高等流体力学
之计算流体力学基础

Computational Fluid Dynamics (CFD)

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课程目的-引

EULER方程组：

无粘流动控制方程，忽略粘性输运、热传导等现象

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

连续方程

$$\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{V}) = -\frac{\partial p}{\partial x} + \rho f_x$$

$$\frac{\partial (\rho v)}{\partial t} + \nabla \cdot (\rho v \mathbf{V}) = -\frac{\partial p}{\partial y} + \rho f_y$$

$$\frac{\partial (\rho w)}{\partial t} + \nabla \cdot (\rho w \mathbf{V}) = -\frac{\partial p}{\partial z} + \rho f_z$$

动量方程

能量方程

$$\frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right] + \nabla \cdot \left[\rho\left(e + \frac{V^2}{2}\right)\mathbf{V}\right] = \rho \dot{q} - \frac{\partial (up)}{\partial x} - \frac{\partial (vp)}{\partial y} - \frac{\partial (wp)}{\partial z} + \rho \mathbf{f} \cdot \mathbf{V}$$

课程目的-引

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}=J$$

通量形式EULER方程

$$U=\begin{Bmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \rho\left(e+\dfrac{V^2}{2}\right)\end{Bmatrix}$$

解向量

$$F=\begin{Bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uw\\ \rho u\left(e+\dfrac{V^2}{2}\right)+pu\end{Bmatrix}$$

通量向量

$$G=\begin{Bmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ \rho wv\\ \rho v\left(e+\dfrac{V^2}{2}\right)+pv\end{Bmatrix}$$

通量向量

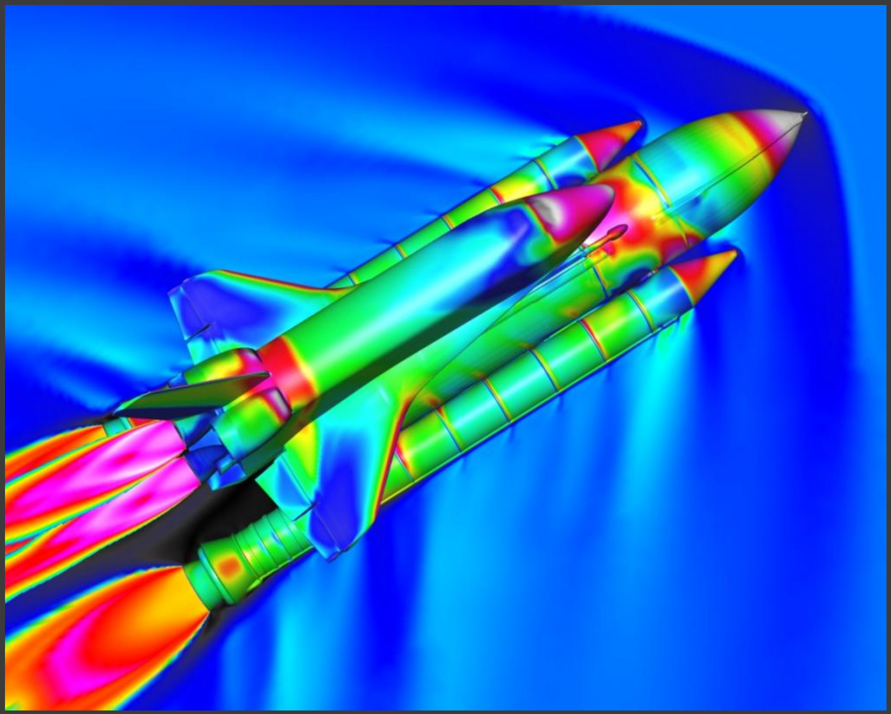
$$H=\begin{Bmatrix}\rho w\\ \rho uw\\ \rho vw\\ \rho w^2+p\\ \rho w\left(e+\dfrac{V^2}{2}\right)+pw\end{Bmatrix}$$

通量向量

$$J=\begin{Bmatrix}0\\ \rho f_x\\ \rho f_y\\ \rho f_z\\ \rho(uf_x+vf_y+wf_z)+\rho\dot{q}\end{Bmatrix}$$

源项

课程目的-引



飞行器表面及包络场内
的静压力云图

忽略航天飞船飞行过程中的粘性及热传导特征，则其包络域内的气体工质热力参数满足EULER方程，即可以用EULER方程组来描述其中各热力参数的关系。

↓

在定解条件下求解EULER方程组

↓

现实问题：拟线性/非线性方程组无解析解或者极难找到！

↓

偏微分方程组如何求解???

课程目的

- 偏微分方程组的求解方法
  - 有限差分法
    - 差分格式
    - 方程离散
    - 网格生成
  - 有限体积法
  - 有限元法

参考书

- John D.Anderson, Computational Fluid Dynamics, McGrawHill Education (English/Chinese Version)
- Klaus A. Hoffmann,Computational Fluid Dynamics,
- Havard Lomax, Thomas H. Pulliam, Fundamentals of Computational Fluid Dynamics, 1999
- David W. Zingg, Fundamentals of Computational Fluid Dynamics.1999
- Pieter Wesseling. Principles of Computational Fluid Dynamics.Springer, 2001
- 韩占忠,FLUENT-流体工程仿真实例与分析. 北京理工大学出版社.2009.8

Why CFD?

- 认识和掌握流动

　　揭示自然界中流动现象的本质、流动特征和流动规律，了解流动现象中不利于流动的因素

- 控制和改进流动

　　通过认识和掌握流动特性，从而控制改进流动，以达到“认识自然、改造自然”的目的。

　　控制和改进流动的方向主要有：提升流动效率（或者功、能转化效率）、提升结构稳定性与安全性（如飞机机翼的设计）、改善环境（例如噪音的控制）等

Why CFD?

流体力学三大分支

- 实验方法
- 理论方法（解析）
- 计算方法（数值模拟）

Why CFD? -- 实验

- 实验

原 理：利用相对运动原理，建立实验设备，如风洞、水洞、水槽等，直接测量流动参数，获取速度、压力、温度以及其它与热力学有关的流体力学方面的数据。

所需设备：动力设备、控制设备、配送设备、实验段、测试设备

地 位：一直以来实验流体力学为流体动力学的一个最重要分支，也是长期以来直至现在流体力学的主要研究手段。

优 点：比较真实可靠（当然也存在实验误差）

缺 点：受实验条件限制，经常需要进行模型实验（按比例缩小或放大模型），便不能满足所有的相似参数、相似定律的要求；同时还受实验设备本身的影响，例如风洞（水洞）洞壁效应、测量设备的干扰等；无法建立真实流动的实验条件（例如高超音速流动）；大量的费用和较长的实验周期，无法全部取得重要的流场参数。

Why CFD? – 理论

- 理论方法（解析法）

原 理：在研究流体运动规律的基础上，建立相应的流动模型（一般有简化），形成描述流动的控制方程，并在一定的假设和条件下，经过解析推导，得到问题的解析解或者简化解。

所需设备：无

地 位：主要用于定性分析或者初步的设计和分析。

优 点：可给出使用范围较广的信息，所需花费的代价非常小便可以给出规律性的结果或者变化规律。

缺 点：工程流体力学相关领域中极少或者没有解析解，因此应用范围受限。

Why CFD? – 数值计算

- 计算方法

原 理：利用计算机，对简化或者非简化过的流体动力学基本方程进行求解并获取各种条件下的流动、热力参数。

所需设备：计算机

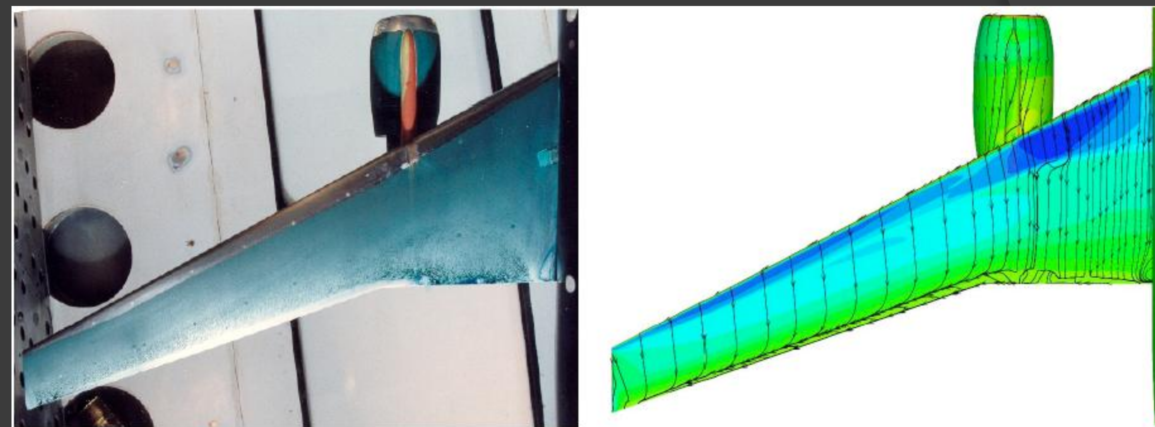
地 位：流体力学领域最新的方法，最活跃同时也是最有生命力，将是流体力学领域最重要的研究手段。

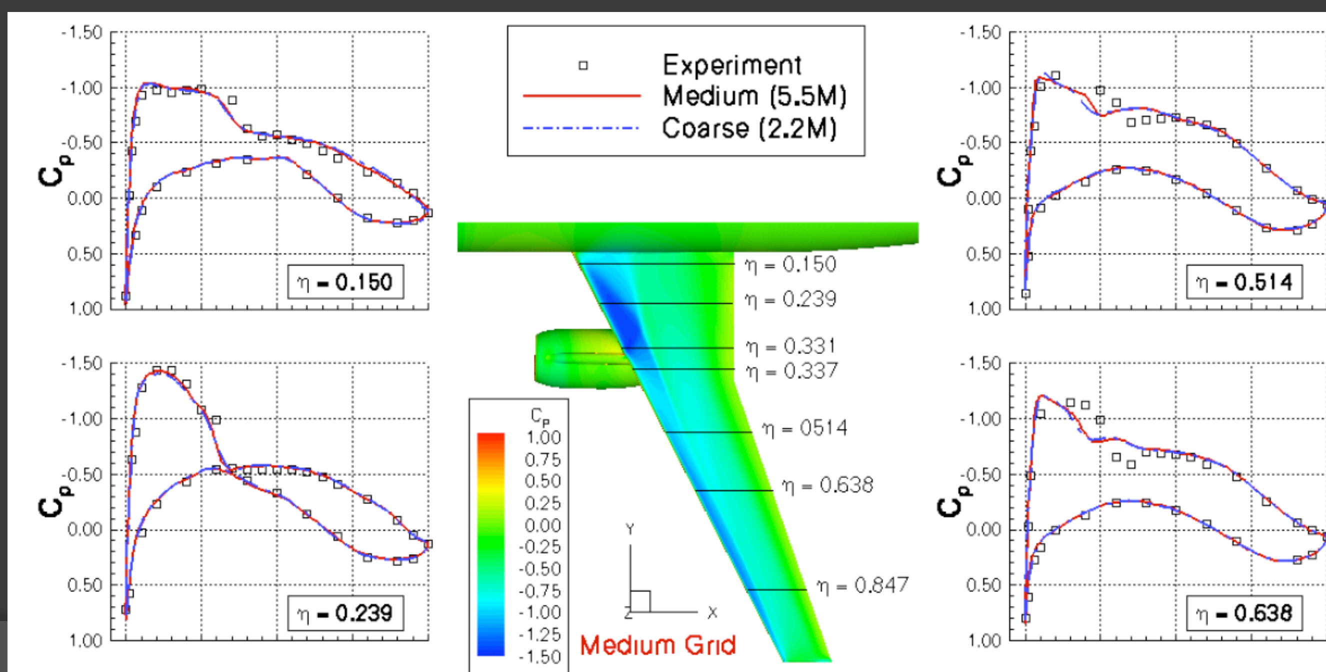
优 点：费用低、周期短；可以非常便捷的实现各种流体条件下的模拟；不存在实验过程中的各种干扰因素；可以详尽、细致的描述任何流动区域的流动细节，可观察到实验所无法观察到的复杂的流动现象。

缺 点：受制于计算机条件;现今的CFD尚处于发展和逐步完善的过程，还存在一定的误差，对复杂问题的模拟精度决定于对特定的流动特征所提出的模型精度，例如湍流流动、燃烧等问题。

Why CFD? – 数值计算

◎ 应用举例





CFD的思想和步骤

- 把空间和时间上连续分布（无穷点）的流动物理量用一系列有限个离散点（节点）上的值来表示，通过一定的方法建立起这些节点上的变量值之间的代数方程组，从而得到所求解变量的近似值。

有限个离散点：网格节点
一定的方法：　差分方法、有限体积法或者有限元法等
代数方程组：　离散方程
近 似 解：　数值解

CFD的思想和步骤

- **物理模型** ← 分析流动的性质，并对流动现象的物理过程进行相应的简化
- **数学模型** ← 建立控制方程组及定解条件
- **区域离散** ← 把流动区域划分为很多子区域(网格)，并从每个网格中选取出节点的过程
- **方程离散** ← 把微分方程组转换为代数方程组，如有限元法，有限体积法，有限差分法
- **代数方程组求解** ← 时间推进法、压力一速度耦合解法等
- **后处理** ← 从计算结果中取出所关心的流动参数

CFD求解的是什么？

- 控制方程

用于描述流体流动过程中各状态参数之间关系的方程。

流体流动状态参数主要包括： 密度(ρ)、压力(P)、速度(V)、温度(T)等。

常规控制方程包括：

连续性方程 （质量守恒定律）

运动方程（动量守恒定律）

能量方程（能量守恒定律）

状态方程（压力、温度以及密度关系方程）

湍流方程（用于描述湍流流动特性的方程）

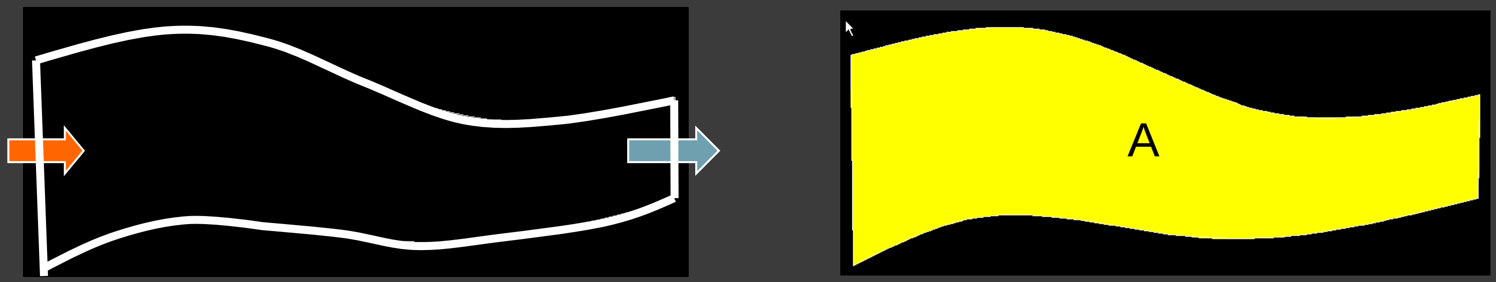
CFD求解的是各种状态参数：密度(ρ)、压力(P)、速度(V)、温度(T)等

第一章　方程离散基本方法

1.1 离散化基本概念

1.计算域的定义

采用计算方法进行流动数值模拟的区域。



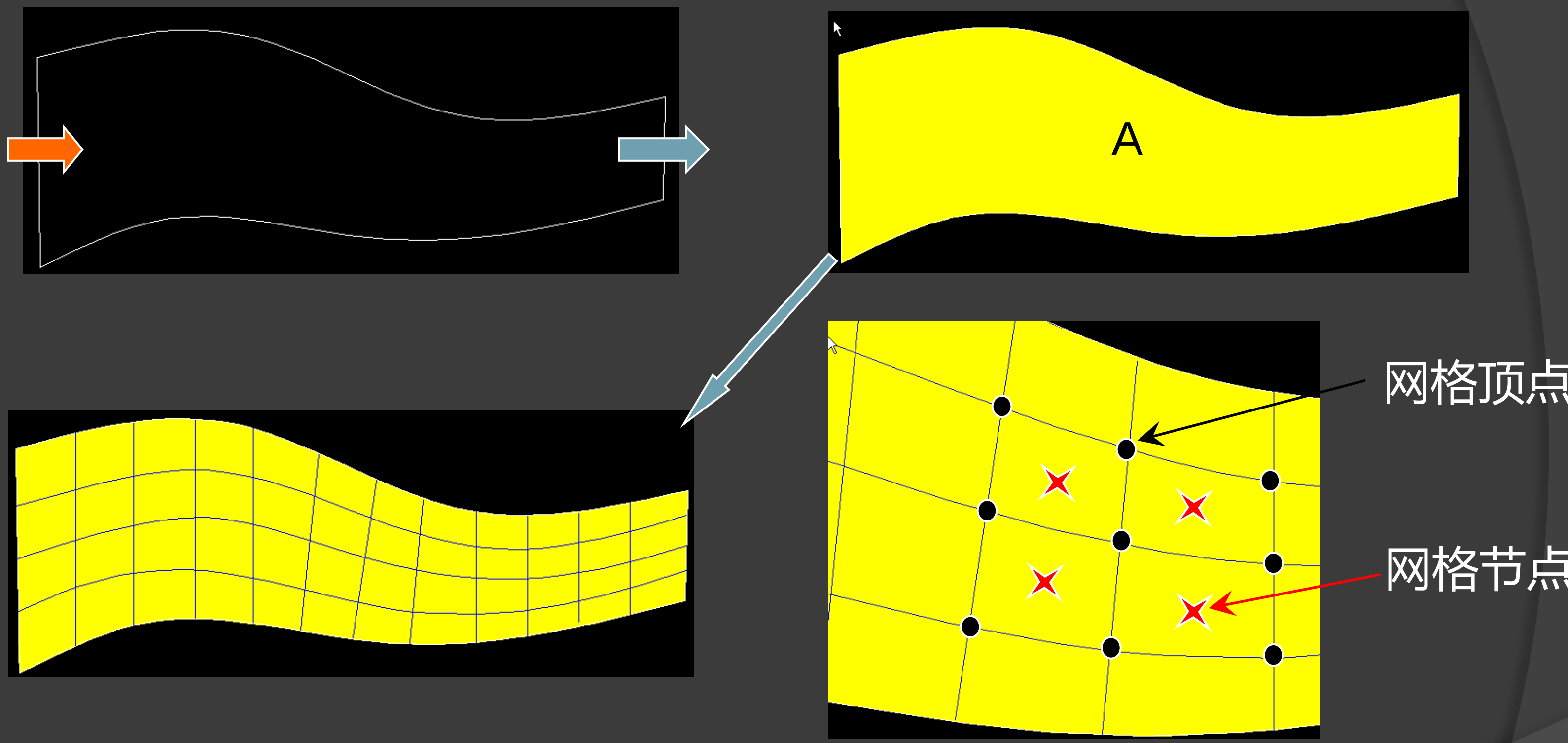
✍对于计算域A，流动参数的分布是连续的，具有无穷个值

✍采用数值计算的方法只能将计算域A的流动参数用有限个点上的分布来描述，这些点上的数值为离散解；

✍将计算域A表述成有限个点的过程成为区域离散（网格划分）

✍在计算域内，求解控制方程，可得到离散点的流场信息。但偏微分方程无法直接求解，需转变为离散方程求解，这个过程称为方程离散；

1.2 区域离散化初步

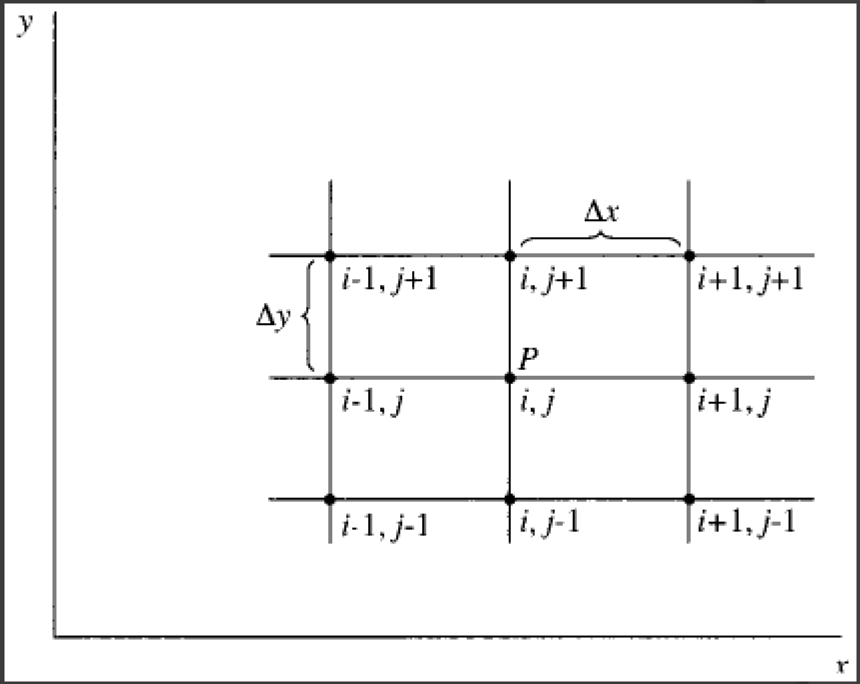


区域离散

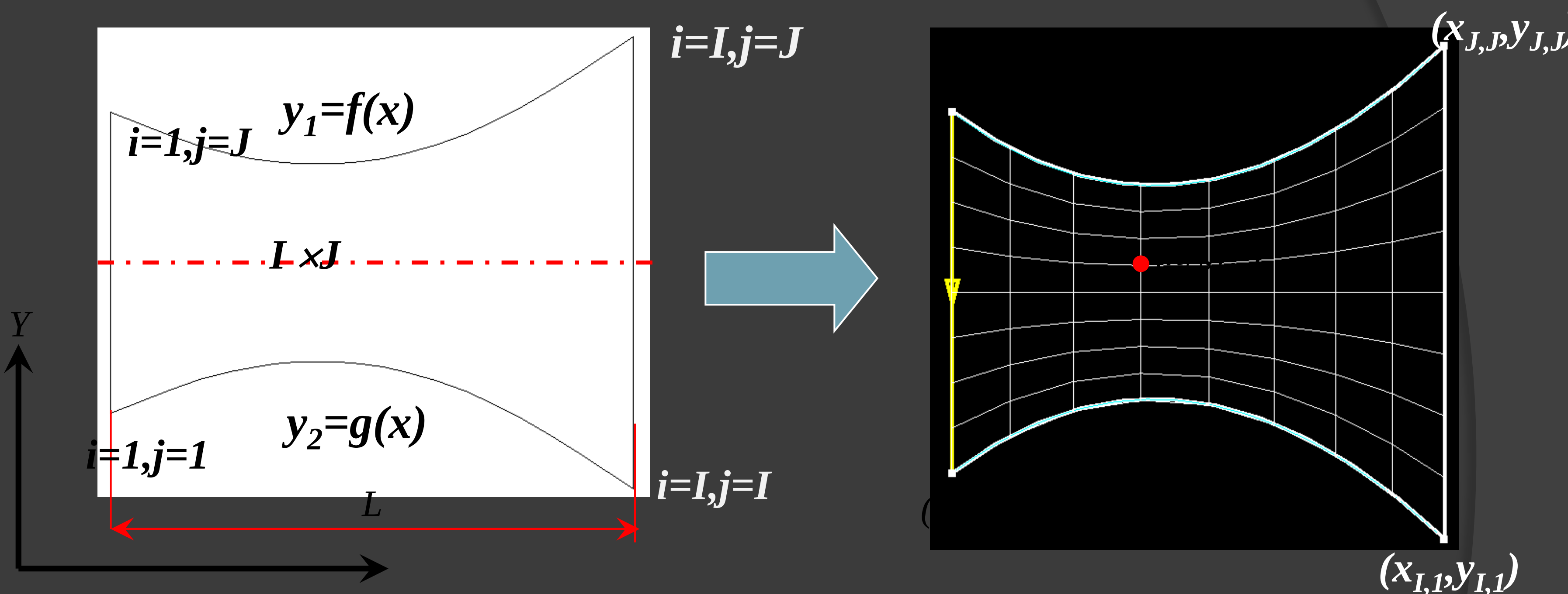
又称网格剖分，是将流动区域划分为网格，并从网格中选取节点的过程。

1.2.1 离散网格

对于N-S方程或者Euler方程的偏微分形式，其未知量的解析解都可以表示成(x,y)的函数，通过该函数可以得到任意位置点的流场信息。但绝大部分情况，找不到解析解；而用将微分形式在右图所示的格点上转换为离散形式后，方程转化为代数方程，可以求解网格点上的流场信息。



1.2.2 网格划分的方法



网格划分方法：

$$x_{i,j} = x_1 + \frac{(x_I - x_1)}{I} \cdot (i-1)$$

$$y_{i,j} = g(x_{i,j}) + (f(x_{i,j}) - g(x_{i,j}))\frac{j-1}{J}$$

$$i = 2 \text{ to } I,\ j = 2 \text{ to } J$$

1.3 有限差分

目的：以代数差分替换偏微分，实现微分方程到代数方程的转换。

1.3.1 泰勒展开

对于x的连续函数$f(x)$，如其在x点有任意阶导数，则f在$x+\Delta x$处的值可以用下式计算：

$$f(x+\Delta x)=f(x)+\frac{\partial f}{\partial x}\Delta x+\frac{\partial^2 f}{\partial x^2}\frac{(\Delta x)^2}{2}+\cdots\frac{\partial^n f}{\partial x^n}\frac{(\Delta x)^n}{n!}+\cdots$$

$$f(x+\Delta x)=\underbrace{f(x)}_{\text{初估值}}+\underbrace{\frac{\partial f}{\partial x}\Delta x}_{\text{斜率的影响}}+\underbrace{\frac{\partial^2 f}{\partial x^2}\frac{(\Delta x)^2}{2}}_{\text{曲率的影响}}+\cdots$$

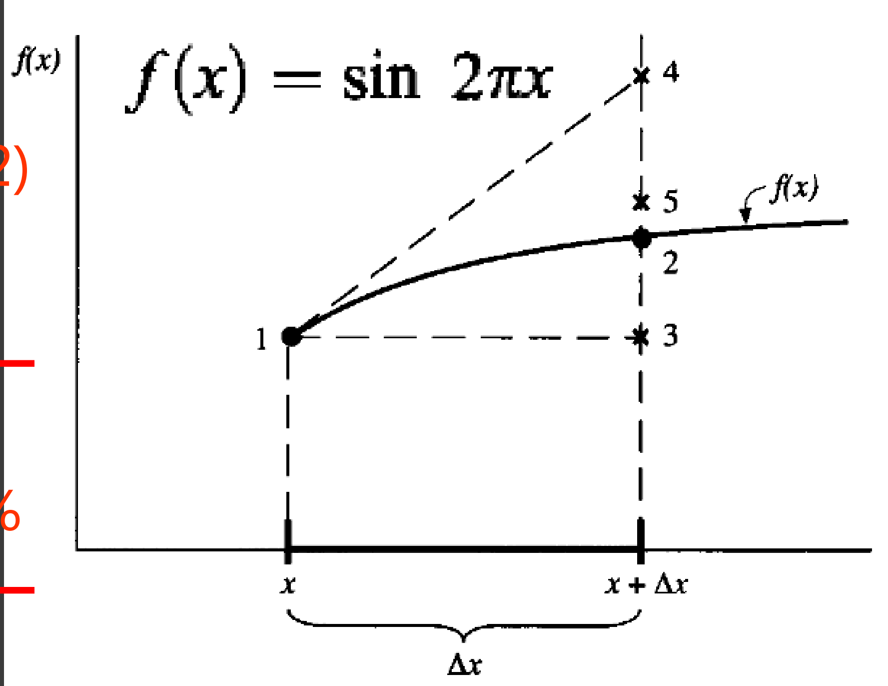
1.3.1 泰勒展开-应用

$x=0.2: \quad f(x)=0.9511$

假设$\Delta x=0.02$:

$x=0.22: \quad f(x)=0.9823$

精确解(点1,2)



初估值(点3)

$f(0.22)\approx f(0.2)=0.9511$

误差：3.167%

引入斜率的影响(点4)

$$f(x+\Delta x)\approx f(x)+\frac{\partial f}{\partial x}\Delta x$$

$$f(0.22)\approx f(0.2)+2\pi\cos[2\pi(0.2)](0.02)$$

$$\approx 0.9511+0.388=0.9899$$

误差：0.775%

1.3.1 泰勒展开-应用

引入曲率的影响(点5)

$$f(x+\Delta x) \approx f(x) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial^2 f}{\partial x^2}\frac{(\Delta x)^2}{2}$$

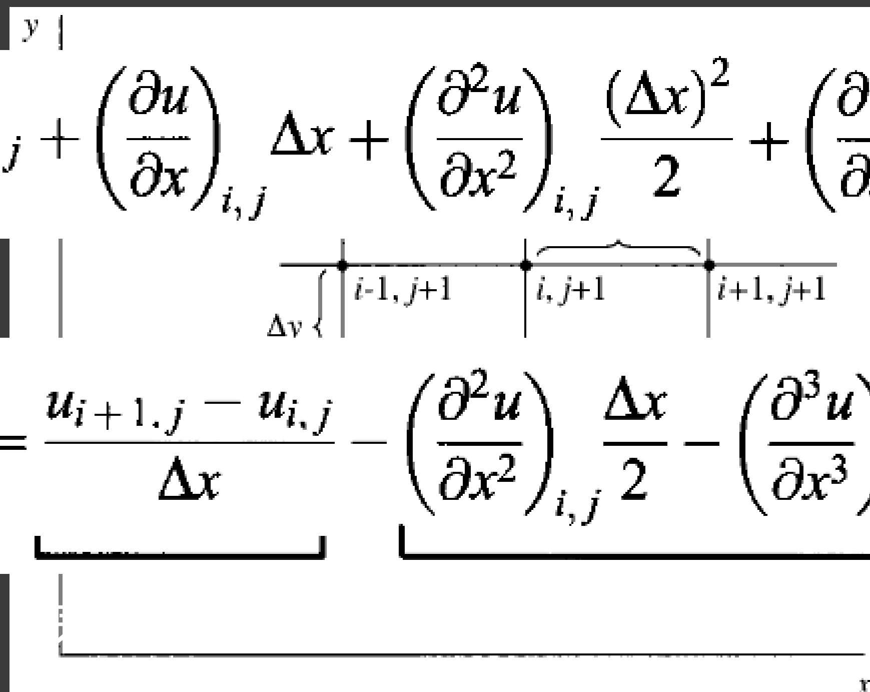
$$\begin{aligned} f(0.22) &\approx f(0.2) + 2\pi\cos[2\pi(0.2)](0.02) - 4\pi^2\sin[2\pi(0.2)]\frac{(0.02)^2}{2} \\ &\approx 0.9511 + 0.0388 - 0.0075 \\ &\approx 0.9824 \end{aligned}$$

误差：
0.01%

阶次越高，误差越小！

1.3.2 差分格式构造-向前差分

在网格节点P上的信息u：???



$$u_{i+1,j} = u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2} + \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^3}{6} + \cdots$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \underbrace{\frac{u_{i+1,j} - u_{i,j}}{\Delta x}}_{} \underbrace{- \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{\Delta x}{2} - \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^2}{6} + \cdots}_{}$$

$$\boxed{\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x)}$$

一阶向前差分

1.3.2 差分格式构造-向后差分

在网格节点上的信息u：

$$u_{i-1,j} = u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j}(-\Delta x) + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}\frac{(-\Delta x)^2}{2} + \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j}\frac{(-\Delta x)^3}{6} + \cdots$$

$$\boxed{\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x} + \mathrm{O}(\Delta x)}$$

一阶向后差分

1.3.3 差分格式构造-中心差分

$$\boxed{\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O(\Delta x)^2}$$ 二阶中心差分

1.3.4 二阶导数的差分

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + O(\Delta x)^2$$

二阶导数的二阶精度中心差分格式

1.3.4 二阶导数的差分

$$u_{i+1,j} = u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2} + \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^3}{6} + \cdots$$

$$u_{i-1,j} = u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j} (-\Delta x) + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(-\Delta x)^2}{2} + \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{(-\Delta x)^3}{6} + \cdots$$

对y求导数相减

$$\boxed{\left(\frac{\partial^2 u}{\partial x\, \partial y}\right)_{i,j} = \frac{(\partial u/\partial y)_{i+1,j} - (\partial u/\partial y)_{i-1,j}}{2\Delta x} - \left(\frac{\partial^4 u}{\partial x^3\, \partial y}\right)_{i,j} \frac{(\Delta x)^2}{12} + \cdots}$$

混合导数的二阶精度中心差分格式

$$\tau_{xx} = \lambda(\nabla \cdot \mathbf{V}) + 2\mu \frac{\partial u}{\partial x}$$

$$\tau_{yy} = \lambda(\nabla \cdot \mathbf{V}) + 2\mu \frac{\partial v}{\partial y}$$

$$\tau_{zz} = \lambda(\nabla \cdot \mathbf{V}) + 2\mu \frac{\partial w}{\partial z}$$

