MATLAB LESSON 1: MATRIX OPERATIONS AND SOLVING LINEAR SYSTEMS

ABSTRACT. In this laboratory session, we introduce some very basic MATLAB commands for performing matrix operations and solving linear systems.

1. STATEMENTS

MATLAB environment behaves like a super-complex calculator. You can enter the commands at the >> command prompt. The answer appears by pressing Enter.

(i) A MATLAB statement is frequently of the form

```
>> variable = expression
```

which assigns the result of `expression` to `variable`. For example,

```
>> a = 3
a =  3
```

(ii) A MATLAB statement may have a simpler form

```
>> expression
```

in which case the result of `expression` is assigned to a special variable called `ans` (which stands for *answer*). For example,

```
>> 3 + 5
ans =  8
```

(iii) You may add a semicolon `;` at the end of the statement; then MATLAB will hide the output. For example,

```
>> b = 3;
>> b ^ 2
ans =  9
```

(iv) The command `help` will give information and usage about the specific `topic`:

```
>> help topic
```

(v) We have defined the symbol `a` as the number 3. We may remove it from the memory by using

```
>> clear a
```

or remove all variables from the memory by using

```
>> clear
```

(vi) If we want to clear the command window, use

```
>> clc
```

2. Precision

By default, MATLAB displays four decimal digits to its answers. But we can change the format for numeric display.

(i) 16 decimal digits:

```
>> format long
>> sqrt(2)
ans = 1.414213562373095
```

(ii) Rational number approximation:

```
>> format rat
>> sqrt(2)
ans = 1393/985
```

MATLAB will approximate decimals with rational numbers when you use `format rat`. Sometimes, this may cause unexpected results. Occasionally, an asterisk $*$ may appear when you expect the quantity to be 0.

(iii) 4 decimal digits (default)

```
>> format short
>> sqrt(2)
ans = 1.4142
```

3. Working with Matrices

Recall that vectors are considered as special matrices. More precisely, a column vector is an $m \times 1$ matrix, and a row vector is a $1 \times n$ matrix.

The entries of a matrix shall be entered row by row, while the entries in each row are separated by a space and the rows are separated by a semi-colon `;`. For example,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

```
>> A = [1 2 3; 4 5 6]
ans =  1  2  3
       4  5  6
```

The size of $\boldsymbol{A}$ is given by

```
>> size(A)
```

```
ans =   2   3
```

and the (i,j)-entry of $\boldsymbol{A}$ is simply given by `A(i,j)`. For example,

```
>>  A(2,3)
ans =   6
```

we can generate special matrices using the following commands:

(i) Zero matrix $\mathbf{0}_{m\times n}$ of size $m \times n$: `zeros(m,n)`.

```
>>  zeros(2,3)
ans =   0   0   0
        0   0   0
```

(ii) Identity matrix $\boldsymbol{I}_n$ of order n: `eye(n)`.

```
>>  eye(2)
ans =   1   0
        0   1
```

(iii) Diagonal matrix with diagonal entries $a_1,\ldots,a_n$: `diag([a1  ...  an])`.

```
>>  diag([2,3])
ans =   2   0
        0   3
```

4. Elementary Row Operations

Let $\boldsymbol{A}$ be a matrix. For example,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{pmatrix}.$$

```
>>  A = [1 2 3 4 5; 2 3 4 5 6; 3 4 5 6 7; 4 5 6 7 8]
ans =   1   2   3   4   5
        2   3   4   5   6
        3   4   5   6   7
        4   5   6   7   8
```

The i^{th} row of $\boldsymbol{A}$ can be abstracted using `A(i,:)`. For example, the 4^{th} row is

```
>>  A(4,:)
ans =   4   5   6   7   8
```

If we need more rows, indicate the indices of the rows in square brackets. For example, the following is the submatrix of $\boldsymbol{A}$ formed by the 2^{nd} and the 4^{th} rows of $\boldsymbol{A}$:

```
>> A([2,4], :)
ans =  2  3  4  5  6
       4  5  6  7  8
```

We can perform the three types of elementary row operations as follows:

(i) Multiplying the i^{th} row by a nonzero constant c: `A(i,:) = c*A(i,:)`.

```
>> A(1,:) = -2*A(1,:);
A =  -2  -4  -6  -8  -10
      2   3   4   5   6
      3   4   5   6   7
      4   5   6   7   8
```

(ii) Interchanging the i^{th} and j^{th} rows: `A([i,j],:) = A([j,i],:)`.

```
>> A([2,3],:) = A([3,2],:);
A =  -2  -4  -6  -8  -10
      3   4   5   6   7
      2   3   4   5   6
      4   5   6   7   8
```

(iii) Adding c times of the j^{th} row to the i^{th} row: `A(i,:) = A(i,:) + c*A(j,:)`.

```
>> A(4,:) = A(4,:) + 2*A(1,:)
A =  -2  -4  -6  -8  -10
      3   4   5   6   7
      2   3   4   5   6
      40 -3  -6  -9  -12
```

5. Matrix Operations

The matrix addition, subtraction and multiplication with scalar can be evaluated using `+`, `-` and `*` respectively. For example,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}.$$

```
>> A = [1 2; 3 4];
>> B = [4 1; 2 5];
```

(i) Addition: $\boldsymbol{A}+\boldsymbol{B}$:

```
>> A + B
ans =  5   3
       5   9
```

(ii) Subtraction: $\boldsymbol{A} - \boldsymbol{B}$:

```
>> A - B
ans =  -3   1
       1   -1
```

(iii) Scalar multiplication: cA:

```
>> 3 * A
ans =  3   6
       9   12
```

We illustrate more operations using the previously defined $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}$.

(iv) Transpose $\boldsymbol{A}^{\mathrm{T}}$:

```
>> A'
ans =  1   3
       2   4
```

(v) Reduced row-echelon form of $\boldsymbol{A}$:

```
>> rref(A)
ans =  1   0
       0   1
```

(vi) Powers $\boldsymbol{A}^n$, provided that $\boldsymbol{A}$ is a square matrix and n is an integer. If $n < 0$, $\boldsymbol{A}$ needs to be invertible (that is, non-singular).

```
>> A ^ 10
ans =  4783807   6972050
       10458075   15241882
```

(vii) If $\boldsymbol{A}$ is invertible, its inverse can be evaluated using either `A^(-1)` or `inv(A)`.

```
>> A ^ (-1)
ans =  -2.0000   1.0000
       1.5000   -0.5000
>> inv(A)
ans =  -2.0000   1.0000
       1.5000   -0.5000
```

(viii) Matrix product $\boldsymbol{AB}$, provided that the sizes are matched.

```
>> A * B
```

```
ans =  8   11
       20  23
```

6. Solve Linear System

Recall that a linear system can be written in the matrix product form $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{A}$ is the coefficient matrix, $\boldsymbol{x}$ is the variable matrix, and $\boldsymbol{b}$ is the constant matrix. Its solution can be found from the (reduced) row-echelon form of the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{b})$.

For example, the linear system

$$\begin{cases} 2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 = -2 \\ x_1 - 2x_2 - 4x_3 + 3x_4 + x_5 = -2 \\ 2x_1 \qquad - 4x_3 + 2x_4 + x_5 = 3 \\ x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 = -7 \end{cases}$$

has augmented matrix

$$(\boldsymbol{A} \mid \boldsymbol{b}) = \left(\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{array}\right)$$

Define the coefficient matrix

```
>> A = [2 -3 -7 5 2; 1 -2 -4 3 1; 2 0 -4 2 1; 1 -5 -7 6 2]
A =  2  -3  -7  5  2
     1  -2  -4  3  1
     2  0  -4  2  1
     1  -5  -7  6  2
```

and the constant matrix

```
>> b = [-2; -2; 3; -7]
b =  -2
     -2
     3
     -7
```

We shall find the reduced row-echelon form (RREF) of $(\boldsymbol{A} \mid \boldsymbol{b})$:

```
>> rref([A b])
ans =  1  0  -2  1  0  1
       0  1  1  -1  0  2
```

```
0   0   0   0   1   1
0   0   0   0   0   0
```

(Here `[A b]` is the matrix obtained by combining `A` and `b` to obtain the augmented matrix. The separator `|` should be omitted in the MATLAB command.)

