CS2040S
Data Structures and Algorithms

Augmented Trees!

Puzzle of the Week:

100 prisoners. Every so often, one is chosen at random to enter a room with a light bulb. You can turn the light bulb on or off.

- WIN if one prisoner announces correctly that all have visited the room.
- LOSE if announcement is incorrect.

What if, initially, the state of the light is unknown, either on or off?

Todays Plan

On the importance of being balanced

- Height-balanced binary search trees
- AVL trees
- Rotations

Tries

- How to handle text?

Data structure design

- How to build new structures on existing ideas?

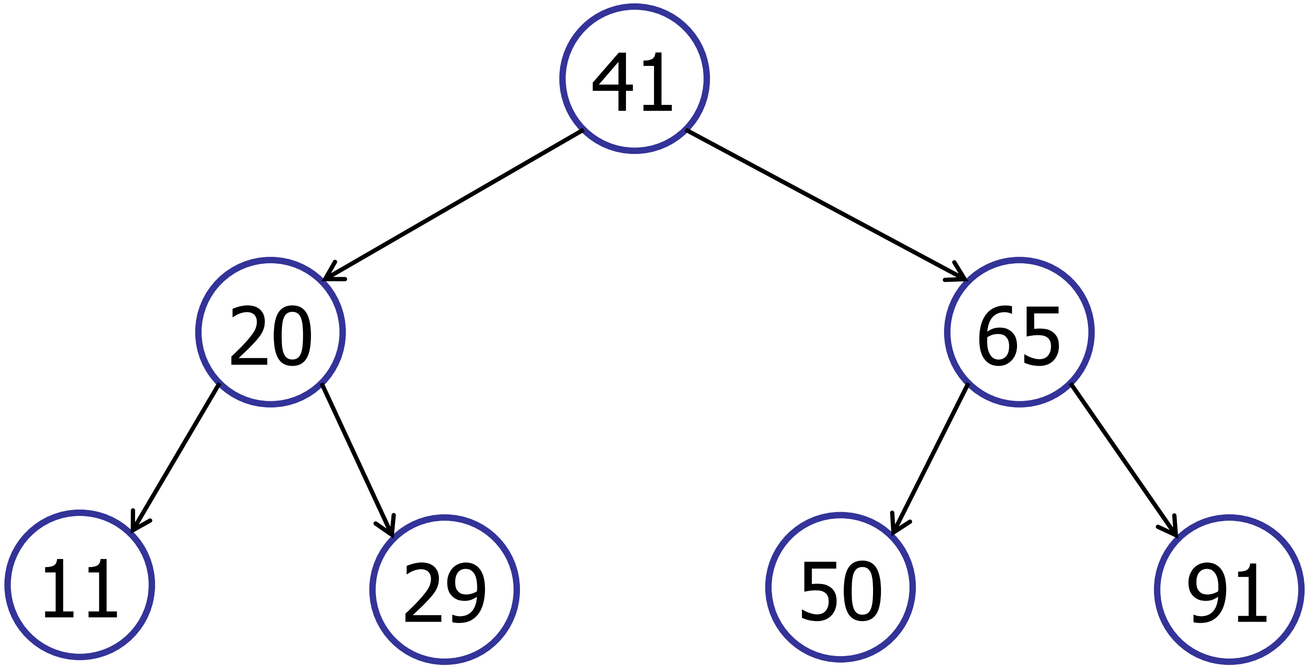
Recap: Dictionary Interface

A collection of (key, value) pairs:

```
interface  IDictionary

   void    insert(Key k, Value v)      insert (k,v) into table
  Value    search(Key k)               get value paired with k
    Key    successor(Key k)            find next key > k
    Key    predecessor(Key k)          find next key < k
   void    delete(Key k)               remove key k (and value)
boolean    contains(Key k)             is there a value for k?
    int    size()                      number of (k,v) pairs
```

Recap: Binary Search Trees



- Two children: v.left, v.right
- Key: v.key
- **BST Property**: all in left sub-tree < key < all in right sub-right

Binary Search Tree

Modifying Operations: O(h)

- insert
- delete

Query Operations: O(h)

- search
- predecessor, successor
- findMax, findMin

Traversals: O(n)

The Importance of Being Balanced

Operations take O(h) time

$\log(n) - 1 \leq h \leq n$

Key definition

A BST is <u>balanced</u> if $h = O(\log n)$

On a balanced BST: all operations run in $O(\log n)$ time.

The Importance of Being Balanced

How to get a balanced tree:

- Define a <u>good property</u> of a tree.
- Show that if the <u>good property</u> holds, then the tree is balanced.
- After every insert/delete, make sure the <u>good property</u> still holds. If not, fix it.

Invariant

AVL Trees [Adelson-Velskii & Landis 1962]

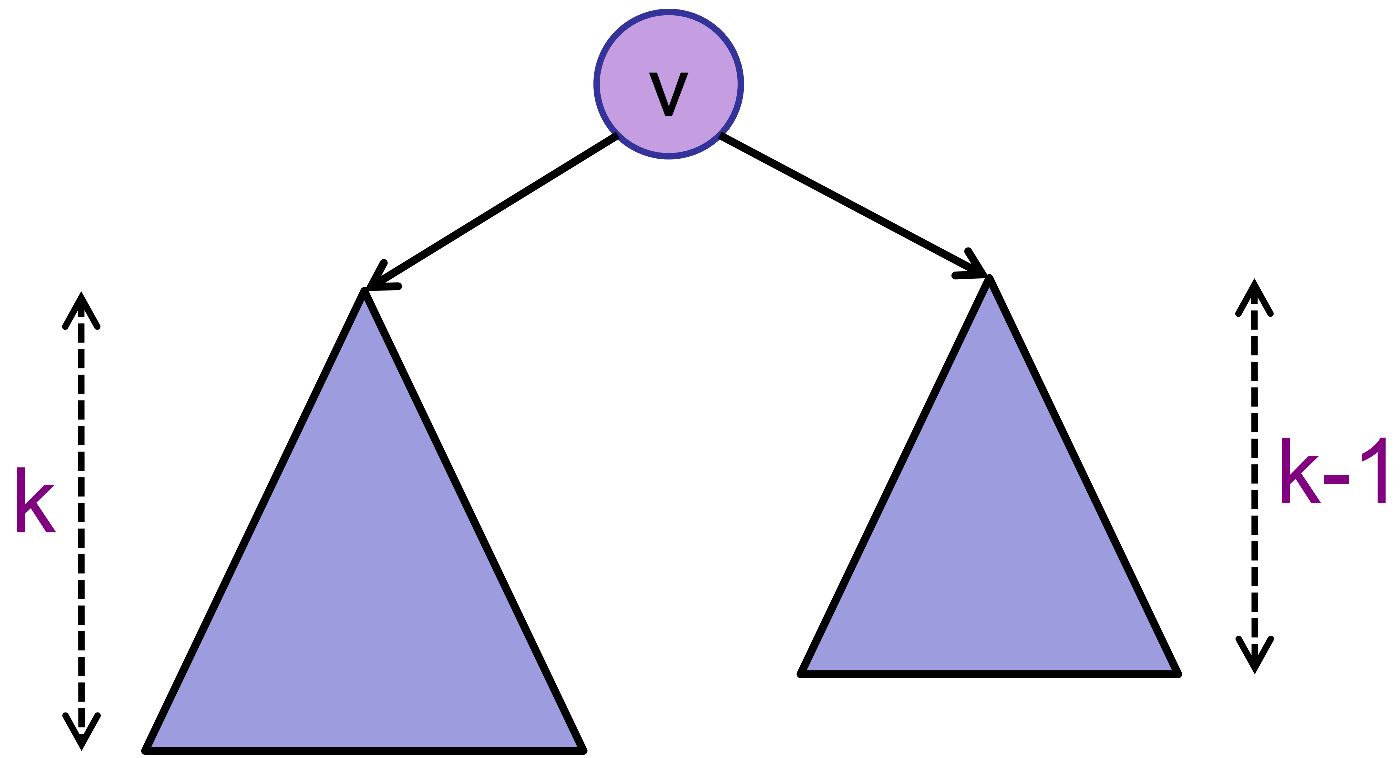
AVL Trees [Adelson-Velskii & Landis 1962]

Key definition

Step 1: Define Invariant

- A node v is **height-balanced** if:

$$|v.left.height - v.right.height| \leq 1$$



Height-Balanced Trees

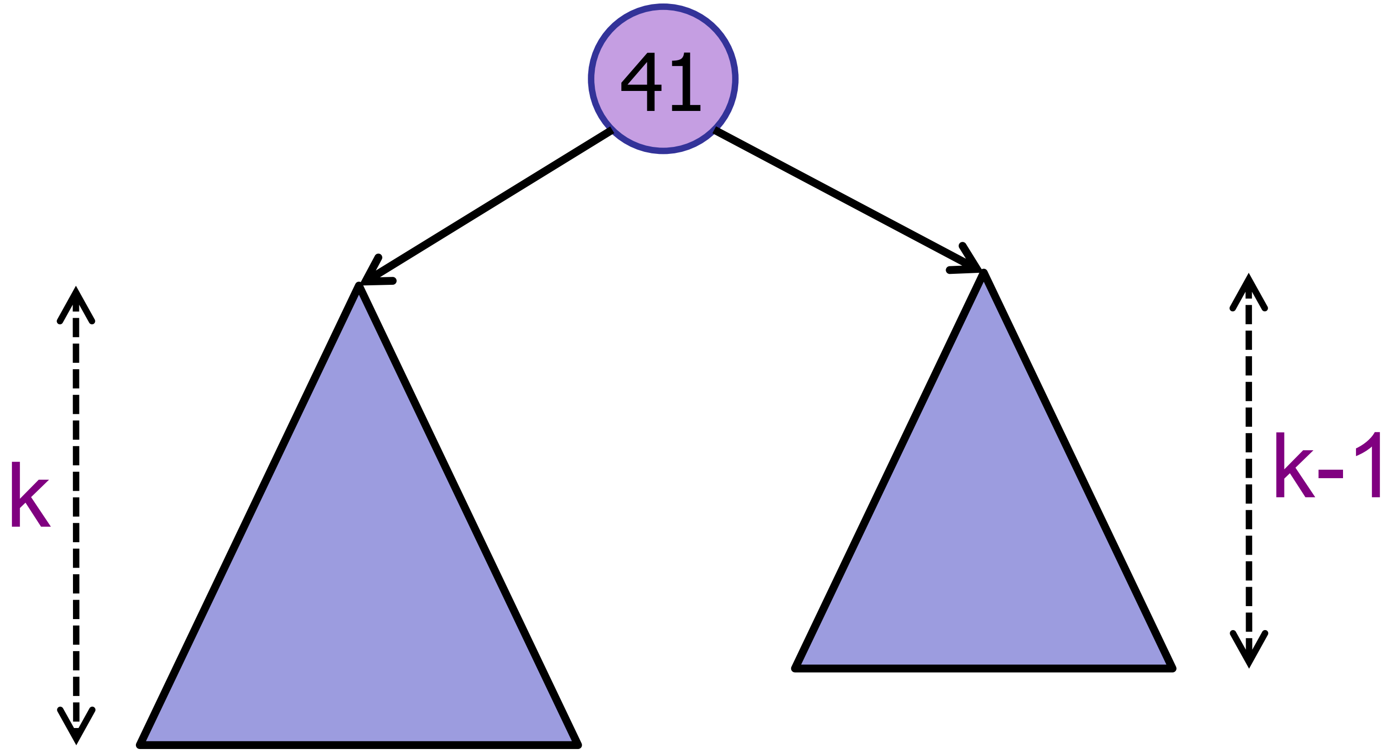
Theorem:

A height-balanced tree with n nodes has **at most** height $h < 2\log(n)$.

→ A height-balanced tree is balanced.

AVL Trees [Adelson-Velskii & Landis 1962]

Step 2: Show how to maintain height-balance



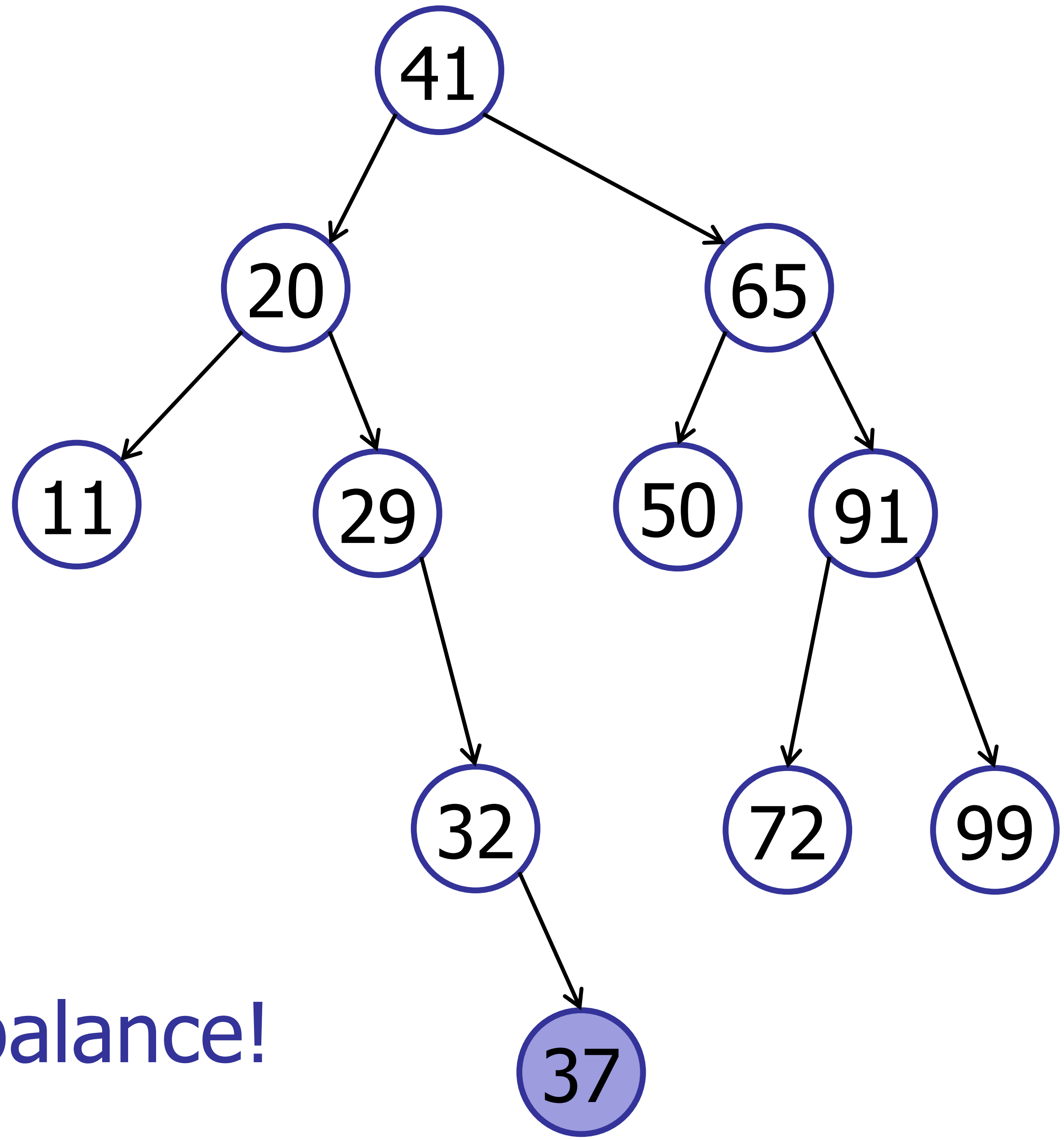
Inserting in an AVL Tree

Initially balanced

insert(37)

No longer balanced after insertion!

Use rotations to rebalance!



Quick review: a rotation costs:

1. $O(1)$
2. $O(\log n)$
3. $O(n)$
4. $O(n^2)$
5. $O(2^n)$

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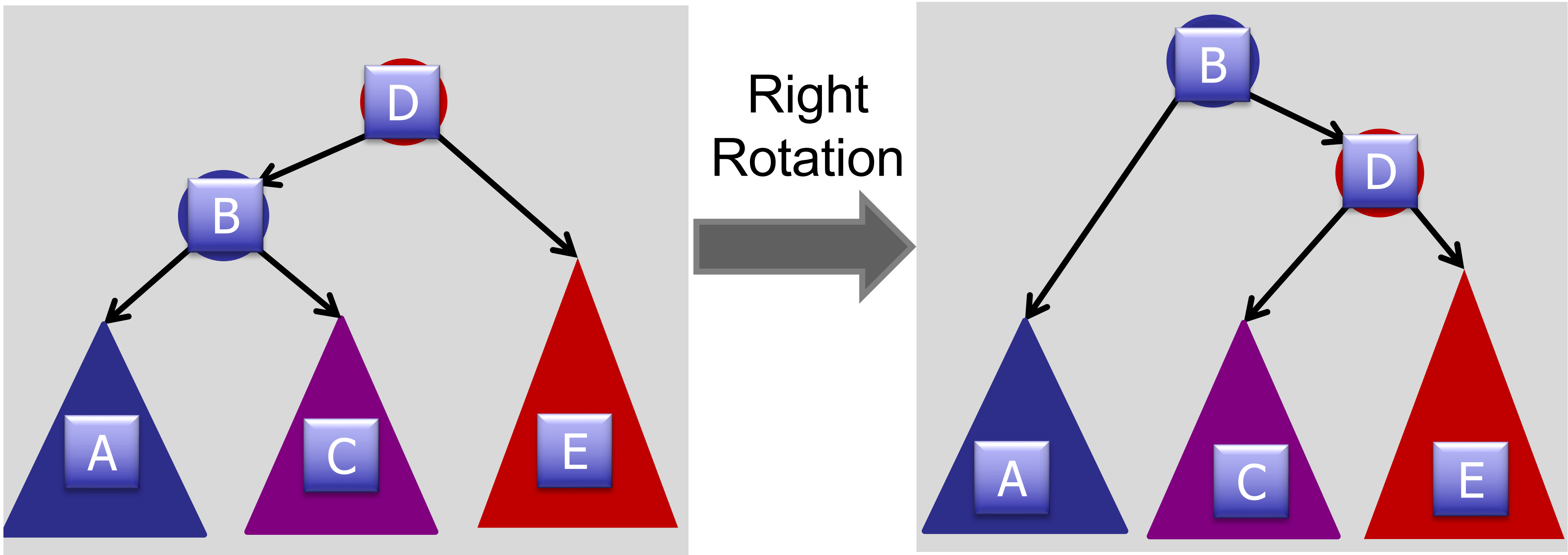
is open

Tree Rotations

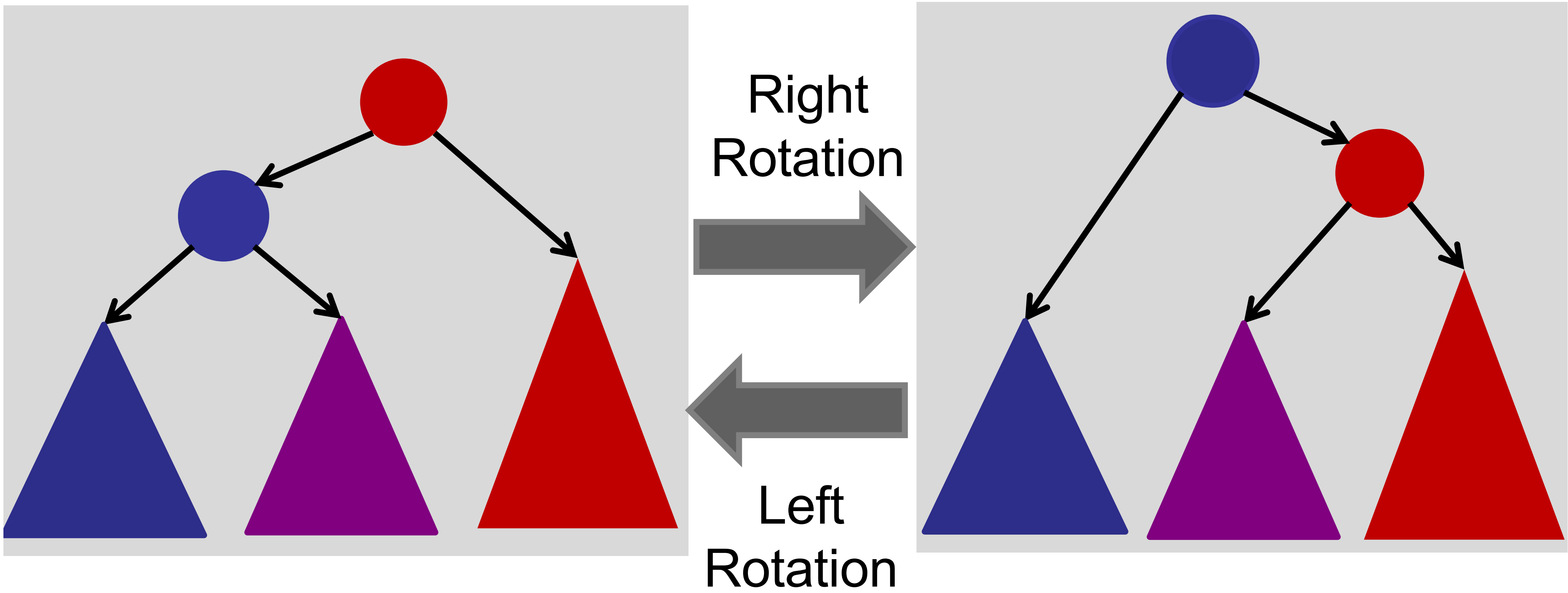


A < B < C < D < E

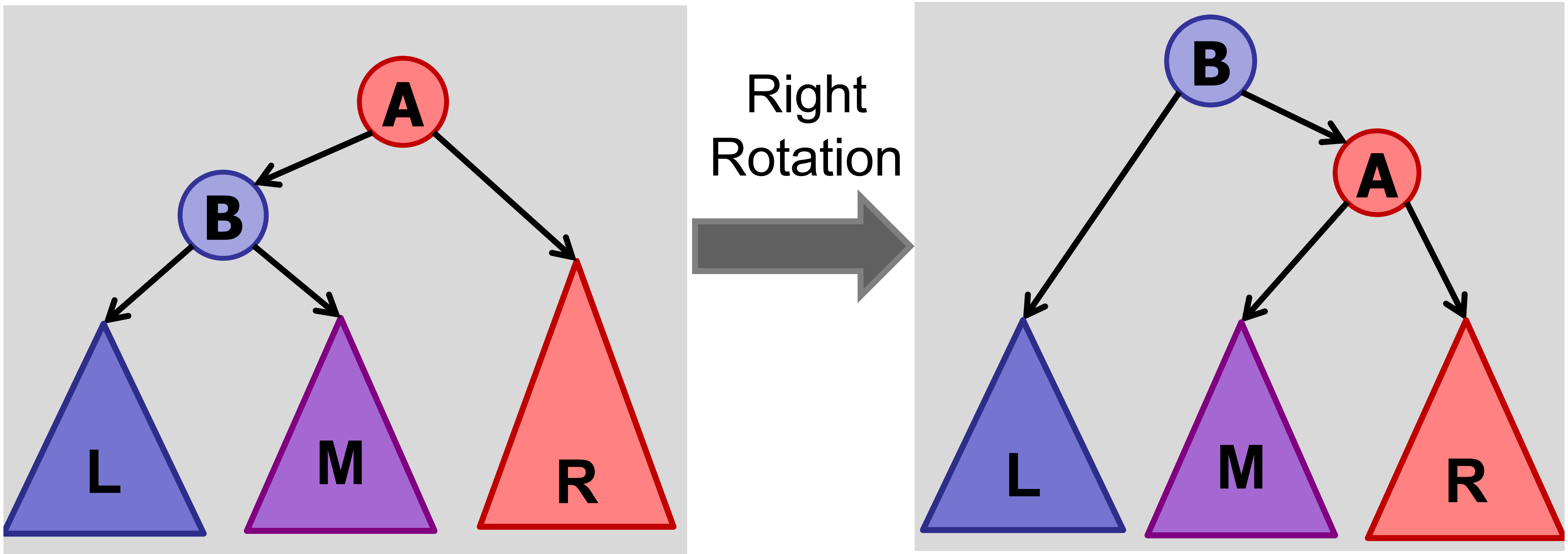
Rotations maintain ordering of keys.

⇒ Maintains BST property.

Tree Rotations



Tree Rotations

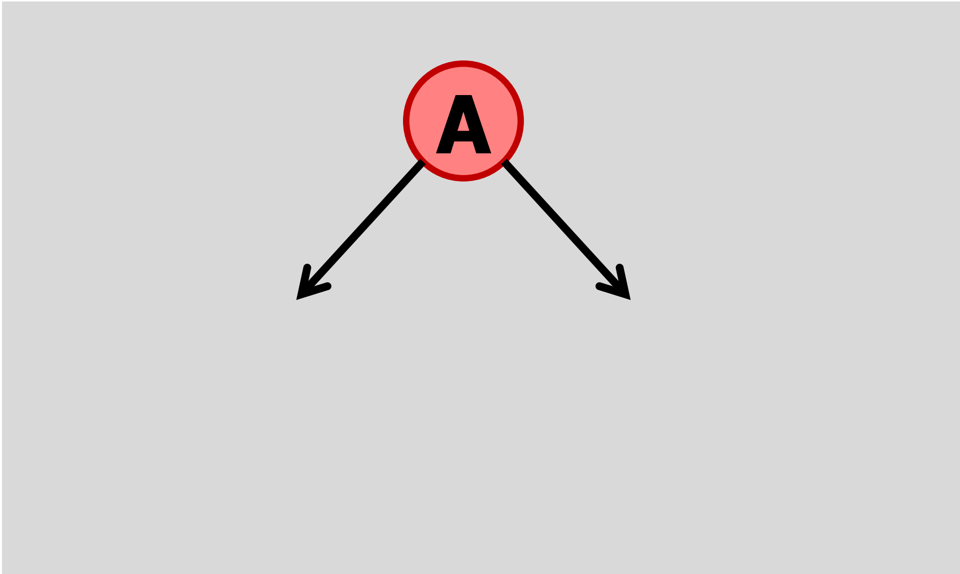


After insert:

Use tree rotations to restore balance.

Height is out-of-balance by 1

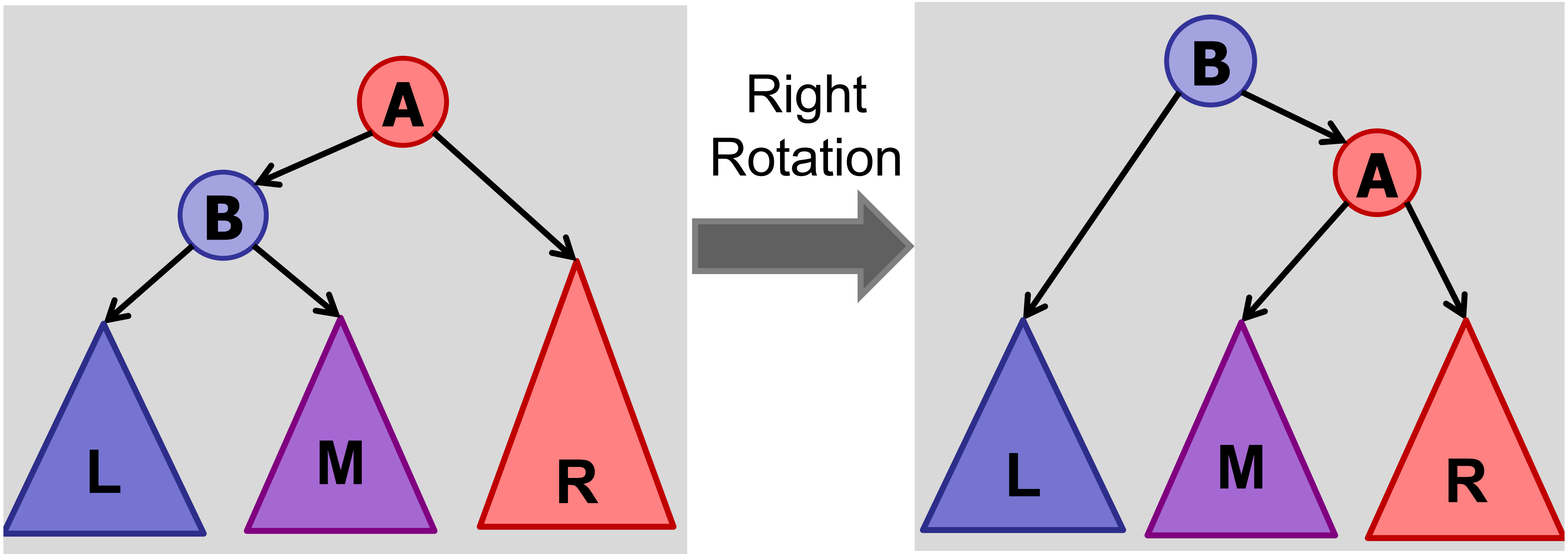
Tree Rotations



A is **LEFT-heavy** if left sub-tree has larger height than right sub-tree.

A is **RIGHT-heavy** if right sub-tree has larger height than left sub-tree.

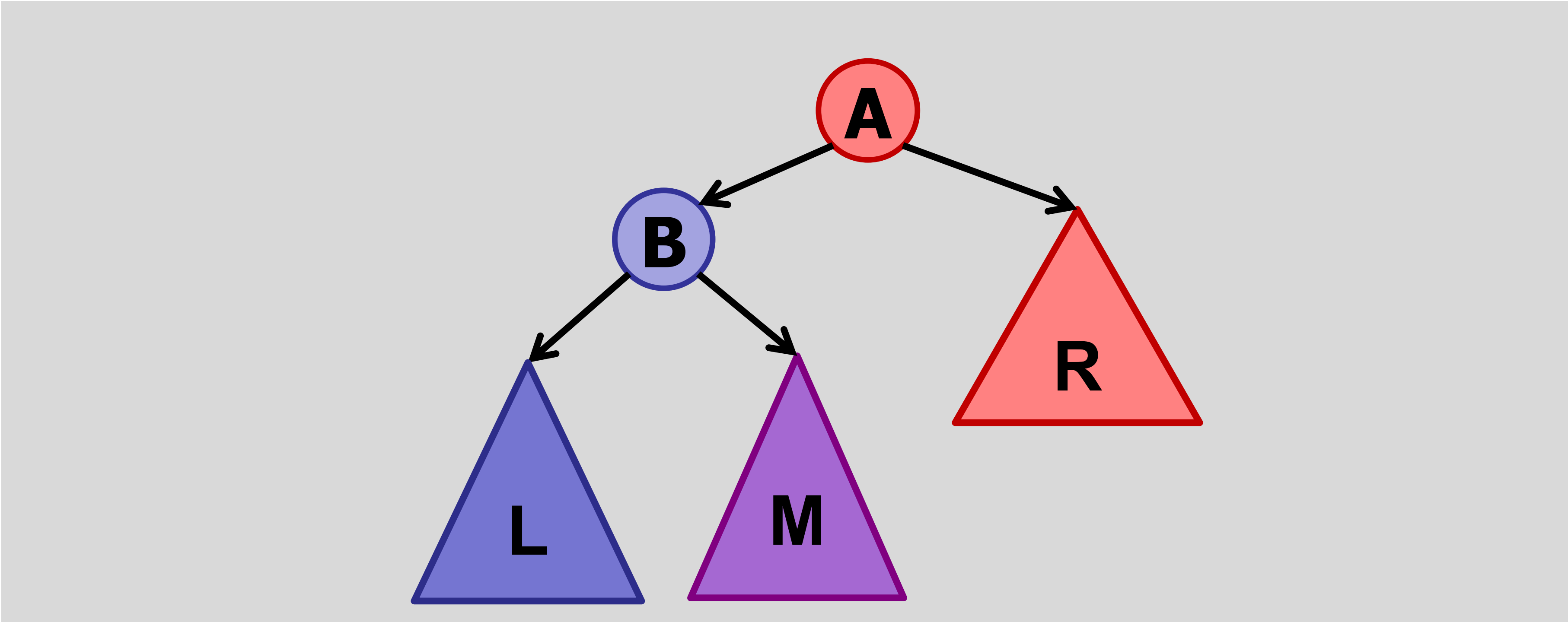
Tree Rotations



Use tree rotations to restore balance.

After insert, start at bottom, work your way up.

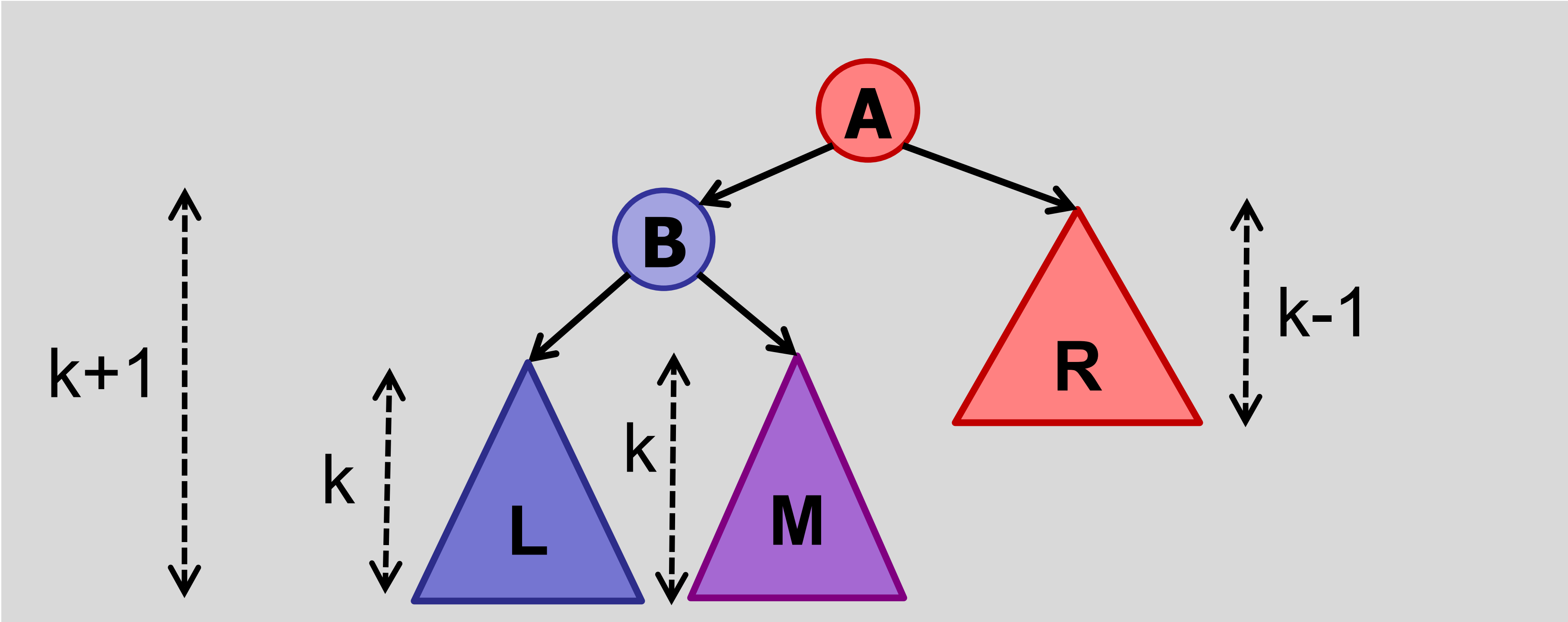
Tree Rotations



Assume **A** is the lowest node in the tree violating balance property.

Assume A is **LEFT-heavy**.

Tree Rotations (Left Heavy)

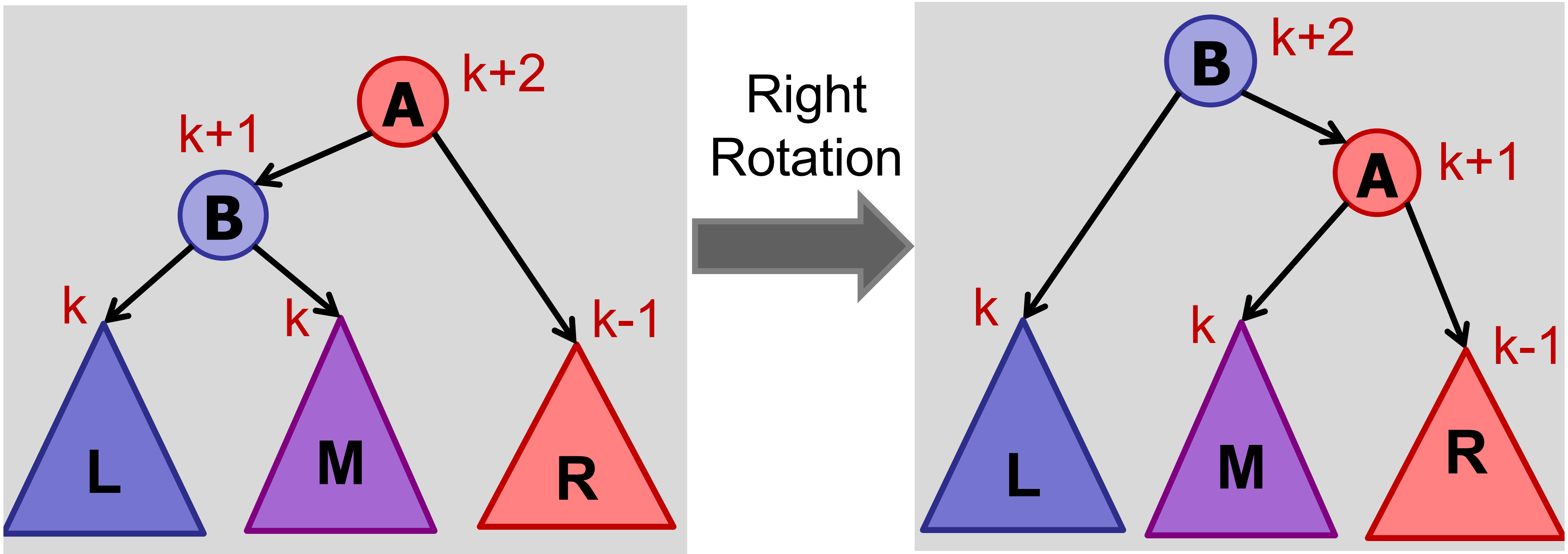


Assume **A** is the lowest node in the tree violating balance property.

