5. Second order partial differential equations in two variables

The general second order partial differential equations in two variables is of the form

$$\boxed{F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}) = 0.}$$

The equation is *quasi-linear* if it is linear in the highest order derivatives (second order), that is if it is of the form

$$\boxed{a(x, y, u, u_x, u_y)u_{xx} + 2\, b(x, y, u, u_x, u_y)u_{xy} + c(x, y, u, u_x, u_y)u_{yy} = d(x, y, u, u_x, u_y)}$$

We say that the equation is *semi-linear* if the coefficients a, b, c are independent of u. That is if it takes the form

$$\boxed{a(x, y)\,)u_{xx} + 2b(x, y)\, u_{xy} + c(x, y)\, u_{yy} = d(x, y, u, u_x, u_y)}$$

Finally, if the equation is semi-linear and d is a *linear* function of u, u_x and u_y, we say that the equation is *linear*. That is, when F is linear in u and all its derivatives.

We will consider the semi-linear equation above and attempt a change of variable to obtain a more convenient form for the equation.

Let $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ be an invertible transformation of coordinates. That is,

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} \neq 0.$$

By the chain rule

$$u_x = u_\xi \xi_x + u_\eta \eta_x, \;\; u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{xx} = u_\xi \xi_{xx} + \xi_x\left(u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x\right) + u_\eta \eta_{xx} + \eta_x\left(u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x\right)$$

$$= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + \text{ first order derivatives of } u$$

Similarly,

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + \text{ first order derivatives of } u$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + u_{\eta\eta}\eta_x\eta_y + \text{ first order derivatives of } u$$

Substituting into the partial differential equation we obtain,

$$\boxed{A(\xi,\eta)u_{\xi\xi} + 2B(\xi,\eta)u_{\xi\eta} + C(\xi,\eta)u_{\eta\eta} = D(\xi,\eta,u,u_\xi,u_\eta)}$$

where

$$A(\xi,\eta) = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2$$

$$B(\xi,\eta) = a\xi_x\eta_x + b(\xi_x\eta_y + \eta_x\xi_y) + c\xi_y\eta_y$$

$$C(\xi,\eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2.$$

It easily follows that

$$B^2 - AC = (b^2 - ac)\left(\frac{\partial(\xi,\eta)}{\partial(x,y)}\right)^2.$$

Therefore $B^2 - AC$ has the same sign as $b^2 - ac$. We will now choose the new coordinates $\xi = \xi(x,y)$, $\eta = \eta(x,y)$ to simplify the partial differential equation.
$\xi(x,y)$ = constant , $\eta(x,y)$ = constant defines two families of curves in $\mathbf{R}^2$. On a member of the family $\xi(x,y)$ = constant, we have that

$$\frac{d\xi}{dx} = \xi_x + \xi_y y' = 0.$$

Therefore substituting in the expression for $A(\xi,\eta)$ we obtain

$$A(\xi,\eta) = a\,\xi_y^2\,y'^2 - 2b\,\xi_y^2\,y' + c\xi_y^2$$

$$= \xi_y^2[a\;y'^2 - 2b\,y' + c\,].$$

We choose the two families of curves given by the two families of solutions of the ordinary differential equation

$$a\, y'^2 - 2b\, y' + c = 0.$$

This nonlinear ordinary differential equation is called *the characteristic equation* of the partial differential equation and provided that $a \neq 0$, $b^2 - ac > 0$ it can be written as

$$\boxed{y' = \frac{b \pm \sqrt{b^2 - ac}}{a}}$$

For this choice of coordinates $A(\xi, \eta) = 0$ and similarly it can be shown that $C(\xi, \eta) = 0$ also. The partial differential equation becomes

$$2\, B(\xi, \eta)\, u_{\xi\eta} = D(\xi, \eta, u, u_{\xi}, u_{\eta})$$

where it is easy to show that $B(\xi, \eta) \neq 0$. Finally, we can write the partial differential equation in the *normal form*

$$\boxed{u_{\xi\eta} = D(\xi, \eta, u, u_{\xi}, u_{\eta})}$$

The two families of curves $\xi(x, y)$ = constant ,$\eta(x, y)$= constant obtained as solutions of the characteristic equation are called *characteristics* and the semi-linear partial differential equation is called *hyperbolic* if $b^2 - ac > 0$ whence it has two families of characteristics and a normal form as given above.

If $b^2 - ac < 0$, then the characteristic equation has *complex* solutions and there are no real characteristics. The functions $\xi(x, y)$, $\eta(x, y)$ are now complex conjugates . A change of variable to the *real* coordinates

$$\alpha = \xi(x, y) + \eta(x, y), \quad \beta = -i(\xi(x, y) - \eta(x, y))$$

results in the partial differential equation where the *mixed* derivative term vanishes,

$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} = D(\alpha, \beta, u, u_{\alpha}, u_{\beta}).}$$

In this case the semi-linear partial differential equation is called *elliptic* if $b^2 - ac < 0$. Notice that the left hand side of the normal form is the Laplacian. Thus Laplaces equation is a special case of an elliptic equation (with $D = 0$).

