有限差分法解偏微分方程式

第一節　偏微分方程

已知二階偏微分方程

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial t}+C\frac{\partial^2 u}{\partial t^2}+f\left(x,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right)=0$$

若 $B^2-4AC=0$，則稱上式爲拋物線型偏微分方程。

若 $B^2-4AC>0$，則稱上式爲**雙曲線型**偏微分方程。

若 $B^2-4AC<0$，則稱上式爲**橢圓型**偏微分方程。

第二節　Heat conduction 偏微分方程之數值解

Heat Conduction Equation ：

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{\alpha^2}\frac{\partial u}{\partial t}\text{，}0<x<L\text{，}t>0$$

邊界條件與初始條件必須給定。

將 x 分成 n 段 $\Delta x=h=\dfrac{L}{n}$，並取 t 方向增量 $k=\Delta t$

【方法一】顯性近似法：

考慮節點 (i,j) 處之差分公式：grid points：(x_i,t_j)，$u_{ij}=u(x_i,t_j)$

二階偏微分爲中央差分近似

$$\frac{\partial^2 u}{\partial x^2}\approx\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{\Delta x^2}\approx\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}$$

一階偏微分爲前向差分近似

$$\frac{\partial u}{\partial t}\approx\frac{u_{i,j+1}-u_{i,j}}{\Delta t}\approx\frac{u_{i,j+1}-u_{i,j}}{k}$$

代入熱傳方程 $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{1}{\alpha^2}\dfrac{\partial u}{\partial t}$，得

$$\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}=\frac{1}{\alpha^2}\frac{u_{i,j+1}-u_{i,j}}{k}$$

移項整理得

$$\frac{\alpha^2 k}{h^2}\left(u_{i-1,j}-2u_{i,j}+u_{i+1,j}\right)=u_{i,j+1}-u_{i,j}$$

令$\alpha^*=\dfrac{\alpha^2 k}{h^2}$代入，得節點$(i,j)$之有限差分近似公式

$$u_{i,j+1}=u_{i,j}+\alpha^*\left(u_{i-1,j}-2u_{i,j}+u_{i+1,j}\right)，i=0,1,2,\cdots,n \quad，\quad j=0,1,2,\cdots$$

其中 $j=0$ 爲初始條件，$u_{i,0}$ 爲給定。

移項得 PDE finite difference formula for the heat equation：

$$u_{i,j+1}=\alpha^* u_{i-1,j}+(1-2\alpha^*)u_{i,j}+\alpha^* u_{i+1,j}$$

$i=0$ 及 $i=n$ 爲邊界條件，$u_{0,j}$ ，$u_{n,j}$ 爲給定。

If $0<\alpha^*\le 0.5$ ， then the approximations $u_{i,j}$ converge to the solution $u(x,t)$

【方法二】隱性近似法：

考慮節點$(i,j+1)$處之差分公式

(注意隱性近(i,j+1)與顯性法(i,j)不同，j+1 line is to-be-determined)

二階偏微分爲中央差分近似，

$$\frac{\partial^2 u}{\partial x^2}\approx\frac{u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1}}{\Delta x^2}\approx\frac{u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1}}{h^2}$$

一階偏微分爲前向差分近似

$$\frac{\partial u}{\partial t}\approx\frac{u_{i,j+1}-u_{i,j}}{\Delta t}\approx\frac{u_{i,j+1}-u_{i,j}}{k}$$

代入熱傳方程$\dfrac{\partial^2 u}{\partial x^2}=\dfrac{1}{\alpha^2}\dfrac{\partial u}{\partial t}$，得

$$\frac{u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1}}{h^2}=\frac{1}{\alpha^2}\frac{u_{i,j+1}-u_{i,j}}{k}$$

移項整理得

$$-\frac{\alpha^2 k}{h^2}\left(u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1}\right)+u_{i,j+1}=u_{i,j}$$

令$\alpha^*=\dfrac{\alpha^2 k}{h^2}$代入，得節點$(i,j+1)$之有限差分近似公式

$$(1+2\alpha^*)u_{i,j+1}-\alpha^*\left(u_{i+1,j+1}+u_{i-1,j+1}\right)=u_{i,j}\text{ ， } i=0,1,2,\cdots,n\text{ ， } j=0,1,2,\cdots$$