Case 1: **B** is equi-height : h(**L**) = h(**M**)

h(**R**) = h(**M**) – 1

Tree Rotations

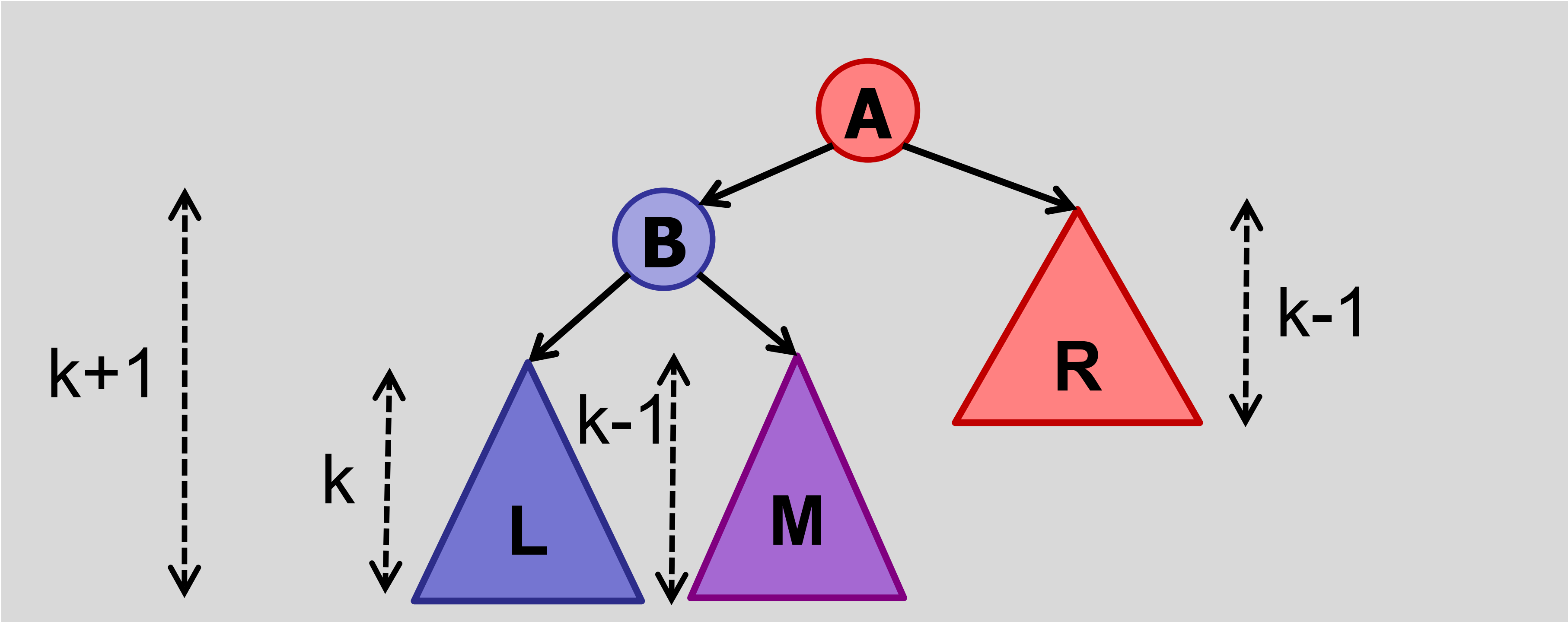


right-rotate:

Case 1: **B** is equi-height : h(**L**) = h(**M**)

h(**R**) = h(**M**) – 1

Tree Rotations (Left Heavy)

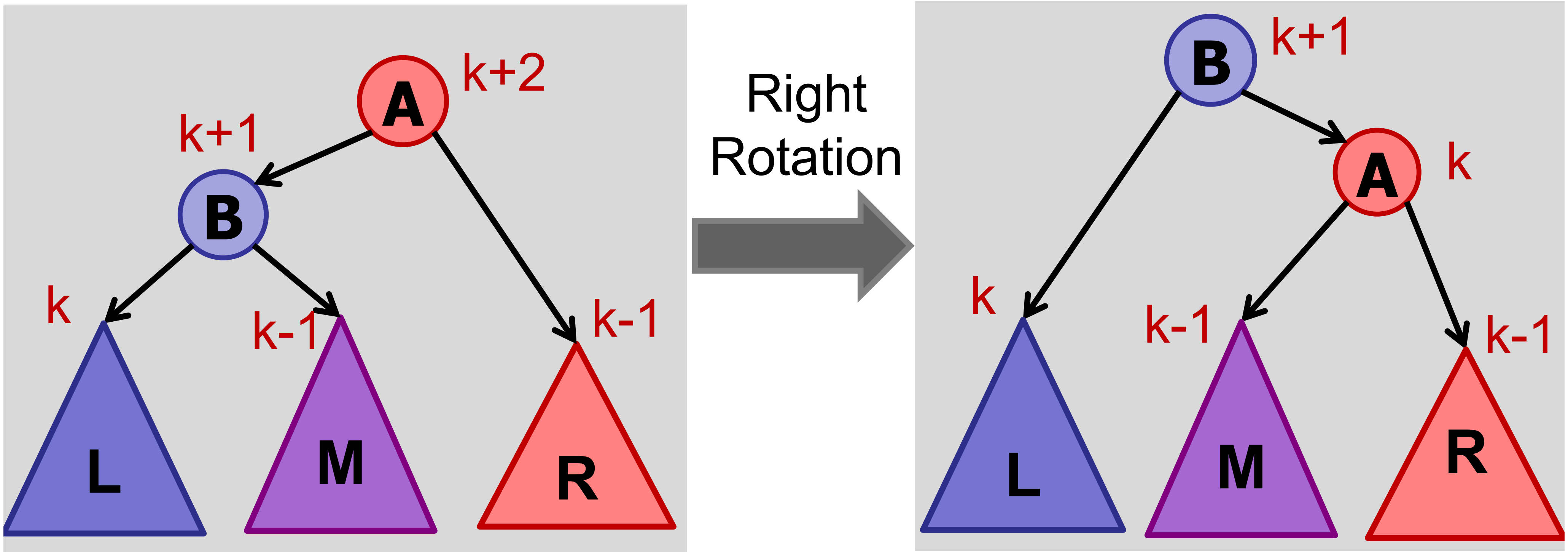


Assume **A** is the lowest node in the tree violating balance property.

Case 2: **B** is left-heavy : h(**L**) = h(**M**) + 1

h(**R**) = h(**M**)

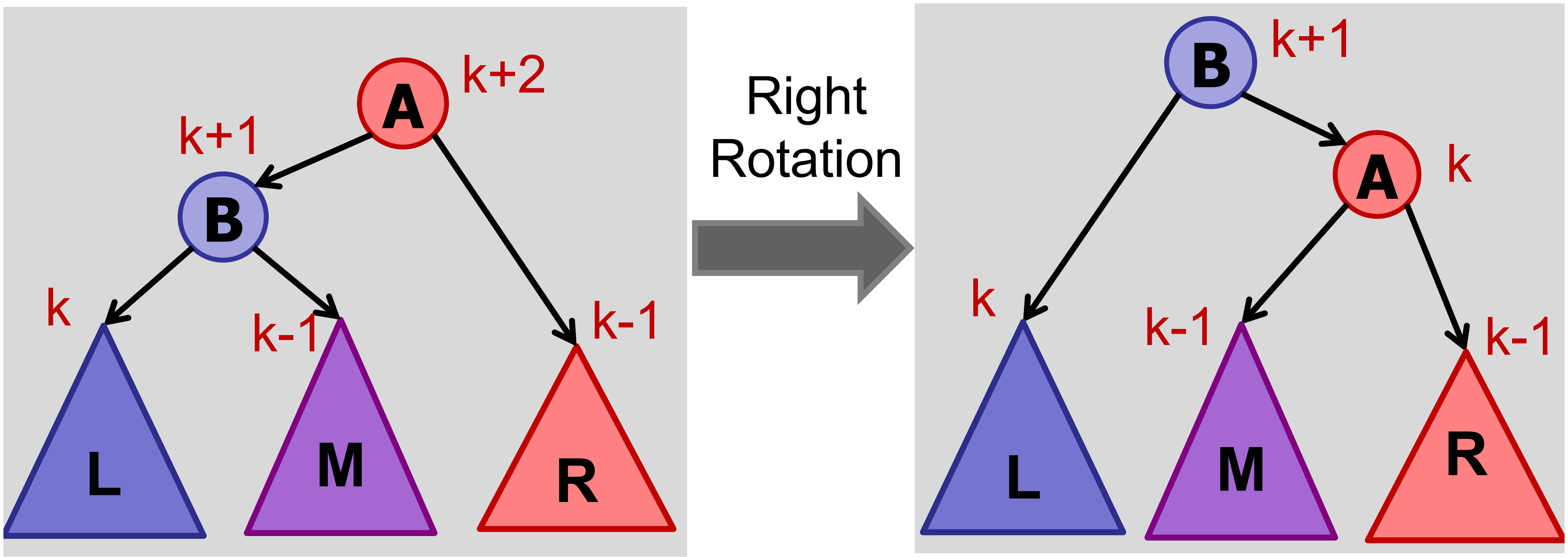
Tree Rotations



right-rotate:

Case 2: **B** is left-heavy: h(**L**) = h(**M**) +1

h(**R**) = h(**M**)

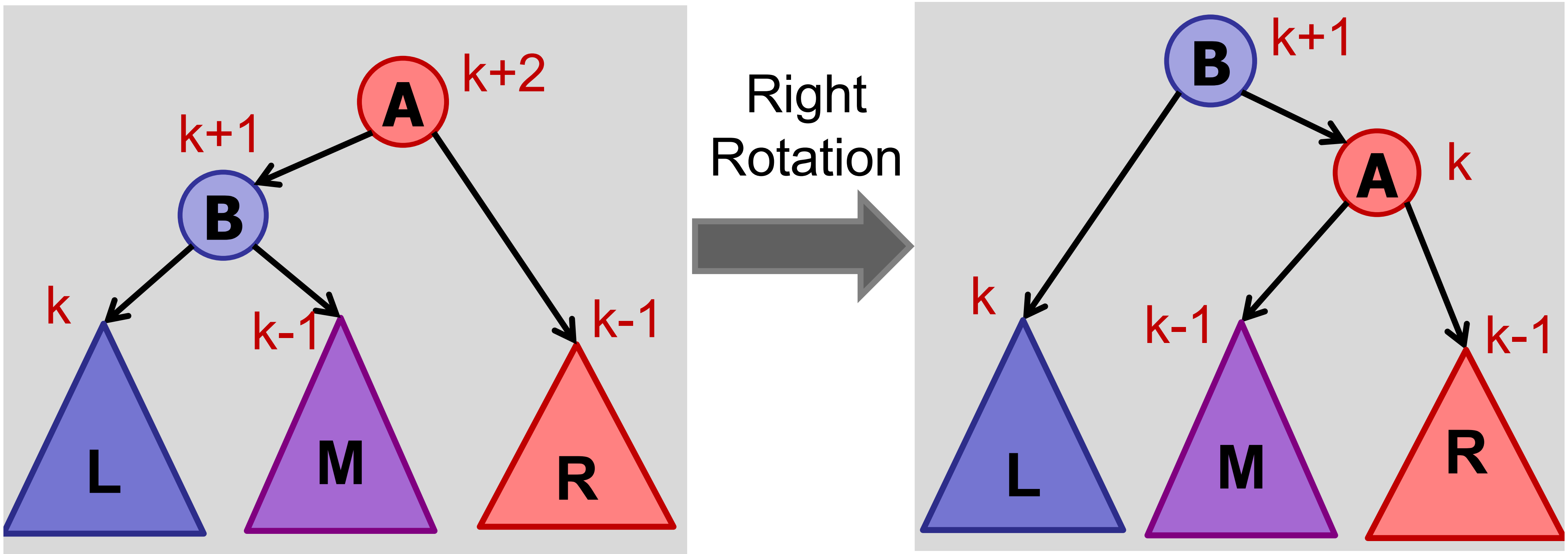


Is it balanced?

1. Yes.
2. No.
3. Maybe.

Yes or No on Zoom.

Tree Rotations

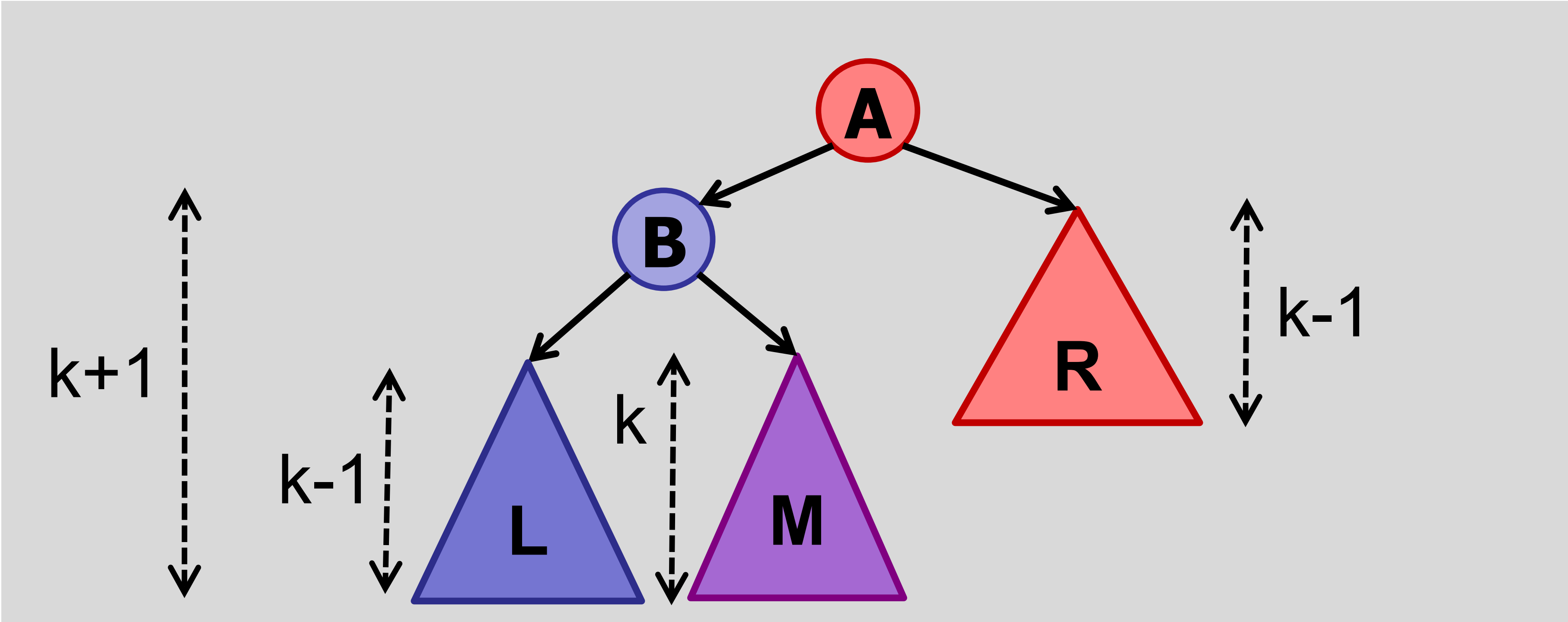


right-rotate:

Case 2: **B** is left-heavy: h(**L**) = h(**M**) +1

h(**R**) = h(**M**)

Tree Rotations (Left Heavy)

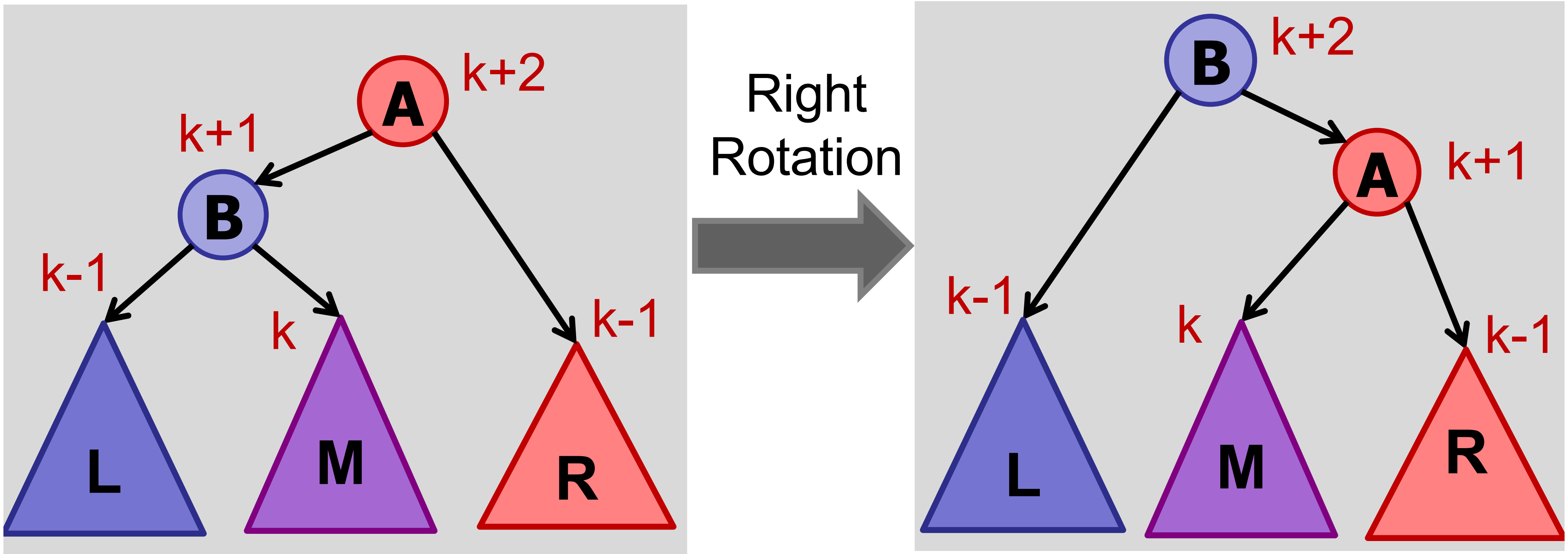


Assume **A** is the lowest node in the tree violating balance property.

Case 3: **B** is right-heavy : h(**L**) = h(**M**) - 1

h(**R**) = h(**L**)

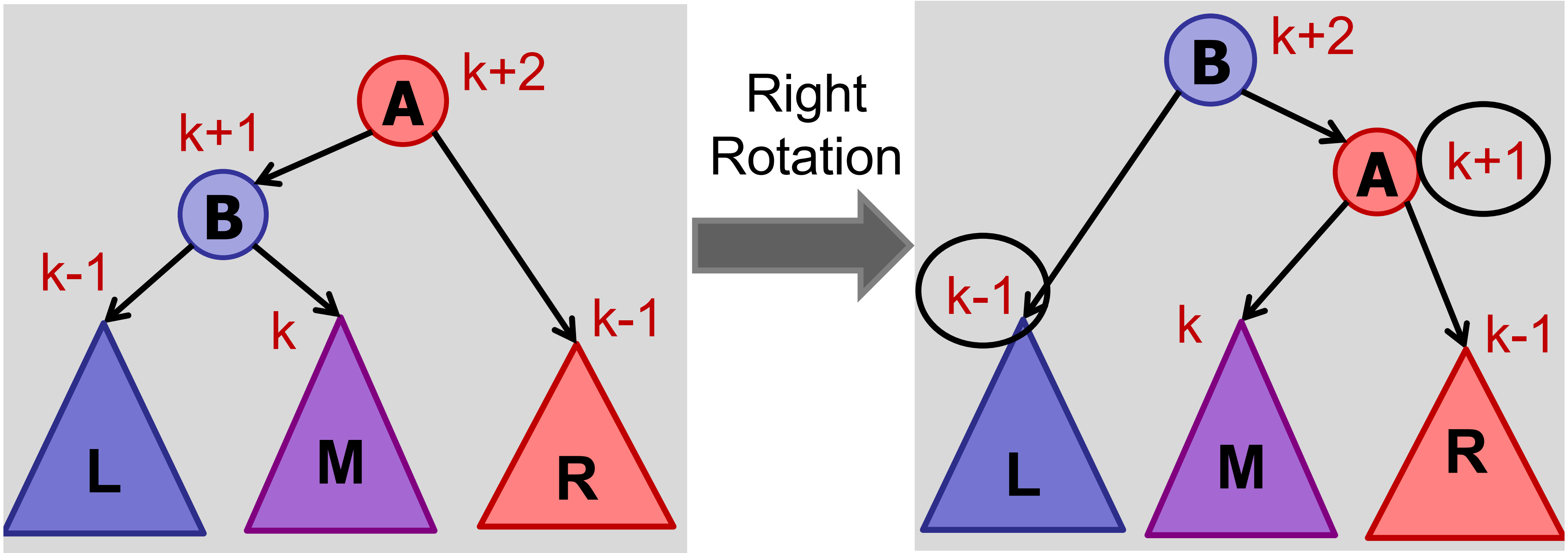
Tree Rotations



right-rotate:

Case 3: **B** is right-heavy: h(**L**) = h(**M**) – 1

h(**R**) = h(**L**)

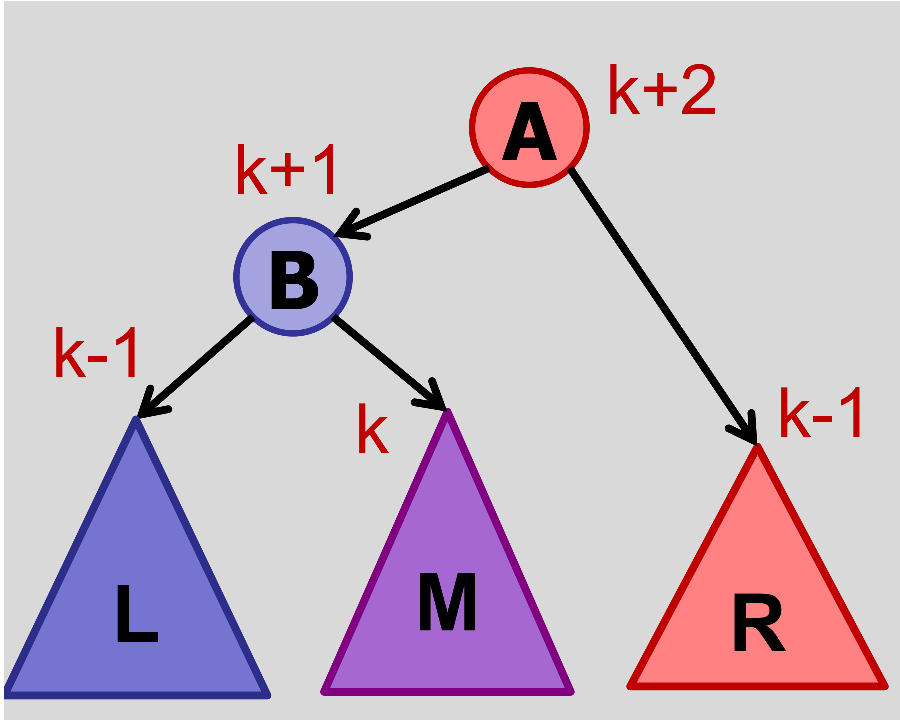


Is it balanced?

1. Yes.
2. No.
3. Maybe.

Yes or No on Zoom.

Tree Rotations



Let's do something first before we right-rotate(A)

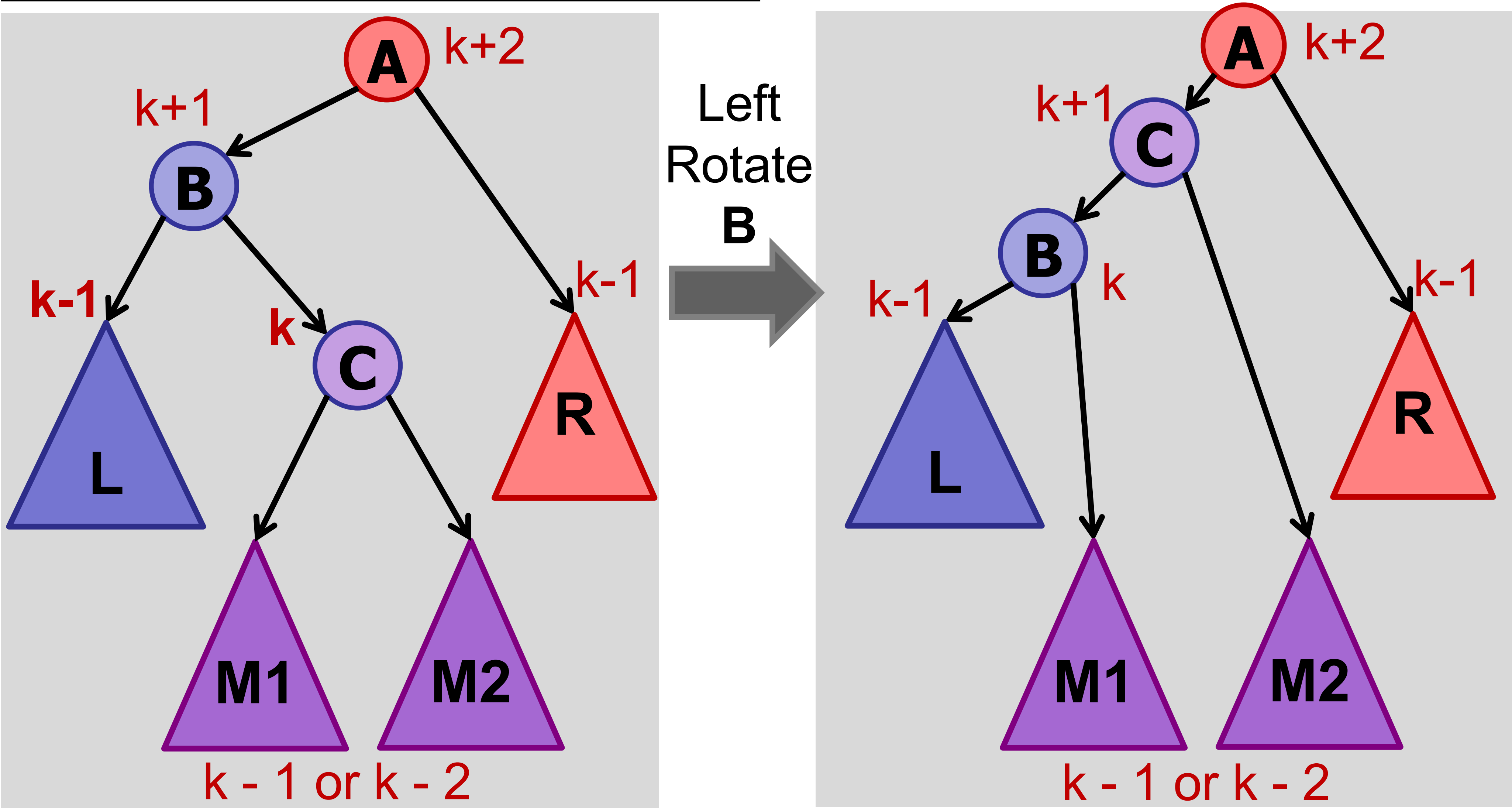
(Reduce it to a problem we have already solved!)

right-rotate:

Case 3: **B** is right-heavy: $h(\mathbf{L}) = h(\mathbf{M}) - 1$

$h(\mathbf{R}) = h(\mathbf{L})$

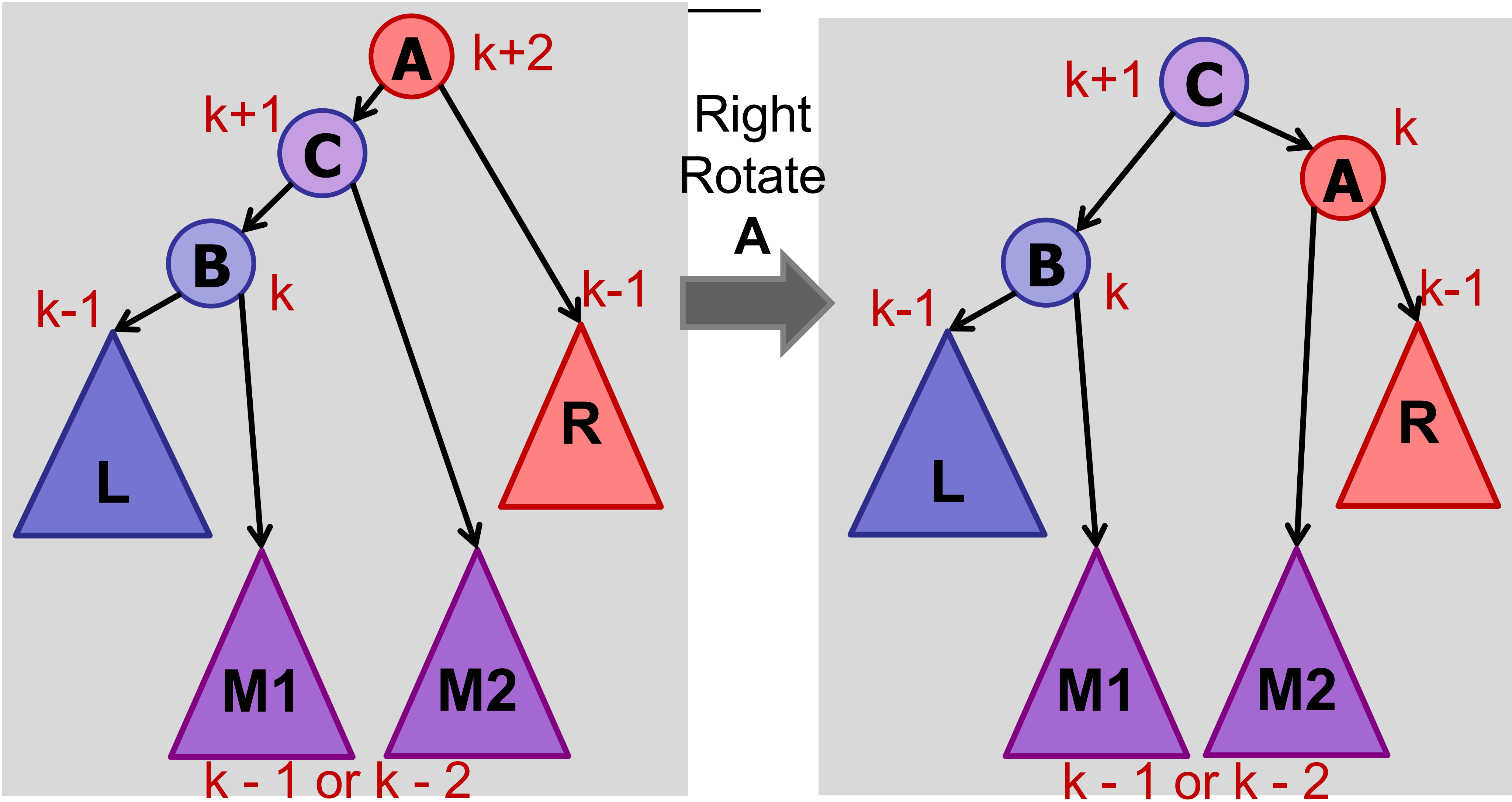
Tree Rotations

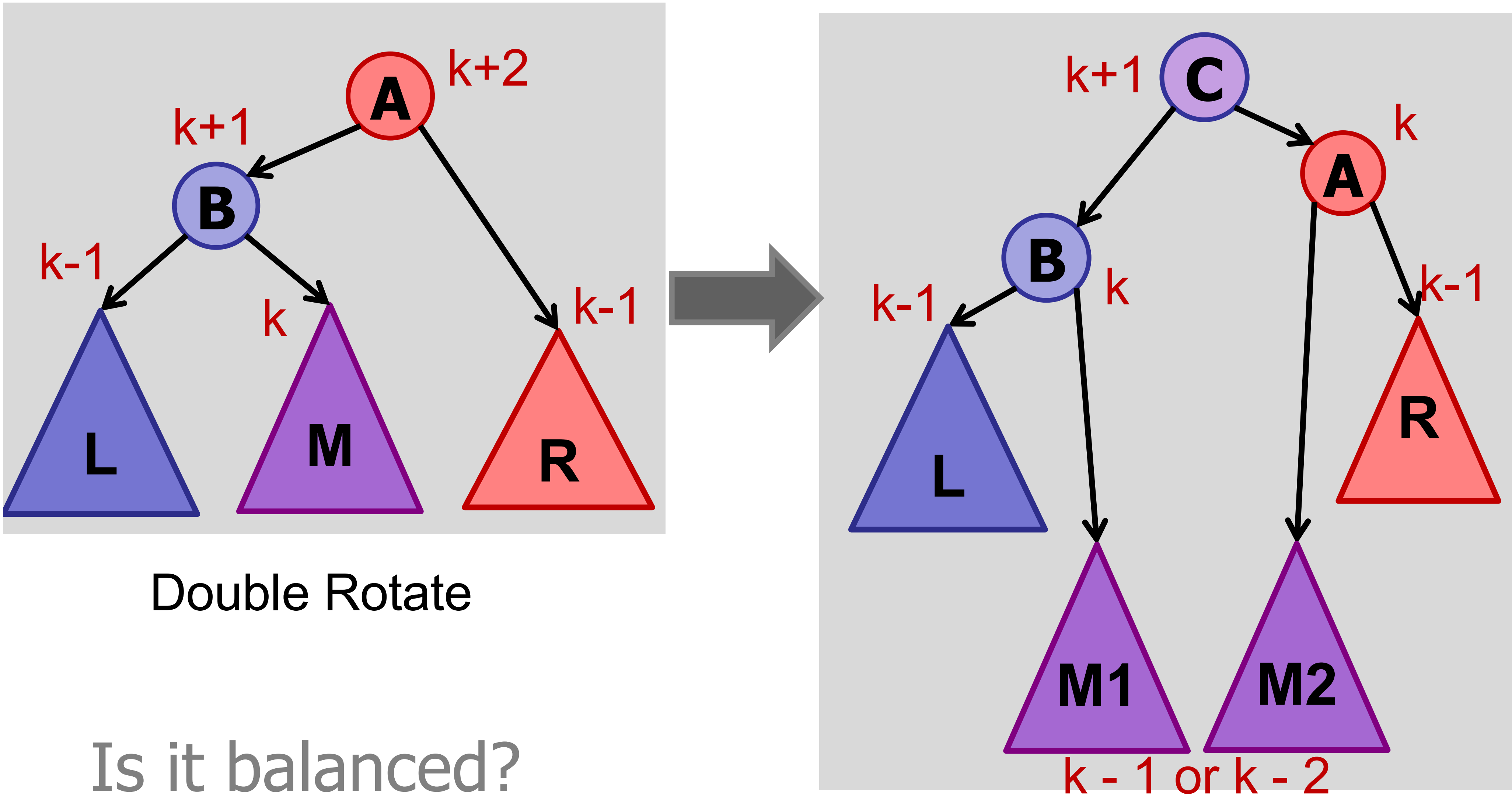


Left-rotate B

After left-rotate B: **A** and **C** still out of balance.

Tree Rotations

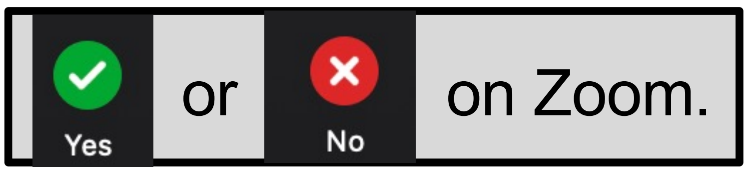




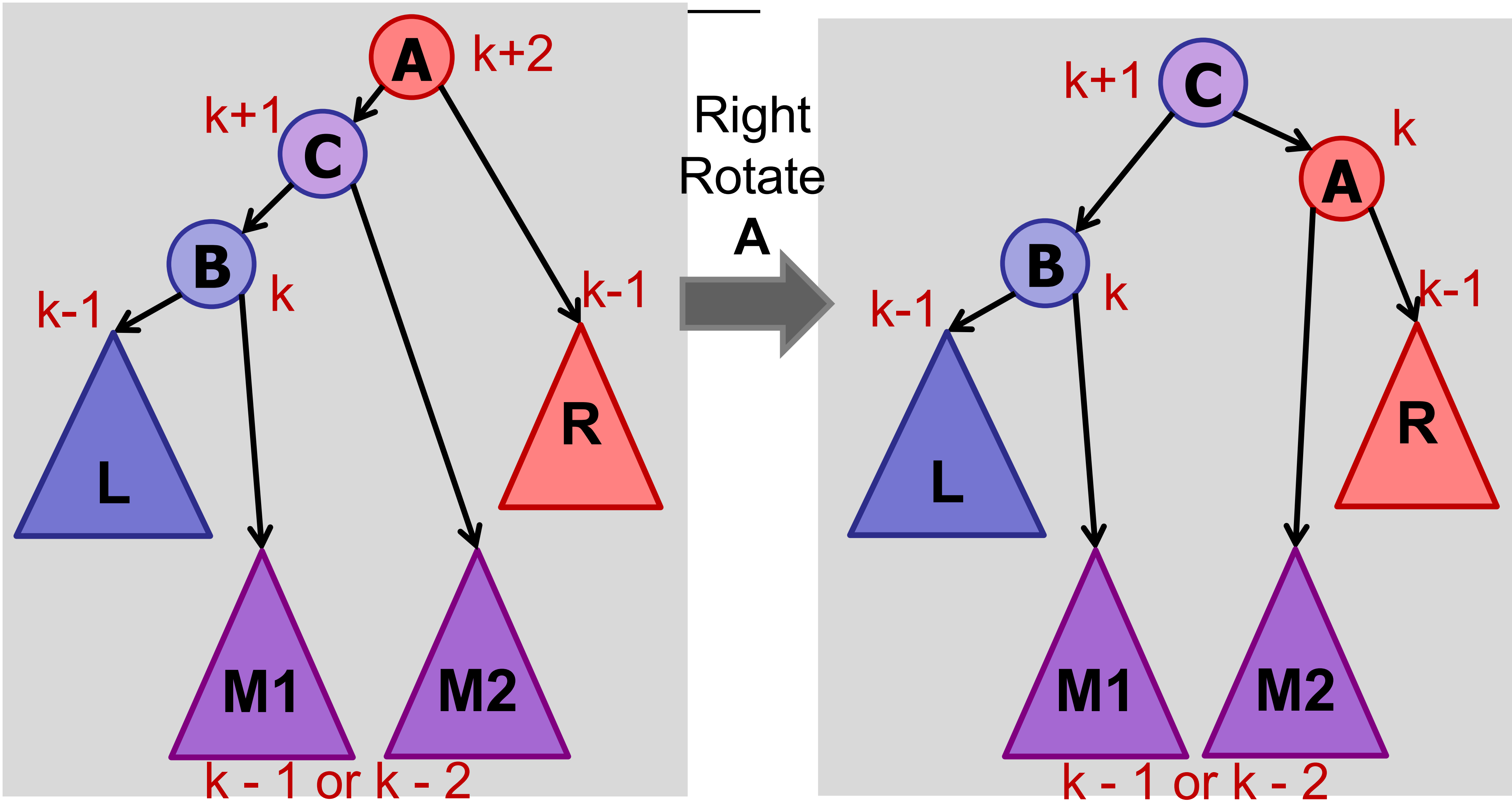
Double Rotate

Is it balanced?

1. Yes.
2. No.
3. Maybe.



Tree Rotations



After right-rotate A: all in balance.

Rotations

Summary:

If v is out of balance and left heavy:

1. v.left is balanced: right-rotate(v)
2. v.left is left-heavy: right-rotate(v)
3. v.left is right-heavy: left-rotate(v.left)
   right-rotate(v)

If v is out of balance and right heavy:

Symmetric three cases….

How many rotations do you need after an insertion (in the worst case)?

