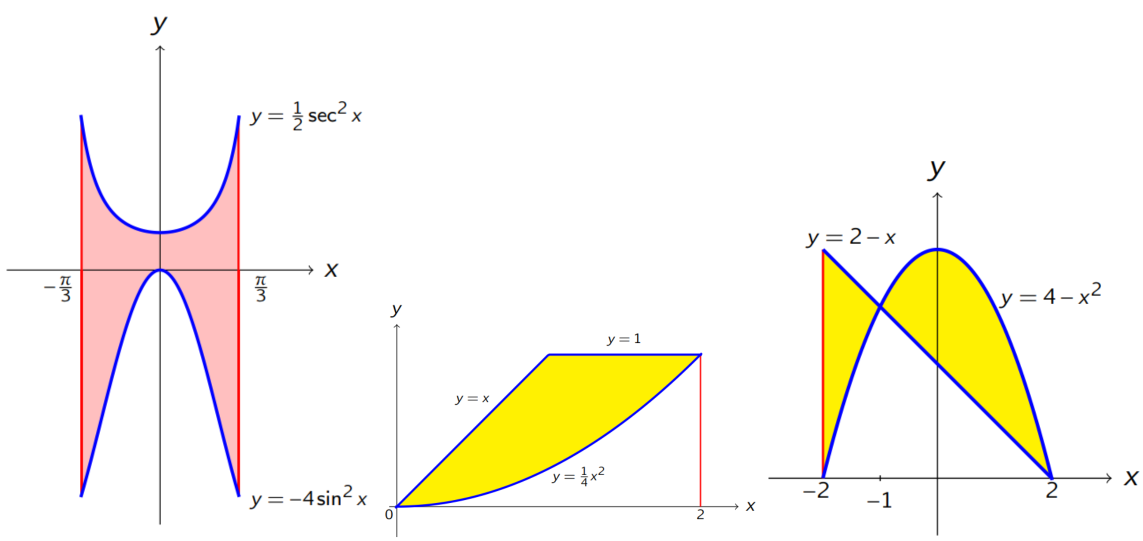
Solutions to Tutorial 5

MA1521 Calculus for Computing

1. (a) Observe that $\sec^2 x > 0$ and $-4\sin^2 x \leq 0$ on $[-\pi/3,\ \pi/3]$.

$$\begin{aligned}
\text{Area} &= \int_{-\pi/3}^{\pi/3} \left[\frac{1}{2}\sec^2 x - (-4\sin^2 x)\right] dx \\
&= \left[\frac{1}{2}\tan x + \int (2 - 2\cos 2x)\, dx\right]_{-\pi/3}^{\pi/3} \\
&= \tan\frac{\pi}{3} + (2x - \sin 2x)\Big|_{-\pi/3}^{\pi/3} \\
&= \sqrt{3} + \frac{4}{3}\pi - 2\sin\frac{\pi}{3} = \frac{4}{3}\pi.
\end{aligned}$$



(a) (b) (c)

(b) The points of intersection: $x = x^2/4$ implies $x = 0$ or $x = 4$. Hence the points of intersection are $(0,0)$ and $(4,4)$.

Note that $y = x^2/4 \Leftrightarrow x = 2\sqrt{y}$.

$$\text{The required area } = \int_0^1 \left[2\sqrt{y} - (y)\right] dy = \left[\frac{4}{3}y^{3/2} - \frac{1}{2}y^2\right]_0^1 = \frac{4}{3} - \frac{1}{2} = \frac{5}{6}.$$

(c) We have that $\quad (2 - x) - (4 - x^2) = x^2 - x - 2 = (x+1)(x-2)$

is negative if and only if $x \in (-1, 2)$.

Hence

$$\begin{aligned}\text{Area } &= \int_{-2}^{2} \left|(2-x)-(4-x^2)\right| dx \\ &= \int_{-2}^{-1} (x^2-x-2)\,dx + \int_{-1}^{2} -(x^2-x-2)\,dx \\ &= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\right]_{-2}^{-1} - \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\right]_{-1}^{2} \\ &= \frac{11}{6} - \frac{-9}{2} = \frac{19}{3}.\end{aligned}$$

2. (a) Given $y = \ln(\cos x)$, $0 \le x \le \frac{\pi}{3}$, we have $y' = -\tan x$. Thus $\sqrt{1+y'^2} = \sqrt{1+(-\tan x)^2}$ $= \sec x$. Therefore, $\int_0^{\frac{\pi}{3}} \sqrt{1+y'^2}\,dx = \int_0^{\frac{\pi}{3}} \sec x\,dx = [\ln|\sec x + \tan x|]_0^{\frac{\pi}{3}} = \ln(2+\sqrt{3})$.

