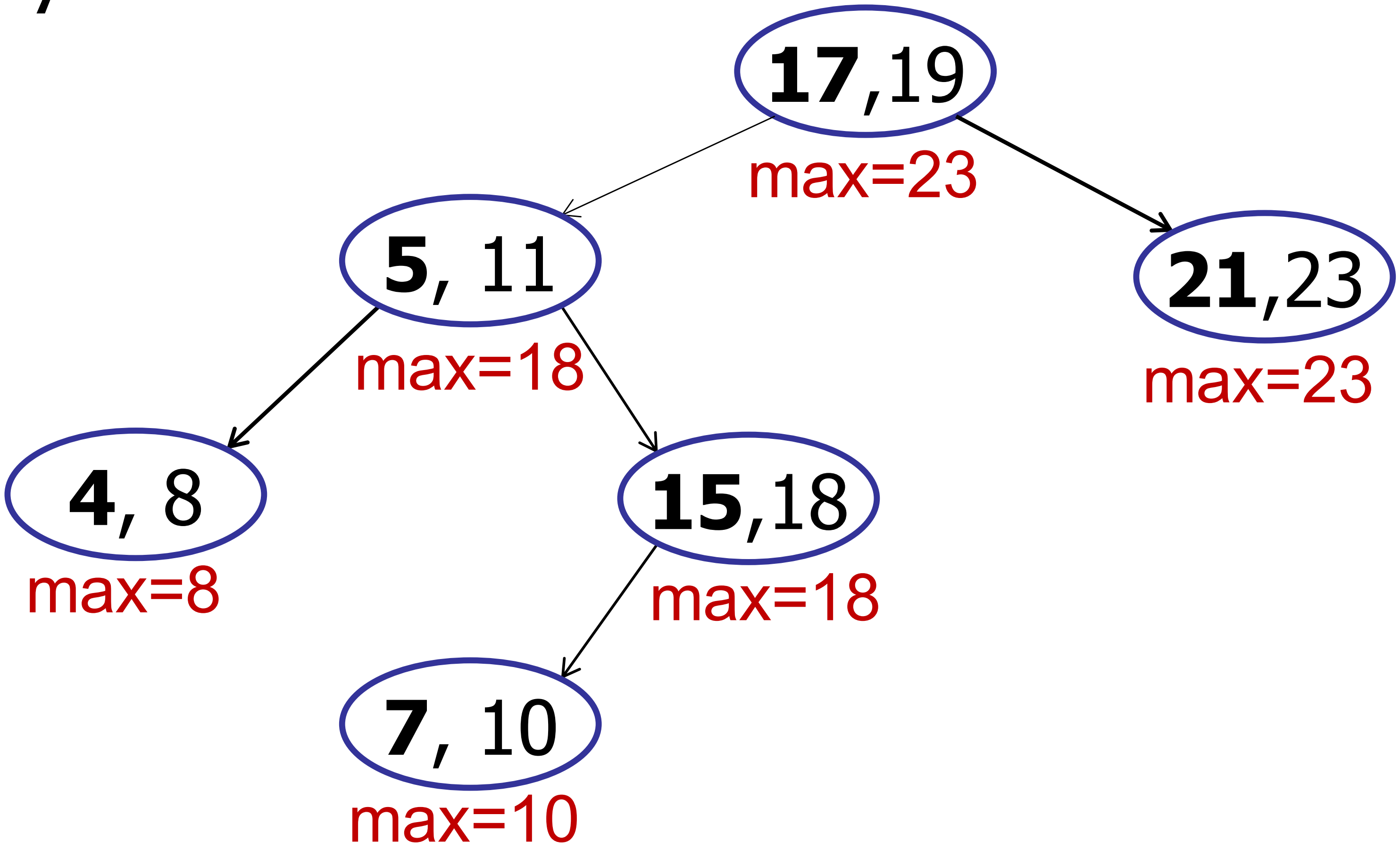
Max in “left sub-tree” is 18:



Safe to go right: 22 is not in the left sub-tree.