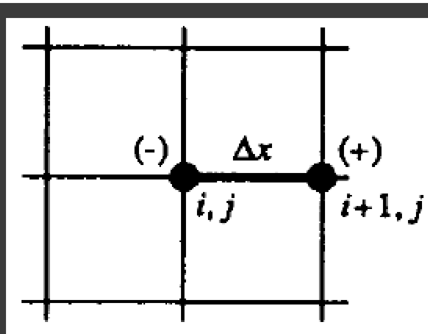
$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

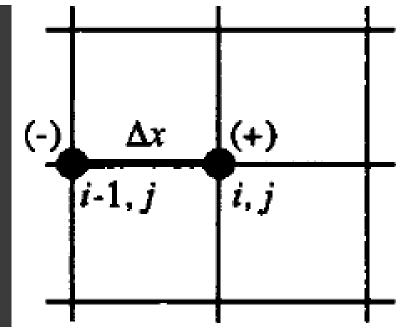
1.3.5 基本差分格式汇总

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} \simeq \frac{u_{i+1,j} - u_{i,j}}{\Delta x}$$



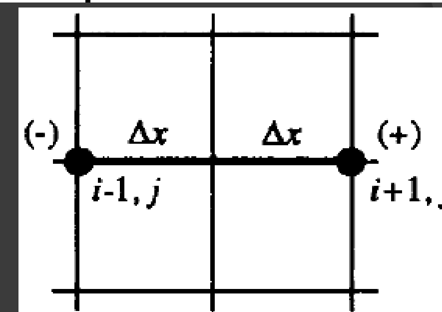
一阶前差

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x}$$



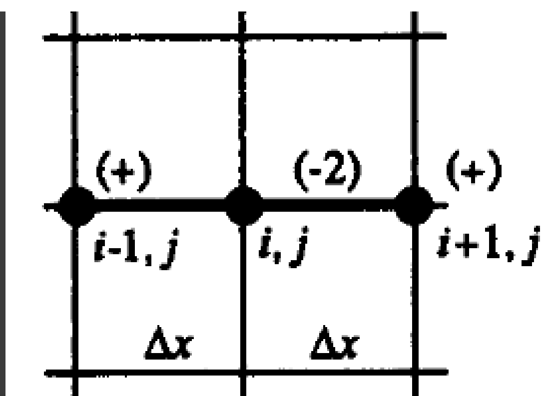
一阶后差

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}$$



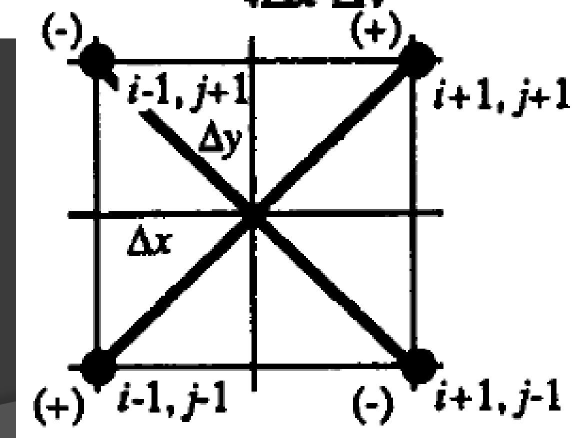
二阶中心差分

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2}$$



二阶导数二阶中心差分

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{u_{i+1,j+1} + u_{i-1,j-1} - u_{i-1,j+1} - u_{i+1,j-1}}{4\Delta x\,\Delta y}$$



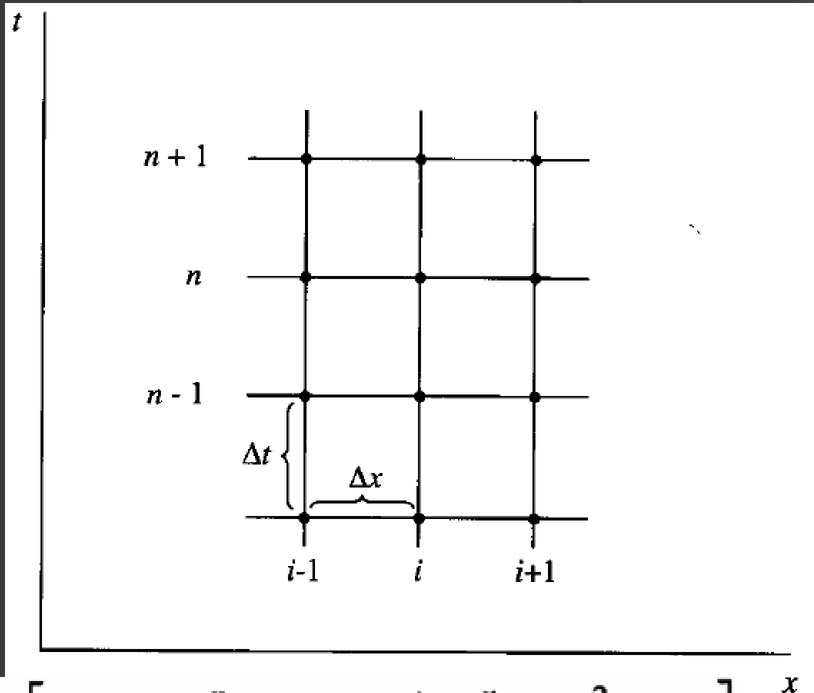
二阶混合导数二阶中心差分

1.3.8 差分方程的构造

一维非定常热传导方程：$\dfrac{\partial T}{\partial t} - \alpha \dfrac{\partial^2 T}{\partial x^2} = 0$

$$\left(\frac{\partial T}{\partial t}\right)_i^n = \frac{T_i^{n+1} - T_i^n}{\Delta t} - \left(\frac{\partial^2 T}{\partial t^2}\right)_i^n \frac{\Delta t}{2} + \cdots$$

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_i^n = \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} - \left(\frac{\partial^4 T}{\partial x^4}\right)_i^n \frac{(\Delta x)^2}{12} + \cdots$$



偏微分方程

$$\overbrace{\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = 0}^{} = \underbrace{\frac{T_i^{n+1} - T_i^n}{\Delta t} - \frac{\alpha(T_{i+1}^n - 2T_i^n + T_{i-1}^n)}{(\Delta x)^2}}_{\text{差分方程}} + \underbrace{\left[-\left(\frac{\partial^2 T}{\partial t^2}\right)_i^n \frac{\Delta t}{2} + \alpha \left(\frac{\partial^4 T}{\partial x^4}\right)_i^n \frac{(\Delta x)^2}{12} + \cdots\right]}_{\text{截断误差}}$$

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{\alpha(T_{i+1}^n - 2T_i^n + T_{i-1}^n)}{(\Delta x)^2}$$

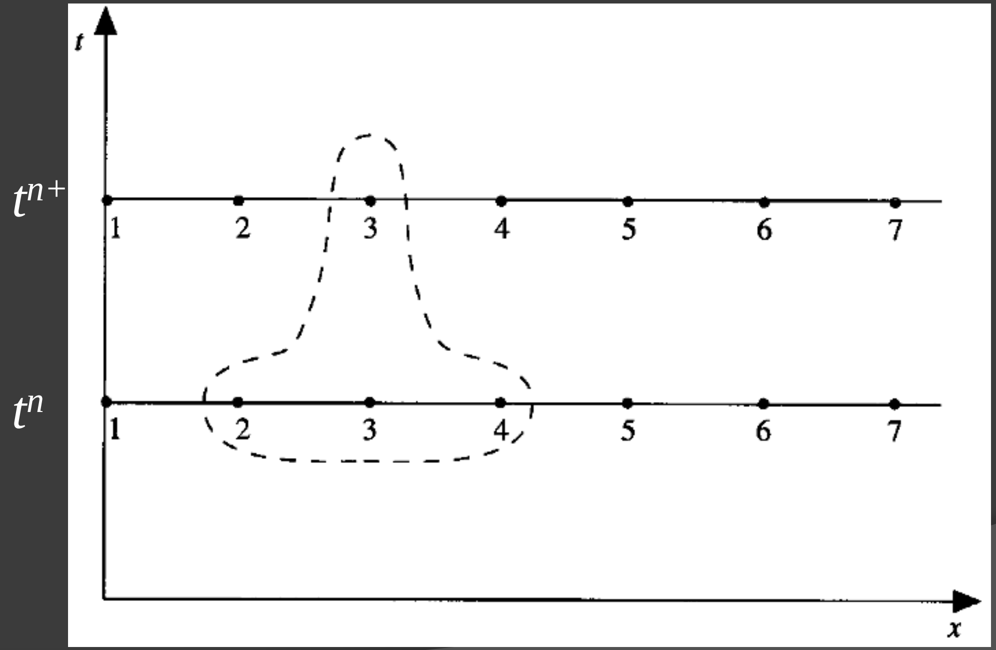
1.3.9 显式格式

一维非定常热传导方程：

$$\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = 0$$

求解表

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{\alpha(T_{i+1}^n - 2T_i^n + T_{i-1}^n)}{(\Delta x)^2}$$



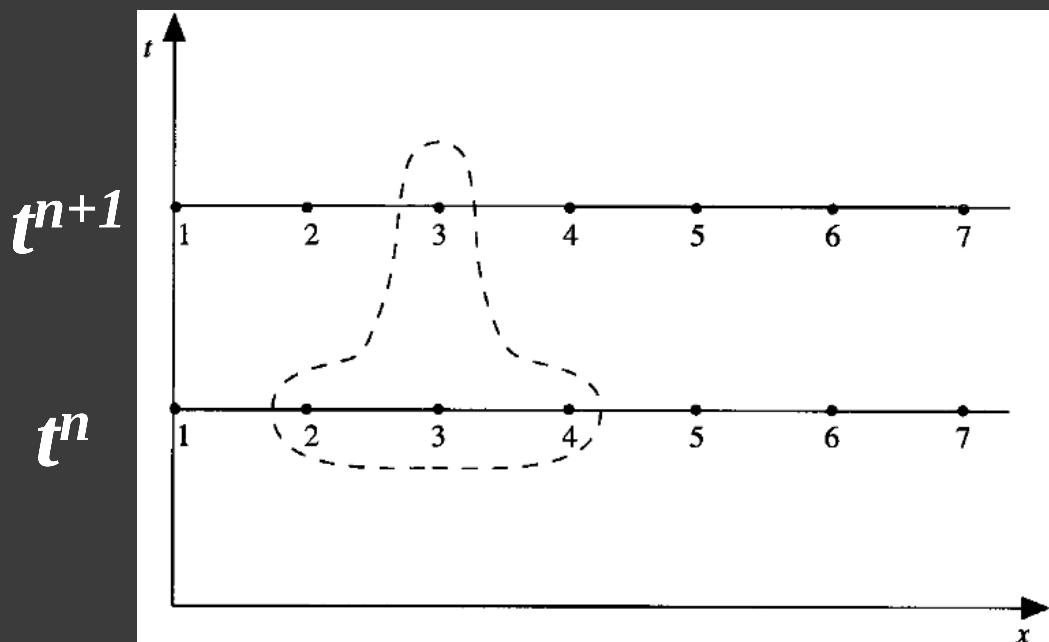
思考：为什么存在计算发散的问题？计算的速度与什么相关？参考Anderson一书。

1.3.10 半隐式格式

一维非定常热传导方程：
$$\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = 0$$

T^{n+1}_{i-1} ？

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{\alpha\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right)}{(\Delta x)^2}$$

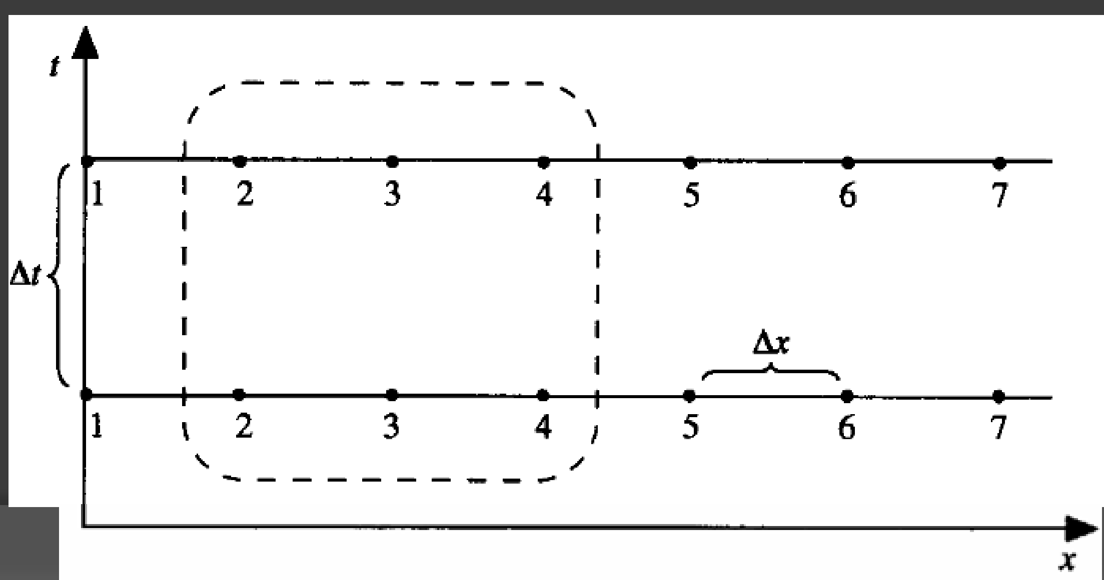


1.3.11 隐式格式

一维非定常热传导方程：$$\frac{\partial T}{\partial t}-\alpha\frac{\partial^2 T}{\partial x^2}=0$$

$$\frac{T_i^{n+1}-T_i^n}{\Delta t}=\alpha\frac{\frac{1}{2}(T_{i+1}^{n+1}+T_{i+1}^n)+\frac{1}{2}(-2T_i^{n+1}-2T_i^n)+\frac{1}{2}(T_{i-1}^{n+1}+T_{i-1}^n)}{(\Delta x)^2}$$

$$\frac{\alpha\,\Delta t}{2(\Delta x)^2}T_{i-1}^{n+1}-\left[1+\frac{\alpha\,\Delta t}{(\Delta x)^2}\right]T_i^{n+1}+\frac{\alpha\,\Delta t}{2(\Delta x)^2}T_{i+1}^{n+1}=-T_i^n-\frac{\alpha\,\Delta t}{2(\Delta t)^2}(T_{i+1}^n-2T_i^n+T_{i-1}^n)$$



1.3.10 隐式格式

$$\boxed{\frac{\alpha\,\Delta t}{2(\Delta x)^2}}T_{i-1}^{n+1}-\boxed{\left[1+\frac{\alpha\,\Delta t}{(\Delta x)^2}\right]}T_i^{n+1}+\frac{\alpha\,\Delta t}{2(\Delta x)^2}T_{i+1}^{n+1}=\boxed{-T_i^n-\frac{\alpha\,\Delta t}{2(\Delta t)^2}(T_{i+1}^n-2T_i^n+T_{i-1}^n)}$$

A B K_i

$$AT_{i-1}^{n+1}-BT_i^{n+1}+AT_{i+1}^{n+1}=K_i$$

K_2-AT_1

网格点2 $AT_1-BT_2+AT_3=K_2$ $\rightarrow$ $-BT_2+AT_3=\boxed{K'_2}$

网格点3 $AT_2-BT_3+AT_4=K_3$

网格点4 $AT_3-BT_4+AT_5=K_4$

网格点5 $AT_4-BT_5+AT_6=K_5$

网格点6 $AT_5-BT_6+AT_7=K_6$ $\rightarrow$ $AT_5-BT_6=K_6-AT_7=K'_6$

1.3.10 隐式格式

$$\begin{bmatrix} -B & A & 0 & 0 & 0 \\ A & -B & A & 0 & 0 \\ 0 & A & -B & A & 0 \\ 0 & 0 & A & -B & A \\ 0 & 0 & 0 & A & -B \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} K'_2 \\ K_3 \\ K_4 \\ K_5 \\ K'_6 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix}$$

1.4 本章小结

关于CFD → What's CFD / Why CFD

离散化基本概念 → 计算域 / 差分格式构造

差分格式 → 前向/后向/中心

方程离散 → 显式/半隐式/隐式 / 时间项/空间项

离散方程稳定性

求解过程

第二章　抛物型控制方程求解

2.1 偏微分方程性质

- 抛物型
- 双曲型
- 椭圆形

定义：自学（Anderson，Hoffman）

不同类型的方程有不同的数值解法！

2.2 抛物型方程显式解法

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

- FTCS(Forward time/ central space)
- Richardson Method
- Dufort-Frankel Method

2.2.1 FTCS – Forward time and Central space

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

$$u_i^{n+1} = u_i^n + \frac{\alpha(\Delta t)}{(\Delta x)^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

精度： $[(\Delta t),\ (\Delta x)^2]$

稳定性条件： $\alpha \Delta t/(\Delta x)^2 \ \leq \ 1/2$

2.2.2 Richardson Method

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

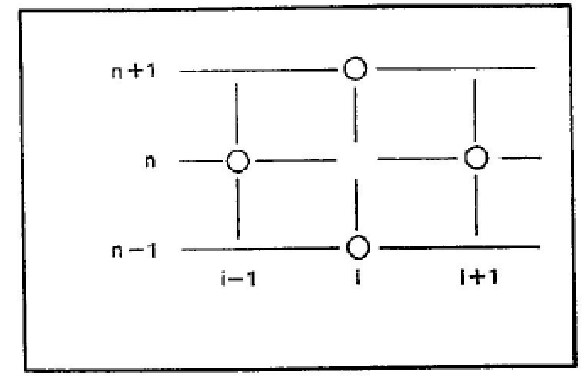
精度： $[(\Delta t)^2, (\Delta x)^2]$

稳定性条件：无条件不稳定！☹

2.2.3 Dufort-Frankel Method

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \alpha \frac{u_{i+1}^n - 2\dfrac{u_i^{n+1} + u_i^{n-1}}{2} + u_{i-1}^n}{(\Delta x)^2}$$



$$u_i^{n+1} = u_i^{n-1} + \frac{2\alpha(\Delta t)}{(\Delta x)^2}\left[u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n\right]$$

精度：$[(\Delta t)^2, (\Delta x)^2, (\Delta t/\Delta x)^2]$

稳定性条件：无条件稳定！☺

✍说明:精度与第一步结果有关

2.3 抛物型方程隐式解法

$$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}$$

- Laasonen Method
- Crank-Nicolson Method
- Beta Formulation

2.3.1 Laasonen Method

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_i^{n+1} - u_i^n}{(\Delta t)} = \alpha \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2}$$

精度： $[(\Delta t), (\Delta x)^2]$

稳定性条件：无条件稳定

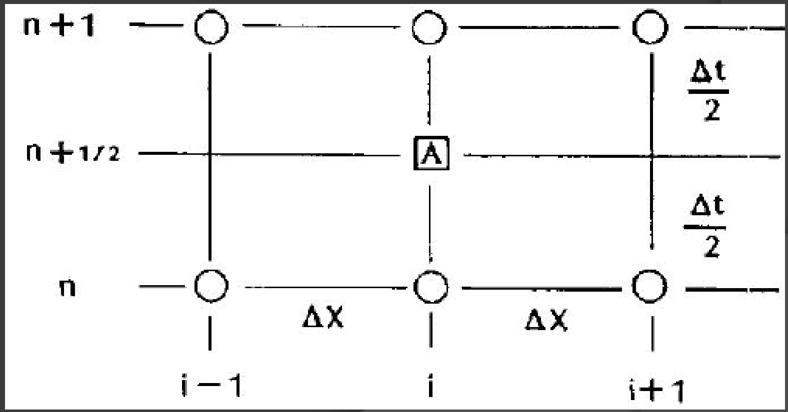
2.3.2 Crank-Nicolson Method

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \left(\frac{1}{2}\right) \left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \right]$$

精度： $[(\Delta t)^2, (\Delta x)^2]$

稳定性条件：无条件稳定！



$$\frac{u_i^{n+\frac{1}{2}} - u_i^n}{\frac{\Delta t}{2}} = \alpha \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{(\Delta x)^2}$$

$$\frac{u_i^{n+1} - u_i^{n+\frac{1}{2}}}{\frac{\Delta t}{2}} = \alpha \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{(\Delta x)^2}$$

2.3.3 Beta Formulation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \left[\beta \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} + (1-\beta) \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \right]$$

稳定性条件：

1/2≤β≤1无条件稳定！(β=1/2→Crank-Nicolson, β=1→Lassnonen）

0≤β<1/2有条件稳定(β=0→FTCS)

二维抛物型方程解法

2.4.1 二维抛物型方程显式解法

$$\frac{\partial u}{\partial t}=\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}=\alpha\left[\frac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{(\Delta x)^2}+\frac{u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n}}{(\Delta y)^2}\right]$$

精度：$[(\Delta t),(\Delta x)^2,(\Delta y)^2]$

稳定性条件：$\left[\frac{\alpha\Delta t}{(\Delta x)^2}+\frac{\alpha\Delta t}{(\Delta y)^2}\right]\le\frac{1}{2}$

定义：$d_x=\frac{\alpha\Delta t}{(\Delta x)^2}$ $d_y=\frac{\alpha\Delta t}{(\Delta y)^2}$ → $(d_x+d_y)\le\frac{1}{2}$

取相同的空间步长，则最大稳定性条件为dx=dy=0.25，求解效率太低！

2.4.2 二维抛物型方程隐式解法

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{(\Delta t)} = \alpha\left[\frac{u_{i+1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{(\Delta x)^2}+\frac{u_{i,j+1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{(\Delta y)^2}\right]$$

$$d_x u_{i+1,j}^{n+1} + d_x u_{i-1,j}^{n+1} - (2d_x+2d_y+1)u_{i,j}^{n+1} + d_y u_{i,j-1}^{n+1} + d_y u_{i,j+1}^{n+1} = -u_{i,j}^{n}$$

$$a_{i,j}u_{i+1,j}^{n+1}+b_{i,j}u_{i-1,j}^{n+1}+c_{i,j}u_{i,j}^{n+1}+d_{i,j}u_{i,j-1}^{n+1}+e_{i,j}u_{i,j+1}^{n+1}=f_{i,j}^{n}$$

$$a_{2,2}u_{3,2}+c_{2,2}u_{2,2}+e_{2,2}u_{2,3}=f_{2,2}-b_{2,2}u_{1,2}-d_{2,2}u_{2,1}$$

$$a_{2,3}u_{3,3}+c_{2,3}u_{2,3}+d_{2,3}u_{2,2}+e_{2,3}u_{2,4}=f_{2,3}-b_{2,3}u_{1,3}$$

$$a_{2,4}u_{3,4}+c_{2,4}u_{2,4}+d_{2,4}u_{2,3}=f_{2,4}-b_{2,4}u_{1,4}-e_{2,4}u_{2,5}$$

$$a_{3,2}u_{4,2}+b_{3,2}u_{2,2}+c_{3,2}u_{3,2}+e_{3,2}u_{3,3}=f_{3,2}+d_{3,2}u_{3,1}$$

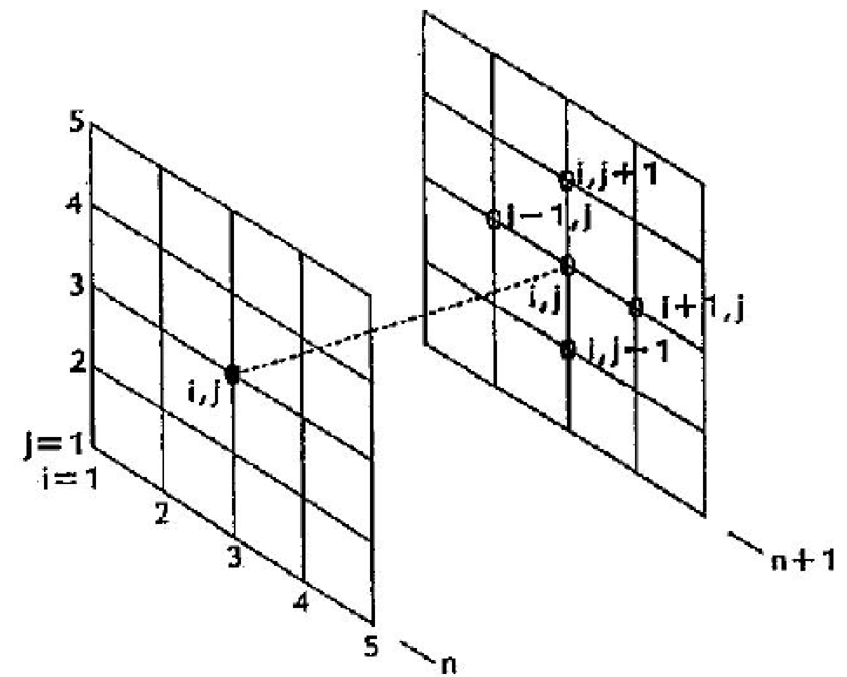
$$a_{3,3}u_{4,3}+b_{3,3}u_{2,3}+c_{3,3}u_{3,3}+d_{3,3}u_{3,2}+e_{3,3}u_{3,4}=f_{3,3}$$

$$a_{3,4}u_{4,4}+b_{3,4}u_{2,4}+c_{3,4}u_{3,4}+d_{3,4}u_{3,3}=f_{3,4}-e_{3,4}u_{3,5}$$

$$b_{4,2}u_{3,2}+c_{4,2}u_{4,2}+e_{4,2}u_{4,3}=f_{4,2}-a_{4,2}u_{5,2}-d_{4,2}u_{4,1}$$

$$b_{4,3}u_{3,3}+c_{4,3}u_{4,3}+d_{4,3}u_{4,2}+e_{4,3}u_{4,4}=f_{4,3}-a_{4,3}u_{5,3}$$

$$b_{4,4}u_{3,4}+c_{4,4}u_{4,4}+d_{4,4}u_{4,3}=f_{4,4}-a_{4,4}u_{5,4}-e_{4,4}u_{4,5}$$



2.4.2 二维抛物型方程隐式解法

$$
\begin{bmatrix}
c_{2,2} & e_{2,2} & 0 & a_{2,2} & 0 & 0 & 0 & 0 & 0 \\
d_{2,3} & c_{2,3} & e_{2,3} & 0 & a_{2,3} & 0 & 0 & 0 & 0 \\
0 & d_{2,4} & c_{2,4} & 0 & 0 & a_{2,4} & 0 & 0 & 0 \\
b_{3,2} & 0 & 0 & c_{3,2} & e_{3,2} & 0 & a_{3,2} & 0 & 0 \\
0 & b_{3,3} & 0 & d_{3,3} & c_{3,3} & e_{3,3} & 0 & a_{3,3} & 0 \\
0 & 0 & b_{3,4} & 0 & d_{3,4} & c_{3,4} & 0 & 0 & a_{3,4} \\
0 & 0 & 0 & b_{4,2} & 0 & 0 & c_{4,2} & e_{4,2} & 0 \\
0 & 0 & 0 & 0 & b_{4,3} & 0 & d_{4,3} & c_{4,3} & e_{4,3} \\
0 & 0 & 0 & 0 & 0 & b_{4,4} & 0 & d_{4,4} & c_{4,4}
\end{bmatrix}
\begin{bmatrix}
u_{2,2} \\ u_{2,3} \\ u_{2,4} \\ u_{3,2} \\ u_{3,3} \\ u_{3,4} \\ u_{4,2} \\ u_{4,3} \\ u_{4,4}
\end{bmatrix}
=
\begin{bmatrix}
f_{2,2} - b_{2,2}u_{1,2} - d_{2,2}u_{2,1} \\
f_{2,3} - b_{2,3}u_{1,3} \\
f_{2,4} - b_{2,4}u_{1,4} - e_{2,4}u_{2,5} \\
f_{3,2} - d_{3,2}u_{3,1} \\
f_{3,3} \\
f_{3,4} - e_{3,4}u_{3,5} \\
f_{4,2} - a_{4,2}u_{5,2} - d_{4,2}u_{4,1} \\
f_{4,3} - a_{4,3}u_{5,3} \\
f_{4,4} - a_{4,4}u_{5,4} - e_{4,4}u_{4,5}
\end{bmatrix}
$$

二维方程为5对角矩阵，求解量太大！

2.4.3 二维抛物型方程ADI解法 Alternating Direction Implicit method

$$\frac{\partial u}{\partial t}=\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{u_{i,j}^{n+\frac{1}{2}}-u_{i,j}^{n}}{\left(\frac{\Delta t}{2}\right)}=\alpha\left[\frac{u_{i+1,j}^{n+\frac{1}{2}}-2u_{i,j}^{n+\frac{1}{2}}+u_{i-1,j}^{n+\frac{1}{2}}}{(\Delta x)^2}+\frac{u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n}}{(\Delta y)^2}\right]$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n+\frac{1}{2}}}{\left(\frac{\Delta t}{2}\right)}=\alpha\left[\frac{u_{i+1,j}^{n+\frac{1}{2}}-2u_{i,j}^{n+\frac{1}{2}}+u_{i-1,j}^{n+\frac{1}{2}}}{(\Delta x)^2}+\frac{u_{i,j+1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{(\Delta y)^2}\right]$$

精度: $[(\Delta t)^2, (\Delta x)^2, (\Delta y)^2]$ 无条件稳定

定义系数: $d_1=\frac{1}{2}d_x=\frac{1}{2}\frac{\alpha\Delta t}{(\Delta x)^2}$ $d_2=\frac{1}{2}d_y=\frac{1}{2}\frac{\alpha\Delta t}{(\Delta y)^2}$

$$-d_1u_{i-1,j}^{n+\frac{1}{2}}+(1+2d_1)u_{i,j}^{n+\frac{1}{2}}-d_1u_{i+1,j}^{n+\frac{1}{2}}=d_2u_{i,j+1}^{n}+(1-2d_2)u_{i,j}^{n}+d_2u_{i,j-1}^{n}$$

$$-d_2u_{i,j-1}^{n+1}+(1+2d_2)u_{i,j}^{n+1}-d_2u_{i,j+1}^{n+1}=d_1u_{i+1,j}^{n+\frac{1}{2}}+(1-2d_1)u_{i,j}^{n+\frac{1}{2}}+d_1u_{i-1,j}^{n+\frac{1}{2}}$$

