

COMPLEX DIFFERENTIAL GEOMETRY

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1. COMPLEX VECTOR BUNDLES

To start, we recall the **abstract definition** of vector bundles on a manifold:

Definition 1.1. *Vector Bundles*

Let M be a smooth manifold. A K –(real/complex) **vector bundle** over M is a smooth

manifold E^1 with a projection map $E \xrightarrow{\pi} M$ such that there exists an open cover $\{U_\alpha\}$ of X and “*trivializations*” of E over these open sets, i.e., diffeomorphisms $\varphi_\alpha : E|_{U_\alpha} := \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times K^r$, such that the following diagram commutes:

$$\begin{array}{ccc} \varphi_\alpha : E|_{U_\alpha} = \pi^{-1}(U_\alpha) & \xrightarrow{\sim} & U_\alpha \times K^r \\ & \searrow \pi & \swarrow \text{id}_{U_\alpha} \times \{0\} \\ & U_\alpha & \end{array}$$

for all U_α (*local triviality condition*). Here, we say that $E \xrightarrow{\pi} M$ is a **K -vector bundle of rank r** , where r is the K -dimension of the vector space in the above isomorphism; equivalently, the K -dimension of the vector space $E_p := \pi^{-1}(p) \subset E$ (the **fiber** of E over p) for **any** point $p \in M$.

Since the maps φ_α preserve the base point p , we can consider, for any point $p \in U_\alpha \cap U_\beta$ in the intersection of two trivializing open sets, the composition:

$$\varphi_\alpha|_{U_\alpha \times K^r} \circ \varphi_\beta^{-1}|_{U_\beta \times K^r} : (U_\alpha \cap U_\beta) \times K^r \rightarrow (U_\alpha \cap U_\beta) \times K^r$$

and it is the **identity** on the first component; thus, for each point $p \in U_\alpha \cap U_\beta$ there exists a map $t_{\alpha\beta}(p) : K^r \rightarrow K^r$, **linear** by the vector bundle properties, and **invertible** by the diffeomorphisms, so that $t_{\alpha\beta}(p) \in \mathrm{GL}(r, K)$, so that:

$$(\varphi_\alpha \circ \varphi_\beta^{-1})(x, v) = (x, t_{\alpha\beta}(x)v)$$

where we are suppressing the subscripts $-|_{U_\alpha \times K^r}$; consequently, these maps $t_{\alpha\beta}$ assemble to a construction $t_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathrm{GL}(r, K)$ of the same smoothness as the original vector bundle.

If, in particular, X is a complex manifold and $E \xrightarrow{\pi} X$ is a complex vector bundle on it, we say that $E \rightarrow X$ is **holomorphic** if so are all the transition maps $\varphi_\alpha|_{U_\alpha \times \mathbb{C}^r} \circ \varphi_\beta^{-1}|_{U_\beta \times \mathbb{C}^r}$ on the overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta$; i.e., if and only if all the transition maps are **holomorphic**. Equivalently, if and only if these define **holomorphic maps** $U_\alpha \rightarrow \mathrm{GL}(r, \mathbb{C})$.

1.1. Sections of Vector Bundles. Moving on, we consider, as before, a (complex) vector bundle of (complex) rank r , i.e., $\pi : E^r \rightarrow X$. A **holomorphic section** s of E over an open set $U \subseteq X$ is a **holomorphic map** $s : U \rightarrow E$ (in particular, $s : U \rightarrow \pi^{-1}(U)$) such that $\boxed{\pi \circ s = \text{id}_U}$; this definition remains valid in **any category** upon replacing “*holomorphic*” by the corresponding term everywhere (e.g. smooth, continuous etc.)

More generally, the notion of a **section** means that the map sends $s(p) \in E_p =: \pi^{-1}(p)$ for each point $p \in U$. A section s is automatically **injective**, and thus provides a copy of X inside E ; conversely, this is true for any such map. Notably, this is always true for the **zero section** $0_E : X \rightarrow E$ of the vector bundle E , which assigns to $x \in X$ the point $0_E(x) := \varphi_U^{-1}(x, 0)$ for any open set $x \in \pi(U)$. Again, this can be generalized as follows: for any K -vector bundle $E^r \rightarrow X$ of rank r , there exists an **action** of $\mathrm{GL}(r, K)$ on E defined as follows: for an element $e \in E$ projecting down to $\pi(e) \in U$ (where U is a trivializing open

¹If we are working with complex vector bundles over a complex manifold, then, we expect E to be a **complex manifold**, as well.

set) an element $A \in \mathrm{GL}(r, K)$ acts on e by:

$$A * e := \varphi_U^{-1}(\pi(e), A\varphi_U(e))$$

which is quickly seen to define a valid $\mathrm{GL}(r, K)$ –action (which is also **smooth**) on E by acting on its fibers.

Here, we will often be considering the interplay between the “holomorphic” (e.g. holomorphic vector bundles, holomorphic sections) and the “smooth” setting (e.g. smooth vector bundles, smooth sections); importantly, we need to observe that **holomorphic sections are stronger than smooth sections**; in particular, there are several vector bundles that are **smoothly trivial**, but not **holomorphically trivial**.

Here, the standard notation for the space of sections of E over U is:

$$\Gamma(U, E) := \{\text{sections of } E \text{ over } U\}$$

which connects to the realization of the vector bundle $E \rightarrow X$ as a **coherent sheaf** over X in the corresponding category, whereby $\Gamma(U, E)$ denotes the corresponding space of sections of this sheaf. Of course, this is meant **in the corresponding category**, i.e., depending on the category we may be working with continuous, smooth, holomorphic etc. sections.

The important observation is that, clearly, due to the vector space structure on the fibers of E , **sections can be added** as well as **multiplied by holomorphic functions** on U . Here, following the sheaf notation, the space of functions on the open set U is denoted by $\mathcal{O}_X(U)$, where \mathcal{O}_X is the **structure sheaf** of X ; in our setting of a topological space (smooth manifold; complex manifold) this corresponds to the sheaf of continuous (respectively, smooth; holomorphic) functions on the open set $U \subset X$. Thus, following this notation, we see that:

$$\Gamma(U, E) \text{ is a } \mathcal{O}_X(U) \text{ – module}$$

In particular, $\Gamma(X, E)$ is called the space of **global sections**; another notation is $H^0(X, E)$, which we will explore later.

Now, we consider the following general setting: let X^n be a general complex manifold of complex dimension n and let $E^r \rightarrow X$ be a complex vector bundle of rank r over X : this means that there is a trivializing local holomorphic coordinate atlas $\{(U_\alpha, \varphi_\alpha)\}$ (i.e., $\varphi_\alpha : U_\alpha \rightarrow \mathbb{C}^n$; essentially, this is a local holomorphic coordinate system $\varphi_\alpha = (z_\alpha^1, \dots, z_\alpha^n)$ on X) of $X = \bigcup_\alpha U_\alpha$ for the vector bundle E . Then, E is **uniquely specified** by the transition functions (matrices) $t_{\mu\nu} : U_\mu \cap U_\nu \rightarrow \mathrm{GL}(r, \mathbb{C})$, which satisfy the cocycle compatibility conditions:

- (i) $t_{\mu\mu} = \mathrm{id}_{\mathrm{GL}(r, \mathbb{C})}$ on U_μ ,
- (ii) $t_{\mu\nu} t_{\nu\mu} = \mathrm{id}_{\mathrm{GL}(r, \mathbb{C})}$ on $U_\mu \cap U_\nu$,
- (iii) $t_{\mu\nu} t_{\nu\rho} t_{\rho\mu} = \mathrm{id}_{\mathrm{GL}(r, \mathbb{C})}$ on $U_\mu \cap U_\nu \cap U_\rho$.

Of course, since all the $t_{\mu\nu}$ are $r \times r$ **complex matrix-valued functions** (in $\mathrm{GL}(r, \mathbb{C})$, by the above) we see that we can write their entries as $t_{\mu\mu}{}^\alpha{}_\beta$ (for the (α, β) –entry) to re-write the above conditions as:

- (i) $t_{\mu\mu}{}^\alpha{}_\beta = \delta^\alpha{}_\beta$ on U_μ ,
- (ii) $t_{\mu\nu}{}^\alpha{}_\beta t_{\nu\mu}{}^\beta{}_\gamma = \delta^\alpha{}_\gamma$,

(iii) $t_{\mu\nu}{}^\alpha{}_\beta t_{\nu\rho}{}^\beta{}_\gamma = t_{\mu\rho}{}^\alpha{}_\gamma$ on $U_\mu \cap U_\nu \cap U_\rho$.

In this setting, the vector bundle $E \rightarrow X$ specified by the Čech (cocycle) data $\{(U_\mu, t_{\mu\nu})\}$ is called **holomorphic**, if each of the transition matrices (functions) $t_{\mu\nu}$ is **holomorphic**.

Following these constructions, **section** $s \in \Gamma(X, E)$ is specified by local data $\{(U_\mu, s_\mu)\}$, i.e., by a uniform choice of **local sections** of E over each element U_μ of the open cover, $s_\mu \in \Gamma(U_\mu, E)$. Then, the fact that $X = \bigcup_\mu U_\mu$ is a trivializing open cover for E amounts to the fact that $\pi^{-1}(U_\mu) \approx U_\mu \times \mathbb{C}^r$, so each such section $s_\mu : U_\mu \rightarrow \pi^{-1}(U_\mu) \subset E$ can equivalently be expressed as a map that maps the point $U_\mu \ni p \mapsto p \in U_\mu$ to itself in $U_\mu \times \mathbb{C}^r$, and it has the **column vector** representation for the functions on the coordinate patch:

$$s_\mu : U_\mu \rightarrow \mathbb{C}^r \text{ smooth map, } s_\mu := \begin{bmatrix} s_\mu^1(z_\mu, \bar{z}_\mu) \\ s_\mu^2(z_\mu, \bar{z}_\mu) \\ \vdots \\ s_\mu^r(z_\mu, \bar{z}_\mu) \end{bmatrix}$$

Here $z_\mu = (z_\mu^1, \dots, z_\mu^r)$ is the local holomorphic coordinate system on the chart U_μ , and the expression of the functions $s_\mu^\rho(z_\mu, \bar{z}_\mu)$ also allows for the possibility that the complex-valued functions s_μ^ρ **may not be holomorphic** (i.e., may have a \bar{z}_μ -dependence, meaning that they may not be holomorphic)

For these local sections $s_\mu \in \Gamma(U_\mu, E)$ to assemble to a **global section** $s \in \Gamma(X, E)$, they need to satisfy the **compatibility condition**:

$$s_\mu = t_{\mu\nu} s_\nu, \quad s_\mu^\alpha = (t_{\mu\nu})^\alpha{}_\beta s_\nu^\beta \quad \text{on } U_\mu \cap U_\nu$$

where i, j are the \mathbb{C}^r -indices (the first equality is one of r -dimensional complex vectors, the second one is a condition on their entries) and we have used the Einstein summation notation. This condition essentially asks that they be “*invariant under change of coordinates*”.

Finally, we say that the bundle E is smooth (respectively, holomorphic) if the transition functions $t_{\mu\nu}$ above are smooth (respectively, holomorphic); and we say that the section $s \in \Gamma(X, E)$ is smooth (respectively, holomorphic) if each of the local sections $s_\mu \in \Gamma(U_\mu, E)$ is smooth (respectively, holomorphic).

Now, using these definitions, we can prove the following key result:

Proposition 1.2. *Identity Theorem for Holomorphic Sections*

Let X^n be a connected complex manifold and let $E^r \rightarrow X$ be a holomorphic vector bundle on X . Let $s, \tilde{s} \in \Gamma(X, E)$ be global sections of E and suppose that there is some non-empty open set $U \subset X$ such that their restrictions $s|_U \equiv \tilde{s}|_U$ agree. Then, we actually have $s \equiv \tilde{s} \in \Gamma(X, E)$, i.e., they agree everywhere on X .

Proof. To see this result, we can consider the section $s' := s - \tilde{s} \in \Gamma(X, E)$, which is also a global holomorphic section of E ; then, the condition $s|_U \equiv \tilde{s}|_U$ translates to $s'|_U \equiv 0$; and we want to prove that $s' \equiv 0 \in \Gamma(X, E)$ everywhere on X ; thus, we might WLOG start by assuming that $s|_U \equiv 0$ to begin. Now, we consider the set $V \subset X$ of points $x \in X$ such that $s|_W \equiv 0$ for some non-empty open set $W \subset X$ containing $W \ni x$. On one hand, denoting

the set $W_x := W$ defined above, we immediately see that $V = \bigcup_{x \in V} W_x$ is the union of open sets, so, it is **open**; on the other hand, we *also* see that on any of the coordinate charts U_α , we have $s|_{U_\alpha} = s_\alpha$. Thus, for any chart U_α such that $U_\alpha \cap U \neq \emptyset$, we get $s_\alpha|_{U_\alpha \cap U} \equiv 0$, whereby the composite map $s_\alpha \circ \varphi_\alpha^{-1}$ is a map from the open subset $\varphi_\alpha^{-1}(U_\alpha) \subset \mathbb{C}^n$ to \mathbb{C}^r ; and, since $s_\alpha|_{U_\alpha \cap U} \equiv 0$, we see that this is identically zero on the non-zero open subset $\varphi_\alpha^{-1}(U_\alpha \cap U) \subset \varphi_\alpha^{-1}(U_\alpha)$.

Consequently, we see that the **identity theorem for holomorphic functions** in several complex variables (applied, for example, to each of the coordinate functions $(s_\alpha \circ \varphi_\alpha^{-1})^i : \mathbb{C}^r \rightarrow \mathbb{C}$) shows that since the function $s_\alpha \circ \varphi_\alpha^{-1}$ **vanishes** on an open subset, then, it must **vanish everywhere** on the domain $\varphi_\alpha^{-1}(U_\alpha)$; meaning that the local section $s_\alpha \equiv 0$ on all of U_α . Moreover, the above result proves that for any chart U_β such that $U_\alpha \cap U_\beta \neq \emptyset$, we **also** get $s_\beta \equiv 0$: indeed, we know that s_β is given on the overlap by $s_\beta = t_{\alpha\beta}^{-1} s_\alpha$, whereby $s_\alpha \equiv 0$ on $U_\alpha \cap U_\beta$ implies that $s_\beta|_{U_\alpha \cap U_\beta} \equiv 0$, as well; and the above result shows that, in fact, $s_\beta \equiv 0$ on all of U_β . Consequently, we may apply **Zorn's lemma** on maximal intersecting chains of open subsets, or we can connect any two points by a sequence of open subsets on which the section s restricts to zero. By either of these approaches, we deduce that $s \equiv 0 \in \Gamma(X, E)$ everywhere on X , as desired; this completes the proof. \square

Likewise, we may now consider the **dual bundle** E^\vee , given by:

$$X = \bigcup_\alpha U_\alpha, \quad \varphi \in \Gamma(X, E^\vee), \quad \varphi \in \{(U_\alpha, \varphi_\alpha)\}$$

Then, this will likewise transform as:

$$(\varphi_\alpha)_i = (\varphi_\beta)_\rho (\psi_{\beta\alpha})^\rho{}_i$$

More generally, the idea is to notice that the **sections of the dual bundle** transform as:

$$ADDDDD$$

Additionally, we see that for sections $s \in \Gamma(X, E)$ and $\varphi \in \Gamma(X, E^\vee)$, we have a **natural pairing** into an inner product $\langle s, \varphi \rangle := \varphi_i s^i$, which is a **scalar function**.

Definition 1.3. The conjugate bundle

Similarly to the above constructions, we may consider a complex manifold X and a vector bundle $E \rightarrow X$. Moreover, we may consider a trivializing open cover $X = \bigcup_\alpha U_\alpha$ of X . Over this, we may define the **conjugate bundle** of E , denoted by \overline{E} , as follows:

$$s \in \Gamma(X, \overline{E}), \quad s \in \{(U_\alpha, s_\alpha)\}, \quad \dot{s}_\mu^i = \overline{(\psi_{\mu\nu})^i}_\rho s^{\bar{\rho}}{}_\nu$$

with transition functions as given above.