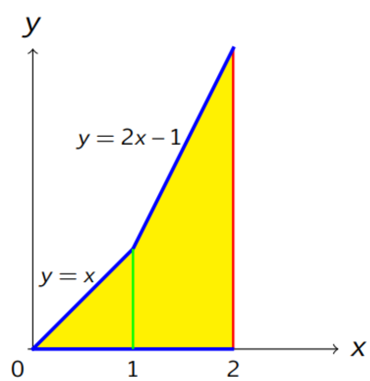
   (b) Given $y = \frac{x^5}{15} + \frac{1}{4x^3}$, $1 \le x \le 2$, we have $y' = \frac{x^4}{3} - \frac{3}{4x^4}$.

   Thus $\sqrt{1+y'^2} = \sqrt{1+(\dfrac{x^4}{3} - \dfrac{3}{4x^4})^2} = \sqrt{(\dfrac{x^4}{3} + \dfrac{3}{4x^4})^2} = \dfrac{x^4}{3} + \dfrac{3}{4x^4}$.

   Hence, $\displaystyle\int_1^2 \sqrt{1+y'^2}\,dx = \int_1^2 \frac{x^4}{3} + \frac{3}{4x^4}\,dx = \left[\frac{x^5}{15} - \frac{1}{4x^3}\right]_1^2 = (\frac{32}{15} - \frac{1}{32}) - (\frac{1}{15} - \frac{1}{4}) = \frac{1097}{480}$.

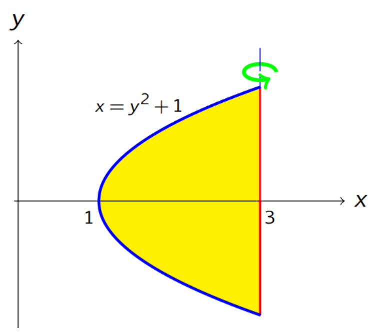
3. The volume of the solid is given by

   $\int_0^1 \pi x^2 dx + \int_1^2 \pi(2x-1)^2 dx = [\pi x^3/3]_0^1 + [\pi(2x-1)^3/6]_1^2 = \pi/3 + [27\pi/6 - \pi/6] = 14\pi/3$.

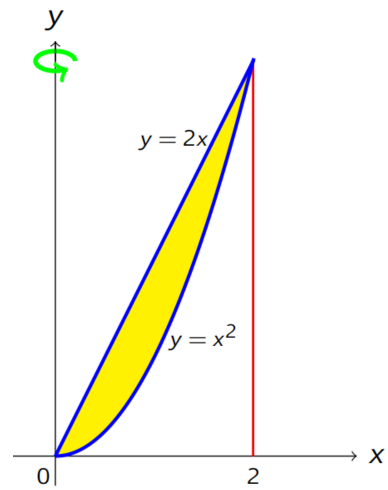


4. (a) The parabola and the line meet at (x,y) with $3 = y^2 + 1$, i.e. at $(3, \pm\sqrt{2})$. Thus

$$\text{Volume} = \int_{-\sqrt{2}}^{\sqrt{2}} \pi\Big[(y^2+1)-3\Big]^2\,dy = \pi\int_{-\sqrt{2}}^{\sqrt{2}}\Big[y^4-4y^2+4\Big]dy$$
$$= \pi\Big[\frac{1}{5}y^5-\frac{4}{3}y^3+4y\Big]_{-\sqrt{2}}^{\sqrt{2}}$$
$$= \pi\, 2\Big[\frac{1}{5}4\sqrt{2}-\frac{4}{3}2\sqrt{2}+4\sqrt{2}\Big]$$
$$= \frac{64}{15}\sqrt{2}\pi.$$



(a)



(b)

(b) The parabola and the line meet at (x,y) with $x^2=2x$, i.e. at $(0,0)$ and $(2,4)$.
Now $y=2x \Leftrightarrow x=y/2$ and $y=x^2 \Leftrightarrow x=\sqrt{y}$, while $\sqrt{y}-(y/2)=\sqrt{y}(1-\sqrt{y}/2)$ is positive for $y\in(0,4)$.
So $x=\sqrt{y}$ is the outer curve and $x=y/2$ is the inner curve.
By the disk method,

