Exercise 3: 5, 7, 8, 10, 12 (a, d), 15 (b, e, g)

5. Let $A = \{\, (1+t, 1+2t, 1+3t) \mid t \in \mathbb{R} \,\}$ be a subset of $\mathbb{R}^3$.

(a) Describe A geometrically.

(b) Show that $A = \{\, (x, y, z) \mid x + y - z = 1 \text{ and } x - 2y + z = 0 \,\}$.

(c) Write down a matrix equation $\boldsymbol{M}\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{M}$ is a 3×3 matrix and $\boldsymbol{b}$ is a 3×1 matrix such that its solution set is A.

a) A line joining the points $(1,1,1)$ and $(2,3,4)$

b) In order to find a implicit form, we need to find two non-parallel planes $ax+by+cz=d$ containing the line. However, in this case, since $x+y-z=1$ and $x-2y+z=0$ are two non-parallel lines, it suffices to show that A lies on both planes.
This is true because $(1+t)+(1+2t)-(1+3t)=1$
$(1+t)-2(1+2t)+(1+3t)=0$

c) Using $x+y-z=1$, $x-2y+z=0$.

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The third equation is not given, hence set to 0.

7. Let P represent a plane in $\mathbb{R}^3$ with equation $x - y + z = 1$ and A, B, C represent three different lines given by the following set notation:

$$A = \{\, (a, a, 1) \mid a \in \mathbb{R} \,\}, \quad B = \{\, (b, 0, 0) \mid b \in \mathbb{R} \,\}, \quad C = \{\, (c, 0, -c) \mid c \in \mathbb{R} \,\}.$$

(a) Express the plane P in explicit set notation.

$x-y+z=1$, its general solution is in the explicit form.

$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \end{array}\right)$ Since, 2^{nd} and 3^{rd} column are non pivot column, set them as arbitary parameters $y=s$, $z=t$

$x-s+t=1$
$x=1+s-t$, $y=s$, $z=t$ where $s,t \in \mathbb{R}$

in set notation: $\{(1+s-t, s, t) \mid s,t \in \mathbb{R}\}$

(b) Does any of the three lines above lie completely on the plane P? Briefly explain your answer.

$A = \{\, (a, a, 1) \mid a \in \mathbb{R} \,\}, \quad B = \{\, (b, 0, 0) \mid b \in \mathbb{R} \,\}, \quad C = \{\, (c, 0, -c) \mid c \in \mathbb{R} \,\}.$

P represent a plane in $\mathbb{R}^3$ with equation $x - y + z = 1$:

A lies on P as $a-a+1=1$
B does not lie on P as $b-0+0 \neq 1$ for some b
C does not lie on P as $c-0-c \neq 1$

(c) Find all the points of intersection of the line B with the plane P.

$A = \{(a,a,1) \mid a \in \mathbb{R}\}$, $B = \{(b,0,0) \mid b \in \mathbb{R}\}$, $C = \{(c,0,-c) \mid c \in \mathbb{R}\}$.

P represent a plane in $\mathbb{R}^3$ with equation $x - y + z = 1$:

B only intersects P when $b=0$. ie Point $(1,0,0)$

(d) Find the equation of another plane that is parallel to (but not overlapping) the plane P, and contains exactly one of the three lines above.

From $x-y+z=1$, the normal vector: $\begin{pmatrix}1\\-1\\1\end{pmatrix}$

$(x-c)-(y-0)+(z+c)=0$

$x-c-y+z+c=0$

$x-y+z=0$ Contains C but not A and B

Alternative solution:

$x-y+z=e$, $C \subseteq$ this plane

Fix $(c,0,-c) \in C$

$c-0+(-c)=e$

$e=0$

(e) Can you find a nonzero linear system whose solution set contains all the three lines? Justify your answer.

No, the solution set of a constant nonzero linear system in three variables represents a point, a line or a plane in $\mathbb{R}^3$. Suppose we have a nonzero linear system whose solution set containes both B and C. Then, the solution set must be a plane. However, the plane containing both B and C is the xz plane which does not contain A. So the solution set cannot contain A.

Have a look at Discussion 1.4.11

8. Let $\boldsymbol{u_1} = (2,1,0,3)$, $\boldsymbol{u_2} = (3,-1,5,2)$, and $\boldsymbol{u_3} = (-1,0,2,1)$. Which of the following vectors are linear combinations of $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, $\boldsymbol{u_3}$?

(a) $(2,3,-7,3)$, (b) $(0,0,0,0)$, (c) $(1,1,1,1)$, (d) $(-4,6,-13,4)$.

$a(2,1,0,3) + b(3,-1,5,2) + c(-1,0,2,1)$

$= (2a+3b-c,\ a-b,\ 5b+2c,\ 3a+2b+c)$

a) $\left(\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{array}\right) \xrightarrow{\text{Gaussian elimination}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ Consistent

b and d are consistent using the same method but not c

Hence, $(2,3,-7,3)$, $(0,0,0,0)$, $(-4,6,-13,4)$ are vectors in Span $\{u_1, u_2, u_3\}$

While $(1,1,1,1)$ is not

10. Let $V = \{ (x, y, z) \mid x - y - z = 0 \}$ be a subset of $\mathbb{R}^3$.

(a) Let $S = \{(1, 1, 0), (5, 2, 3)\}$. Show that $\text{span}(S) = V$.