1.2. Connections on Vector Bundles. In the previous setting of a (complex) **vector bundle** $E \rightarrow X$, we may consider a connection on E , given by $\nabla = d + A$. For our manifold X coming with a trivializing open cover $X = \bigcup_\alpha U_\alpha$ for the vector bundle E , we may consider the connection on each local chart as $A = \{(U_\mu, A_\mu)\}$. Then, the **connection A is a matrix-valued 1-form**, such that:

$$A_\mu = \psi_{\mu\nu} A_\nu \psi_{\mu\nu}^{-1} - d\psi_{\mu\nu} \psi_{\mu\nu}^{-1}, \quad \text{on } U_\mu \cap U_\nu$$

This is an important connection for the transformation of connections on local charts.

On the other hand, the **covariant derivative** ∇ transforms as:

$$\nabla_i s^\rho = \partial_i s^\rho + A_i^\rho{}_\ell s^\ell$$

Thus, on any local chart, we will get the transformation:

$$\nabla_i s^\rho{}_\mu = t_{\mu\nu}{}^\rho{}_\ell \nabla_i s^\ell{}_\nu$$

Indeed, for this one we recall that the covariant derivative $\nabla_i s$ is **again a section** of E ; here, we used the conventions:

$$\nabla_i = \nabla_{\frac{\partial}{\partial z^i}}, \quad \nabla_{\bar{i}} = \nabla_{\frac{\partial}{\partial \bar{z}^i}}$$

The important point here is that when we have a **section of the vector bundle and a section of its dual**, we may always **pair them to get a scalar function**.

Next, we may likewise consider **the Nabla (Covariant Derivative) on the Dual Bundle**.

Moving on, we will work with **the covariant derivative on the Conjugate Bundle \overline{E}** corresponding to a vector bundle $E \rightarrow X$. This is given by first taking sections $s \in \Gamma(X, E)$ and $\bar{s} \in \Gamma(X, \overline{E})$, whereby this will transform as:

$$\nabla_i s^{\bar{k}} := \overline{\nabla_{\bar{i}} s^k}, \quad \nabla_i s^{\bar{k}} = \partial_i s^{\bar{k}} + \overline{A_i^k}{}_\ell s^\ell$$

Finally, we know how to define a **connection on the tensor bundle**. This can be done by just pasting together the actions, so that:

$$\nabla_i h_{\bar{k}\bar{j}} := \partial_i h_{\bar{k}\bar{j}} - \overline{A_{\bar{i}}{}^\rho{}_k} h_{\bar{p}\bar{j}} - A_{i\rho}{}_{\bar{j}} h_{\bar{k}\rho}$$

Here, we are adopting the “*Physicist notation*,” whereby the Hermitian metric h is **sesquilinear on the first term**. Putting together all the terms of the metric, we may define:

$$h := h_{\bar{k}\bar{j}} dz^j \otimes d\bar{z}^k$$

Thus, in general, we deduce that for any two sections $V, W \in \Gamma(X, E)$, we get their **inner product in terms of the metric h** as:

$$\langle V, W \rangle_h := h(V, W) = h_{\bar{k}\bar{j}} V^j \overline{W^k}$$

ADDD

1.3. Holomorphic Vector Bundles from Smooth Vector Bundles. Now, we will work to develop a view of holomorphic vector bundles as **smooth bundles with a preferred class of sections**: the holomorphic sections.

Let $E \rightarrow X$ be a smooth complex (i.e., the fibers are \mathbb{C}^r) vector bundle over a complex manifold X . We say that a \mathbb{C} -linear operator $\bar{\partial}_E : \Gamma(E) \rightarrow \Gamma(T^\vee X^{0,1} \otimes E)$ is a “*partial connection*” on E , if it satisfies a “*Leibniz rule*” of the form:

$$\boxed{\bar{\partial}_E(fs) = \bar{\partial}f \otimes s + f\bar{\partial}_E s}$$

for any smooth complex-valued function $f \in C^\infty(X, \mathbb{C})$ on X and a section $s \in \Gamma(E)$; here, $\bar{\partial}$ denotes the usual differential operator on X . Then, we may show that a partial connection

extends to a linear map:

$$\bar{\partial}_E : \mathcal{A}^{p,q}(X; E) \rightarrow \mathcal{A}^{p,q+1}(X; E)$$

on E -valued (“ E -”twisted) (p, q) -forms, i.e.:

$$\mathcal{A}^{p,q}(X; E) := \Gamma^\infty(T^\vee X^{p,q} \otimes E) \equiv \Gamma^\infty(\Lambda^{p,q} X \otimes E)$$

This linear map $\bar{\partial}_E$ should satisfy:

$$\boxed{\bar{\partial}_E(\alpha \wedge \sigma) = \bar{\partial}\alpha \wedge \sigma + (-1)^{k+\ell}\alpha \wedge \bar{\partial}_E\sigma}, \quad \alpha \in \mathcal{A}^{k,\ell}(X), \sigma \in \mathcal{A}^{p,q}(X; E)$$

In particular, this map satisfies:

$$(\bar{\partial}_E)^2 s = \Phi s, \quad \Phi \in \mathcal{A}^{0,2}(X; \text{End } E), \quad \forall s \in \Gamma(E)$$

i.e., Φ is an endomorphism-valued $(0, 2)$ -form on X .

Now, we may define a **holomorphic structure** on E as a partial connection (per the above) satisfying $\boxed{(\bar{\partial}_E)^2 = 0}$. Then, a section $s \in \Gamma(E)$ is called a **holomorphic section**, if it satisfies $\bar{\partial}_E s = 0$. Here, we recall the following result:

Theorem 1.4. *Koszul-Malgrange Integrability theorem*

*ADDD. Then, if $\bar{\partial}_E^2 = 0$, the complex vector bundle E admits local trivializations on which $\bar{\partial}_E$ is expressed as $\bar{\partial}$; these are called the **holomorphic local trivializations** relative to $\bar{\partial}_E$.*

Thus, we may define a **holomorphic vector bundle** $\mathcal{E} := (E, \bar{\partial}_E)$ as a smooth complex vector bundle endowed with a holomorphic structure $\bar{\partial}_E$. By the Kozsul-Malgrange theorem above, this definition is equivalent to having **holomorphic transition functions**.

Example 1.5. *The bundle of holomorphic p -forms*

Consider a complex manifold X . For any p , we may consider the vector bundle $E := \Lambda^{p,0} T^\vee X$; the holomorphic sections of this bundle are precisely the holomorphic p -forms on X , i.e., the $(p, 0)$ -forms α satisfying $\bar{\partial}\alpha = 0$.

In this setting, we may identify a suitable operator $\bar{\partial}_E$ and holomorphic local trivializations, so that the holomorphic sections of E are as above.

Remark 1.6. While this approach worked for $(p, 0)$, we see that **it does not work** for any (p, q) with $q > 0$; that is, we cannot use $\bar{\partial}$ to define holomorphic sections of $\Lambda^{p,q} T^\vee X$, when $q \neq 0$.

1.4. Line bundles over projective space.

2. CHERN CONNECTIONS AND CHERN CURVATURE

Definition 2.1 (Hermitian Metric on a Vector Space). Let V be a complex vector space (over \mathbb{C} ; or, equivalently, a **real vector space** equipped with a complex structure $J : V \rightarrow V$) with complex dimension $\dim_{\mathbb{C}} = n$. A *Hermitian metric* h on V is a \mathbb{R} -bilinear map $h : V \times V \rightarrow \mathbb{C}$, such that

$$\boxed{h(\alpha v, \beta u) = \alpha \bar{\beta} \overline{h(u, v)}},$$

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Equivalently, this means that h may be viewed as a **C–bilinear pairing** on $V \times \overline{V}$ (V with its **conjugate vector space** \overline{V}).² This property makes h into a **symmetric sesquilinear pairing**. In particular, we have:

$$\begin{aligned} h(\alpha_1 v_1 + \alpha_2 v_2, u) &= \alpha_1 h(v_1, u) + \alpha_2 h(v_2, u) \\ h(v, \beta_1 u_1 + \beta_2 u_2) &= \overline{\beta_1} h(v, u_1) + \overline{\beta_2} h(v, u_2) \\ h(v, u) &= \overline{h(u, v)}, \quad h(v, v) = \overline{h(v, v)} \in \mathbb{R} \end{aligned}$$

for all $u, v, u_i, v_i \in V$ and $\alpha, \beta, \alpha_i, \beta_i \in \mathbb{C}$. In particular, we say that h is **positive-definite**, if it satisfies $h(u, u) > 0$ for all $u \neq 0 \in V$.³ A complex vector space (V, J) equipped with a (positive-definite) Hermitian form h is called a (positive-definite) *Hermitian space*.

Now, let (V, J, h) be a Hermitian space, as above. Then, the **real part of the Hermitian form**:

$$g := \operatorname{Re} h, \quad g(-, -) := \operatorname{Re} h(-, -)$$

defines (descends to) a **symmetric \mathbb{R} –bilinear form** $g : V \otimes V \rightarrow \mathbb{R}$; in fact, if h is positive-definite (as above), then g is a **Riemannian metric**, i.e., an **inner product** (positive-definite symmetric \mathbb{R} –bilinear form). Moreover, the **imaginary part** of the Hermitian form:

$$\omega := -\operatorname{Im} h, \quad \omega(-, -) := -\operatorname{Im} h(-, -)$$

defines (descends to) a **skew-symmetric \mathbb{R} –bilinear form** $\omega : V \wedge V \rightarrow \mathbb{R}$; in fact, if h is positive-definite (as above), then ω is a **symplectic form** on V .

Independently of positive-definiteness, the above construction allows us to write:

$$h = g - i\omega$$

as an equality of \mathbb{R} –bilinear forms on \mathbb{C} , where g is fully symmetric, ω is fully antisymmetric, and both are **real-valued (real forms)**; it is immediate to verify these via $h(v, u) = \overline{h(u, v)}$.

Now, we should also study what it means for h to be **sesquilinear**: we recall that for a real vector space equipped with a complex structure (V, J) , we have a natural action of $\mathbb{C} \curvearrowright V$ with i “acting as J ”, i.e., $z = (\alpha + \beta i) * v := \alpha v + \beta Jv \in V$. Thus, we now see that for $\zeta = \alpha_1 + i\alpha_2, \eta = \beta_1 + i\beta_2 \in \mathbb{C}$, we have:

$$\begin{aligned} [(\alpha_1\beta_1 + \alpha_2\beta_2) + i(\alpha_2\beta_1 - \alpha_1\beta_2)]h(v, w) &= \zeta\bar{\eta}h(v, w) = h((\alpha_1 + i\alpha_2) * v, (\beta_1 + i\beta_2) * w) \\ &= h(\alpha_1 v + \alpha_2 Jv, \beta_1 w + \beta_2 Jw) \\ &= \alpha_1\beta_1 h(v, w) + \alpha_2\beta_1 h(Jv, w) + \alpha_1\beta_2 h(v, Jw) + \alpha_2\beta_2 h(Jv, Jw) \end{aligned}$$

whereby setting $\alpha_1 = \beta_1 = 0$ and comparing terms here gives $h(Jv, Jw) = h(v, w)$; and substituting this above leaves us with:

$$i(\alpha_2\beta_1 - \alpha_1\beta_2)h(v, w) = \alpha_2\beta_1 h(Jv, w) + \alpha_1\beta_2 h(v, Jw)$$

so that $h(Jv, w) = ih(v, w)$, $h(v, Jw) = -ih(v, w)$; this is immediate by \mathbb{C} –sesquilinearity.

²On this vector space, we have the action of complex numbers by $\alpha * v := \bar{\alpha}v$, viewed as an element of the underlying abelian group V .

³It is possible to give such a definition, since we showed above that $h(v, v) = \overline{h(v, v)} \in \mathbb{R}$ is always **real**.

Using these results, we can easily see now that the components g, ω are related by:⁴

$$\boxed{\omega(v, w) = g(Jv, w) = -g(v, Jw), \quad g(v, w) = \omega(v, Jw) = -\omega(Jv, w)}$$

which means, in particular, that the (semi-)Riemannian metrics g on V arising from (and, thus, fully determining) Hermitian forms h on V via $h = g - i\omega$ are precisely the ones that are **compatible** with the complex structure J , i.e., $\boxed{g(J-, J-) = g(-, -)}$; the same applies to the compatible symplectic forms ω with $\boxed{\omega(J-, J-) = \omega(-, -)}$. Here, the content of the above result is precisely that if $h : V \otimes V \rightarrow \mathbb{R}$ is **any** positive-definite Hermitian form on V , then, its real part $g := \operatorname{Re} h$ is a compatible Riemannian metric on V . For this reason, we say that a Riemannian metric $g : V \otimes V \rightarrow \mathbb{R}$ is **Hermitian**, if it satisfies the compatibility relation $g(J-, J-) = g(-, -)$, i.e., if it arises from a Hermitian form.

In particular, the positive-definiteness of the resulting Riemannian metric is immediate from that of h , since we showed that $h(v, v) \in \mathbb{R}$ for all $v \in V$, so that $g(v, v) = \operatorname{Re} h(v, v) = h(v, v) > 0$, if and only if h is **positive-definite**.

Remark 2.2. Using these properties, it is easy to associate Hermitian forms to **Kähler vector space structures**. Indeed, given a real vector space V with a linear complex structure J , then, the construction of a positive-definite Hermitian metric $g : V \otimes V \rightarrow \mathbb{R}$ is equivalent to defining a **linear Kähler structure** $\omega \in \wedge^2 V^\vee$.

Moving on, we assume now that V is finite-dimensional, so that we may fix a basis $\{\mathbf{e}_j\}$ for it, and set:

$$\boxed{h_{i\bar{j}} := h(\mathbf{e}_i, \mathbf{e}_{\bar{j}})}$$

In this setting, the previous properties imply that $h_{i\bar{j}} = h_{\bar{j}i} = \overline{h_{ij}} = \overline{h_{ji}}$. Then, our Hermitian form h is evidently given with respect to this basis by:

$$\boxed{h = h_{i\bar{j}} \mathbf{e}^i \otimes \overline{\mathbf{e}^j} \in V^\vee \otimes \overline{V}^\vee}$$

Here, $\{\mathbf{e}^i\} \in V^\vee$ is the **dual basis** of $\{\mathbf{e}_i\}$, that is:

$$\mathbf{e}^i(\mathbf{e}_k) = \delta_k^i, \quad \mathbf{e}^i(\alpha^k \mathbf{e}_k) = \delta_k^i \alpha^k$$

Analogously, $\overline{\mathbf{e}^j} \in \overline{V}^\vee$ is the **conjugate dual** of \mathbf{e}_i , that is:

$$\mathbf{e}^i(\mathbf{e}_k) = \delta_k^i, \quad \overline{\mathbf{e}^i}(\alpha^k \mathbf{e}_k) = \delta_k^i \overline{\alpha^k}$$

Evidently, these definitions make $(h_{i\bar{j}})$ into a **positive-definite Hermitian matrix**.

Example 2.3. Standard Kähler Vector space

For us, the standard example of a Kähler vector space arises in the following setting: we let $V := \mathbb{R}^{2n}$ be the $2n$ -dimensional real vector space equipped with the complex structure J , which is given by the canonical identification $\mathbb{R}^{2n} \simeq \mathbb{C}^n$; hence, in terms of the canonical

⁴Indeed, for this, we need only observe that $\omega(v, w) = -\operatorname{Im} h(v, w) = \operatorname{Re}(ih(v, w)) = \operatorname{Re} h(Jv, w) =: g(Jv, w)$, as desired; and everything else follows by successful, repeated applications of J .

linear basis $\{\mathbf{e}_i\}$ of \mathbb{R}^{2n} , this is:

$$J = (J^k_\ell) := \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}$$

where Id is the $n \times n$ identity matrix; and we let:

$$\omega = (\omega_{ij}) := \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}, \quad g = (g_{ij}) := \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix}$$

so that these are, respectively, the symplectic form and the Riemannian metric; meaning that (V, J, ω, g) is a **Kähler vector space**. In fact, the corresponding smooth manifold \mathbb{R}^{2n} becomes endowed with the structure of a **smooth manifold** in the standard way, as it is equipped with the bilinear forms J, ω, g , extended as constant rank-2 tensors over the manifold.