1. 1
2. 2
3. 4
4. log(n)
5. 2log(n)
6. n

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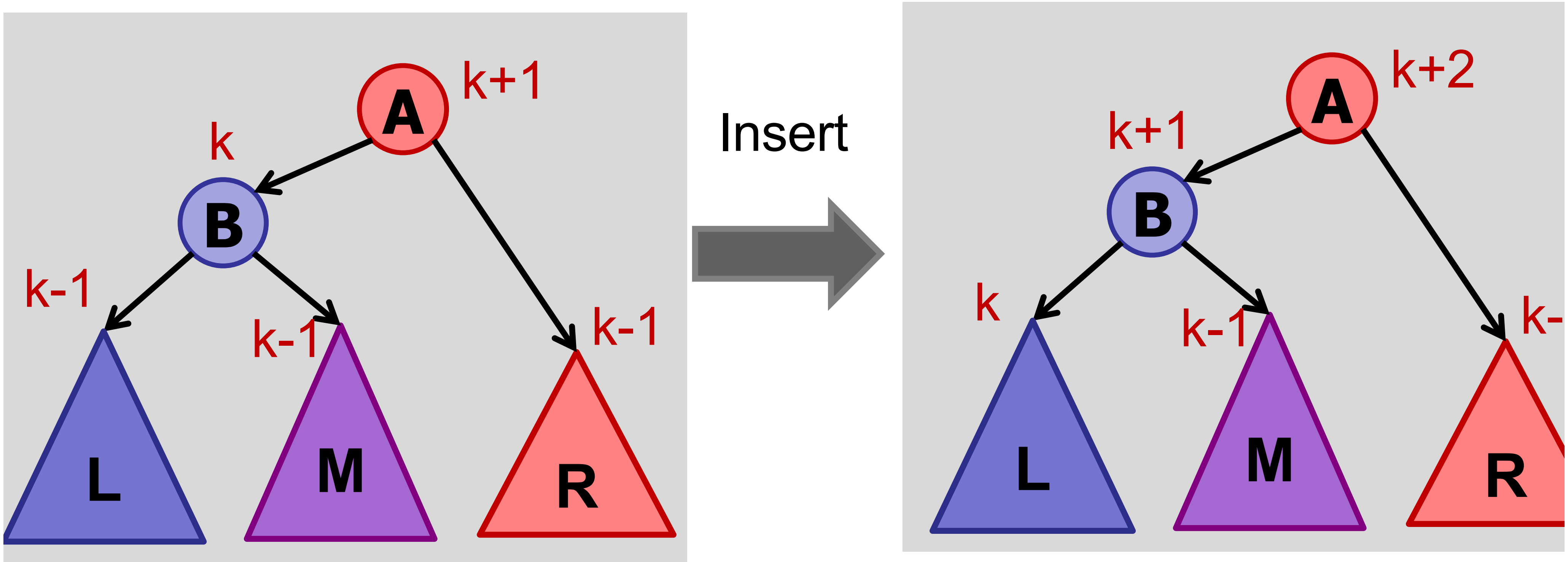
How many rotations do you need after an insertion (in the worst case)?

1. 1
2. 2
3. 4
4. log(n)
5. 2log(n)
6. n

Question:

Why isn't it 2log(n)?

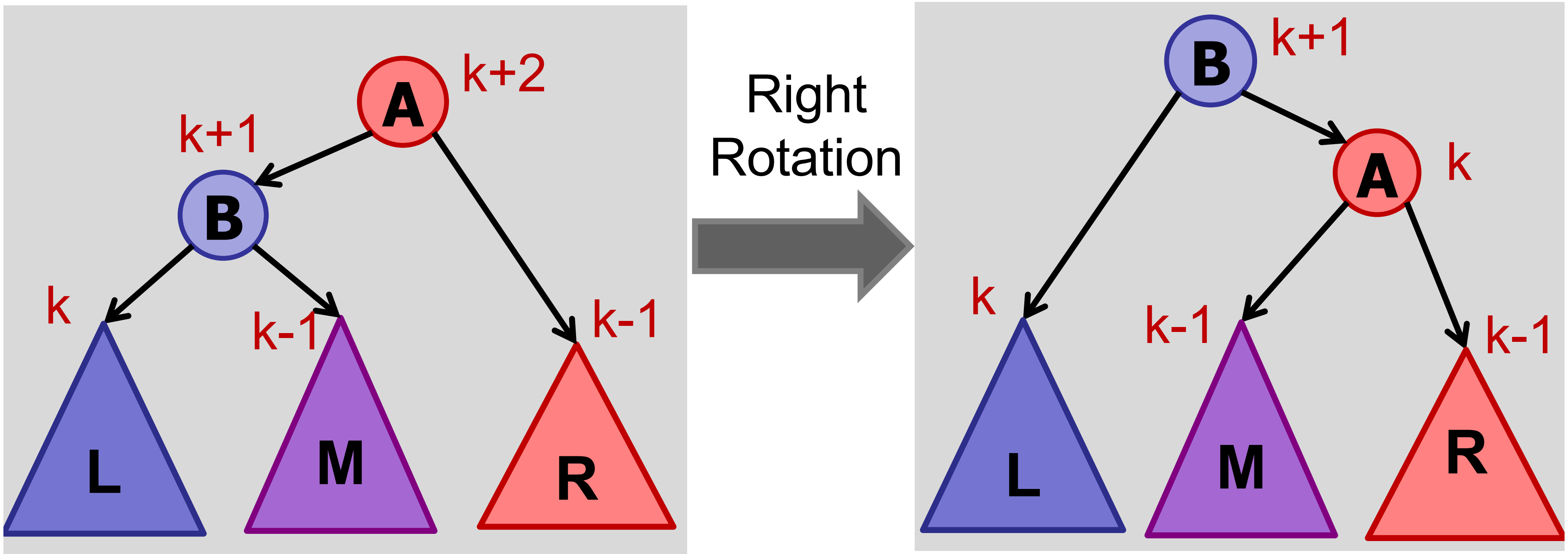
How many rotations?



Case 2: **B** is left-heavy

Insert increased heights by 1.

How many rotations?

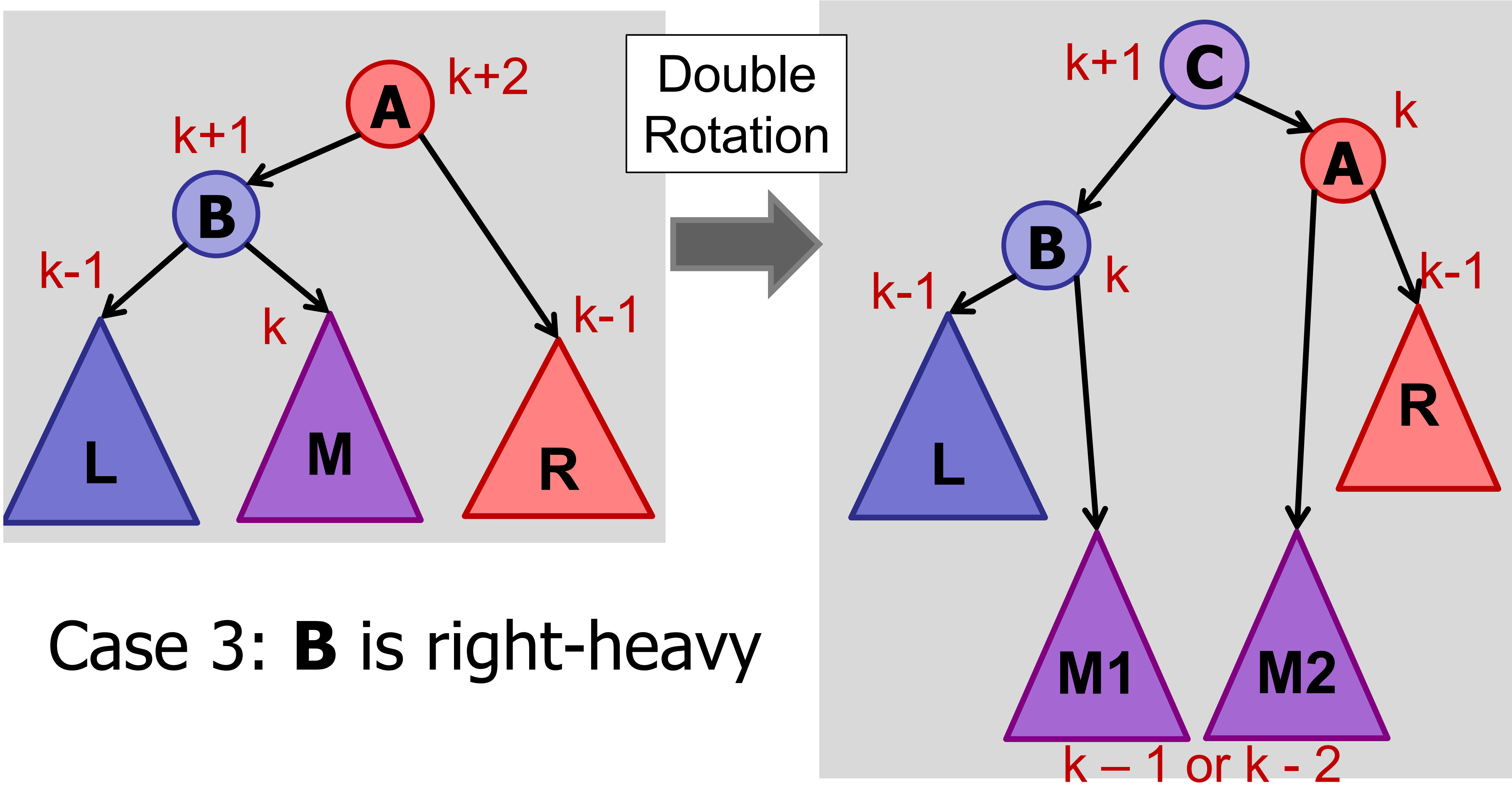


Case 2: **B** is left-heavy

Rotation reduces root height by 1.

(Everything higher in tree is unchanged!)

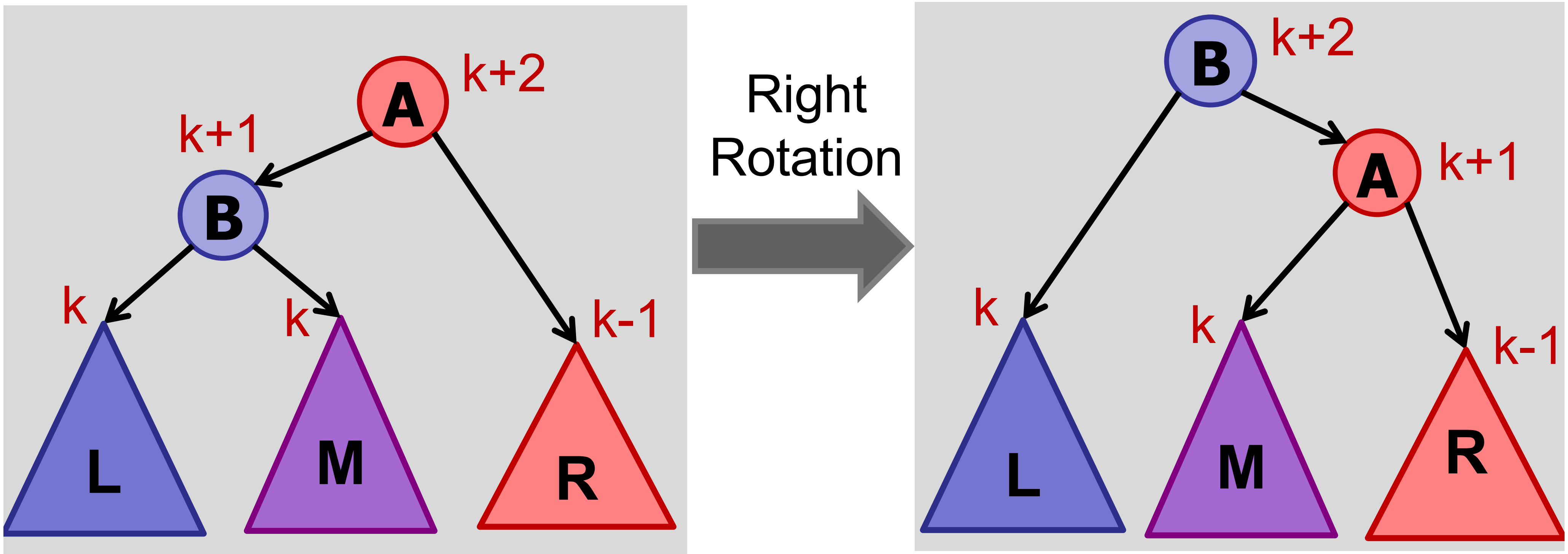
How many rotations?



Case 3: **B** is right-heavy

Rotation reduces root height by 1.

How many rotations?



Case 1: **B** is balanced

Rotation does *not* reduce height by 1.

Challenge: figure out why this is okay!

Insert in AVL Tree

Summary:

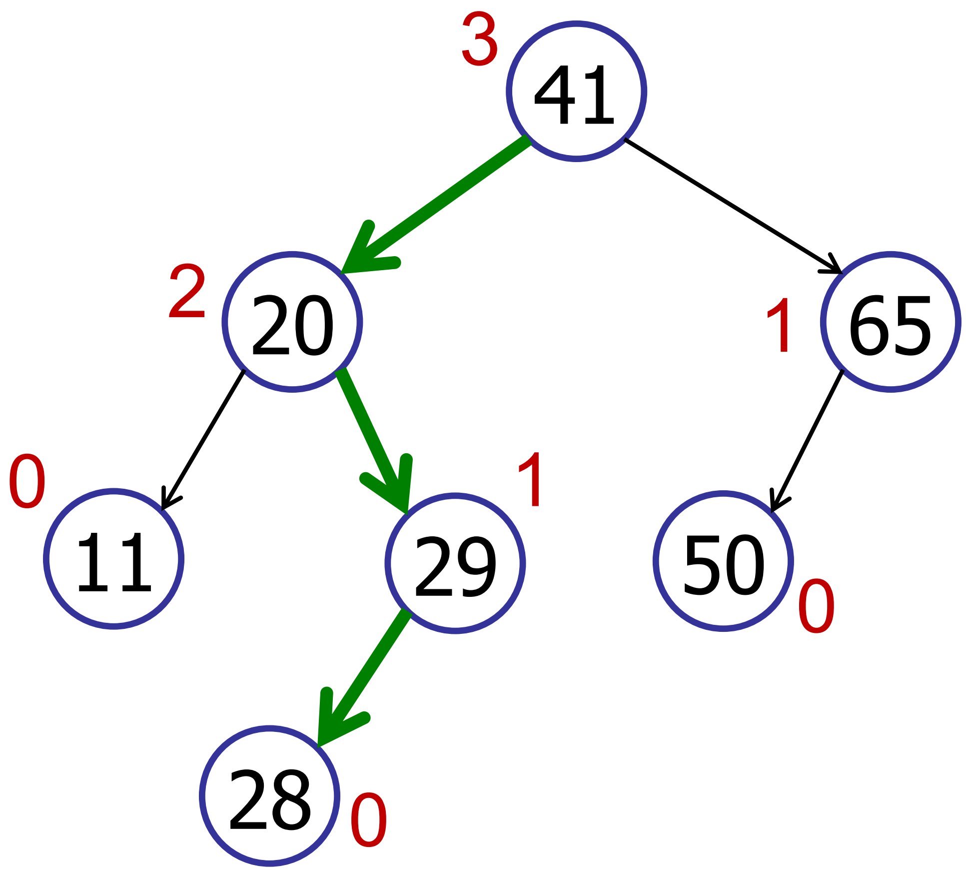
- Insert key in BST.
- Walk up tree:
  - At every step, check for balance.
  - If out-of-balance, use rotations to rebalance and return.

Key observation:

- Only need to fix *lowest* out-of-balance node.
- Only need at most two rotations to fix.

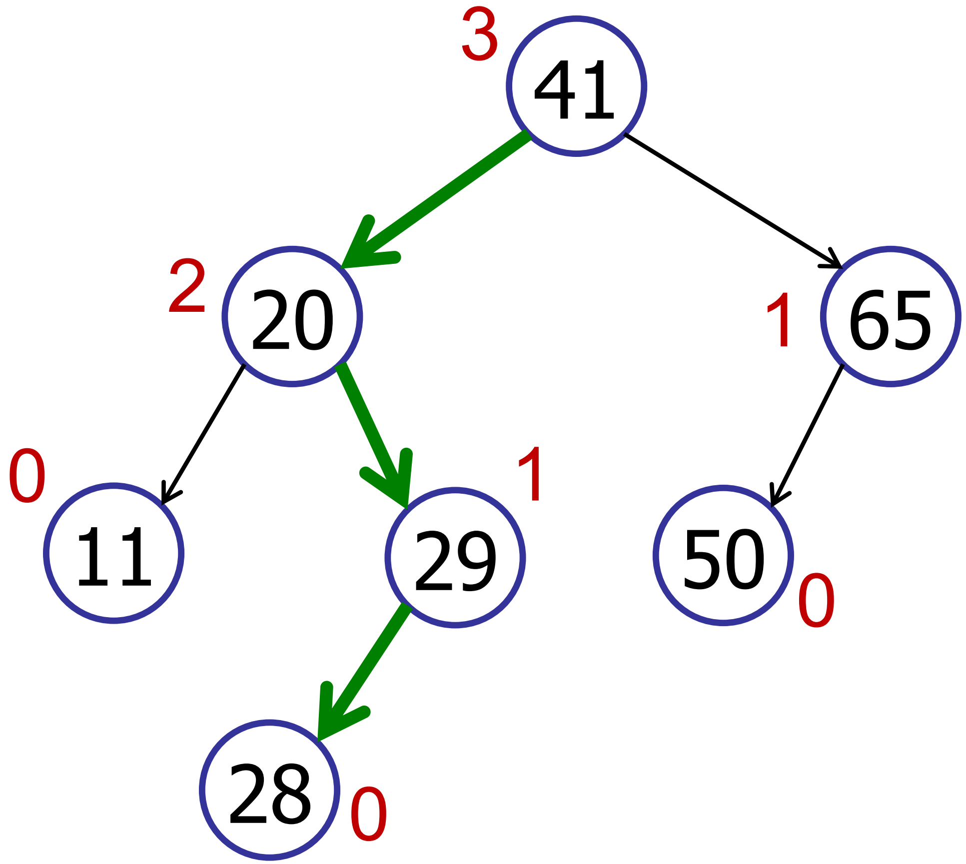
Example

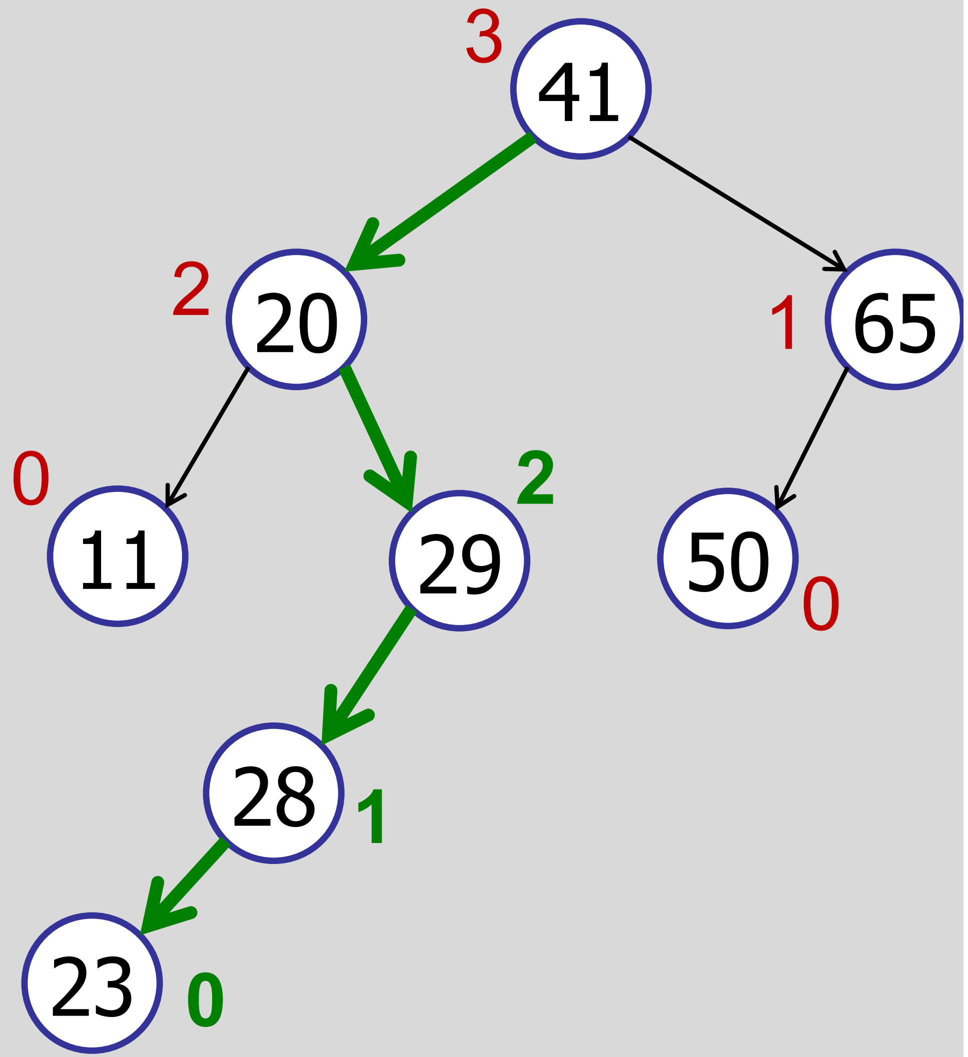
insert(23)



Example

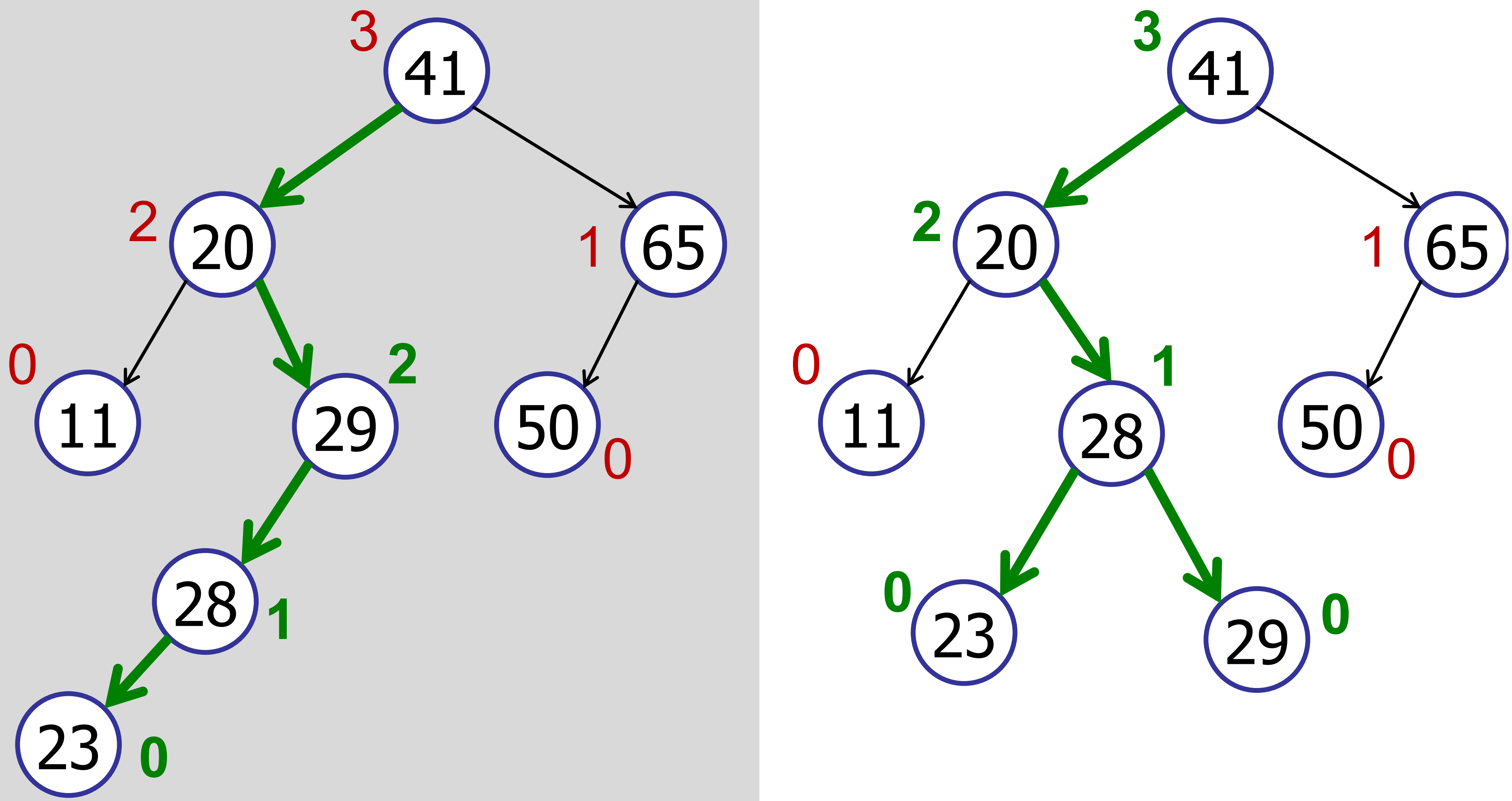
insert(23)





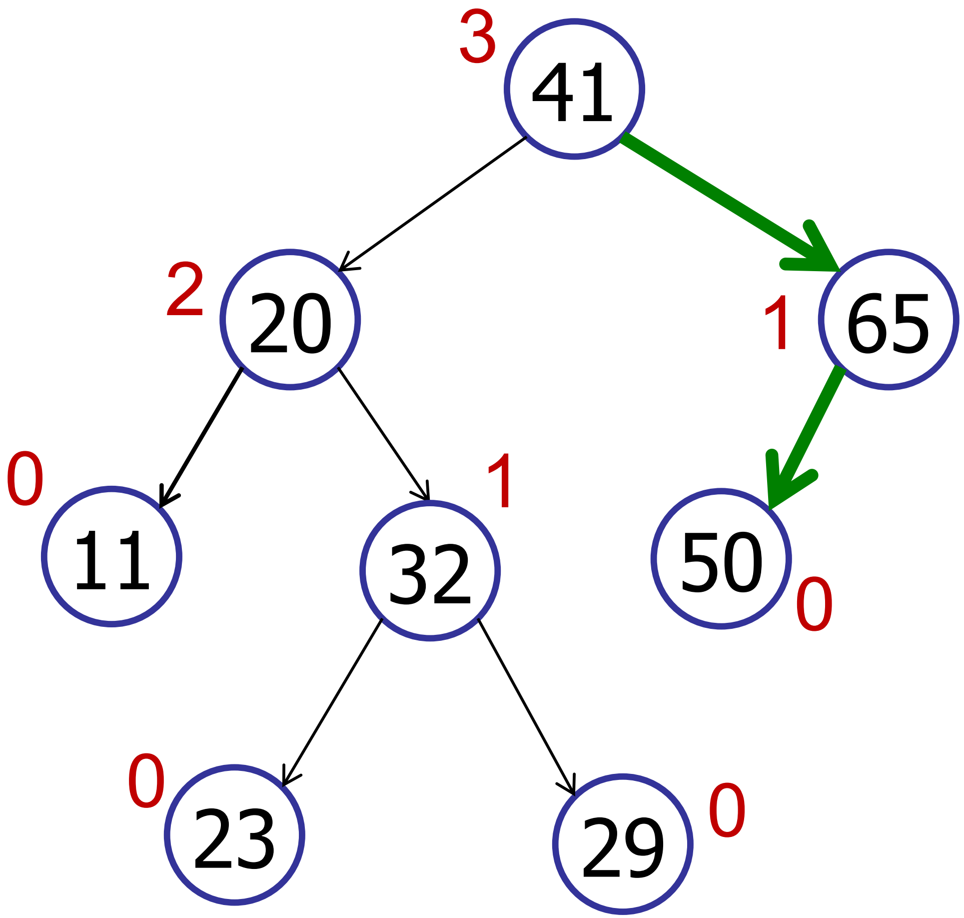
Example

right-rotate(29)



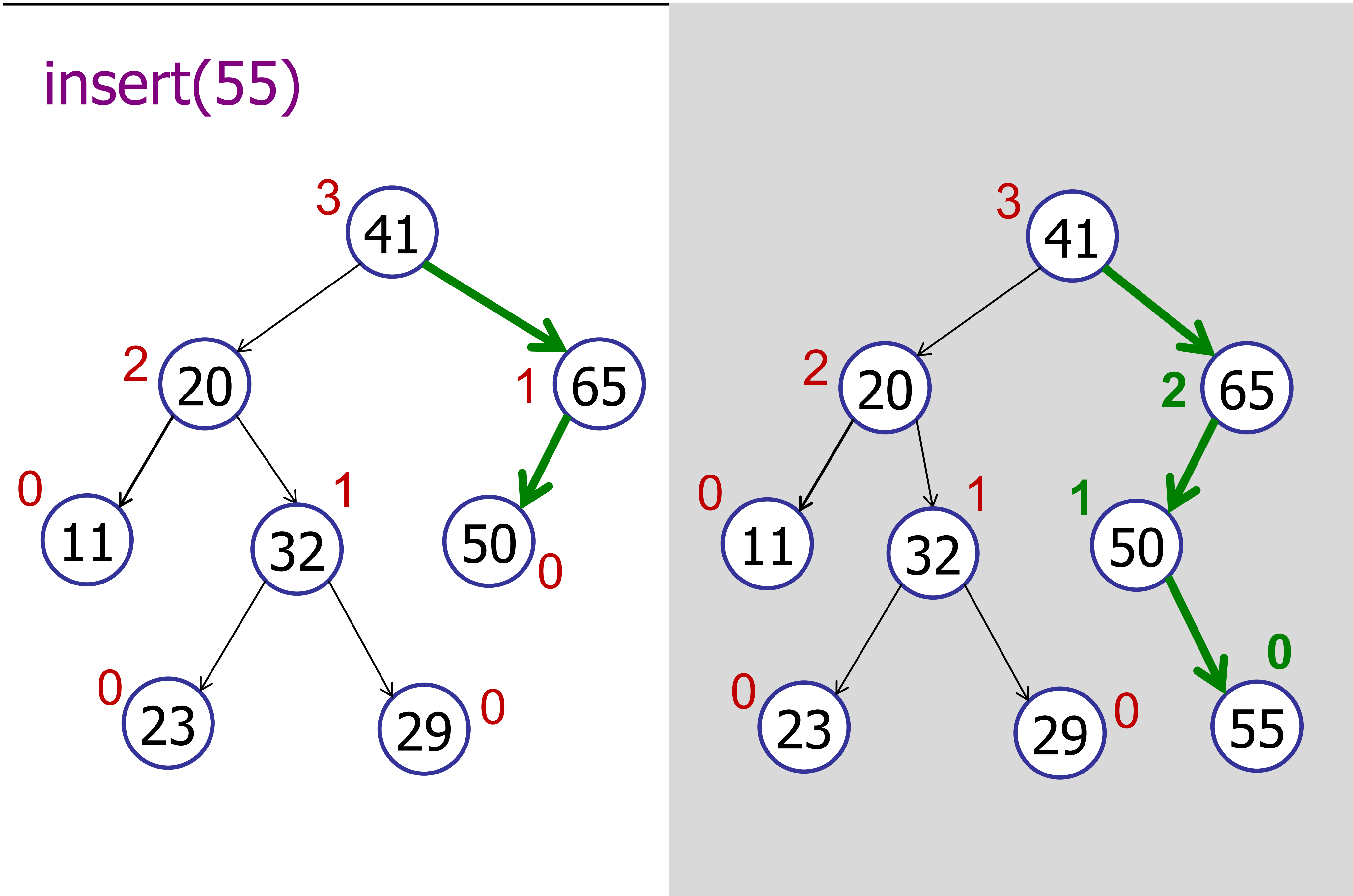
Example

insert(55)



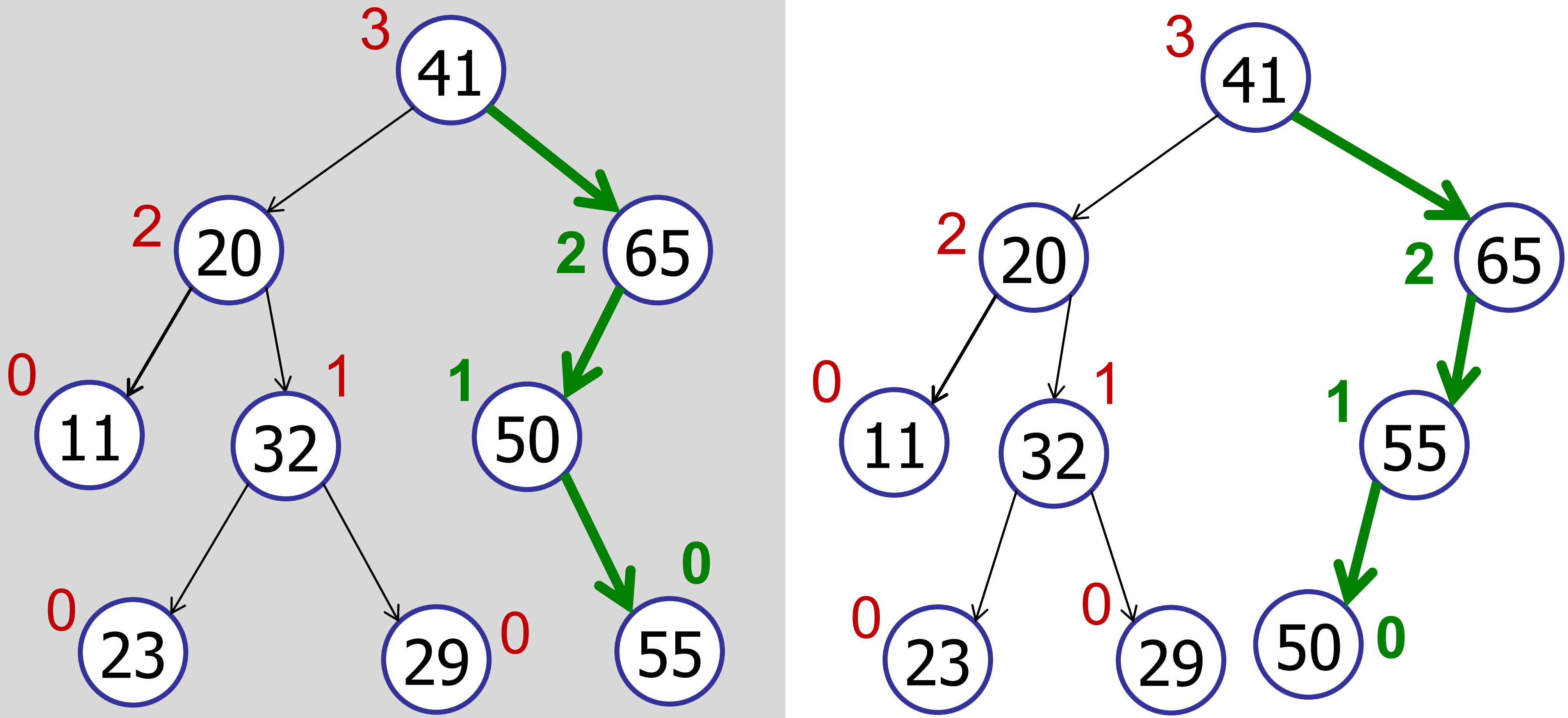
Example

insert(55)

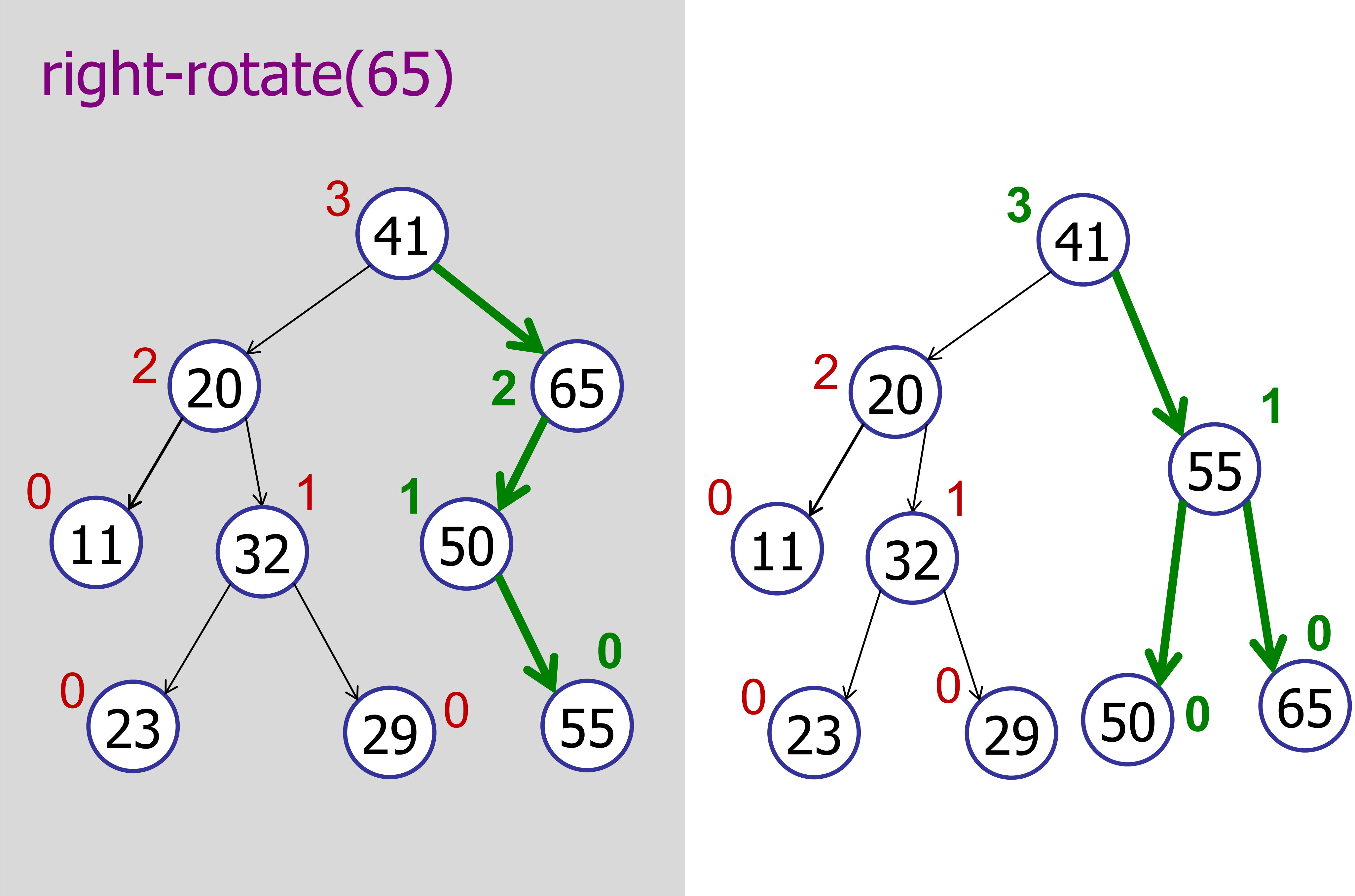


Example

left-rotate(50)