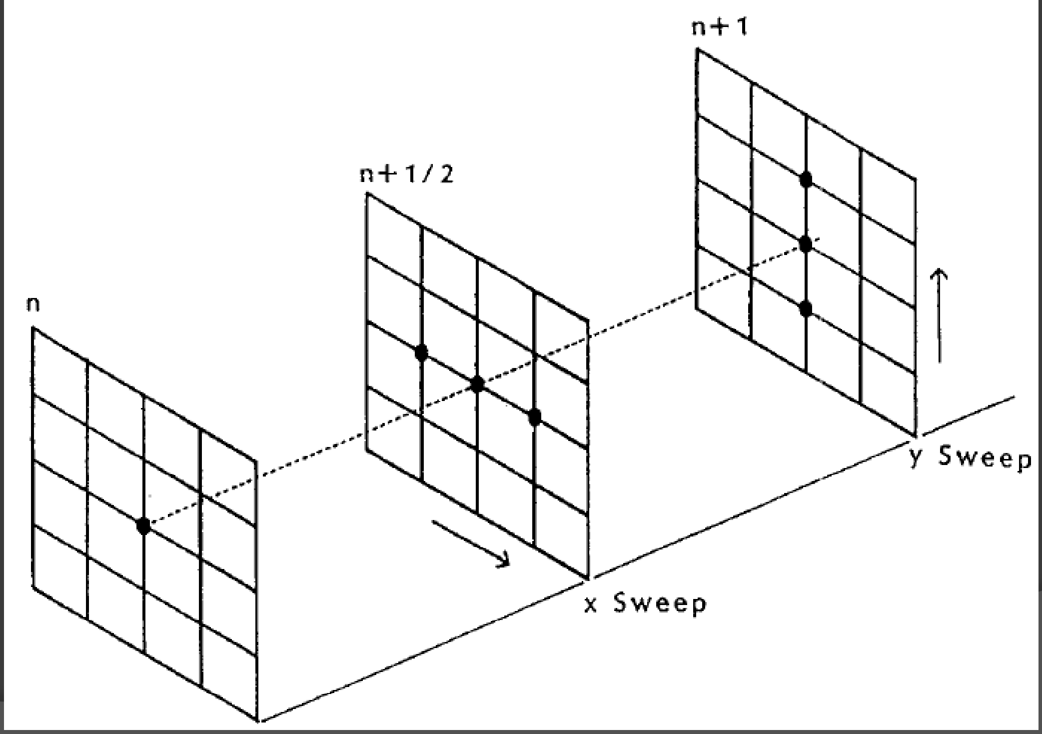
2.4.3 二维抛物型方程ADI解法

$$-d_1u_{i-1,j}^{n+\frac{1}{2}}+(1+2d_1)u_{i,j}^{n+\frac{1}{2}}-d_1u_{i+1,j}^{n+\frac{1}{2}}=d_2u_{i,j+1}^{n}+(1-2d_2)u_{i,j}^{n}+d_2u_{i,j-1}^{n}$$

x方向上为隐式，y方向上为显式→三对角矩阵

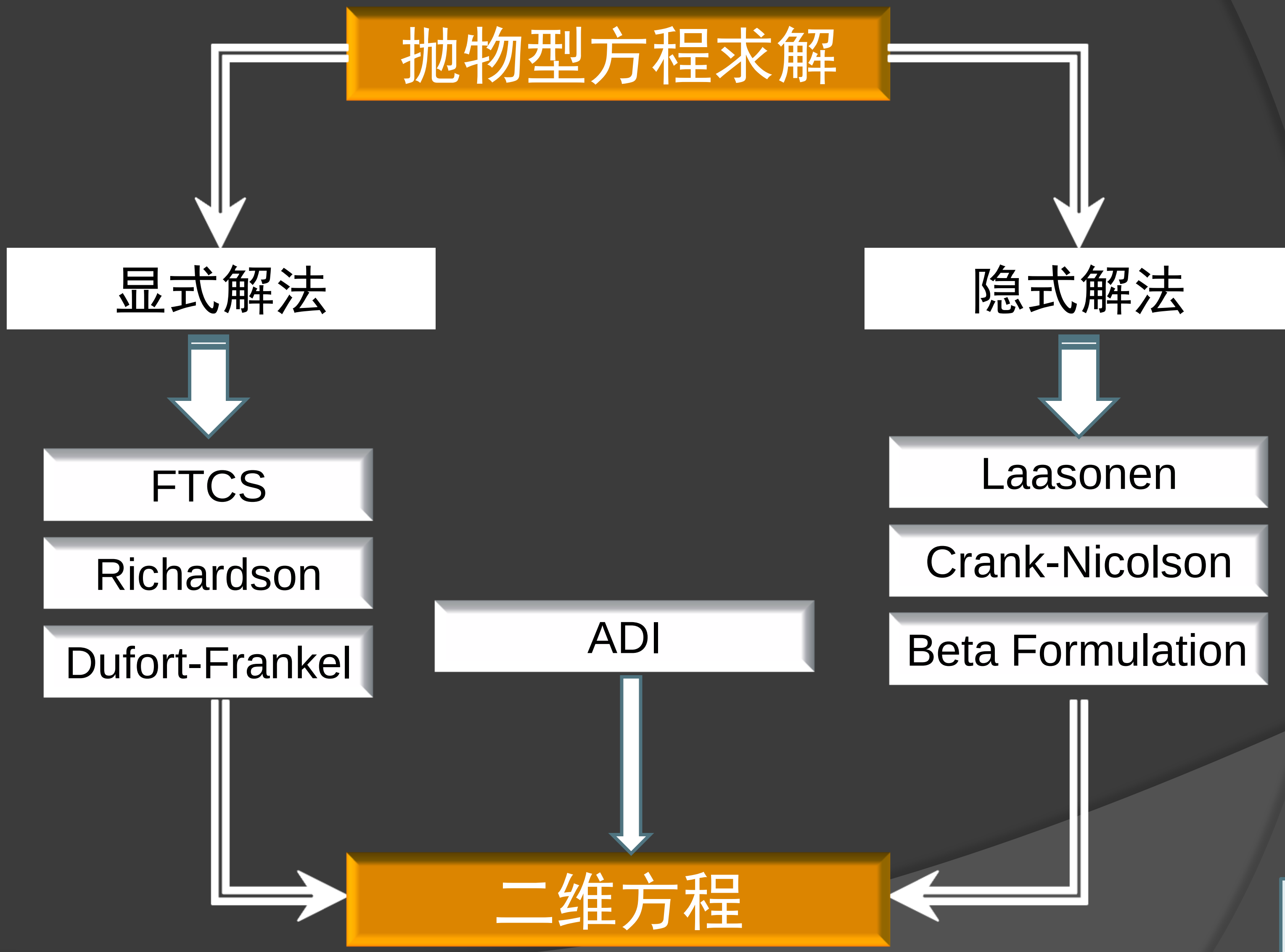
$$-d_2u_{i,j-1}^{n+1}+(1+2d_2)u_{i,j}^{n+1}-d_2u_{i,j+1}^{n+1}=d_1u_{i+1,j}^{n+\frac{1}{2}}+(1-2d_1)u_{i,j}^{n+\frac{1}{2}}+d_1u_{i-1,j}^{n+\frac{1}{2}}$$

y方向上为隐式，x方向上为显式→三对角矩阵



应用举例：Hoffman, Vol I, Page80

2.5 本章小结



抛物型方程求解
显式解法
隐式解法
FTCS
Richardson
Dufort-Frankel
Laasonen
Crank-Nicolson
Beta Formulation
ADI
二维方程

FTCS(Forward Time-Central Space)

$$u_i^{n+1} = u_i^n + \frac{\alpha(\Delta t)}{(\Delta x)^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

$$[(\Delta t),\ (\Delta x)^2]$$

$$\alpha \Delta t/(\Delta x)^2 \ \leq \ 1/2$$

Richardson

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

$$[(\Delta t)^2, (\Delta x)^2]$$

无条件不稳定！

Dufort-Frankel

$$u_i^{n+1} = u_i^{n-1} + \frac{2\alpha(\Delta t)}{(\Delta x)^2}\left[u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n\right]$$

$$\left[(\Delta t)^2,\ (\Delta x)^2,\ (\Delta t/\Delta x)^2\right]$$

无条件稳定！

Laasonen

$$\frac{u_i^{n+1} - u_i^n}{(\Delta t)} = \alpha \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2}$$

$$[(\Delta t),\ (\Delta x)^2]$$

无条件稳定

Crank-Nicolson

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\alpha\left(\frac{1}{2}\right)\left[\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{(\Delta x)^2}+\frac{u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n}}{(\Delta x)^2}\right]$$

$$[(\Delta t)^2, (\Delta x)^2]$$

无条件稳定

Beta Formulation

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\alpha\left[\beta\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{(\Delta x)^2}+(1-\beta)\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{(\Delta x)^2}\right]$$

稳定性条件：

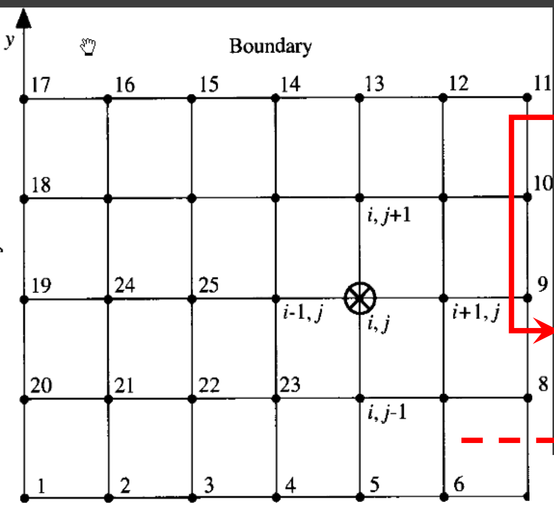
1/2≤β≤1无条件稳定！(β=1/2→Crank-Nicolson, β=1→Lassnonen）

0≤β<1/2有条件稳定(β=0→FTCS)

第三章　椭圆型控制方程求解

3.1松弛迭代法

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0$$



$$\frac{\Phi_{i+1,j}-2\Phi_{i,j}+\Phi_{i-1,j}}{(\Delta x)^2}+\frac{\Phi_{i,j+1}-2\Phi_{i,j}+\Phi_{i,j-1}}{(\Delta y)^2}=0$$

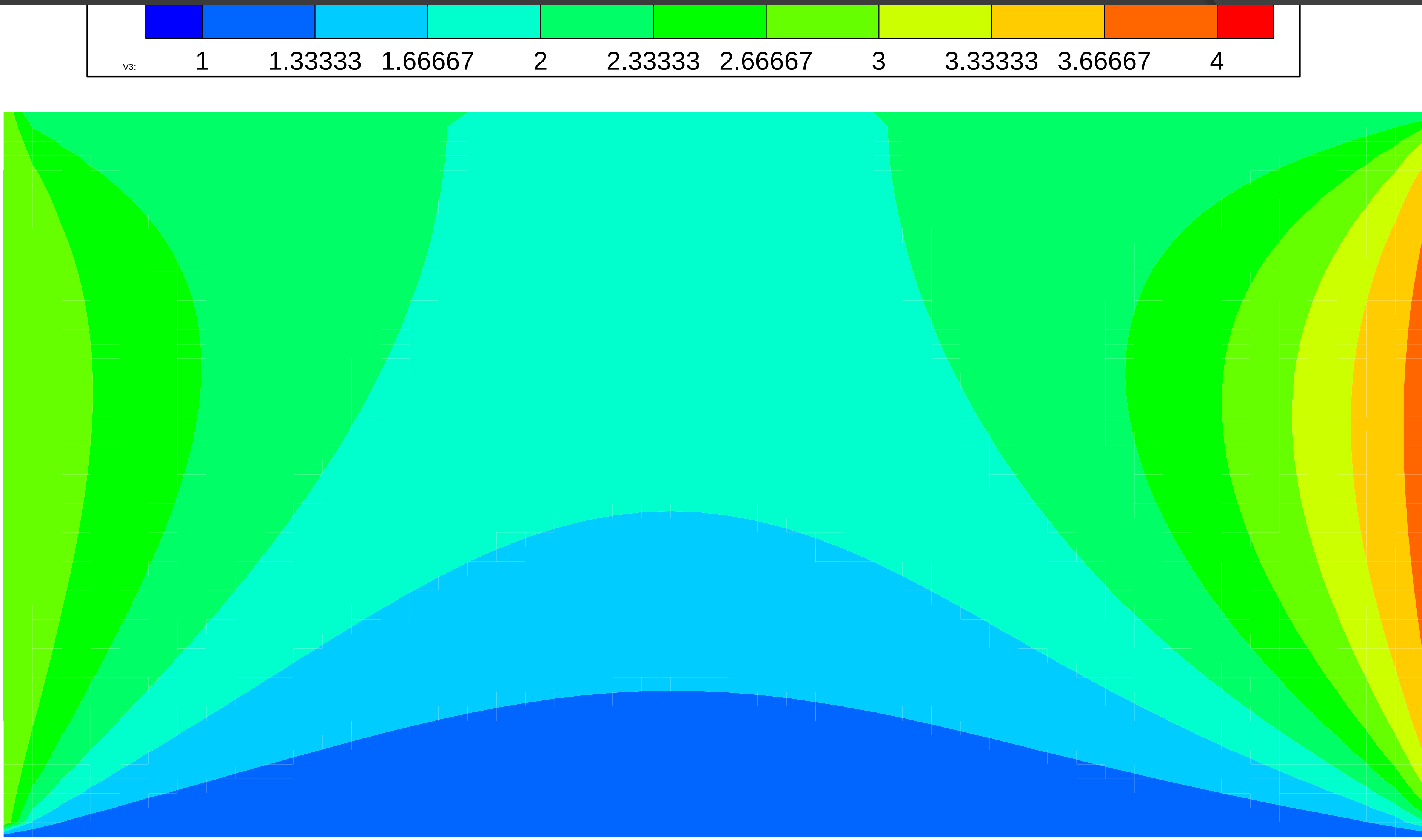
$$\Phi_{i,j}^{n+1}=\frac{(\Delta x)^2(\Delta y)^2}{2(\Delta y)^2+2(\Delta x)^2}\left[\frac{\Phi_{i+1,j}^{n}+\Phi_{i-1,j}^{n}}{(\Delta x)^2}+\frac{\Phi_{i,j+1}^{n}+\Phi_{i,j-1}^{n}}{(\Delta y)^2}\right]$$

$$\overline{\Phi_{i,j}^{n+1}}=\frac{(\Delta x)^2(\Delta y)^2}{2(\Delta y)^2+2(\Delta x)^2}\left[\frac{\Phi_{i+1,j}^{n}+\Phi_{i-1,j}^{n+1}}{(\Delta x)^2}+\frac{\Phi_{i,j+1}^{n}+\Phi_{i,j-1}^{n+1}}{(\Delta y)^2}\right]$$

$$\Phi_{i,j}^{n+1}=\Phi_{i,j}^{n}+\omega\left(\overline{\Phi_{i,j}^{n+1}}-\Phi_{i,j}^{n}\right)$$

ω<1:亚松弛
ω>1:超松弛

程序演示


V3:
1
1.33333
1.66667
2
2.33333
2.66667
3
3.33333
3.66667
4

第四章　双曲型控制方程求解

4.1 显式方法

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} \qquad a > 0$$

Euler's FTFS $\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_{i+1}^n - u_i^n}{\Delta x}$ 无条件不稳定?? $O(\Delta t,\ \Delta x)$

Euler's FTCS $\frac{u_i^{n+1} - \boxed{u_i^n}}{\Delta t} = -a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$ 无条件不稳定?? $[(\Delta t),\ (\Delta x)^2]$

Euler's FTBS $\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_i^n - u_{i-1}^n}{\Delta x}$ $a\Delta t/\Delta x \leq 1$ $O(\Delta t,\ \Delta x)$

Lax method $u_i^{n+1} = \boxed{\frac{1}{2}(u_{i+1}^n + u_{i-1}^n)} - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$ $a\Delta t/\Delta x \leq 1$

Midpoint leapfrog $\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = -a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$ $a\Delta t/\Delta x \leq 1$ $[(\Delta t)^2,\ (\Delta x)^2]$

4.1 显式方法

Lax-Wendroff method

$$u_i^{n+1} = u_i^n + \frac{\partial u}{\partial t}\Delta t + \frac{\partial^2 u}{\partial t^2}\frac{(\Delta t)^2}{2!} + O(\Delta t)^3$$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 u}{\partial t^2} = -a\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = -a\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = a^2\frac{\partial^2 u}{\partial x^2}$$

$$u_i^{n+1} = u_i^n + \left(-a\frac{\partial u}{\partial x}\right)\Delta t + \frac{(\Delta t)^2}{2}\left(a^2\frac{\partial^2 u}{\partial x^2}\right)$$

$$u_i^{n+1} = u_i^n - a\Delta t\left[\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\right] + \frac{1}{2}a^2(\Delta t)^2\left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}\right]$$

$$[(\Delta t)^2,\ (\Delta x)^2]$$

$$a\Delta t/\Delta x \leq 1$$

4.2 隐式方法

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} \qquad a > 0$$

Euler's FTCS

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{a}{2\Delta x}\left[u_{i+1}^{n+1} - u_{i-1}^{n+1}\right] \Rightarrow \frac{1}{2}c\,u_{i-1}^{n+1} - u_i^{n+1} - \frac{1}{2}c\,u_{i+1}^{n+1} = -u_i^n$$

三对角矩阵 $O[(\Delta t, \Delta x)^2]$

Implicit first upwind differencing method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x} \Rightarrow c\,u_{i-1}^{n+1} - (1+c)u_i^{n+1} = -u_i^n$$

$$A_i u_{i-1}^{n+1} + B_i u_i^{n+1} = D_i$$

$$u_i^{n+1} = \frac{D_i - A_i u_{i-1}^{n+1}}{B_i}$$

二对角矩阵 $O(\Delta t, \Delta x)$

4.2 隐式方法

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} \qquad a > 0$$

Crank-Nicolson

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{1}{2}\left[\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} + \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\right]$$

$$\frac{1}{4}c\,u_{i-1}^{n+1} - u_i^{n+1} - \frac{1}{4}c\,u_{i+1}^{n+1} = -u_i^n + \frac{1}{4}c\left(u_{i+1}^n - u_{i-1}^n\right)$$

三对角矩阵 $[(\Delta t)^2,\ (\Delta x)^2]$

多维问题，矩阵求解计算量过大 $\longrightarrow$ ADI

4.3 多步解法—MacCormack Method

$$\frac{u_i^* - u_i^n}{\Delta t} = -a\frac{u_{i+1}^n - u_i^n}{\Delta x}$$

$$\frac{u_i^{n+1} - u_i^{n+\frac{1}{2}}}{\frac{1}{2}\Delta t} = -a\frac{u_i^* - u_{i-1}^*}{\Delta x}$$

$$u_i^{n+\frac{1}{2}} = \frac{1}{2}\left(u_i^n + u_i^*\right)$$

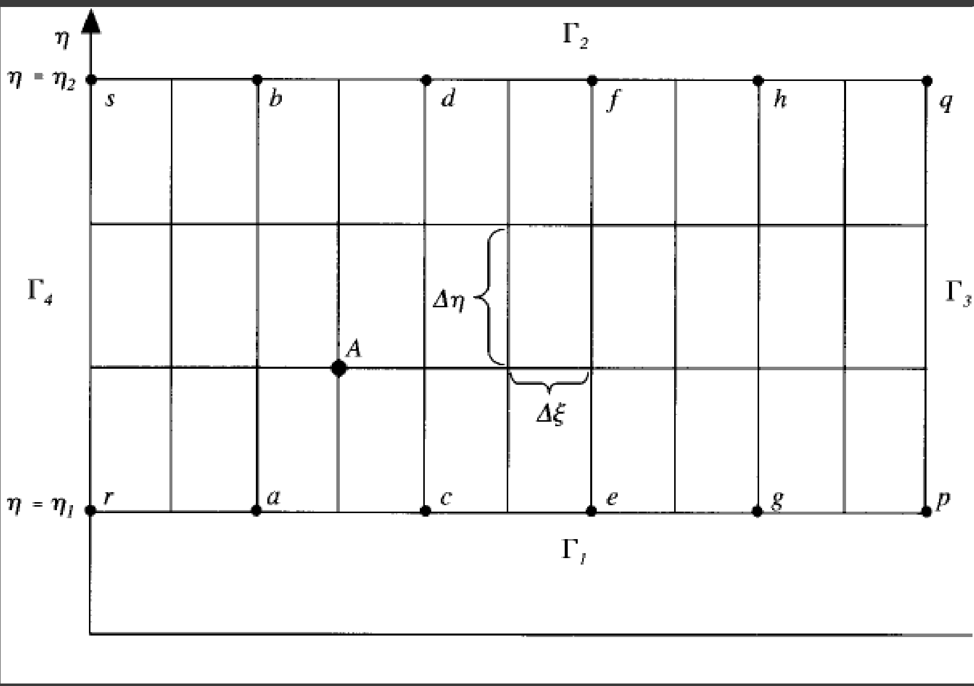
预估步 $$u_i^* = u_i^n - \frac{a\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right)$$

校正步 $$u_i^{n+1} = \frac{1}{2}\left[\left(u_i^n + u_i^*\right) - \frac{a\Delta t}{\Delta x}\left(u_i^* - u_{i-1}^*\right)\right]$$

$$\left[(\Delta t)^2,\ (\Delta x)^2\right]$$

4.4 二维流动控制方程应用

例：求解二维Euler流动



$$\frac{\partial \rho}{\partial t} = -\left(\rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y}\right)$$

$$\frac{\partial u}{\partial t} = -\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x}\right)$$

$$\frac{\partial v}{\partial t} = -\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y}\right)$$

$$\frac{\partial e}{\partial t} = -\left(u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y} + \frac{p}{\rho}\frac{\partial u}{\partial x} + \frac{p}{\rho}\frac{\partial v}{\partial y}\right)$$

非守恒形式

4.4.1 Lax-Wendroff方法

$$\rho_{i,j}^{t+\Delta t}=\rho_{i,j}^{t}+\left(\frac{\partial \rho}{\partial t}\right)_{i,j}^{t}\Delta t+\left(\frac{\partial^2 \rho}{\partial t^2}\right)_{i,j}^{t}\frac{(\Delta t)^2}{2}+\cdots$$

$$u_{i,j}^{t+\Delta t}=u_{i,j}^{t}+\left(\frac{\partial u}{\partial t}\right)_{i,j}^{t}\Delta t+\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j}^{t}\frac{(\Delta t)^2}{2}+\cdots$$

$$v_{i,j}^{t+\Delta t}=v_{i,j}^{t}+\left(\frac{\partial v}{\partial t}\right)_{i,j}^{t}\Delta t+\left(\frac{\partial^2 v}{\partial t^2}\right)_{i,j}^{t}\frac{(\Delta t)^2}{2}+\cdots$$

$$e_{i,j}^{t+\Delta t}=e_{i,j}^{t}+\left(\frac{\partial e}{\partial t}\right)_{i,j}^{t}\Delta t+\left(\frac{\partial^2 e}{\partial t^2}\right)_{i,j}^{t}\frac{(\Delta t)^2}{2}+\cdots$$

u,v一阶偏导
可由动量方程
得到

由连续性方程:

$$\left(\frac{\partial \rho}{\partial t}\right)_{i,j}^{t}=-\left(\rho_{i,j}^{t}\frac{u_{i+1,j}^{t}-u_{i-1,j}^{t}}{2\Delta x}+u_{i,j}^{t}\frac{\rho_{i+1,j}^{t}-\rho_{i-1,j}^{t}}{2\Delta x}\right.$$
$$\left.+\rho_{i,j}^{t}\frac{v_{i,j+1}^{t}-v_{i,j-1}^{t}}{2\Delta y}+v_{i,j}^{t}\frac{\rho_{i,j+1}^{t}-\rho_{i,j-1}^{t}}{2\Delta y}\right)$$

4.4.1 Lax-Wendroff方法

但对于二阶导非常繁琐：

$$\boxed{\frac{\partial^2 \rho}{\partial t^2}} = -\rho \frac{\partial^2 u}{\partial x\, \partial t} + \frac{\partial u}{\partial x}\frac{\partial \rho}{\partial t} + u \frac{\partial^2 \rho}{\partial x\, \partial t} + \frac{\partial \rho}{\partial x}\frac{\partial u}{\partial t} + \rho \frac{\partial^2 v}{\partial y\, \partial t} + \frac{\partial v}{\partial y}\frac{\partial \rho}{\partial t} + v \frac{\partial^2 \rho}{\partial y\, \partial t} + \frac{\partial \rho}{\partial y}\frac{\partial v}{\partial t}$$

求出来，太繁琐！

其中：

$$\frac{\partial^2 u}{\partial x\, \partial t} = -u \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v \frac{\partial^2 u}{\partial x\, \partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} - \frac{1}{\rho^2}\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial x}$$

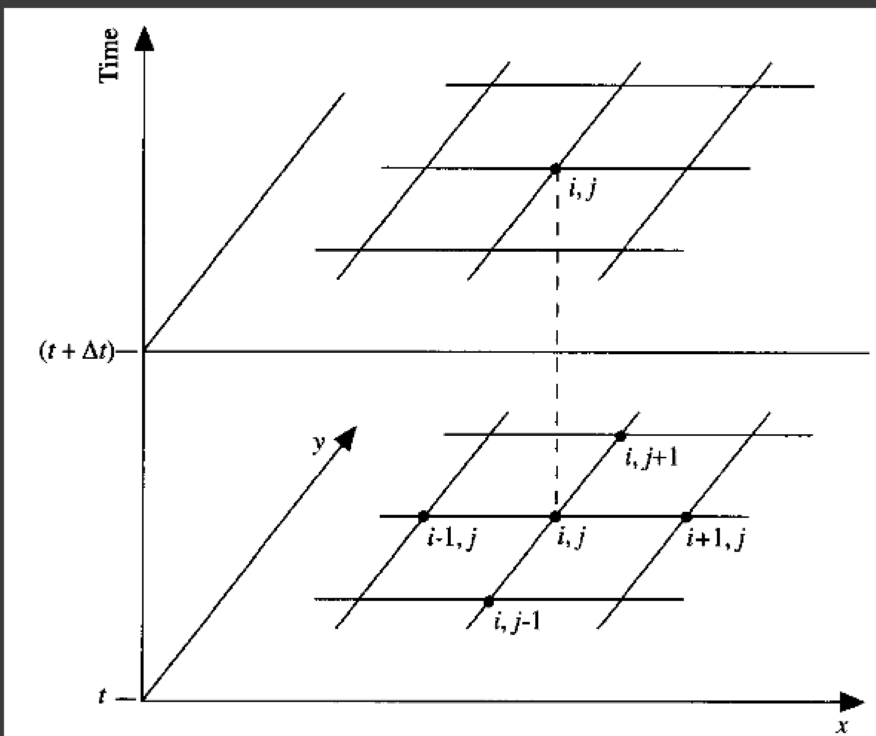
$$= -u_{i,j}^t \frac{u_{i+1,j}^t - 2u_{i,j}^t + u_{i-1,j}^t}{(\Delta x)^2} + \left(\frac{u_{i+1,j}^t - u_{i-1,j}^t}{2\Delta x}\right)^2 + v_{i,j}^t \frac{u_{i+1,j+1}^t + u_{i-1,j-1}^t - u_{i-1,j+1}^t - u_{i+1,j-1}^t}{4(\Delta x)(\Delta y)}$$

$$+ \frac{u_{i,j+1}^t - u_{i,j-1}^t}{2\Delta y}\frac{v_{i+1,j}^t - v_{i-1,j}^t}{2\Delta x} + \frac{1}{\rho_{i,j}^t}\frac{p_{i+1,j}^t - 2p_{i,j}^t + p_{i-1,j}^t}{(\Delta x)^2}$$

$$- \frac{1}{(\rho_{i,j}^t)^2}\frac{p_{i+1,j}^t - p_{i-1,j}^t}{2\Delta x}\frac{\rho_{i+1,j}^t - \rho_{i-1,j}^t}{2\Delta x}$$

4.4.1 Lax-Wendroff方法

同样方法可以求出u,v,e（都要用到二阶导，以得到二阶精度）



Lax-Wendroff方法：

1. 时空二阶精度
2. 求解方法直观但复杂，表达形式繁琐。

4.4.2 MacCormack Method

例：求解二维Euler流动

$$\frac{\partial \rho}{\partial t} = -\left(\rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y}\right)$$

$$\frac{\partial u}{\partial t} = -\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x}\right)$$

$$\frac{\partial v}{\partial t} = -\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y}\right)$$

$$\frac{\partial e}{\partial t} = -\left(u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y} + \frac{p}{\rho}\frac{\partial u}{\partial x} + \frac{p}{\rho}\frac{\partial v}{\partial y}\right)$$

非守恒形式

4.4.2 MacCormack Method

预估一校正两步格式

$$\text{预估步}\left\{\begin{aligned}
(\bar{\rho})^{t+\Delta t}_{i,j} &= \rho^t_{i,j} + \left(\frac{\partial \rho}{\partial t}\right)^t_{i,j}\Delta t \\
(\bar{u})^{t+\Delta t}_{i,j} &= u^t_{i,j} + \left(\frac{\partial u}{\partial t}\right)^t_{i,j}\Delta t \\
(\bar{v})^{t+\Delta t}_{i,j} &= v^t_{i,j} + \left(\frac{\partial v}{\partial t}\right)^t_{i,j}\Delta t \\
(\bar{e})^{t+\Delta t}_{i,j} &= e^t_{i,j} + \left(\frac{\partial e}{\partial t}\right)^t_{i,j}\Delta t
\end{aligned}\right.$$

$$\left(\frac{\partial \rho}{\partial t}\right)^t_{i,j} = -\left(\rho^t_{i,j}\frac{u^t_{i+1,j}-u^t_{i,j}}{\Delta x} + u^t_{i,j}\frac{\rho^t_{i+1,j}-\rho^t_{i,j}}{\Delta x} + \rho^t_{i,j}\frac{v^t_{i,j+1}-\rho^t_{i,j}}{\Delta y} + v^t_{i,j}\frac{\rho^t_{i,j+1}-\rho^t_{i,j}}{\Delta y}\right)$$

一阶精度

4.4.2 MacCormack Method

校正步

$$\left(\overline{\frac{\partial \rho}{\partial t}}\right)_{i,j}^{t+\Delta t} = -\left[(\bar{\rho})_{i,j}^{t+\Delta t}\frac{(\bar{u})_{i,j}^{t+\Delta t}-(\bar{u})_{i-1,j}^{t+\Delta t}}{\Delta x} + (\bar{u})_{i,j}^{t+\Delta t}\frac{(\bar{\rho})_{i,j}^{t+\Delta t}-(\bar{\rho})_{i-1,j}^{t+\Delta t}}{\Delta x} + (\bar{\rho})_{i,j}^{t+\Delta t}\frac{(\bar{v})_{i,j}^{t+\Delta t}-(\bar{v})_{i,j-1}^{t+\Delta t}}{\Delta y} + (\bar{v})_{i,j}^{t+\Delta t}\frac{(\bar{\rho})_{i,j}^{t+\Delta t}-(\bar{\rho})_{i,j-1}^{t+\Delta t}}{\Delta y}\right]$$

$$\left(\frac{\partial \rho}{\partial t}\right)_{\mathrm{av}} = \frac{1}{2}\left[\underbrace{\left(\frac{\partial \rho}{\partial t}\right)_{i,j}^{t}} + \underbrace{\left(\overline{\frac{\partial \rho}{\partial t}}\right)_{i,j}^{t+\Delta t}}\right] \longrightarrow \rho_{i,j}^{t+\Delta t} = \rho_{i,j}^{t} + \left(\frac{\partial \rho}{\partial t}\right)_{\mathrm{av}}\Delta t$$

二阶精度→？

4.4.2 MacCormack Method

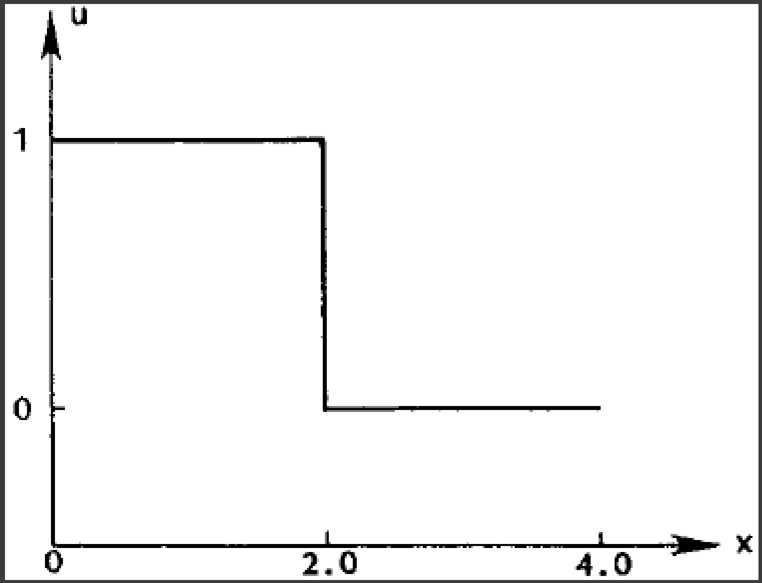
校正步

同样方法可以求出$u^{t+\Delta t}, v^{t+\Delta t}, e^{t+\Delta t}$

4.5 非线性方程求解

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$$ 在不同空间点具有传播速度的波动方程

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)$$ 或 $$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x}$$



$$u(x,0) = 1 \qquad 0 \le x \le 2.0$$

$$u(x,0) = 0 \qquad 2.0 \le x \le 4.0$$

在以上初值及边界条件下波的传播规律

4.5.1 Lax Method

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \xrightarrow{\text{FTCS}} \frac{u_i^{n+1} - \boxed{u_i^n}}{\Delta t} = -\frac{E_{i+1}^n - E_{i-1}^n}{2\Delta x}$$

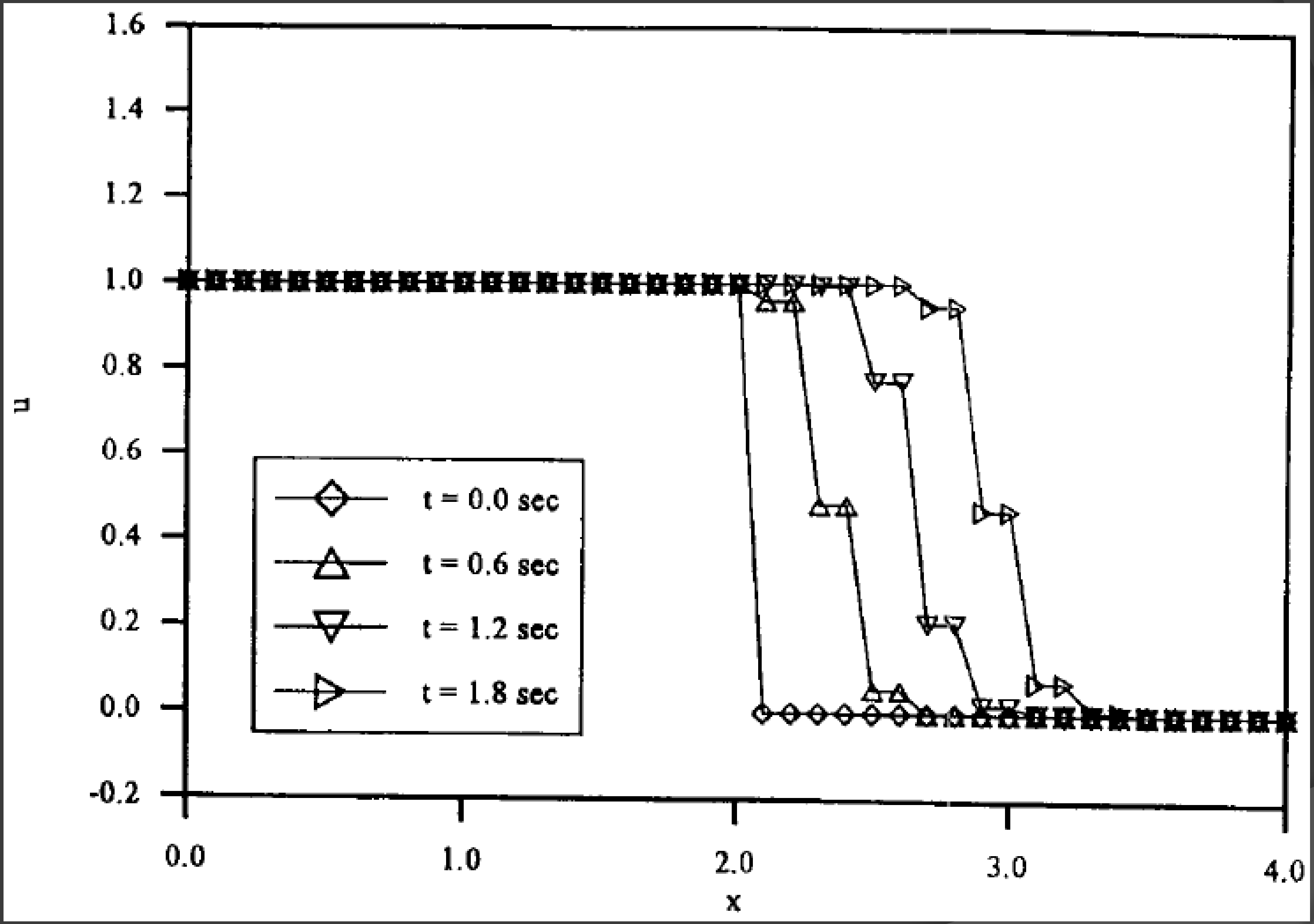
$$u_i^{n+1} = \boxed{\frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right)} - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right)$$

