NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2015-2016

MA1101R Linear Algebra I

November 2015 — Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **Six (6)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions.

3. This is a **closed book** examination but each candidate is allowed to bring in **TWO (2)** double-sided A4-sized handwritten helpsheets.

4. Calculators can be used. However, various steps in the calculations should be laid out systematically.

5. Write down your matriculation/registration number on the cover page of each answer book used.

Question 1 [12 Marks]

Let $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 5 & 0 \end{pmatrix}$.

(i) Use the Gauss-Jordan Elimination to reduce $\boldsymbol{A}$ to the reduced row-echelon form. (Indicate the elementary row operations used in each step.)

(ii) Let $T : \mathbb{R}^6 \to \mathbb{R}^4$ be a linear transformation such that $\boldsymbol{A}$ is the standard matrix for T. Write down a basis for the kernel of T and a basis for the range of T.

Question 2 [12 Marks]

Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ and $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ be two bases for $\mathbb{R}^3$.

Suppose $\boldsymbol{P} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ is the transition matrix from S to T.

(i) Find the transition matrix from T to S.

(ii) Suppose $\boldsymbol{u_1} = (1, 1, 1)$, $\boldsymbol{u_2} = (0, 1, 1)$ and $\boldsymbol{u_3} = (0, 0, 1)$. Find $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$.

Question 3 [21 Marks]

Let $\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$ where a is a constant.

(i) Find all the eigenvalues of $\boldsymbol{A}$.

(ii) For each of the eigenvalues λ of $\boldsymbol{A}$, find a basis for the eigenspace associated with λ.

(iii) Determine the value of a so that $\boldsymbol{A}$ is diagonalizable.

(iv) When $\boldsymbol{A}$ is a diagonalizable, find an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{D} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$.

Question 4 [21 Marks]

(a) Let $V = \text{span}\{\boldsymbol{u_1},\ \boldsymbol{u_2},\ \boldsymbol{u_3}\}$ where

$$\boldsymbol{u_1} = (1,1,0,0), \quad \boldsymbol{u_2} = (0,2,1,1) \quad \text{and} \quad \boldsymbol{u_3} = (1,1,3,1).$$

(i) Use the Gram-Schmidt Process to transform $\{\boldsymbol{u_1},\ \boldsymbol{u_2},\ \boldsymbol{u_3}\}$ to an orthonormal basis for V.

(ii) Find the projection of $\boldsymbol{w} = (1,0,0,1)$ onto V.

(b) Let W be a subspace of $\mathbb{R}^n$ and $W^\perp = \{\boldsymbol{w} \in \mathbb{R}^n \mid \boldsymbol{w} \text{ is orthogonal to } W\}$.

Prove that $\dim(W) + \dim(W^\perp) = n$.

Question 5 [17 Marks]

(All vectors in this question are written as column vectors.)

Let $\boldsymbol{A}$ be an $n \times n$ matrix such that $\boldsymbol{A}^n = \boldsymbol{0}$. Suppose there exists a nonzero vector $\boldsymbol{v} \in \mathbb{R}^n$ such that $\boldsymbol{A}^{n-1}\boldsymbol{v} \neq \boldsymbol{0}$.

(a) Give an example of a 2×2 matrix $\boldsymbol{A}$ such that $\boldsymbol{A} \neq \boldsymbol{0}$ but $\boldsymbol{A}^2 = \boldsymbol{0}$.

(b) Prove that $\{\boldsymbol{v},\ \boldsymbol{Av},\ \ldots,\ \boldsymbol{A}^{n-1}\boldsymbol{v}\}$ is a basis for $\mathbb{R}^n$.

(c) Let $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{A}^{n-1}\boldsymbol{v} & \cdots & \boldsymbol{Av} & \boldsymbol{v} \end{pmatrix}$ which is an invertible matrix of order n.

Show that

$$\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Question 6 [17 Marks]

(All vectors in this question are written as column vectors.)

Let $\boldsymbol{A}$ be an invertible matrix of order $\boldsymbol{n}$ such that for any nonzero vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is always equal to the angle between $\boldsymbol{Au}$ and $\boldsymbol{Av}$.

(a) Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \cdots & \boldsymbol{a_n} \end{pmatrix}$ where $\boldsymbol{a_i}$ is the ith column of $\boldsymbol{A}$.

Show that $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n}\}$ is an orthogonal basis for $\mathbb{R}^n$.

(Hint: Use the standard basis $E = \{\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_n}\}$ and consider vectors $\boldsymbol{Ae_i}$ for $i = 1, 2, \ldots, n$.)

(b) Prove that $\boldsymbol{A} = c\boldsymbol{P}$ for some scalar c and orthogonal matrix $\boldsymbol{P}$.

[END OF PAPER]

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Question 1 [14 Marks]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & k & 1 & k \\ k & k & 2 & 0 \\ k & 0 & k & 0 \end{pmatrix}$ where k is a constant.