Consequently, if we write $x^1, \dots, x^n : \mathbb{R}^n \rightarrow \mathbb{R}$ for the standard coordinate functions on \mathbb{R}^n , with each $z^k := x^k + iy^k$, $\bar{z}^k := x^k - iy^k : bR^{2n} \rightarrow \mathbb{C}$ being the corresponding **complex coordinates**, then, we get $\omega \in \Omega^2(\mathbb{R}^n)$, a differential 2-form on \mathbb{R}^{2n} given by $\omega := \sum_{k=1}^n dx^k \wedge dy^k = -\frac{i}{2} \sum_{k=1}^n dz^k \wedge d\bar{z}^k$; and we see that $\omega_{jk} J^k_\ell = g_{j\ell}$ and, similarly, $\omega(J_-, J_-)_{ij} = \omega_{k\ell} J^k_i J^\ell_j = \omega_{ij}$. Here, the Riemannian metric tensor is given by $g = \sum_{k=1}^n (dx^k)^2 + (dy^k)^2$, where we have defined $(dx^k)^2 := dx^k \otimes dx^k$, etc. Thus, the **Hermitian form** is given by $h := g - i\omega = \sum_{k=1}^n dz^k \otimes d\bar{z}^k$.

Finally, we now have the following useful result, which tells us that, under certain conditions, we can induce a **positive-definite Hermitian form** on a vector space:

Proposition 2.4. Positive-Definite Hermitian Form from Short Exact Sequence
Consider the following short exact sequence:

$$0 \longrightarrow V \xrightarrow{j} W \xrightarrow{\pi} Z \longrightarrow 0$$

which is a short exact sequence of finite-dimensional \mathbb{C} -vector spaces. Let h_W and h_Z be Hermitian forms on W and Z , respectively, such that $h_W|_V$ (i.e., j^*h_W on V) and h_Z are **positive-definite**. Then, if $\lambda \gg 0$ is any sufficiently large positive real number, then the Hermitian form $h_W + \lambda\pi^*h_Z$ is **positive-definite** on W .

Proof. Using the **finite-dimensionality** of the spaces V, W, Z , the properties of short exact sequences suggest that we may write $k := \dim_{\mathbb{C}} V$ and $\dim_{\mathbb{C}} W = k + \ell$, so that $\dim_{\mathbb{C}} Z = \ell$. Moreover, since j is injective, we see that if $\{v_1, \dots, v_k\}$ is a \mathbb{C} -basis of the complex vector space V ,⁵ and since $j : V \hookrightarrow W$ is **injective**, we deduce that the elements $\{jv_1, \dots, jv_k\}$ are **\mathbb{C} -linearly independent** on W ; consequently, since W has complex dimension $k + \ell$, we see that we may extend these to a basis of W by adding ℓ elements, which are thus linearly independent from $j(V)$. In fact, we know that a **Hermitian form on a complex vector space always admits an orthonormal basis**; indeed, this happens because such transformations satisfy $h = h^\dagger := \overline{h^T}$, (i.e., they are their own conjugate transpose,⁶) so, the

⁵Essentially, this means that $\{v_1, Jv_1, \dots, v_k, Jv_k\}$ is an \mathbb{R} -**basis** of V .

⁶Indeed, we see this because the associated matrix H satisfies $h(x, y) = y^\dagger H x = (y^\dagger H^\dagger) x = (Hy)^\dagger x = \overline{h(y, x)}$; and, conversely, $y^\dagger H x : h(x, y) = \overline{h(y, x)} = (Hy)^\dagger x = y^\dagger H^\dagger x$, meaning that $H = H^\dagger$.

Spectral Theorem for complex operators guarantees the existence of an orthonormal basis $\{v_i\}$ with respect to the positive-definite Hermitian form j^*h_W on V , meaning that the $\{v_i\}$ can be selected in this way.

In fact, applying this result similarly to the positive-definite Hermitian form h_Z on the vector space Z of complex dimension ℓ , we can produce an orthonormal \mathbb{C} -basis $\{z_1, \dots, z_\ell\}$ of ℓ elements for Z . Thus, since the map $\pi : W \rightarrow Z$ is **surjective** by the properties of the SES, we may select $w_i \in \pi^{-1}(z_i)$ to be *any* elements in the corresponding preimages of the z_i (i.e., such that $\pi(w_i) = z_i$; this exists by the surjectivity of the map π) we deduce that the $\{jv_1, \dots, jv_k, w_1, \dots, w_\ell\}$ assemble to a \mathbb{C} -basis of W . Indeed, they are **linearly independent**, since applying π to any linear relation between them will give us:

$$\sum_{r=1}^k a_r j(v_r) + \sum_{s=1}^\ell b_s w_s = 0 \implies \sum_{r=1}^k a_r \underbrace{\pi(j(v_r))}_{=0} + \sum_{s=1}^\ell b_s \pi(w_s) = \sum_{s=1}^\ell b_s z_s = 0, \implies b_s = 0$$

because the $\{z_s\}$ form a **basis** of Z , thus, are **linearly independent**; consequently, going back to the original expression gives:

$$\sum_{r=1}^k a_r j(v_r) + \underbrace{\sum_{s=1}^\ell b_s w_s}_{=0} = 0 \implies j \left(\sum_{r=1}^k a_r v_r \right) = 0 \implies \sum_{r=1}^k a_r v_r = 0 \implies a_r = 0$$

by the injectivity of j and by the fact that the $\{v_r\}$ form a basis of V ; consequently, we see that any such linear relation between them must be **trivial**, meaning that these vectors are **linearly independent**; and since there are $k + \ell = \dim_{\mathbb{C}} W$ of them, we deduce that these form a \mathbb{C} -basis of W , as desired.

Now, using these constructions for the basis, we see that any element $w \in W$ is expressible in the form $w = a^\rho j(v_\rho) + b^\mu w_\mu$ (in Einstein summation notation) for $a^\rho, b^\mu \in \mathbb{C}$; moreover, we can now consider the complex constants $h_{\rho\mu}^v := h_W(j(v_\rho), w_\mu)$ and $h_{\mu\nu} := h_W(w_\mu, w_\nu)$. Using these, we compute:

$$\begin{aligned} h_W(w, w) &= h_W(a^\rho j(v_\rho) + b^\mu w_\mu, a^\sigma j(v_\sigma) + b^\nu w_\nu) \\ &= a^\rho \overline{a^\sigma} h_W(j(v_\rho), j(v_\sigma)) + a^\rho \overline{b^\nu} h_W(j(v_\rho), w_\nu) + b^\mu \overline{a^\sigma} h_W(w_\mu, j(v_\sigma)) + b^\mu \overline{b^\nu} h_W(w_\mu, w_\nu) \\ &= a^\rho \overline{a^\sigma} \delta_{\rho\sigma} + a^\rho \overline{b^\mu} h_W(j(v_\rho), w_\mu) + b^\mu \overline{a^\rho} \overline{h_W(j(v_\rho), w_\mu)} + b^\mu \overline{b^\nu} h_W(w_\mu, w_\nu) \\ &= \sum_{i=1}^k |a^i|^2 + 2 \operatorname{Re} \left(a^\rho \overline{b^\mu} h_W(j(v_\rho), w_\mu) \right) + b^\mu \overline{b^\nu} h_W(w_\mu, w_\nu) \\ &= \sum_{i=1}^k |a^i|^2 + 2 \operatorname{Re} \left(a^\rho \overline{b^\mu} h_{\rho\mu}^v \right) + b^\mu \overline{b^\nu} h_{\mu\nu} \\ &= \sum_{i=1}^k |a^i|^2 + 2 \operatorname{Re} \left(a^\rho \overline{b^\mu} h_{\rho\mu}^v \right) + \sum_{j=1}^\ell |b^j|^2 h_{jj} + 2 \sum_{i < j} \operatorname{Re}(b^i \overline{b^j} h_{ij}) \end{aligned}$$

because $h_W(j(v_\rho), j(v_\sigma)) = j^*h_W(v_\rho, v_\sigma) = \delta_{\rho\sigma}$, by the assumption that $\{v_i\}$ is an orthonormal basis of V with respect to the positive-definite Hermitian form j^*h_W .

Now, we clearly see that for all a, b, h , we can bound the real part below by:

$$\operatorname{Re}(a\bar{b}h) \geq -|a| \cdot |b| \cdot |h|, \quad \operatorname{Re}(b\bar{b}'h) \geq -|b| \cdot |b'| \cdot |h|$$

and, since there are only finitely many coefficients $h_{\rho\mu}^v, h_{\mu\nu}$, we can select some finite $C > 0$ such that $2|h_{\rho\mu}^v|, 2|h_{\mu\nu}| < C$ for all indices ρ, μ, ν ; consequently, we get:

$$\begin{aligned} 2\operatorname{Re}\left(a^\rho\bar{b}^\mu h_{\rho\mu}^v\right) + \sum_{j=1}^{\ell} |b^j|^2 h_{jj} + 2 \sum_{i < j} \operatorname{Re}(b^i\bar{b}^j h_{ij}) &\geq -C \left(\sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell}} |a^i| \cdot |b^j| - \sum_{j=1}^{\ell} |b^j|^2 - 2 \sum_{i < j} |b^i| \cdot |b^j| \right) \\ &= -C \cdot \left[\left(\sum_{i=1}^k |a^i| \right) \cdot \left(\sum_{j=1}^{\ell} |b^j| \right) + \left(\sum_{j=1}^{\ell} |b^j| \right)^2 \right] \end{aligned}$$

and, on the other hand, we can also use the **Cauchy-Schwarz inequality** to write:

$$\sum_{i=1}^k |a^i|^2 = \frac{1}{k} \left(\sum_{i=1}^k 1 \right) \cdot \left(\sum_{i=1}^k |a^i|^2 \right) \geq \frac{1}{k} \left(\sum_{i=1}^k |a^i| \right)^2$$

which finally gives us, for $w = a^\rho j(v_\rho) + b^\mu w_\mu$, the inequality:

$$h_W(w, w) \geq \frac{1}{k} \left(\sum_{i=1}^k |a^i| \right)^2 - C \cdot \left[\left(\sum_{i=1}^k |a^i| \right) \cdot \left(\sum_{j=1}^{\ell} |b^j| \right) + \left(\sum_{j=1}^{\ell} |b^j| \right)^2 \right] = \frac{1}{k} A^2 - C(AB + B^2)$$

where we set $A := \sum_{i=1}^k |a^i|$ and $B := \sum_{j=1}^{\ell} |b^j|$. On the other hand, we can *also* compute now that for any element of the worm $u = w + j(v)$, we have:

$$\begin{aligned} \pi^* h_Z(u, u) &= h_Z(\pi(u), \pi(u)) = h_Z(\pi(w) + \pi(j(v)), \pi(w) + \pi(j(v))) = h_Z(\pi(w), \pi(w)) \\ &= \pi^* h_Z(w, w) \end{aligned}$$

due to $\pi \circ j = 0$; consequently, for any $u = a^\rho j(v_\rho) + b^\mu w_\mu = j(v) + w$ for some $v \in V, w \in W$, we can compute that $\pi^* h_Z(u, u) = \pi^* h_Z(w, w) \geq B^2/\ell$. Thus, it is a direct application of the Cauchy-Schwarz inequality that selecting the constant $\lambda_0 = \ell(kC^2 + C + 1)$ works, where C is the constant bounding the entries of $h_{\rho\mu}^v, h_{\mu\nu}$ that we constructed; indeed, it gives $(h_W + \lambda\pi^* h_Z)(w, w) > 0$, since equality can only hold if $A = B = 0$, i.e., all $a^i = b^j = 0$, meaning that $w = 0$; this proves that $h_W + \lambda\pi^* h_Z$ is a **positive-definite Hermitian form** on W , as desired; this completes the proof. \square

Definition 2.5. Hermitian Vector Bundles

Let X be a complex manifold and $E \rightarrow X$ be a (smooth) complex vector bundle, as defined above. We say that $E \rightarrow X$ is **Hermitian**, if it is possible to define a positive-definite Hermitian (metric) tensor h on E , i.e., a smoothly-varying Hermitian form h_x on the fiber $E_x \cong \mathbb{C}^r$ of each point on X .

We can also define this concept in a more structured way:

Definition 2.6. A Hermitian metric h for a smooth complex vector bundle $E \rightarrow X$ is a smooth section h of the vector bundle $E^\vee \otimes \overline{E^\vee} \rightarrow X$, such that the corresponding function

$h : E \otimes \overline{E} \rightarrow \mathbb{C}$ (to the constant line bundle) can be defined (with abuse of notation) by:

$$h(v, \overline{w}) := \langle h, v \otimes \overline{w} \rangle, \quad v, w \in H_x$$

and satisfies $h(v, \overline{w}) = \overline{h(w, \overline{v})}$, is **positive-definite** (i.e., $h(v, \overline{v}) \geq 0$ with equality iff $v = 0$) and **R-linear**: \mathbb{C} -linear in the first entry and \mathbb{C} -antilinear in the second one.

An important result in this setting is that **any complex vector bundle $E \rightarrow X$ admits a Hermitian metric**; this is a standard argument by partitions of unity, completely analogous to the one for Riemannian metrics. Indeed, we can consider a locally finite trivializing open coordinate cover $\{U_\alpha\}$ of X for the vector bundle E , so that we have local Hermitian metrics given by:

$$h_\alpha(z^i \mathbf{e}_{\alpha,i}, w^j \mathbf{e}_{\alpha,j}) =: z^i \overline{w^j} (h_\alpha)_{i\bar{j}}$$

where $\{\mathbf{e}_{\alpha,i}\}$ is a frame for the vector bundle E over the open set U_α . Then, we can take a partition of unity $\{\rho_\alpha\}$ subordinate to this cover, so that the combination
$$h = \sum_\alpha \rho_\alpha h_\alpha$$
 will be a Hermitian metric.

Definition 2.7. The Angle Pairing

As in the above setting, we let $E \rightarrow X$ be a Hermitian vector bundle. Then, we get a natural sesquilinear map, given by:

$$\mathcal{A}^p(X, E) \times \mathcal{A}^q(X, E) \rightarrow \mathcal{A}^{p+q}(X, E), \quad (s, t) \mapsto \{s, t\}$$

In local coordinates, we may write these sections as $s := \sigma^i \otimes \mathbf{e}_i$ and $t := \tau^j \otimes \mathbf{e}_j$. Then, the pairing is locally given by:

$$\{s, t\} := (\sigma^i \wedge \overline{\tau^j}) \langle \mathbf{e}_i, \mathbf{e}_j \rangle = h_{i\bar{j}} \sigma^i \wedge \overline{\tau^j}, \quad \boxed{\{s, t\} = h_{i\bar{j}} \sigma^i \wedge \overline{\tau^j}}$$

where \wedge denotes the usual wedge product of differential forms and $h_{i\bar{j}} := \langle \mathbf{e}_i, \mathbf{e}_j \rangle$ is the local coordinate expression of the Hermitian metric tensor.

Moreover, we can now examine the concept of **metric connections** on a Hermitian vector bundles:

Definition 2.8. Metric Connection on a Hermitian Vector Bundle

Let $E \rightarrow X$; we recall that the definition of a **connection** ∇ on E is defined as ADDDDDD.

In particular, we say that the connection ∇ is **compatible with the Hermitian structure** of E , or a **metric connection**, if for any $s \in \mathcal{A}^p(X, E)$ and any $t \in \mathcal{A}^q(X, E)$, we actually have:

$$\boxed{d\{s, t\} = \{\nabla s, t\} + (-1)^p \{s, \nabla t\}}$$

for the angle pairing. Importantly, for any such metric connection, we have for the curvature ADDDD:

$$\boxed{\Theta^* = -\Theta}$$

where Θ^* is used to denote ADDDDDD.

Proof. To see this result, we may use the property $d^2 = 0$ to write:

$$\begin{aligned} 0 = d^2\{s, t\} &= d(d\{s, t\}) = d(\{\nabla s, t\} + (-1)^p\{s, \nabla t\}) \\ &= \{\nabla^2 s, t\} + (-1)^p \{\nabla s, \nabla t\} + (-1)^{p+1} \{\nabla s, \nabla t\} + (-1)^{p(p+1)} \{s, \nabla^2 t\} \\ &= \{\nabla^2 s, t\} + \{s, \nabla^2 t\} \end{aligned}$$

Thus, we see that ADDDD

□

Now, in particular, we may again consider local holomorphic coordinates $\{z^i\}$ on X ; in terms of these we have, for $\partial_i := \frac{\partial}{\partial z^i}$ and $\nabla_i := \nabla_{\frac{\partial}{\partial z^i}}$:

$$\boxed{\partial_i\{s, t\} = \{\nabla_i s, t\} + (-1)^p\{s, \nabla_i t\}}$$

this result will become useful later.