Interval Trees

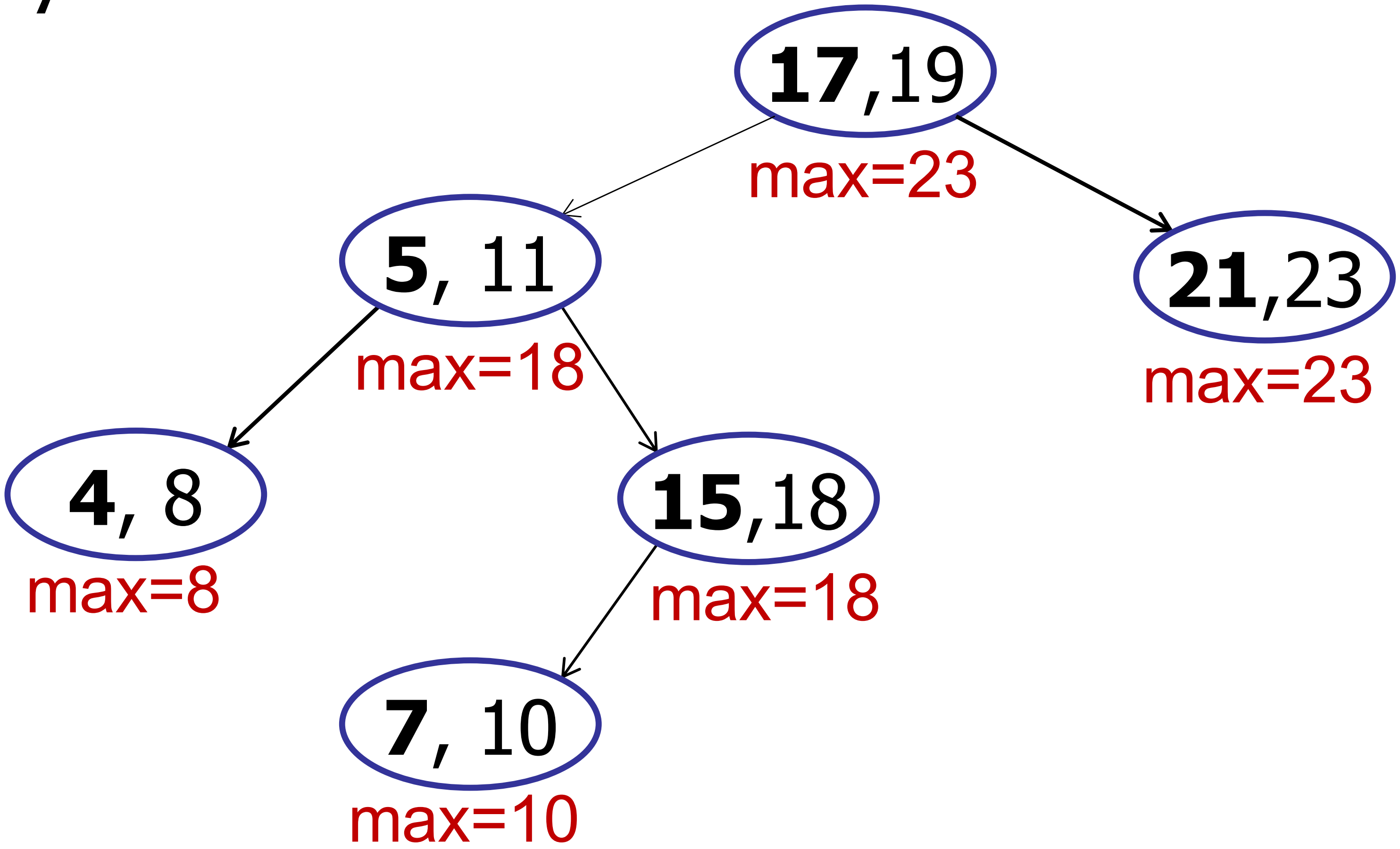
Why does it work?