或

$$u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\Delta t}{4\Delta x}\left[\left(u_{i+1}^n\right)^2 - \left(u_{i-1}^n\right)^2\right]$$

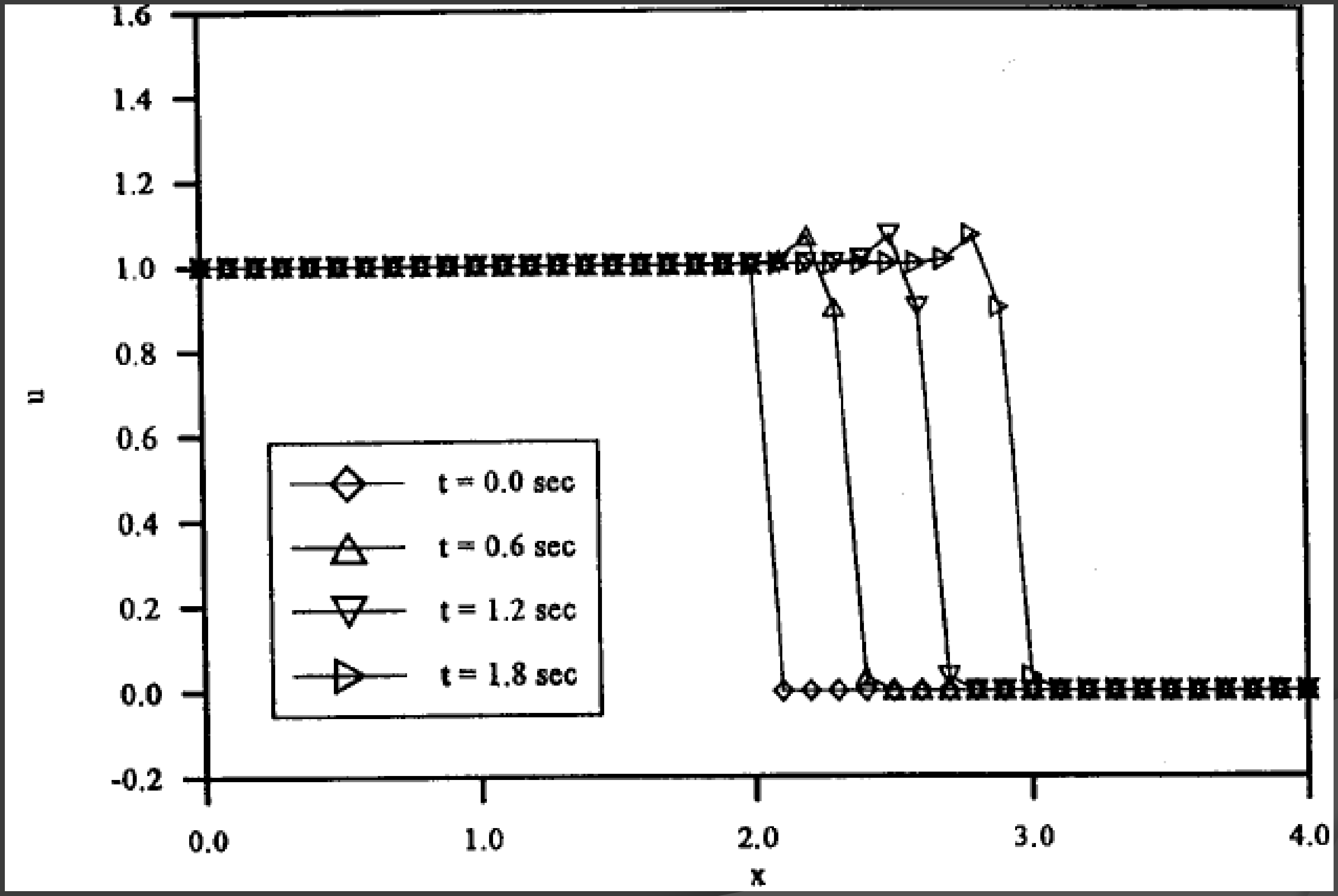
$$\left|\frac{\Delta t}{\Delta x} u_{\max}\right| \le 1$$

4.5.1 Lax Method



$\Delta x = 0.1$ and $\Delta t = 0.1$

4.5.2 Lax-Wendroff method

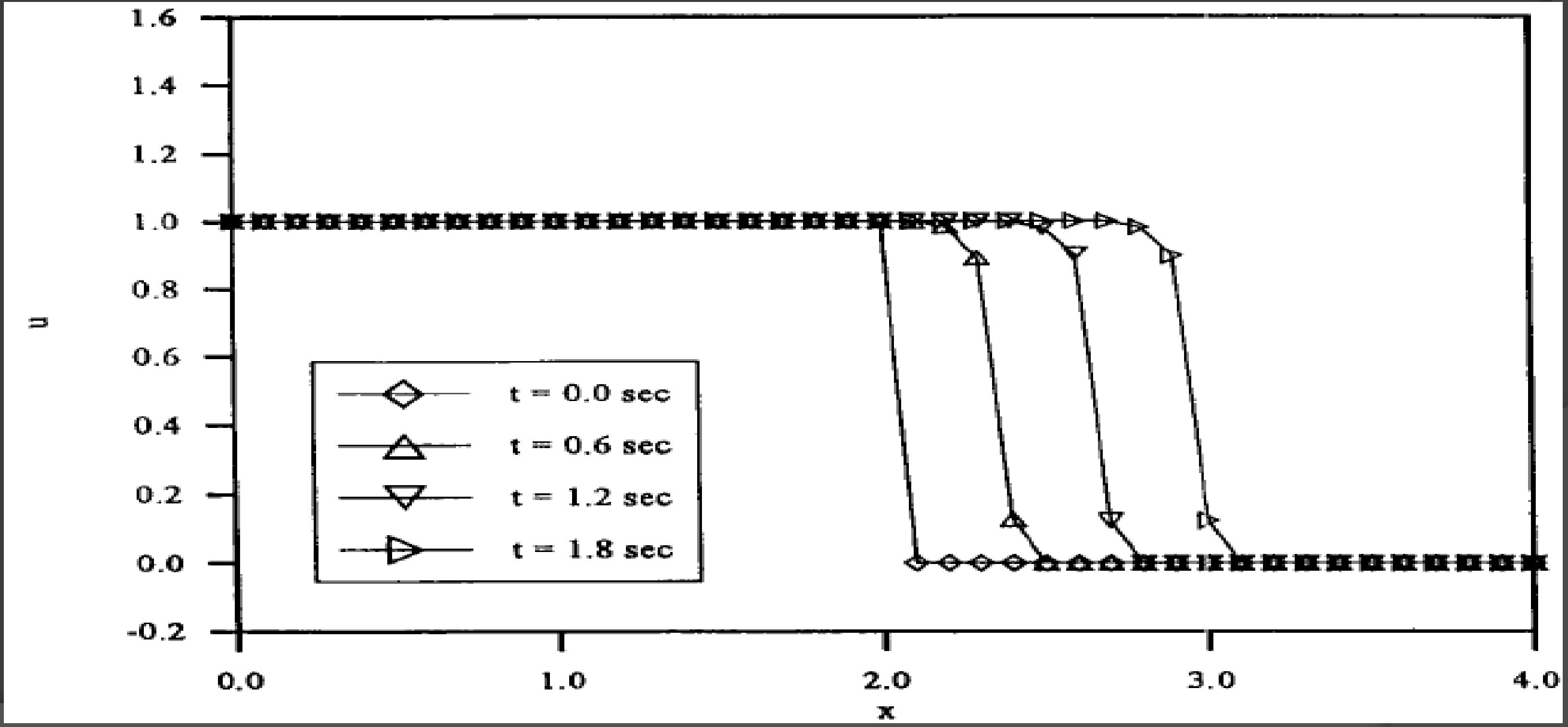


4.5.3 MacCormack

$$u_i^* = u_i^n - \frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_i^n\right)$$

$$u_i^{n+1} = \frac{1}{2}\left[u_i^n + u_i^* - \frac{\Delta t}{\Delta x}\left(E_i^* - E_{i-1}^*\right)\right]$$

$$\left|u_{\max}\Delta t/\Delta x\right| \le 1$$



4.5.4 Runge-Kutta Method

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x}$$

一阶前差

$$u_i^{n+1} = u_i^n - \Delta t \left(\frac{\partial E}{\partial x}\right)^n$$

$$u_i^{(1)} = u_i^n$$

$$u_i^{(2)} = u_i^n - \frac{\Delta t}{2} \boxed{\left(\frac{\partial E}{\partial x}\right)_i^n}$$

使用步长 Δt/2

可任意空间格式来表述

$$u_i^{n+1} = u_i^n - \Delta t \left\{ \frac{1}{2} \left[\boxed{\left(\frac{\partial E}{\partial x}\right)^{(1)}} + \boxed{\left(\frac{\partial E}{\partial x}\right)^{(2)}} \right] \right\}$$

E=u^2/2 ！

4.5.4 Runge-Kutta Method

$$u_i^{n+1} = u_i^n - \Delta t \left[a \left(\frac{\partial E}{\partial x} \right)^{(1)} + b \left(\frac{\partial E}{\partial x} \right)^{(2)} \right]$$

a+b=1→多种二阶RK格式

$$
\begin{aligned}
u_i^{(1)} &= u_i^n \\
u_i^{(2)} &= u_i^n - \alpha_2 \Delta t \left(\frac{\partial E}{\partial x} \right)_i^{(1)} \\
u_i^{(3)} &= u_i^n - \alpha_3 \Delta t \left(\frac{\partial E}{\partial x} \right)_i^{(2)} \\
u_i^{(4)} &= u_i^n - \alpha_4 \Delta t \left(\frac{\partial E}{\partial x} \right)_i^{(3)} \\
u_i^{(Q)} &= u_i^n - \alpha_Q \Delta t \left(\frac{\partial E}{\partial x} \right)_i^{(Q-1)}
\end{aligned}
$$

$$u_i^{n+1} = u_i^n - \Delta t \left[\sum_{q=1}^{q=Q} \beta_q \left(\frac{\partial E}{\partial x} \right)^q \right]$$

$$\sum_{q=1}^{q=Q} \beta_q = 1$$

四阶RK格式最常用，其中$\beta_{1.2.3.4}$分别为1/6、1/3、1/3、1/6

4.5.4 Runge-Kutta Method

- RK格式应用时多用显式格式
- 对于线性方程，RK稳定性强于普通显式格式；对于非线性方程，容易引起解震荡，需加阻尼项；

阻尼项(damping)添加方法：

$$u_i^{n+1} = u_i^{n+1} + D$$

$$D = -\varepsilon_e\left(u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n\right)$$

4.5.4 Runge-Kutta Method

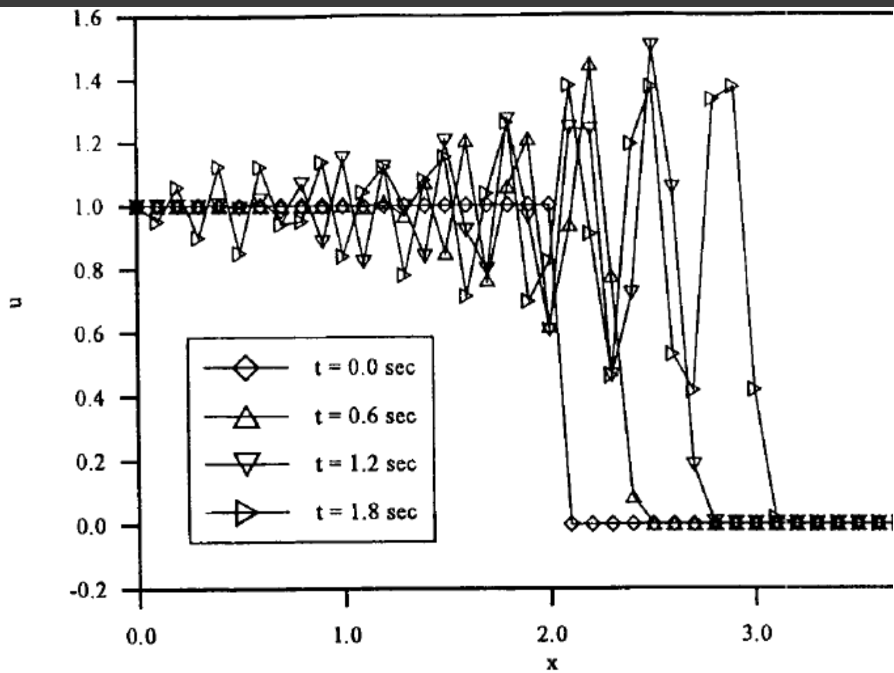


Figure 6-31. Solution of the inviscid Burgers equation by the four Runge-Kutta method, $\Delta x = 0.1$ and $\Delta t = 0.1$.

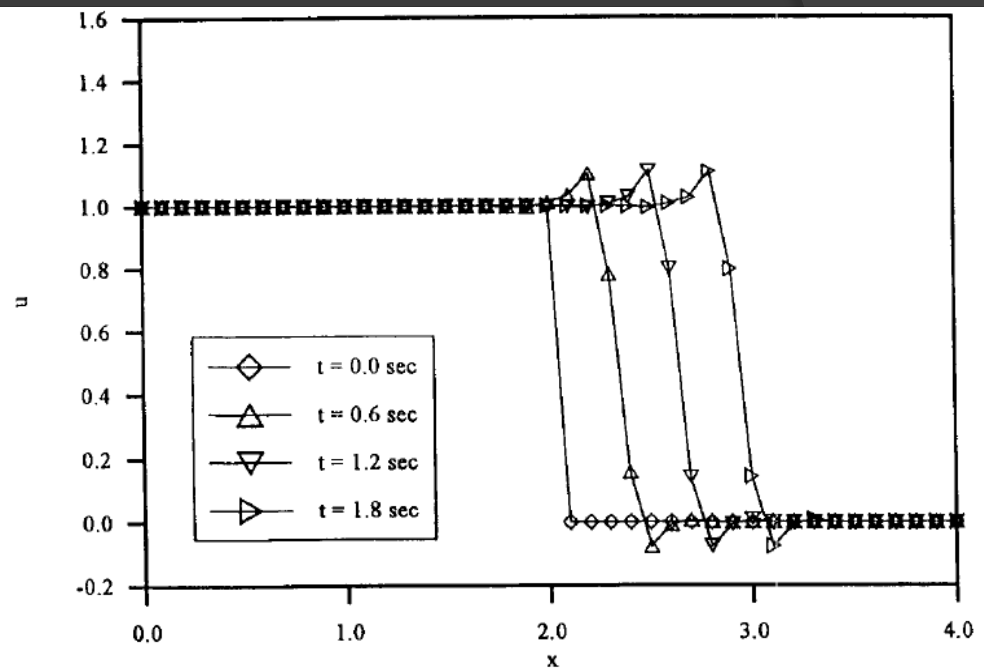


Figure 6-32. Solution of the inviscid Burgers equation by the fourth-order Runge-Kutta method with $\varepsilon_e = 0.1$, $\Delta x = 0.1$, and $\Delta t = 0.1$.

空间中心，无Damping

空间中心，有Damping

4.5.5 Modified Runge-Kutta Method

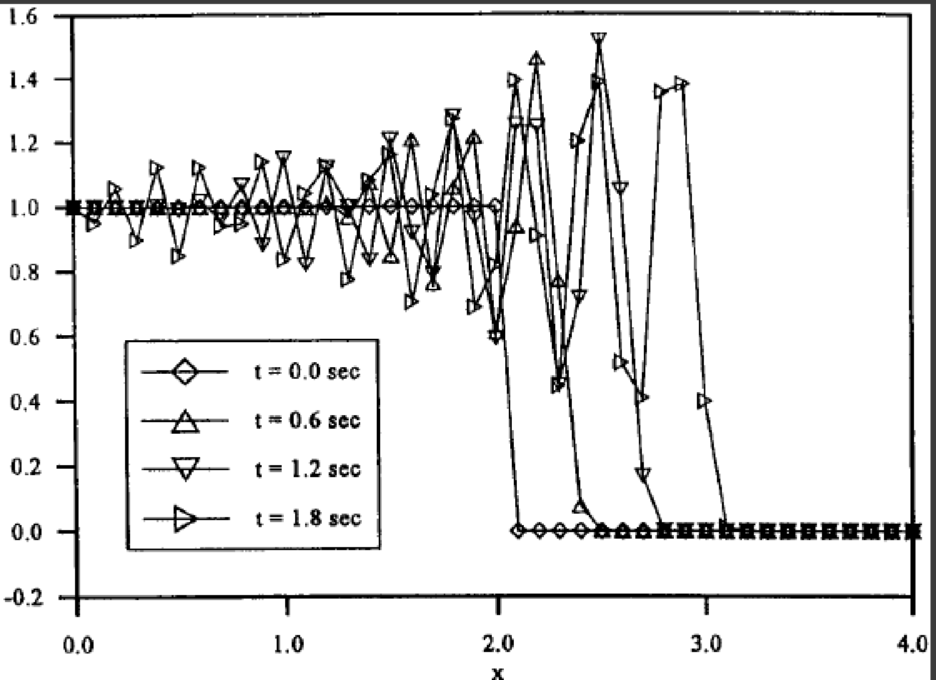
四阶

$$
\begin{aligned}
u_i^{(1)} &= u_i^n \\
u_i^{(2)} &= u_i^n - \frac{\Delta t}{4}\left(\frac{\partial E}{\partial x}\right)_i^{(1)} \\
u_i^{(3)} &= u_i^n - \frac{\Delta t}{3}\left(\frac{\partial E}{\partial x}\right)_i^{(2)} \\
u_i^{(4)} &= u_i^n - \frac{\Delta t}{2}\left(\frac{\partial E}{\partial x}\right)_i^{(3)} \\
u_i^{n+1} &= u_i^n - \Delta t\left(\frac{\partial E}{\partial x}\right)_i^{(4)}
\end{aligned}
$$

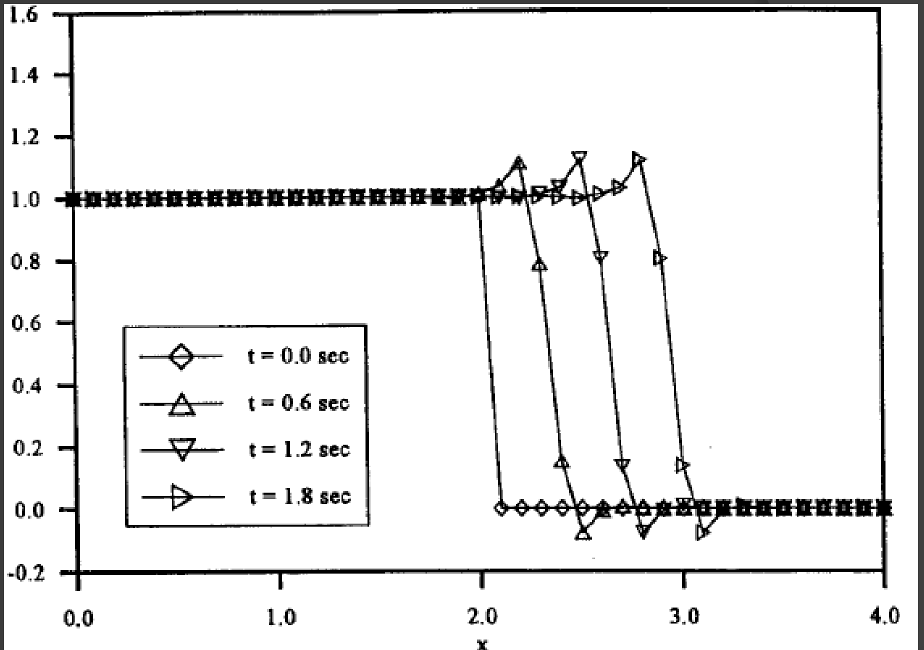
$$
\begin{aligned}
u_i^{(1)} &= u_i^n \\
u_i^{(2)} &= u_i^n - \frac{\Delta t}{2}\left(\frac{\partial E}{\partial x}\right)_i^{(1)} \\
u_i^{n+1} &= u_i^n - \Delta t\left(\frac{\partial E}{\partial x}\right)_i^{(2)}
\end{aligned}
$$

二阶

4.5.5 Modified Runge-Kutta Method



Solution of the inviscid Burgers equation by the modified fourth-order Runge-Kutta method, $\Delta x = 0.1$ and $\Delta t = 0.1$.



Solution of the inviscid Burgers equation by the modified fourth-order Runge-Kutta method with damping of $\varepsilon_e = 0.1$, $\Delta x = 0.1$, and $\Delta t = 0.1$.

空间中心，无Damping

空间中心，有Damping

4.6 本章小结

椭圆型方程求解

松弛迭代法

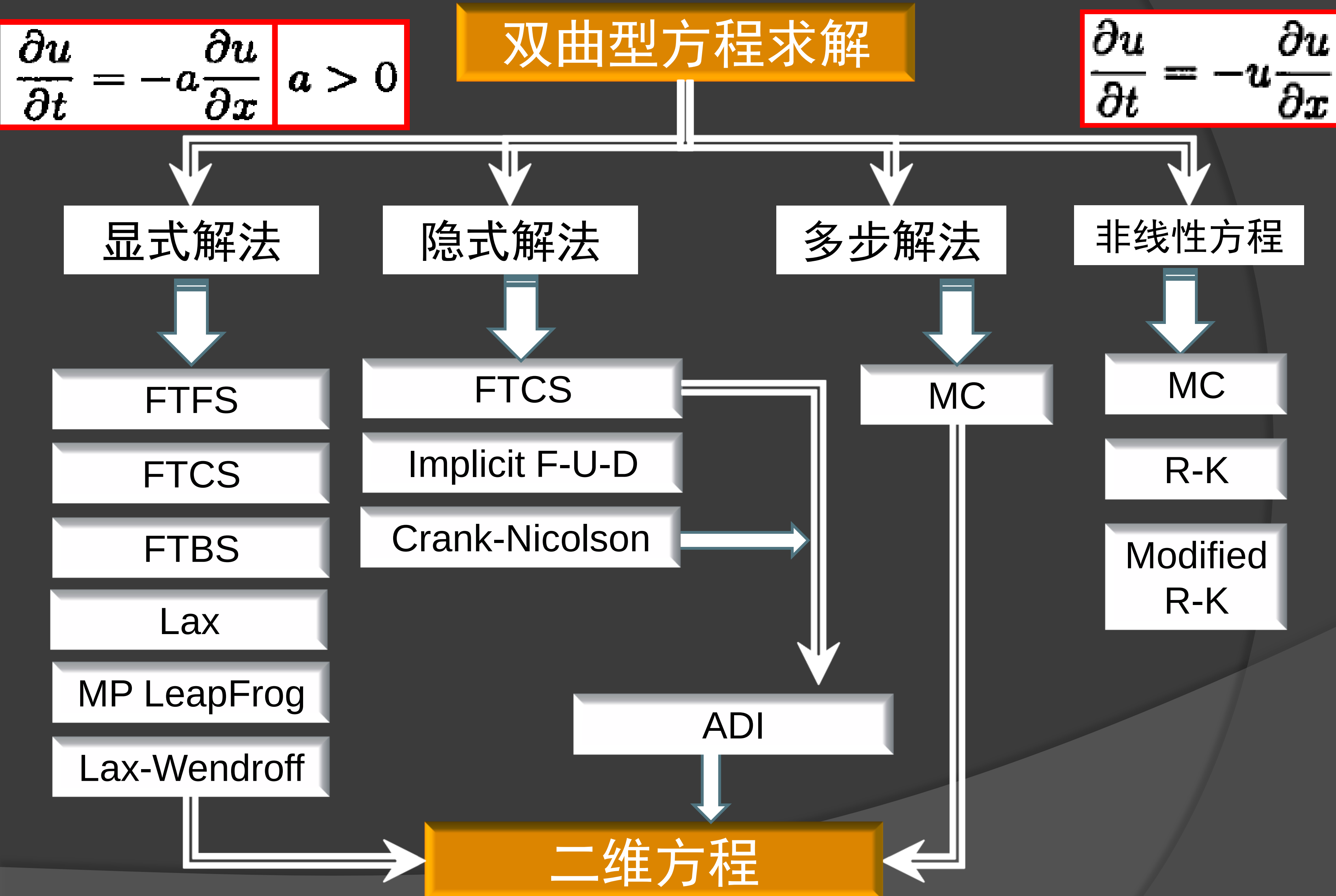
超松弛/亚松弛

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0$$

$$\overline{\Phi_{i,j}^{n+1}}=\frac{(\Delta x)^2(\Delta y)^2}{2(\Delta y)^2+2(\Delta x)^2}\left[\frac{\Phi_{i+1,j}^{n}+\Phi_{i-1,j}^{n+1}}{(\Delta x)^2}+\frac{\Phi_{i,j+1}^{n}+\Phi_{i,j-1}^{n+1}}{(\Delta y)^2}\right]$$

$$\Phi_{i,j}^{n+1}=\Phi_{i,j}^{n}+\omega\left(\overline{\Phi_{i,j}^{n+1}}-\Phi_{i,j}^{n}\right)$$

4.6 本章小结



$\frac{\partial u}{\partial t}=-a\frac{\partial u}{\partial x}$
$a>0$
双曲型方程求解
$\frac{\partial u}{\partial t}=-u\frac{\partial u}{\partial x}$
显式解法
隐式解法
多步解法
非线性方程
FTFS
FTCS
FTBS
Lax
MP LeapFrog
Lax-Wendroff
FTCS
Implicit F-U-D
Crank-Nicolson
MC
MC
R-K
Modified R-K
ADI
二维方程

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} \qquad a > 0$$

Euler's FTFS	$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_{i+1}^n - u_i^n}{\Delta x}$	无条件不稳定	$O(\Delta t,\ \Delta x)$
Euler's FTCS	$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$	无条件不稳定	$[(\Delta t),\ (\Delta x)^2]$
Euler's FTBS	$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_i^n - u_{i-1}^n}{\Delta x}$	$a\Delta t/\Delta x \leq 1$	$O(\Delta t,\ \Delta x)$
Lax method	$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$	$a\Delta t/\Delta x \leq 1$	$[(\Delta t)^2,\ (\Delta x)^2]$
Midpoint leapfrog	$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = -a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$	$a\Delta t/\Delta x \leq 1$	$[(\Delta t)^2,\ (\Delta x)^2]$
Lax-Wendroff	$u_i^{n+1} = u_i^n - a\Delta t\left[\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\right] + \frac{1}{2}a^2(\Delta t)^2\left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}\right]$	$a\Delta t/\Delta x \leq 1$	$[(\Delta t)^2,\ (\Delta x)^2]$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} \qquad a > 0$$

Euler's FTCS

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{a}{2\Delta x}\left[u_{i+1}^{n+1} - u_{i-1}^{n+1}\right] \Longrightarrow \frac{1}{2}c\,u_{i-1}^{n+1} - u_i^{n+1} - \frac{1}{2}c\,u_{i+1}^{n+1} = -u_i^n$$

三对角矩阵 $O(\Delta t, \Delta x)^2]$

Implicit first upwind differencing method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x} \Longrightarrow u_i^{n+1} = \frac{D_i - A_i u_{i-1}^{n+1}}{B_i}$$

二对角矩阵 $O(\Delta t, \ \Delta x)$

Crank-Nicolson

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{1}{2}\left[\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} + \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\right]$$

$$\frac{1}{4}c\,u_{i-1}^{n+1} - u_i^{n+1} - \frac{1}{4}c\,u_{i+1}^{n+1} = -u_i^n + \frac{1}{4}c\left(u_{i+1}^n - u_{i-1}^n\right)$$

三对角矩阵 $[(\Delta t)^2, \ (\Delta x)^2]$

MC

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} \qquad a > 0$$

预估步 $u_i^* = u_i^n - \frac{a\Delta t}{\Delta x}(u_{i+1}^n - u_i^n)$

校正步 $u_i^{n+1} = \frac{1}{2}\left[(u_i^n + u_i^*) - \frac{a\Delta t}{\Delta x}(u_i^* - u_{i-1}^*)\right]$

$[(\Delta t)^2, (\Delta x)^2]$

R-K

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$$

$$\begin{aligned}
u_i^{(1)} &= u_i^n \\
u_i^{(2)} &= u_i^n - \alpha_2 \Delta t \left(\frac{\partial E}{\partial x}\right)_i^{(1)} \\
u_i^{(3)} &= u_i^n - \alpha_3 \Delta t \left(\frac{\partial E}{\partial x}\right)_i^{(2)} \\
u_i^{(4)} &= u_i^n - \alpha_4 \Delta t \left(\frac{\partial E}{\partial x}\right)_i^{(3)} \\
u_i^{(Q)} &= u_i^n - \alpha_Q \Delta t \left(\frac{\partial E}{\partial x}\right)_i^{(Q-1)}
\end{aligned}$$

$$u_i^{n+1} = u_i^n - \Delta t \left[\sum_{q=1}^{q=Q} \beta_q \left(\frac{\partial E}{\partial x}\right)^q\right]$$

$$\Sigma_{q=1}^{q=Q} \beta_q = 1$$

四阶RK格式最常用，其中$\beta_{1.2.3.4}$分别为1/6、1/3、1/3、1/6

Damping: $u_i^{n+1} = u_i^{n+1} + D$

$$D = -\varepsilon_e \left(u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n\right)$$

Modified R-K

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$$

四阶

$$u_i^{(1)} = u_i^n$$

$$u_i^{(2)} = u_i^n - \frac{\Delta t}{4}\left(\frac{\partial E}{\partial x}\right)_i^{(1)}$$

$$u_i^{(3)} = u_i^n - \frac{\Delta t}{3}\left(\frac{\partial E}{\partial x}\right)_i^{(2)}$$

$$u_i^{(4)} = u_i^n - \frac{\Delta t}{2}\left(\frac{\partial E}{\partial x}\right)_i^{(3)}$$

$$u_i^{n+1} = u_i^n - \Delta t\left(\frac{\partial E}{\partial x}\right)_i^{(4)}$$

Damping:

$$u_i^{n+1} = u_i^{n+1} + D$$

$$D = -\varepsilon_e\left(u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n\right)$$

第五章 混合型方程求解方法

抛物型、双曲型：时间推进法或空间推进法
椭圆形：松弛迭代法
兼有多种方程特性→混合型方程：??→压力修正法

不可压缩粘性流动：不可压N-S方程组→椭圆形+混合型

5.1 不可压缩流动N-S方程

连续方程

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

*x*方向动量方程

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

*y*方向动量方程

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho vu)}{\partial x}+\frac{\partial(\rho v^2)}{\partial y}=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

如果采用显式时间推进法，则需满足稳定性条件：

$$\Delta t \le \frac{1}{|u|/\Delta x+|v|/\Delta y+a\sqrt{1/(\Delta x)^2+1/(\Delta y)^2}}$$

不可压缩流动，音速*a*→∞→*Δt*→0!!