Example



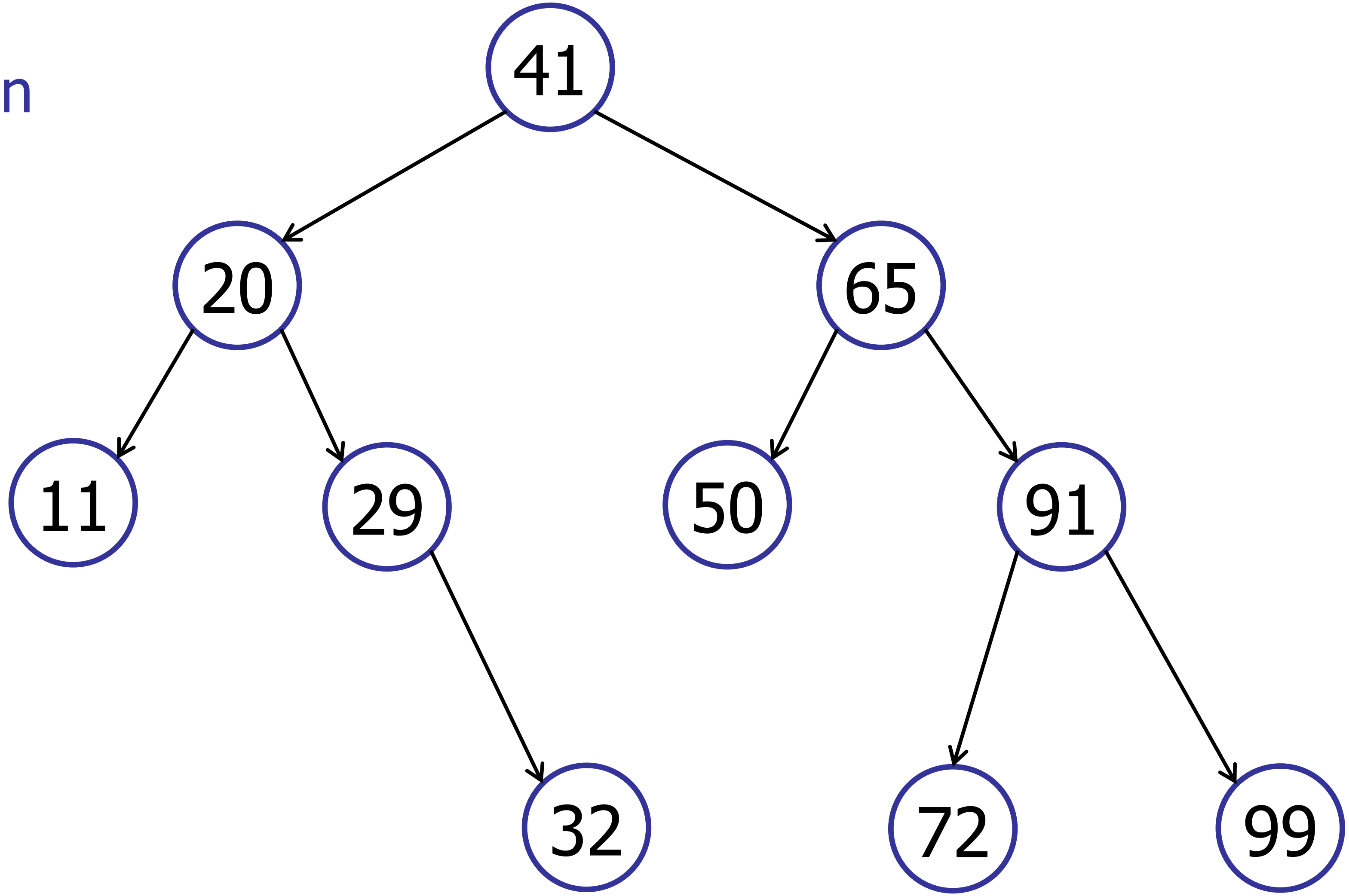
right-rotate(65)
3
41
2
20
2
65
0
11
1
32
1
50
0
23
0
29
0
55
3
41
2
20
1
55
0
11
1
32
0
23
0
29
50
0
0
65

Binary Search Tree

delete(v)

Three cases:

1. No children
2. 1 child
3. 2 children



Binary Search Tree

delete(v)

1. If v has two children, swap it with its successor.

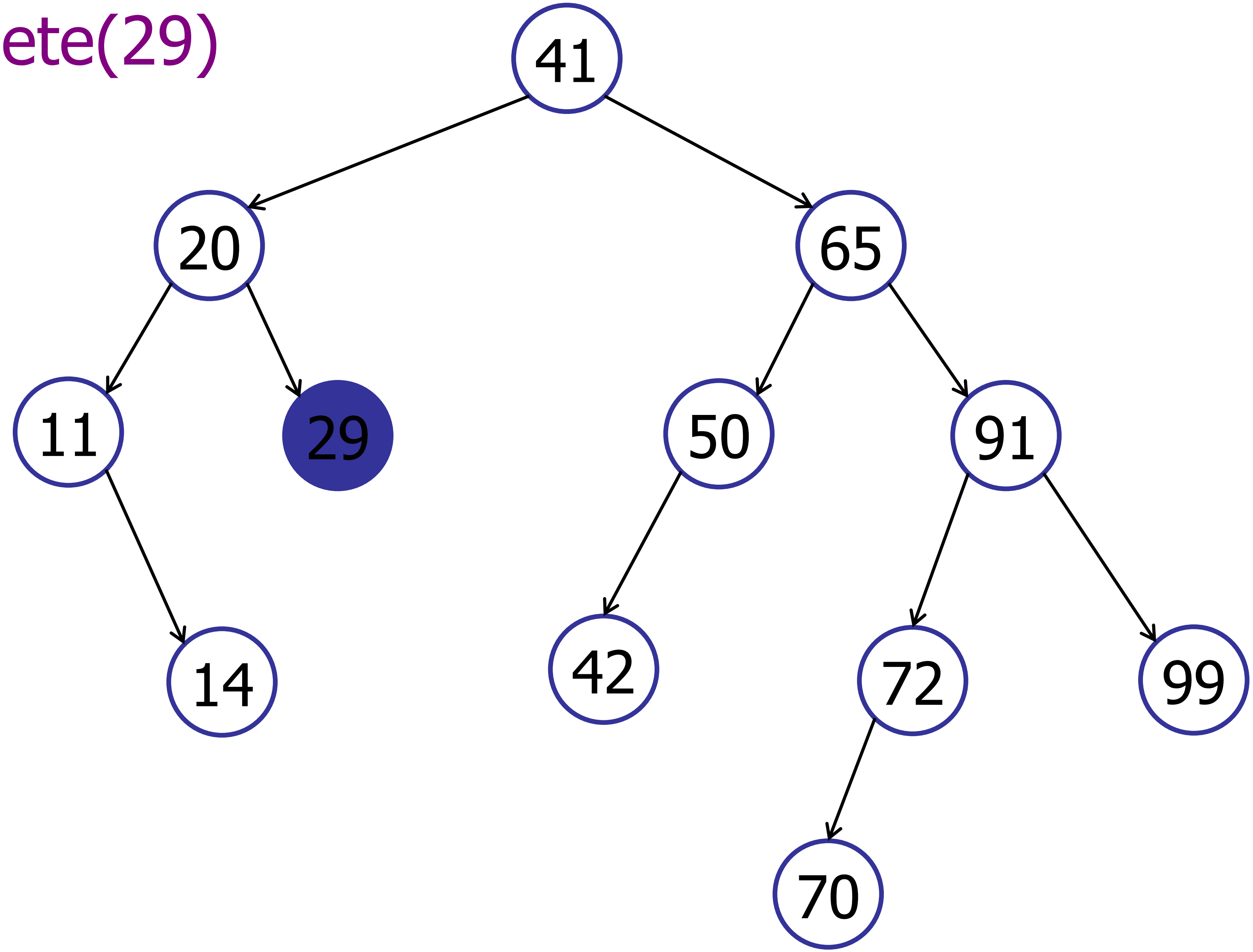
2. Delete node v from binary tree (and reconnect children).

3. For every ancestor of the deleted node:
   - Check if it is height-balanced.
   - If not, perform a rotation.
   - Continue to the root.

Deletion may take up to O(log(n)) rotations.

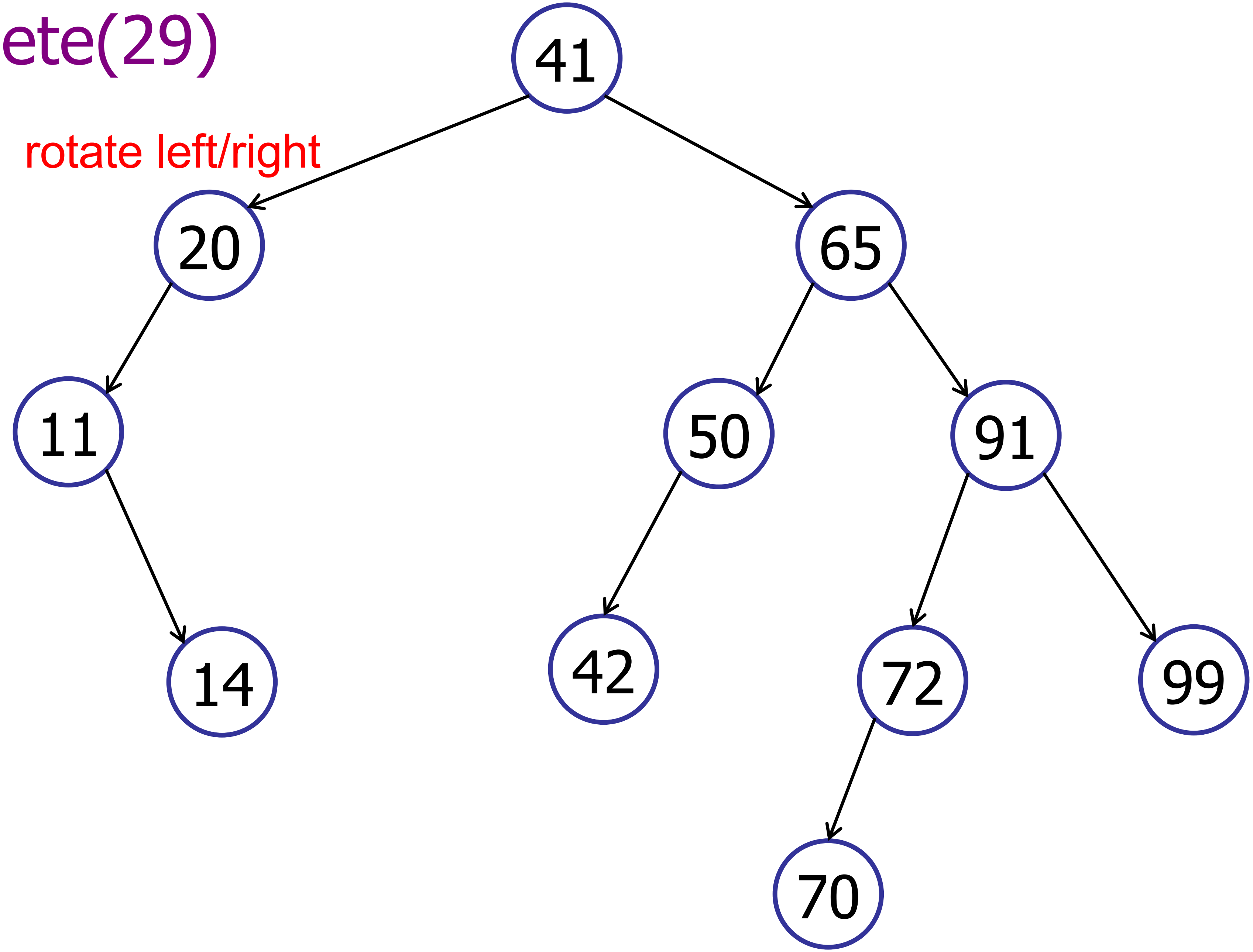
Binary Search Tree

delete(29)



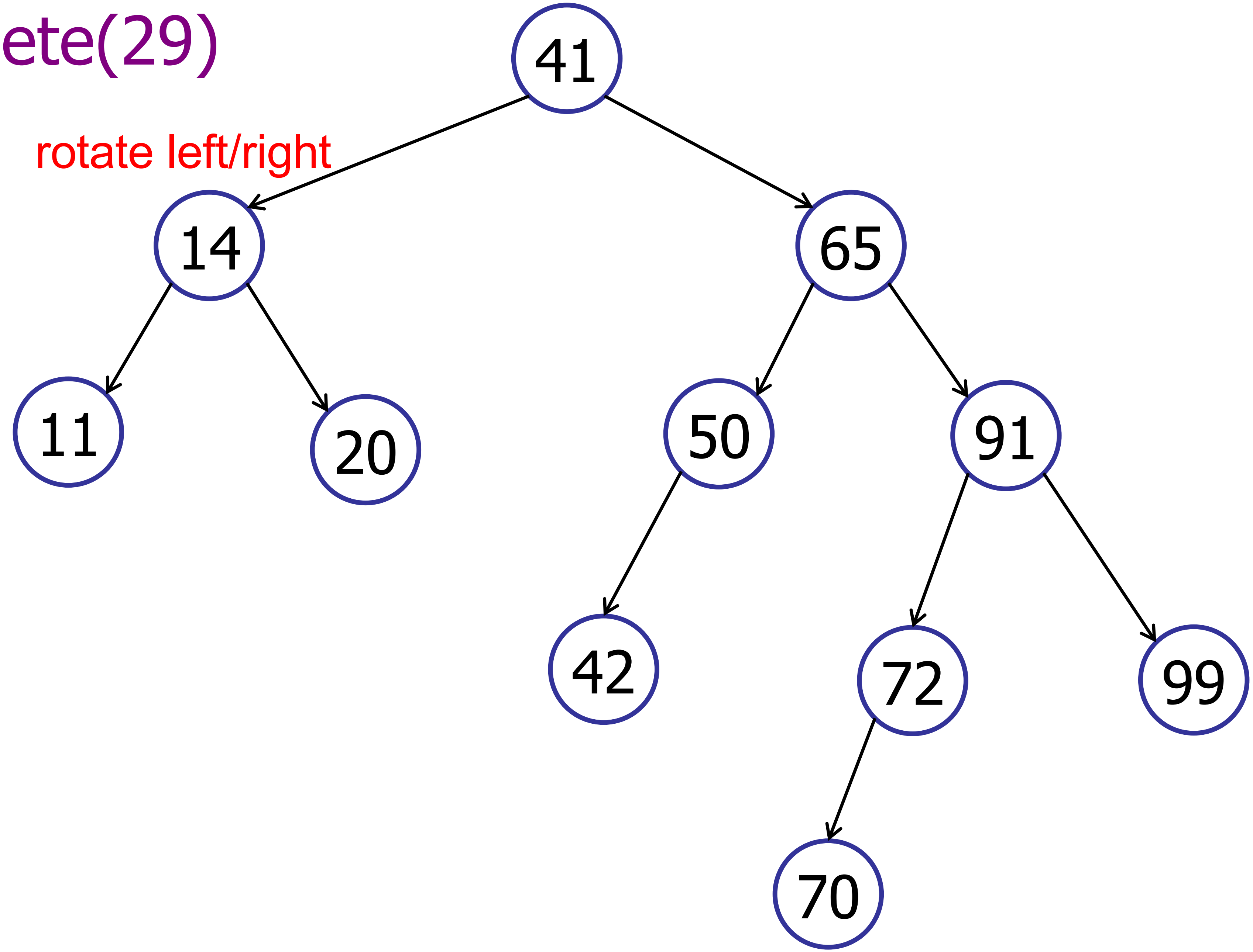
Binary Search Tree

delete(29)



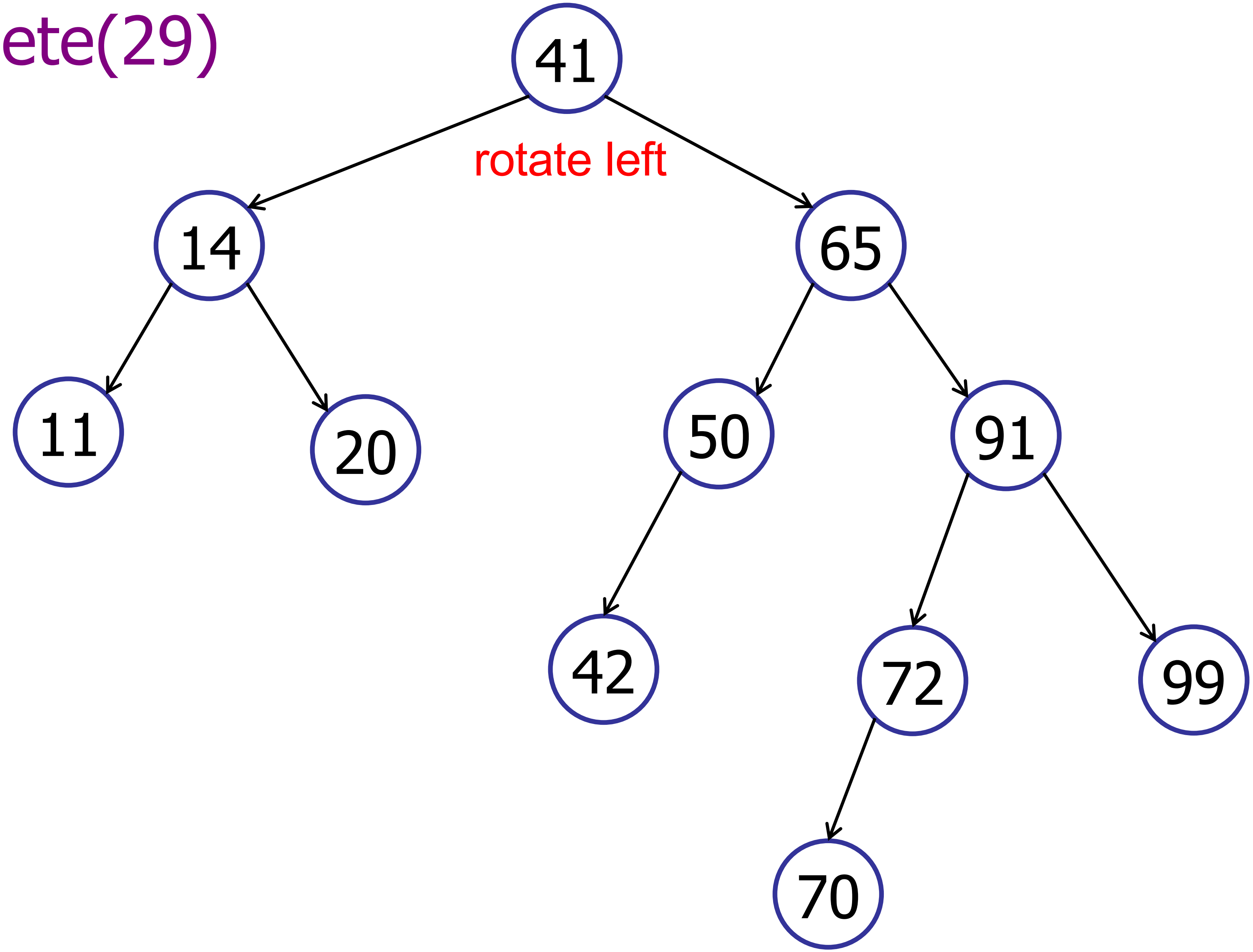
Binary Search Tree

delete(29)



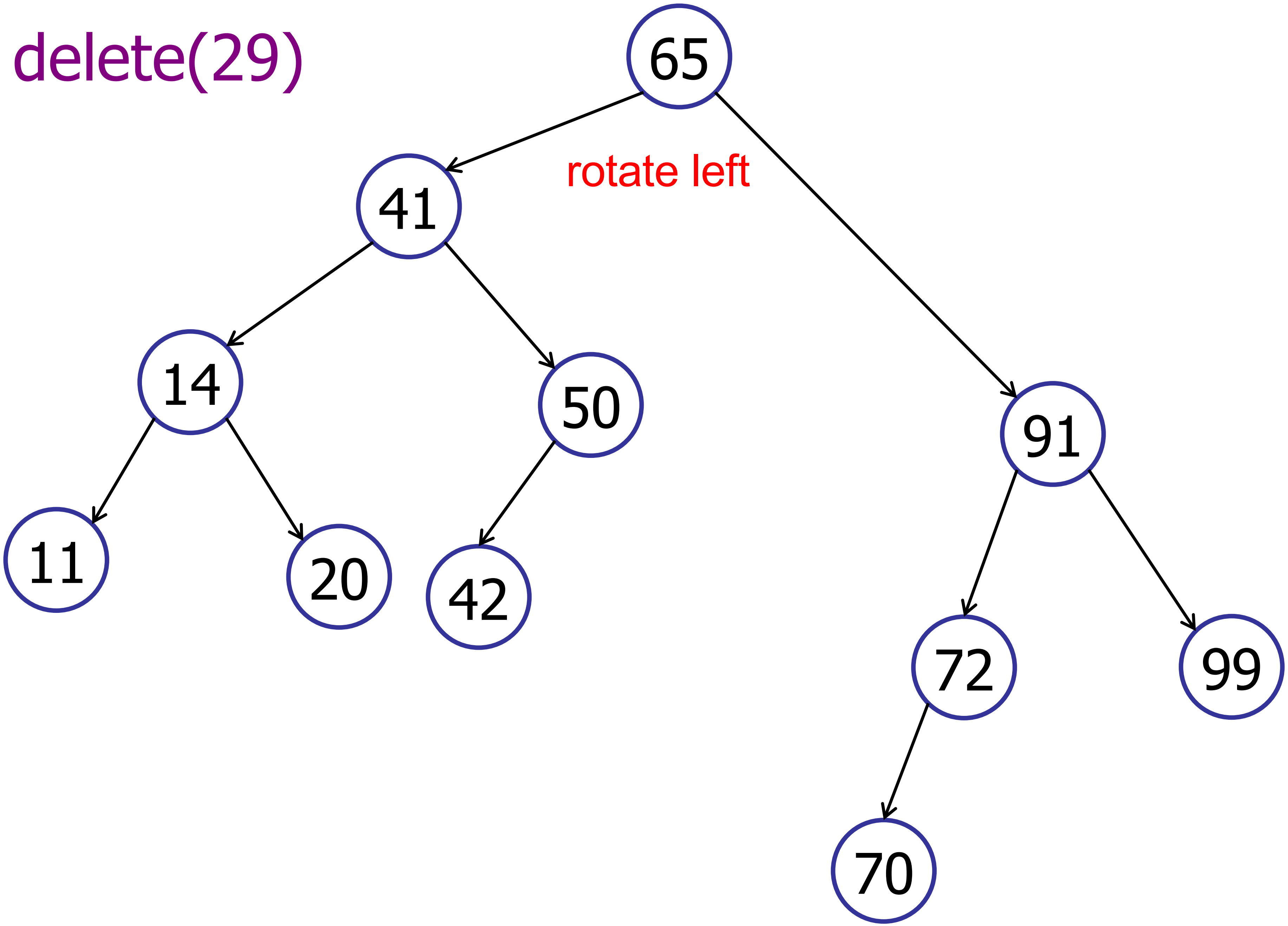
Binary Search Tree

delete(29)



Binary Search Tree

delete(29)



How many rebalances?

Why are two rotations not enough?

- Delete reduced height.
- Rotations (to rebalance) reduce height!

Key observation:

- Rebalancing does not “undo” the change in height caused by insertion.

Delete in AVL Tree

Summary:

- Delete key from BST.
- Walk up tree:
  - At every step, check for balance.
  - If out-of-balance, use rotations to rebalance.
  - Continue to root.

Key observation:

- It is *not* sufficient to only fix lowest out-of-balance node in tree.

Every insertion requires 1 or 2 rotations?

1. Yes
2. No
3. I don't know

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A tree is **balanced** if every node's children differ in height be at most 1?

1. Yes
2. No
3. I don't know

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A tree is **balanced** if every node either has two children or zero children?

1. Yes
2. No
3. I don't know

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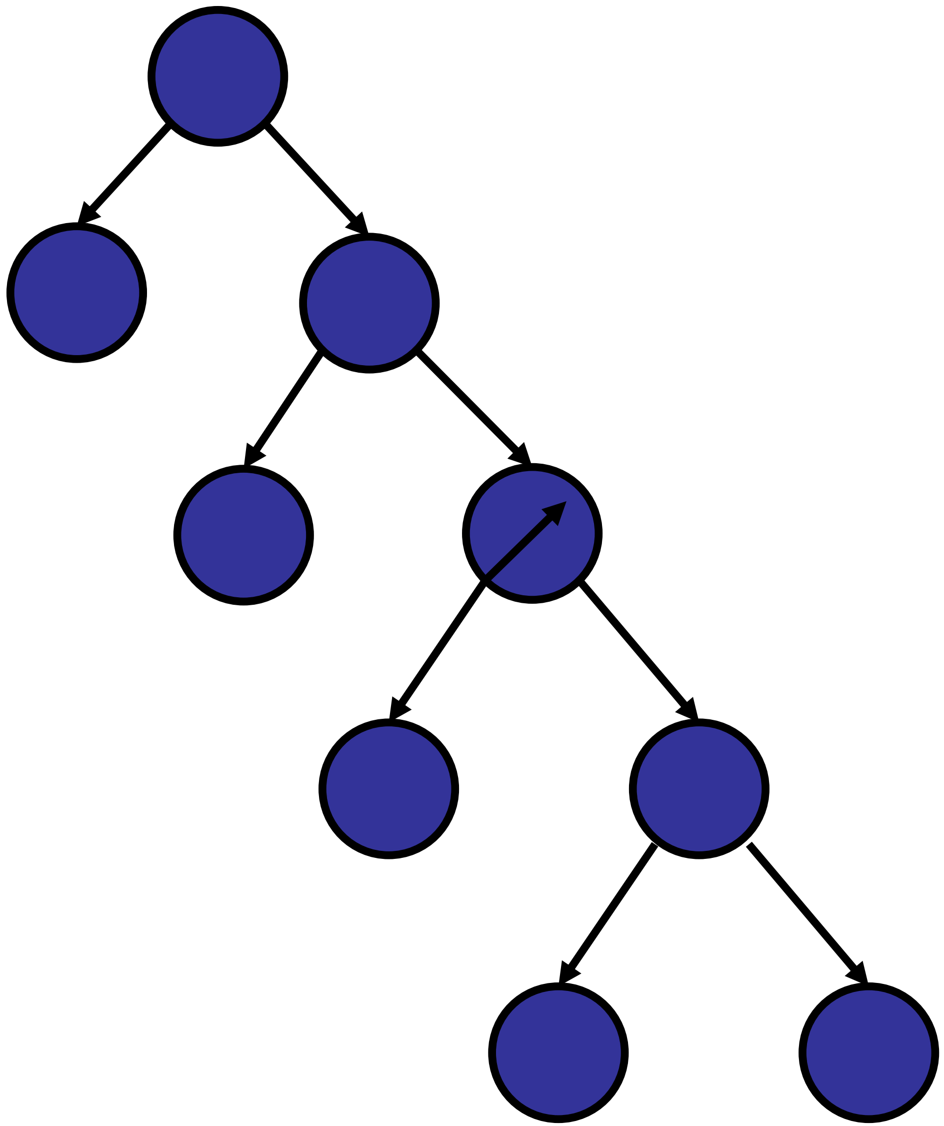
A tree is balanced if every node either has two children or zero children?

1. Yes
2. No
3. I don't know



Using rotations, you can create every possible "tree shape."

1. True
2. False
3. I don't know

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AVL Trees

What if you do not remove deleted nodes?

- Mark a node “deleted” and leave it in the tree.

Logical deletes:

- Performance degrades over time.
- Clean up later? (Amortized performance…)

AVL Trees

What if you do not want to store the height in every node?

- Only store difference in height from parent.

Todays Plan

On the importance of being balanced

- Height-balanced binary search trees
- AVL trees
- Rotations

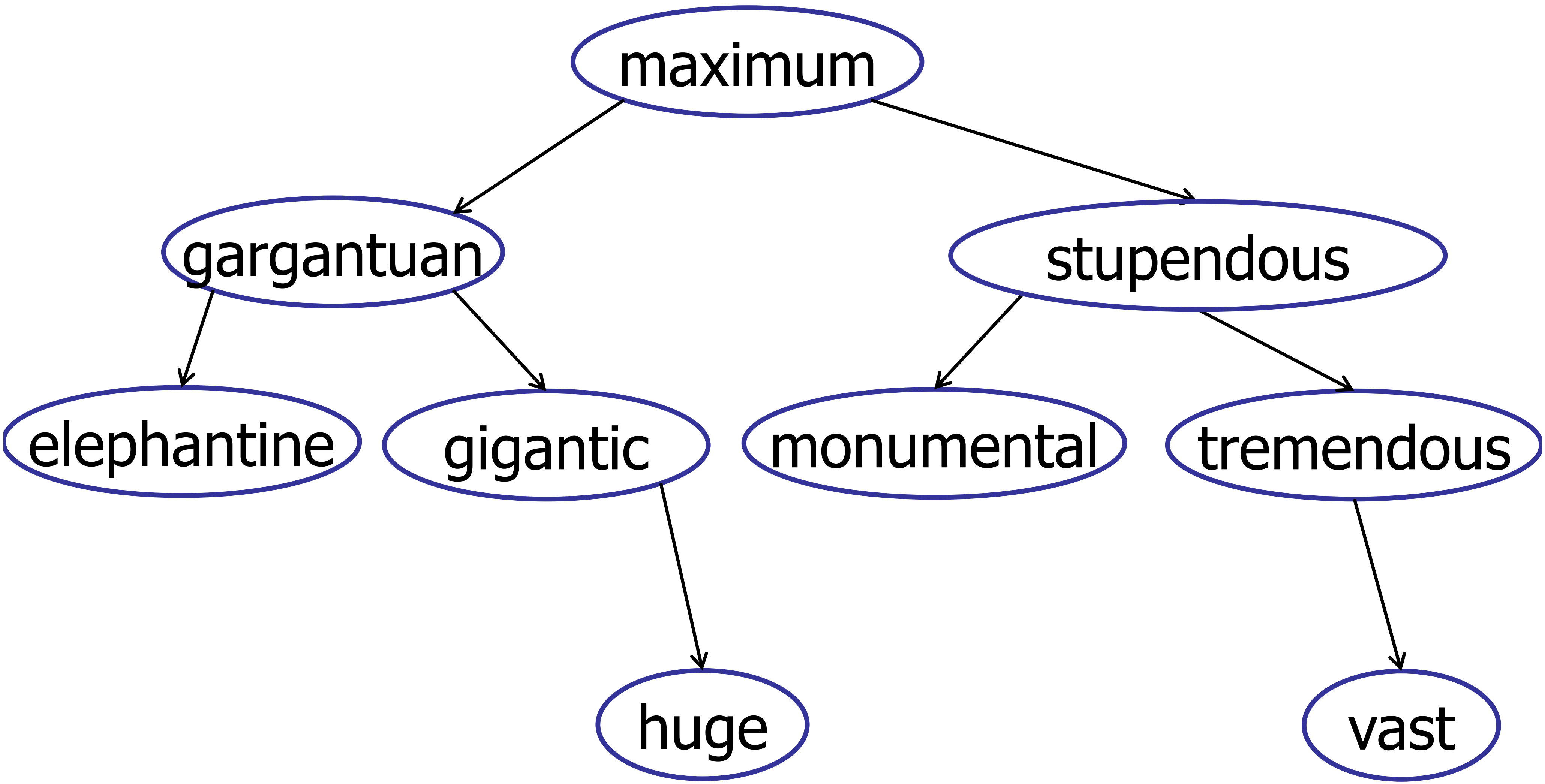
Tries

- How to handle text?

Data structure design

- How to build new structures on existing ideas?

What about text strings?



Implement a searchable dictionary!

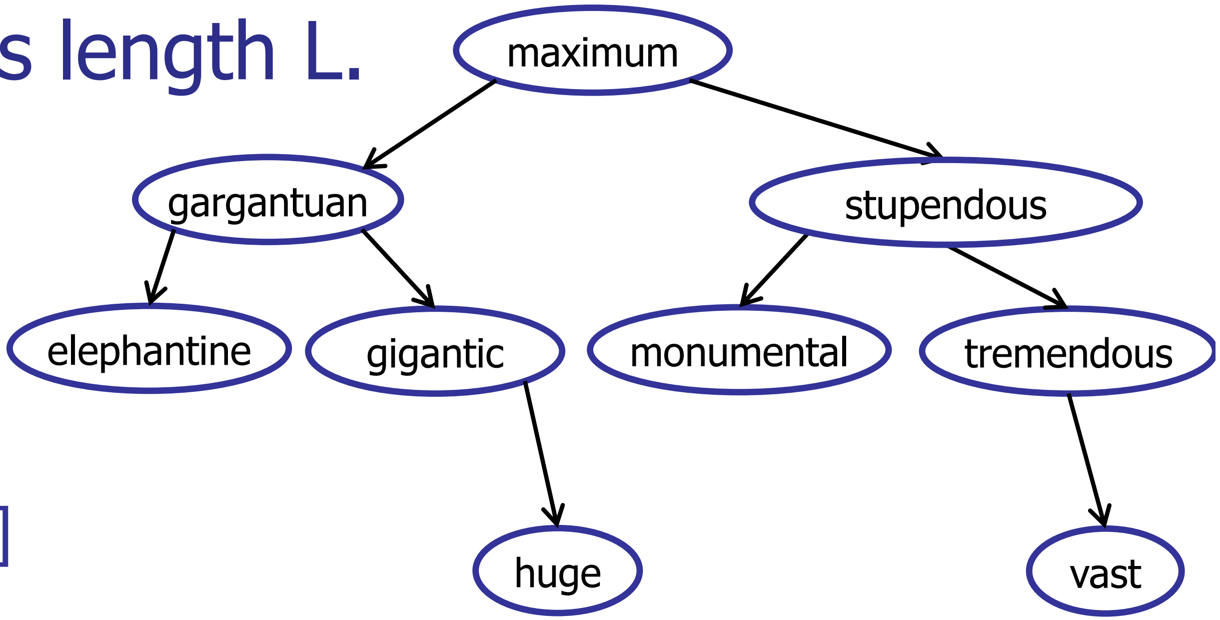
What about text strings?

Cost of comparing two strings:

- Cost[A ?= B] = min(A.length, B.length)
- Compare strings letter by letter

Cost of tree operation:

- Assume string has length L.
- Cost: O(hL)

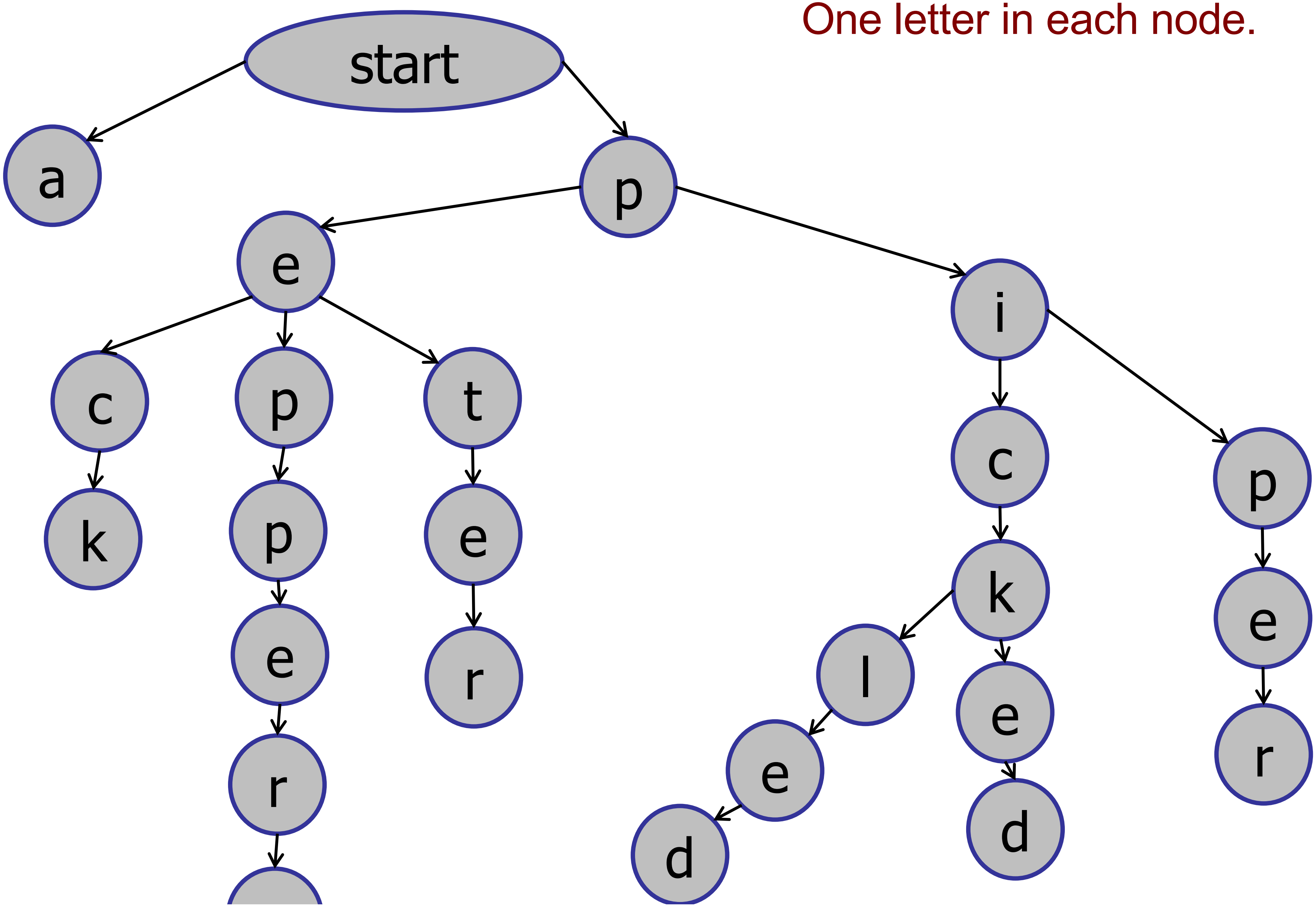


[In the worst case.]

[Optimizations are possible.]

