Logic of Quantified Statements

Predicates can be obtained by removing some or all of the nouns from a statement

let P stand for "is a student at NUS" and let Q stand for "is a student at"
Then both P and Q are predicate symbols

$P(x) =$ "x is a student at NUS"
$Q(x, y) =$ "x is a student at y" } Predicate Variables

Predicate : predicate symbol + predicate variables

A Predicate is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.

The domain of a predicate variable is the set of all values that may be substituted in place of the variable

Truth set : If $P(x)$ is a predicate and x has domain D, the truth set is the set of all elements of D that make $P(x)$ true when they are substituted for x.

The truth set of $P(x)$ is denoted $\{x \in D \mid P(x)\}$

such that

Universal Quantifier : $\forall$

one sure way to change predicates into statements is to assign specific values to all their variables

If x represents the number 35, the sentence "x is divisible by 5" is a true statement

Another method : add quantifiers
↳ words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true

$\forall$: universal

Existential Quantifier : $\exists$

Example: "There is a student in CS1231S" can be written a

$\exists$ a person p such that p is a student in CS1231S.

Or, more formally,

$\exists p \in P$ such that p is a student in CS1231S.

where P is the set of all people.

- The words *such that* are inserted just before the predicate. If the context is clear, sometimes the abbreviation s.t. is used.
- Some alternative expressions for "there exists" are "there is a", "we can find a", "there is at least one", "for some", and "for at least one".

Definition 3.1.3 (Universal Statement)

Let $Q(x)$ be a predicate and D the domain of x. A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$".

- It is defined to be true iff $Q(x)$ is true for every x in D.
- It is defined to be false iff $Q(x)$ is false for at least one x in D.

A value for x for which $Q(x)$ is false is called a **counterexample.**

Definition 3.1.4 (Existential Statement)

Let $Q(x)$ be a predicate and D the domain of x.
An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$".

- It is defined to be true iff $Q(x)$ is true for at least one x in D.
- It is defined to be false iff $Q(x)$ is false for all x in D.

$\exists!$ is the **uniqueness quantifier symbol**. It means "there exists a unique" or "there is one and only one".

Negation of a Universal statement

$$\sim(\forall x \in D, P(x)) \equiv \exists x \in D \text{ such that } \sim P(x)$$

not-(for all statement) = (there exists) not-statement

Negation of an Existential Statement

$$\sim(\exists x \in D \text{ such that } P(x)) \equiv \forall x \in D, \sim P(x)$$

not-(there exists statement) = (for all) not-statement

Negation of Universal Conditional Statements

Of special importance in mathematics.

$\sim(\forall x\,(P(x) \to Q(x))) \equiv \exists x$ such that $\sim(P(x) \to Q(x))$ ... (A)

$\sim(P(x) \to Q(x)) \equiv P(x) \wedge \sim Q(x)$ ... (B)

Substituting (B) into (A):

$\sim(\forall x\,(P(x) \to Q(x))) \equiv \exists x$ such that $(P(x) \wedge \sim Q(x))$

a. $\forall$ people p, if p is blond then p has blue eyes.

$\exists$a person p such that p is blond and p does not have blue eyes.

Vacous Truth of Universal Statements

The statement is false if, and only if, its negation is true.

"All the balls in the bowl are blue"

If there are no balls in the bowl

negation: There exists a ball in the bowl that is not blue is false

Hence the statement is true "by default"

- A vacuous truth is a conditional or universal statement that is only true because the hypothesis (antecedent) cannot be satisfied.
- For this reason, sometimes we say a statement is vacuously true only because it does not really say anything.

$\forall a \in X, P(a)$ is vacuously true if X is an empty set.
(Eg: All mooloomeelees are mammals.)

In general, a statement of the form
$\forall x \in D\,(P(x) \to Q(x))$
is called vacuously true or true by default
if, and only if, $P(x)$ is false for every x in D.

Definition: A set A is a subset of set B, denoted as $A \subseteq B$, if every element in A is an element in B.
Proof that the empty set $\emptyset$ is a subset of every set.
Proof: Since $\forall x, (x \notin \emptyset)$, the argument holds vacuously.
(Alternatively can prove by contradiction, but is longer.)

Definition 3.2.1 (Contrapositive, converse, inverse)

Consider a statement of the form: $\forall x \in D\ (P(x) \to Q(x))$.

1. Its **contrapositive** is: $\forall x \in D\ (\sim Q(x) \to \sim P(x))$.
2. Its **converse** is: $\forall x \in D\ (Q(x) \to P(x))$.
3. Its **inverse** is: $\forall x \in D\ (\sim P(x) \to \sim Q(x))$.

Necessary and Sufficient Conditions, Only if

Definition 3.2.2 (Necessary and Sufficient conditions, Only if)

- "$\forall x$, $r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x\ (r(x) \to s(x))$".
- "$\forall x$, $r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x\ (\sim r(x) \to \sim s(x))$" or, equivalently, "$\forall x\ (s(x) \to r(x))$".
- "$\forall x$, $r(x)$ **only if** $s(x)$" means "$\forall x\ (\sim s(x) \to \sim r(x))$" or, equivalently, "$\forall x\ (r(x) \to s(x))$".