We can also think of this construction via the following perspective: since our Hermitian metric is defined as $h(s, \bar{t}) := h \otimes s \otimes \bar{t}$ (when we think of the metric h as a section of the vector bundle $E^\vee \otimes \overline{E^\vee}$) we may compute that:

$$d(h(s, \bar{t})) = h(\nabla s, \bar{t}) + h(s, \bar{\nabla t}) + \nabla h(s, \bar{t})$$

i.e., the same result as above; here, the ∇ in the third term of the RHS is precisely the connection for the vector bundle $E^\vee \otimes \overline{E^\vee}$ coming from the connection for E , as we saw earlier. In fact, we now say:

Definition 2.9. *Connection compatible with Hermitian metric*

Let $E \rightarrow X$ be a complex vector bundle. A connection ∇ on E is said to be **compatible with a Hermitian metric h** on E , or simply a **metric connection** for the Hermitian vector bundle (E, h) , if for any sections s, t and tangent vector v , we have:

$$d(h(s, t))v = h((\nabla s)(v), \bar{t}) + h(s, \bar{(\nabla t)v})$$

or, equivalently, $d(h(s, t))X = h(\nabla_X s, \bar{t}) + h(s, \bar{\nabla_X t})$, for any vector field X . Importantly, this is equivalent to saying that the associated connection on the vector bundle $E^\vee \otimes \overline{E^\vee}$ **annihilates h** , i.e., $\boxed{\nabla h = 0}$ is **parallel** (*a covariantly constant tensor*) wrt. this connection.

Proposition 2.10. *Normal Frames on Holomorphic Vector Bundles*

Let X be a complex (Hermitian) manifold with a complex holomorphic vector bundle $E \rightarrow X$ over it. Now, let $x \in X$ be any point on X a local holomorphic coordinate chart $(U; z^1, \dots, z^n)$ around it. Then, there exists a holomorphic frame $\{\mathbf{e}_i\}_{i=1}^r$ on $U \ni x$, such that:

$$\boxed{h_{i\bar{j}} := \langle \mathbf{e}_i(x), \mathbf{e}_j(x) \rangle = \delta_{ij} - R_{k\bar{\ell}i\bar{j}} z^k \bar{z}^\ell + O(|z|^3)}$$

Here, the $(R_{j\bar{k}\alpha\bar{\beta}})$ are the coefficients of the Chern curvature tensor $\Theta_E|_x$ for E around x , in terms of the above local frame.

Proof. To see this result, we start by ADDDD

□

Again, we recall that a connection ∇ on a vector bundle is equivalent to a sheaf map $\mathcal{A}_X^0(E) \rightarrow \mathcal{A}_X^1(E)$, from the sheaf of smooth sections into the sheaf of E -valued 1-forms. Using the decomposition $T^\vee X = T^{1,0}X \oplus T^{0,1}X$ of the cotangent bundle (and the corresponding forms) we can decompose $\mathcal{A}_X^1(E) = \mathcal{A}_X^{1,0}(E) \oplus \mathcal{A}_X^{0,1}(E)$, which enables us to decompose the connection ∇ into its $(1,0)$ - and $(0,1)$ -parts as $\boxed{\nabla = \nabla^{1,0} + \nabla^{0,1}}$. Both of these satisfy the Leibniz formulas: for any sections $\alpha \in \mathcal{A}^\bullet(X)$ and $\sigma \in \mathcal{A}^\bullet(X, E)$, we have:

$$\begin{aligned}\nabla^{1,0}(\alpha \wedge \sigma) &= \partial\alpha \wedge \sigma + (-1)^{|\alpha|}\alpha \wedge \nabla^{1,0}\sigma, \\ \nabla^{0,1}(\alpha \wedge \sigma) &= \bar{\partial}\alpha \wedge \sigma + (-1)^{|\alpha|}\alpha \wedge \nabla^{0,1}\sigma,\end{aligned}$$

It turns out that for a general complex vector bundle, this splitting is not particularly helpful; however, for a **holomorphic vector bundle**, this becomes crucial in enabling us to define the $\bar{\partial}$ -operator for sections of holomorphic vector bundles.

Definition 2.11. The $\bar{\partial}$ -operator on a holomorphic vector bundle

Let $E \rightarrow X$ be a holomorphic vector bundle. Then, we may define the $\bar{\partial}_E$ -operator on E by $\bar{\partial}_E : \mathcal{A}_X^0(E) \rightarrow \mathcal{A}_X^{0,1}(E)$ locally, as follows: let $\{e_\alpha\}$ be a holomorphic local frame of sections of E , and let $s = s^\alpha e_\alpha$ be a local section of E . Then, we define:

$$\boxed{\bar{\partial}_E(s) = \bar{\partial}_E(s^\alpha e_\alpha) = (\bar{\partial}s^\alpha) \otimes e_\alpha}$$

on the trivializing open set $U \subset X$.

To see that this construction is well-defined, we simply notice that if $\{\tilde{e}_\mu\}$ is another holomorphic local frame on an open set V with nowhere-zero **holomorphic** transition functions (since E is, and since so are the two frames) $\tilde{e}_\mu = t^\alpha{}_\mu e_\alpha$ on U , then we may express $s = \tilde{s}^\mu \tilde{e}_\mu$ in this frame, so that $s^\alpha = t^\alpha{}_\mu \tilde{s}^\mu$. Thus, we compute, by the holomorphicity of $t^\alpha{}_\mu$:

$$\bar{\partial}s^\alpha = \bar{\partial}(t^\alpha{}_\mu \tilde{s}^\mu) = t^\alpha{}_\mu \bar{\partial}\tilde{s}^\mu \implies (\bar{\partial}s^\alpha) \otimes e_\alpha = (\bar{\partial}\tilde{s}^\mu) \otimes t^\alpha{}_\mu e_\alpha = (\bar{\partial}\tilde{s}^\mu) \otimes \tilde{e}_\mu$$

which proves that the expression $\bar{\partial}_E(s)$ is well-defined and frame-independent, as desired.

Definition 2.12. $(1,0)$ -connection on a holomorphic vector bundle

A connection ∇ on a holomorphic vector bundle $E \rightarrow X$ is said to be a $(1,0)$ -connection if in some (and hence, in any) holomorphic frame, all the connection 1-forms $\omega^\beta{}_\alpha$ are $(1,0)$ -forms; equivalently, the $(0,1)$ -part of ∇ is precisely $\boxed{\nabla^{0,1} = \bar{\partial}_E}$.

We have seen that if $E \rightarrow X$ is a holomorphic vector bundle over X , then there is a unique operator $\bar{\partial}_E : \mathcal{A}^{p,q}(X, E) \rightarrow \mathcal{A}^{p,q+1}(X, E)$ such that in each coordinate patch of a holomorphic (complex analytic) chart (i.e., with holomorphic transition functions) this is given by:

$$\bar{\partial}_E = \varepsilon(d\bar{z}^k)\partial_{\bar{k}} := \varepsilon(d\bar{z}^k)\frac{\partial}{\partial\bar{z}^k}$$

where the notation means that ADDDDD In particular, the sheaf $\Gamma_O(E)$ of holomorphic sections of the holomorphic vector bundle V is the **kernel** of the operator $\bar{\partial}_E$ acting on the sheaf $\Gamma^\infty(E) := \mathcal{A}^{0,0}(E)$ of smooth sections of E . Consequently, we also say that a covariant derivative (covariant) ∇ on E is holomorphic, if its $(0,1)$ -part satisfies $\nabla^{0,1} = \bar{\partial}_E$. In particular, we see that:

Proposition 2.13. Metric-compatible holomorphic connection

Let $E \rightarrow X$ be a holomorphic vector bundle on the manifold X equipped with a Hermitian metric h . Then, there exists a **unique** holomorphic, metric-compatible connection (covariant derivative) on E ; that is, ∇ preserves the metric, i.e., $\nabla h = 0$.

Proof. Of course, if ∇ is both holomorphic and metric-compatible (preserves the metric), then taking the component of $dh(s, t)$ lying in $\mathcal{A}^{0,1}(M)$, for any (local) sections s, t of E , gives us:

$$\bar{\partial}h(s, t) = h(\nabla^{0,1}s, t) + h(s, \nabla^{1,0}t)$$

and moreover, the holomorphicity assumption requires that $\nabla^{0,1} = \bar{\partial}$, so that this equation uniquely determines $\nabla^{1,0}$ (if it exists). On the other hand, it is easy to check that this $\nabla = \nabla^{1,0} + \nabla^{0,1}$ is a covariant derivative, as desired. \square

2.1. Chern connection on a Line Bundle. Here is the most important instance of this construction: let $L \rightarrow X$ be a complex line bundle and let $s_1, \dots, s_k \in \Gamma(X, L)$ be global (smooth) sections of L with **no common zeroes**. Choose a frame ξ for L , so that there are smooth functions f_i with $s_i = f_i\xi$ for each i . Then, we define a metric h on E by:

$$h(v, \bar{w}) := \frac{a\bar{b}}{|f_1|^2 + \dots + |f_k|^2}, \quad \text{where } v = a\xi, w = b\xi$$

which is clearly well-defined, independently of the choice of frame. Thus, one makes the standard abuse of notation to write:

$$h(v, \bar{w}) := \frac{v\bar{w}}{|s_1|^2 + \dots + |s_k|^2}$$

For example, if $X = \mathbb{CP}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow \mathbb{CP}^n$ is the **hyperplane bundle**, then we can take the sections z_0, z_1, \dots, z_n on \mathbb{CP}^n , where the $[z_0 : z_1 : \dots : z_n]$ are the homogeneous coordinates; this gives rise to the metric:

$$h_{FS}(v, \bar{w}) := \frac{v\bar{w}}{|z_0|^2 + |z_1|^2 + \dots + |z_n|^2}$$

called the **Fubini-Study metric**; we will study this later.

Remark 2.14 (Kähler potentials.). Let ξ be a frame for the line bundle $L \rightarrow X$ over an open set U , so that the function $h(\xi, \bar{\xi})$ is **positive, real, nowhere-vanishing** on U , by the properties of the Hermitian metric and since the frame ξ is nowhere-vanishing. Then, we may define the function φ (in this frame) as $\boxed{\varphi^{(\xi)} := -\log h(\xi, \bar{\xi})}$, so that for any section $s = f\xi$, we have $h(s, \bar{s}) = |f|^2 e^{-\varphi}$. Again, a standard abuse of notation is to omit the reference frame and simply write $\boxed{h(s, \bar{s}) = |s|^2 e^{-\varphi}}$, i.e., to write that $h = e^{-\varphi}$ is our Hermitian metric.

Theorem 2.15. Hermitian Metric on Line Bundle via Transitions

Let $\pi : L \rightarrow X$ be a complex holomorphic line bundle over the complex manifold X with trivializing open cover $\{(U_\alpha, \varphi_\alpha)\}$, where the trivializing functions are given by:

$$\varphi_\alpha : L|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{C}$$

where $L|_{U_\alpha} := \pi^{-1}(U_\alpha)$. We also have transition functions $t_{\mu\nu} : U_\mu \cap U_\nu \rightarrow \mathbb{C}^\times$, given by:

$$t_{\mu\nu} : U_\mu \cap U_\nu \rightarrow \mathbb{C}^\times, \quad t_{\mu\nu} := \varphi_\mu \circ \varphi_\nu^{-1}$$

Now, consider a collection of positive functions $\lambda_\alpha : U_\alpha \rightarrow \mathbb{R}^+$; these assemble into a Hermitian metric on L , if and only if they satisfy:

$$\boxed{\lambda_\beta = \lambda_\alpha |t_{\alpha\beta}|^2}, \quad \text{on } U_\alpha \cap U_\beta$$

which is given as follows: for any $x \in U_\alpha$ and elements $v_1, v_2 \in L_x := \pi^{-1}(x)$ such that $v_i := \varphi_\alpha^{-1}(x, z_i)$, we define:

$$\boxed{(v_1, v_2)_L := \lambda_\alpha(x) z_1 \bar{z}_2}$$

Conversely, for any Hermitian metric $(-, -)_L$ on L , we define functions $\lambda_\alpha : U_\alpha \rightarrow \mathbb{R}^+$ by:

$$\boxed{\lambda_\alpha(x) := \left\| \varphi_\alpha^{-1}(x, 1) \right\|_L^2}$$

then, these λ_α satisfy the compatibility conditions above.

Of course, the above construction $\lambda_\beta = \lambda_\alpha |t_{\alpha\beta}|^2$ can also be translated as follows: in the above definitions, we showed that **local sections** $s_\alpha \in \Gamma(U_\alpha, L)$ of a line bundle L assemble to a global section $s \in \Gamma(X, L)$ if and only if they satisfy the transition relation $s_\alpha = t_{\alpha\beta} s_\beta$.⁷ Consequently, we see that for any such section, we must have $|s_\alpha|^2 = |t_{\alpha\beta}|^2 |s_\beta|^2$, i.e., $\boxed{|h_\alpha| |s_\alpha|^2 = h_\beta |s_\beta|^2}$; this is an expression of the above coordinate-invariance, as this simply says that the norm of a section of E (with respect to this metric) is **well-defined**, independently of the parametrization on the chart.

Proof. In the one direction, we see now that for any point $x \in U_\ell$ and any two vectors $v_1, v_2 \in L_x := \pi^{-1}(x)$, we may then express these elements as $v_i := \varphi_\alpha^{-1}(x, z_i)$, for $i = 1, 2$ and some $z_i \in \mathbb{C}$; then, we may define a Hermitian metric $(-, -)_L$ on L , given by:

$$(v_1, v_2)_L := \lambda_\alpha(x) z_1 \bar{z}_2$$

This metric is certainly **Hermitian**, since it clearly satisfies the necessary properties; moreover, we should now show that it is **well-defined**. That is, for any point on the contesting parametrization $x \in U_\alpha \cap U_\beta$, we get, by the defining properties, some $w_1, w_2 \in \mathbb{C}$ such that:

$$v_1 := \varphi_\alpha^{-1}(x, z_1) = \varphi_\beta^{-1}(x, w_1), \quad v_2 := \varphi_\alpha^{-1}(x, z_2) = \varphi_\beta^{-1}(x, w_2)$$

This indeed happens because x lies on the overlap, so that $L_x \subset L|_{U_\alpha \cap U_\beta} := L|_{U_\alpha} \cap L|_{U_\beta}$ is contained in both trivializations; consequently, since the transition functions are defined as $t_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1}$, meaning that $\varphi_\alpha = \varphi_\beta \circ t_{\alpha\beta}$, we get:

$$\begin{aligned} (x, z_i) &= \varphi_\alpha(v_i) = (\varphi_\beta \circ t_{\alpha\beta})(x, z_i) = \varphi_\beta(x, t_{\alpha\beta}(x)w_i), \\ (x, w_i) &= \varphi_\beta(v_i), \quad \implies (x, t_{\alpha\beta}(x)w_i) = (x, z_i) \end{aligned}$$

Thus, we now conclude that $z_i = t_{\alpha\beta}(x)w_i$. As a result, we now obtain, in terms of the other trivializing open set U_β :

$$\begin{aligned} (v_1, v_2)_L &:= \lambda_\beta(x) w_1 \bar{w}_2 = \lambda_\beta(x) \frac{v_1}{t_{\alpha\beta}(x)} \cdot \frac{\bar{v}_2}{\bar{t}_{\alpha\beta}(x)} = \frac{\lambda_\beta(x)}{|t_{\alpha\beta}(x)|^2} z_1 \bar{z}_2 \\ &= \lambda_\alpha(x) z_1 \bar{z}_2 \end{aligned}$$

⁷In general, this relation holds, but might need us to consider $r \times r$ **function-valued matrices**, where r is the rank of the vector bundle; here, we can clearly take $r = 1$ for a line bundle.

precisely as defined above; this establishes our desired conclusion that **our Hermitian metric is well-defined**, completing the proof.