其中　$j=0$ 爲初始條件，$u_{i,0}$ 爲給定。

　　$i=0$　及 $i=n$ 爲邊界條件，$u_{0,j}$ ，$u_{n,j}$ 爲給定。

空間網格點： $i=0$ 到 $i=n$ 各節點代入展開得

1. 當 $i=1$ ，代入差分通式 $(1+2\alpha^*)u_{i,j+1}-\alpha^*\left(u_{i+1,j+1}+u_{i-1,j+1}\right)=u_{i,j}$ ，得

$$(1+2\alpha^*)u_{1,j+1}-\alpha^*\left(u_{2,j+1}+u_{0,j+1}\right)=u_{1,j}$$

移項得

$$(1+2\alpha^*)u_{1,j+1}-\alpha^*u_{2,j+1}=u_{1,j}+\alpha^*u_{0,j+1}=u_{1,j}+\alpha^*u_0$$

2. 當 $i=n-1$ ，代入差分通式 $(1+2\alpha^*)u_{i,j+1}-\alpha^*\left(u_{i+1,j+1}+u_{i-1,j+1}\right)=u_{i,j}$ ，得

$$(1+2\alpha^*)u_{n-1,j+1}-\alpha^*\left(u_{n,j+1}+u_{n-2,j+1}\right)=u_{n-1,j}$$

移項得

$$(1+2\alpha^*)u_{n-1,j+1}-\alpha^*u_{n-2,j+1}=u_{n-1,j}+\alpha^*u_{n,j+1}=u_{n-1,j}+\alpha^*u_n$$

最後代入整理得矩陣形式如下：

$$\begin{bmatrix} 1+2\alpha^* & -\alpha^* & 0 & & 0 \\ -\alpha^* & 1+2\alpha^* & -\alpha^* & \cdots & 0 \\ 0 & -\alpha^* & 1+2\alpha^* & & 0 \\ & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1+2\alpha^* \end{bmatrix}\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ u_{n-1,j+1} \end{bmatrix}=\begin{bmatrix} u_{1,j}+\alpha^*u_0 \\ u_{2,j} \\ u_{3,j} \\ \vdots \\ u_{n-1,j}+\alpha^*u_n \end{bmatrix}$$

【觀念】

※ 隱性法爲無條件穩定

※ 顯性法爲條件穩定（ $\alpha^*\le 0.5$ ），時間網格距愈小愈好。

第三節　雙曲線型偏微分方程

Wave Equation ：

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \text{，} 0 < x < L \text{，} t > 0$$

邊界條件與初始條件必須給定。

將 x 分成 n 段 $\Delta x = h = \frac{L}{n}$，並取 t 方向增量 $k = \Delta t$

考慮節點 (i, j) 處之差分公式

二階偏微分爲中央差分近似

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

及

$$\frac{\partial^2 u}{\partial t^2} \approx \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta t^2} \approx \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

代入波動方程 $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$，得

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} = \frac{1}{c^2}\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

移項整理得

$$\frac{c^2k^2}{h^2}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) = u_{i,j-1} - 2u_{i,j} + u_{i,j+1}$$

令 $\alpha^2 = \frac{c^2k^2}{h^2}$ 代入，得節點 (i, j) 之有限差分近似公式

$$u_{i,j+1} = -u_{i,j-1} + 2u_{i,j} + \alpha^2\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right)$$

或

$$u_{i,j+1} = \alpha^2\left(u_{i+1,j} + u_{i-1,j}\right) + 2\left(1-\alpha^2\right)u_{i,j} - u_{i,j-1} \text{，} i = 0,1,2,\cdots,n \text{，} j = 0,1,2,\cdots$$

其中 $j = 0$ 爲初始條件，$u(x_i,0) = u_{i,0} = U_i$ 及 $\frac{\partial u}{\partial t}(x_i,0) = V_i$ 爲給定。代入得

$$u_{i,1} = \alpha^2\left(u_{i+1,0} + u_{i-1,0}\right) + 2\left(1-\alpha^2\right)u_{i,0} - u_{i,-1}$$

上式中有$u_{i,-1}$項，此項計算可由初始條件$\frac{\partial u}{\partial t}(t_0)=V_i$，利用中央差分近似一階微分值 ，即

$$\frac{\partial u}{\partial t}(t_0)\approx\frac{-u_{i,-1}+u_{i,1}}{2\Delta t}=\frac{-u_{i,-1}+u_{i,1}}{2k}=V_i$$