Claim: If search goes left and there is no overlap in the left subtree…

Interval Trees

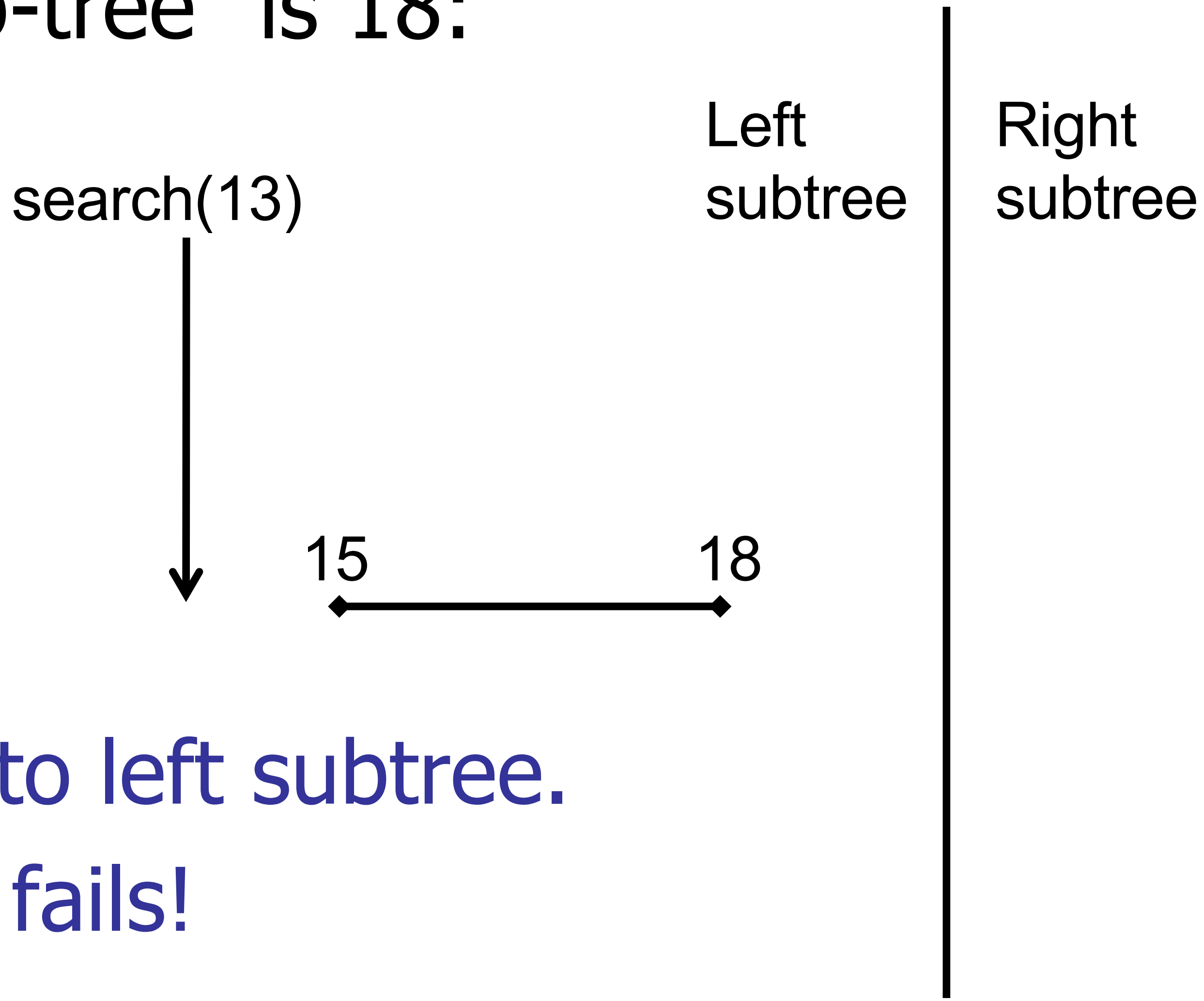
Why does it work?



Claim: If search goes left, then safe to go left.

Interval Trees

Max in “left sub-tree” is 18:

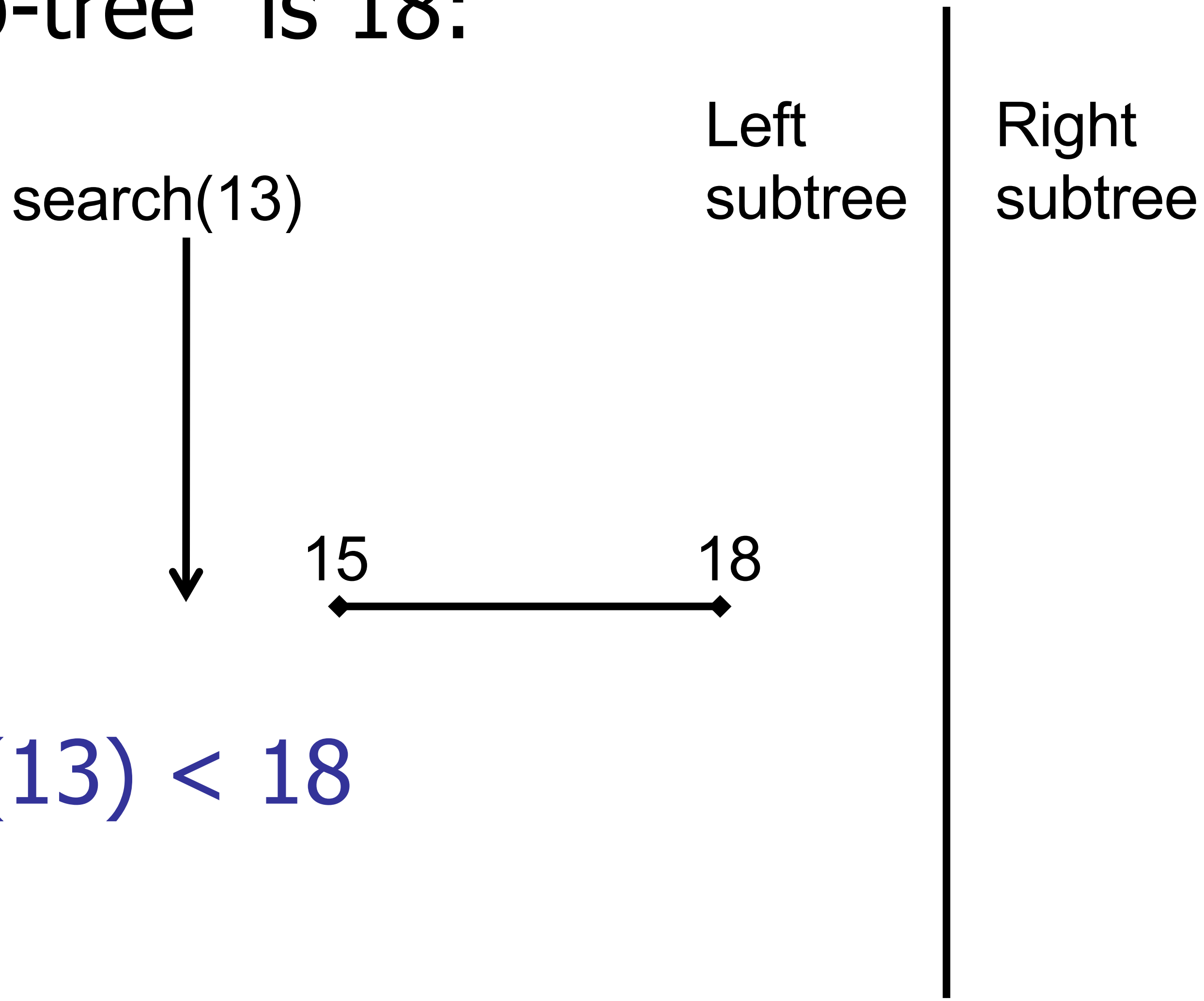


Assume we go to left subtree.

Assume search fails!

Interval Trees

Max in "left sub-tree" is 18:



Go left: search(13) < 18

Interval Trees

Max in “left sub-tree” is 18:

Left subtree

Right subtree

search(13)