In the opposite direction, we can see that the existence of the Hermitian metric $(-, -)_L$ helps us define:

$$\text{On } L_x := \pi^{-1}(x), \quad \|t_{\mu\nu}(x)\|_L^2 := (t_{\mu\nu}(x), t_{\mu\nu}(x))_L$$

Using this, we have, for example $\|f(x, v)\| = \|v\|_L \cdot \|f(x, 1)\|$. Consequently, we obtain:

$$\lambda_\beta = \lambda_\alpha |t_{\alpha\beta}|^2, \quad \text{on } U_\alpha \cap U_\beta$$

Indeed, we may confirm this result at any point $x \in U_\alpha \cap U_\beta$, whereby:

$$\begin{aligned} \lambda_\beta(x) &:= \|\varphi_\beta^{-1}(x, 1)\|_L^2 = \left\| (\varphi_\alpha^{-1} \circ \varphi_\alpha \circ \varphi_\beta^{-1})(x, 1) \right\|_L^2 = \left\| (\varphi_\alpha^{-1} \circ t_{\alpha\beta})(x, 1) \right\|_L^2 \\ &= \left\| (\varphi_\alpha^{-1} \circ t_{\alpha\beta})(x, 1) \right\|_L^2 = \left\| \varphi_\alpha^{-1}(t_{\alpha\beta}(x, 1)) \right\|_L^2 \\ &= \left\| \varphi_\alpha^{-1}(x, t_{\alpha\beta}(x)) \right\|_L^2 = \|t_{\alpha\beta}(x)\|_L^2 \cdot \left\| \varphi_\alpha^{-1}(x, 1) \right\|_L^2 \\ &= \|t_{\alpha\beta}(x)\|_L^2 \cdot \lambda_\alpha(x) \end{aligned}$$

which means that $\lambda_\beta = \lambda_\alpha |t_{\alpha\beta}|_L^2$ as desired; this completes our proof of the result. \square

Proposition 2.16. Curvature of the Chern Connection on a Line Bundle

Let $L \rightarrow X$ be a complex holomorphic line bundle over the complex manifold X with trivializing open cover $\{(U_\alpha, \varphi_\alpha)\}$ together with positive functions $\lambda_\alpha : U_\alpha \rightarrow \mathbb{R}^+$ satisfying the compatibility conditions $\lambda_\beta = \lambda_\alpha |t_{\alpha\beta}|^2$ as above. In this setting, we know that the curvature of the (unique) Chern connection ∇ on L , defined as $\Theta := \partial\bar{\partial}\log h$, is given by:

$$\boxed{\Theta = -i\partial\bar{\partial}\log\lambda_\alpha}, \quad \text{on } U_\alpha$$

and this is well-defined on the charts.

Proof. For this result, we need only prove that **the curvature matrix is well-defined**. To see this, we equivalently want to prove that:

$$\partial\bar{\partial}\log\lambda_\alpha = \partial\bar{\partial}\log\lambda_\beta, \quad \text{on } U_\alpha \cap U_\beta$$

To see why this result holds, we need only write, as above:

$$\lambda_\beta = \lambda_\alpha |t_{\alpha\beta}|^2 \implies \log\lambda_\beta = \log\lambda_\alpha + \log|t_{\alpha\beta}|^2$$

On the other hand, we now observe that, in general, since $\bar{\partial}(\bar{f}) = \bar{\partial}f$, we obtain:

$$\partial_i \partial_{\bar{j}} |f|^2 = \partial_i \partial_{\bar{j}} (f\bar{f}) = \partial_i (\bar{f} \partial_j f + f \partial_{\bar{j}} \bar{f}) = \partial_i f \partial_{\bar{j}} \bar{f} + f \partial_i \partial_{\bar{j}} \bar{f} + \partial_i \bar{f} \partial_{\bar{j}} f + \bar{f} \partial_i \partial_{\bar{j}} f$$

Of course, we know that for any function f we have, since $d = \partial + \bar{\partial}$ and $\bar{\partial}^2 = 0$, that:

$$\partial\bar{\partial}f = (\partial + \bar{\partial})\bar{\partial}f = d(\bar{\partial}f)$$

i.e., d -closed; thus, the curvature Θ is **well-defined** as a cohomology class, as desired; this completes the proof. \square

Example 2.17. The Tautological Line Bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ of $\mathbb{C}\mathbb{P}^n$

Over $\mathbb{C}\mathbb{P}^n$, we have the completely analogous construction of a line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$, which is

called the **tautological line bundle**. This is given by $L := J^\vee$, i.e., L is the **dual bundle** to the **universal line bundle** $J := \mathcal{O}_{\mathbb{P}^n}(-1)$ defined above.

The total space of the tautological line bundle is given by:

$$L := \left\{ ([z^0 : z^1 : \cdots : z^n], \lambda(z^0, z^1, \dots, z^n)) \in \mathbb{C}\mathbb{P}^n \times \mathbb{C}^{n+1} \mid \lambda \in \mathbb{C} \right\}$$

For this, the corresponding transition functions as before are instead given by:

$$t_{ij}([z^0 : z^1 : \cdots : z^n]) = \frac{z^j}{z^i}$$

In particular, we know that **the dual bundle can always be trivialized using the same open cover as the original one**, that is, we again use the $U_i := \{z^i \neq 0\} \subset \mathbb{C}\mathbb{P}^n$.

Now, we construct a **Hermitian metric** on L . Over each chart U_i , we define:

$$\lambda_i = \left(\sum_{k=0}^n \left| \frac{z^k}{z^i} \right|^2 \right)^{-1}$$

Then, in particular, we see that:

$$\lambda_j = \left(\sum_{k=0}^n \left| \frac{z^k}{z^j} \right|^2 \right)^{-1} = \left| \frac{z^j}{z^i} \right|^2 \left(\sum_{k=0}^n \left| \frac{z^k}{z^i} \right|^2 \right)^{-1} = \lambda_i |t_{ij}|^2$$

upon recalling the definition of the transition function $t_{ij} = z^j/z^i$ given above. Consequently, our previous lemma shows that this condition is both necessary and sufficient for the collection of maps λ to **define a Hermitian metric on the line bundle** $L \rightarrow X$.

To compute the **curvature** of this metric, we need only write out the previous expression $\Theta = -\partial\bar{\partial} \log h$, so that:

$$\omega = \partial\bar{\partial} \log \left(\sum_{k=0}^n \left| \frac{z^k}{z^i} \right|^2 \right)$$

expressed on the local chart U_i . For an example, we may choose $i = 0$ and normalize $z^i := Z^i/Z^0$ for all $1 \leq i \leq n$; then, our expression for the curvature becomes:

$$\omega = \partial\bar{\partial} \log \left(1 + \sum_{k=1}^n |z^k|^2 \right)$$

From our previous considerations, we know that this is precisely the **Fubini-Study metric** on the holomorphic tangent bundle $T^{1,0}\mathbb{P}^n$.

For example, this local expression will now help us determine the **curvature of the Fubini-Study metric** on the holomorphic tangent bundle $T^{1,0}\mathbb{P}^n$ of $\mathbb{C}\mathbb{P}^n$. Indeed, we saw above that on the affine open chart $U_0 \subset \mathbb{C}\mathbb{P}^n$, we may consider the components of the Hermitian metric:

$$h_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log \left(1 + \sum_{k=1}^n |z^k|^2 \right)$$

Now, we may fix the standard point $P = [1 : 0 : \cdots : 0] = (0, 0, \dots, 0) \in U_0$. For this point, we can now use **Taylor's expansion** to obtain, up to first order:

$$\log(1+x) = x - x^2 + O(x^3) \implies$$

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$$\implies \log \left(1 + \sum_{k=1}^n |z^k|^2 \right) = \sum_{k=1}^n |z^k|^2 - \left(\sum_{k=1}^n |z^k|^2 \right)^2 + O(|z|^6)$$

On the other hand, considering **normal coordinates** here shows us that:

$$h_{i\bar{j}} := \langle \mathbf{e}_i(x), \mathbf{e}_{\bar{j}}(x) \rangle = \delta_{ij} - R_{k\bar{\ell}i\bar{j}} z^k \bar{z}^{\ell} + O(|z|^3)$$

Thus, we compare these local expressions to obtain, around the point P :

$$\begin{aligned} \delta_{ij} - R_{k\bar{\ell}i\bar{j}} z^k \bar{z}^{\ell} + O(|z|^3) &= \partial \partial_{\bar{j}} \left(\sum_{k=1}^n |z^k|^2 - \left(\sum_{k=1}^n |z^k|^2 \right)^2 + O(|z|^6) \right) \\ &= \partial_i \partial_{\bar{j}} \left(\sum_{k=1}^n |z^k|^2 \right) - \partial_i \partial_{\bar{j}} \left(\sum_{k=1}^n |z^k|^2 \right)^2 + \partial_i \partial_{\bar{j}} O(|z|^6) \\ &= \delta_{ij} - (\delta_{ij} \delta_{k\ell} + \delta_{i\ell} \delta_{jk}) z^k \bar{z}^{\ell} + O(|z|^4) \end{aligned}$$

Thus, comparing the two expressions now leaves us with:

$$R_{k\bar{\ell}i\bar{j}}(P) = \delta_{ij} \delta_{k\ell} + \delta_{i\ell} \delta_{jk} \equiv h_{i\bar{j}}(P) h_{k\bar{\ell}}(P) + h_{i\bar{\ell}}(P) h_{k\bar{j}}(P)$$

whereby it turns out that up to a linear transformation we may obtain, for **any** $P \in U_0$:

$$R_{k\bar{\ell}i\bar{j}} = h_{i\bar{j}} h_{k\bar{\ell}} + h_{i\bar{\ell}} h_{k\bar{j}}$$

We will see more on this result later.

2.2. Curvature on Hermitian Vector Bundles. Now, let ∇ be the Chern connection on a Hermitian vector bundle with metric $(E, h) \rightarrow X$, i.e., the unique metric-compatible holomorphic connection (covariant derivative) on E . Since the curvature tensor is **skew-adjoint** as a section of $\Lambda^2 T^{\vee} X \otimes \text{End}(E)$, we see that $(\nabla^{1,0})^2 = (\nabla^{0,1})^2 = 0$; consequently, the curvature of ∇ restricts to just a section of $\mathcal{A}^{1,1}(X, \text{End}(E))$, i.e., an $\text{End}(E)$ -valued $(1,1)$ -form. In fact, this will be equal to the commutator $[\nabla^{1,0}, \nabla^{0,1}]$, due to this vanishing (which also makes the corresponding commutators vanish)

Let us see concretely how the Hermitian metric and the holomorphic connection corresponding to it are related in a chart: if the Hermitian metric on E is given at a point $x \in X$ by a $r \times r$ Hermitian matrix $h(x)$, then the corresponding covariant derivative ∇_E equals:

$$\nabla_E = d + h^{-1} \partial h$$

where $h^{-1} \partial h$ is interpreted as a matrix of 1-forms; here, the operator d only acts on the **form part** of the E -valued forms. The curvature of ∇_E is then the $\text{End}(E)$ -valued $(1,1)$ -form:

$$\Theta_E := F_h := F_{\nabla_E} := h^{-1} \bar{\partial} \partial h - (h^{-1} \bar{\partial} h) \wedge (h^{-1} \partial h)$$

Of course, we know that for $E = L$ a **line bundle**, the curvature h is simply a smooth, positive real function and that the endomorphism bundle $\text{End}(L) \cong \mathbb{C}$ is **globally holomorphically trivial**; consequently, the curvature is simply equal to:

$$\Theta_L := F_h = \bar{\partial} \log h \equiv \bar{\partial} \log(\|s\|^2)$$

where, in the last part, s is any non-vanishing holomorphic section of L over the chart.

Let $E \rightarrow X$ be a Hermitian holomorphic vector bundle over a complex manifold, so that the curvature Θ_E of the vector bundle E is a section of the $(1,1)$ -twisted endomorphism bundle of E , i.e., $\Lambda^{1,1}X \otimes \text{End}(E)$, where $\Lambda^{1,1}X := T^\vee X^{1,0} \otimes T^\vee X^{0,1} = T^\vee X^{1,0} \otimes \overline{T^\vee X^{1,0}}$. For this, we recall the isomorphism:

$$\text{End}(E) \otimes \Lambda^{1,1}X \cong (T^{1,0}X \otimes E \otimes \overline{T^{1,0}X \otimes E})^\vee$$

which realizes the endomorphism (of E)-valued curvature $(1,1)$ -form Θ_E as a Hermitian form θ_E on $T^{1,0}X \otimes E$. We can use this to define **Griffiths-Nakano positivity** on E . This is defined by:

$$\boxed{\Theta_E(v \otimes \sigma, w \otimes \tau) := h(\Theta_E(v, \bar{w})\sigma, \tau)}$$

where h is a Hermitian metric on E , the $v, w \in \Gamma^\infty(X, TX)$ are sections of TX , and the $\sigma, \tau \in \Gamma^\infty(X, E)$ are sections of E . This formula applies to decomposable (rank one) tensors and extends to all tensors by sesquilinearity.

Now, let $(E, h_E), (F, h_F) \rightarrow X$ be two holomorphic Hermitian vector bundles over X ; then, we can define their **tensor product** as the vector bundle having transition functions given by the product $t_{\mu\nu}^E t_{\mu\nu}^F$ of the corresponding transition functions on a common refinement of the trivializing open sets that define them by Čech data; consequently, we see that we can also define the **product metric** $h_{E \times F} := h_E \otimes h_F$ on their tensor product by extending the corresponding definitions on E, F , first locally and then using the same partitions of unity that we used for the construction of h_E, h_F . From an alternative, abstract perspective, we can also recall that a Hermitian metric on a vector bundle E is a smooth global section of the vector bundle $E^\vee \otimes \overline{E}^\vee$; and we get:

$$(E^\vee \otimes \overline{E}^\vee) \otimes (F^\vee \otimes \overline{F}^\vee) \simeq (E \otimes F)^\vee \otimes \overline{E \otimes F}^\vee$$

which gives the natural correspondence between $h_E \otimes h_F$ and $h_{E \otimes F}$; this gives us the following property of the curvature:

$$\boxed{\Theta_{E \otimes F} = \Theta_E \otimes \text{id}_F + \text{id}_E \otimes \Theta_F}$$

Using this, we now compute for sections v, σ, τ of TX, E, F , respectively:

$$\begin{aligned} \theta_{E \otimes F}(v \otimes \sigma \otimes \tau, v \otimes \sigma \otimes \tau) &= h_{E \otimes F}(\Theta_{E \otimes F}(v, \bar{v}) \cdot (\sigma \otimes \tau), \sigma \otimes \tau) \\ &= h_{E \otimes F}((\Theta(E)(v, \bar{v}) \otimes \text{Id}_F + \text{Id}_E \otimes \Theta(F)(v, \bar{v})) \cdot (\sigma \otimes \tau), \sigma \otimes \tau) \\ &= h_{E \otimes F}((\Theta(E)(v, \bar{v}) \cdot \sigma) \otimes \tau + \sigma \otimes (\Theta(F)(v, \bar{v}) \cdot \tau), \sigma \otimes \tau) \\ &= h_E(\Theta(v, \bar{v})\sigma, \sigma)h_F(\tau, \tau) + h_E(\sigma, \sigma)h_F(\Theta(v, \bar{v})\tau, \tau) \\ &=: \theta_E(v \otimes \sigma, v \otimes \sigma)h_F(\tau, \tau) + h_E(\sigma, \sigma)\theta_F(v \otimes \tau, v \otimes \tau) \end{aligned}$$

which gives rise to the following decomposition for θ_E :

$$\boxed{\theta_{E \otimes F} = \theta_E \otimes h_F + h_E \otimes \theta_F}$$

In fact, we can use this decomposition to show that for any Hermitian holomorphic vector bundle E on a compact complex manifold with a positive line bundle L (meaning that it is **algebraic**, by the Kodaira embedding theorem) there is some k_0 such that $E \otimes L^k$ is **Griffiths-positive** for all $k \geq k_0$; in fact, it is **Nakano positive**, as can be seen by using

the Cauchy-Schwarz inequality together with compactness, alongside the obvious results $\theta_{L^m} = m\theta_L$.

Let us recall the notation Θ_E for the curvature of the (Chern connection on the) holomorphic vector bundle $E \rightarrow X$. For example, we have the canonical line bundle $K_X \rightarrow X$, given by $K_X = \Lambda^n T^\vee X^{1,0}$, as well as the anticanonical bundle $K_X^\vee := \Lambda^n T^{1,0} X$. The latter has curvature $\Theta_{K_X^\vee} = \sum_i \langle R\partial_k, \partial_{\bar{k}} \rangle$, where R is the Riemannian curvature of X ; again, this is simply an $(1, 1)$ -form, since we are working with a line bundle.

In this setting, we obtain the following result:

Proposition 2.18. The Bochner-Kodaira formula

Let $(E, h) \rightarrow X$ be a Hermitian holomorphic vector bundle over the Kähler manifold X , with operator $\bar{\partial}_E : \mathcal{A}^{p,q}(X, E) \rightarrow \mathcal{A}^{p,q+1}(X, E)$. Then, in a local holomorphic coordinate system, as have:

$$\bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E = \Delta^{0,\bullet} + \varepsilon(d\bar{z}^k) \iota(dz^\ell) \Theta_{E \otimes K_X^\vee}(\partial_\ell, \partial_{\bar{k}})$$

which is called the **Bochner-Kodaira formula**; here, the notation $\Delta^{0,\bullet}$ refers to the connection Laplacian $(\nabla^{0,1})^* \nabla^{0,1}$ formed by the $(0, 1)$ -part of the connection.