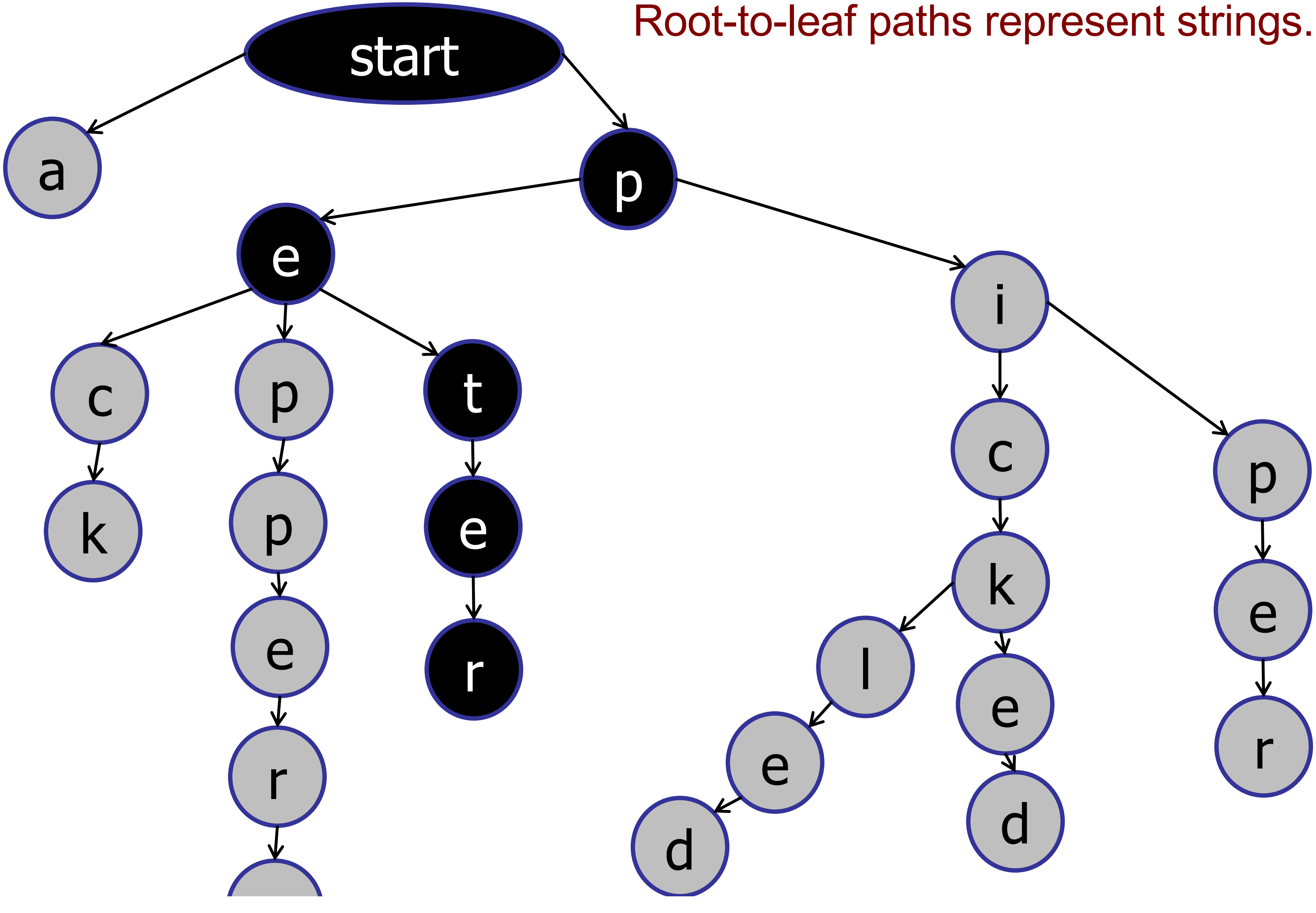
Trie [prounounced: try]

One letter in each node.



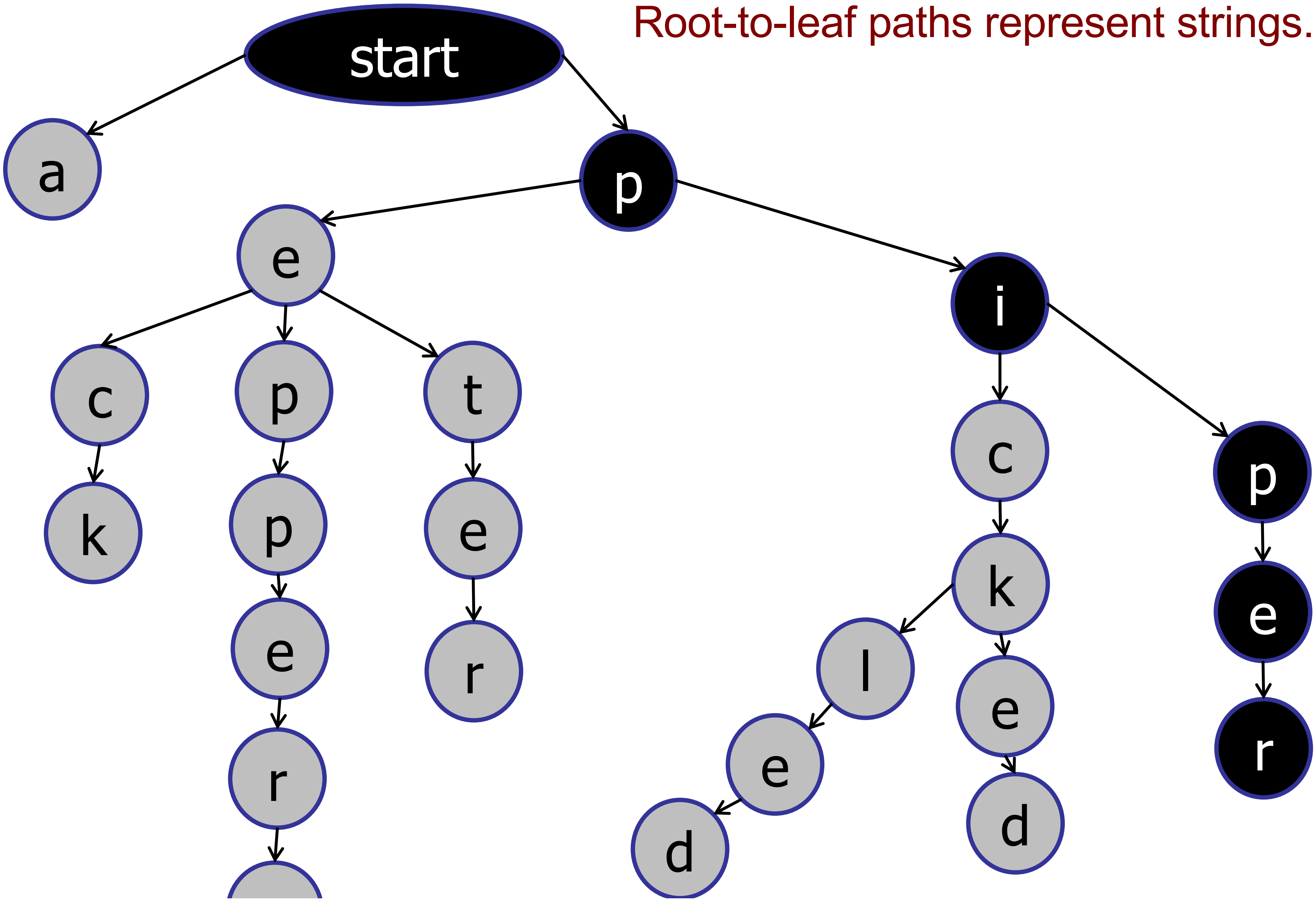
Trie [prounounced: try]

Root-to-leaf paths represent strings.



Trie [prounounced: try]

Root-to-leaf paths represent strings.



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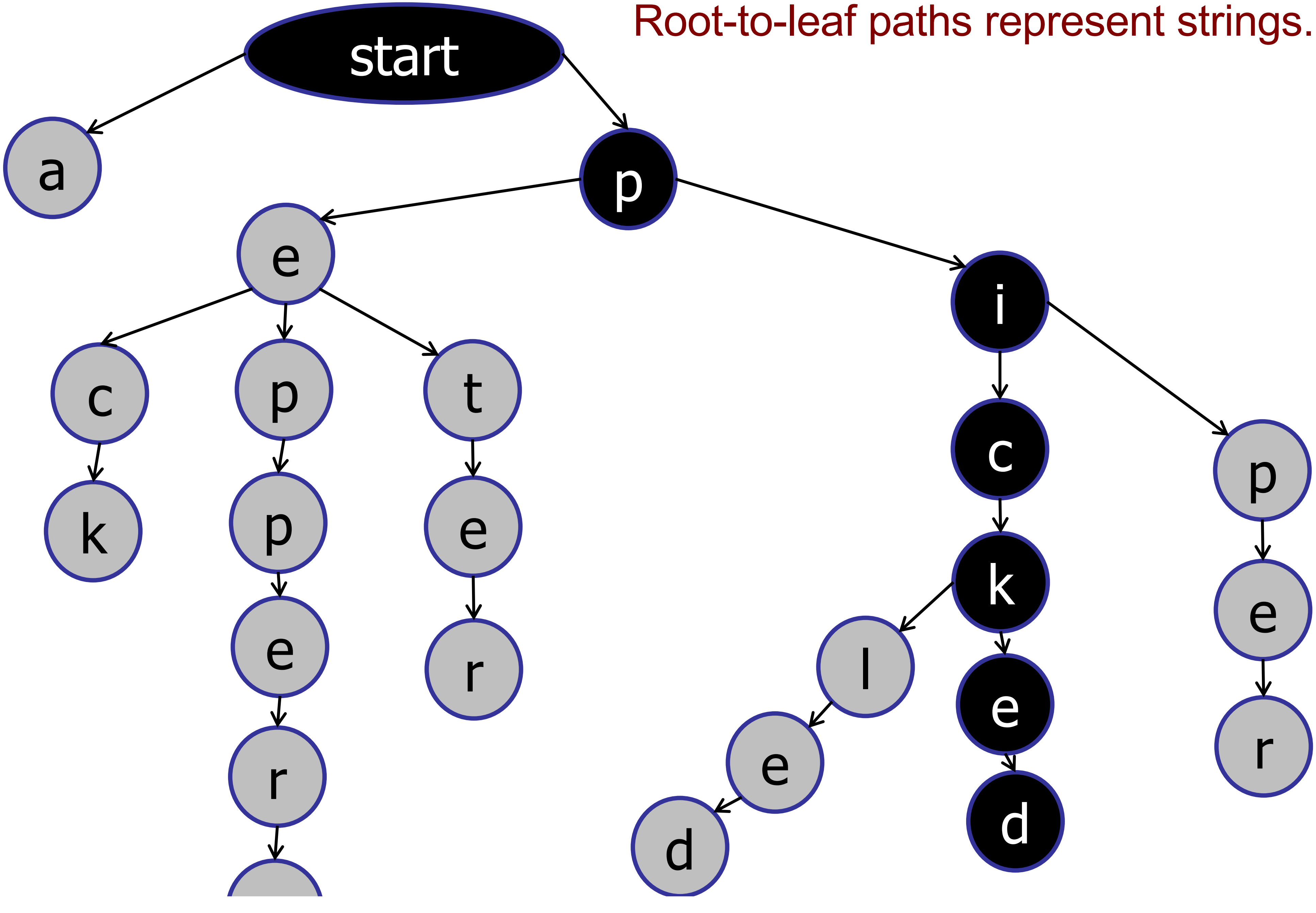
Root-to-leaf paths represent strings.



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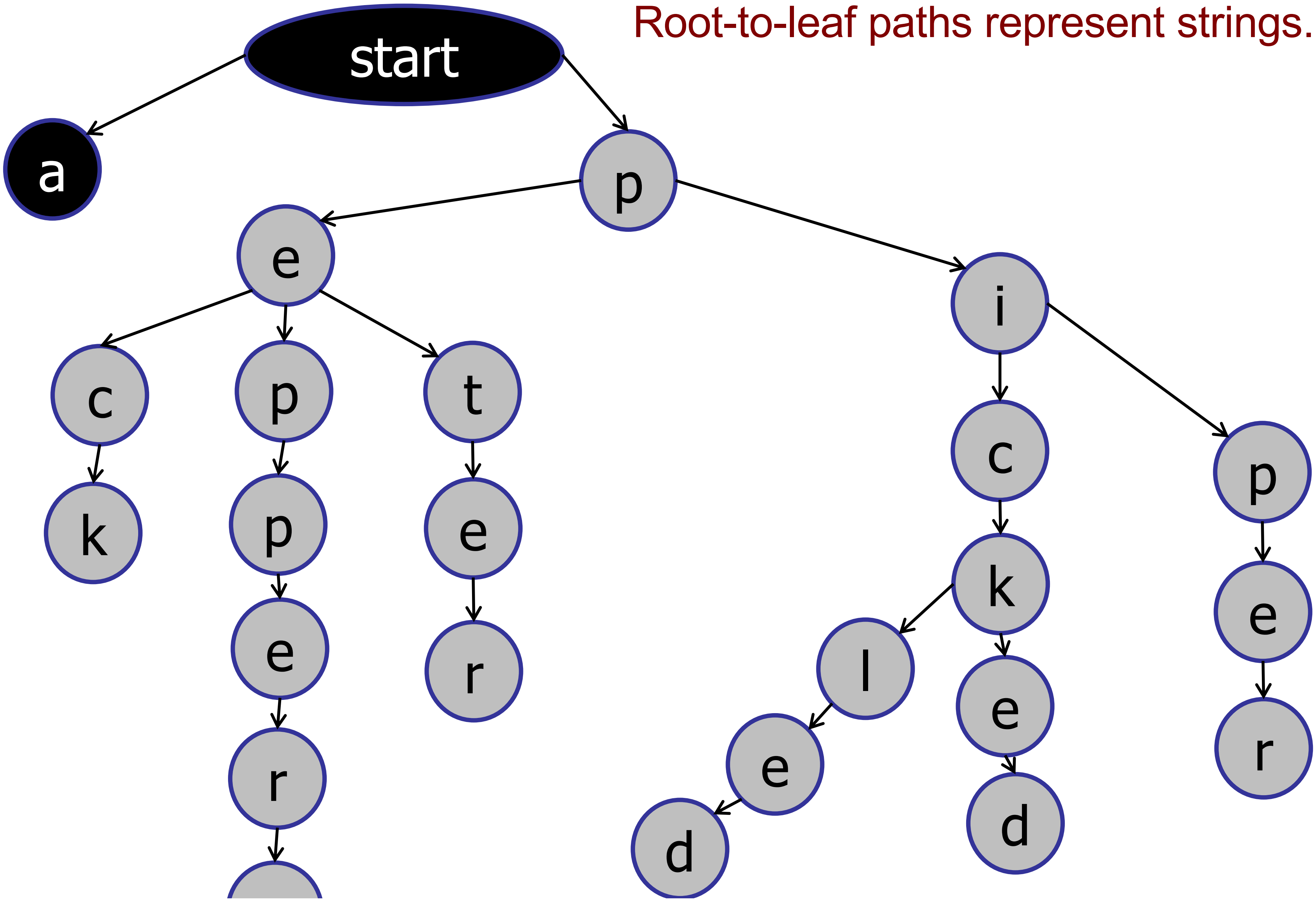
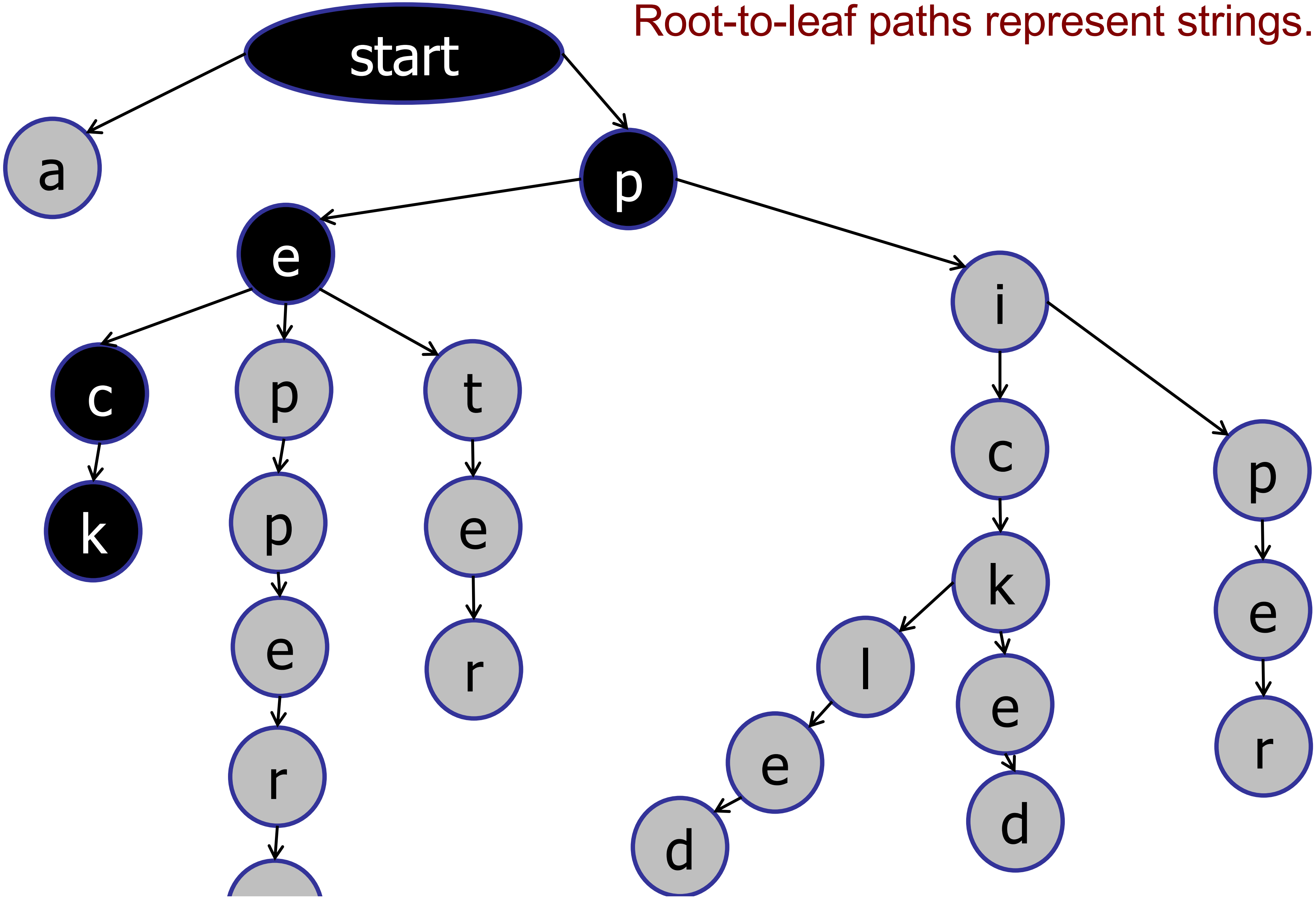
Root-to-leaf paths represent strings.



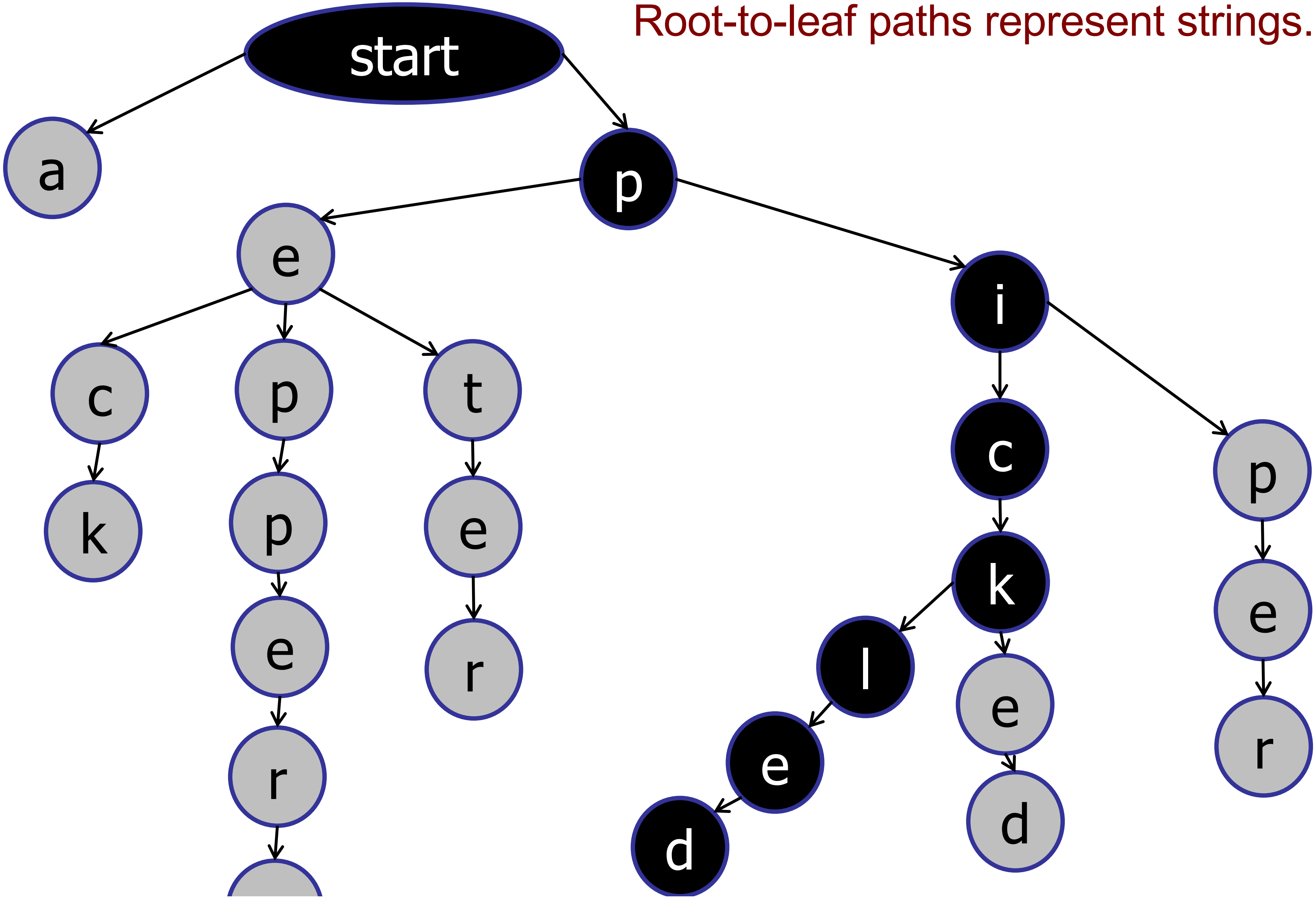
Trie [prounounced: try]

Root-to-leaf paths represent strings.



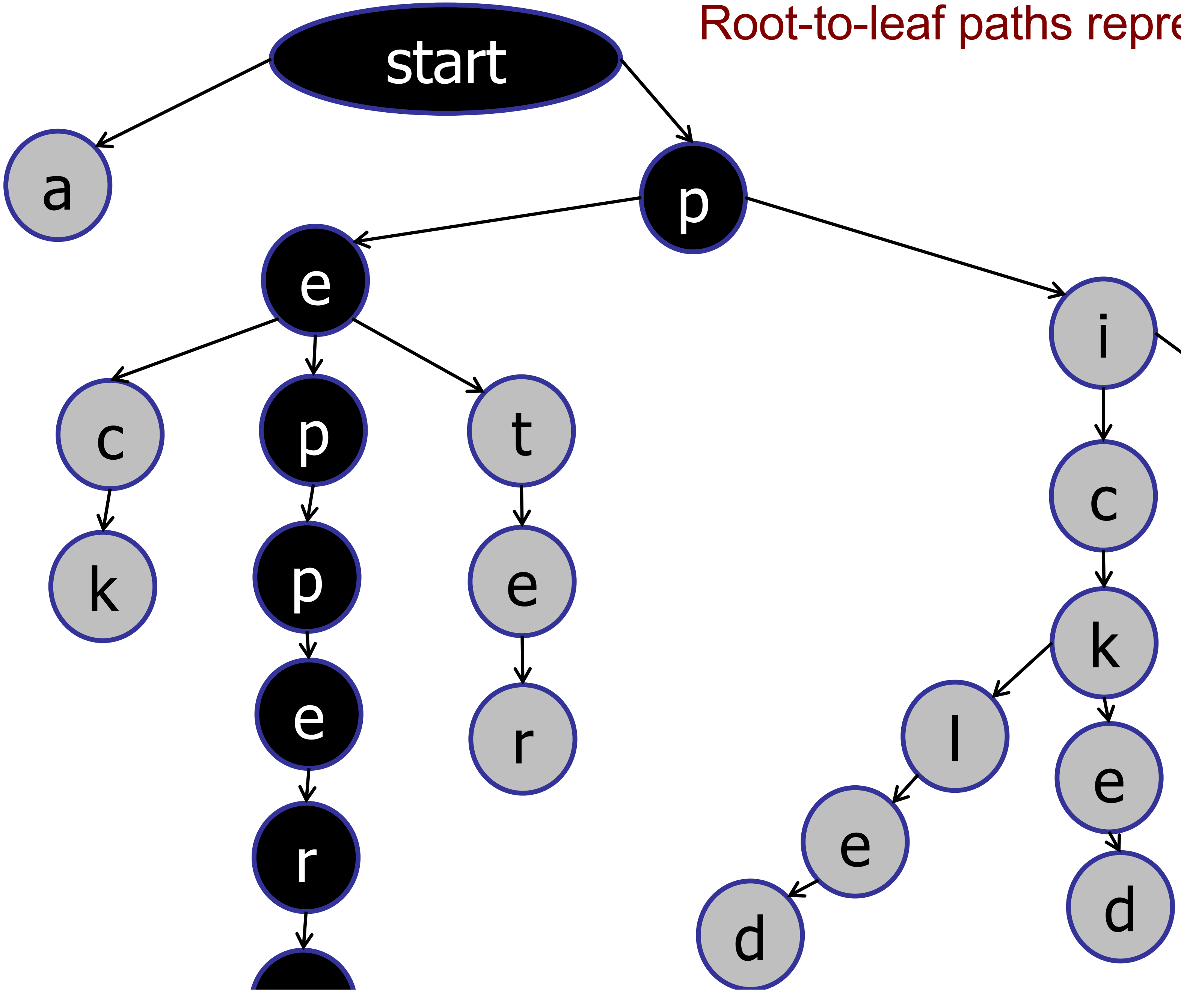
Trie [prounounced: try]

Root-to-leaf paths represent strings.

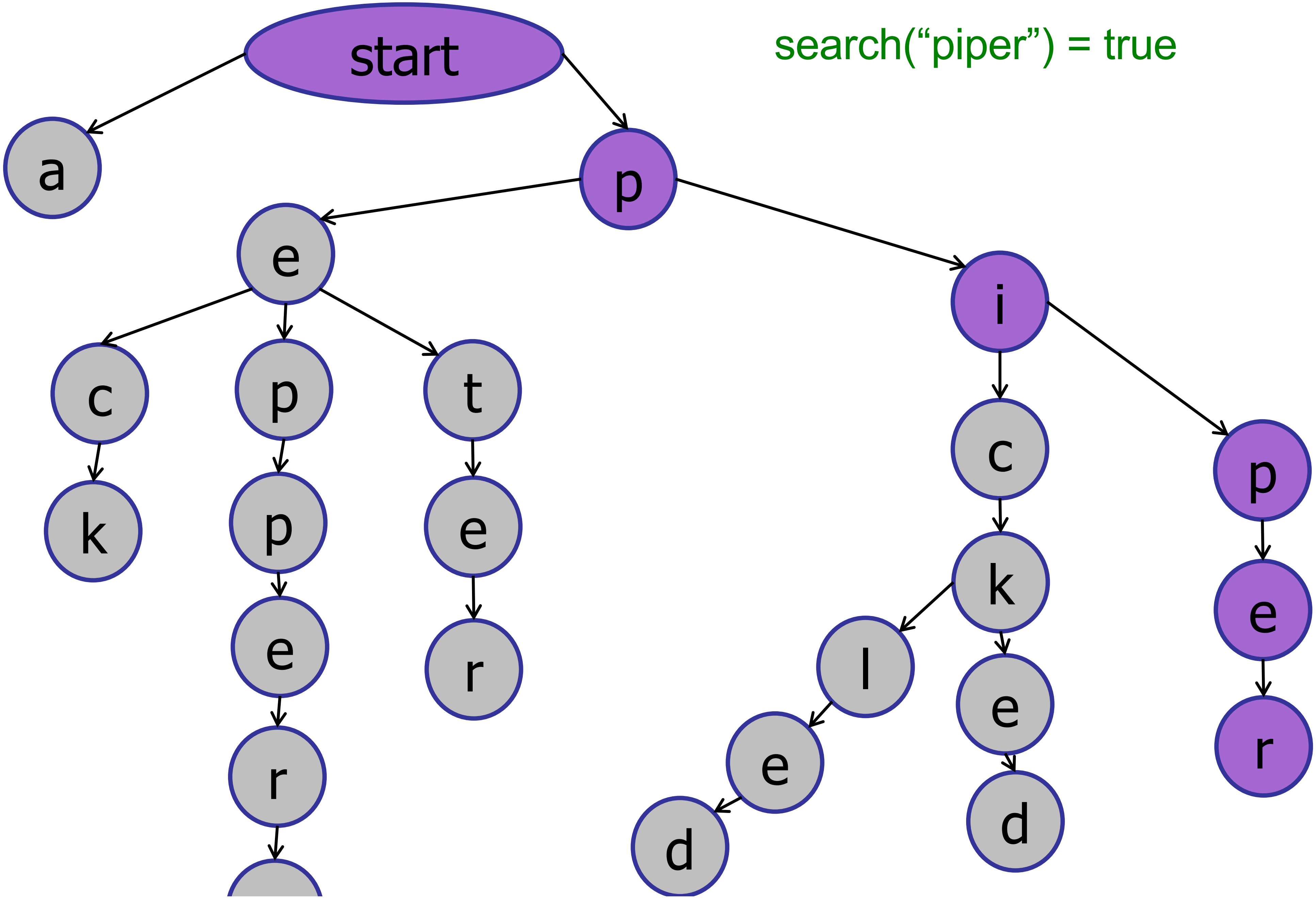


Trie [prounounced: try]

Root-to-leaf paths represent strings.



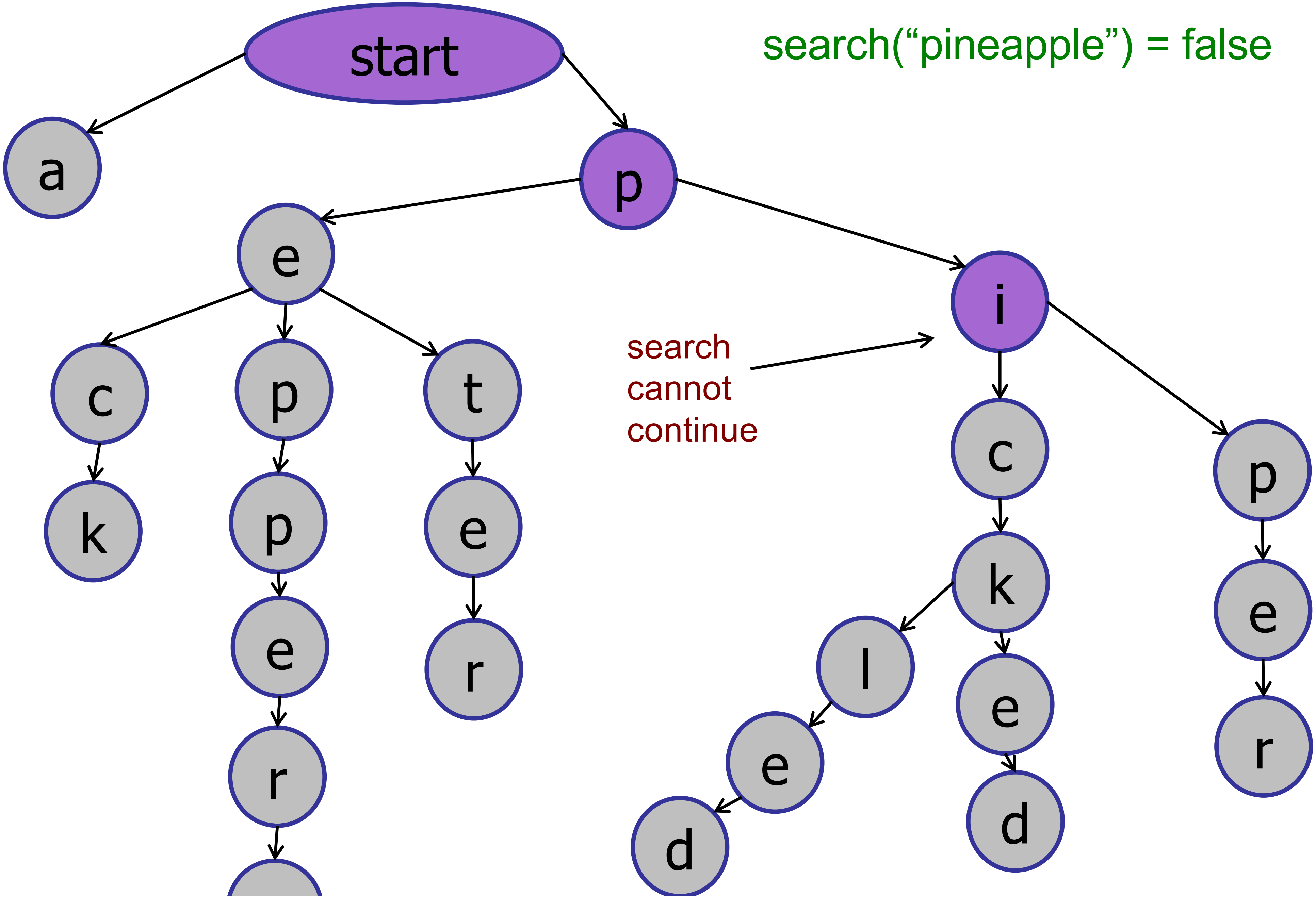
Searching a Trie



search("piper") = true

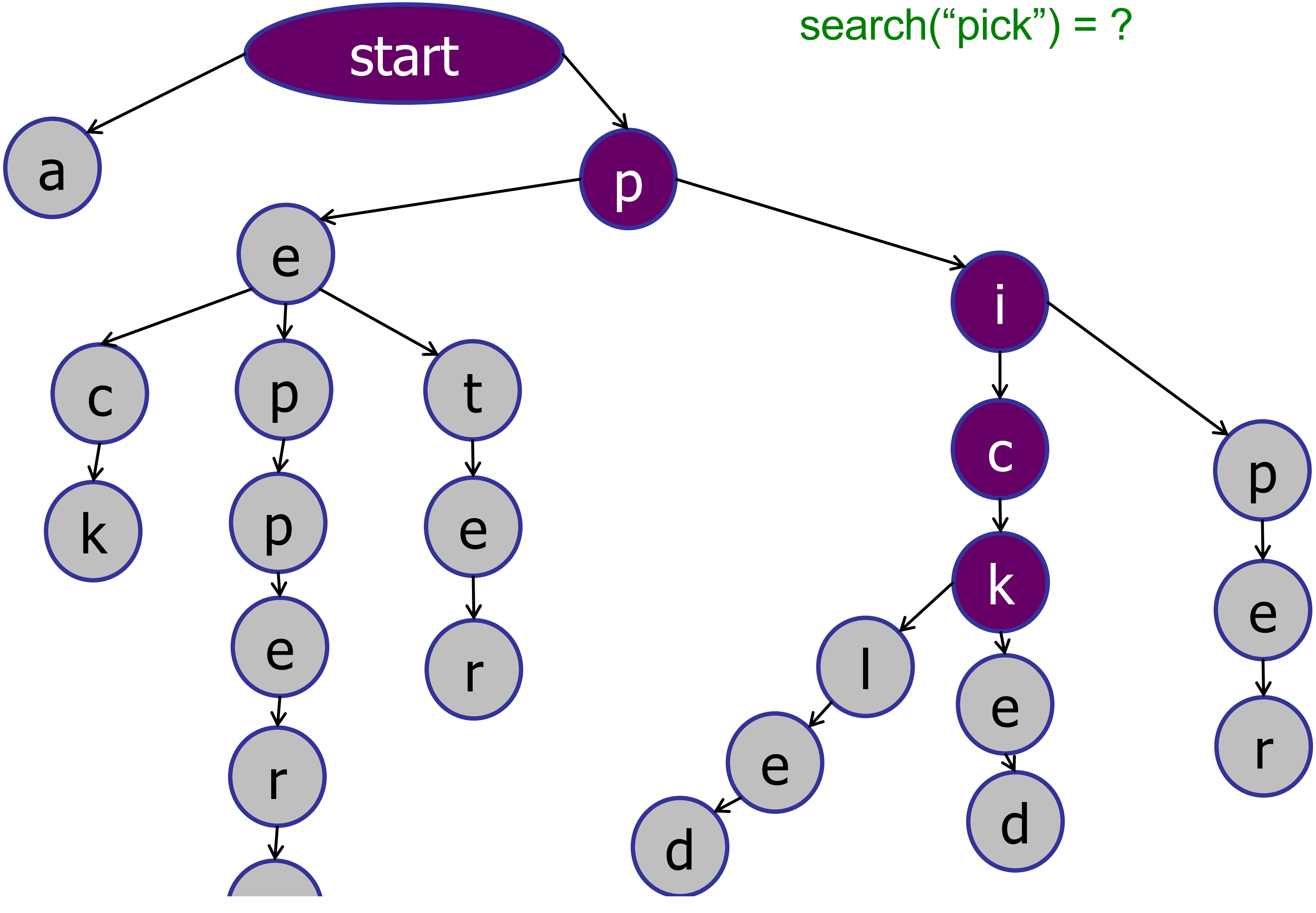
Searching a Trie

search(“pineapple”) = false



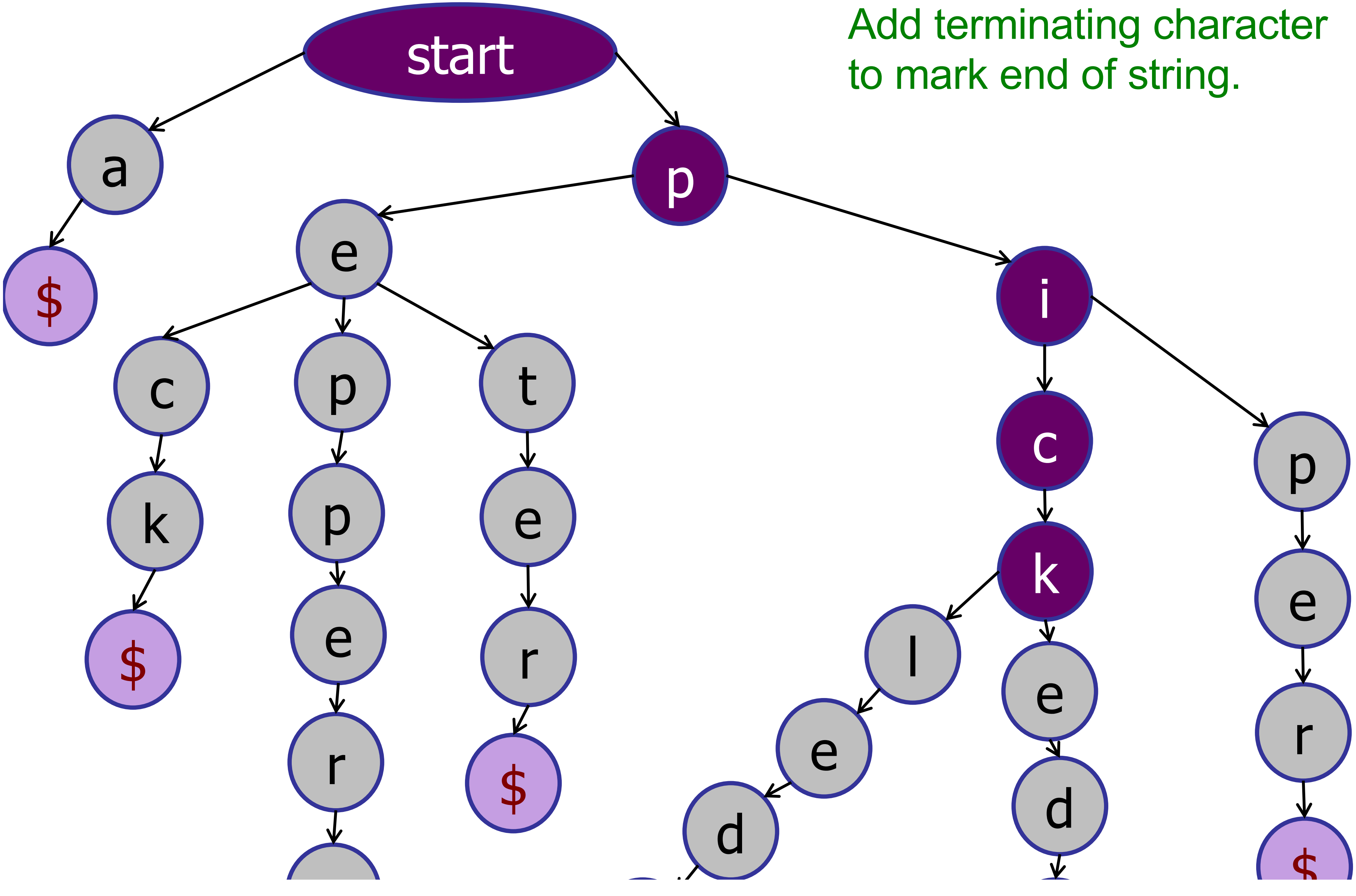
Trie Details

search(“pick”) = ?