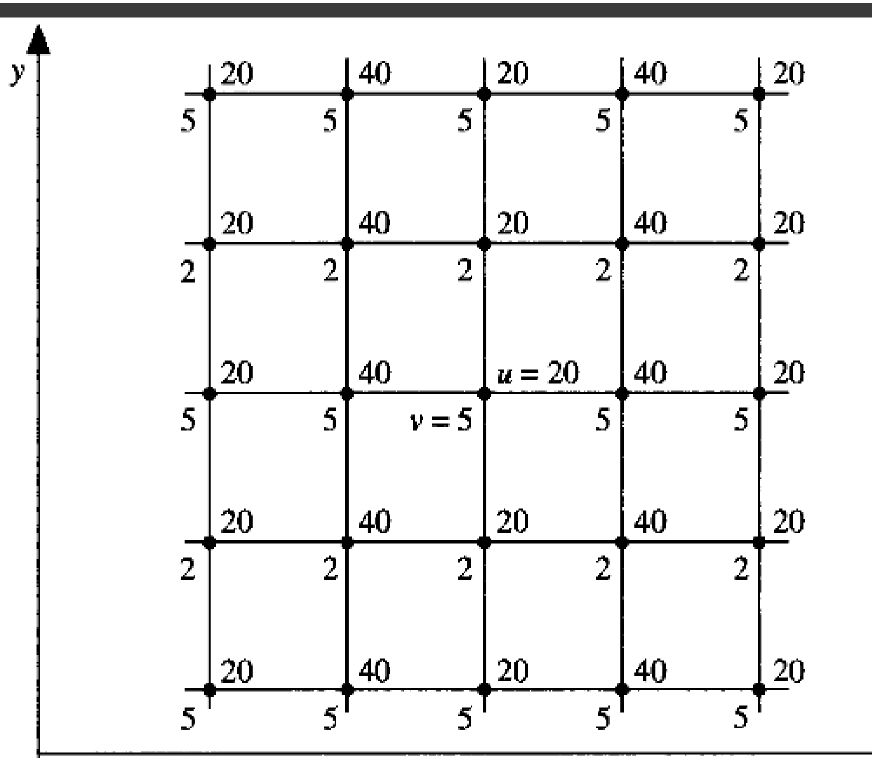
5.2 压力修正法-差分谬解及交错网格

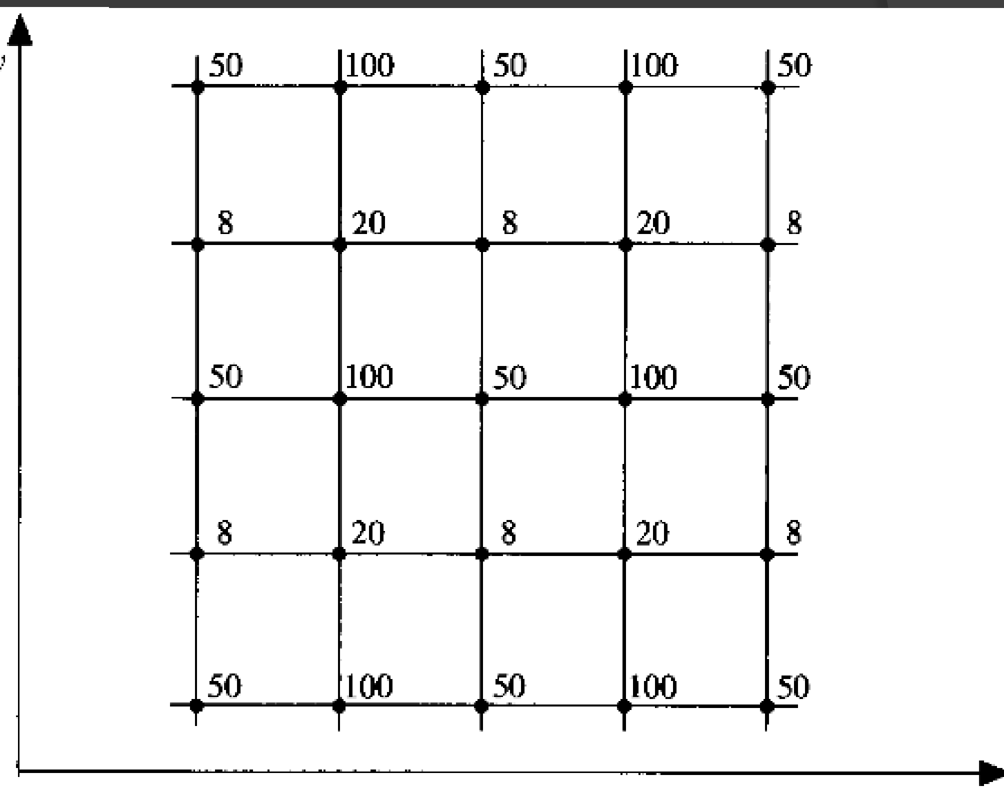
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}+\frac{v_{i,j+1}-v_{i,j-1}}{2\Delta y}=0$$

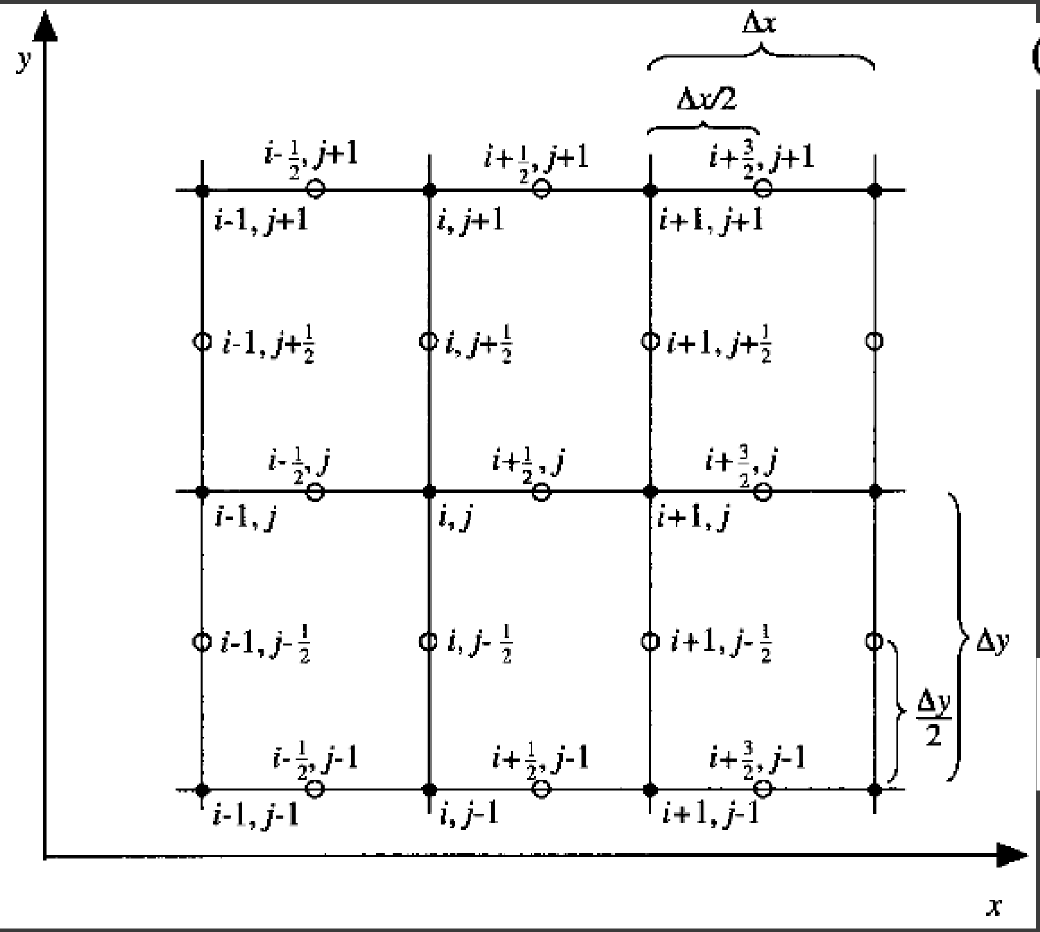
$$\frac{\partial p}{\partial x}=\frac{p_{i+1,j}-p_{i-1,j}}{2\Delta x}$$

$$\frac{\partial p}{\partial y}=\frac{p_{i,j+1}-p_{i,j-1}}{2\Delta y}$$





5.2 压力修正法-差分谬解及交错网格



$(i-1, j), (i, j), (i+1, j), (i, j+1), (i, j-1)$

↓

P

$(i-\frac{1}{2}, j), (i+\frac{1}{2}, j),$

↓

u

$(i, j+\frac{1}{2}), (i, j-\frac{1}{2})$

↓

v

$$\frac{u_{i+1/2,j}-u_{i-1/2,j}}{\Delta x}+\frac{v_{i,j+1/2}-v_{i,j-1/2}}{\Delta y}=0$$

5.3 压力修正法

压力修正法原理及基本思想：

原理：迭代求解

基本思想：

1. 给定流场初始压力 P^*
2. 根据动量方程求解速度 u, v
3. 使用 u, v 根据连续性方程构建压力的修正方程，并求得压力修正量 P'
4. 根据压力修正量求得 u, v 的修正量，并对 u, v 进行修正
5. 重复步骤2-4，直至计算所得的速度场满足连续性方程。

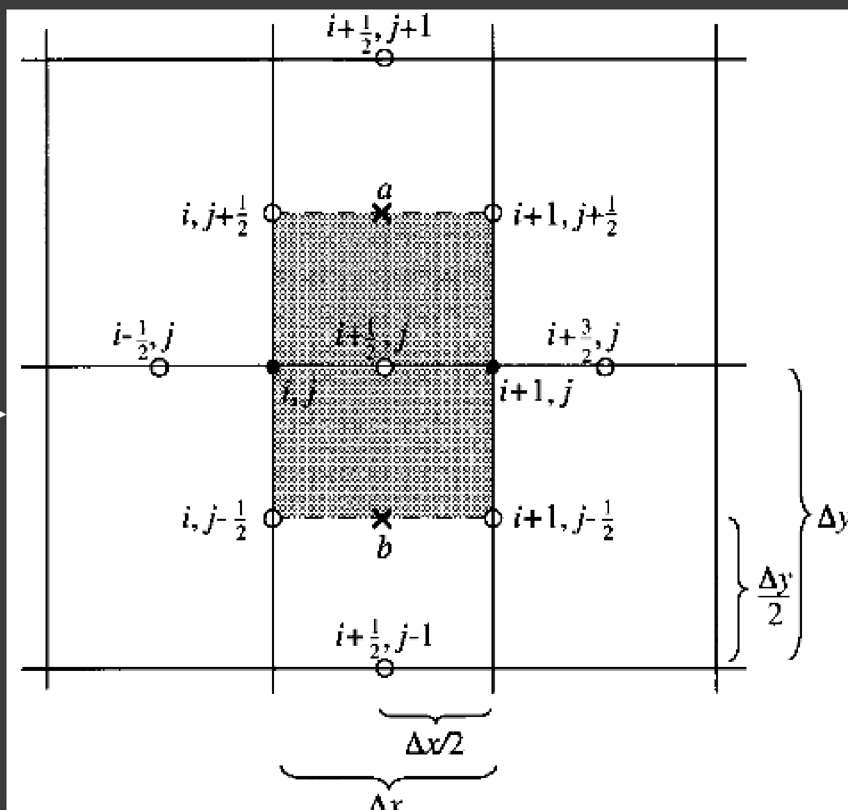
$$p^* \rightarrow u^*,\ v^* \rightarrow p' \rightarrow p = p^* + p' \rightarrow \begin{cases} u = u^* + u' \\ v = v^* + v' \end{cases}$$

u,v不满足连续性方程

5.3 压力修正法

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho vu)}{\partial x}+\frac{\partial(\rho v^2)}{\partial y}=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$



$a:\ \bar{v}_{j+1/2}\equiv\frac{1}{2}\left(v_{i,j+1/2}+v_{i+1,j+1/2}\right)$

$b:\ v_{j-1/2}\equiv\frac{1}{2}\left(v_{i,j-1/2}+v_{i+1,j-1/2}\right)$

5.3 压力修正法

以(i+1/2,j)为中心,x方向动量方程：

$$\frac{(\rho u)^{n+1}_{i+1/2,j}-(\rho u)^{n}_{i+1/2,j}}{\Delta t}=-\left[\frac{(\rho u^2)^n_{i+3/2,j}-(\rho u^2)^n_{i-1/2,j}}{2\Delta x}\right.$$

$$\left.+\frac{(\rho u\bar{v})^n_{i+1/2,j+1}-(\rho uv)^n_{i+1/2,j-1}}{2\Delta y}\right]-\frac{p^n_{i+1,j}-p^n_{i,j}}{\Delta x}$$

$$+\mu\left[\frac{u^n_{i+3/2,j}-2u^n_{i+1/2,j}+u^n_{i-1/2,j}}{(\Delta x)^2}+\frac{u^n_{i+1/2,j+1}-2u^n_{i+1/2,j}+u_{i+1/2,j-1}}{(\Delta y)^2}\right]$$

$$\boxed{(\rho u)^{n+1}_{i+1/2,j}=(\rho u)^n_{i+1/2,j}+A\,\Delta t-\frac{\Delta t}{\Delta x}(p^n_{i+1,j}-p^n_{i,j})} \qquad \text{E5.3.1}$$

$$A=-\left[\frac{(\rho u^2)^n_{i+3/2,j}-(\rho u^2)^n_{i-1/2,j}}{2\Delta x}+\frac{(\rho u\bar{v})^n_{i+1/2,j+1}-(\rho uv)^n_{i+1/2,j-1}}{2\Delta y}\right]$$

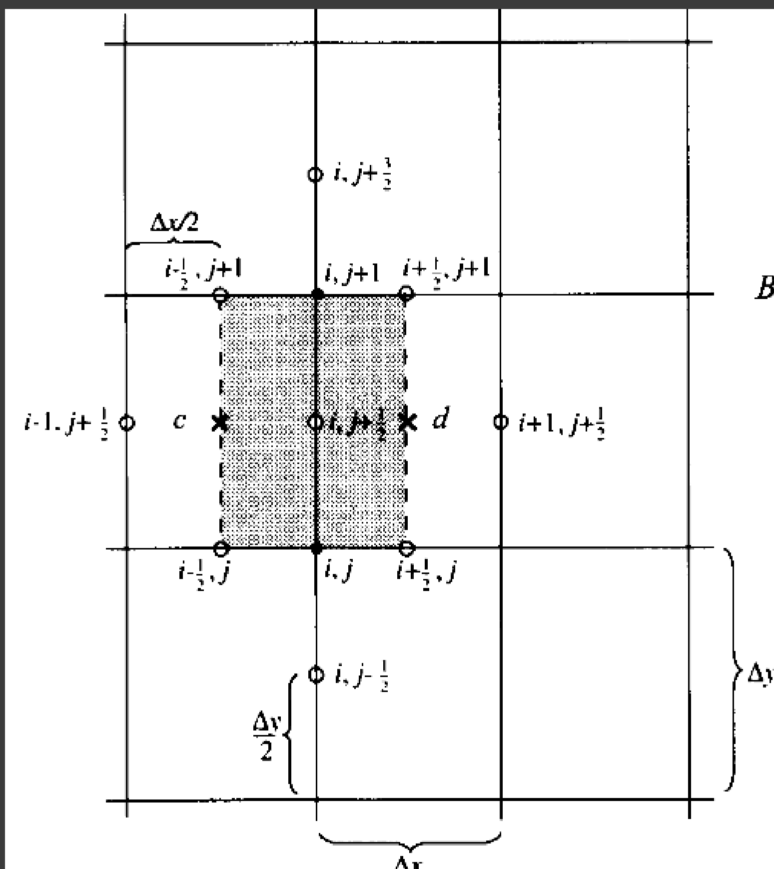
$$+\mu\left[\frac{u^n_{i+3/2,j}-2u^n_{i+1/2,j}+u^n_{i-1/2,j}}{(\Delta x)^2}+\frac{u^n_{i+1/2,j+1}-2u^n_{i+1/2,j}+u_{i+1/2,j-1}}{(\Delta y)^2}\right]$$

5.3 压力修正法

以(i,j+1/2)为中心,y方向动量方程：

$$c:\ u=\frac{1}{2}\left(u_{i-1/2,j}+u_{i-1/2,j+1}\right)$$

$$d:\ \bar{u}=\frac{1}{2}\left(u_{i+1/2,j}+u_{i+1/2,j+1}\right)$$



$$(\rho v)^{n+1}_{i,j+1/2}=(\rho v)^{n}_{i,j+1/2}+B\,\Delta t-\frac{\Delta t}{\Delta x}\left(p^{n}_{i,j+1}-p^{n}_{i,j}\right)$$

E5.3.2

$$B=-\left[\frac{(\rho v\bar{u})^{n}_{i+1,j+1/2}-(\rho v u)^{n}_{i-1,j+1/2}}{2\Delta x}+\frac{(\rho v^2)^{n}_{i,j+3/2}-(\rho v^2)_{i,j-1/2}}{2\Delta y}\right]$$
$$+\mu\left[\frac{v^{n}_{i+1,j+1/2}-2v^{n}_{i,j+1/2}+v_{i-1,j+1/2}}{(\Delta x)^2}+\frac{v^{n}_{i,j+3/2}-2v^{n}_{i,j+1/2}+v^{n}_{i,j-1/2}}{(\Delta x)^2}\right]$$

5.3 压力修正法

Step 1: p = p*

Step 2: 求u*,v*

$$(\rho u^*)_{i+1/2,j}^{n+1} = (\rho u^*)_{i+1/2,j}^{n} + A^*\ \Delta t - \frac{\Delta t}{\Delta x}(p_{i+1,j}^* - p_{i,j}^*)$$ **E5.3.3**

$$(\rho v^*)_{i,j+1/2}^{n+1} = (\rho v^*)_{i,j+1/2}^{n} + B^*\ \Delta t - \frac{\Delta t}{\Delta y}(p_{i,j+1}^* - p_{i,j}^*)$$ **E5.3.4**

E5.3.1-E5.3.3

$$(\rho u')_{i+1/2,j}^{n+1} = (\rho u')_{i+1/2,j}^{n} + A'\ \Delta t - \frac{\Delta t}{\Delta x}(p'_{i+1,j} - p'_{i,j})^n$$

$$\left.\begin{aligned}
(\rho u')_{i+1/2,j}^{n+1} &= (\rho u)_{i+1/2,j}^{n+1} - (\rho u^*)_{i+1/2,j}^{n+1} \\
(\rho u')_{i+1/2,j}^{n} &= (\rho u)_{i+1/2,j}^{n} - (\rho u^*)_{i+1/2,j}^{n} \\
A' &= A - A^* \\
p'_{i+1,j} &= p_{i+1,j} - p_{i+1,j}^* \\
p'_{i,j} &= p_{i,j} - p_{i,j}^*
\end{aligned}\right\}$$ **E5.3.5**

5.3 压力修正法

E5.3.2-E5.3.4

$$(\rho v')^{n+1}_{i,j+1/2} = (\rho v')^{n}_{i,j+1/2} + B'\,\Delta t - \frac{\Delta t}{\Delta y}\left(p'_{i,j+1} - p'_{i,j}\right)$$

$$\left.\begin{aligned}
(\rho v')^{n+1}_{i,j+1/2} &= (\rho v)^{n+1}_{i,j+1/2} - (\rho v^{*})^{n+1}_{i,j+1/2} \\
(\rho v')^{n}_{i,j+1/2} &= (\rho v)^{n}_{i,j+1/2} - (\rho v^{*})^{n}_{i,j+1/2} \\
B' &= B - B^{*} \\
p'_{i,j+1} &= p_{i,j+1} - p^{*}_{i,j+1} \\
p'_{i,j} &= p_{i,j} - p^{*}_{i,j}
\end{aligned}\right\} \text{E5.3.6}$$

E5.3.5、E5.3.6为修正量形式的动量方程。根据迭代最终收敛时u,v(u’,v’)必须满足连续方程的要求，可构建基于压力修正p’的公式。

5.3 压力修正法

Step 3: 构建p′ 压力修正方程(Patankar方法)

$$\cancel{(\rho u')^n \quad (\rho v')^n} \quad 0$$

A’,B’=0

$$\left.\begin{aligned}(\rho u')^{n+1}_{i+1/2,j} &= -\frac{\Delta t}{\Delta x}(p'_{i+1,j}-p'_{i,j})^n \\ (\rho v')^{n+1}_{i,j+1/2} &= -\frac{\Delta t}{\Delta x}(p'_{i,j+1}-p'_{i,j})^n\end{aligned}\right\}$$

$$(\rho u')^{n+1}_{i+1/2,j} = (\rho u)^{n+1}_{i+1/2,j} - (\rho u^*)^{n+1}_{i+1/2,j}$$

$$(\rho u)^{n+1}_{i+1/2,j} = (\rho u^*)^{n+1}_{i+1/2,j} - \frac{\Delta t}{\Delta x}(p'_{i+1,j}-p'_{i,j})^n$$

$$(\rho v)^{n+1}_{i,j+1/2} = (\rho v^*)^{n+1}_{i,j+1/2} - \frac{\Delta t}{\Delta y}(p'_{i,j+1}-p'_{i,j})^n$$

E5.3.7

5.3 压力修正法

Step 3: 构建p’压力修正方程(Patankar方法)

u,v满足连续性方程 $$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0$$

$$\frac{(\rho u)_{i+1/2,j}-(\rho u)_{i-1/2,j}}{\Delta x}+\frac{(\rho v)_{i,j+1/2}-(\rho v)_{i,j-1/2}}{\Delta y}=0$$

E5.3.7去掉时间标代入上式：

$$\frac{(\rho u^{*})_{i+1/2,j}-\Delta t/\Delta x(p'_{i+1,j}-p'_{i,j})-(\rho u^{*})_{i-1/2,j}+\Delta t/\Delta x(p'_{i,j}-p'_{i-1,j})}{\Delta x}$$
$$+\frac{(\rho v^{*})_{i,j+1/2}-\Delta t/\Delta y(p'_{i,j+1}-p'_{i,j})-(\rho v^{*})_{i,j-1/2}+\Delta t/\Delta x(p'_{i,j}-p'_{i,j-1})}{\Delta y}=0$$

5.3 压力修正法

$$a p'_{i,j} + b p'_{i+1,j} + b p'_{i-1,j} + c p'_{i,j+1} + c p'_{i,j-1} + d = 0$$

E5.3.8

$$a = 2\left[\frac{\Delta t}{(\Delta x)^2} + \frac{\Delta t}{(\Delta y)^2}\right]$$

$$b = -\frac{\Delta t}{(\Delta x)^2}$$

$$c = -\frac{\Delta t}{(\Delta y)^2}$$

$$d = \frac{1}{\Delta x}\left[(\rho u^*)_{i+1/2,j} - (\rho u^*)_{i-1/2,j}\right] + \frac{1}{\Delta y}\left[(\rho v^*)_{i,j+1/2} - (\rho v^*)_{i,j-1/2}\right]$$

中心格式

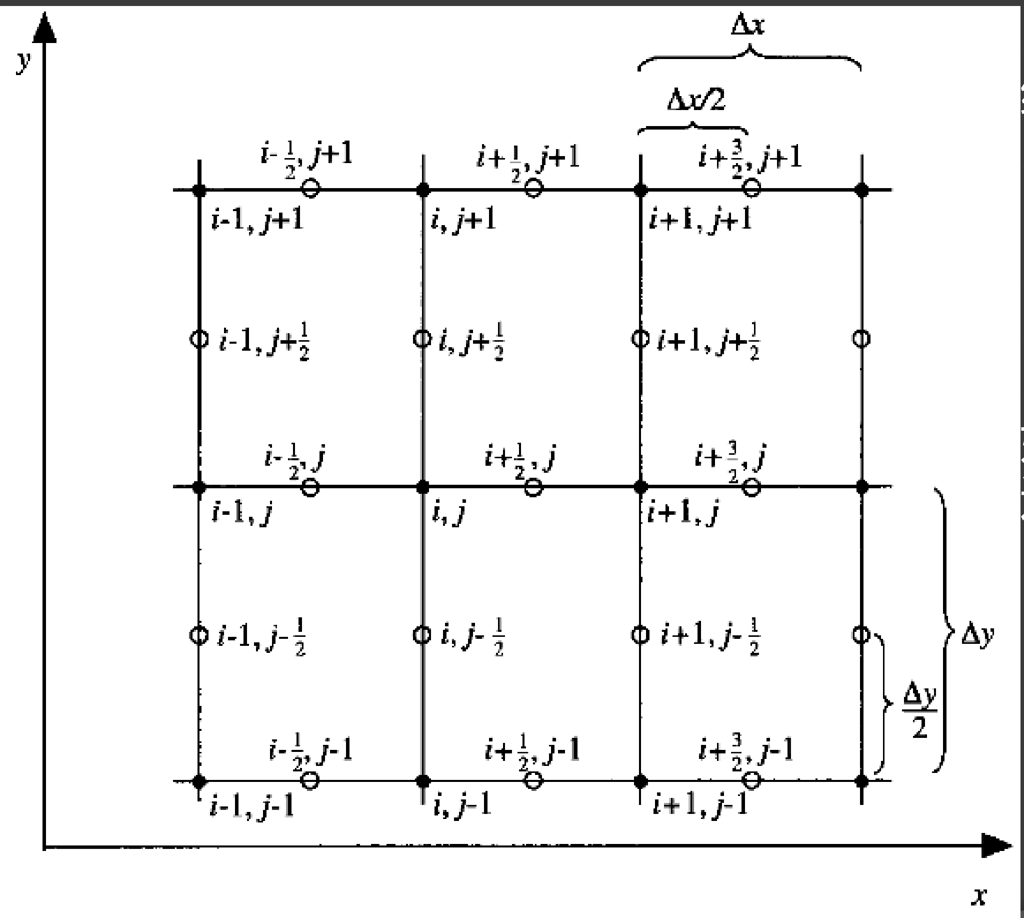
泊松方程

$$\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = Q$$

松弛法可解！

5.4 SIMPLE算法

SIMPLE: Semi-Implicit Method for Pressure-Linked Equations



Step 1: 猜测 实心格点上的$(P^*)^n$值，同时给定空心格点上的$(\rho u^*, \rho v^*)^n$值

Step 2:通过E5.3.3和E5.3.4计算$(\rho u^*, \rho v^*)^{n+1}$，

Step 3:使用E5.3.8求解p'（松弛法）

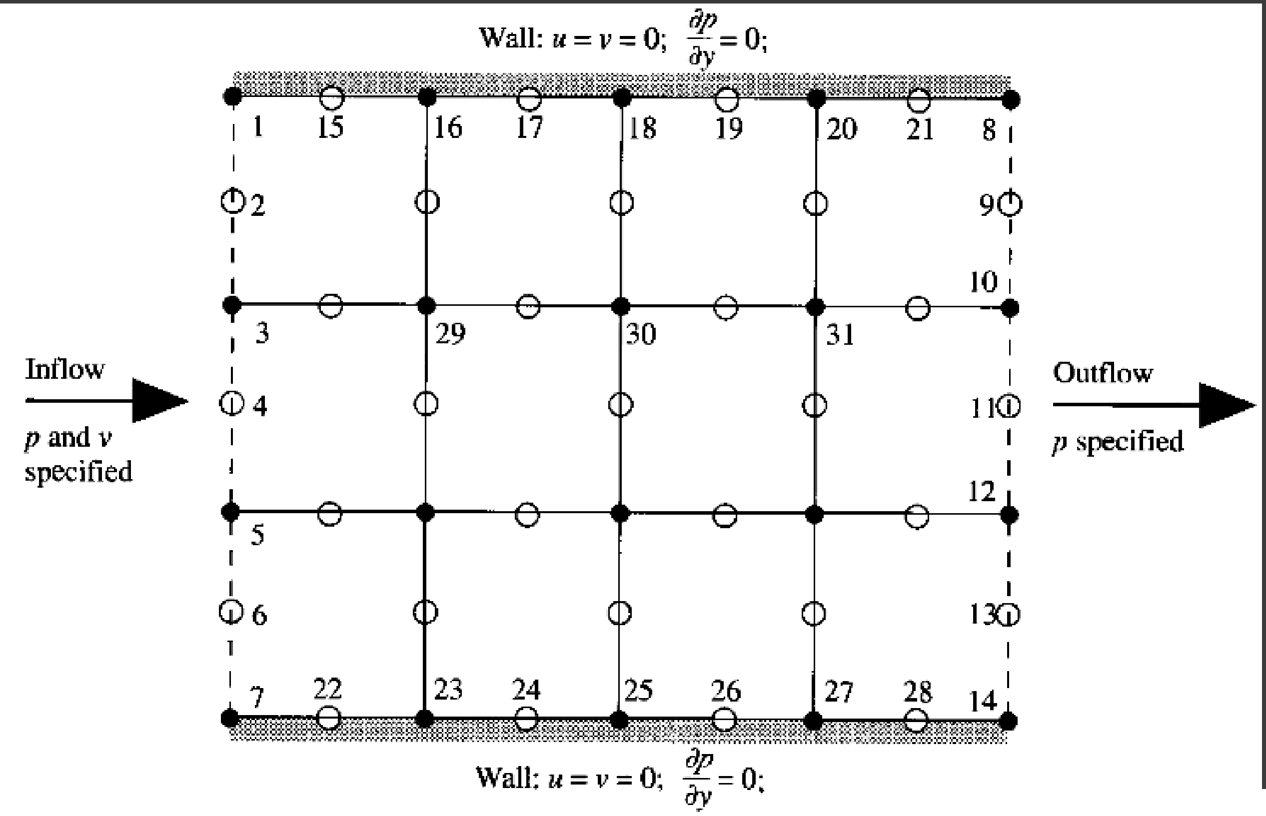
Step 4：更新$P^{n+1} = (p^*)^n + p'$

Step 5：用P^{n+1}替换P^*，返回step 2，开始新一轮迭代。跟踪连续方程项的误差，达到足够小，计算收敛。

对于迭代中可能出现发散的问题，可以在Step 4中使用亚松弛法,α_p可取为0.8。

$$p^{n+1} = (p^*)^n + \alpha_p p'$$

5.4 SIMPLE算法



进口条件: p,v给定，u未定

$$p_1' = p_3' = p_5' = p_7' = 0$$

出口条件: p给定，u,v未定

$$p_8' = p_{10}' = p_{12}' = p_{14}' = 0$$

壁面条件:无滑移u,v

$$u_{15} = u_{17} = u_{19} = u_{21} = u_{22}$$

$$= u_{24} = u_{26} = u_{28} = 0$$

0 0

$$\left(\frac{\partial p}{\partial y}\right)_w = \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)_w$$

$$p_1 = p_3 \qquad p_{16} = p_{29} \qquad p_5 = p_7$$

第六章　结构网格坐标变换与网格生成方法

6.1 进行坐标变换的原因

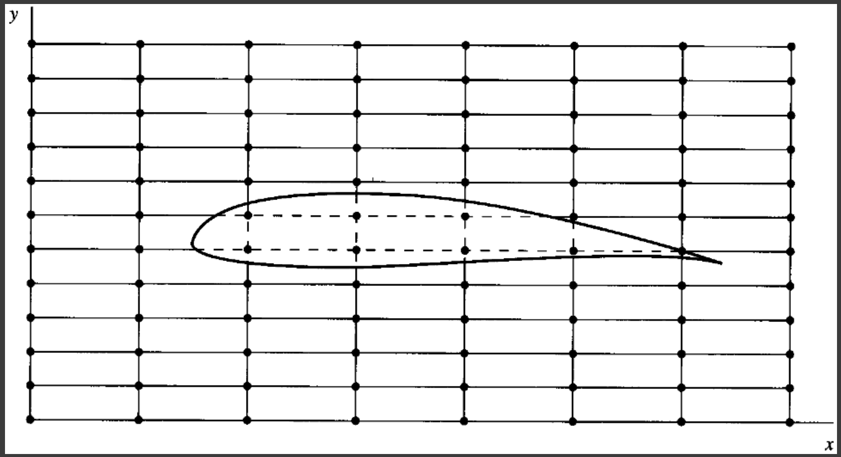
坐标变换原因及目的：

原因：有限差分法无法在非均匀网格进行计算求解；

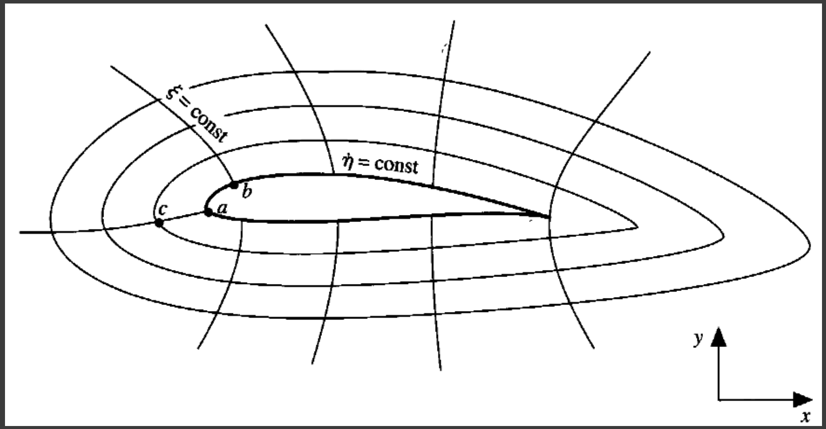
目的：通过坐标变换，将非均匀网格转变成为均匀网格；

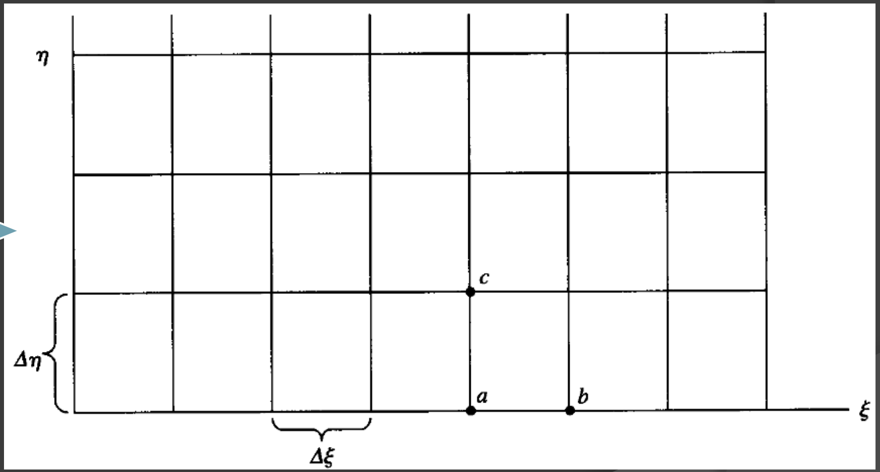
思路：构建计算平面，并建立计算平面与物理平面之间的点对点关系。

✍ 坐标变换仅在有限差分法中才会用到！



均匀网格
问题：翼型内部网格为固体，网格点流场信息无法处理；如翼型表面为固体边界，无法进行边界识别。





非均匀网格
问题：有限差分法无法直接求解

计算平面上的均匀网格

6.2 结构网格的坐标变换

直接变换：

建立新的坐标系（计算坐标系）

$$\xi = \xi(x, y, t)$$
$$\eta = \eta(x, y, t)$$
$$\tau = \tau(t)$$

ξ、η为计算平面上均匀分布的网格

$$\left(\frac{\partial}{\partial x}\right)_{y,t} = \left(\frac{\partial}{\partial \xi}\right)_{\eta,\tau}\left(\frac{\partial \xi}{\partial x}\right)_{y,t} + \left(\frac{\partial}{\partial \eta}\right)_{\xi,\tau}\left(\frac{\partial \eta}{\partial x}\right)_{y,t} + \left(\frac{\partial}{\partial \tau}\right)_{\xi,\eta}\left(\frac{\partial \tau}{\partial x}\right)_{y,t}$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right)$$

$$\frac{\partial}{\partial y} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right)$$

$$\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial t}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial t}\right) + \left(\frac{\partial}{\partial \tau}\right)\left(\frac{d\tau}{dt}\right)$$