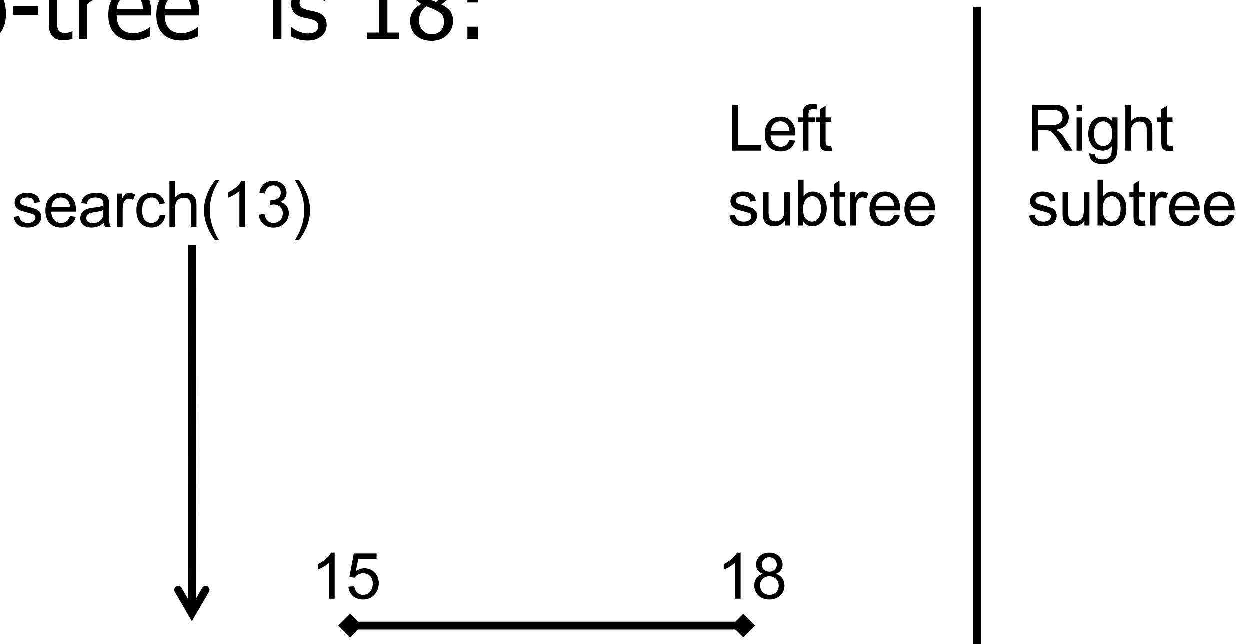
15

18

Go left: search(13) < 15 < 18

Interval Trees

Max in “left sub-tree” is 18:

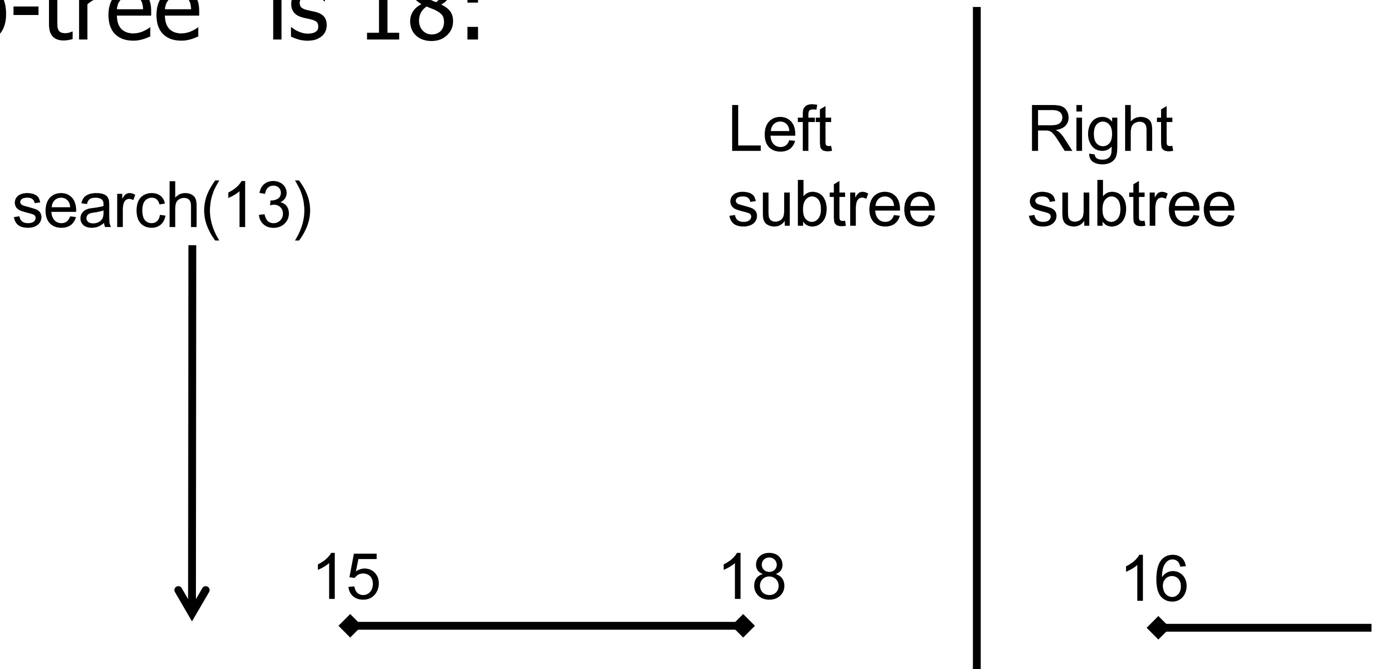


Go left: search(13) < 15 < 18

Tree sorted by left endpoint.