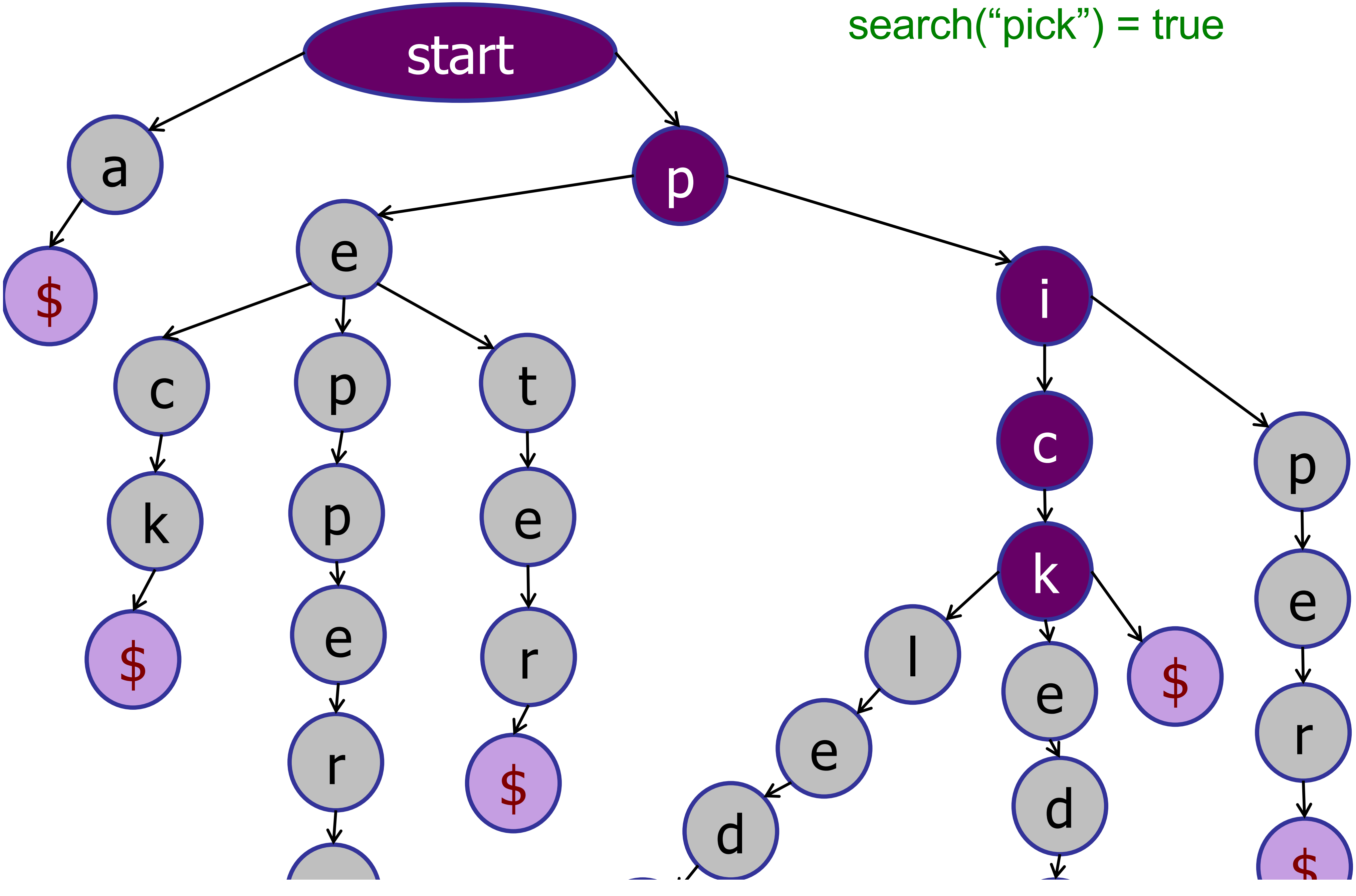
Trie Details

Add terminating character to mark end of string.

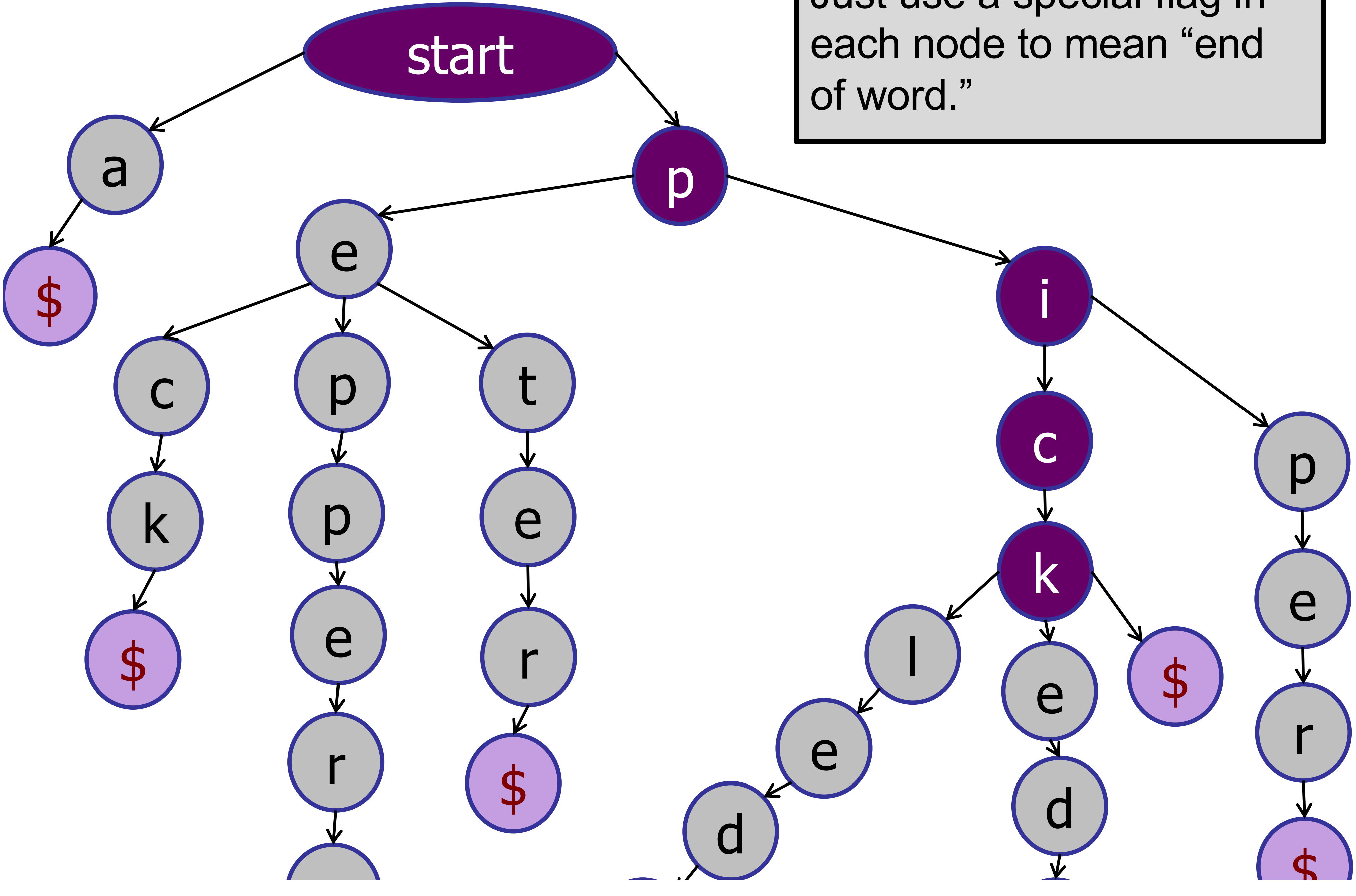


Trie Details

search(“pick”) = true



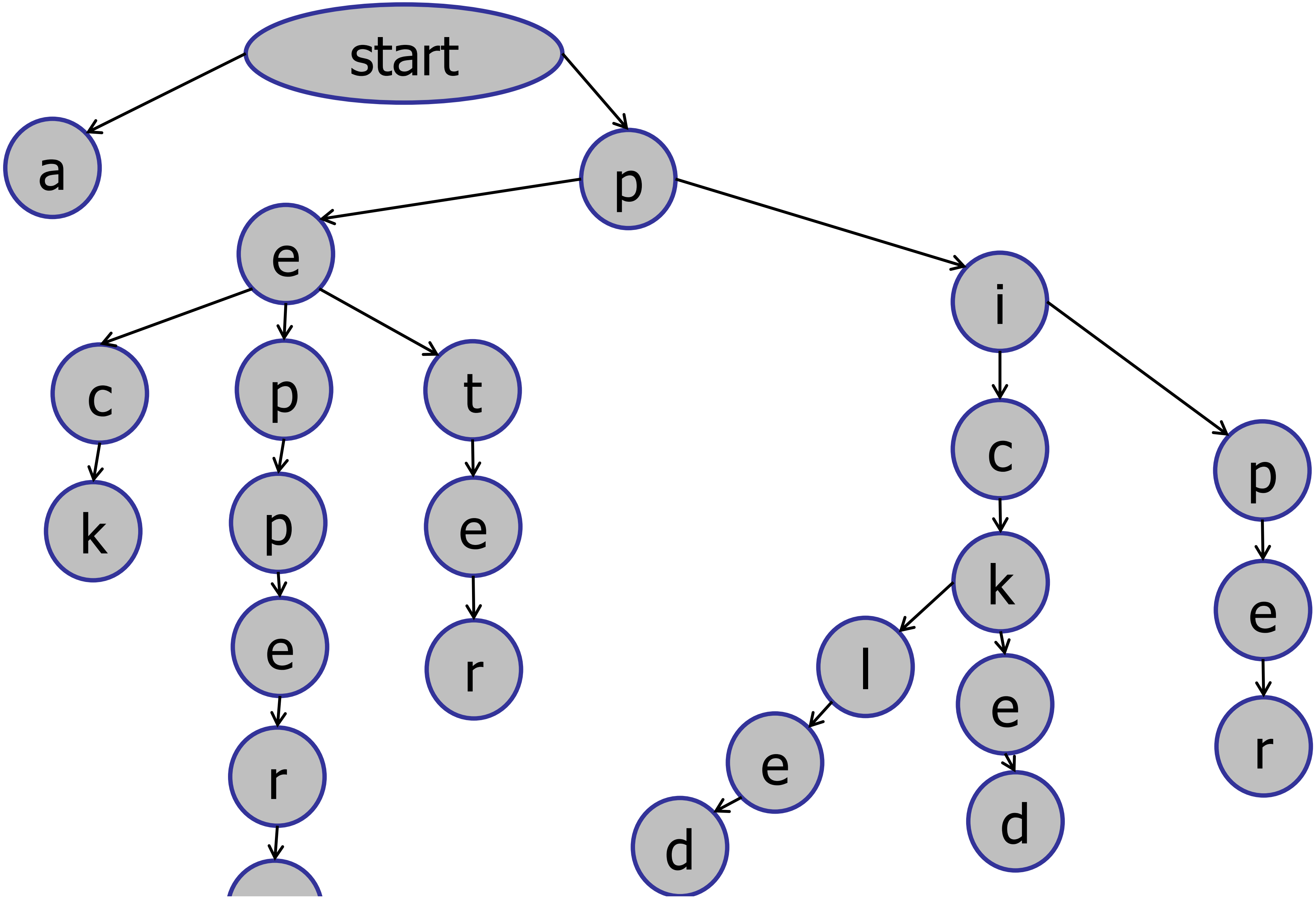
Trie Details



Or:
Just use a special flag in each node to mean "end of word."

Trie

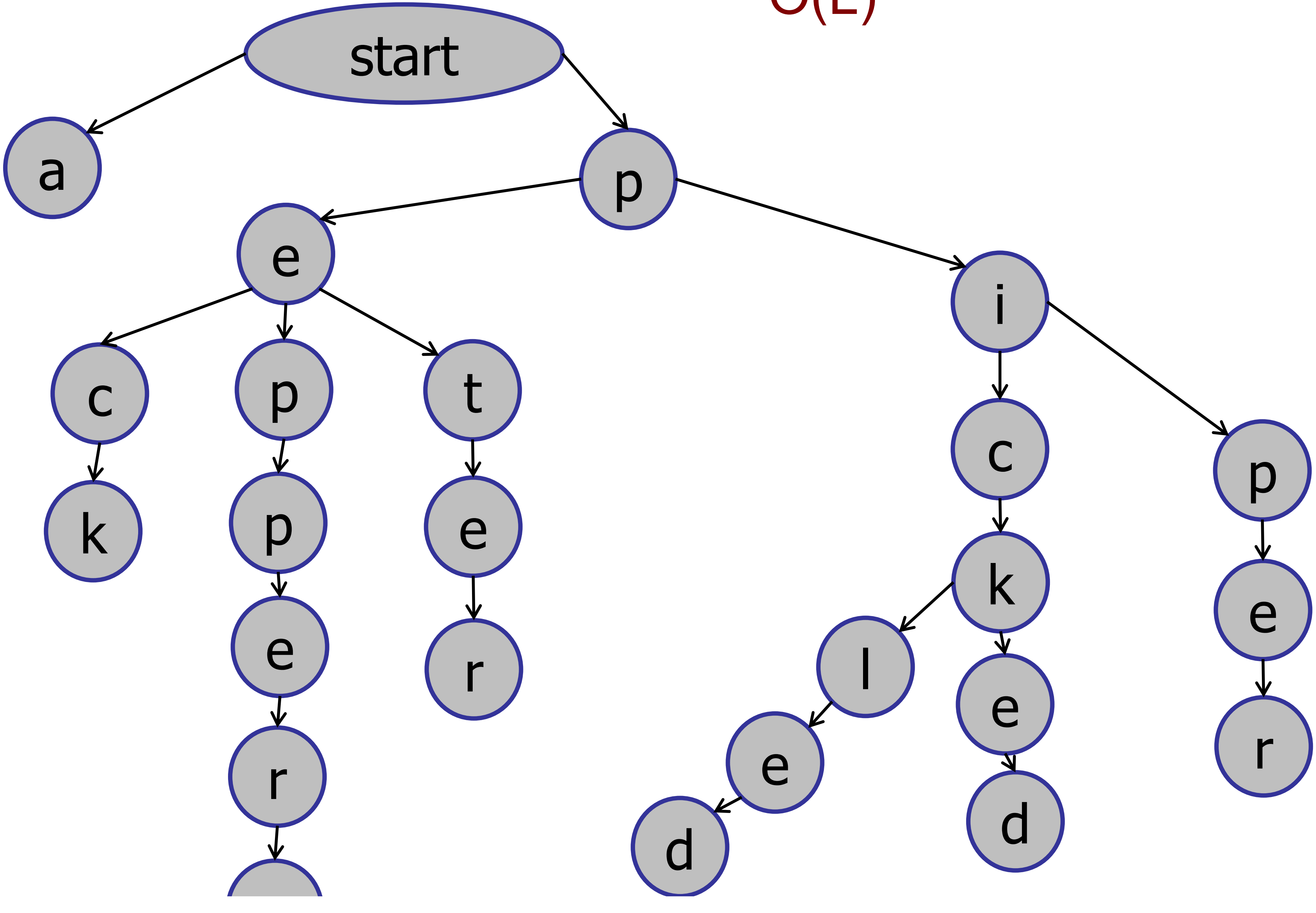
Cost to search for a string of length L?



Trie

Cost to search for a string of length L?

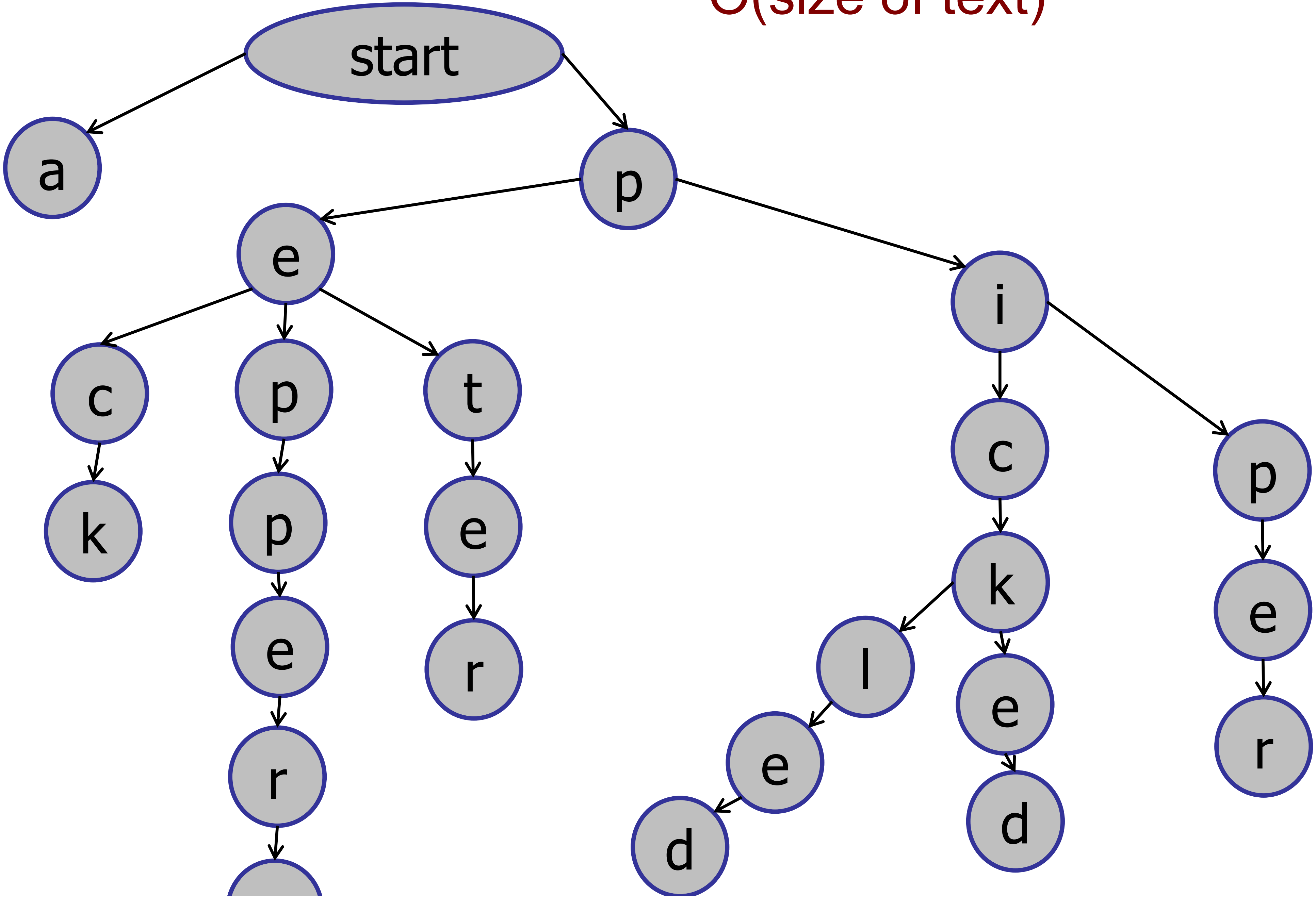
O(L)



Trie

Space for storing a try?

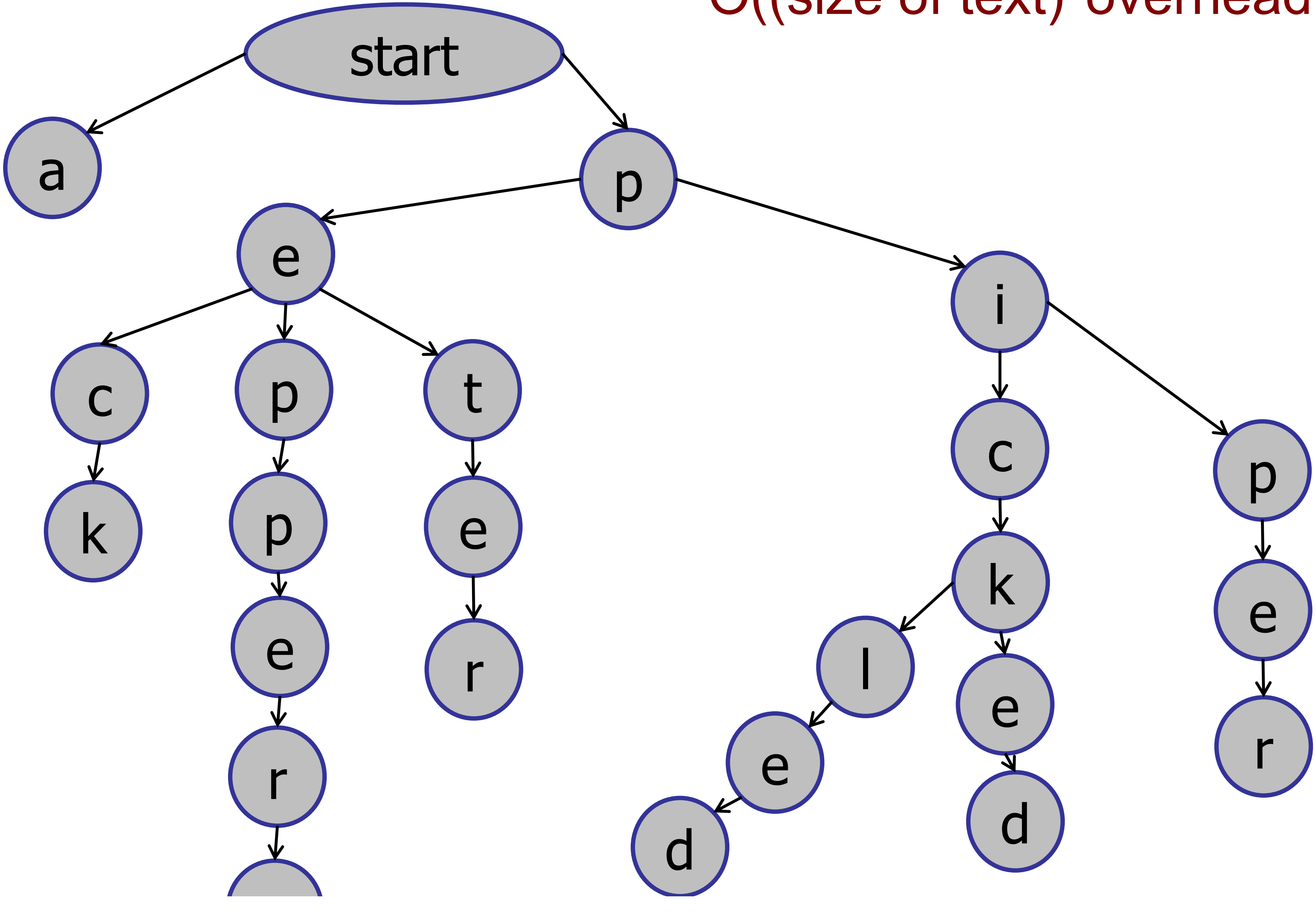
O(size of text)



Trie

Space for storing a try?

O((size of text)*overhead)



Trie Tradeoffs

Time:

- Trie tends to be faster: O(L) vs. O(Lh).
- Does not depend on number of strings.

Even faster if string is not in trie!

Trie Tradeoffs

Time:

- Trie tends to be faster: O(L).
- Does not depend on size of total text.
- Does not depend on number of strings.

Space:

- Trie tends to use more space.
- BST and Trie use O(text size) space.
- But Trie has more nodes and more overhead.

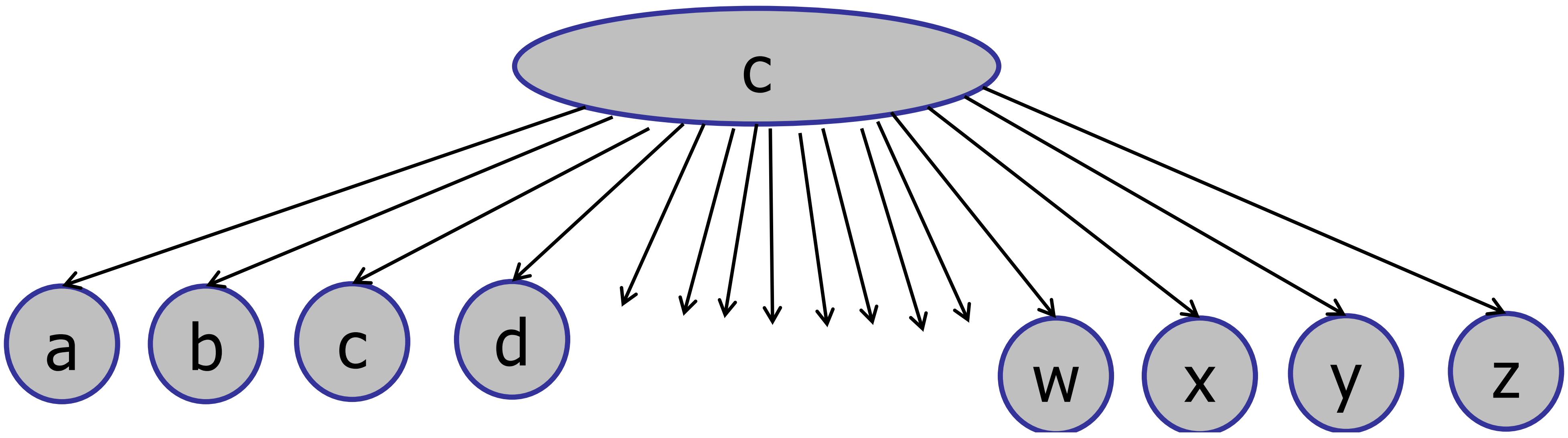
Trie Space

Trie node:

- Has many children.
- For strings: fixed degree.
- Ascii character set: 256

wasted space?

```
TrieNode children[] = new TrieNode[256];
```



Trie Applications

String dictionaries

- Searching
- Sorting / enumerating strings

Partial string operations:

- Prefix queries: find all the strings that start with pi.
- Long prefix: what is the longest prefix of “pickling” in the trie?
- Wildcards: find a string of the form “pi??le” in the trie.

Todays Plan

On the importance of being balanced

- Height-balanced binary search trees
- AVL trees
- Rotations

Tries

- How to handle text?

Data structure design

- How to build new structures on existing ideas?

Dynamic Data Structures

1. Maintain a set of items

2. Modify the set of items

3. Answer queries.

Big picture idea:

Trees are a good way to store, summarize, and search dynamic data.

Dynamic Data Structures

- Operations that create a data structure
  - build (preprocess)
- Operations that modify the structure
  - insert
  - delete
- Query operations
  - search, select, etc.

“Why do we need to learn how an AVL tree works?”

Just use a Java TreeMap, right?

"Why do we need to learn how an AVL tree works?"

1. Learn how to think like a computer scientist.

“Why do we need to learn how an AVL tree works?”

1. Learn how to think like a computer scientist.
2. Learn to modify existing data structures to solve new problems.

Augmented Data Structures

Many problems require storing additional data in a standard data structure.

Augment more frequently than invent…

Plan

Three examples of augmenting balanced BSTs

1. Order Statistics

2. Interval Queries

3. Orthogonal Range Searching

Augmenting data structures

Basic methodology:

1. Choose underlying data structure

   (tree, hash table, linked list, stack, etc.)

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   (tree, hash table, linked list, stack, etc.)

2. Determine additional info needed.

3. Modify data structure to *maintain* additional info when the structure changes.

   (subject to insert/delete/etc.)

Augmenting data structures

Basic methodology:

1. Choose underlying data structure

   (tree, hash table, linked list, stack, etc.)

2. Determine additional info needed.

3. Modify data structure to *maintain* additional info when the structure changes.

   (subject to insert/delete/etc.)

4. Develop new operations.

Plan

Three examples of augmenting balanced BSTs

1. Order Statistics

2. Interval Queries

3. Orthogonal Range Searching

Order Statistics

Input

A set of integers.

Output: select(k)

The k^{th} item in the set.

52	**7**	**13**	**43**	**22**	**92**	**18**	**9**	**65**	**67**	**87**	**25**

select(4)

select(2) returns:

52	**7**	**13**	**43**	**22**	**92**	**18**	**9**	**65**	**67**	**87**	**25**

1. 52
2. 9
3. 13
4. 43
5. 25

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Order Statistics

Input

A set of integers.

Output: select(k)

The k^{th} item in the set.

52	**7**	**13**	**43**	**22**	**92**	**18**	**9**	**65**	**67**	**87**	**25**

select(4)

Order Statistics

Input

A set of integers.

Output: select(k)

The k^th item in the set.

7	**9**	**13**	**18**	**22**	**25**	**43**	**52**	**65**	**67**	**87**	**92**

select(4)

Order Statistics

Input

A set of integers.

Output: select(k) Sort: $O(n \log n)$

The k^{th} item in the set.

7	**9**	**13**	**18**	**22**	**25**	**43**	**52**	**65**	**67**	**87**	**92**

select(4)

Order Statistics

Input

A set of integers.

Output: select(k) $\longrightarrow$ QuickSelect: O(n)

The k^{th} item in the set.

7	**9**	**13**	**18**	**22**	**25**	**43**	**52**	**65**	**67**	**87**	**92**

select(4)

Order Statistics

Solution 1:

Sort: O(n log n)

Solution 2:

QuickSelect: O(n)

7	**9**	**13**	**18**	**22**	**25**	**43**	**52**	**65**	**67**	**87**	**92**

select(4)

Order Statistics

Solution 1:

Preprocess: sort --- $O(n \log n)$

Select: $O(1)$

Solution 2:

Preprocess: nothing --- $O(1)$

QuickSelect: $O(n)$

Order Statistics

Solution 1:

Preprocess: sort --- O(n log n)

Select: O(1)

Solution 2:

Preprocess: nothing --- O(1)

QuickSelect: O(n)

Trade-off: how many items to select?

Dynamic Order Statistics

Implement a data structure that supports:

- insert(int key)
- delete(int key)

and also:

- select(int k)

7	**9**	**13**	**18**	**22**	**25**	**43**	**52**	**65**	**67**	**87**	**92**

select(4)

Dynamic Order Statistics

Solution 1:

Basic structure: sorted array A.

insert(int item): add item to sorted array A.

select(int k): return A[k]

7	**9**	**13**	**18**	**22**	**25**	**43**	**52**	**65**	**67**	**87**	**92**

Dynamic Order Statistics

Solution 2:

Basic structure: unsorted array A.

insert(int item): add item to end of array A.

select(int k): run QuickSelect(k)

7	**9**	**13**	**18**	**22**	**25**	**43**	**52**	**65**	**67**	**87**	**92**

When is it more efficient to maintain a sorted array (Solution 1)?

A. Always

B. When there are more inserts than selects.

C. When there are more selects than inserts.

D. Never

E. I'm confused.

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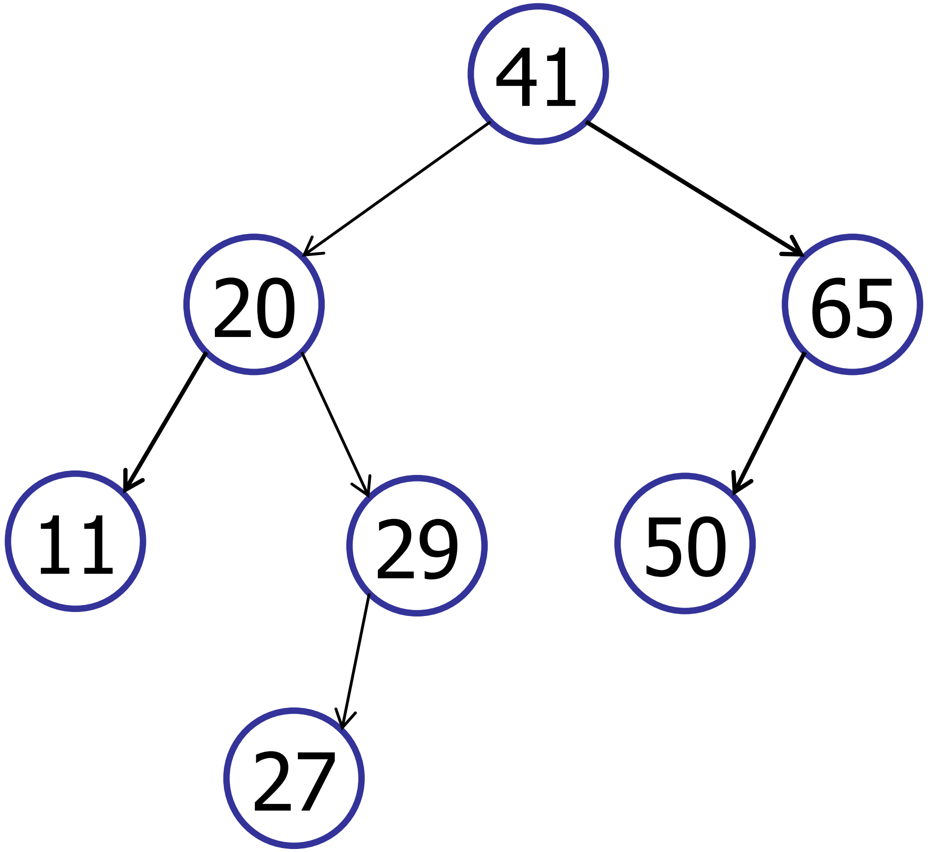
Dynamic Order Statistics

	Insert	Select
Solution 1: Sorted Array	O(n)	O(1)
Solution 2: Unsorted Array	O(1)	O(n)

7	**9**	**13**	**18**	**22**	**25**	**43**	**52**	**65**	**67**	**87**	**92**

Dynamic Order Statistics

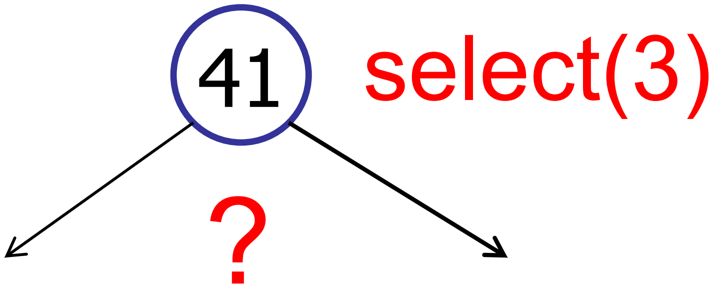
Today: use a (balanced) tree



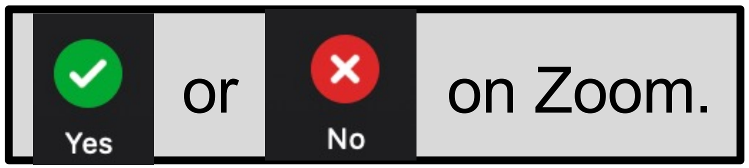
11	**20**	**27**	**29**	**41**	**50**	**65**

Dynamic Order Statistics

How to find the right item?



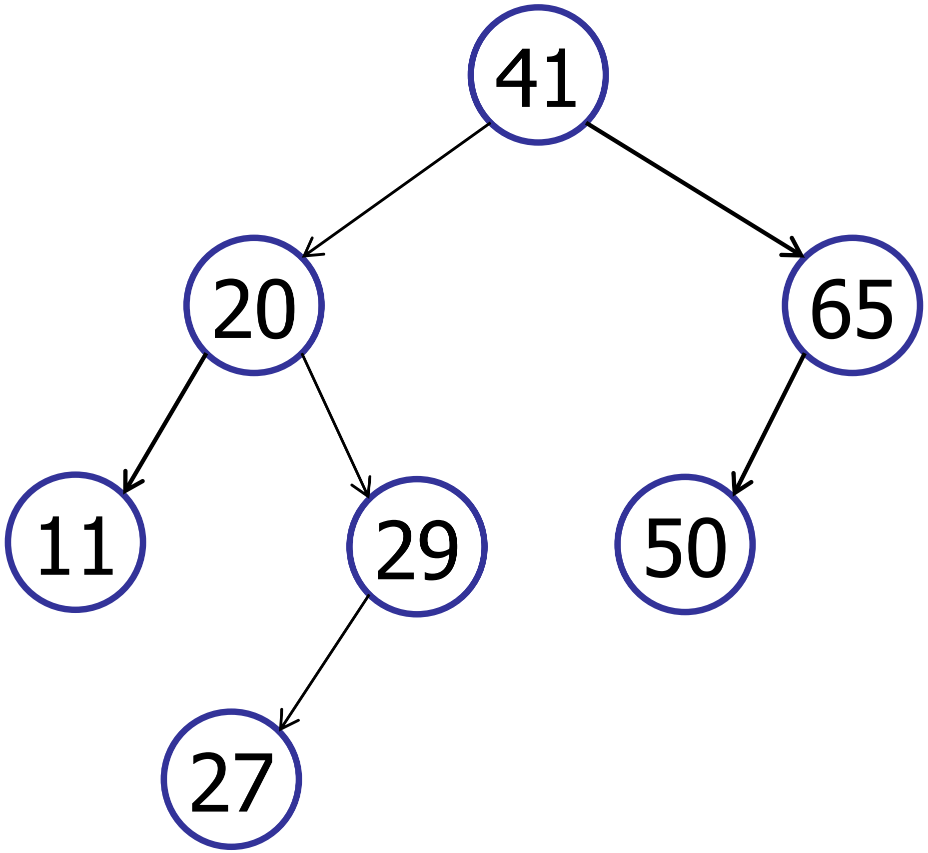
Left Right



Dynamic Order Statistics

Simple solution: traversal

select(k): O(k)
in-order
traversal

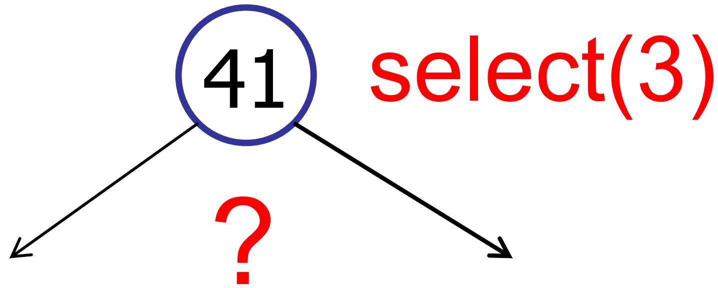


11	**20**	**27**	**29**	**41**	**50**	**65**

Dynamic Order Statistics

Augment!