(a) Use Gauss-Jordan Elimination to reduce $\boldsymbol{A}$ to the reduced row-echelon form. (Write down the elementary row operations clearly.)

(b) Find a basis for the nullspace space of $\boldsymbol{A}$.

(*Warning*: The value of k will affect your answers.)

Question 2 [12 Marks]

Let $V = \{ (a+b-2c,\ 2b-c,\ 3c+d,\ a+3b+d) \mid a,b,c,d \in \mathbb{R} \}$.

(a) Show that V is a subspace of $\mathbb{R}^4$.

(b) Find a basis for V and determine the dimension of V.

(c) Let $W = \{ (1+a+b-2c,\ 2b-c,\ 3c+d,\ 1+a+3b+d) \mid a,b,c,d \in \mathbb{R} \}$.

(i) Is the zero vector contained in W? Justify your answer.

(ii) Is W a subspace of $\mathbb{R}^4$?

Question 3 [18 Marks]

Let $\boldsymbol{B} = \begin{pmatrix} 4 & 0 & 2 & -2 \\ 0 & 4 & -2 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}$.

(a) Find an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ so that $\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} = \boldsymbol{D}$.

(b) Write down a matrix $\boldsymbol{C}$ such that $\boldsymbol{C}^2 = \boldsymbol{B}$.

(You can express your answer in the form $\boldsymbol{P}\boldsymbol{X}\boldsymbol{P}^{-1}$ where $\boldsymbol{X}$ is a 4×4 matrix and $\boldsymbol{P}$ is the invertible matrix obtained in (a).)

Question 4 [22 Marks]

(All vectors in this question are written as column vectors.)

Let $\boldsymbol{u_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{u_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{u_3} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, and $\boldsymbol{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\boldsymbol{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(a) (i) Find the reduced row-echelon form of the following 3×6 matrix

$$\begin{pmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{u_3} & \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \end{pmatrix}.$$

(ii) Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$. You can assume that S is a basis for $\mathbb{R}^3$.

Write down the coordinate vectors $(\boldsymbol{e_1})_S$, $(\boldsymbol{e_2})_S$, $(\boldsymbol{e_3})_S$.

(b) Define a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T(\boldsymbol{x}) = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} \quad \text{for } \boldsymbol{x} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + c_3\boldsymbol{u_3} \in \mathbb{R}^3.$$

(i) Find the standard matrix for T.

(ii) Determine the rank and nullity of T.

(iii) Explain why $T(\boldsymbol{x})$ is the orthogonal projection of $\boldsymbol{x}$ onto $V = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$.

Question 5 [17 Marks]

(All vectors in this question are written as column vectors.)

Let $\boldsymbol{A}$ be a square matrix of order n.

(a) Let $E = \{\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_n}\}$ be the standard basis for $\mathbb{R}^n$.

Show that $\boldsymbol{A}\boldsymbol{e_j} =$ the jth column of $\boldsymbol{A}$.

(b) Suppose $\boldsymbol{A}^m = \boldsymbol{0}$ and $\boldsymbol{A}^{m-1} \neq \boldsymbol{0}$ for some integer $m \geq 2$.

(i) Show that there exists at least one vector $\boldsymbol{u} \in \mathbb{R}^n$ such that $\boldsymbol{A}^{m-1}\boldsymbol{u} \neq \boldsymbol{0}$.

(ii) Show that $\{\boldsymbol{u}, \boldsymbol{A}\boldsymbol{u}, \ldots, \boldsymbol{A}^{m-1}\boldsymbol{u}\}$ is linearly independent where $\boldsymbol{u}$ is the vector obtained in Part (i).

(c) Prove that if $\boldsymbol{A}^{n+1} = \boldsymbol{0}$, then $\boldsymbol{A}^n = \boldsymbol{0}$.

Question 6 [17 Marks]

(All vectors in this question are written as column vectors.)

(a) Let $\boldsymbol{B}$ be a 2×2 symmetric matrix and let $\boldsymbol{u}$, $\boldsymbol{v}$ be two eigenvectors of $\boldsymbol{B}$ associated with the eigenvalues λ and μ respectively.

(i) Show that if $\lambda \neq \mu$, then $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

(ii) Suppose $\boldsymbol{u} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $\lambda = 1$ and $\mu = 3$.

Find $\boldsymbol{B}$. (Hint: $\{\boldsymbol{u}, \boldsymbol{v}\}$ is an orthonormal basis for $\mathbb{R}^2$.)

(b) Let $\boldsymbol{C}$ be a symmetric matrix of order n with a characteristic polynomial

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

Prove that for any nonzero vector $\boldsymbol{x} \in \mathbb{R}^n$, $\lambda_1 \leq \dfrac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}} \leq \lambda_n$.

(Hint for (a)(i) and (b): For $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v}$.)