Interval Trees

Max in “left sub-tree” is 18:



Go left: search(13) < 15 < 18

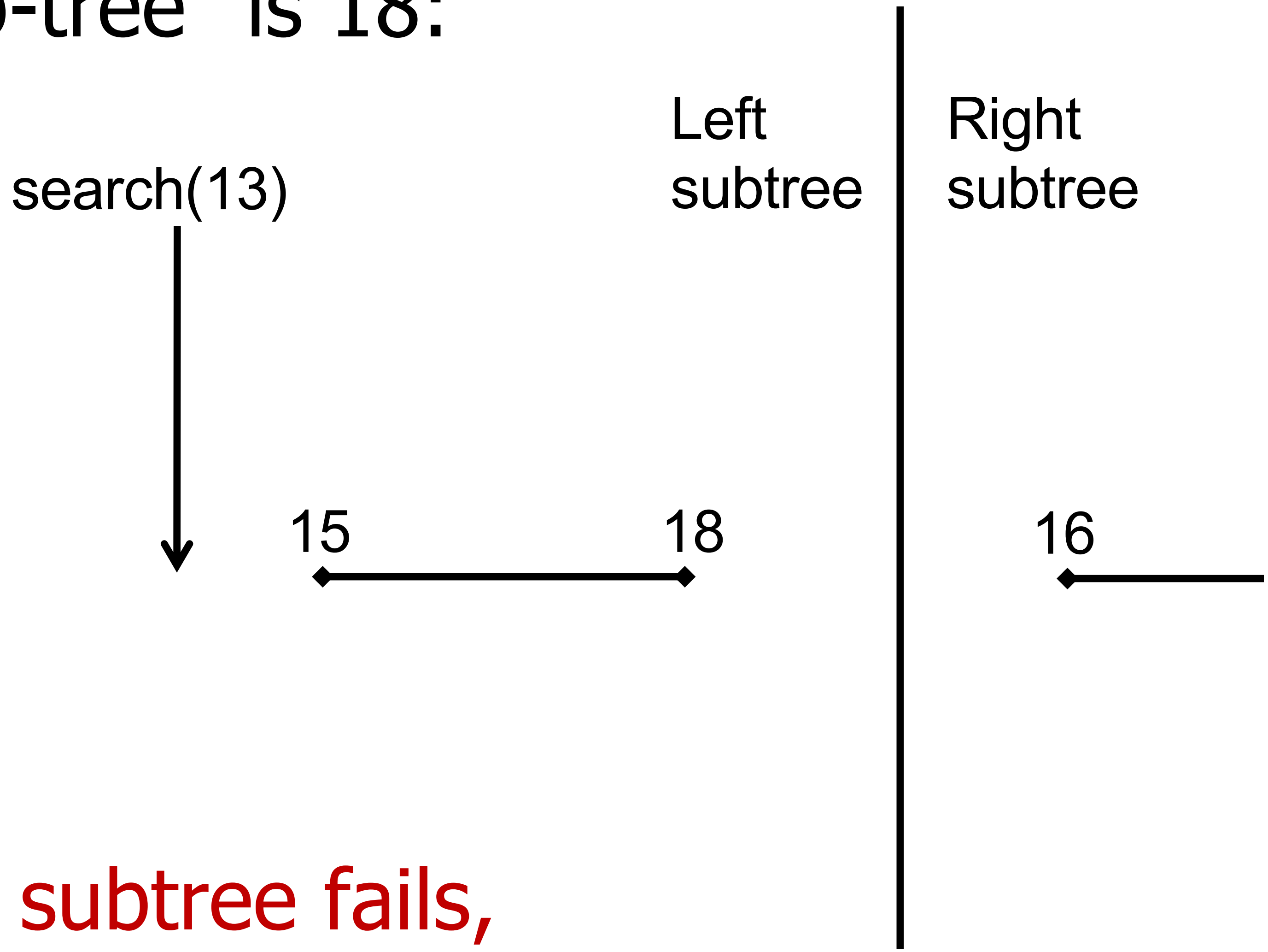
Tree sorted by left endpoint.

13 < every interval in right subtree

➔ Search also would fail in right subtree

Interval Trees

Max in "left sub-tree" is 18:

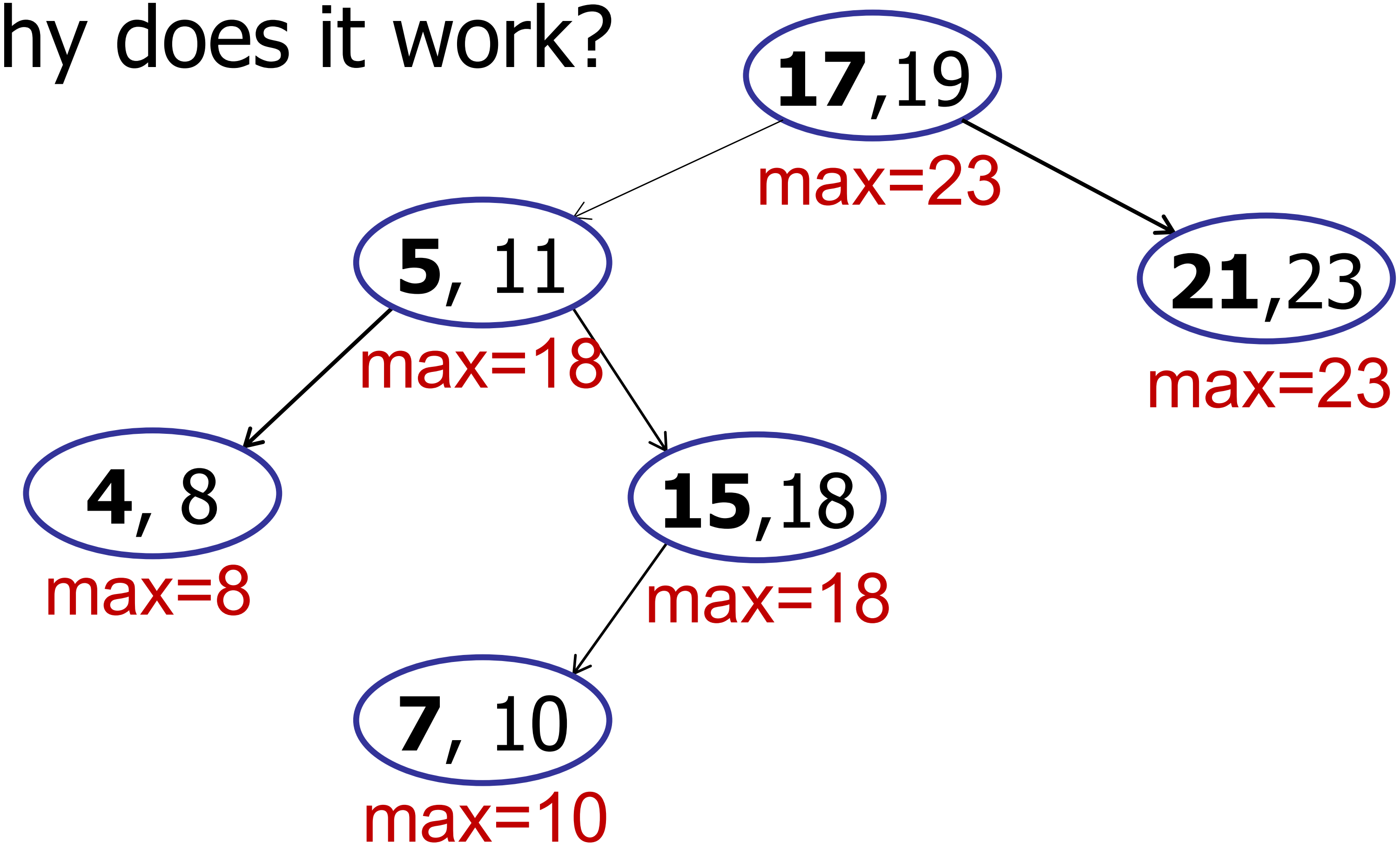


If search in left subtree fails,

Then search also would fail in right subtree!

Interval Trees

Why does it work?



Claim: If search goes left and fails, then key < every interval in right sub-tree.

Interval Trees

If search goes right: then no interval in left subtree.

➔ Either search finds key in right subtree or it is not in the tree.

If search goes left: if there is no interval in left subtree, then there is no interval in right subtree either.

➔ Either search finds key in left subtree or it is not in the tree.

Conclusion: search finds an overlapping interval, if it exists.

The running time of interval-search is:

1. $O(1)$
2. $\boxed{O(\log n)}$
3. $O(n)$
4. $O(n \log n)$
5. $O(n^2)$
6. Can’t say.