Proof. The proof is similar to the Weitzenböck-Lichnerowicz formula considered previously; we work in **normal coordinates**. Using the fact that the connection ∇ on E is holomorphic, we may write the dbar-operator on E as $\bar{\partial}_E = \varepsilon(d\bar{z}^k) \nabla_{\bar{k}}$, whereby its adjoint will be $\bar{\partial}_E^* = -\iota(dz^\ell) \nabla_\ell$. Consequently:

$$\bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E = \Delta^{0,\bullet} + \varepsilon(d\bar{z}^k) \iota(dz^\ell) \left(R^+(\partial_\ell, \partial_{\bar{k}}) + \Theta_E(\partial_\ell, \partial_{\bar{k}}) \right)$$

by the definition of the curvature Θ_E as $\Theta_E = [\nabla^{1,0}, \nabla^{0,1}]$ for this holomorphic connection, as we know that $(\nabla^{1,0})^2 = (\nabla^{0,1})^2 = 0$. Here, R^+ denotes the curvature of $\Lambda^n T^\vee X^{0,1}$ and of course, we know from the above that for a **line bundle** L and any vector bundle E , we have $\boxed{\Theta_{E \otimes L} = \Theta_E + \Theta_L}$ (with Θ_L being viewed as inducing the identity on the $\text{End}(E \otimes L)$ -part of the form, via $\text{End}(L) \cong \mathbb{C}$ (holomorphically trivial) and $\text{End}(E \otimes L) \cong \text{End}(E) \otimes \text{End}(L) \cong \text{End}(E)$). Consequently, we see that the last expression will have the desired form $\Theta_{E \otimes K_X^\vee} = \Theta_E + \Theta_{K_X^\vee}$, if and only if:

$$\varepsilon(d\bar{z}^k) \iota(dz^\ell) R^+(\partial_\ell, \partial_{\bar{k}}) = \varepsilon(d\bar{z}^k) \iota(dz^\ell) \Theta_{K_X^\vee}(\partial_\ell, \partial_{\bar{k}})$$

Indeed, the LHS equals:

$$\begin{aligned} \varepsilon(d\bar{z}^k) \iota(dz^\ell) R^+(\partial_\ell, \partial_{\bar{k}}) &= \varepsilon(d\bar{z}^k) \iota(dz^\ell) \varepsilon(d\bar{z}^j) \iota(dz^j) \langle R(\partial_\ell, \partial_{\bar{k}}) \partial_j, \partial_{\bar{i}} \rangle \\ &= \varepsilon(d\bar{z}^k) \iota(dz^j) \langle R(\partial_\ell, \partial_{\bar{k}}) \partial_j, \partial_{\bar{i}} \rangle \end{aligned}$$

Now, using the **torsion-free** property of the Levi-Civita connection, we have:

$$R(\partial_k, \partial_{\bar{j}}) \partial_\ell + R(\partial_\ell, \partial_k) \partial_{\bar{j}} + R(\partial_{\bar{j}}, \partial_\ell) \partial_k = 0$$

and since also $R(\partial_k, \partial_\ell) = 0$, we deduce that $R(\partial_k, \partial_{\bar{j}}) \partial_\ell = R(\partial_\ell, \partial_{\bar{j}}) \partial_k$, whereby:

$$\begin{aligned} \varepsilon(d\bar{z}^k) \iota(dz^\ell) R^+(\partial_\ell, \partial_{\bar{k}}) &= \varepsilon(d\bar{z}^k) \iota(dz^j) \langle R(\partial_\ell, \partial_{\bar{k}}) \partial_j, \partial_{\bar{\ell}} \rangle \\ &= \varepsilon(d\bar{z}^k) \iota(dz^j) \langle R(\partial_j, \partial_{\bar{k}}) \partial_\ell, \partial_{\bar{\ell}} \rangle \end{aligned}$$

$$\begin{aligned}
&= \varepsilon(d\bar{z}^k)\iota(dz^j)R(\partial_j, \partial_{\bar{k}}) \\
&= \varepsilon(d\bar{z}^k)\iota(dz^j)\Theta_{K_X^\vee}(\partial_j, \partial_{\bar{k}})
\end{aligned}$$

precisely as desired; this completes the proof of the theorem. \square

This formula has some important *vanishing* results: if $L \rightarrow X$ is a Hermitian holomorphic line bundle with curvature $\Theta_L = \sum F_{i\bar{j}} dz^i \wedge d\bar{z}^j$, we say that L is **positive** if the Hermitian form $v \mapsto \Theta_L(v, \bar{v})$ on $T^{1,0}X$ is **positive**. This positivity condition is related to the existence of global holomorphic sections of L : for example, if L is a global holomorphic section of L , and x is a point in X at which $|s|^2$ attains a strict maximum, then it is easy to see that $\Theta_L|_x = \bar{\partial}\partial \log |s|^2$ is **positive** on $T_x^{1,0}X$.

In what follows, let $E \rightarrow X$ be a holomorphic vector bundle over X equipped with a Hermitian metric. Now, a smooth E -valued (n, q) -form can be viewed in two ways: first, it can be viewed as a (n, q) -form taking values in (a section) of the holomorphic vector bundle; on the other hand, it can also be viewed as a section of the vector bundle $\Lambda^{n,q} T^\vee X \otimes E \rightarrow X$, with the important difference that this complex bundle is now **not holomorphic**. Thus, it makes sense to consider the natural $\bar{\partial}$ -operators for these two types of sections.

Starting from the first perspective, we fix a basis $\{e_\alpha\}$ of local holomorphic sections of $E \rightarrow X$ (i.e., a local frame); to indicate the type of forms we are using, we will use subscripts for the multi-indices. Then, denoting by ∇ the Chern connection associated to the E -valued (p, q) -forms on the Kähler manifold (X, g) , we see that:

Lemma 2.19. The $\bar{\partial}$ -operator on E -valued forms

Let $s = s^\alpha_{\bar{J}_q} dz^N \wedge d\bar{z}^{J_q} \otimes e_\alpha$ be the local expression of an E -valued (n, q) -form, where we are applying the Einstein summation convention over multi-indices of degree $|I_q| = q$. Then, the $\bar{\partial}_E$ -operator acts on s by:

$$\bar{\partial}_E s = (-1)^n \sum_{k=0}^q (-1)^k (\nabla_{\bar{J}_k} s)^\alpha_{\bar{j}_0 \cdots \hat{\bar{j}_k} \cdots \bar{j}_q} dz^N \wedge d\bar{z}^J \otimes e_\alpha$$

where $\hat{}$ denotes the omission of the corresponding index.

Proof. In what follows, let us denote $dz^N := dz^1 \wedge \cdots \wedge dz^n$, since we will use it extensively.

By the definition of the $\bar{\partial}_E$ -operator, we know that:

$$\begin{aligned}
\bar{\partial}_E s &= (\bar{\partial} s^\alpha_{\bar{J}_q}) \wedge dz^N \wedge d\bar{z}^{J_q} \otimes e_\alpha \\
&= (-1)^n \bar{\partial}_{\bar{J}} s^\alpha_{\bar{j}_0 \cdots \bar{j}_q} \wedge dz^N \wedge d\bar{z}^j \wedge d\bar{z}^{J_q} \otimes e_\alpha \\
&= (-1)^n \sum_{k=0}^q (-1)^k \bar{\partial}_{\bar{J}_k} s^\alpha_{\bar{j}_0 \cdots \hat{\bar{j}_k} \cdots \bar{j}_q} dz^N \wedge d\bar{z}^{j_0} \wedge \cdots \wedge d\bar{z}^{j_q} \otimes e_\alpha \\
&= (-1)^n \sum_{k=0}^q (-1)^k (\nabla_{\bar{J}_k} s)^\alpha_{\bar{j}_0 \cdots \hat{\bar{j}_k} \cdots \bar{j}_q} dz^N \wedge d\bar{z}^J \otimes e_\alpha
\end{aligned}$$

where the last equality holds since we know that $\nabla^{0,1} = \bar{\partial}_E$ for the Chern connection. \square

Now, let us recall the notation $\overline{\partial}_E^*$ for the formal adjoint of $\overline{\partial}_E$, defined by the relation:

$$(\overline{\partial}_E^* s, t) = (s, \overline{\partial}_E t)$$

for all smooth twisted forms s and for all compactly supported smooth twisted forms t ; in particular, $\overline{\partial}_E^*$ sends smooth E -valued (n, q) -forms to smooth E -valued $(n, q-1)$ -forms.

Importantly, we see now that the operator $\overline{\partial}_E^*$ is locally expressible as:

Lemma 2.20. The $\overline{\partial}^*$ -operator on E -valued forms

Let $s = s^\alpha_{\bar{J}_q} dz^N \wedge d\bar{z}^{J_q} \otimes e_\alpha$ be the local expression of an E -valued (n, q) -form, where we are applying the Einstein summation convention over multi-indices of degree $|I_q| = q$. Then, the $\overline{\partial}_E^*$ -operator acts on s by:

$$\overline{\partial}_E s = (-1)^{n+1} g^{i\bar{j}} (\nabla_i s)^\alpha_{\bar{j}\bar{j}_1 \cdots \widehat{\bar{j}_{q-1}}} dz^N \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_{q-1}} \otimes e_\alpha$$

in the same conventions as above

Proof. We do this by using the local form (from the Lemma 2.19 above) of the $\overline{\partial}_E$ operator: for any local E -valued $(n, q-1)$ -form $t = t^\alpha_{\bar{J}_{q-1}} dz^N \wedge d\bar{z}^{J_{q-1}} \otimes e_\alpha$, we have:

$$\begin{aligned} (\overline{\partial}_E^* s, t) &= (s, \overline{\partial}_E t) \\ &= \frac{1}{n!q!} \int_X s^\alpha_{\bar{J}_1 \cdots \bar{J}_q} \cdot (-1)^n \sum_{k=1}^q (-1)^{k+1} \overline{(\nabla_{\bar{j}'_k} t)}^{\beta}_{\bar{j}'_1 \cdots \widehat{\bar{j}'_k} \cdots \bar{j}'_q} g^{J'_q \bar{J}_q} h_{\alpha\bar{\beta}} \\ &= \frac{1}{n!q!} \int_X (-1)^{n+1} \sum_{k=1}^q (-1)^{k+1} g^{j'_k \bar{J}_k} (\nabla_{j'_k} s)^\alpha_{\bar{j}_1 \cdots \bar{j}_q} \overline{t^{\beta}_{\bar{j}'_1 \cdots \widehat{\bar{j}'_k} \cdots \bar{j}'_q}} g^{j'_1 \bar{j}_1} \cdots \widehat{g^{j'_k \bar{j}_k}} \cdots g^{j'_q \bar{j}_q} h_{\alpha\bar{\beta}} \\ &= \frac{1}{n!(q-1)!} \int_X \left((-1)^{n+1} g^{i\bar{j}} (\nabla_i s)^\alpha_{\bar{j}\bar{j}_1 \cdots \widehat{\bar{j}_{q-1}}} \right) \overline{t^{\beta}_{\bar{j}'_1 \cdots \widehat{\bar{j}'_k} \cdots \bar{j}'_q}} g^{J'_{q-1} \bar{J}_{q-1}} h_{\alpha\bar{\beta}} \end{aligned}$$

where the second-to-last inequality follows from the **metric compatibility** of the Chern connection on E , i.e., $\nabla \langle s, t \rangle = \langle \nabla s, t \rangle + \langle s, \nabla t \rangle$; this completes the proof. \square

Now, let us examine an alternative approach to E -valued (n, q) -forms, or, equivalently, since $\Lambda^{n,q} T^\vee X = K_X \otimes \Lambda^q T^\vee X^{0,1}$, these are sections of the complex vector bundle $K_X \otimes \Lambda^q T^\vee X^{0,1} \otimes E \rightarrow X$, which is of course **not holomorphic**. However, because X is equipped with a metric (coming from the Kähler form ω) there is a natural way to map sections of $K_X \otimes \Lambda^q T^\vee X^{0,1} \otimes E \rightarrow X$ to sections of the **holomorphic vector bundle** $K_X \otimes \Lambda^q T^{1,0} X \otimes E \rightarrow X$, i.e., by passing via the *musical isomorphism* $T^\vee X^{0,1} \xrightarrow{\sim} T^{1,0} X$ given by the metric.

Let us denote this map by Φ_ω ; then, in terms of local data, we may write the section as $s = s^\alpha_{\bar{J}} dz^N \otimes d\bar{z}^J \otimes e_\alpha$, so that:

$$\begin{aligned} \Phi_\omega s &:= g^{I\bar{J}} s^\alpha_{\bar{J}} dz^N \otimes \frac{\partial}{\partial z^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial z^{i_q}} \otimes e_\alpha \\ &:= g^{I\bar{J}} s^\alpha_{\bar{J}} dz^N \otimes \frac{\partial}{\partial z^I} \otimes e_\alpha \end{aligned}$$

i.e., $\Phi_\omega s$ is a section of $K_X \otimes \Lambda^q T^{1,0} X \otimes E \rightarrow X$, and clearly the map $s \mapsto \Phi_\omega s$ is a bijective correspondence that depends only on the pointwise values of s .

This map Φ_ω extends naturally to a bijective correspondence Φ_ω of $K_X \otimes \Lambda^q T^{1,0} X \otimes E$ -valued $(0, k)$ -forms, i.e., sections of $\Lambda^k T^\vee X^{0,1} \otimes K_X \otimes \Lambda^q T^{1,0} X \otimes E \rightarrow X$, by acting trivially on the last factor; i.e., for $J = (j_1, \dots, j_k)$, we get:

$$\Phi_\omega(d\bar{z}^J \otimes (dz^N) \wedge d\bar{z}^I \otimes e_\alpha) := d\bar{z}^J \otimes \Phi_\omega(dz^N \wedge d\bar{z}^I \otimes e_\alpha)$$

Importantly, we see that since now $K_X \otimes \Lambda^q T^{1,0} X \otimes E \rightarrow X$ is a **holomorphic vector bundle**, it comes equipped with a natural $\bar{\partial}$ -operator, and we thus get a well-defined $(K_X \otimes \Lambda^q T^{1,0} X \otimes E)$ -valued $(0, 1)$ -form $\bar{\partial}\Phi_\omega s$.

Using these properties, we may now define:

Definition 2.21. *The $(0, 1)$ -connection*

The operator $\nabla^{0,1} : \mathcal{A}(X, \Lambda_X^{n,q} \otimes E) \rightarrow \Gamma(X, T^\vee X^{0,1} \otimes \Lambda_X^{n,q} \otimes E)$ is defined by:

$$\boxed{\nabla^{0,1} s := \Phi_\omega^{-1} \bar{\partial} \Phi_\omega s}$$

In terms of local data, this is given by:

$$\begin{aligned} \nabla^{0,1} s &:= g_{I\bar{L}} \frac{\partial}{\partial \bar{z}^k} \left(g^{I\bar{J}} s^\alpha_{\bar{J}} \right) d\bar{z}^k \otimes dz^N \wedge d\bar{z}^L \otimes e_\alpha \\ &= (\nabla_{\bar{k}} s)^\alpha_{\bar{J}} d\bar{z}^k \otimes dz^N \wedge d\bar{z}^I \otimes e_\alpha \end{aligned}$$

where ∇ denotes the Chern connection for E -valued tensors.