All birds can fly: $\forall x\ (\mathrm{Bird}(x) \to \mathrm{Fly}(x))$

There is a bird that can fly: $\exists x\ (\mathrm{Bird}(x) \wedge \mathrm{Fly}(x))$

Negations of Multiply-Quantified Statements

Recall in 3.2.1: $\sim(\forall x \in D, P(x)) \equiv \exists x \in D$ such that $\sim P(x)$

$\sim(\exists x \in D$ such that $P(x)) \equiv \forall x \in D, \sim P(x)$

(A) So, to find: $\sim(\forall x \in D, \exists y \in E$ such that $P(x, y))$

➔ $\exists x \in D$ such that $\sim(\exists y \in E$ such that $P(x, y))$

➔ $\exists x \in D$ such that $\forall y \in E, \sim P(x, y)$.

$\sim(\forall x \in D, \exists y \in E$ such that $P(x, y))$ ≡ $\exists x \in D$ such that $\forall y \in E, \sim P(x, y)$

(B) Similarly, to find: $\sim(\exists x \in D$ such that $\forall y \in E, P(x, y))$

➔ $\forall x \in D, \sim(\forall y \in E, P(x, y))$

➔ $\forall x \in D, \exists y \in E$ such that $\sim P(x, y)$.

$\sim(\exists x \in D$ such that $\forall y \in E, P(x, y))$ ≡ $\forall x \in D, \exists y \in E$ such that $\sim P(x, y)$

In a statement containing both $\forall$ and $\exists$, changing the order of the quantifiers usually changes the meaning of the statement.

However, if one quantifier immediately follows another quantifier *of the same type*, then the order of the quantifiers does not affect the meaning.

Examples:

- $\forall x\ \forall y$ is equivalent to $\forall y\ \forall x$ (likewise for $\exists$)
- $\forall x\ \forall y$ *may* be written as $\forall x,y$ (likewise for $\exists$)

Formal Logical Notation

In some areas of computer science, logical statements are expressed in purely symbolic notation.

The notation involves using predicates to describe all properties of variables and omitting the words *such as* in existential statements.

"$\forall x \in D, P(x)$" written as $\forall x\ (x \in D \to P(x))$

"$\exists x \in D$ such that $P(x)$" written as $\exists x\ (x \in D \wedge P(x))$

We will follow this way of writing.

Use formal, logical notation to write the following statements, and write a formal negation for each statement.

d. There is a square x such that for all triangles y, x is to right of y.

Statement: $\exists x\ (\mathrm{Square}(x) \wedge \forall y\ (\mathrm{Triangle}(y) \to \mathrm{RightOf}(x, y)))$

Negation: $\sim(\exists x\ (\mathrm{Square}(x) \wedge \forall y\ (\mathrm{Triangle}(y) \to \mathrm{RightOf}(x, y))))$

$\equiv \forall x \sim(\mathrm{Square}(x) \wedge \forall y\ (\mathrm{Triangle}(y) \to \mathrm{RightOf}(x, y)))$

$\equiv \forall x\ (\sim\mathrm{Square}(x) \vee \sim(\forall y\ (\mathrm{Triangle}(y) \to \mathrm{RightOf}(x, y))))$

$\equiv \forall x\ (\sim\mathrm{Square}(x) \vee \exists y\ (\sim(\mathrm{Triangle}(y) \to \mathrm{RightOf}(x, y))))$

$\equiv \forall x\ (\sim \mathrm{Square}(x) \vee \exists y\ (\mathrm{Triangle}(y) \wedge \sim\mathrm{RightOf}(x, y)))$

Universal Instantiation

If some property is true of *everything* in the set, then it is true of *any particular* thing in the set.

Universal instantiation is the fundamental tool of deductive reasoning.

Universal Modus Ponens

Formal version	Informal version
$\forall x\ (P(x) \to Q(x))$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$P(a)$ for a particular a.	a makes $P(x)$ true.
• $Q(a)$.	• a makes $Q(x)$ true.

Universal Modus Tollens

Formal version	Informal version
$\forall x\ (P(x) \to Q(x))$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim Q(a)$ for a particular a.	a does not make $Q(x)$ true.
• $\sim P(a)$.	• a does not makes $P(x)$ true.

Converse Error (Quantified Form)

Formal version	Informal version
$\forall x\ (P(x) \to Q(x))$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$Q(a)$ for a particular a.	a makes $Q(x)$ true.
• $P(a)$.	• a makes $P(x)$ true.

Inverse Error (Quantified Form)

Formal version	Informal version
$\forall x\ (P(x) \to Q(x))$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim P(a)$ for a particular a.	a does not make $P(x)$ true.
• $\sim Q(a)$.	• a does not make $Q(x)$ true.

Universal Transitivity

Formal version	Informal version
$\forall x\ (P(x) \to Q(x))$.	Any x that makes $P(x)$ true makes $Q(x)$ true.
$\forall x\ (Q(x) \to R(x))$.	Any x that makes $Q(x)$ true makes $R(x)$ true.
• $\forall x\ (P(x) \to R(x))$.	• Any x that makes $P(x)$ true makes $R(x)$ true.

Definition 3.4.1 (Valid Argument Form)

To say that **an argument form is valid** means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true.

An **argument is called valid** if, and only if, its form is valid.

3.4.8. Rules of Inference for Quantified Statements

Rule of Inference	Name
$\forall x \in D\ P(x)$ $\therefore P(a)$ if $a \in D$	Universal instantiation
$P(a)$ for every $a \in D$ $\therefore \forall x \in D\ P(x)$	Universal generalization
$\exists x \in D\ P(x)$ $\therefore P(a)$ for some $a \in D$	Existential instantiation
$P(a)$ for some $a \in D$ $\therefore \exists x \in D\ P(x)$	Existential generalization