If $b^2 - ac = 0$, the characteristic equation $y' = \frac{b}{a}$ has only one family of solutions $\eta(x, y)$ = constant. We make the change of variable

$$\xi = x, \ \ \eta = \eta(x, y).$$

Then

$$A(\xi, \eta) = a$$

$$B(\xi, \eta) = a\eta_x + b\eta_y$$

$$C(\xi, \eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = \frac{(a\eta_x + b\eta_y)^2 - (b^2 - ac)\eta_y^2}{a} = \frac{B(\xi,\eta)^2}{a}$$

Also since $\eta(x, y)$ = constant,

$$0 = \eta_x + \eta_y y' = \eta_x + \eta_y \frac{b}{a} = \frac{a\eta_x + b\eta_y}{a} = \frac{B(\xi,\eta)}{a}$$

Therefore $B(\xi, \eta) = 0$, $C(\xi, \eta) = 0$, $A(\xi, \eta) \neq 0$ and the normal form in the case $b^2 - ac = 0$ is

$$A(\xi, \eta)\, u_{\xi\xi} = D(\xi, \eta, u, u_\xi, u_\eta)$$

or finally

$$\boxed{u_{\xi\xi} = D(\xi, \eta, u, u_\xi, u_\eta)}$$

The partial differential equation is called *parabolic* in the case $b^2 - a = 0$. An example of a parabolic partial differential equation is the equation of *heat conduction*

$$\frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} = 0 \ \text{ where } \ u = u(x, t).$$

Example 1. Classify the following linear second order partial differential equation and find its general solution .

$$xyu_{xx} + x^2 u_{xy} - yu_x = 0.$$

In this example $b^2 - ac = \left(\frac{x^2}{2}\right)^2 \geq 0 \Rightarrow$ the partial differential equation is hyperbolic provided $x \neq 0$, and parabolic for $x = 0$.

For $x \neq 0$ the characteristic equations are

$$y' = \frac{b \pm \sqrt{b^2 - ac}}{a} = \frac{\frac{x^2}{2} \pm \frac{x^2}{2}}{xy} = 0 \text{ or } \frac{x}{y}$$

If $y' = 0$, $y =$ constant.

If $y' = \frac{x}{y}$, $x^2 - y^2$ = constant. Therefore two families of characteristics are

$$\xi = x^2 - y^2,\ \eta = y.$$

Using the chain rule a number of times we calculate the partial derivatives

$$u_x = u_\xi\, 2x + u_\eta\, 0 = 2x u_\xi$$

$$u_{xx} = 2u_\xi + 2x(u_{\xi\xi}\, 2x + u_{\xi\eta}\, 0) = 2u_\xi + 4x^2 u_{\xi\xi}$$

$$u_{xy} = 2x\left(u_{\xi\xi}(-2y) + u_{\xi\eta} 1\right) = -4xy u_{\xi\xi} + 2x u_{\xi\eta}\ .$$

Substituting into the partial differential equation we obtain the normal form

$$u_{\xi\eta} = 0 \quad (\text{provided } x \neq 0).$$

Integrating this equation with respect to η

$$u_\xi = f(\xi),$$

where f is an arbitrary function of one real variable. Integrating again with respect to ξ

$$u(\xi, \eta) = \int f(\xi)\, d\xi + G(\eta) = F(\xi) + G(\eta)$$

where F, G are arbitrary functions of one real variable. Reverting to the original coordinates we find the general solution

$$u(x, y) = F(x^2 - y^2) + G(y)$$

Example 2. Classify, reduce to normal form and obtain the general solution of the partial differential equation

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 4x^2$$

For this equation $b^2 - ac = (xy)^2 - x^2 y^2 = 0 \Rightarrow$ the equation is parabolic everywhere in the plane (x, y). The characteristic equation is

$$y' = \frac{b}{a} = \frac{xy}{x^2} = \frac{y}{x} .$$

Therefore there is one family of characteristics $\frac{y}{x}$ = constant.

Let $\xi = x$ and $\eta = \frac{y}{x}$. Then using the chain rule,

$$u_x = u_\xi \, 1 + u_\eta \left(-\frac{y}{x^2} \right) = u_\xi - \frac{y}{x^2} u_\eta$$

$$u_y = u_\xi \, 0 + u_\eta \left(\frac{1}{x} \right) = \frac{1}{x} u_\eta$$

$$u_{xx} = u_{\xi\xi} \, 1 + u_{\xi\eta} \left(-\frac{y}{x^2} \right) + \frac{2y}{x^3} u_\eta - \frac{y}{x^2} \left(u_{\eta\xi} 1 + u_{\eta\eta} \left(-\frac{y}{x^2} \right) \right)$$

$$= u_{\xi\xi} - \frac{2y}{x^2} u_{\xi\eta} + \frac{y^2}{x^4} u_{\eta\eta} + \frac{2y}{x^3} u_\eta$$

$$u_{yy} = \frac{1}{x} \left(u_{\eta\xi} 0 + u_{\eta\eta} \left(\frac{1}{x} \right) \right) = \frac{1}{x^2} u_{\eta\eta}$$

$$u_{yx} = -\frac{1}{x^2} u_\eta + \frac{1}{x} \left(u_{\eta\xi} 1 + u_{\eta\eta} \left(-\frac{y}{x^2} \right) \right)$$

$$= \frac{1}{x} u_{\xi\eta} - \frac{y}{x^3} u_{\eta\eta} - \frac{1}{x^2} u_\eta .$$

Substituting into the partial differential equation we obtain the normal form

$$u_{\xi\xi} = 4 .$$

Integrating with respect to ξ

$$u_\xi = 4\xi + f(\eta)$$

where f is an arbitrary function of a real variable. Integrating again with respect to ξ

$$u(\xi, \eta) = 2\xi^2 + \xi f(\eta) + g(\eta),$$

Therefore the general solution is given by

$$u(x, y) = 2x^2 + xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$$

where f, g are arbitrary functions of a real variable.