[END OF PAPER]

National University of Singapore
Department of Mathematics

Semester 1, 2017/18

MA1101R Linear Algebra I

November 2017 — Time allowed: 2 hours

Student Number: ____________________

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of 6 questions, for a total of 80 points. Excluding the cover page, there are 12 printed pages.
2. Answer all 6 questions.
3. This is a closed book examination but you are allowed to bring in one A4-size and double-sided helpsheet.
4. You can use any kind of calculators (except devices which can be used for communication and/or web-surfing). However, various steps in the calculations should be laid out systematically.
5. Write down your student number on the cover page of this booklet.
6. Write your answers in the space below each question. This booklet will be collected at the end of the examination.
7. The left-hand pages can be used for rough work.

Question	Points	Score
1	11	
2	8	
3	9	
4	18	
5	17	
6	17	
Total:	80	

1. (11 points) Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$.

(a) Use the **Gauss-Jordan Elimination** to reduce $\boldsymbol{A}$ to its reduced row-echelon form. (Write down the steps of your computations.)

(b) Write down a basis for the row space of $\boldsymbol{A}$.

(c) Write down a basis for the column space of $\boldsymbol{A}$.

(d) Write down a basis for the nullspace of $\boldsymbol{A}$.

2. (8 points) Let $V = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ where $\boldsymbol{u_1} = (1,1,0,0)$, $\boldsymbol{u_2} = (1,1,-1,-1)$ and $\boldsymbol{u_3} = (1,a,1,a)$ where a is an unknown constant.

   Apply the Gram-Schmidt Process to $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ to obtain an orthonormal basis for V.

   (Warning: The value of a may affect your answer.)

3. (9 points) Let W be a vector space with a basis $S = \{ \boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3} \}$. Let $T = \{ \boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3} \}$ where

$$\boldsymbol{w_1} = \boldsymbol{v_1} + 2\boldsymbol{v_2}, \quad \boldsymbol{w_2} = \boldsymbol{v_2} + 2\boldsymbol{v_3} \ \text{ and } \ \boldsymbol{w_3} = \boldsymbol{v_3}.$$

(a) Show that T is a basis for W.

(b) Find the transition matrix from S to T.

4. (18 points) Let $\boldsymbol{B} = \begin{pmatrix} -2 & 0 & -2 & 1 \\ -1 & -1 & -2 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

(a) Find the characteristic polynomial of $\boldsymbol{B}$ and verify that the eigenvalues of $\boldsymbol{B}$ are -1 and 0.

(b) Find a basis for the eigenspace E_{-1} of $\boldsymbol{B}$.

(c) Find a basis for the eigenspace E_0 of $\boldsymbol{B}$.

(d) Write down an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} = \boldsymbol{D}$.

(e) Find $\boldsymbol{B}^{1101}$.

5. (17 points) Let $\boldsymbol{C}$ be a square matrix.

(a) Show that the nullspace of $\boldsymbol{C}$ is a subset of the nullspace of $\boldsymbol{C}^2$.

(b) If $\operatorname{rank}(\boldsymbol{C}^2) = \operatorname{rank}(\boldsymbol{C})$, show that the nullspace of $\boldsymbol{C}^2$ is equal to the nullspace of $\boldsymbol{C}$.

(c) Give an example of a 2×2 matrix $\boldsymbol{C}$ with $\text{rank}(\boldsymbol{C}^2) = \text{rank}(\boldsymbol{C})$.

(d) Give an example of a 2×2 matrix $\boldsymbol{C}$ with $\text{rank}(\boldsymbol{C}^2) < \text{rank}(\boldsymbol{C})$.

(e) Can $\text{rank}(\boldsymbol{C}^2) > \text{rank}(\boldsymbol{C})$? Why?

6. (17 points) Let $\boldsymbol{A}$ be an $n \times n$ matrix.

For each $\lambda \in \mathbb{R}$, we define a linear transformation $T_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$T_\lambda(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u} - \lambda\boldsymbol{u} \quad \text{for} \quad \boldsymbol{u} \in \mathbb{R}^n.$$

(a) Write down the standard matrix for T_λ.

(b) For any $\lambda, \mu \in \mathbb{R}$, show that

$$(\boldsymbol{A} - \lambda\boldsymbol{I})(\boldsymbol{A} - \mu\boldsymbol{I}) = (\boldsymbol{A} - \mu\boldsymbol{I})(\boldsymbol{A} - \lambda\boldsymbol{I}).$$

(c) Suppose $\boldsymbol{A}$ is diagonalizable and the eigenvalues of $\boldsymbol{A}$ are λ_1, λ_2, $\ldots$, λ_k.

(i) If $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$, say, $\boldsymbol{A}\boldsymbol{v} = \lambda_i \boldsymbol{v}$ for some i, show that $(\boldsymbol{A} - \lambda_1 \boldsymbol{I})(\boldsymbol{A} - \lambda_2 \boldsymbol{I}) \cdots (\boldsymbol{A} - \lambda_k \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}$.

(Hint: First, show that $(\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}$ and then use the result in part (b).)

(ii) Define $S = T_{\lambda_1} \circ T_{\lambda_2} \circ \cdots \circ T_{\lambda_k}$. Prove that S is the zero transformation.