One sees that the 1^{st}, the 2^{nd} and the 5^{th} columns are pivot columns.

Set $x_3 = s$ and $x_4 = t$ to be arbitrary parameters, and solve other variables:

$$x_1 = 2s - t + 1, \quad x_2 = -s + t + 2, \quad x_5 = 1.$$

Indeed, we can verify that $\boldsymbol{x} = \begin{pmatrix} 2s - t + 1 \\ -s + t + 2 \\ s \\ t \\ 1 \end{pmatrix}$ is a solution.

We first declare that s and t are parameters.

```
>>  syms s t
```

Then define

```
>>  x = [2*s-t+1; -s+t+2; s; t; 1]
x =  2*s - t + 1
     t - s + 2
     s
     t
```

Note that $\boldsymbol{x}$ is a solution if and only if $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. So we evaluate $\boldsymbol{A}\boldsymbol{x}$ and compare it with $\boldsymbol{b}$:

```
>>  A * x
ans =  -2
       -2
       3
       -7
```

7. Activities

7.1. **Activity 1.** Enter the following commands in MATLAB window and observe the outputs. Describe what MATLAB has done.

```
>>  x = [1 2 3]
>>  b = [1; 2; 3]
```

```
>> A = [1 2 pi; 0.1 5 6; 7 8 1/2]
>> format rat
>> A
>> format short
>> A
>> format long
>> A
>> format
>> A
>> A1 = [1 0; 2 1; 3 2; 4 3]
>> 2 * 3
>> ans
>> 0.3 * A
>> ans
>> B = ans * b
>> A * x
>> A * x'
```

7.2. **Activity 2.** Input the following three matrices.

$$\boldsymbol{A} = \begin{pmatrix} 7 & -2 & 0 & -2 \\ -3 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 16 & -3 & 0 & -2 \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} -2 & 5 & 17 & -2 \\ 2 & 6 & -15 & -1 \\ -2 & 6 & 19 & -2 \\ 1 & 2 & -6 & 0 \end{pmatrix}, \qquad \boldsymbol{C} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) Using MATLAB, compute the products $(\boldsymbol{AB})^{-1}$, $(\boldsymbol{BA})^{-1}$, $\boldsymbol{A}^{-1}\boldsymbol{B}^{-1}$ and $\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$. What are the relations between these matrices?

(ii) Compute $\boldsymbol{C}^{-1}$. What result do you get? Why?

(iii) Compute $(\boldsymbol{A}+\boldsymbol{B})^{-1}$ and $\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1}$. Are these matrices equal?

(iv) Compute the products $(\boldsymbol{AB})^{\mathrm{T}}$, $(\boldsymbol{BA})^{\mathrm{T}}$, $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}$ and $\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$. What are the relations between these matrices?

(v) Compute $(\boldsymbol{A}^{\mathrm{T}})^{-1}$ and $(\boldsymbol{A}^{-1})^{\mathrm{T}}$. Are these matrices equal? Is this relation true for any invertible matrix $\boldsymbol{A}$?

(vi) Compute $\boldsymbol{C}^2, \boldsymbol{C}^3$ and $\boldsymbol{C}^4$. What do you observe? Can you generalize this observation to upper triangular matrices of order n with all the diagonal entries 0?

7.3. **Activity 3.**

1. Consider the following linear system:

$$\begin{cases} x + y + 2z = 1 \\ 3x + 6y - 5z = -1 \\ 2x + 4y + 3z = 0 \end{cases}$$

(i) Enter the coefficient matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 6 & -5 \\ 2 & 4 & 3 \end{pmatrix}$ and constant matrix $\boldsymbol{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

(ii) Type `inv(A) * b` to get a solution for this system.

(iii) Type `rref([A b])` to get the reduced row-echelon form of the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{b})$, and then solve the system.

2. Try to use the two different methods to solve the following linear system:

$$\begin{cases} x + 2y + z = 1 \\ x + 2y + 2z = 1 \\ 2x + 4y + z = 2 \end{cases}$$

(*Hint*: Only one method works.)