移項得

$$-u_{i,-1}=2kV_i-u_{i,1}$$

$i=0$ 及$i=n$ 爲邊界條件，$u_{0,j}$ ，$u_{n,j}$爲給定。

代回原差分近似式得

$$u_{i,1}=\alpha^2(U_{i+1}+U_{i-1})+2(1-\alpha^2)U_i+2kV_i-u_{i,1}$$

移項得

$$2u_{i,1}=\alpha^2(U_{i+1}+U_{i-1})+2(1-\alpha^2)U_i+2kV_i$$

或

$$u_{i,1}=\frac{\alpha^2}{2}(U_{i+1}+U_{i-1})+(1-\alpha^2)U_i+kV_i$$

第四節　　橢圓型偏微分方程

Lapace Equation ：

$$\nabla^2u=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=0\text{ ， }0<x<a\text{ ， }0<y<b$$

Poisson's Equation ：

$$\nabla^2u=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=f(x,y)\text{，}0<x<a\text{，}0<y<b$$，$f(x,y)$爲給定之 source function。

Helmholtz's Equation ：

$$\nabla^2u=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+g(x,y)u=f(x,y)\text{，}0<x<a\text{ ，}0<y<b$$

將 x 分成 n 段 $\Delta x = h = \dfrac{L}{n}$，並取 y 方向增量 $k = \Delta y$

考慮節點 (i, j) 處之差分公式

二階偏微分爲中央差分近似

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

及

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2} \approx \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

代入 Laplace 方程 $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$，得

$$\nabla^2 u_{i,j} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

移項整理得

$$\nabla^2 u_{i,j} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

令 $r = \dfrac{k}{h}$ 代入，得節點 (i, j) 之有限差分近似公式

$$\nabla^2 u_{i,j} = \frac{r^2 u_{i-1,j} - 2r^2 u_{i,j} + r^2 u_{i+1,j}}{r^2 h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{r^2 h^2}$$

或

$$\nabla^2 u_{i,j} = \frac{r^2 u_{i-1,j} + r^2 u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 2\left(1 + r^2\right) u_{i,j}}{r^2 h^2}$$

$i = 0,1,2,\cdots,n$ ， $j = 0,1,2,\cdots,n$

當 $\Delta x = \Delta y$ 及 $r = 1$，代入得

$$\nabla^2 u_{i,j} = \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2}$$

Therefore, $u_{i,j} = \dfrac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}}{4}$

The “relaxation” method can be used to find the results.

Boundary Conditions

Dirichlet: The values of only the function, u, are specified on a boundary.

Neumann: The values of only the normal derivatives of the function are gives on a boundary.

Cauchy: The values of both the function and its normal derivative are specified on the same boundary.

Type of boundary condition		TYPE OF PDE: Elliptic (*Poisson's equation*) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x, y)$ Equilibrium distribution	Parabolic (*Heat conduction*) $\frac{\partial T}{\partial t} = -D\frac{\partial^2 T}{\partial x^2}$ Diffusion	Hyperbolic *The wave equation* $\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$ Wave propagation
Dirichlet or *Neumann* on	OPEN Surface	*Not enough*	**UNIQUE, STABLE solution in one direction only**	*Not enough*
	CLOSED Surface	**UNIQUE, STABLE solution**	*Too much*	*Too much*
Cauchy on	OPEN Surface	*Unphysical results*	*Too much*	**UNIQUE, STABLE solution**
	CLOSED Surface	*Too much*	*Too much*	*Too much*

Table 22.1 *Boundary conditions appropriate to PDEs*

Finite difference expressions:

Forward difference

$$\left.\frac{\partial u}{\partial x}\right|_{(x_i y_j)} \approx \frac{(u_{i+1,j} - u_{ij})}{\Delta x} + \mathcal{O}(\Delta x)$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_i y_j)} \approx \frac{(u_{i+2,j} - 2u_{i+1,j} + u_{ij})}{(\Delta x)^2} + \mathcal{O}(\Delta x)$$

Central difference

$$\left.\frac{\partial u}{\partial x}\right|_{(x_i y_j)} \approx \frac{(u_{i+1,j} - u_{i-1,j})}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_i y_j)} \approx \frac{(u_{i+1,j} - 2u_{ij} + u_{i-1,j})}{(\Delta x)^2} + \mathcal{O}(\Delta x^2)$$