6.2 结构网格的坐标变换-例

$$\frac{\partial u}{\partial x}+a\frac{\partial u}{\partial y}=0$$

$$\xi_x\frac{\partial u}{\partial \xi}+\eta_x\frac{\partial u}{\partial \eta}+a\left(\xi_y\frac{\partial u}{\partial \xi}+\eta_y\frac{\partial u}{\partial \eta}\right)=0$$

$$(\xi_x+a\xi_y)\frac{\partial u}{\partial \xi}+(\eta_x+a\eta_y)\frac{\partial u}{\partial \eta}=0$$

方程的离散及求解都要在 ξ、η 平面上进行！

6.2 结构网格的坐标变换

$$\frac{\partial^2}{\partial x^2}=\left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x^2}\right)+\left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right)+\left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial^2}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right)^2+2\left(\frac{\partial^2}{\partial \eta\,\partial \xi}\right)\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \xi}{\partial x}\right)$$

$$\frac{\partial^2}{\partial y^2}=\left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial y^2}\right)+\left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial y^2}\right)+\left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial^2}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right)^2+2\left(\frac{\partial^2}{\partial \eta\,\partial \xi}\right)\left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \xi}{\partial y}\right)$$

$$\frac{\partial^2}{\partial x\,\partial y}=\left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x\,\partial y}\right)+\left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x\,\partial y}\right)+\left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right)+\left(\frac{\partial^2}{\partial \xi\,\partial \eta}\right)\left[\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right)+\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right)\right]$$

6.2 结构网格的坐标变换

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

$$\frac{\partial^2\phi}{\partial\xi^2}\left[\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\xi}{\partial y}\right)^2\right]+\frac{\partial^2\phi}{\partial\eta^2}\left[\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2\right]$$
$$+2\frac{\partial^2\phi}{\partial\xi\,\partial\eta}\left[\left(\frac{\partial\eta}{\partial x}\right)\left(\frac{\partial\xi}{\partial x}\right)+\left(\frac{\partial\eta}{\partial y}\right)\left(\frac{\partial\xi}{\partial y}\right)\right]$$
$$+\frac{\partial\phi}{\partial\xi}\left(\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}\right)+\frac{\partial\phi}{\partial\eta}\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}\right)=0$$

6.3 结构网格的坐标逆变换

逆变换：

$$x = x(\xi, \eta, \tau)$$
$$y = y(\xi, \eta, \tau)$$
$$t = t(\tau)$$

$$\frac{\partial}{\partial x} = \frac{1}{J}\left[\left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial y}{\partial \eta}\right) - \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial y}{\partial \xi}\right)\right]$$

$$\frac{\partial}{\partial y} = \frac{1}{J}\left[\left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial x}{\partial \xi}\right) - \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial x}{\partial \eta}\right)\right]$$

$$J \equiv \frac{\partial(x, y)}{\partial(\xi, \eta)} \equiv \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix}$$

$$J = \frac{1}{x_\xi y_\eta - y_\xi x_\eta}$$

6.3 结构网格的坐标逆变换

$$\begin{aligned} \xi &= \xi(x,y) \\ \eta &= \eta(x,y) \end{aligned} \Rightarrow \begin{aligned} d\xi &= \xi_x dx + \xi_y dy \\ d\eta &= \eta_x dx + \eta_y dy \end{aligned} \Rightarrow \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$\begin{aligned} x &= x(\xi,\eta) \\ y &= y(\xi,\eta) \end{aligned} \Rightarrow \begin{aligned} dx &= x_\xi d\xi + x_\eta d\eta \\ dy &= y_\xi d\xi + y_\eta d\eta \end{aligned} \Rightarrow \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix}$$

$$\begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix}^{-1}$$

$$\begin{aligned} \xi_x &= J y_\eta \\ \xi_y &= -J x_\eta \\ \eta_x &= -J y_\xi \\ \eta_y &= J x_\xi \end{aligned}$$

$$J = \frac{1}{x_\xi y_\eta - y_\xi x_\eta}$$

6.3 结构网格的坐标逆变换

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=0$$

$$\Downarrow$$

$$\frac{\partial U_1}{\partial t}+\frac{\partial F_1}{\partial \xi}+\frac{\partial G_1}{\partial \eta}=0$$

$$U_1=JU$$

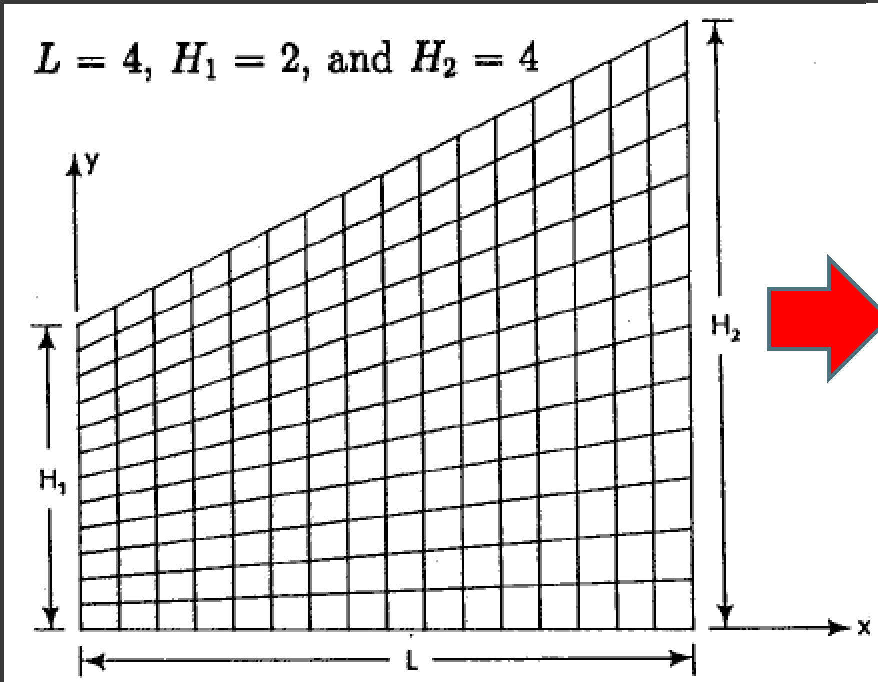
$$F_1=JF\frac{\partial \xi}{\partial x}+JG\frac{\partial \xi}{\partial y}=F\frac{\partial y}{\partial \eta}-G\frac{\partial x}{\partial \eta}$$

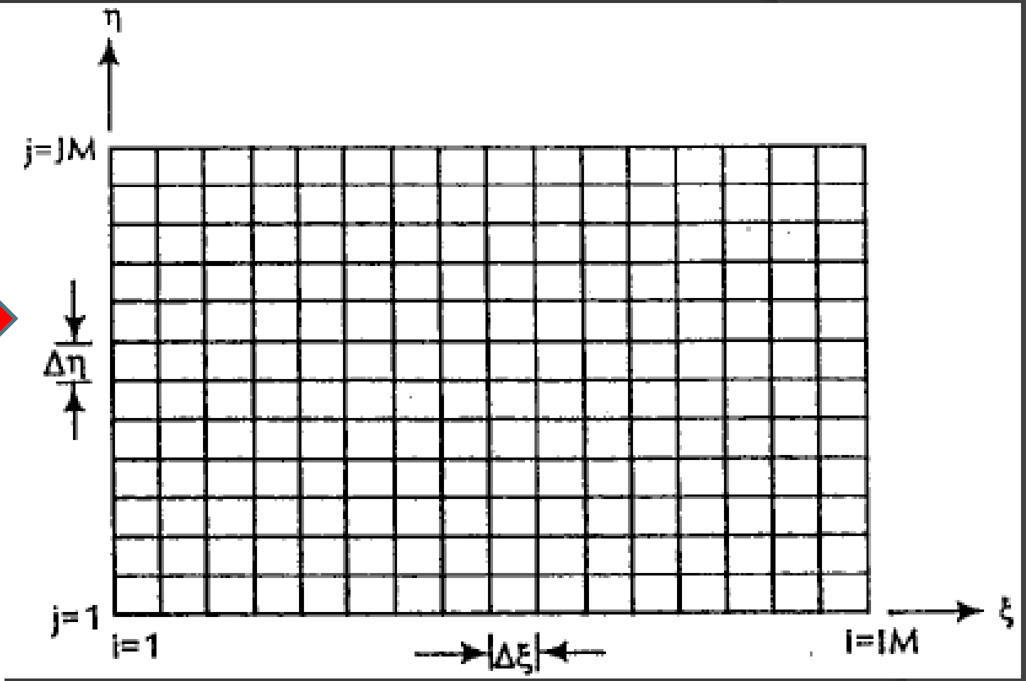
$$G_1=JF\frac{\partial \eta}{\partial x}+JG\frac{\partial \eta}{\partial y}=-F\frac{\partial y}{\partial \xi}+G\frac{\partial x}{\partial \xi}$$

6.4 结构化网格生成方法

1. 代数方法
2. 几何方法
3. 保角变换
4. Thompson法

6.4.1 代数法网格生成及求解





物理域

$$\xi = \xi(x, y)$$
$$\eta = \eta(x, y)$$

计算域

$$\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 0$$

$$(\xi_x + a\xi_y)\frac{\partial u}{\partial \xi} + (\eta_x + a\eta_y)\frac{\partial u}{\partial \eta} = 0$$

6.4.1 代数法网格生成及求解

$$\xi = x$$
$$\eta = \frac{y}{y_t}$$
$$y_t = H_1 + \frac{H_2 - H_1}{L} x$$

$$\xi = x$$
$$\eta = \frac{y}{H_1 + \frac{H_2 - H_1}{L} x}$$

$$x = \xi$$
$$y = \left(H_1 + \frac{H_2 - H_1}{L} \xi \right) \eta$$

$$\Delta\xi = \frac{L}{IM - 1}$$
$$\Delta\eta = \frac{1.0}{JM - 1}$$

$$\xi_x = 1$$
$$\xi_y = 0$$

$$\eta_x = -\frac{(H_2 - H_1) y / L}{[H_1 + (H_2 - H_1) x / L]^2} \quad \text{or}$$
$$\eta_x = -\frac{H_2 - H_1}{L} \frac{\eta}{[H_1 + (H_2 - H_1)\xi / L]}$$

$$\eta_y = \frac{1}{H_1 + (H_2 - H_1) x / L} \quad \text{or}$$
$$\eta_y = \frac{1}{H_1 + (H_2 - H_1)\xi / L}$$

6.4.1 代数法网格生成及求解

$$\begin{aligned}\xi_x &= \boxed{J y_\eta} \\ \xi_y &= -J x_\eta \\ \eta_x &= -J y_\xi \\ \eta_y &= J x_\xi\end{aligned}$$

$$J = \frac{1}{x_\xi y_\eta - y_\xi x_\eta}$$

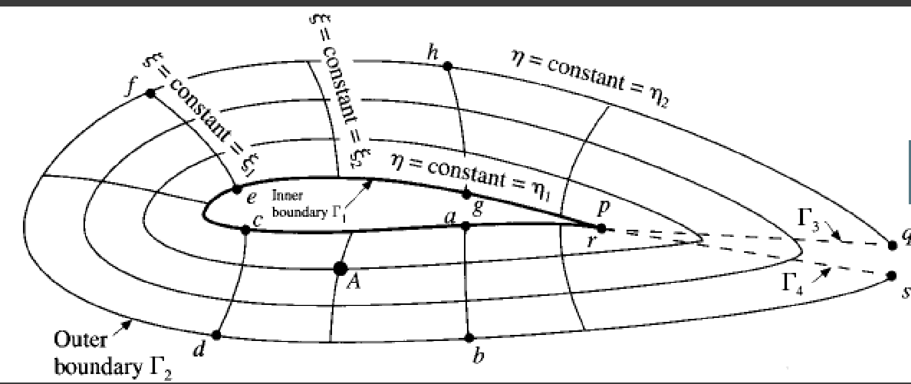
亦可通过二阶中心格式求得

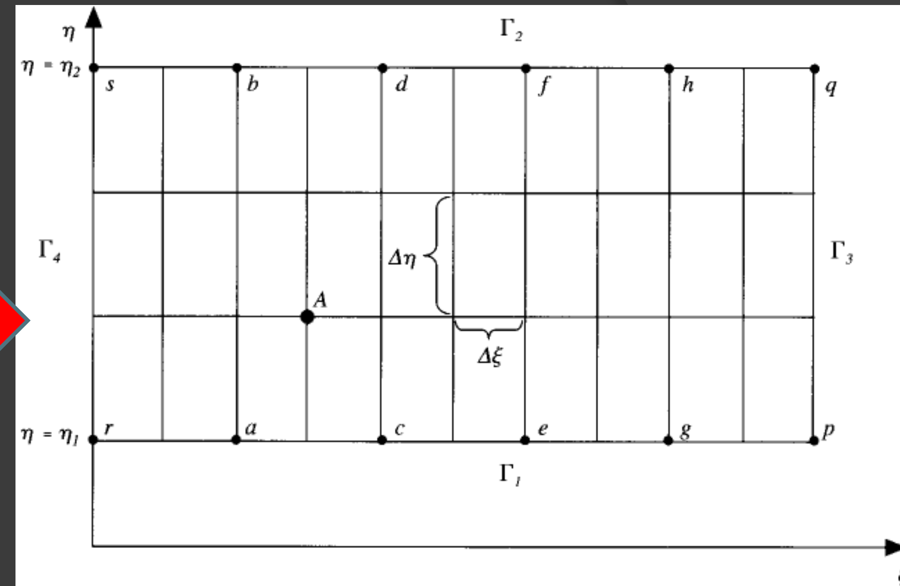
$$J = \frac{1}{x_\xi y_\eta - y_\xi x_\eta}$$

$$\boxed{(\xi_x + a\xi_y)}\frac{\partial u}{\partial \xi} + \boxed{(\eta_x + a\eta_y)}\frac{\partial u}{\partial \eta} = 0$$

皆为已知数，方程可使用已学格式离散及求解方法进行求解!

6.4.2 PDE型网格生成及求解





物理域

计算域

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} = 0$$

$$\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = 0$$

$$\alpha \frac{\partial^2 x}{\partial \xi^2} - 2\beta \frac{\partial^2 x}{\partial \xi \, \partial \eta} + \gamma \frac{\partial^2 x}{\partial \eta^2} = 0$$

$$\alpha \frac{\partial^2 y}{\partial \xi^2} - 2\beta \frac{\partial^2 y}{\partial \xi \, \partial \eta} + \alpha \frac{\partial^2 y}{\partial \eta^2} = 0$$

$$\alpha = \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2$$

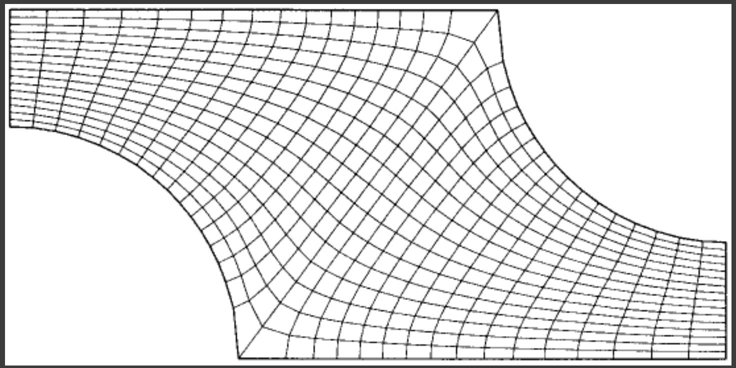
$$\beta = \left(\frac{\partial x}{\partial \xi}\right)\left(\frac{\partial x}{\partial \eta}\right) + \left(\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial y}{\partial \eta}\right)$$

$$\gamma = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2$$

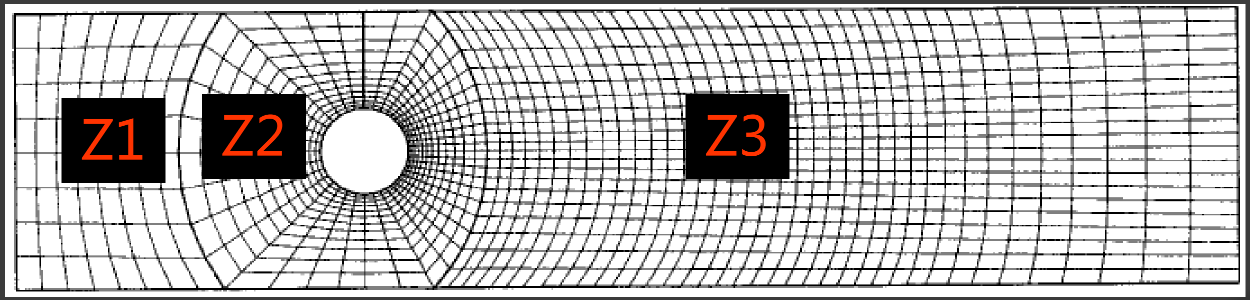
6.5 网格类型及划分方法

- 网格类型
  - 结构化
    - 结构化网格
    - 多块网格 H/O/C/I
    - 优点：计算量小/存储简便/容易实现网格剖分
    - 缺点：复杂通道难以实现网格剖分且无法保证网格质量
  - 非结构网格
    - 前沿推进
    - Delaunay三角化
    - 笛卡尔网格
    - 优点：可对任意复杂区域进行网格剖分，网格质量较好
    - 缺点：存储复杂/生成步骤繁琐/生成速度慢
  - 混和网格
    - 结构化 + 非结构化