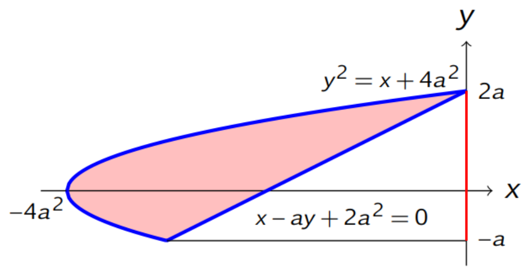
$$\text{Volume} = \int_0^4 \pi\sqrt{y}^2\,dy - \int_0^4 \pi\Big(\frac{y}{2}\Big)^2\,dy = \pi\frac{1}{2}\Big[4^2-0^2\Big]-\pi\frac{1}{4}\frac{1}{3}\Big[4^3-0^3\Big]=\frac{8}{3}\pi.$$

Alternatively, we can use the shell method.

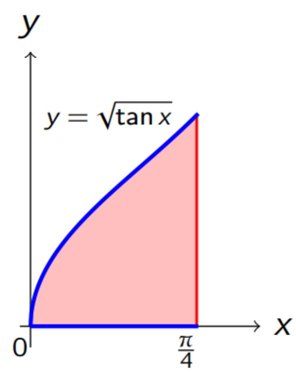
$$\text{Volume} = 2\pi\int_0^2 x(2x-x^2)\,dx = 2\pi[\frac{2x^3}{3}-\frac{x^4}{4}]_0^2 = 2\pi[\frac{16}{3}-4]=\frac{8\pi}{3}.$$

5. Solving the simultaneous equations $y^2=x+4a^2$ and $x-ay+2a^2=0$ by eliminating x we have $y^2-ay-2a^2=0$ and so $y=-a$ or $y=2a$.

$$\text{Area} = \int_{-a}^{2a}\Big[\Big(ay-2a^2\Big)-\Big(y^2-4a^2\Big)\Big]dy = \Big[\tfrac{1}{2}ay^2+2a^2y-\tfrac{1}{3}y^3\Big]_{-a}^{2a}=\tfrac{9}{2}a^3.$$



6. Volume $= \int_0^{\frac{\pi}{4}} \pi\left(\sqrt{\tan x}\right)^2 dx = [-\pi \ln\cos x]_0^{\frac{\pi}{4}} = \frac{\pi}{2}\ln 2$.



Solutions to Further Exercises

1. $\int_1^2 \frac{1}{x^7+x}\,dx = \int_1^2 \frac{1}{x} - \frac{x^5}{x^6+1}\,dx = \left[\ln|x| - \frac{1}{6}\ln|x^6+1|\right]_1^2 = (\ln 2 - \frac{1}{6}\ln(65)) - (0 - \frac{1}{6}\ln 2) = \frac{1}{6}(\ln 2^7 - \ln(65)) = \frac{1}{6}\ln(\frac{128}{65})$.

2. $\sum_{k=1}^{n} \frac{1}{\sqrt{3kn+n^2}} = \sum_{k=1}^{n} \frac{1}{n\sqrt{\frac{3k}{n}+1}} = \sum_{k=1}^{n} \frac{1}{3}\frac{3}{n}\frac{1}{\sqrt{k\frac{3}{n}+1}} = \frac{1}{3}\sum_{k=1}^{n}\frac{3}{n}\frac{1}{\sqrt{k\frac{3}{n}+1}}$. From this we see that $b-a=3$ and $a=1$ so that $b=4$, and the function is $\frac{1}{\sqrt{x}}$.

   Consequently,
$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{3kn+n^2}} = \frac{1}{3}\int_1^4 \frac{1}{\sqrt{x}}\,dx = \frac{2}{3}.$$

3. Given $x = \frac{1}{8}(4e^{2y} + e^{-2y})$, $\ln 2 \le y \le \ln 3$, we have $\frac{dx}{dy} = e^{2y} - \frac{1}{4}e^{-2y}$.

   Thus $\sqrt{1+(\frac{dx}{dy})^2} = \sqrt{1+(e^{2y}-\frac{1}{4}e^{-2y})^2} = \sqrt{(e^{2y}+\frac{1}{4}e^{-2y})^2} = e^{2y}+\frac{1}{4}e^{-2y}$.

   Therefore, $\int_{\ln 2}^{\ln 3}\sqrt{1+(\frac{dx}{dy})^2}\,dx = \int_{\ln 2}^{\ln 3} e^{2y}+\frac{1}{4}e^{-2y}\,dy = \left[\frac{1}{2}e^{2y} - \frac{1}{8e^{2y}}\right]_{\ln 2}^{\ln 3} = (\frac{9}{2}-\frac{1}{72}) - (2-\frac{1}{32}) = \frac{725}{288}$.