Now, we can compute the formal adjoint $(\nabla^{0,1})^*$ of $\nabla^{0,1}$ with respect to the inner products induced on the sections of $\Lambda^{n,q} T^\vee X \otimes E$ and $T^\vee X^{0,1} \otimes \Lambda^{n,q} T^\vee X \otimes E$ by the metrics h (on E) and g (on X). For smooth sections:

$$s \in \Gamma^\infty(K_X \otimes \Lambda^q T^\vee X^{0,1} \otimes T^\vee X^{0,1} \otimes E), \quad t \in \Gamma_c^\infty(K_X \otimes \Lambda^q T^\vee X^{0,1} \otimes E)$$

where the latter is compactly supported, we may write these as $s = s^\alpha_{\bar{J}, \bar{j}} d\bar{z}^j \otimes dz^N \wedge d\bar{z}^J \otimes e_\alpha$ (and similarly for t) we may compute:

$$\begin{aligned} ((\nabla^{0,1})^* s, t) &= (s, \nabla^{0,1} t) = \int_X g^{k\bar{j}} s^\alpha_{\bar{J}, \bar{j}} \cdot \overline{(\nabla_{\bar{k}} t)^\bar{J}_\alpha} d\text{Vol}_g \\ &= \int_X (-g^{k\bar{j}} (\nabla_k s)^\alpha_{\bar{J}, \bar{j}}) \cdot \overline{s^J_\alpha} d\text{Vol}_g \end{aligned}$$

where $d\text{Vol}_g = \left(\frac{\sqrt{-1}}{2}\right)^n dz^1 \wedge d\bar{z}^1 \wedge \dots \wedge dz^n \wedge d\bar{z}^n$; importantly, we used here the **metric compatibility** of the Chern connection in order to exchange the integrals inside via $\nabla \langle s, t \rangle = \langle \nabla s, t \rangle + \langle s, \nabla t \rangle$. Consequently, we obtain:

$$(2.2.1) \quad ((\nabla^{0,1})^* s)^\alpha_{\bar{J}} = -g^{k\bar{j}} (\nabla_k s)^\alpha_{\bar{J}, \bar{j}}$$

where, of course, we are viewing s as a *form* (if we were to view this as a *section*, we would just get 0). Consequently, we may also associate to $\nabla^{0,1}$ its Laplace-Beltrami operator $(\nabla^{0,1})^* \nabla^{0,1}$ on sections of the vector bundle $\Lambda^{n,q} T^\vee X \otimes E \rightarrow X$; and these computations prove that:

$$(2.2.2) \quad \boxed{(\nabla^{0,1})^* \nabla^{0,1} = -g^{k\bar{j}} \nabla_k \nabla_{\bar{j}}}$$

Using this result, we finally arrive at:

Theorem 2.22. The Bochner-Kodaira identity

Let $(E, h) \rightarrow X$ be a Hermitian holomorphic vector bundle over a complex Kähler manifold (X, ω) . Then, we have the (formal) identity:

$$\bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E = (\nabla^{0,1})^* \nabla^{0,1} + \Theta_E(h)$$

where Θ_E is the operator on the E -valued (p, q) -valued forms induced by the curvature of the metric h .

Proof. Let us use the propositions 2.19 and 2.20 to compute the LHS of this expression. For the first one, we see that a local E -valued (n, q) -form $s = s^\alpha_{\bar{J}_q} dz^N \wedge d\bar{z}^{J_q} \otimes e_\alpha$ has:

$$\begin{aligned} (\bar{\partial}_E \bar{\partial}_E^* s)_{\bar{j}_1 \dots \bar{j}_q}^\alpha &= (-1)^{n+1} g^{i\bar{j}} \nabla_i \left((\bar{\partial}s)_{\bar{j}\bar{J}}^\alpha dz^1 \wedge \dots \wedge dz^n \wedge d\bar{z}^J \otimes e_\alpha \right) \\ &= (-1)^{n+1} g^{i\bar{j}} \nabla_i \left((-1)^n (\nabla_{\bar{j}} s)_{\bar{J}}^\alpha dz^N \wedge d\bar{z}^J \otimes e_\alpha \right) - \\ &\quad - g^{i\bar{j}} \nabla_i \sum_{k=1}^q (-1)^k (\nabla_{\bar{j}_k} s)_{\bar{j}_1 \dots \hat{\bar{j}_k} \dots \bar{j}_q}^\alpha dz^N \wedge d\bar{z}^J \otimes e_\alpha \\ &= \left((-g^{i\bar{j}} \nabla_i \nabla_{\bar{j}} s)_{\bar{j}_1 \dots \bar{j}_q}^\alpha + \sum_{k=1}^q \left(g^{i\bar{j}} \nabla_i \nabla_{\bar{j}_k} s \right)_{\bar{j}_1 \dots (\bar{j})_k \dots \bar{j}_q}^\alpha \right) dz^N \wedge d\bar{z}^J \otimes e_\alpha \end{aligned}$$

where, again, the notation $(\bar{j})_k$ means that in the k -th entry, the index \bar{j}_k has been replaced by \bar{j} . Moving on to the other expression, we see that:

$$\begin{aligned} (\bar{\partial}_E \bar{\partial}_E^* s)_{\bar{j}_1 \dots \bar{j}_q}^\alpha &= (-1)^n \sum_{k=1}^q (-1)^{k+1} \left(\nabla_{\bar{j}_k} (\bar{\partial}_E s) \right)_{\bar{j}_1 \dots \hat{\bar{j}_k} \dots \bar{j}_q}^\alpha \\ &= (-1)^n \sum_{k=1}^q (-1)^{k+1} \left(\nabla_{\bar{j}_k} \left((-1)^{n+1} g^{i\bar{j}} \nabla_i s \right) \right)_{\bar{j}_1 \dots \hat{\bar{j}_k} \dots \bar{j}_q}^\alpha \\ &= - \sum_{k=1}^q \left(\nabla_{\bar{j}_k} \left(g^{i\bar{j}} \nabla_i s^\alpha \right) \right)_{\bar{j}_1 \dots (\bar{j})_k \dots \bar{j}_q} \\ &= - \sum_{k=1}^q g^{i\bar{j}} \left(\nabla_{\bar{j}_k} \nabla_i s \right)_{\bar{j}_1 \dots (\bar{j})_k \dots \bar{j}_q}^\alpha \end{aligned}$$

where, in the middle step, we isolated s^α by looking at its *form components*. Thus, we may now sum these two expressions over multi-indices $\bar{J}_q = (\bar{j}_1, \dots, \bar{j}_q)$ to obtain:

$$((\bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E) s)_{\bar{J}_q}^\alpha = -g^{i\bar{j}} \left(\nabla_i \nabla_{\bar{j}} s \right)_{\bar{J}_q}^\alpha + \sum_{k=1}^q g^{i\bar{\ell}} \left([\nabla_i, \nabla_{\bar{j}_k}] s \right)_{\bar{j}_1 \dots (\bar{\ell})_k \dots \bar{j}_q}^\alpha$$

so, by the equality 2.2.2 from above, we see that the first summand of this last expression is precisely $(\nabla^{0,1})^* \nabla^{0,1} s$; and we need to study the second one.

For this, we know by definition that $[\nabla_i, \nabla_{\bar{j}}] = \Theta_E(h)_{i\bar{j}}$ is the curvature, we get:

$$\sum_{k=1}^q g^{i\bar{\ell}} [\nabla_i, \nabla_{\bar{j}_k}] s_{\bar{j}_1 \dots (\bar{\ell})_k \dots \bar{j}_q}^\alpha = \sum_{k=1}^q g^{i\bar{\ell}} \Theta_E(h)_{\beta i \bar{j}_k}^\alpha s_{\bar{j}_1 \dots (\bar{\ell})_k \dots \bar{j}_q}^\beta$$

where, again, the notation $(\bar{\ell})_k$ means that in the k -th entry, the index \bar{j}_k has been replaced by $\bar{\ell}$. Finally, we observe now that the RHS of the above expression is *by definition* the

operator $\Theta_E(h)$ induced in E -valued (n, q) -forms by the curvature of h , as:

$$\Theta_E(h)s := \sum_{k=1}^q g^{i\bar{\ell}} \Theta_E(h)^\alpha{}_{\beta i\bar{j}_k} s^\beta{}_{\bar{j}_1 \dots (\bar{\ell})_k \dots \bar{j}_q} dz^N \wedge d\bar{z}^{j_1} \wedge d\bar{z}^{j_q} \otimes e_\alpha$$

(Notice here that the notation $\Theta_E(h)$ was adopted instead of Θ_E primarily for the notational convenience of avoiding clashing multi-indices and subscripts when examining the form-valued index components of the curvature)

This enables us to reassemble the desired identity; this completes the proof. \square

For example, we may now prove that:

Theorem 2.23. Kodaira Vanishing Theorems

Let X be a compact Kähler manifold and $E \rightarrow X$ a holomorphic vector bundle on X . Denote by $\mathcal{O}(E)$ its sheaf of holomorphic sections. In what follows, all the vector bundles on X are assumed Hermitian and holomorphic.

- (1) Let $L \rightarrow X$ be a line bundle such that $L \otimes K_X^\vee$ is **positive**. Then, we have $H^i(X, \mathcal{O}(L)) = 0$ for $i > 0$.
- (2) Let $L \rightarrow X$ be a positive line bundle and $E \rightarrow X$ a vector bundle. Then, for all sufficiently large m , we have:

$$H^i(X, \mathcal{O}(L^m \otimes E)) = 0, \quad \text{for } i > 0$$

- (1) *Proof.* We denote by $\Theta_{L \otimes K_X^\vee} = F_{ij}^{L \otimes K_X^\vee} dz^i \wedge d\bar{z}^j$ the curvature of $L \otimes K_X^\vee$, and by $\lambda(\Theta_{L \otimes K_X^\vee})$ the endomorphism $\varepsilon(d\bar{z}^i)\iota(dz^j)F_{ij}^{L \otimes K_X^\vee}$ of $\wedge^n T^\vee X^{0,1}$; this is just $\varepsilon(d\bar{z}^i)\iota(dz^j)\Theta_{L \otimes K_X^\vee}(\partial_j, \partial_i)$. Then, the Bochner-Kodaira formula is expressible as:

$$\bar{\partial}_L \bar{\partial}_L^* + \bar{\partial}_L^* \bar{\partial}_L = \Delta^{0,\bullet} + \lambda(\Theta_{L \otimes K_X^\vee})$$

and thus, for any L -valued $(0, i)$ -form α , we get:

$$\begin{aligned} \int_X \langle (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\alpha, \alpha \rangle &= \int_X \underbrace{\langle \nabla^{0,1}\alpha, \nabla^{0,1}\alpha \rangle}_{=\|\nabla^{0,1}\alpha\|^2 \geq 0} + \int_X \langle \lambda(\Theta_{L \otimes K_X^\vee})\alpha, \alpha \rangle \\ &\geq \int_X \langle \lambda(\Theta_{L \otimes K_X^\vee})\alpha, \alpha \rangle \end{aligned}$$

Of course, the **positivity condition** on the line bundle $L \otimes L_X^\vee$ equivalently means (by our previous constructions) that the endomorphism $\lambda(\Theta_{L \otimes K_X^\vee})$ is **positive-definite**, i.e., that $\langle \lambda(\Theta_{L \otimes K_X^\vee})\alpha, \alpha \rangle > 0$ for all $i > 0$ and $\alpha \neq 0$. On the other hand, we recall from the general Hodge theory of elliptic complexes that $H^i(X, \mathcal{O}(L))$ can be identified with the space $\mathcal{H}^{0,i}(X, L)$ of harmonic L -valued $(0, i)$ -forms on X , which must in particular satisfy $\Delta_{\bar{\partial}_L}\alpha = (\bar{\partial}_L \bar{\partial}_L^* + \bar{\partial}_L^* \bar{\partial}_L)\alpha = 0$; but this is impossible as the RHS is always **positive** in this case, so that the LHS can never be zero, meaning that there exists no such harmonic form and thus $H^i(X, \mathcal{O}(L)) = 0$, as desired. \square

- (2) *Proof.* Let us follow the same notation as before, $\lambda(-)$, for the endomorphism induced on the vector bundle by the curvature form, so that the general Bochner-Kodaira formula is:

$$\bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E = \Delta_E^{0,\bullet} + \lambda(\Theta_{E \otimes K^\vee})$$

Now, since L is a line bundle, the previous arguments show that $\Theta_{L^m \otimes E} = \Theta_E + m\Theta_L$; thus, the Bochner-Kodaira formula applied to the vector bundle $L^m \otimes E$ gives us:

$$\bar{\partial}_{L^m E} \bar{\partial}_{L^m E}^\star + \bar{\partial}_{L^m E}^\star \bar{\partial}_{L^m E} = \Delta_{L^m E}^{0,\bullet} + m\lambda(\Theta_L) + (\text{curvature terms independent of } m)$$

so, by again plugging this formula into the above integral computation, we see that for m sufficiently large (here, we are using the compactness of X to see that the curvature terms that are independent of m are **uniformly bounded** on X) the term $m\lambda(\Theta_L)$ dominates the RHS and is **positive**, since L is (i.e., the endomorphism $\lambda(\Theta_L)$ is) so we may again run the exact same argument as above to obtain this vanishing. \square

This result is also important for another reason: it shows that for m sufficiently large, the Euler characteristic of the vector bundle $L^m \otimes E$ (equivalently, its sheaf of holomorphic sections) is given by:

$$\text{Eul}(L^m \otimes E) = \dim H^0(X, \mathcal{O}(L^m \otimes E)) = h^0(L^m \otimes E)$$

Here, the Hirzebruch-Riemann-Roch theorem gives an explicit formula for the Euler number $\text{Eul}(L^m \otimes E)$, and hence also for $h^0(L^m \otimes E)$; thus, it combines an **index theorem** with a **vanishing theorem**.

3. HERMITIAN MANIFOLDS

Definition 3.1. *Hermitian structure on a manifold*

Let (X, g, J) be a Riemannian manifold equipped with an almost-complex structure J . This is called a **Hermitian manifold** if the metric g is **compatible** with the ACS J , i.e., $J^*g = g$.⁸ The induced **real** $(1, 1)$ -form $\omega_J := g(J-, -)$ is called the **fundamental form**. Thus, a **Hermitian manifold** (X, g, ω) is a complex manifold X endowed with a Hermitian structure g ; more generally, an **almost Hermitian manifold** (X, J, g, ω_J) is an **almost-complex manifold** (X, J) with compatible Hermitian metric g and fundamental real $(1, 1)$ -form ω .⁹

Indeed, the fact that ω is **real** follows from the corresponding fiber-wise properties established earlier; while the fact that this is a $(1, 1)$ -**form** follows, equivalently, from the fact that it is J^* -**invariant**, since g is.¹⁰ However, in order to consider a **splitting** $d = \partial + \bar{\partial}$ of the differential, we need to work with an **integrable ACS**.

Thanks to our previous work, we can see that **any complex manifold admits a Hermitian structure**. Indeed, this is a direct analogue of the fact that every smooth manifold admits a Riemannian metric, and it follows directly from that fact: given **any** arbitrary Riemannian metric g on an almost-complex manifold (M, J) , we can construct a new metric

⁸Equivalently, this means that $g(J-, J-) = g(-, -)$, i.e., for any point $x \in X$, the scalar product g_x on $T_x X$ is compatible with the ACS $J_x \in \text{End}(T_x X)$.

⁹In this setting, we can say that **a Hermitian manifold is an almost-Hermitian manifold whose ACS J is integrable**

¹⁰Explicitly, this can be computed by noticing that $J^2 = -\text{id}$, so that $\omega(-, J-) = g(J-, J-) = g(-, -)$ by the J -invariance of the Hermitian metric; consequently, $g(-, J-) = \omega(-, J^2-) = -\omega(-, -)$, whereby $J^*\omega = J^*g(J-, -) = g(J^2-, J-) = -g(-, J-) = \omega(-, -)$, as desired.

\tilde{g} compatible with the ACS J in an obvious manner:

$$\tilde{g} := \frac{1}{2}(g + J^*g), \quad \tilde{g}(u, v) := \frac{1}{2}(g(u, v) + g(Ju, Jv))$$

In fact, choosing a Hermitian metric on an almost-complex manifold (M, J) is equivalent to a choice of a $U(n)$ -structure on M , i.e., a **reduction** of the structure group of the frame bundle of M from $\mathrm{GL}(n, \mathbb{C})$ to the unitary group $U(n)$. In fact, a **unitary frame** on an almost-Hermitian manifold is a complex linear frame that is **orthonormal** with respect to any Hermitian frame. Consequently, this result simply says that the **unitary frame bundle** of M is the **principal $U(n)$ -bundle** in all unitary frames.