6.5.1 块结构化网格

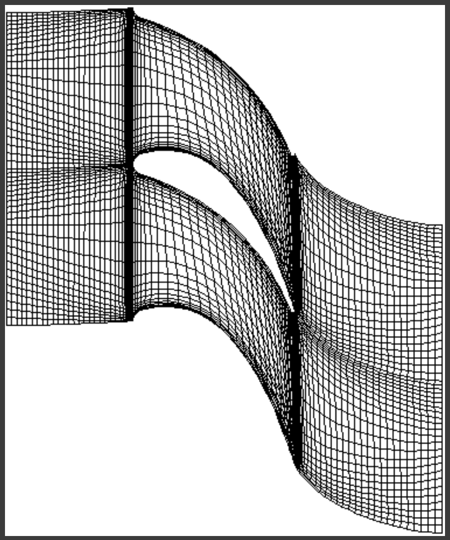


单块结构化网格

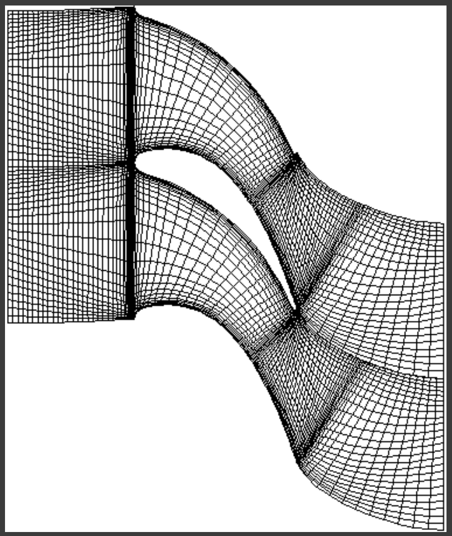


多块结构化网格

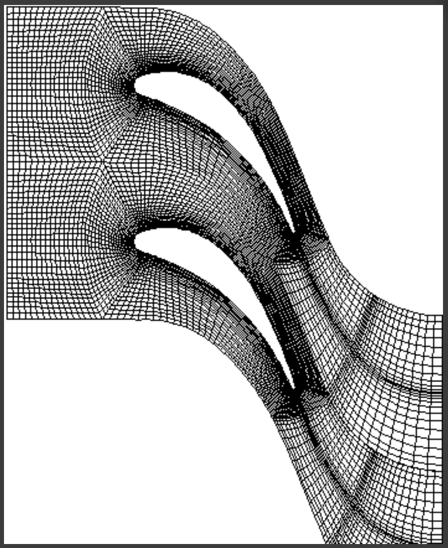
6.2.2 块结构化网格



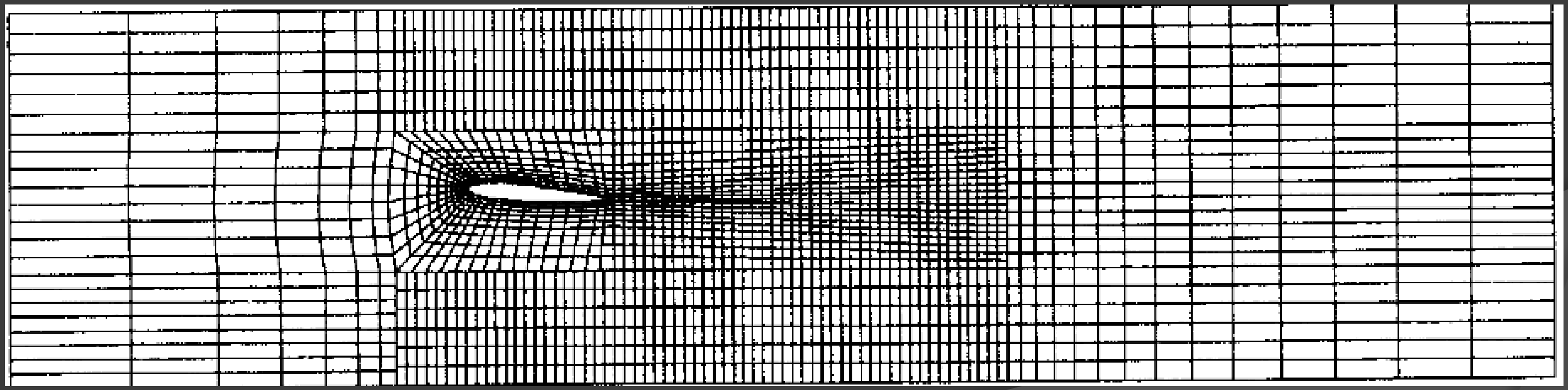
H型



H-I型



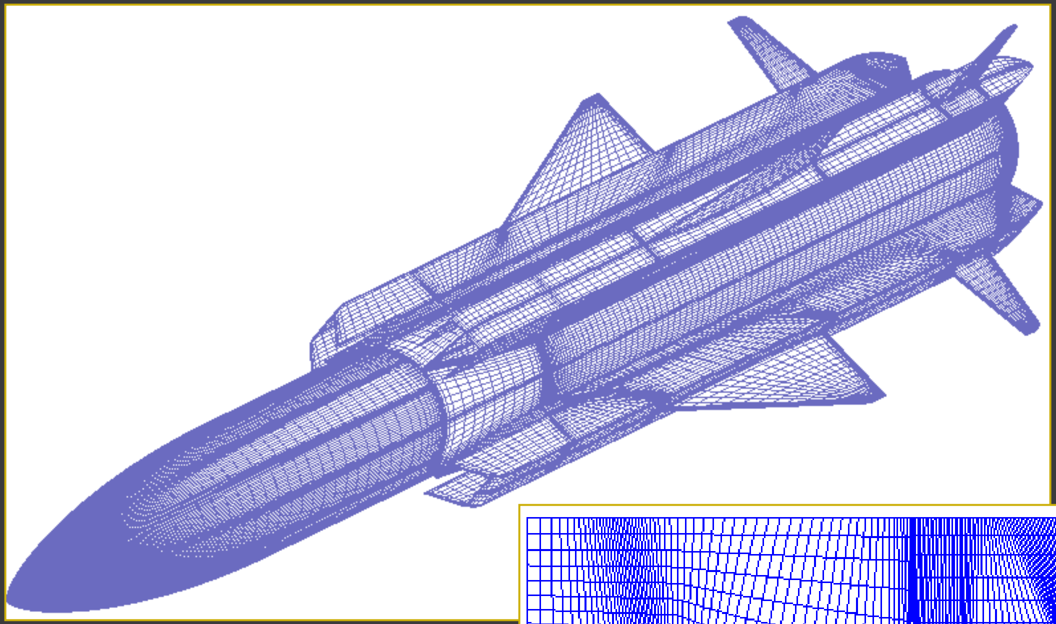
H-O-H型

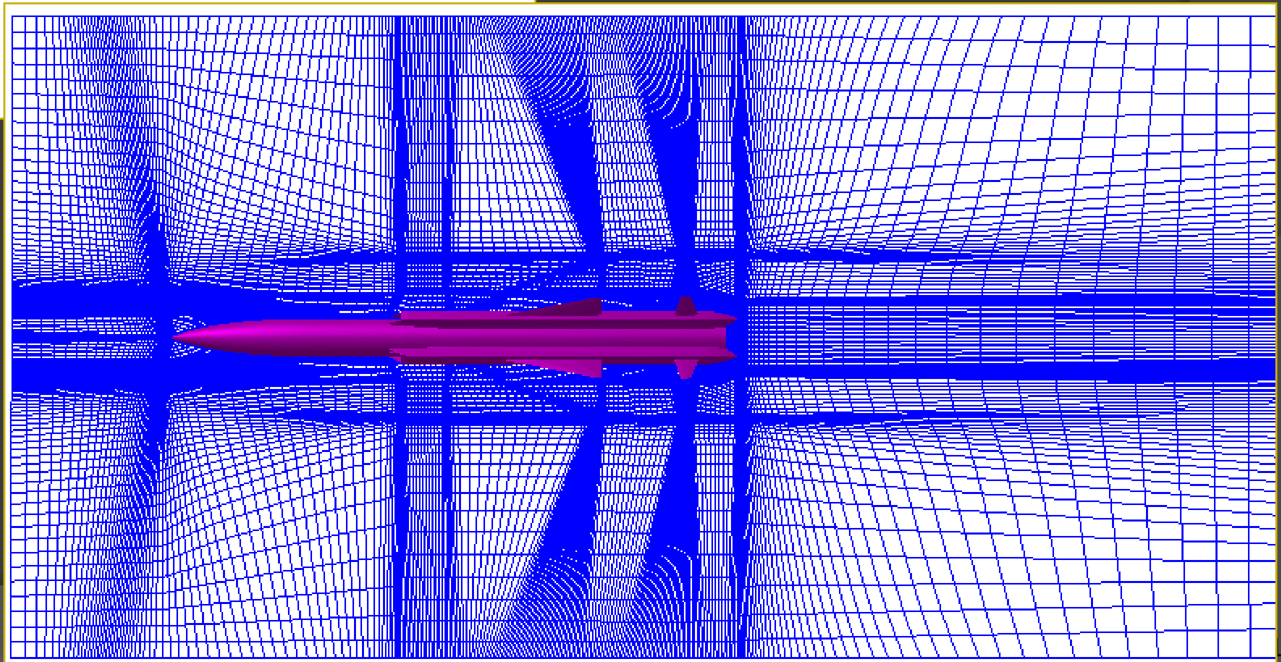


H-C-H型

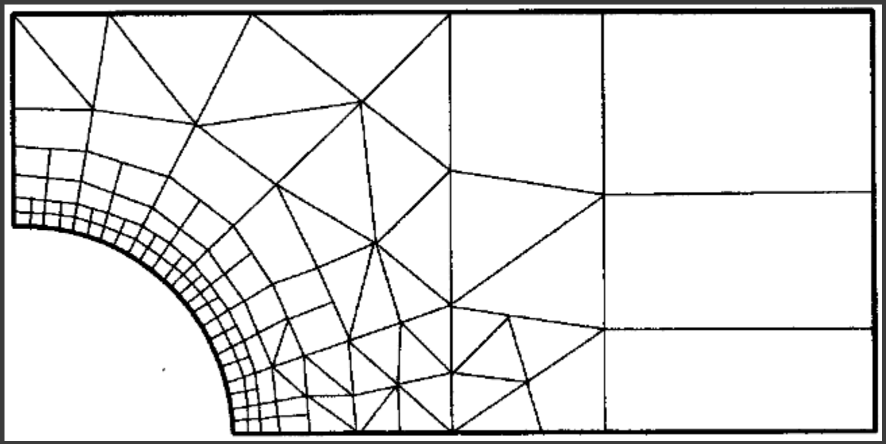
- 分块网格可较方便的实现局部加密和自适应

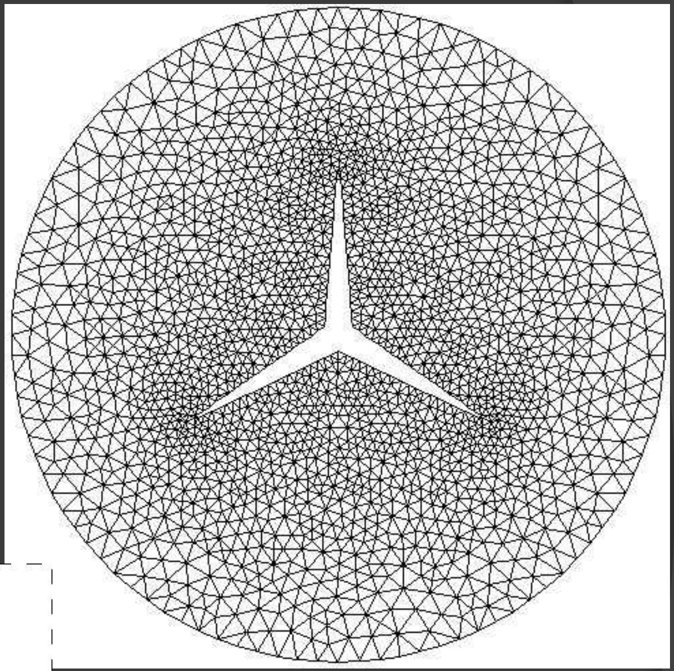
6.5.2 块结构化网格

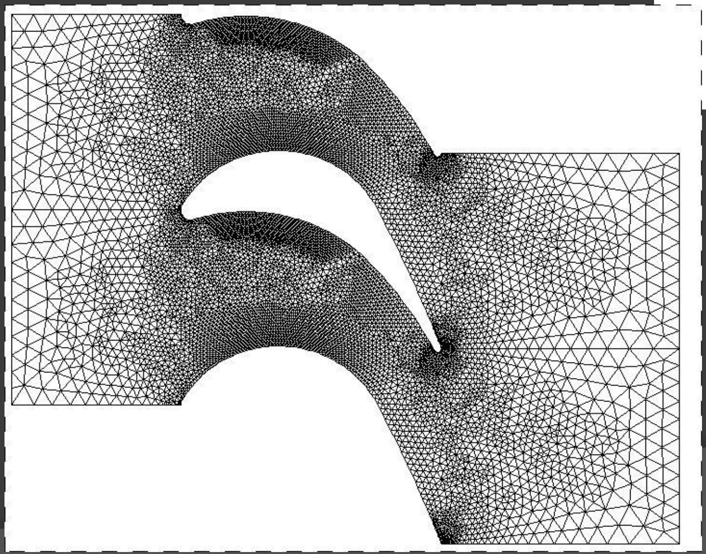




6.5.3 非结构网格

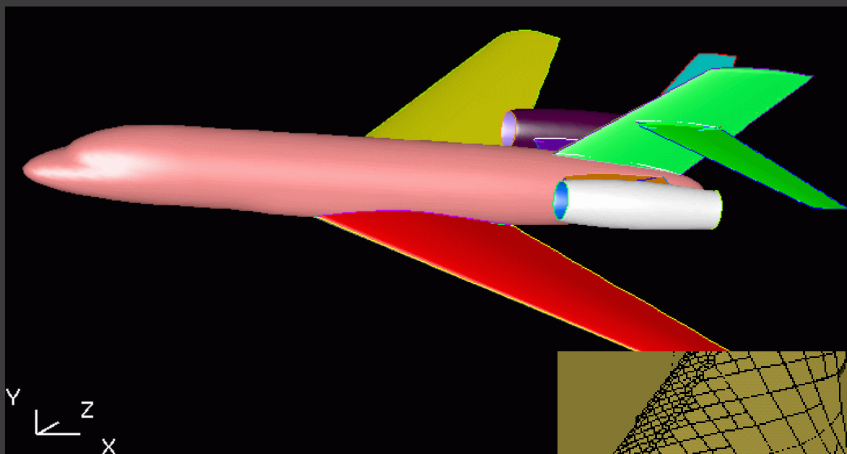


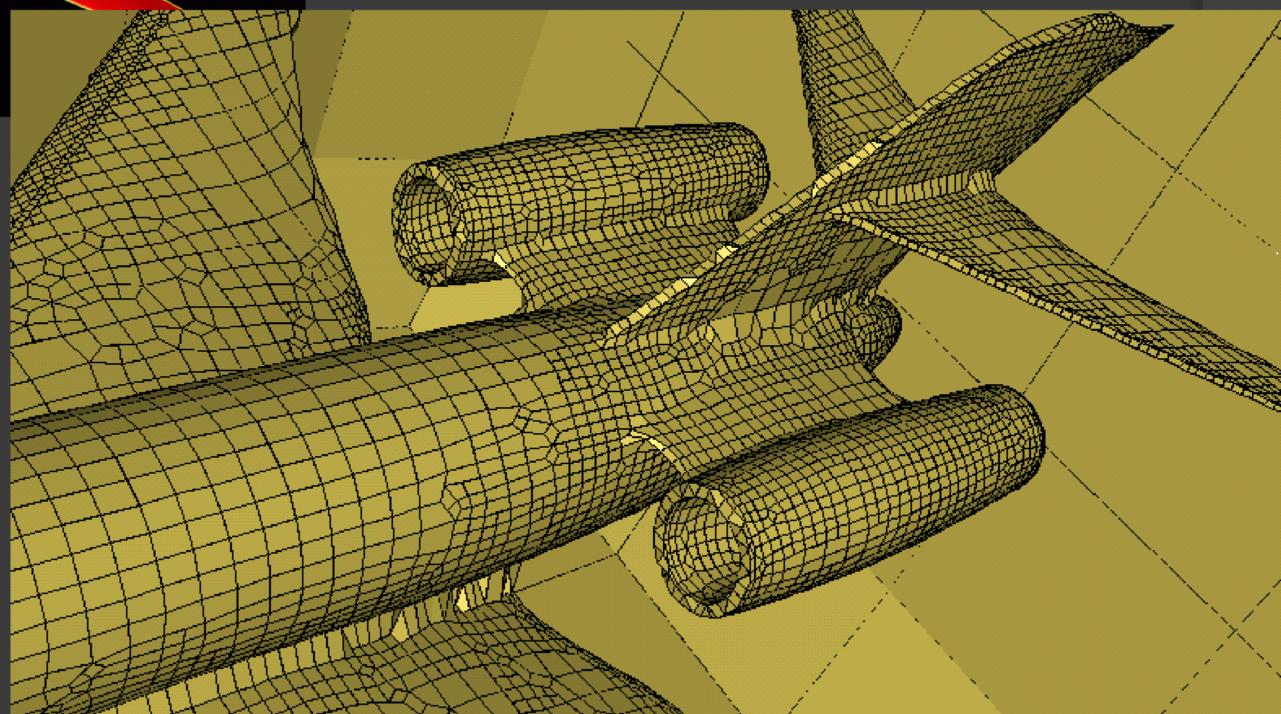




三角网格

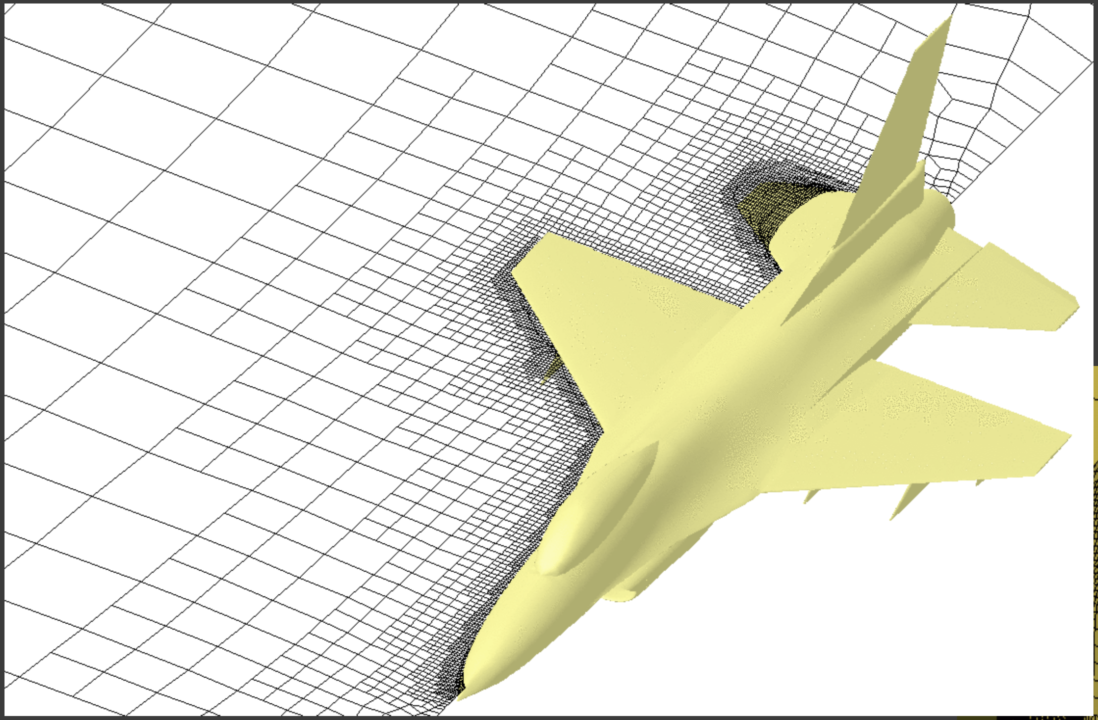
6.5.3 非结构网格

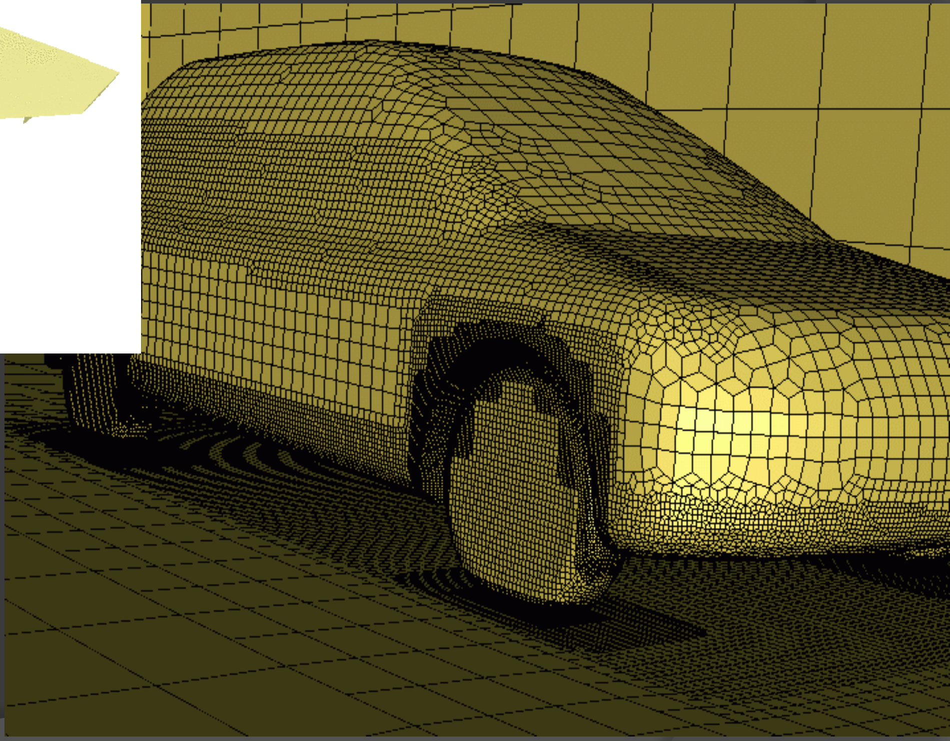




笛卡尔网格

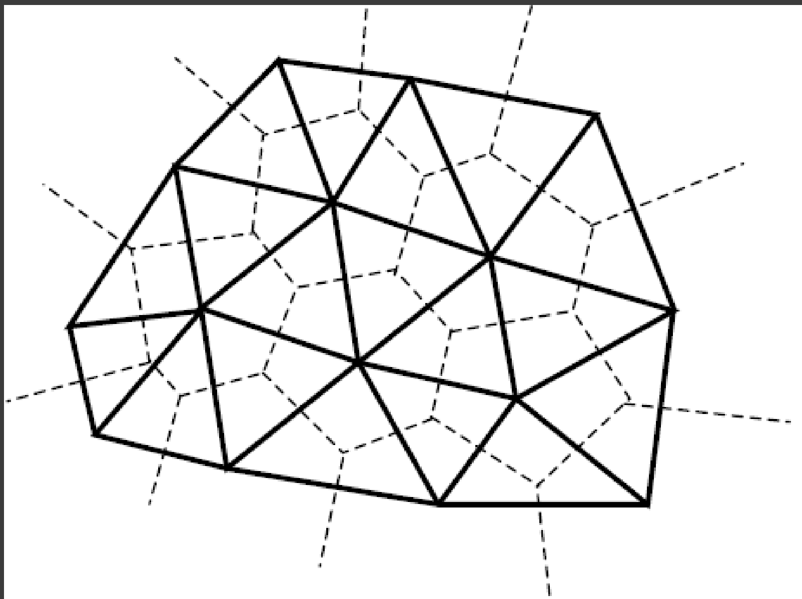
6.5.3 非结构网格





笛卡尔网格

6.5.4 非结构网格-Delaunay

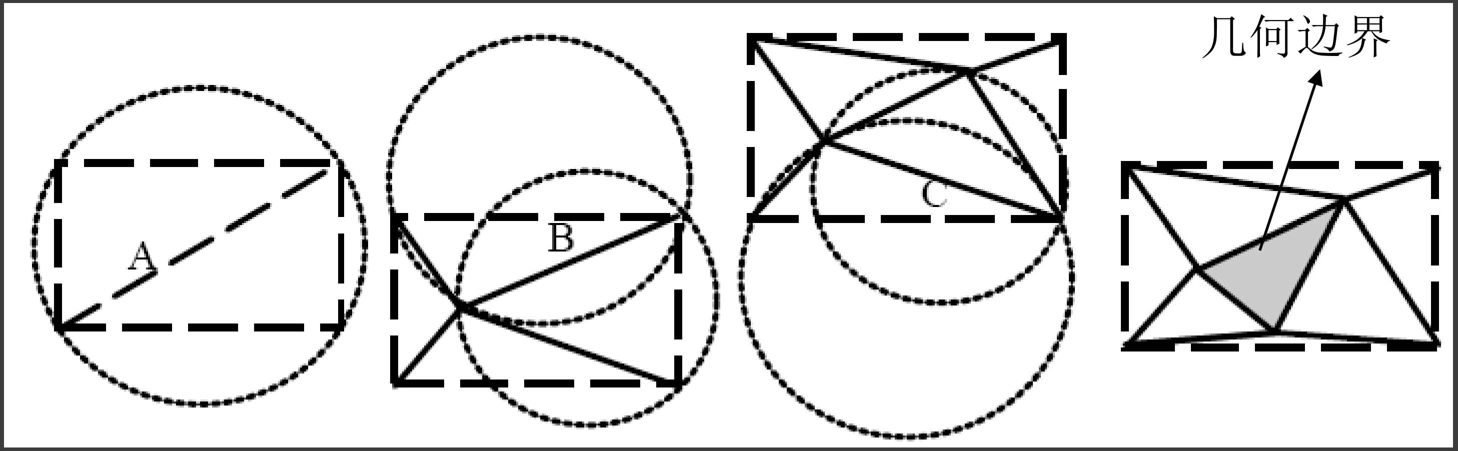


Delaunay方法基于Dirichlet思想，具有严格的数学及几何基础：任何一个三角形单元的外接圆不得包含其他三角形单元的非共顶点，也即所有三角形单元的外接圆圆心之间互相连接，所组成的任何多边形应该为凸多边形

网格生成步骤：计算域外边界→初始网格→内部加点→网格光顺

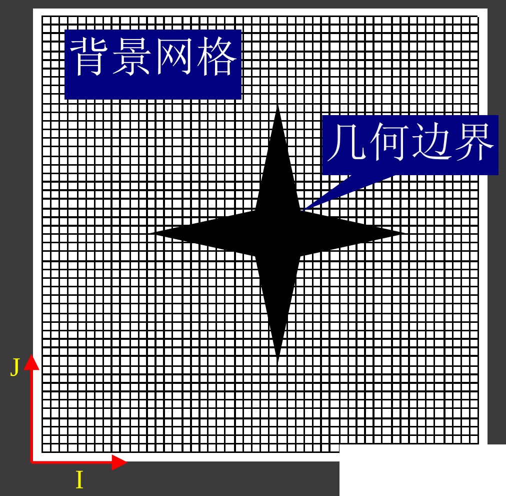
6.5.4 非结构网格-Delaunay

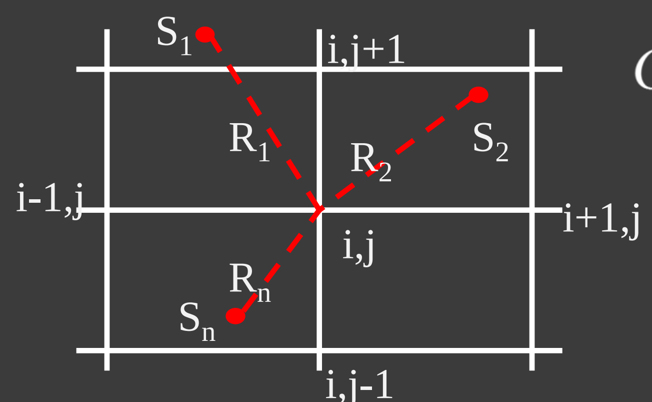
初始网格的生成:



6.5.5 非结构网格-Advancing Front

初始网格的生成:





结构化背景网格中的
源分布示意

任一节点(i, j)处源项：

$$G_{i,j}=\sum_{k=1}^{n}\psi_k\left(S_{i,j}A_k-B_k\right)$$

$$A_k=\frac{1}{R_k^2}\qquad B_k=\frac{S_k}{R_k^2}$$

ψ_k： 源强度

$$S_{i,j}^{m+1}=(1-\omega)S_{i,j}^{m}+\frac{\omega\left(\Delta Y^2(S_{i-1,j}^{m+1}+S_{i+1,j}^{m})+\Delta X^2(S_{i,j-1}^{m+1}+S_{i,j+1}^{m})+\Delta X^2\Delta Y^2\sum_{k=1}^{N}\psi_k B_k\right)}{2\left(\Delta X^2+\Delta Y^2\right)+\Delta X^2\Delta Y^2\sum_{k=1}^{N}\psi_k A_k}$$

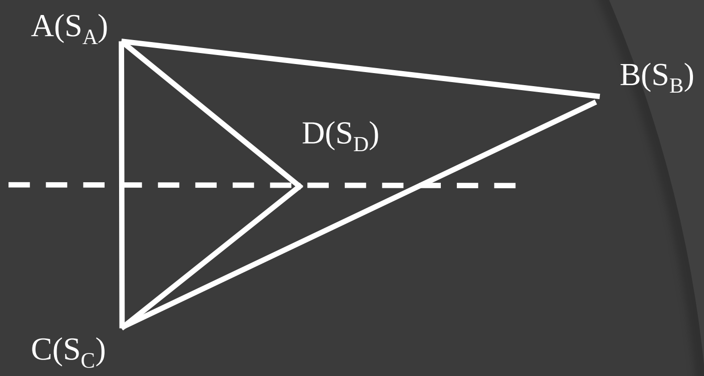
- 三角形分类
  - 待删除三角形
  - 合格三角形
  - 活动三角形

$$\begin{cases} L_{I,J}\le \dfrac{S_I+S_J}{2}\cdot Coef & \text{合格三角形} \\ \text{其它} & \text{活动三角形} \end{cases}$$