What extra information would help?



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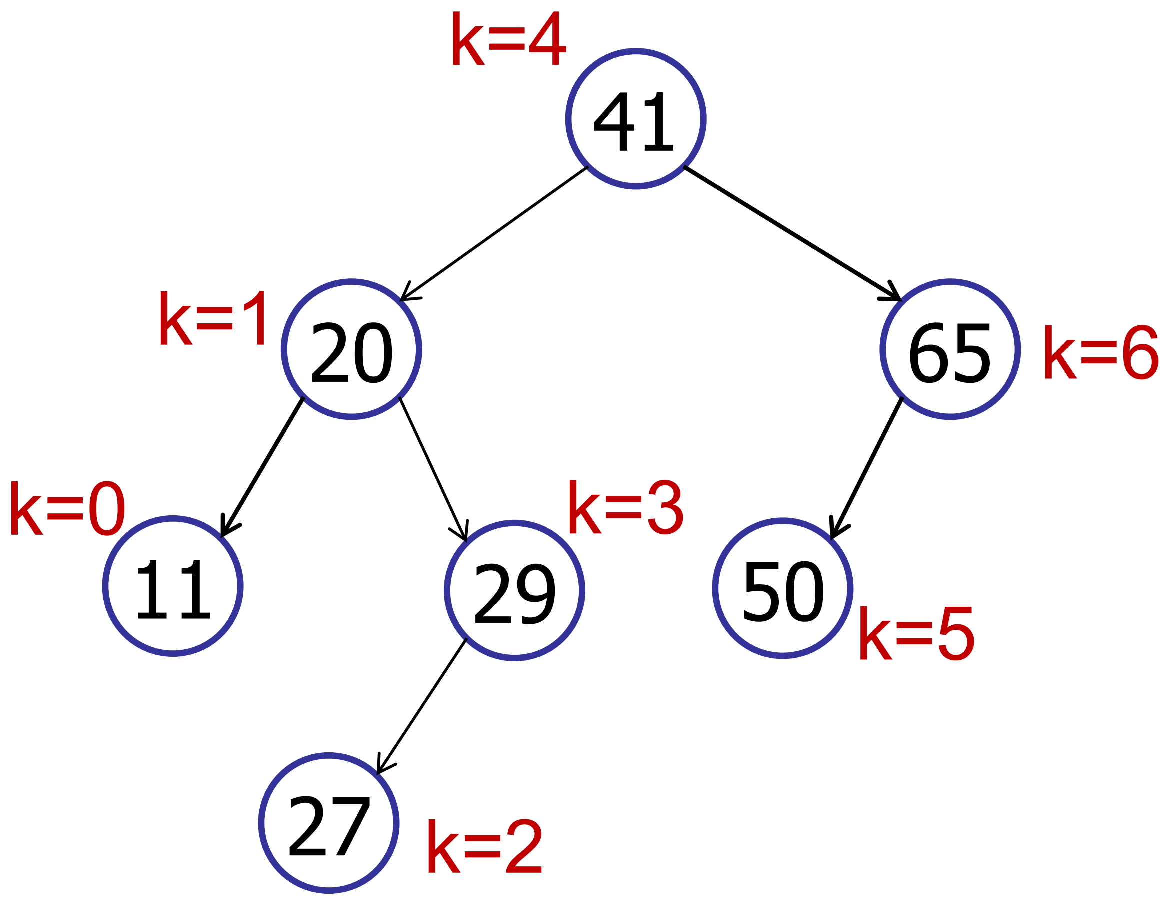
Dynamic Order Statistics

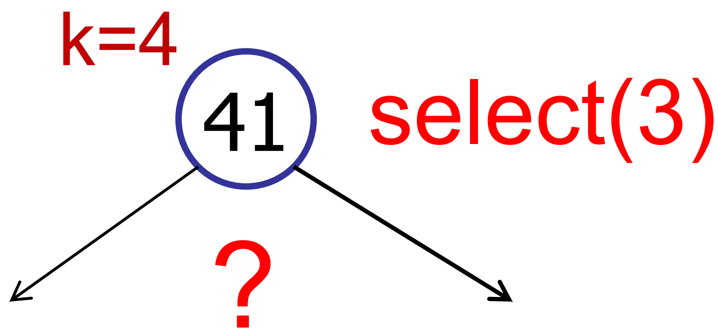
Idea: store rank in every node



11	**20**	**27**	**29**	**41**	**50**	**65**

Dynamic Order Statistics

Idea: store rank in every node

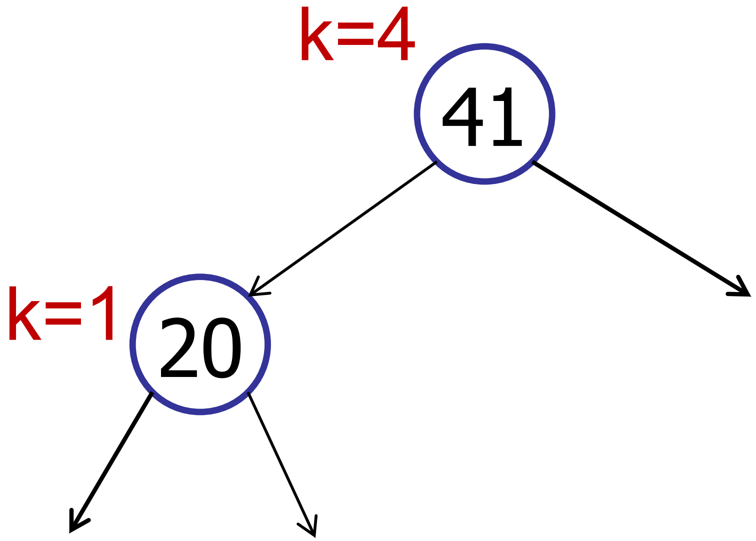


11	**20**	**27**	**29**	**41**	**50**	**65**

Dynamic Order Statistics

Idea: store rank in every node

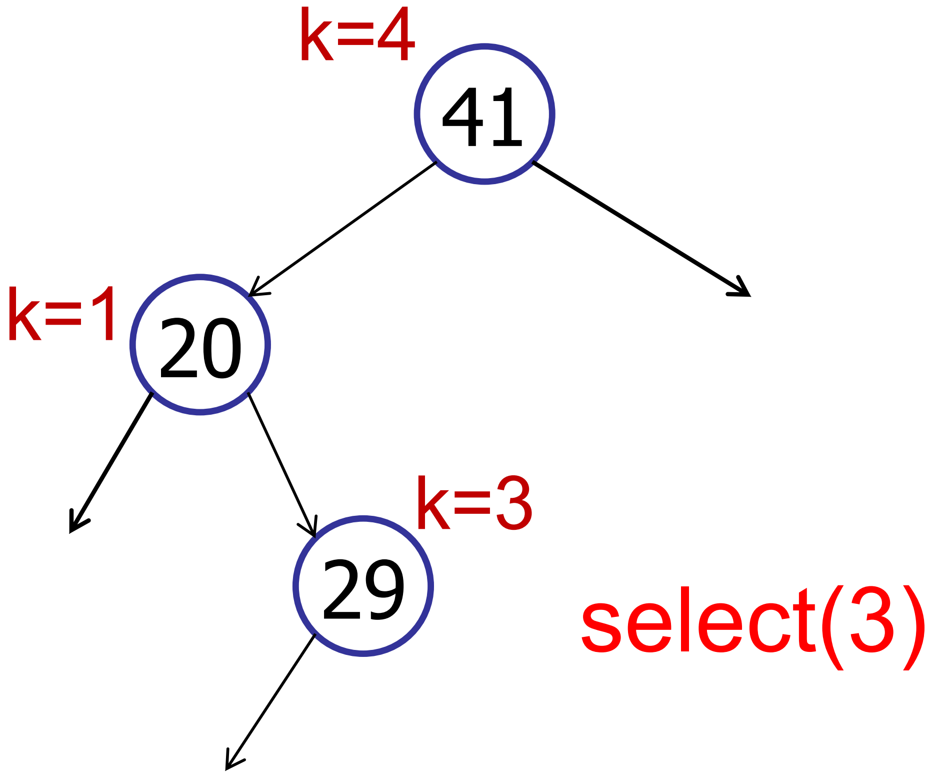
select(3)



11	**20**	**27**	**29**	**41**	**50**	**65**

Dynamic Order Statistics

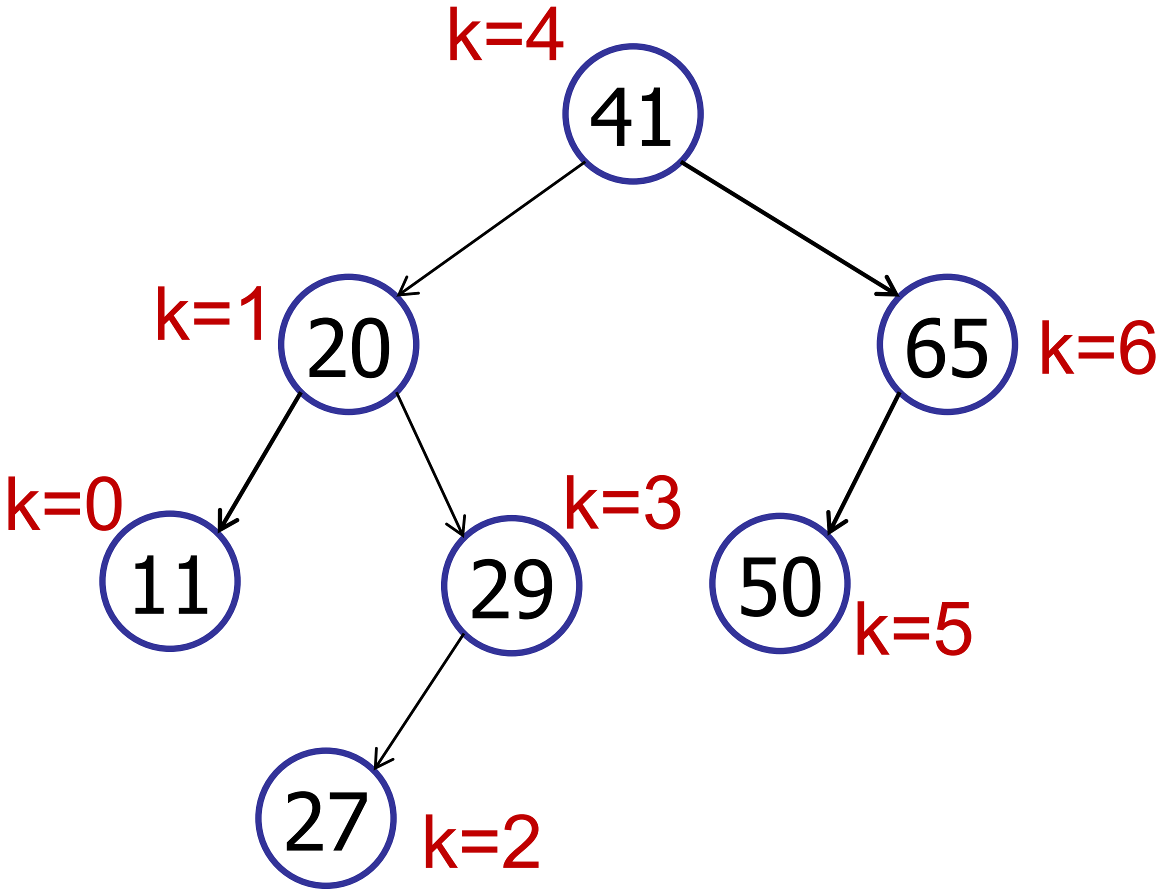
Idea: store rank in every node



11	**20**	**27**	**29**	**41**	**50**	**65**

Dynamic Order Statistics

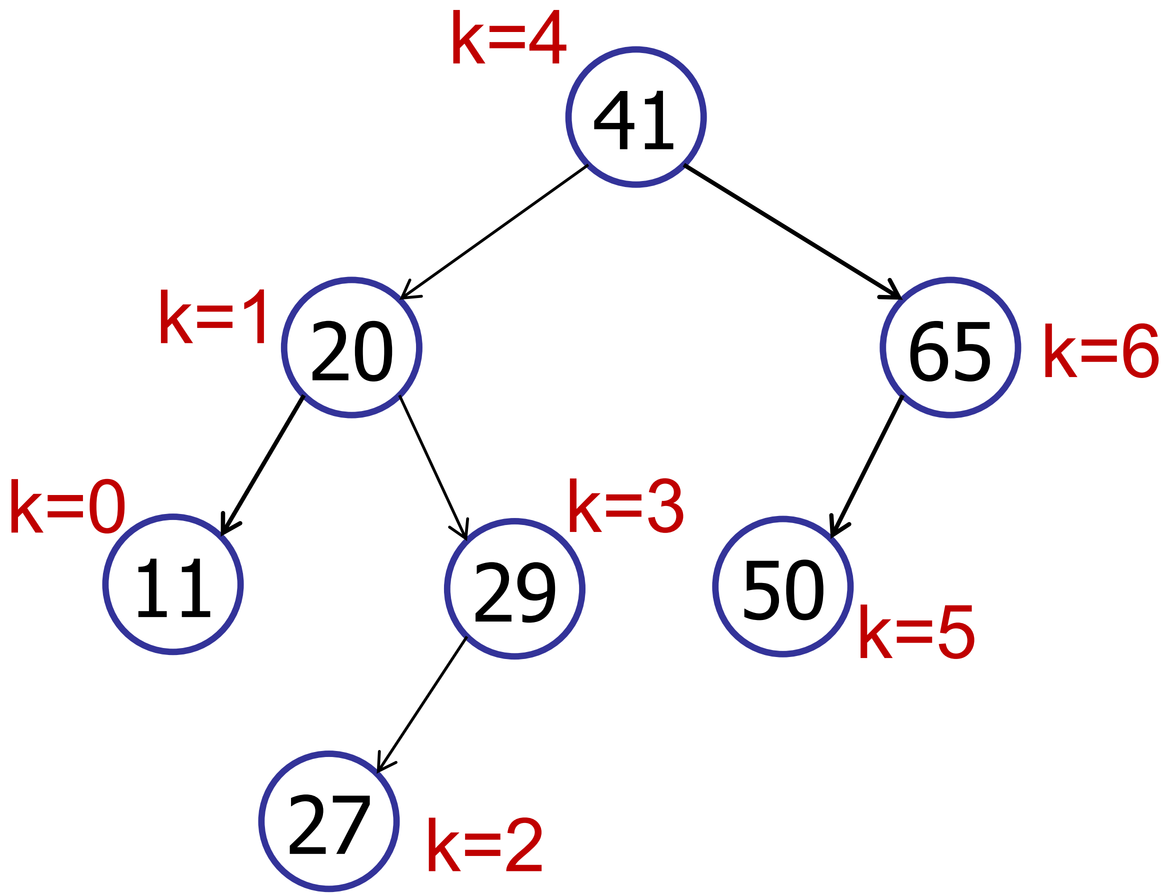
Idea: store rank in every node



11	**20**	**27**	**29**	**41**	**50**	**65**

Dynamic Order Statistics

Question: What goes wrong if you store ranks on every node??



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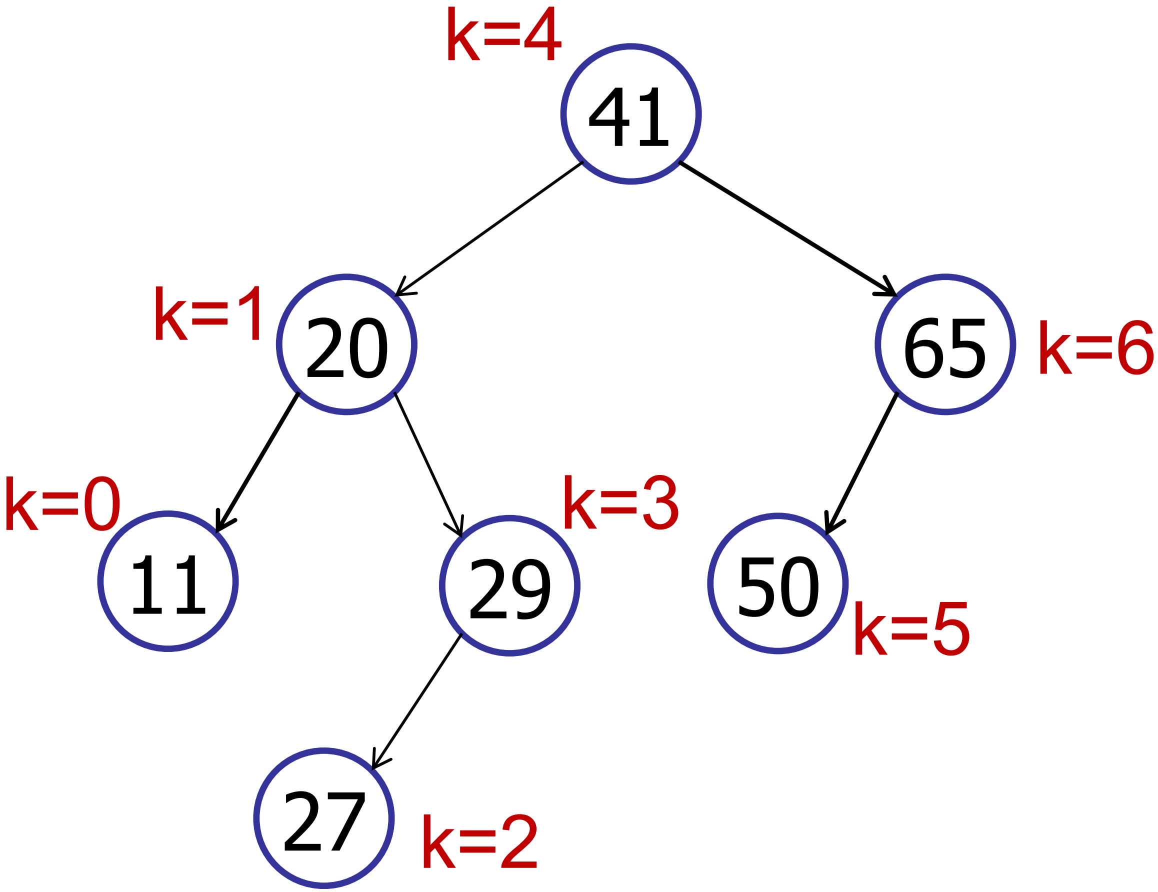
Dynamic Order Statistics

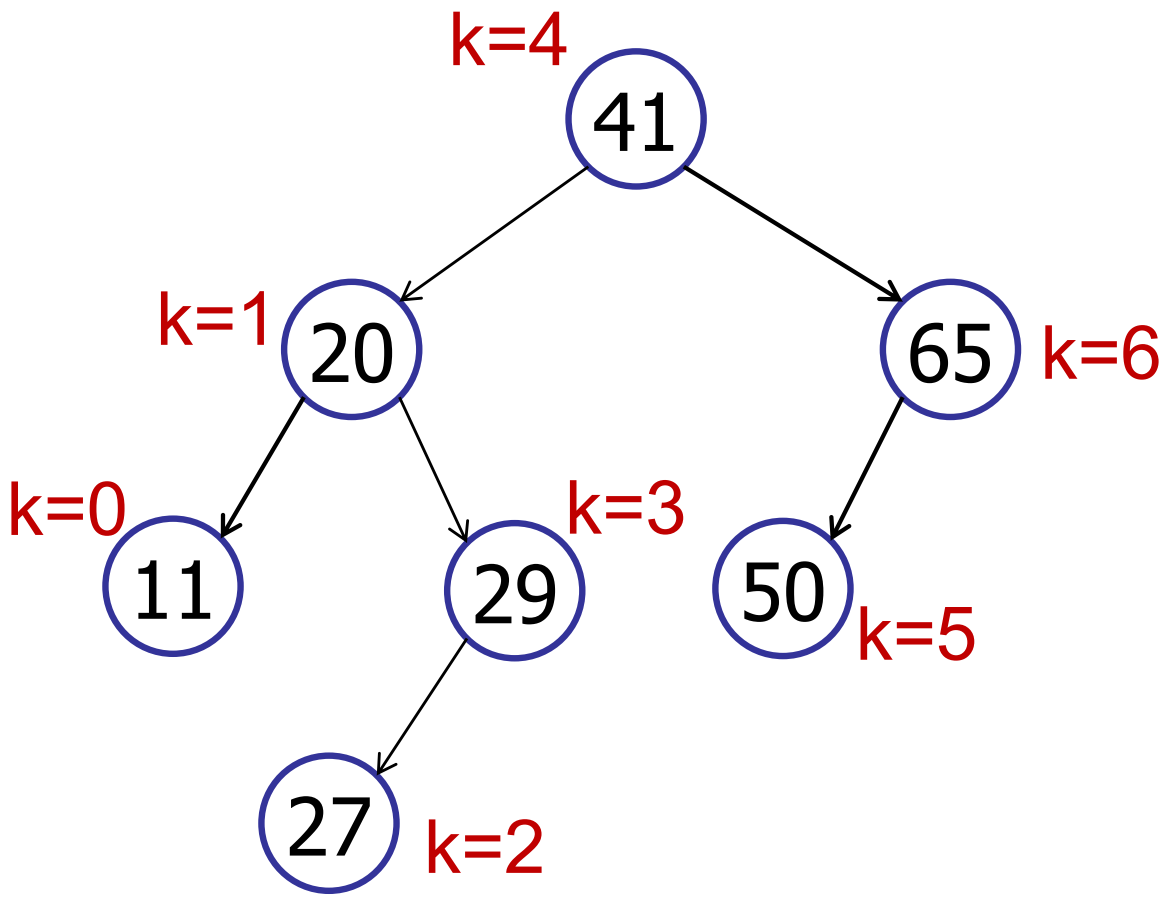
Idea: store rank in every node



Problem: insert(5)

Dynamic Order Statistics

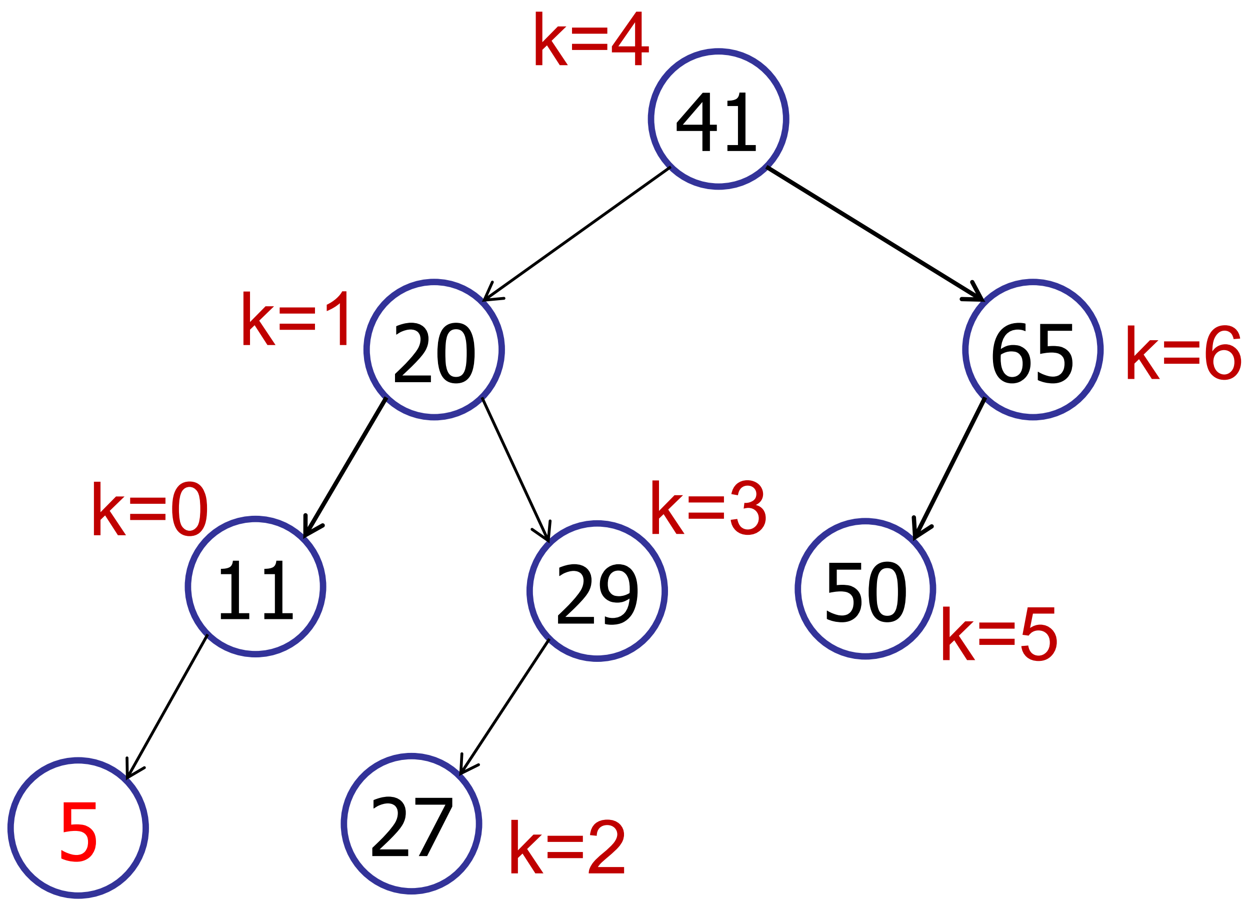
Idea: store rank in every node



Problem: insert(5) requires updating *all* the ranks!

Dynamic Order Statistics

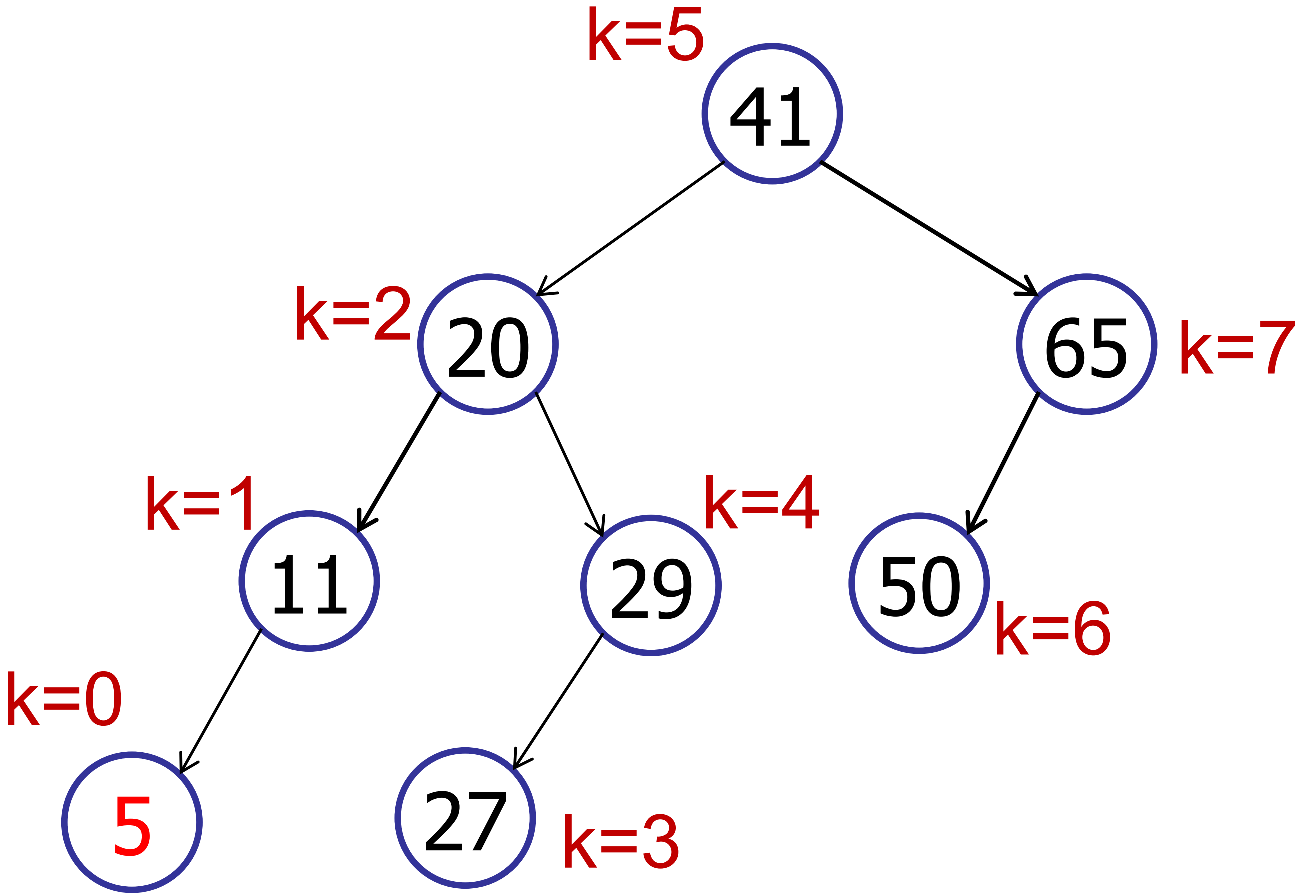
Idea: store rank in every node



5	**11**	**20**	**27**	**29**	**41**	**50**	**65**

Dynamic Order Statistics

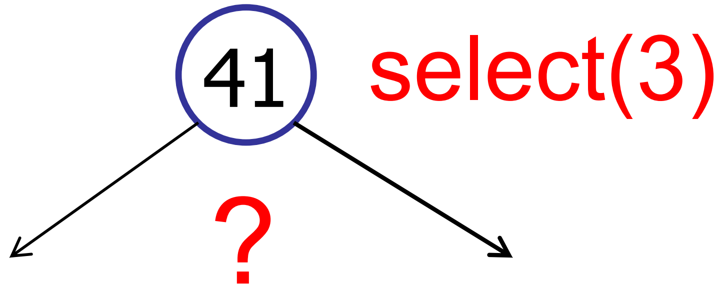
Conclusion: too expensive to store rank in every node!



5	**11**	**20**	**27**	**29**	**41**	**50**	**65**

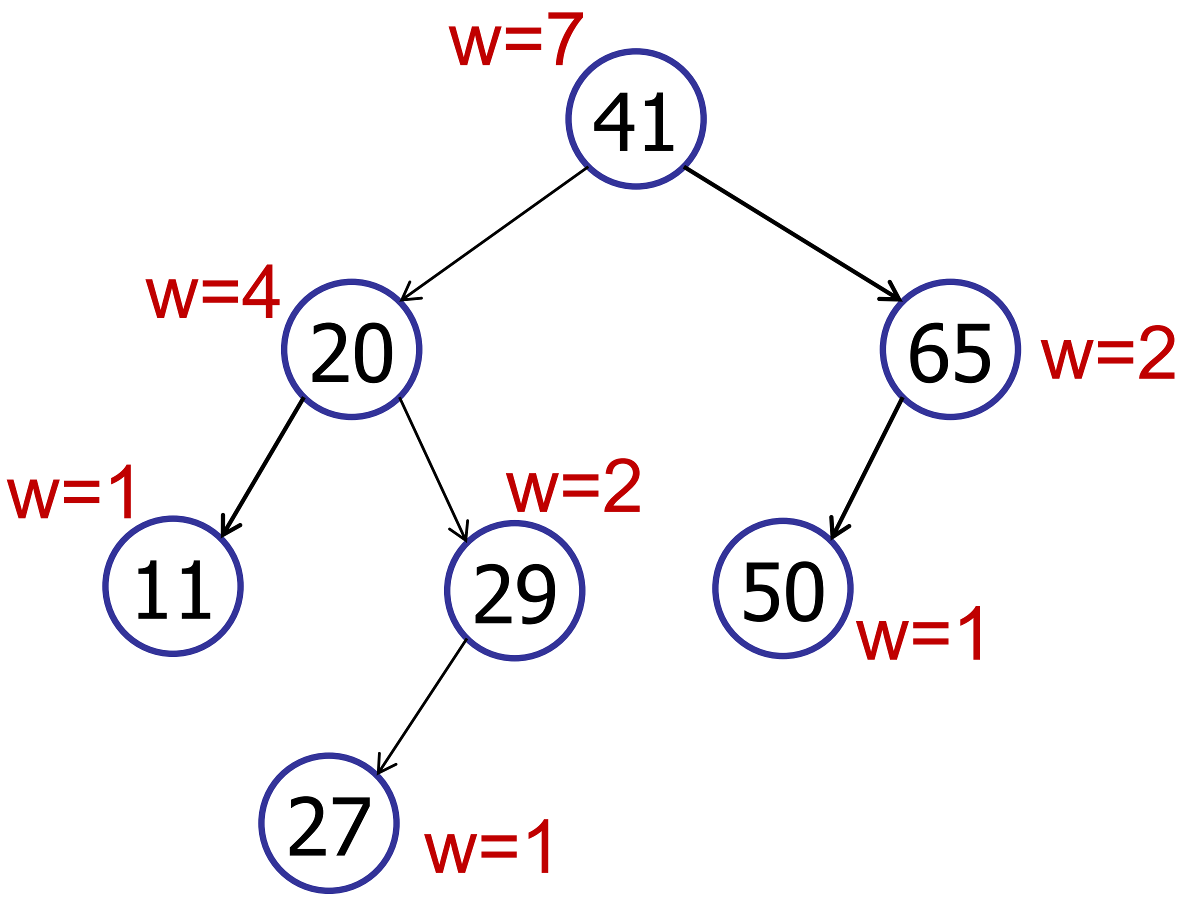
Dynamic Order Statistics

What should we store in each node?



Dynamic Order Statistics

Idea: store *size* of sub-tree in every node



Dynamic Order Statistics

Idea: store size of sub-tree in every node

The weight of a node is the size of the tree rooted at that node.

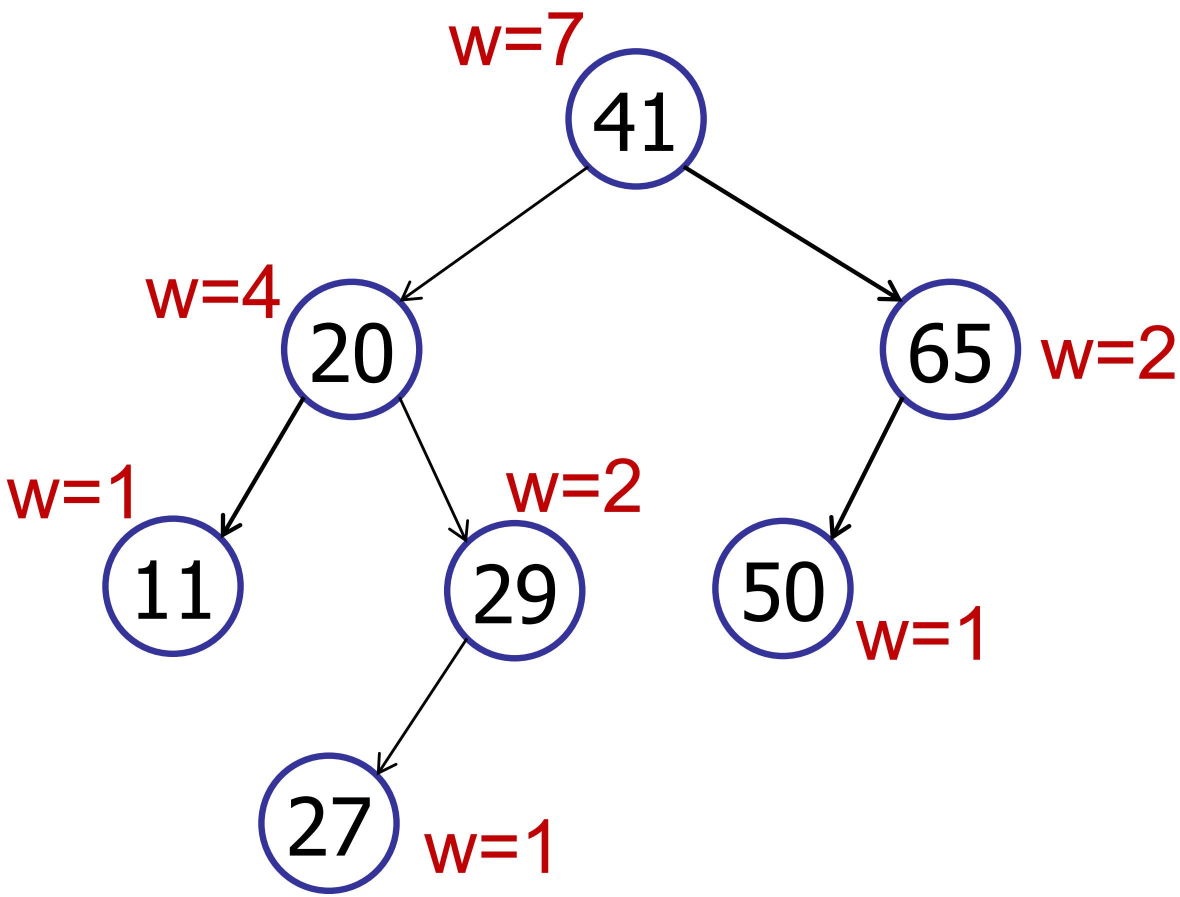
Define weight:

$w(\text{leaf}) = 1$

$w(v) = w(v.\text{left}) + w(v.\text{right}) + 1$

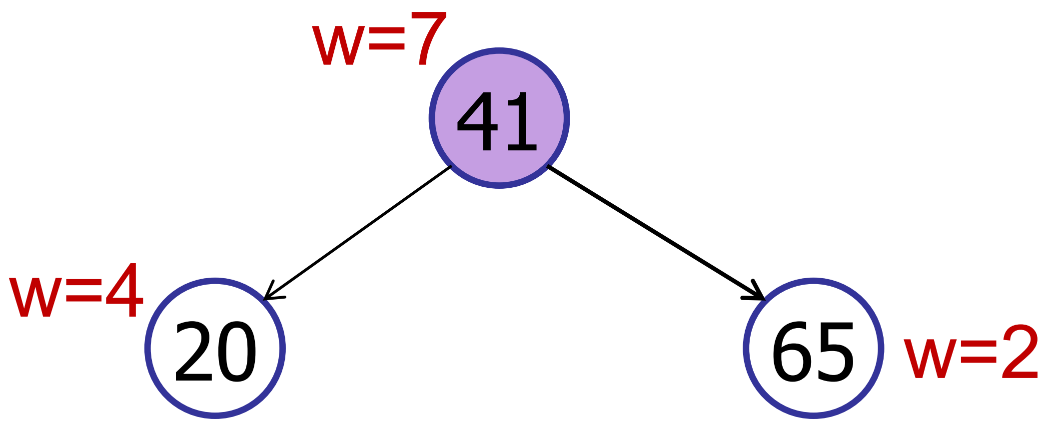
Dynamic Order Statistics

Idea: store *size* of sub-tree in every node



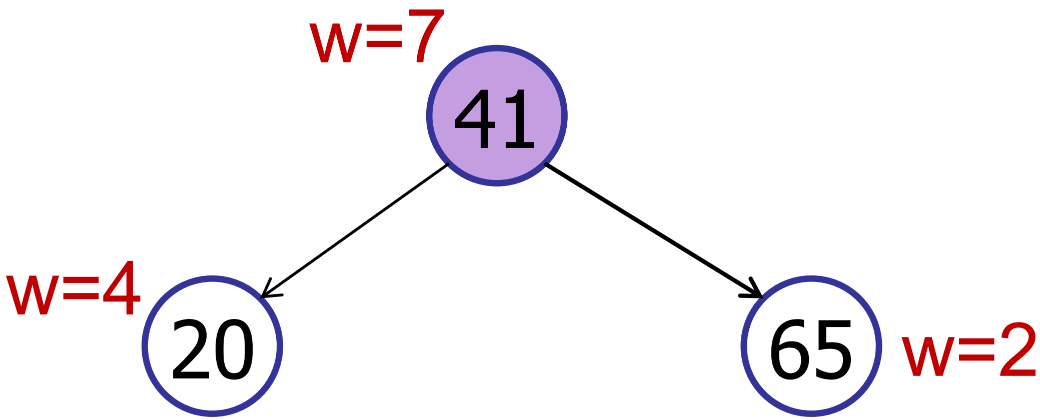
Dynamic Order Statistics

Example: select(3)



What is the rank of 41?