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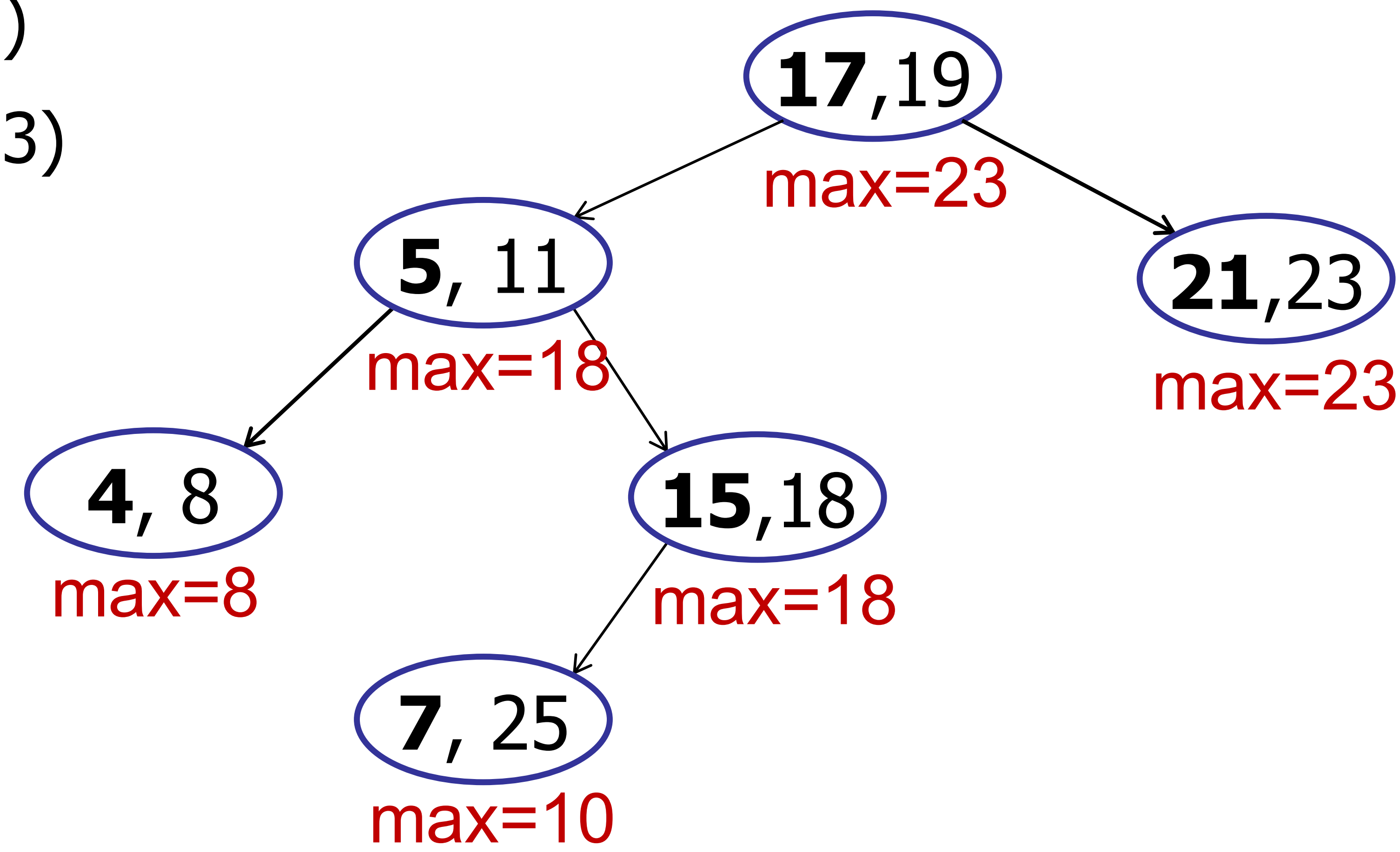
is open

Interval Trees

Extension: List all intervals that overlap with point?

E.g.: search(22) returns:

- (7,25)
- (21,23)



Interval Trees

Extension: List all intervals that overlap with point?

All-Overlaps Algorithm:

Repeat until no more intervals:

- Search for interval.
- Add to list.
- Delete interval.

Repeat for all intervals on list:

- Add interval back to tree.

The running time of All-Overlaps, if there are k overlapping intervals?

1. O(1)
2. O(k)
3. $\boxed{\text{O(k log n)}}$
4. O(k + log n)
5. O(kn)
6. O(kn log n)

ARCHIPELAGO

is open

Interval Trees

Extension: List all intervals that overlap with point?

All-Overlaps Algorithm: O(k log n)

Repeat until no more intervals:

- Search for interval.
- Add to list.
- Delete interval.

Repeat for all intervals on list:

- Add interval back to tree.

Best known solution: O(k + log n)

Today

Two examples of augmenting BSTs

1. Order Statistics

2. Intervals