6.5.5 非结构网格-Advancing Front

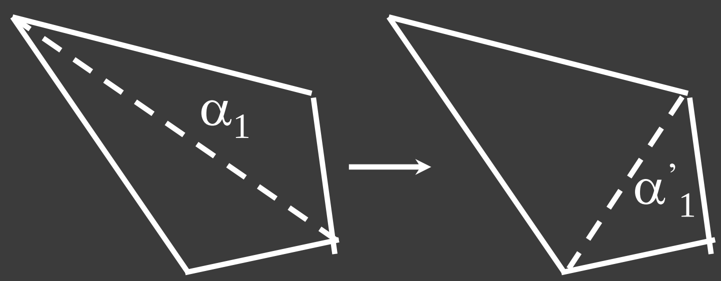
➢ 加点位置准则：

$$\frac{L_{AD/CD}}{S_D} - \frac{2\times L_{AC}}{S_A+S_C} \le O(0)$$

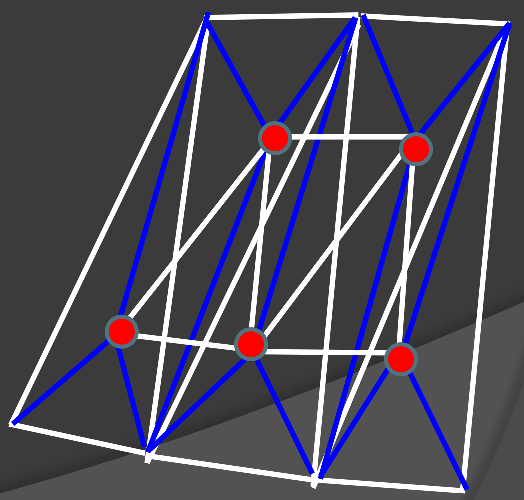


活动三角形插入点示意图

- 局部重组

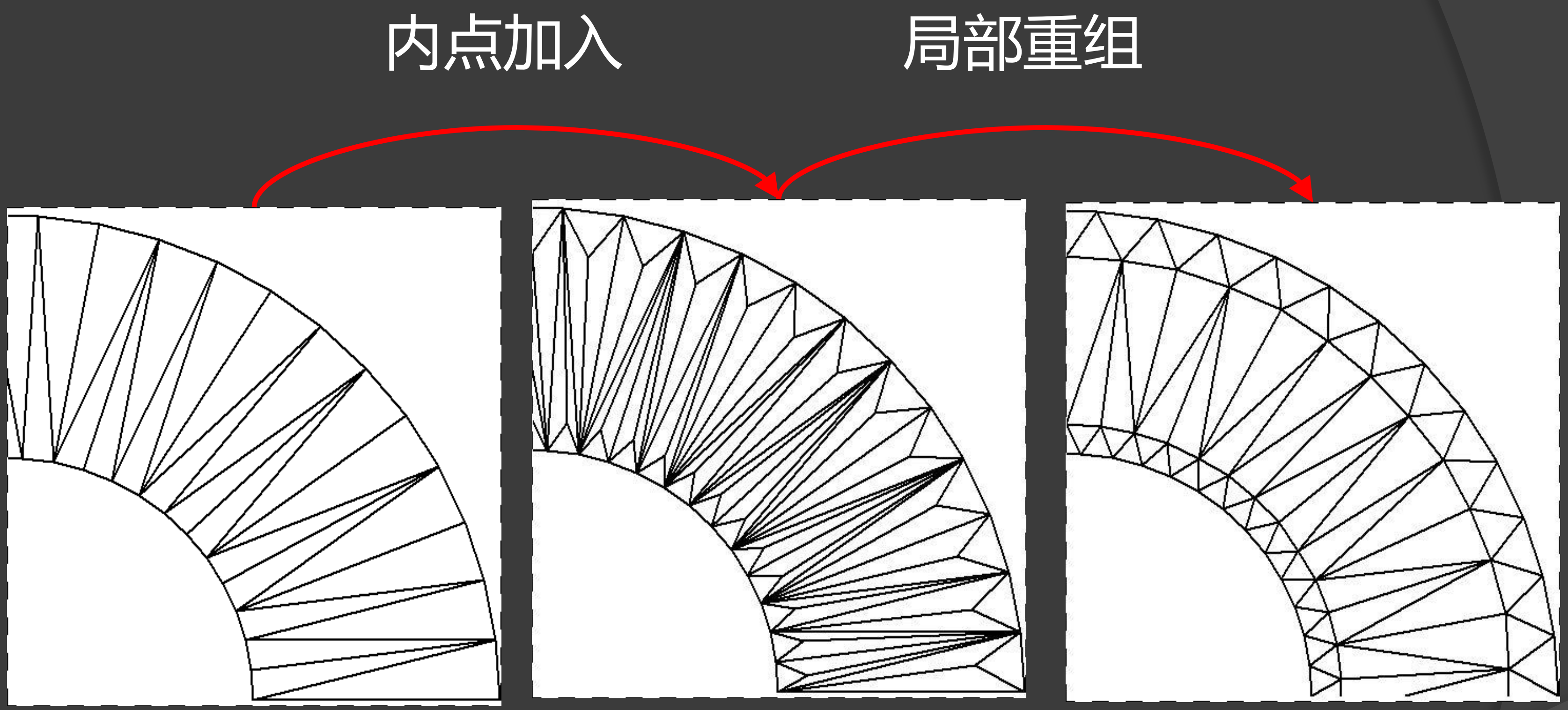


双单元网格局部重组示意图



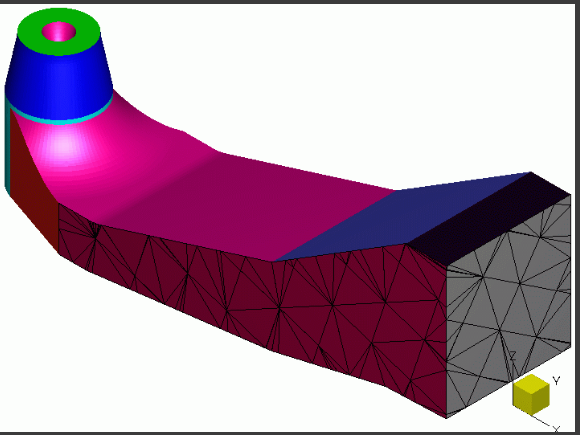
多单元网格局部重组示意图

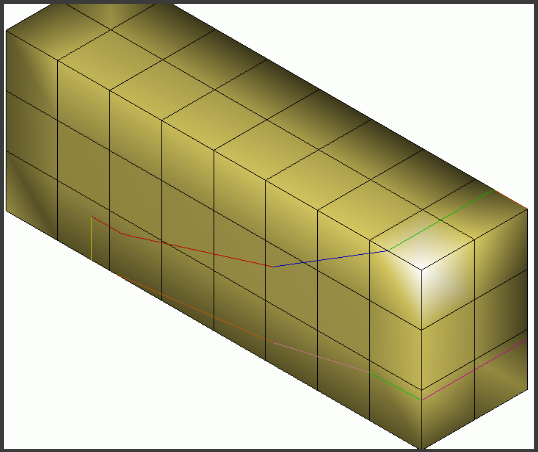
6.5.5 非结构网格-Advancing Front

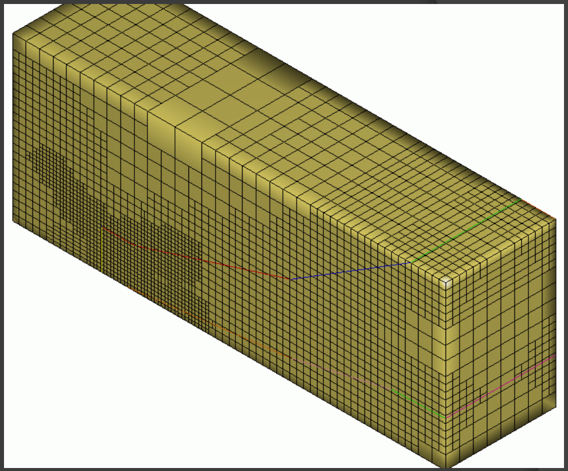


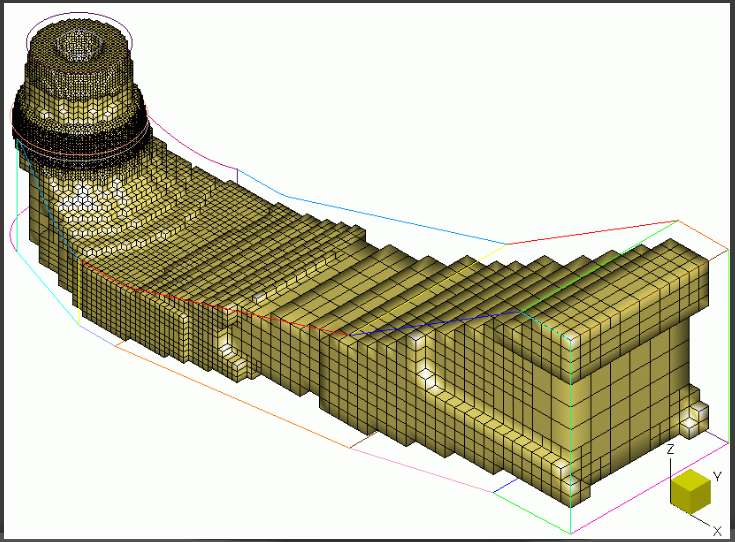
网格局部重组过程

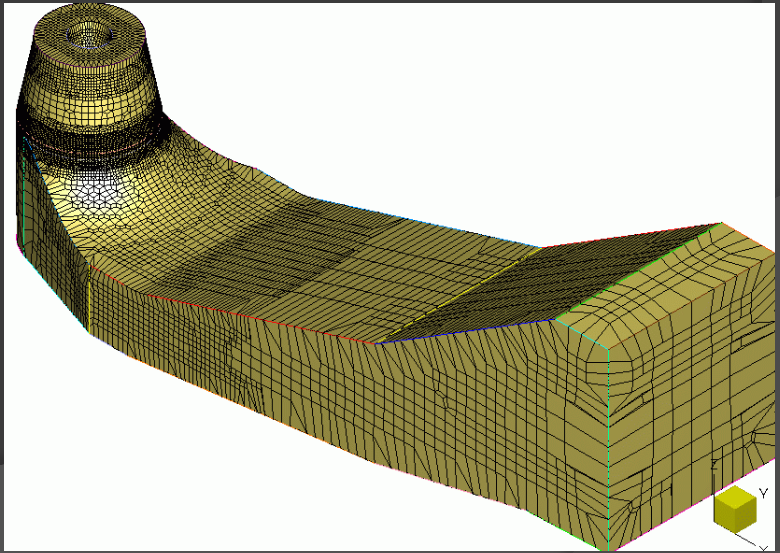
6.5.6 非结构网格-Cartisan Mesh



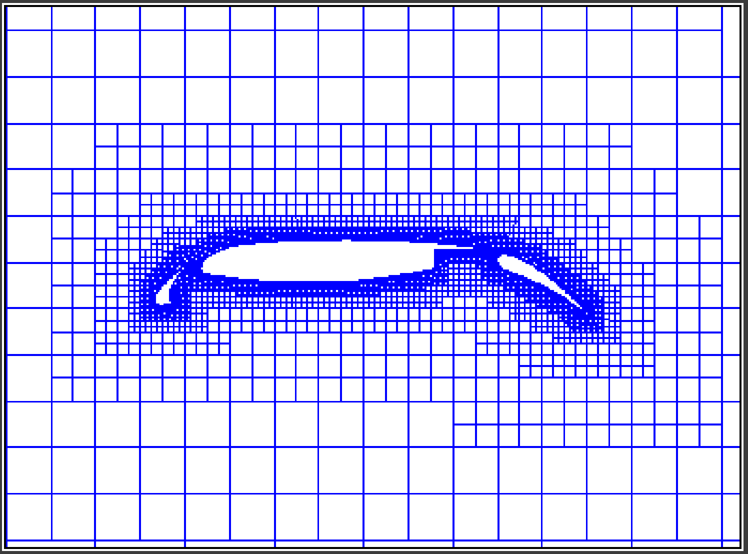


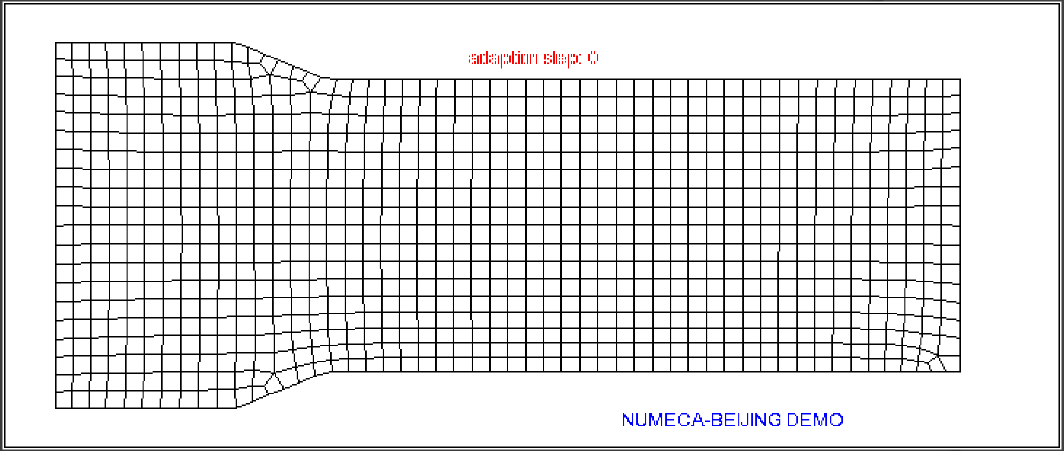


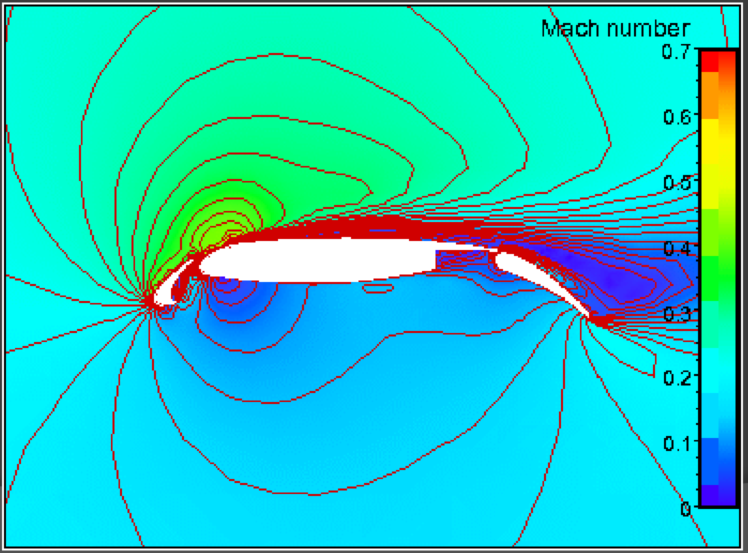


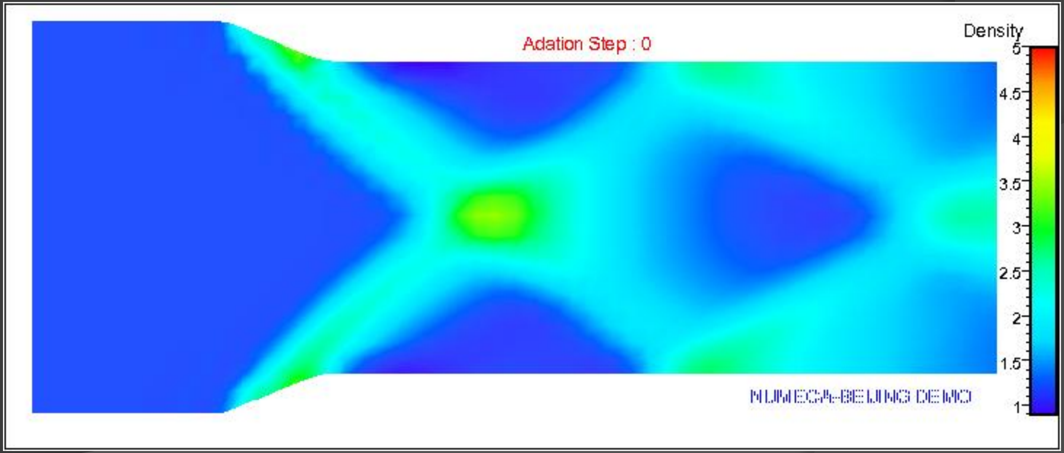


6.5.7 非结构网格-自适应网格

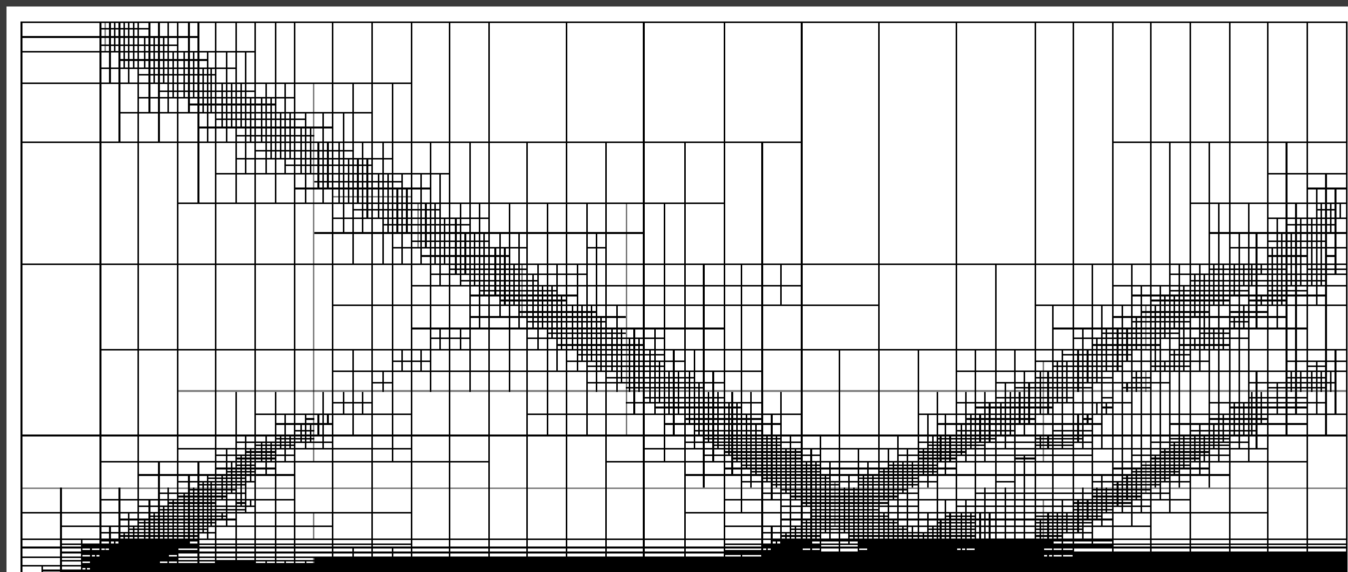


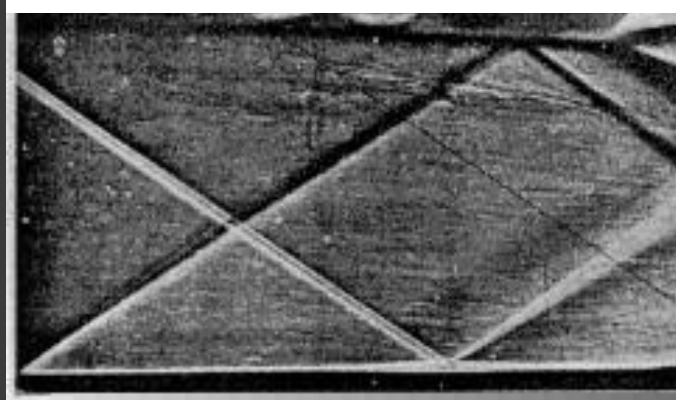


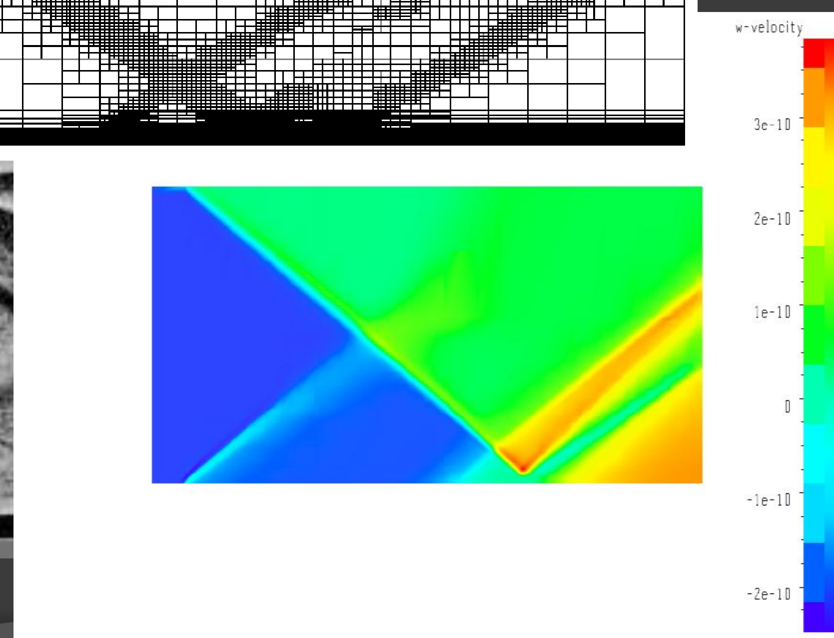




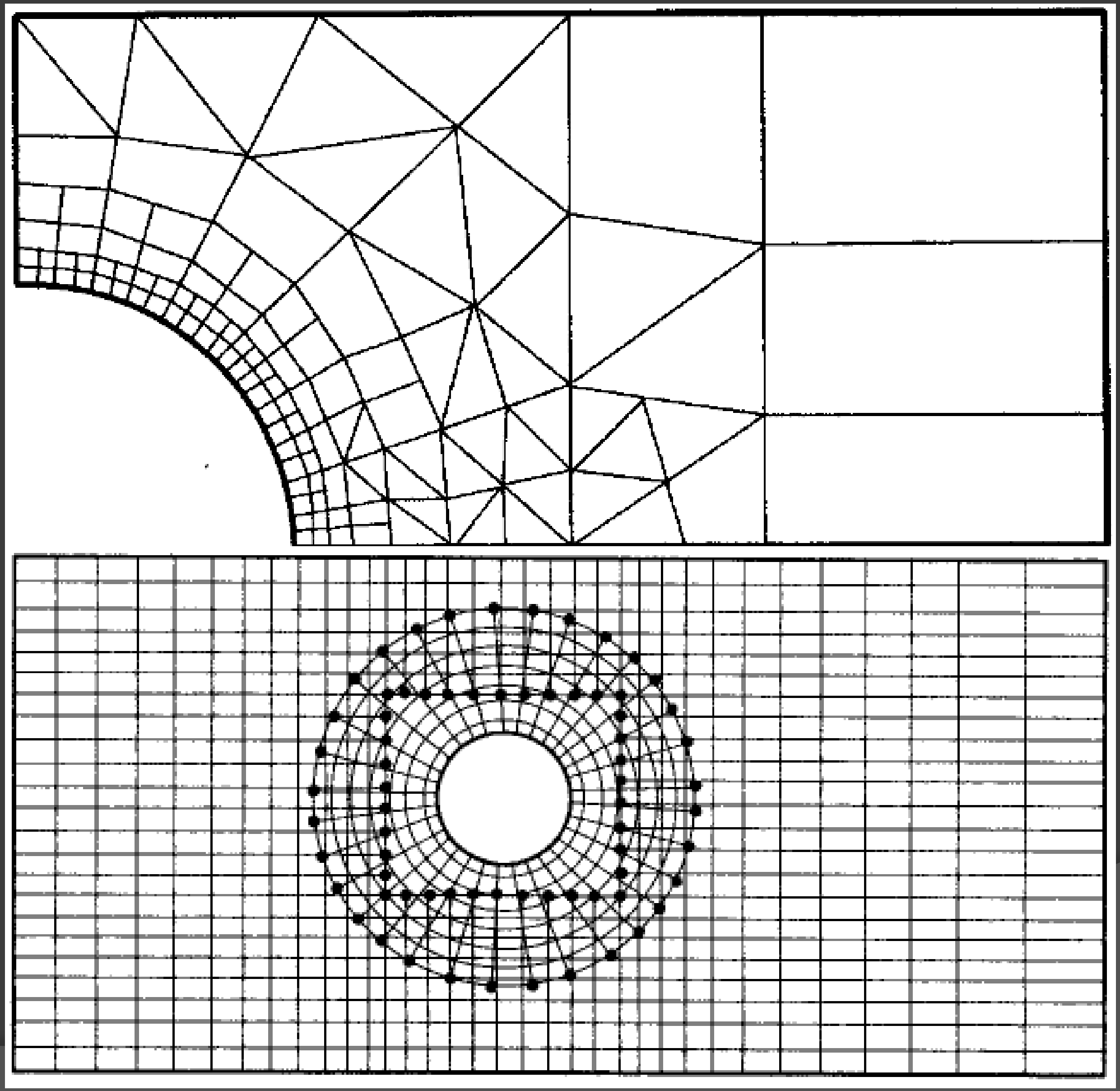
6.5.8 非结构网格-自适应网格







6.5.9 复合网格



6.6 本章小结

$$\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 0$$

$$(\xi_x + a\xi_y)\frac{\partial u}{\partial \xi} + (\eta_x + a\eta_y)\frac{\partial u}{\partial \eta} = 0$$

坐标变换

变换原因？

变化方法

网格分类与生成

结构化

代数法

Thompson

多块网格

非结构化

三角形/四面体

Delaunay

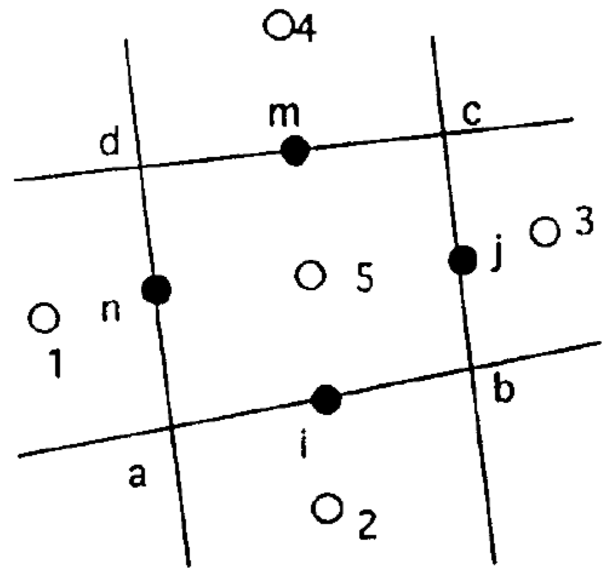
前沿推进

笛卡尔/六面体

第七章　有限体积法基础

7.1 有限体积法的基本原理

$$\frac{\partial Q}{\partial t}+\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}=0$$



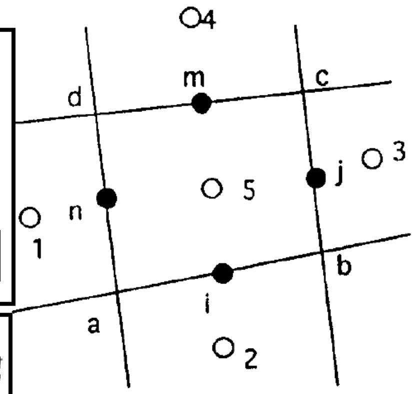
$$\int_{abcd}\left(\frac{\partial Q}{\partial t}\right)dxdy=-\int_{abcd}\left(\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}\right)dxdy$$

格林定理

$$\int_{abcd}\left(\frac{\partial Q}{\partial t}\right)dxdy=-\oint_{abcd}(Edy-Fdx)$$

7.1.1 有限体积法-格子中心式

$$\left(\frac{Q_5^{n+1}-Q_5^n}{\Delta t}\right)A_{abcd} = -\left[E_i\Delta y_{ab}+E_j\Delta y_{bc}+E_m\Delta y_{cd}+E_n\Delta y_{da}\right]$$
$$+\left[F_i\Delta x_{ab}+F_j\Delta x_{bc}+F_m\Delta x_{cd}+F_n\Delta x_{da}\right]$$



$$\Delta x_{ab}=x_b-x_a\ ,\quad \Delta x_{bc}=x_c-x_b\ ,\quad \Delta x_{cd}=x_d-x_c\ ,\quad \Delta x_{da}=x_a-x_d$$

$$\Delta y_{ab}=y_b-y_a\ ,\quad \Delta y_{bc}=y_c-y_b\ ,\quad \Delta y_{cd}=y_d-y_c\ ,\quad \Delta y_{da}=y_a-y_d$$

$$E_i=\frac{1}{2}(E_5^*+E_2^*),\qquad E_j=\frac{1}{2}(E_5^*+E_3^*)\ F_i=\frac{1}{2}(F_5^*+F_2^*),\qquad F_j=\frac{1}{2}(F_5^*+F_3^*)$$
$$E_m=\frac{1}{2}(E_5^*+E_4^*),\qquad E_n=\frac{1}{2}(E_5^*+E_1^*)\ F_m=\frac{1}{2}(F_5^*+F_4^*),\qquad F_n=\frac{1}{2}(F_5^*+F_1^*)$$

- 式中的*如果代表着n，则为显式求解；如代表n+1，则为隐式求解；
- 如果abcd为矩形，则对上式等同于有限差分法；
- E与F的表示亦可用其它算法表示求解；

7.1.2 有限体积法-格子顶点式

$$\left[\frac{(Q_a+Q_b+Q_c+Q_d)^{n+1}-(Q_a+Q_b+Q_c+Q_d)^n}{4\Delta t}\right]A_{abcd}=$$

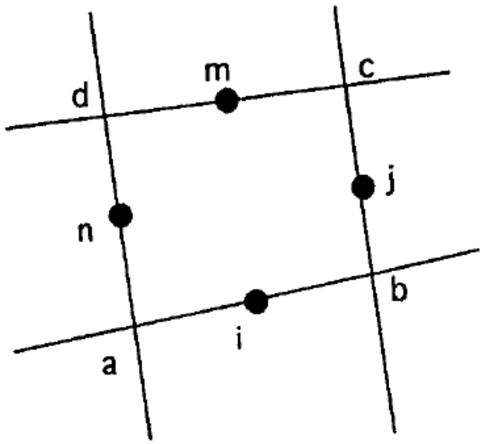
$$-(E_i\Delta y_{ab}+E_j\Delta y_{bc}+E_m\Delta y_{cd}+E_n\Delta y_{da})+(F_i\Delta x_{ab}+F_j\Delta x_{bc}+F_m\Delta x_{cd}+F_n\Delta x_{da})$$

$$E_i=\frac{1}{2}(E_a^*+E_b^*),\quad E_j=\frac{1}{2}(E_b^*+E_c^*)$$

$$E_m=\frac{1}{2}(E_c^*+E_d^*),\quad E_n=\frac{1}{2}(E_d^*+E_a^*)$$

$$F_i=\frac{1}{2}(F_a^*+F_b^*),\quad F_j=\frac{1}{2}(F_b^*+F_c^*)$$

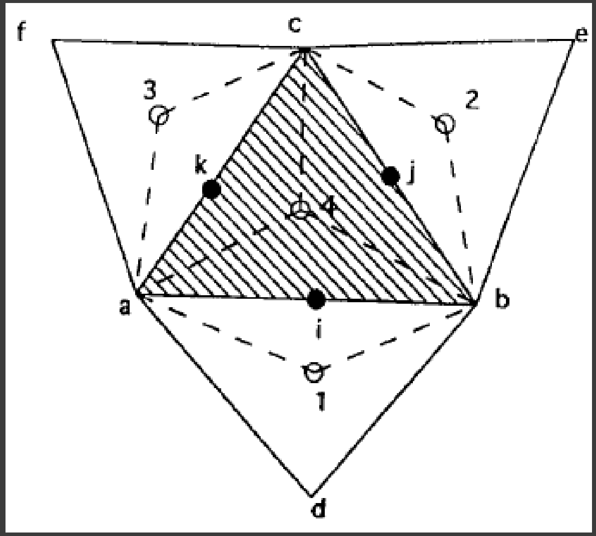
$$F_m=\frac{1}{2}(F_c^*+F_d^*),\quad F_n=\frac{1}{2}(F_d^*+F_a^*)$$

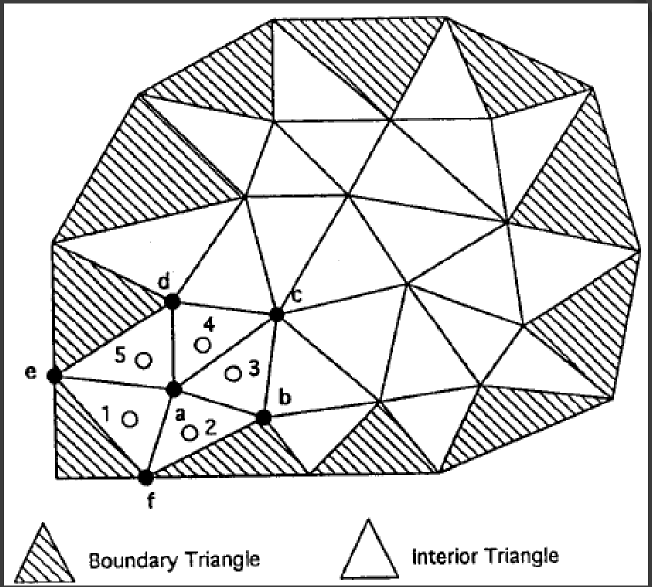


- 式中的*如果代表着n，则为显式求解；如代表n+1，则为隐式求解；
- E与F的表示亦可用其它算法表示求解；

7.1.3 有限体积法例-二维导热问题

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

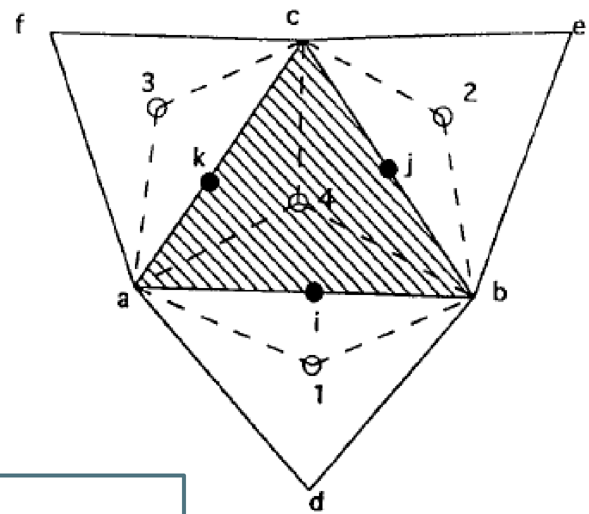




单元编号形式Counter-clockwise (CCW)

7.1.3 有限体积法例-二维导热问题

$$\boxed{\int_{abc}\left(\frac{\partial T}{\partial t}\right)dxdy} = \alpha\int_{abc}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)dxdy$$



$$\int_{abc}\left(\frac{\partial T}{\partial t}\right)dxdy = \left(\frac{T_4^{n+1}-T_4^n}{\Delta t}\right)A_{abc}$$

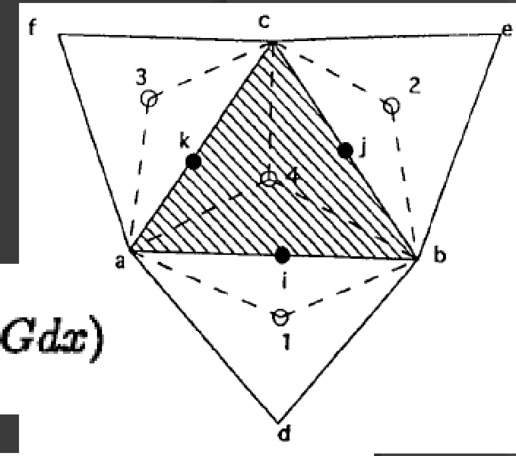
$$x_4 = \frac{1}{3}(x_a + x_b + x_c)$$

$$y_4 = \frac{1}{3}(y_a + y_b + y_c)$$

$$A_{abc} = \frac{1}{2}(x_b y_c + x_a y_b + x_c y_a - x_b y_a - x_c y_b - x_a y_c)$$

$$T_4^n = \left(\frac{T_a^n}{L_{4a}}+\frac{T_b^n}{L_{4b}}+\frac{T_c^n}{L_{4c}}\right)\Big/\left(\frac{1}{L_{4a}}+\frac{1}{L_{4b}}+\frac{1}{L_{4c}}\right)$$

7.1.3 有限体积法例-二维导热问题



令 $$F = \frac{\partial T}{\partial x} \quad \text{and} \quad G = \frac{\partial T}{\partial y}$$ 则方程右端积分为：

$$\int_{abc} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) dx dy = \int_{abc} \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right) dx dy = \oint_{abc} (F dy - G dx)$$

取abc单元

$$\begin{aligned} \text{RHS} &= \oint_{abc} (F dy - G dx) \\ &= (F_i \Delta y_{ab} + F_j \Delta y_{bc} + F_k \Delta y_{ca}) - (G_i \Delta x_{ab} + G_j \Delta x_{bc} + G_k \Delta x_{ca}) \end{aligned}$$

其中

$$\Delta x_{ab} = x_b - x_a, \quad \Delta x_{bc} = x_c - x_b, \quad \Delta x_{ca} = x_a - x_c$$

$$\Delta y_{ab} = y_b - y_a, \quad \Delta y_{bc} = y_c - y_b, \quad \Delta y_{ca} = y_a - y_c$$

显式算法

$$T_4^{n+1} = T_4^n + \alpha \frac{\Delta t}{A_{abc}} (\text{RHS})$$

RHS中的F和G如何确定？？？

7.1.3 有限体积法例-二维导热问题

$$F_i = \left(\frac{\partial T}{\partial x}\right)_i = \left[\int_{a1b4}\left(\frac{\partial T}{\partial x}\right)dxdy\right] \Big/ A_{a1b4}$$

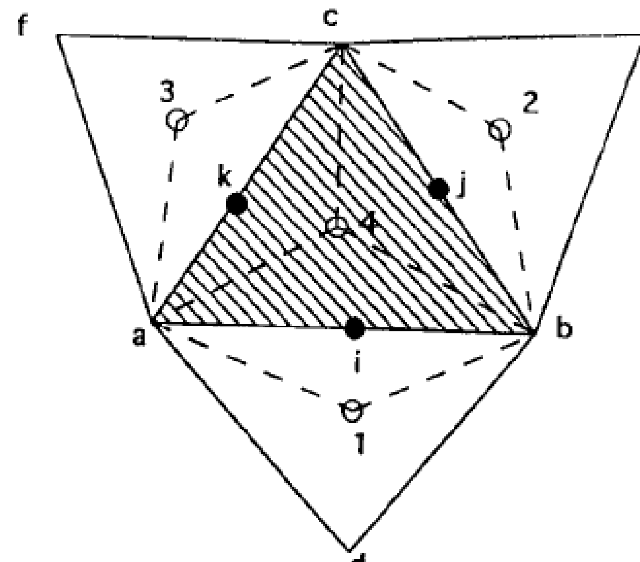
$$= \left[(T_a^n + T_1^n)\Delta y_{a1} + (T_1^n + T_b^n)\Delta y_{1b} + (T_b^n + T_4^n)\Delta y_{b4} + (T_4^n + T_a^n)\Delta y_{4a}\right] \Big/ (2A_{a1b4})$$

$$F_j = \left(\frac{\partial T}{\partial x}\right)_j = \left[\int_{b2c4}\left(\frac{\partial T}{\partial x}\right)dxdy\right] \Big/ A_{b2c4}$$

$$= \left[(T_b^n + T_2^n)\Delta y_{b2} + (T_2^n + T_c^n)\Delta y_{2c} + (T_c^n + T_4^n)\Delta y_{c4} + (T_4^n + T_b^n)\Delta y_{4b}\right] \Big/ (2A_{b2c4})$$

$$F_k = \left(\frac{\partial T}{\partial x}\right)_k = \left[\int_{c3a4}\left(\frac{\partial T}{\partial x}\right)dxdy\right] \Big/ A_{c3a4}$$

$$= \left[(T_c^n + T_3^n)\Delta y_{c3} + (T_3^n + T_a^n)\Delta y_{3a} + (T_a^n + T_4^n)\Delta y_{a4} + (T_4^n + T_c^n)\Delta y_{4c}\right] \Big/ (2A_{c3a4})$$



$$\boxed{\Delta y_{mn} = y_n - y_m}$$

$$A_{a1b4} = \frac{1}{3}(A_{abc} + A_{adb})$$

$$A_{b2c4} = \frac{1}{3}(A_{abc} + A_{bec})$$

$$A_{c3a4} = \frac{1}{3}(A_{abc} + A_{cfa})$$

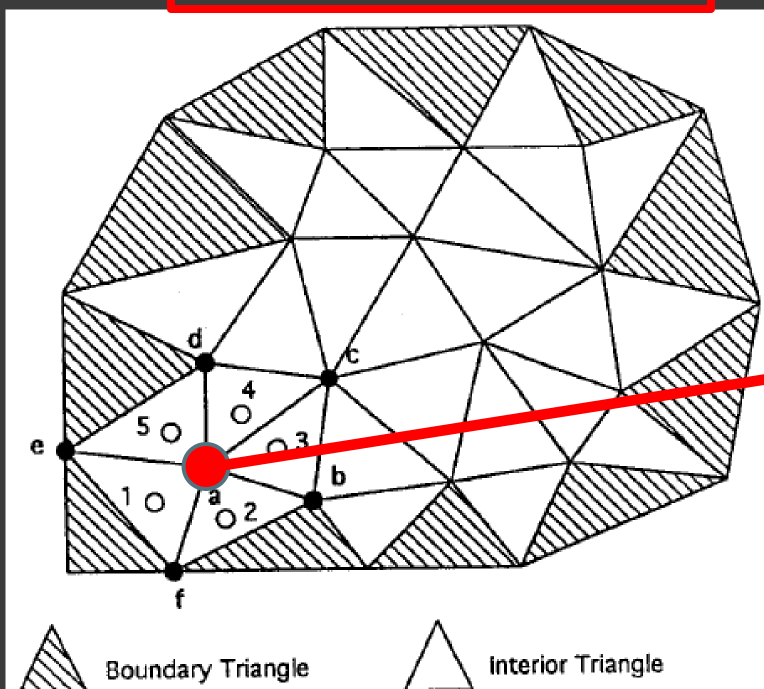
G可用类似的方法确定!

7.1.3 有限体积法例-二维导热问题

F、G确定 → RHS确定 → T_4^{n+1} 确定

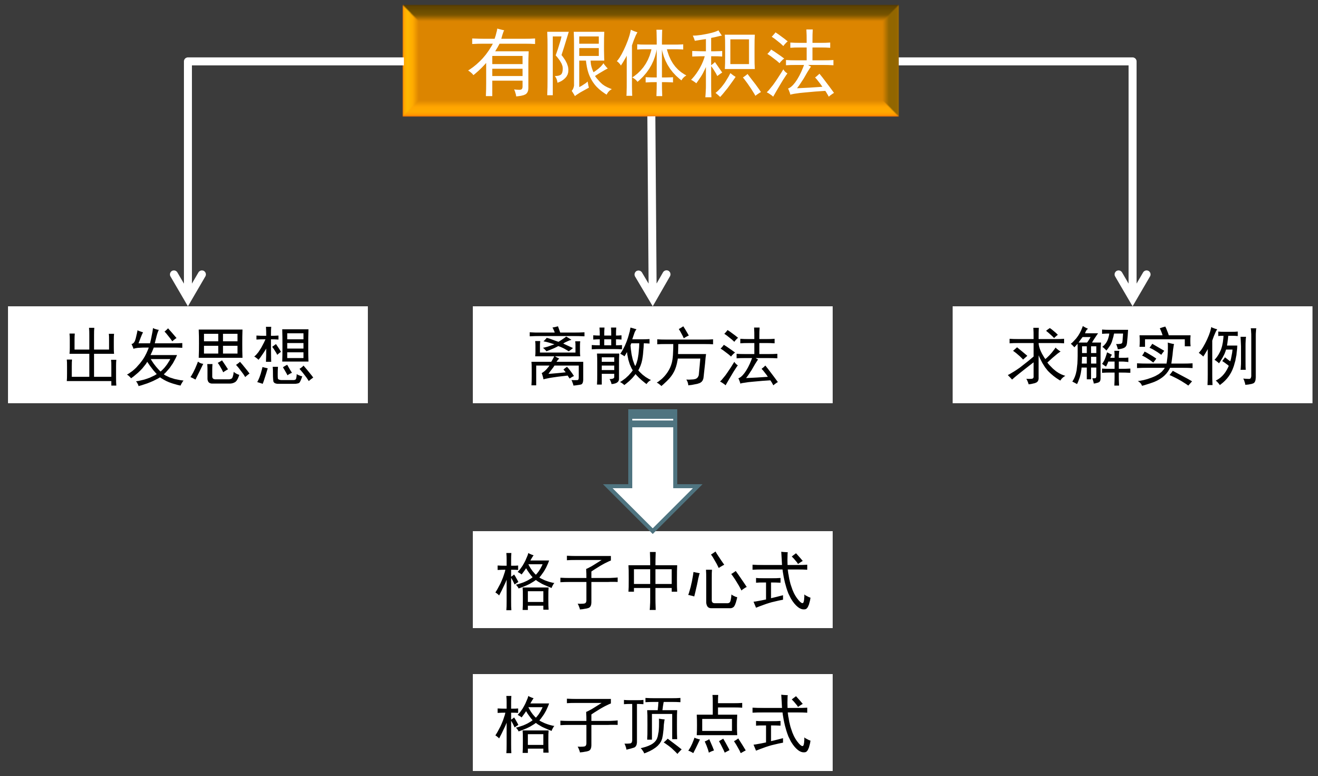
↓ 整场内部三角单元计算

T_i^{n+1} 确定 → 所有单元顶点T确定 → F、G确定



$$T_a^{n+1} = \frac{\sum_{m=1}^{5} \frac{T_m^{n+1}}{L_{ma}}}{\sum_{m=1}^{5} \frac{1}{L_{ma}}}$$

7.2 本章小结



课程总结

- **物理模型** ← 分析流动的性质，并对流动现象的物理过程进行相应的简化
- **数学模型** ← 建立控制方程组及定解条件
- **区域离散** ← 把流动区域划分为很多子区域(网格)，并从每个网格中选取出节点的过程
- **方程离散** ← 把微分方程组转换为代数方程组，如有限元法，有限体积法，有限差分法
- **代数方程组求解** ← 迭代法、时间推进法、压力—速度耦合解法等
- **后处理** ← 从计算结果中取出所关心的流动参数

区域离散/网格生成

- 结构化
  - 代数法
  - Thompson
  - 多块网格
- 非结构化
  - 三角形/四面体
    - Delaunay
    - 前沿推进
  - 笛卡尔/六面体

差分格式

差分格式构造

前向/后向/中心

显式/半隐式/隐式

一阶导/二阶导

格式稳定性

方程离散

时间项/空间项

模型方程求解过程

离散方程稳定性

需掌握：差分格式是如何推导出来的？
精度又是如何确定的？
显式方法和隐式方法的优缺点各是什么？

抛物型方程求解

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

显式解法

- FTCS
- Richardson
- Dufort-Frankel

隐式解法

- Laasonen
- Crank-Nicolson
- Beta Formulation

ADI

二维方程

需掌握：求解时差分方法的使用方式？稳定性条件？精度？ADI的作用，如何构造？如何工作？

椭圆型方程求解

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0$$

松弛迭代法

超松弛/亚松弛

需掌握：松弛因子对计算的影响

$$\overline{\Phi_{i,j}^{n+1}}=\frac{(\Delta x)^2(\Delta y)^2}{2(\Delta y)^2+2(\Delta x)^2}\left[\frac{\Phi_{i+1,j}^{n}+\Phi_{i-1,j}^{n+1}}{(\Delta x)^2}+\frac{\Phi_{i,j+1}^{n}+\Phi_{i,j-1}^{n+1}}{(\Delta y)^2}\right]$$

$$\Phi_{i,j}^{n+1}=\Phi_{i,j}^{n}+\omega\left(\overline{\Phi_{i,j}^{n+1}}-\Phi_{i,j}^{n}\right)$$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} \qquad a > 0$$

双曲型方程求解

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$$

显式解法

- FTFS
- FTCS
- FTBS
- Lax
- MP LeapFrog
- Lax-Wendroff

隐式解法

- FTCS
- Implicit F-U-D
- Crank-Nicolson

ADI

多步解法

- MC

非线性方程

- MC
- R-K
- Modified R-K

二维方程

需掌握：离散方程的形式？精度？稳定性条件？MC两步方法的构造过程？2/4阶RK方法的构造过程及形式？

感谢您的坚持！