Harmonic Numbers and Their p-Adic Structure

Doruk Üstündağ

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Abstract

In this talk, we explore the arithmetic nature of harmonic numbers through the lens of p-adic analysis. Beginning with a brief introduction to the p-adic valuation and metric, we investigate how these tools reveal deep structural properties of harmonic numbers. We discuss classic results such as the non-integrality of H_n, p-adic formulations of their differences, and the formation and growth of $J(p)$ sets introduced by Eswarathasan and Levine. Building on this foundation, we examine Wolstenholme's and Leudesdorf's theorems and their p-adic interpretations. The talk then extends to generalized, hyperharmonic, and Dedekind harmonic numbers, highlighting recent developments and open problems regarding their integrality. Finally, we conclude with a connection to a real-world application—the book stacking problem—where harmonic numbers naturally describe the maximal overhang of stacked blocks.

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Address: *Department of Mathematics, Izmir Institute of Technology, Urla, Izmir, Turkey*

References

[1] Emre Alkan, Haydar Göral, and D. C. Sertbaş. Hyperharmonic numbers can rarely be integers. *INTEGERS: Electronic Journal of Combinatorial Number Theory*, 18:A43, 2018.

[2] Çağatay Altuntaş. On the finiteness of some p-divisible sets. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 73:1011–1039, 2024.

[3] Çağatay Altuntaş and Haydar Göral. Dedekind harmonic numbers. *Proceedings of the Indian Academy of Sciences: Mathematical Sciences*, 131(2):Article 46, 2021.

[4] David W. Boyd. A p-adic study of the partial sums of the harmonic series. *Exp. Math.*, 3:287–302, 1994.

[5] Keith Conrad. *p*-adic growth of harmonic numbers. https://kconrad.math.uconn.edu/blurbs/, 2006. Available on Keith Conrad's expository papers webpage.

[6] John H. Conway and Richard K. Guy. *The Book of Numbers*. Springer-Verlag, New York, 1996.

[7] Arulappah Eswarathasan and Eugene Levine. p-integral harmonic sums. *Discrete Mathematics*, 91(3):249–257, 1991.

[8] Haydar Göral and Doğa Can Sertbaş. Almost all hyperharmonic numbers are not integers. *Journal of Number Theory*, 171:495–526, 2017.

[9] J Kürschák. On the harmonic series. *Matematikai és Fizikai Lapok*, 27:299–300, 1918.

[10] C. Leudesdorf. Some results in the elementary theory of numbers. *Proceedings of the London Mathematical Society*, s1-20(1):199–212, 1888.

[11] Romeo Mestrovic. Wolstenholme's theorem: Its generalizations and extensions in the last hundred and fifty years (1862–2012), 2011.

[12] Alexa Pomerantz. An introduction to the p-adic numbers. 2020.

[13] Doğa Can Sertbaş. Hyperharmonic integers exist. *Comptes Rendus. Mathématique*, 358:1179–1185, 01 2021.

[14] J. Wolstenholme. On certain properties of prime numbers. *The Quarterly Journal of Pure and Applied Mathematics*, 5:35–39, 1862.