(b) Let $S' = \{(1, 1, 0), (5, 2, 3), (0, 0, 1)\}$. Show that $\text{span}(S') = \mathbb{R}^3$.

a) To show $\text{Span}(V) = V$, $\text{Span}(V) \subseteq V$ and $\text{Span}(V) \supseteq V$

Since $(1,1,0)$ and $(5,2,3)$ satisfy the equation $x-y-z=0$
$(1,1,0), (5,2,3) \in V$ and hence $\text{Span}(S) \subseteq V$

The explicit form of V is $x = s+t$, $y = s$, $z = t$ where $s, t \in \mathbb{R}$
Let $(s+t, s, t)$ be any vector in V. Considering the following equation:

$$a(1,1,0) + b(5,2,3) = (s+t, s, t) \iff \begin{cases} a + 5b = s+t \\ a + 2b = s \\ 3b = t \end{cases}$$

$$\left(\begin{array}{cc|c} 1 & 5 & s+t \\ 1 & 2 & s \\ 0 & 3 & t \end{array}\right) \xrightarrow[\text{elimination}]{\text{Gaussian}} \left(\begin{array}{cc|c} 1 & 5 & s+t \\ 0 & 3 & t \\ 0 & 0 & 0 \end{array}\right)$$

The System is consistent for all $s, t \in \mathbb{R}$. So $V \subseteq \text{Span}(S)$
Therefore, $\text{Span}(S) = V$

b) $$\begin{pmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix} \xrightarrow[\text{elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 5 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The System is Consistent. $V \subseteq \text{Span}(S')$
$V \supseteq \text{Span}(S')$ is done as a) linearly independent

12. Let $\boldsymbol{u_1} = (2, 0, 2, -4)$, $\boldsymbol{u_2} = (1, 0, 2, 5)$, $\boldsymbol{u_3} = (0, 3, 6, 9)$, $\boldsymbol{u_4} = (1, 1, 2, -1)$, $\boldsymbol{v_1} = (-1, 2, 1, 0)$, $\boldsymbol{v_2} = (3, 1, 4, 0)$, $\boldsymbol{v_3} = (0, 1, 1, 3)$, $\boldsymbol{v_4} = (-4, 3, -1, 6)$. Determine if the following are true.

(a) $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} \subseteq \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$.

(b) $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\} \subseteq \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$.

(c) $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} = \mathbb{R}^4$.

(d) $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\} = \mathbb{R}^4$.

a) $$\left(\begin{array}{cccc|c} -1 & 3 & 0 & -4 & 1 \\ 2 & 1 & 1 & 3 & 0 \\ 1 & 4 & 1 & -1 & 2 \\ 0 & 0 & 3 & 6 & 5 \end{array}\right) \xrightarrow[\text{elimination}]{\text{Gaussian}} \left(\begin{array}{cccc|c} -1 & 3 & 0 & -4 & 1 \\ 0 & 7 & 1 & -5 & 2 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

The System is inconsistent. Since $u_2 \notin \text{Span}\{v_1, v_2, v_3, v_4\}$, $\text{Span}\{u_1, u_2, u_3, u_4\} \not\subseteq \text{Span}\{v_1, v_2, v_3, v_4\}$

d) As shown in a), the system is inconsistent
Hence $\text{span}\{v_1, v_2, v_3, v_4\} \neq \mathbb{R}^4$

16. Determine which of the following are subspaces of $\mathbb{R}^4$. Justify your answers.

(a) $\{(w, x, y, z) \mid w + x = y + z\}$.

(b) $\{(w, x, y, z) \mid wx = yz\}$.

(c) $\{(w, x, y, z) \mid w + x + y = z^2\}$.

(d) $\{(w, x, y, z) \mid w = 0 \text{ and } y = 0\}$.

(e) $\{(w, x, y, z) \mid w = 0 \text{ or } y = 0\}$.

(f) $\{(w, x, y, z) \mid w = 1 \text{ and } y = 0\}$.

(g) $\{(w, x, y, z) \mid w + z = 0 \text{ and } x + y - 4z = 0 \text{ and } 4w + y - z = 0\}$.

(h) $\{(w, x, y, z) \mid w + z = 0 \text{ or } x + y - 4z = 0 \text{ or } 4w + y - z = 0\}$.

b) No $(1,0,0,1)$ and $(0,2,0,1)$ belong to the set
but $(1,0,0,1) + (0,2,0,1) = (1,2,0,1)$ does not

$wx = yz$ is not a linear equation.
This gives a hint.

e) No. $(1,0,0,0)$ and $(0,0,1,0)$ belong to the set but $(1,0,0,0) + (0,0,1,0) = (1,0,1,0)$ does not

g) Yes. It is a solution set of a homogeneous linear system

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -4 & 0 \\ 0 & 1 & 4 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$ is consistent.