1. 1
2. 3
3. 5
4. 7
5. 9
6. Can’t tell.



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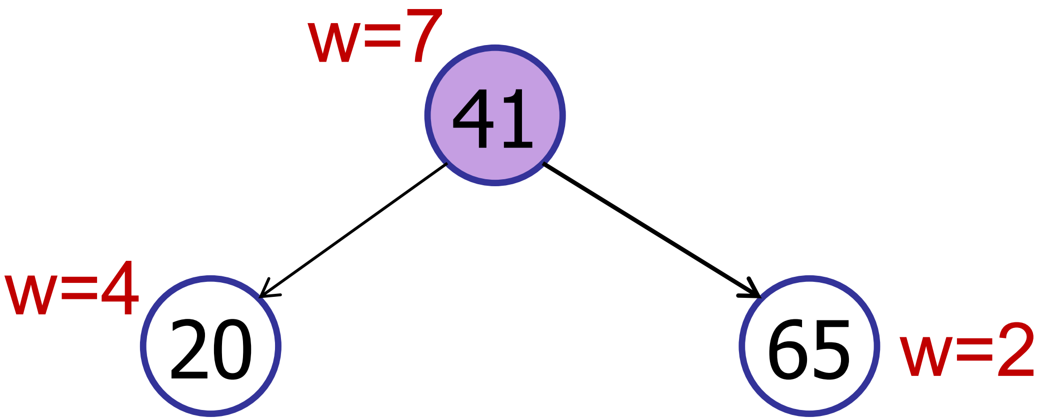
Dynamic Order Statistics

Example: select(3)



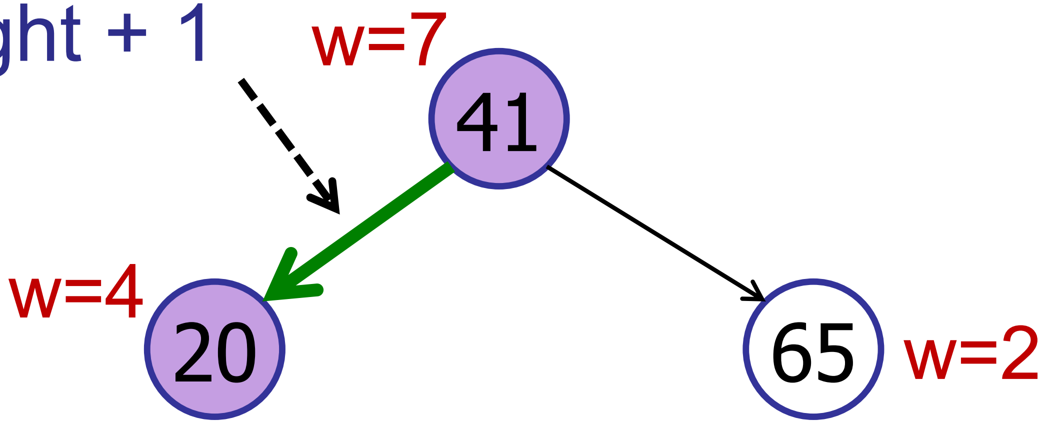
“rank in subtree” = left.weight + 1

Dynamic Order Statistics

Example: select(3)

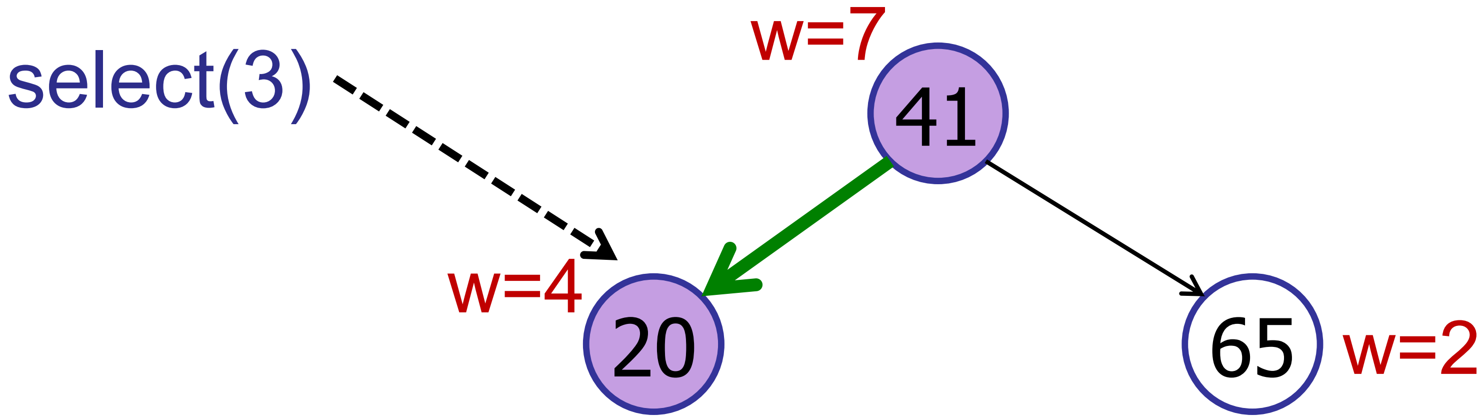
3 < left.weight + 1

Go left!



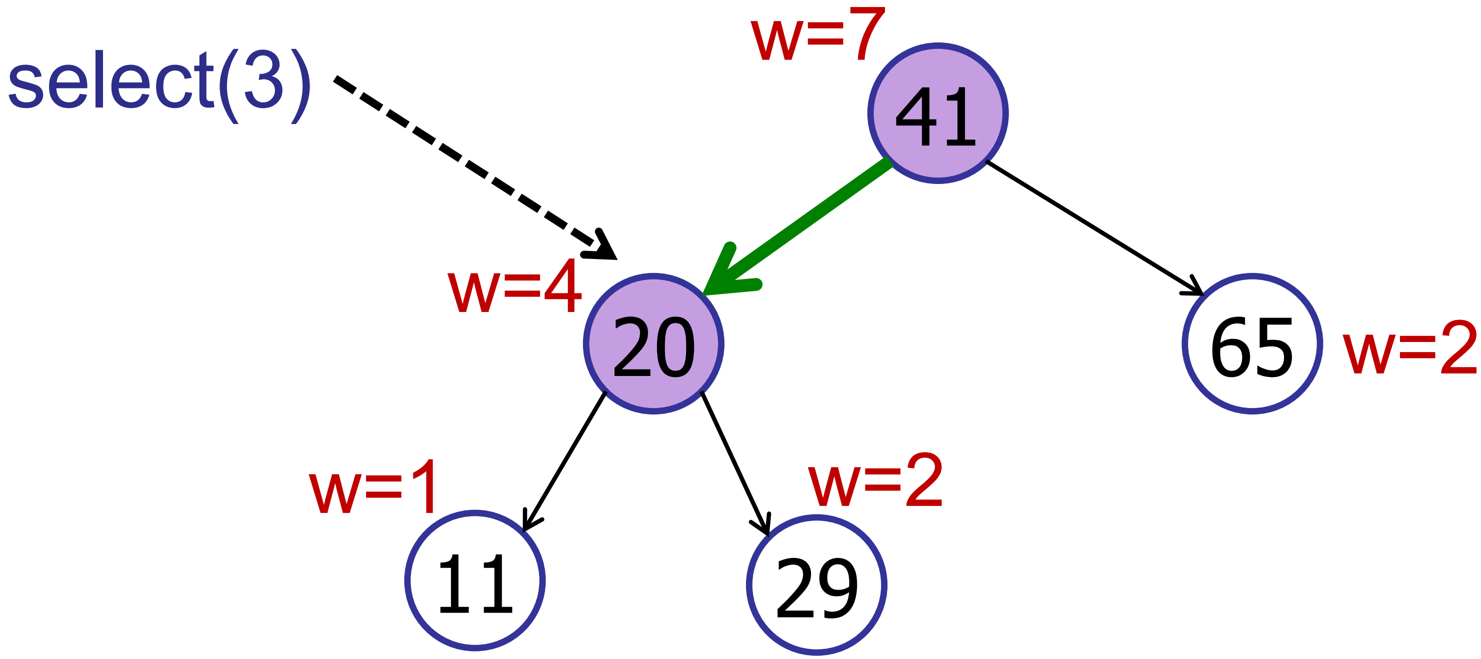
Dynamic Order Statistics

Example: select(3)



Dynamic Order Statistics

Example: select(3)

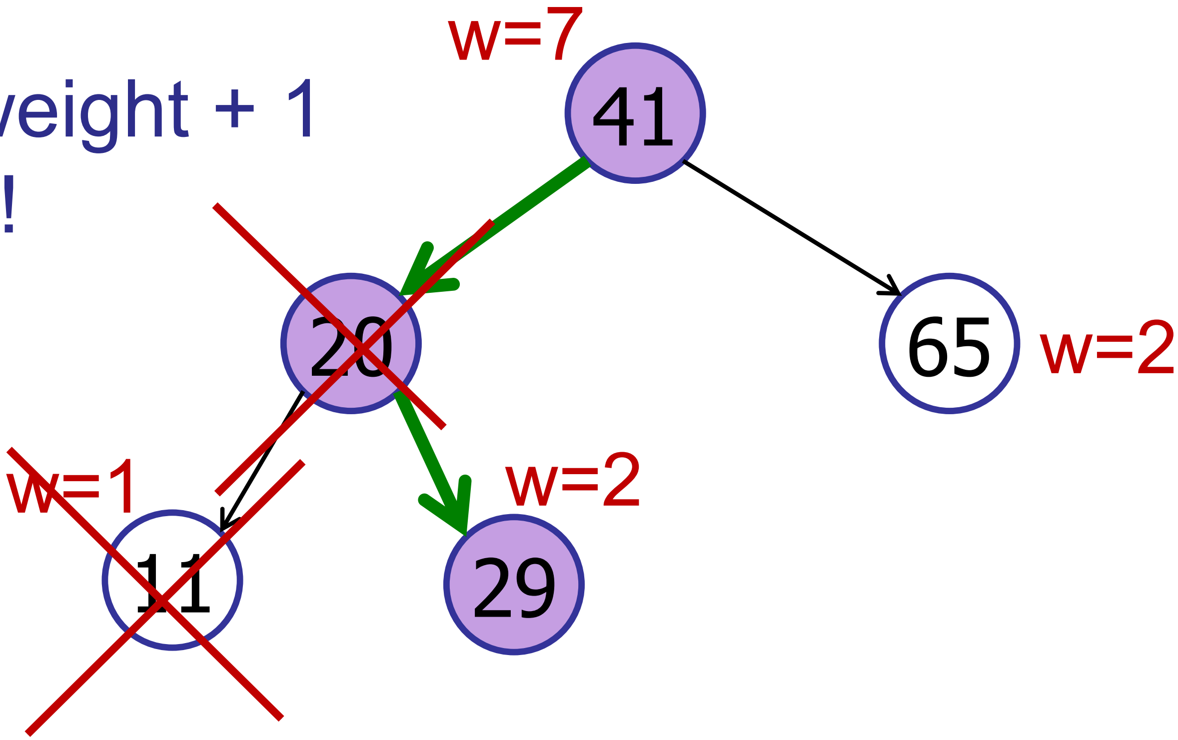


Dynamic Order Statistics

Example: select(3)

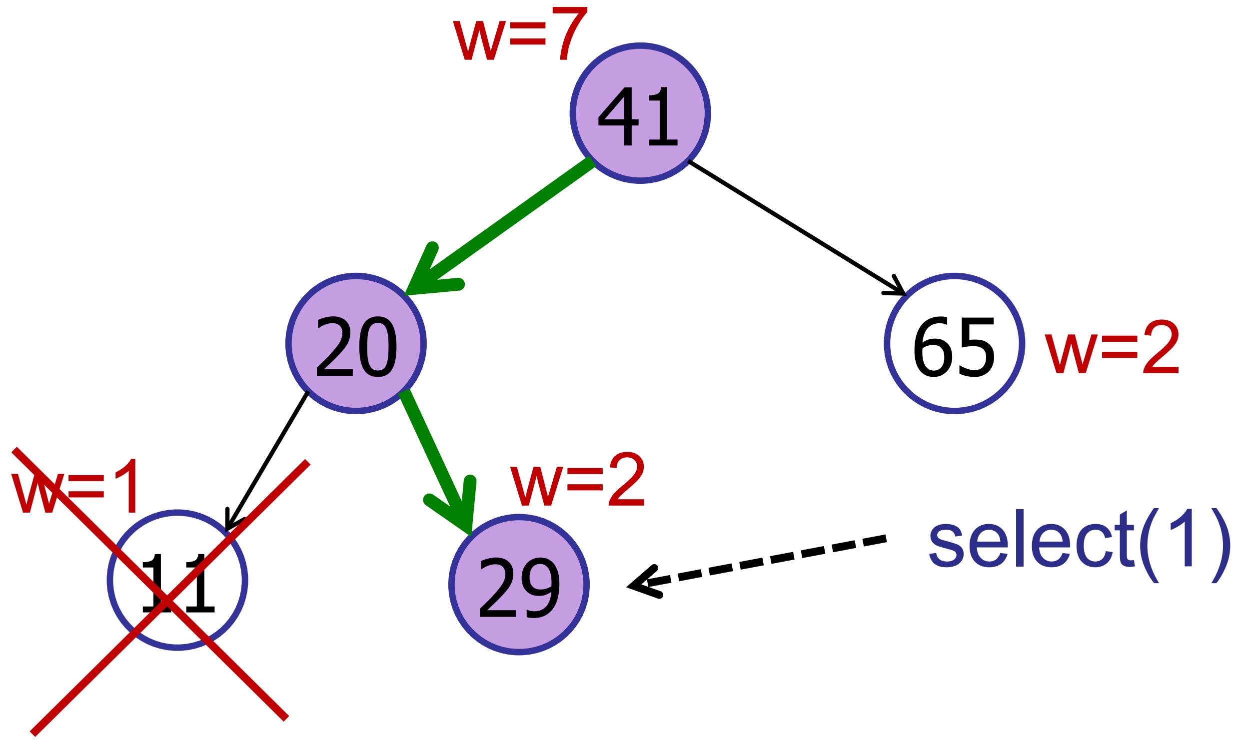
3 > left.weight + 1

Go right!



Dynamic Order Statistics

Example: select(3)



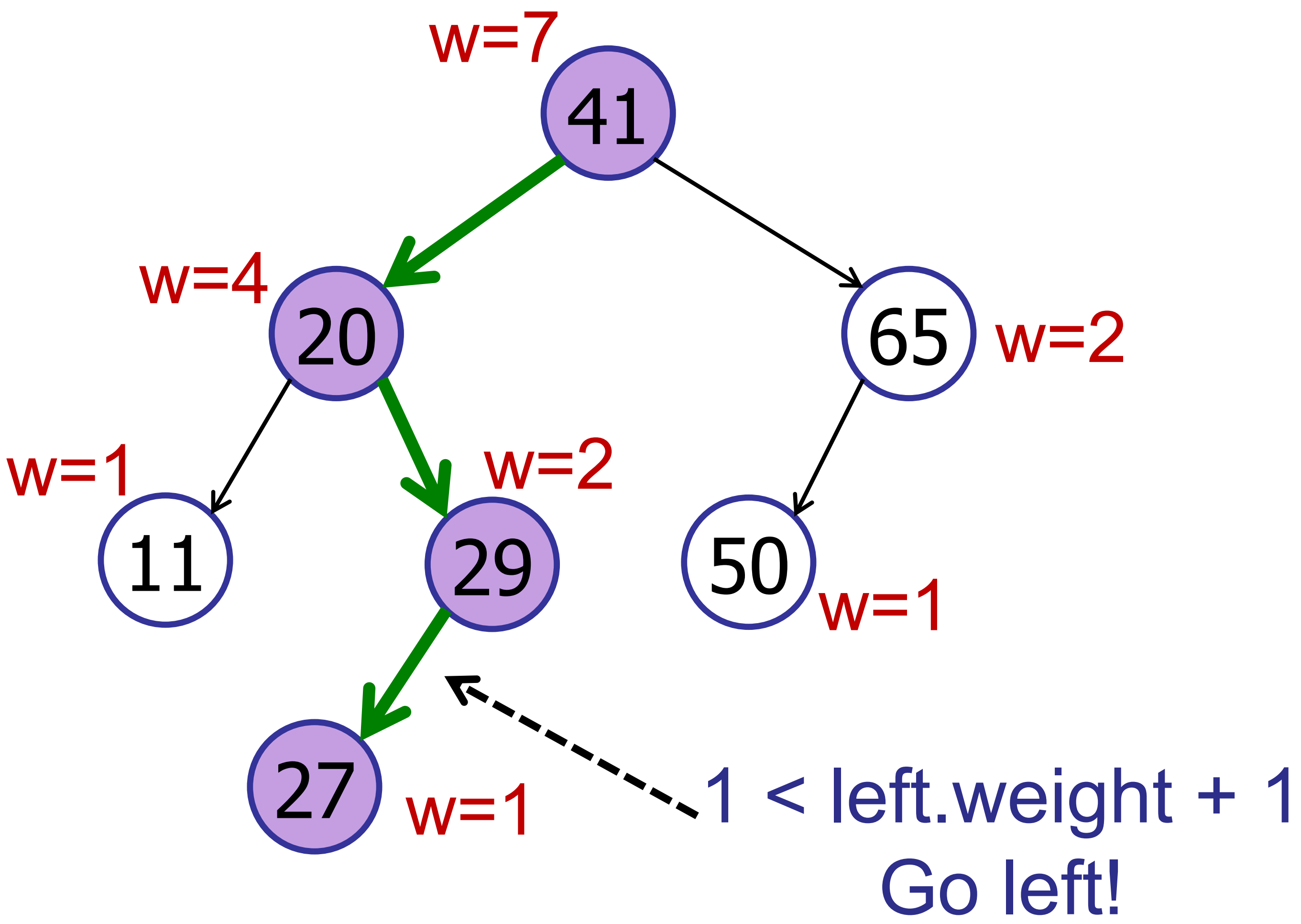
Item to select:

3 – (left.weight + 1) =

3 – (1 + 1) = 1

Dynamic Order Statistics

Example: select(3)

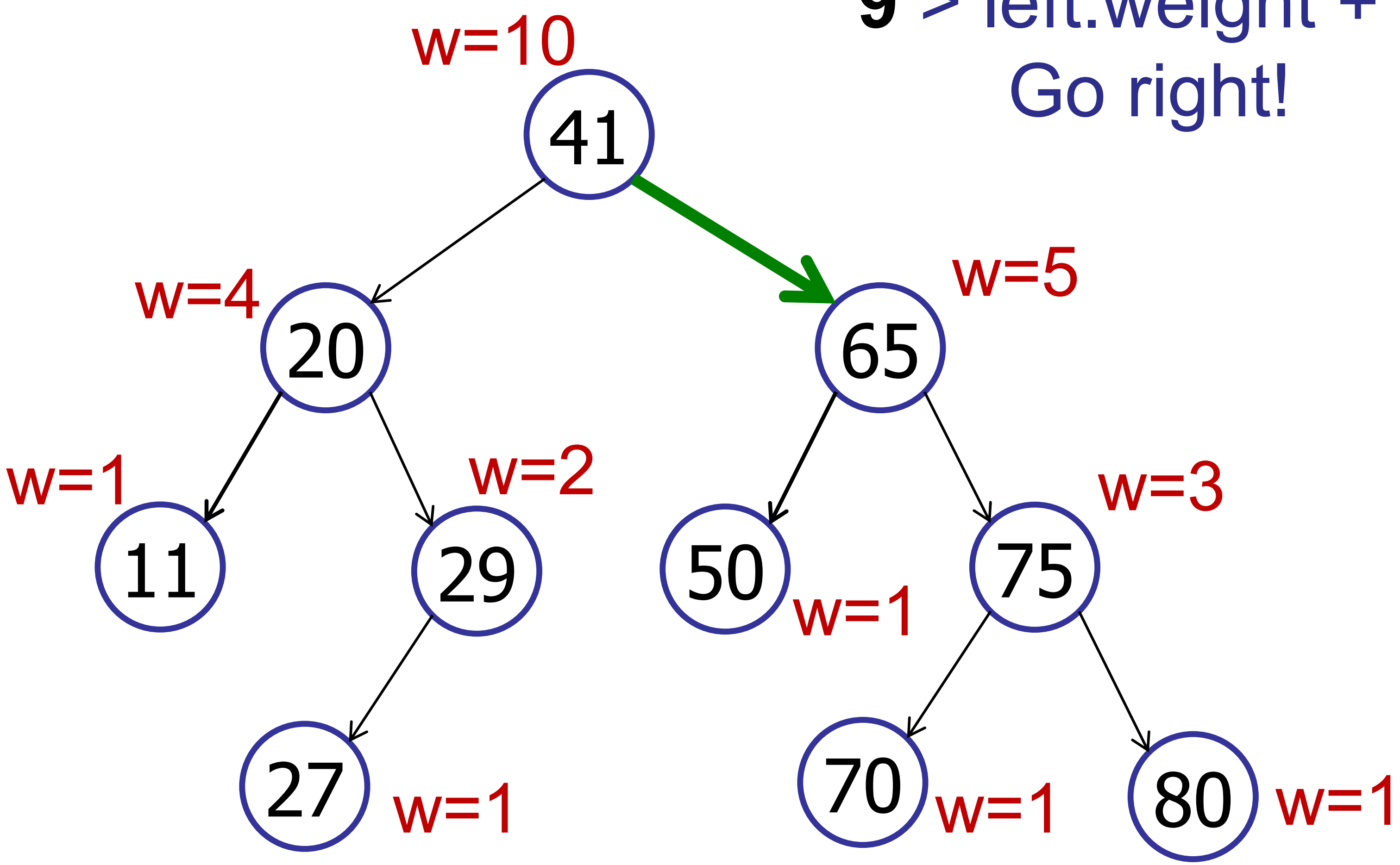


Dynamic Order Statistics

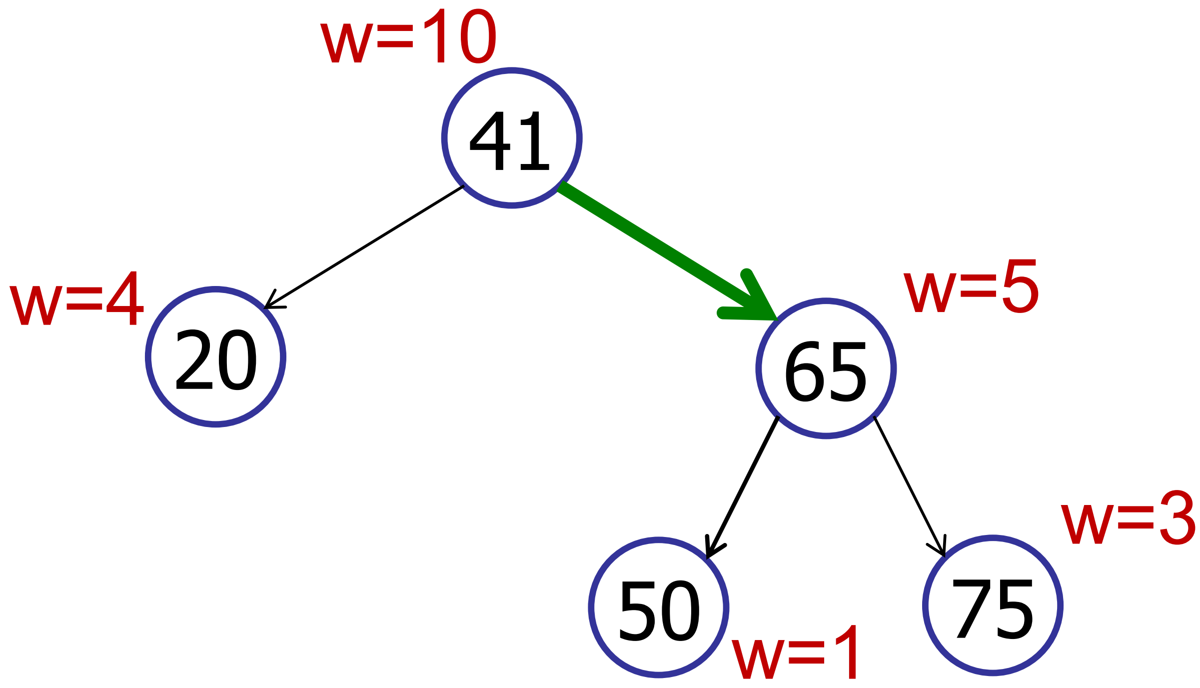
Example: select(9)

9 > left.weight + 1

Go right!



select(9)

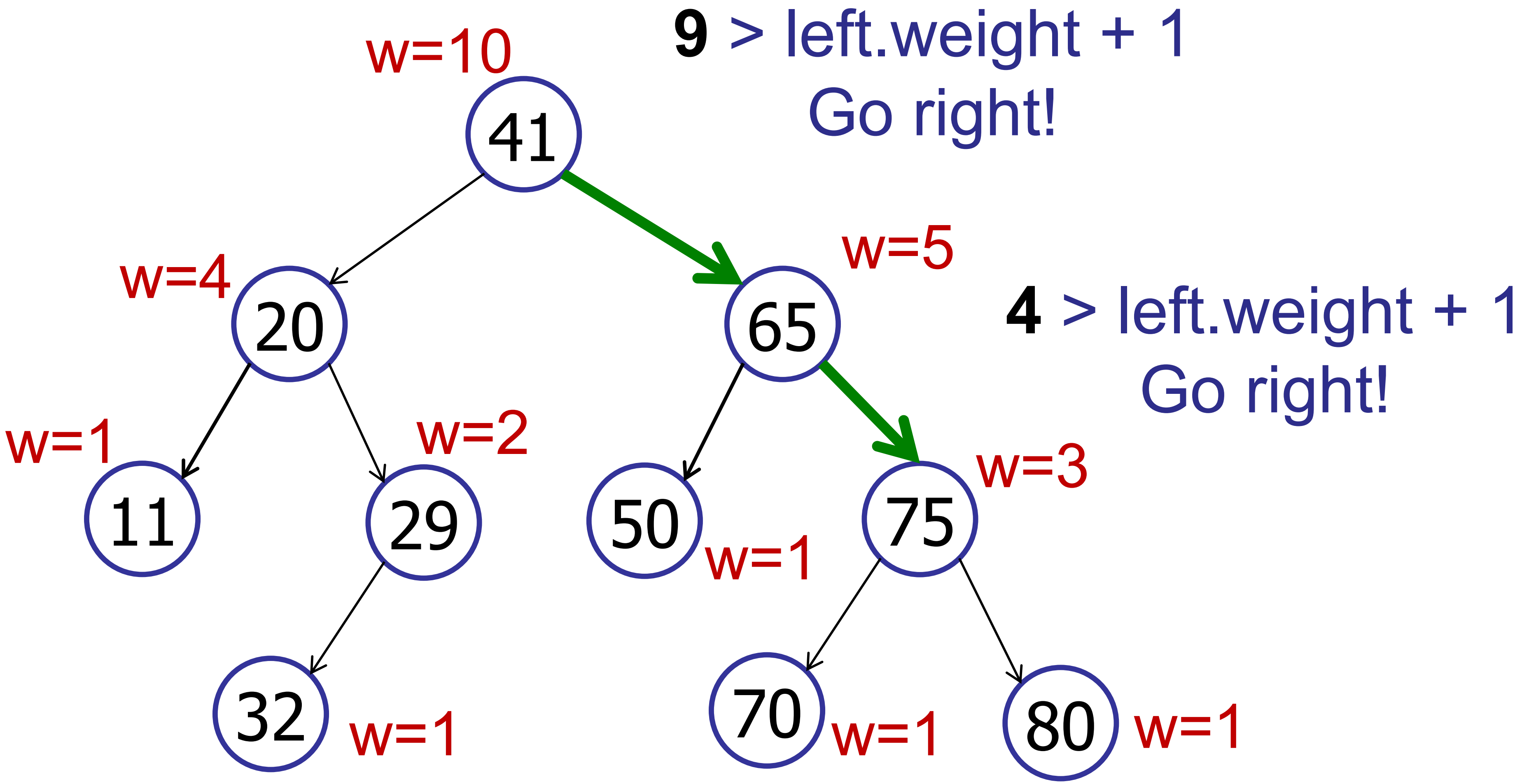


1. Go left at 65
2. Go right at 65
3. Stop at 65
4. I'm confused

ARCHIPELAGO
is open

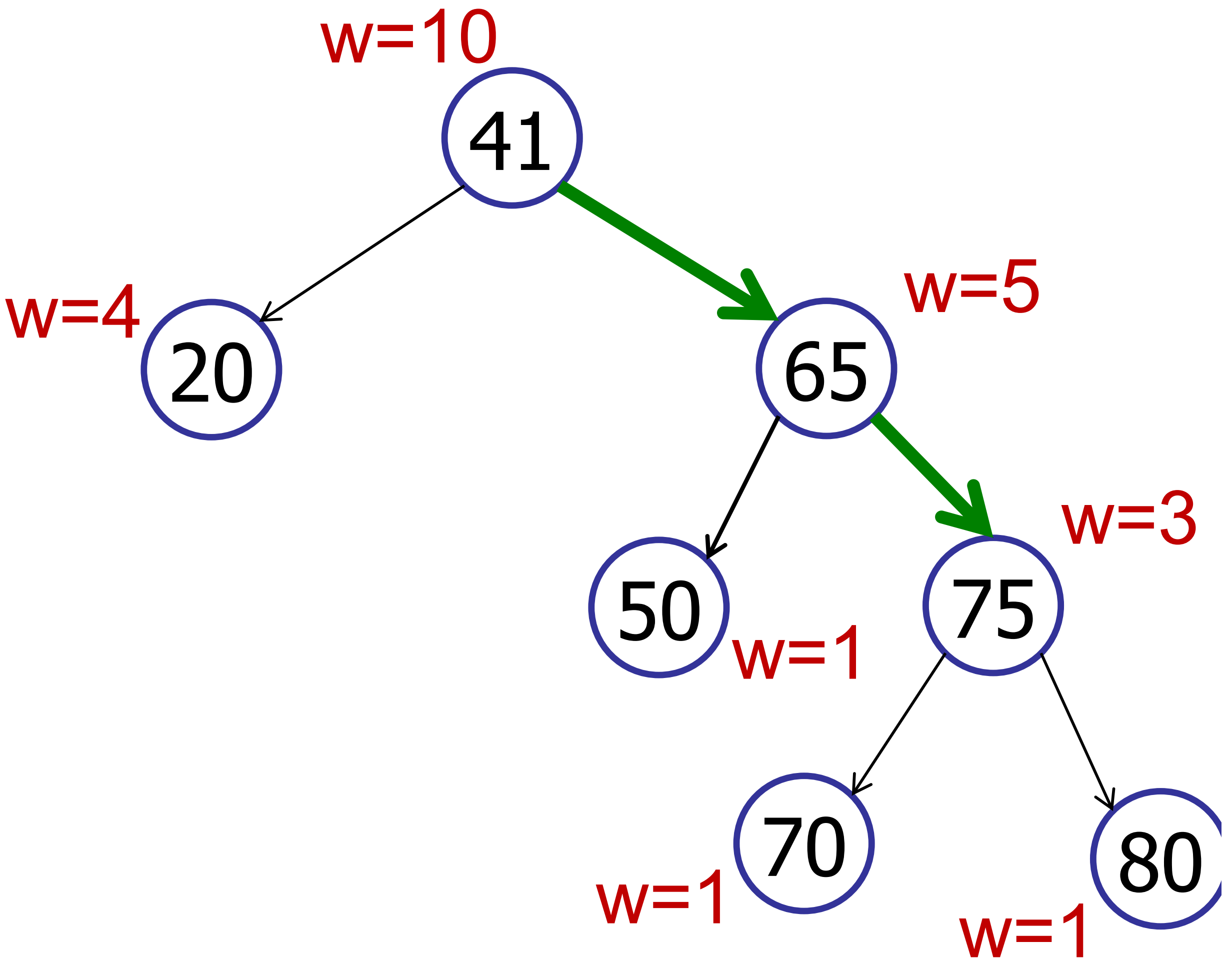
Dynamic Order Statistics

select(9)



select(9)

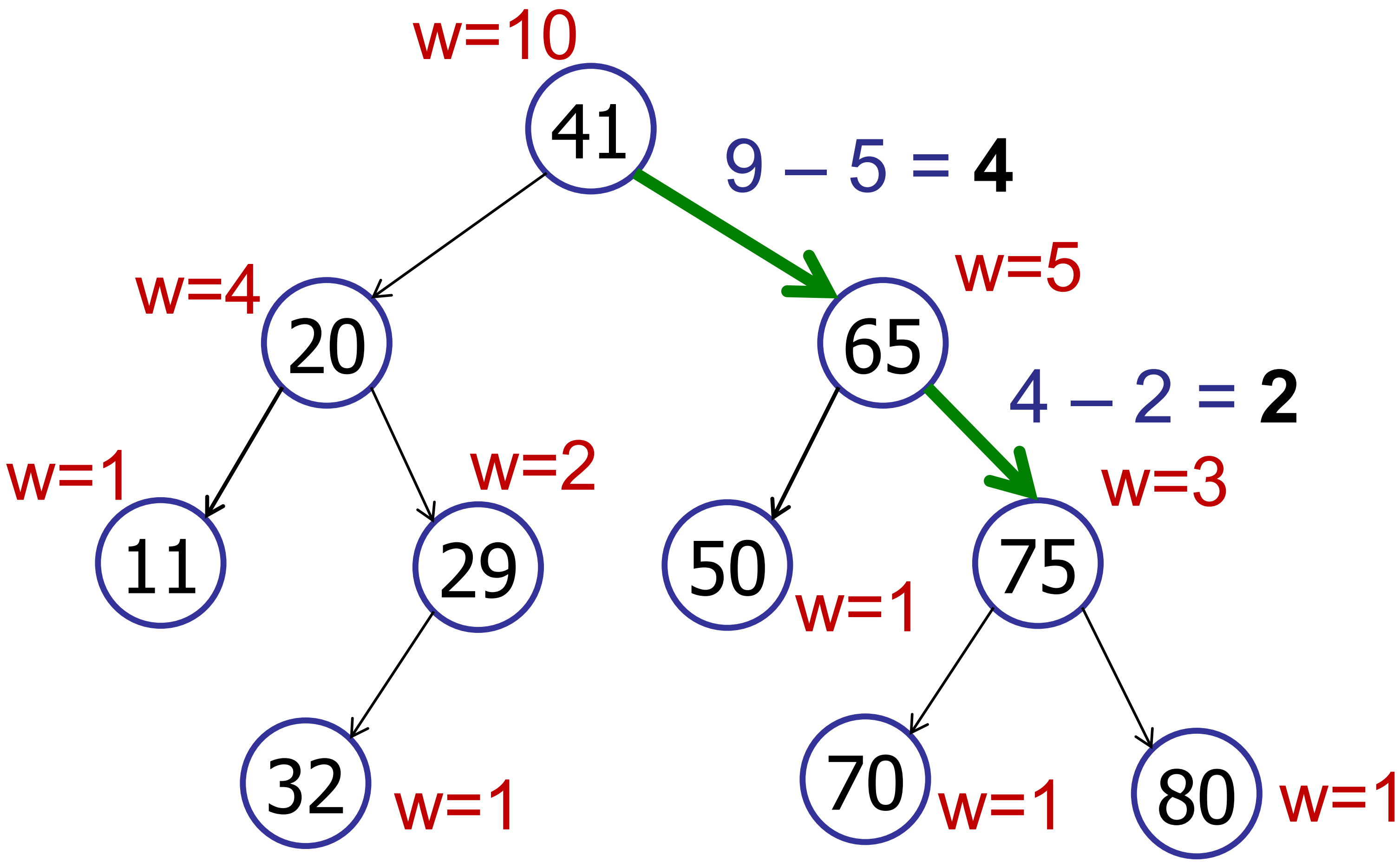
1. Go left at 75
2. Go right at 75
3. Stop at 75
4. I’m confused



ARCHIPELAGO
is open

Dynamic Order Statistics

select(9)



Dynamic Order Statistics

select(k)

```
rank = m_left.weight + 1;
if (k == rank) then
    return v;
else if (k < rank) then
    return m_left.select(k);
else if (k > rank) then
    return m_right.select(k–rank);
```

Dynamic Order Statistics

select(k) : finds the node with rank k

Example: find the 10th tallest student in the class.

Dynamic Order Statistics

select(k) : finds the node with rank k

Example: find the 10th tallest student in the class.

---

rank(v) : computes the rank of a node v

Example: determine the percentile of Johnny's height.
Is Johnny in the 10^{th} percentile or the 90^{th} percentile?