Locally, the Hermitian form is expressible as $h = h_{\alpha\beta}dz^\alpha \otimes dz^\beta = h_{j\bar{k}}dz^j \otimes d\bar{z}^k$: indeed, the **sesquilinearity** of h (as a bilinear form) shows that it can only have mixed $(1, 1)$ -entries in order to locally satisfy $h(\alpha v, \beta w) = \alpha\bar{\beta}h(v, w)$; moreover, upon local examination, it shows that we must actually have $h_{k\bar{j}} = \overline{h_{j\bar{k}}}$. Consequently, the fundamental form ω is expressible

$$\text{in the form } \omega = -\mathrm{Im}(h) = \frac{i}{2}(h - \bar{h})$$

$$\begin{aligned} \omega = -\mathrm{Im} h &= \frac{i}{2} \left(h_{j\bar{k}}dz^j \otimes d\bar{z}^k - \overline{h_{j\bar{k}}dz^j \otimes d\bar{z}^k} \right) = \frac{i}{2} \left(h_{j\bar{k}}dz^j \otimes d\bar{z}^k - h_{k\bar{j}}d\bar{z}^j \otimes dz^k \right) \\ &= \frac{i}{2} \left(h_{j\bar{k}}dz^j \otimes d\bar{z}^k - h_{j\bar{k}}d\bar{z}^k \otimes dz^j \right) = \frac{i}{2} h_{j\bar{k}} \left(dz^j \otimes d\bar{z}^k - d\bar{z}^k \otimes dz^j \right) \\ &= \frac{i}{2} h_{j\bar{k}} dz^j \wedge d\bar{z}^k \end{aligned}$$

where, in the middle step, we **changed the indices** in the summation to make them match; and we used the definition of $v \wedge w := v \otimes w - w \otimes v$. Thus, locally:

$$\omega = \frac{i}{2} h_{j\bar{k}} dz^j \wedge d\bar{z}^k$$

for the fundamental form (in Einstein summation notation). In particular, the fact that g is a Riemannian metric; equivalently, this means that the matrix $(h_{i\bar{j}}(x))$ is a **positive-definite Hermitian matrix**. In particular, as before, we see that since ω is **self-conjugate**, it must arise on $T_{\mathbb{C}}X$ as the **complexification of a real form** on TX . More generally, it is evident that any two of the data (g, J, ω) of the Hermitian structure determines the third, i.e.:

Definition 3.2. (*Almost*) Hermitian structure on a(n almost) complex manifold

Let (X, J) be an (almost) complex manifold. Then, a(n almost) Hermitian structure on X can be specified by any one of the following:

- (i) a Hermitian metric h as above,
- (ii) a Riemannian metric g that preserves the ACS J ($J^*g = g$),¹¹
- (iii) equivalently, an ACS J that is **compatible** with the Riemannian metric g ,
- (iv) a non-degenerate 2-form ω that **preserves** J ($J^*\omega = \omega$) and is **positive-definite** in the sense that $g(u, u) = \omega(u, Ju) > 0$ for all non-zero tangent vectors u .

¹¹Somewhat confusingly, this is sometimes itself called a **Hermitian metric**.

Here, we are observing that a Hermitian metric h on an almost-complex manifold X defines a **Riemannian metric** g on the underlying smooth manifold, which is defined (as noticed above) to be the **real part** of h , i.e., $g = \text{Re}(h) = \frac{1}{2}(h + \bar{h})$. The form g is a **real symmetric bilinear form** on the complexified tangent bundle $T_{\mathbb{C}}X$ and, since it is equal to its own conjugate, we know that it must be the **complexification of a real symmetric bilinear form** on the *real* tangent bundle $T_{\mathbb{R}}X$. Then, the symmetry and positive-definiteness of g on $T_{\mathbb{R}}X$ follow from the corresponding properties of h . Thus, in local holomorphic coordinates:

$$g = \frac{1}{2}h_{j\bar{k}}(dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j)$$

Evidently, this construction means that all three forms h, g, ω **preserve (are compatible with) the almost-complex structure** J , i.e., $J^*h = h$, $J^*g = g$ and $J^*\omega = \omega$ acting on all complex tangent vectors.

Finally, we see that every almost-Hermitian manifold (M, J) also admits a **canonical volume form**, which is precisely the Riemannian volume form determined by its corresponding Riemannian metric g . This is given in terms of the associated $(1, 1)$ -form ω by:

$$d\text{Vol}_M := \frac{\omega^n}{n!} \in \mathcal{A}^{n,n}(M)$$

where $\omega^n := \omega^{\wedge n}$ is the n -fold wedge product of ω with itself. Thus, we see that the volume form is a **real (n, n) -form** on M given (if M is a complex manifold) in local holomorphic coordinates by:

$$d\text{Vol}_M = \left(\frac{i}{2}\right)^n \det(h_{j\bar{k}}) dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$$

where the multiplication by $\left(\frac{i}{2}\right)^n$ guarantees that the expression $\frac{i}{2}dz^k \wedge d\bar{z}^k = dx^k \wedge dy^k$ is real, while the **positive-definiteness** of the Hermitian metric $(h_{j\bar{k}})$ guarantees that this is, indeed, a **positive-definite volume form** because it is everywhere positive.

Remark 3.3. Thus, we arrive at the following conclusion: if X is a complex manifold with a Hermitian metric h , then the real part g of h restricted to TX is a Riemannian metric on X , while the imaginary part ω of h restricted to TX is a $(1, 1)$ -form on X .

In particular, these constructions imply that if X, Y are vector fields on the Hermitian manifold (X, h, J) , then:

$$\omega(X, Y) = -\text{Im } h(X, Y) = -\text{Im } \overline{h(Y, X)} = \text{Im } h(Y, X) = -\omega(Y, X)$$

which confirms that this is a differential form; in particular, we get $g(X, Y) = \omega(X, JY)$. In this setting, the complex manifold X is called **Kähler**, if ω is moreover a **closed** 2-form.

In general, Hermitian manifolds make it quite natural to **translate to the complex setting** (i.e., z, \bar{z}) many of the natural results from Riemannian geometry. For example, we have:

Proposition 3.4. Hermitian Christoffel Symbols of Levi-Civita Connection

Consider the Levi-Civita connection ∇^{LC} on a Hermitian (complex) manifold X as in the above setting. Then, its Christoffel symbols are given by:

$$\boxed{\Gamma_{\alpha\beta}^\gamma = \frac{1}{2}g^{\gamma\delta}(\partial_\beta g_{\alpha\delta} + \partial_\alpha g_{\beta\delta} - \partial_\delta g_{\alpha\beta})}$$

where the indices are allowed to run over $\{1, 2, \dots, n, \bar{1}, \bar{2}, \dots, \bar{n}\}$ - that is, over barred as well as over unbarred indices, for $\partial_i := \frac{\partial}{\partial z^i}$, $\partial_{\bar{i}} := \frac{\partial}{\partial \bar{z}^i}$.

Proof. This is the exact same proof as for the Riemannian case; we provide it here in the interest of completeness. First, recall the definition of the Christoffel symbols:

$$\nabla_\alpha^{\text{LC}} \partial_\beta = \Gamma_{\alpha\beta}^\gamma \partial_\gamma$$

where, again, the indices $\alpha, \beta, \gamma, \delta$ may be barred as well as unbarred. This **Einstein summation** result comes from the property that the connection maps $\nabla_\alpha^{\text{LC}} : TX \rightarrow TX$, so that, for each $\partial_\beta \in TX$, we have $\nabla_\alpha \partial_\beta \in TX$ as well; thus, it belongs to the **span of the basis elements** $\{\partial_\gamma\}$, with coefficient $\Gamma_{\alpha\beta}^\gamma$.

First of all, the **torsion-free** property of the Levi-Civita connection implies that:

$$\Gamma_{\alpha\beta}^\gamma = \Gamma_{\beta\alpha}^\gamma, \quad \forall \alpha, \beta, \gamma$$

Next, the fact that the connection ∇^{LC} is metric-compatible implies:

$$\begin{aligned} \partial_\alpha g_{\beta\gamma} &= g(\nabla_\alpha^{\text{LC}} \partial_\beta, \partial_\gamma) + g(\partial_\beta, \nabla_\alpha^{\text{LC}} \partial_\gamma) = g(\Gamma_{\alpha\beta}^\delta \partial_\delta, \partial_\gamma) + g(\partial_\beta, \Gamma_{\alpha\gamma}^\delta \partial_\delta) = \\ &= \Gamma_{\alpha\beta}^\delta g(\partial_\delta, \partial_\gamma) + \Gamma_{\alpha\gamma}^\delta g(\partial_\beta, \partial_\delta) = \Gamma_{\alpha\beta}^\delta g_{\delta\gamma} + \Gamma_{\alpha\gamma}^\delta g_{\beta\delta} \end{aligned}$$

where the last part just comes from the definition of the components of the metric tensor, as $g_{\alpha\beta} := g(\partial_\alpha, \partial_\beta)$. Now, then, we may write out our expresions as:

$$(3.0.1) \quad \partial_\alpha g_{\beta\delta} = \Gamma_{\alpha\beta}^\gamma g_{\gamma\delta} + \Gamma_{\alpha\delta}^\gamma g_{\beta\gamma}$$

$$(3.0.2) \quad \partial_\beta g_{\alpha\delta} = \Gamma_{\alpha\beta}^\gamma g_{\gamma\delta} + \Gamma_{\beta\delta}^\gamma g_{\alpha\gamma}$$

$$(3.0.3) \quad \partial_\delta g_{\alpha\beta} = \Gamma_{\alpha\delta}^\gamma g_{\gamma\beta} + \Gamma_{\beta\delta}^\gamma g_{\alpha\gamma}$$

Thus, subtracting (5) + (6) - (7), we see that:

$$\partial_\beta g_{\alpha\delta} + \partial_\alpha g_{\beta\delta} - \partial_\delta g_{\alpha\beta} = 2\Gamma_{\alpha\beta}^\gamma g_{\gamma\delta}$$

Now, dividing both sides by 2 and multiplying by the inverse metric $g^{\gamma\delta}$ gives the proof; evidently, the resulting expression for the Christoffel symbol is **torsion-free**. \square

In particular, if the metric g is J -compatible, i.e., $g(X, Y) = \Omega(X, JY)$ for the associated 2-form, we may recall that the **J - compatible metric only has mixed terms**, that is, $g_{ij} = g_{i\bar{j}} = 0$ (and same for the inverse metric $g^{i\bar{j}}$), we obtain the expressions:

$$\tilde{\Gamma}_{ij}^k = \frac{1}{2}g^{k\bar{\ell}}(\partial_i g_{\bar{\ell}j} + \partial_j g_{\bar{\ell}i}), \quad \tilde{\Gamma}_{i\bar{j}}^k = \frac{1}{2}g^{k\bar{\ell}}(\partial_{\bar{j}} g_{\bar{\ell}i} - \partial_{\bar{i}} g_{\bar{\ell}j})$$

because all the other terms fall out. In addition, we see that:

$$\tilde{\Gamma}_{\bar{k}\bar{\ell}}^i = \frac{1}{2} g^{i\bar{j}} (\partial_{\bar{\ell}} g_{\bar{k}\bar{j}} + \partial_{\bar{k}} g_{\bar{j}\bar{\ell}} - \partial_{\bar{j}} g_{\bar{k}\bar{\ell}}) = 0$$

because all the double-barred terms also vanish. That is, we get:

$$\tilde{\Gamma}_{ij}^k = \tilde{\Gamma}_{ji}^k = \frac{1}{2} g^{k\bar{\ell}} (\partial_i g_{\bar{\ell}j} + \partial_j g_{\bar{\ell}i}), \quad \tilde{\Gamma}_{i\bar{j}}^k = \tilde{\Gamma}_{\bar{j}i}^k = \frac{1}{2} g^{k\bar{\ell}} (\partial_{\bar{j}} g_{\bar{\ell}i} - \partial_i g_{\bar{\ell}i}), \quad \tilde{\Gamma}_{\bar{i}\bar{j}}^k = 0$$

which, in particular, means that we may use the expressions of the Christoffel symbols of the Chern connection, whereby our expression is transformed into:

$$\boxed{\tilde{\Gamma}_{ij}^k = \frac{1}{2} (\Gamma_{ij}^k + \Gamma_{ji}^k), \quad \tilde{\Gamma}_{i\bar{j}}^k = \frac{1}{2} (\Gamma_{i\bar{j}}^k + \Gamma_{\bar{j}i}^k)}$$

Consequently, noticing the symmetry in these expressions, we see, for example, that:

$$g_{k\bar{\ell}} \tilde{\Gamma}_{i\bar{j}}^k = -g_{k\bar{j}} \tilde{\Gamma}_{i\bar{\ell}}^k, \quad g_{\bar{k}\ell} \tilde{\Gamma}_{i\bar{j}}^k = -g_{\bar{k}i} \tilde{\Gamma}_{\ell\bar{j}}^k$$

In particular, our observation from earlier shows that since $g_{ij} = g_{i\bar{j}} = 0$, we get ω as an $(1, 1)$ -form with expression:

$$\begin{aligned} \omega &= ig_{j\bar{j}} dz^j \wedge d\bar{z}^k, \quad d\omega = \partial\omega + \bar{\partial}\omega \\ &= i \left(\underbrace{\partial_\ell g_{j\bar{k}} dz^\ell \wedge dz^j \wedge d\bar{z}^k}_{=\partial\omega} + \underbrace{\partial_{\bar{\ell}} g_{j\bar{k}} d\bar{z}^\ell \wedge dz^j \wedge d\bar{z}^k}_{=\bar{\partial}\omega} \right) \end{aligned}$$

Now, we may reorganize the terms inside $\partial\omega$ to recover:

$$\partial_\ell g_{j\bar{k}} dz^\ell \wedge dz^j \wedge d\bar{z}^k = \sum_{j<\ell} (\partial_j g_{\ell\bar{k}} - \partial_\ell g_{j\bar{k}}) dz^j \wedge dz^\ell \wedge d\bar{z}^k$$

Similarly, for the computation of $\bar{\partial}\omega$, we are left with:

$$\partial_{\bar{\ell}} g_{j\bar{k}} d\bar{z}^\ell \wedge dz^j \wedge d\bar{z}^k = \sum_{k<\ell} (\partial_{\bar{\ell}} g_{j\bar{k}} - \partial_{\bar{k}} g_{j\bar{\ell}}) d\bar{z}^\ell \wedge dz^j \wedge d\bar{z}^k$$

Next, the observations from above allows us to recall the identities:

$$\partial_j g_{\ell\bar{k}} - \partial_\ell g_{j\bar{k}} = 2g_{\ell\bar{m}} \tilde{\Gamma}_{j\bar{k}}^{\bar{m}}, \quad \partial_{\bar{\ell}} g_{j\bar{k}} - \partial_{\bar{k}} g_{j\bar{\ell}} = -2g_{m\bar{\ell}} \tilde{\Gamma}_{j\bar{k}}^m$$

Thus, we get, recalling that $\partial\omega$ is of type $(2, 1)$ and $\bar{\partial}\omega$ is of type $(1, 2)$:

$$(\partial\omega)_{j\ell\bar{k}} = 2ig_{\ell\bar{m}} \tilde{\Gamma}_{j\bar{k}}^{\bar{m}}, \quad (\bar{\partial}\omega)_{j\bar{\ell}\bar{k}} = -2ig_{m\bar{\ell}} \tilde{\Gamma}_{j\bar{k}}^m$$

In particular, this shows that:

$$\boxed{d\omega = 0 \iff \tilde{\Gamma}_{j\bar{k}}^{\bar{m}} = \tilde{\Gamma}_{j\bar{k}}^m = 0}$$

Conversely, we may assume that the compatibility condition $\tilde{\Gamma}_{j\bar{k}}^{\bar{m}} = \tilde{\Gamma}_{j\bar{k}}^m = 0$ is true; then, we obtain, as observed above:

$$\partial_j g_{\ell\bar{k}} = \partial_\ell g_{j\bar{k}}, \quad \partial_{\bar{\ell}} g_{j\bar{k}} = \partial_{\bar{k}} g_{j\bar{\ell}}$$

that is, we may **freely commute indices** inside the partials. Thus, we get:

where, in the last part, we used the **inverse metric** to simplify the terms. Thus, we are left with:

$$d\omega = 0 \iff \tilde{\Gamma}_{ij}^k = \tilde{\Gamma}_{ji}^k = g^{k\bar{\ell}} \partial_i g_{\bar{\ell}j}, \quad \tilde{\Gamma}_{i\bar{j}}^{\bar{k}} = \tilde{\Gamma}_{\bar{j}\bar{i}}^{\bar{k}} = g^{\bar{k}\ell} \partial_{\bar{i}} g_{\ell\bar{j}}$$

Of course, just one of these results is necessary; the second part will immediately follow upon taking conjugates. For example, we recall that the Wirtinger derivative acts as $\partial_{\bar{i}} \bar{f} = \overline{\partial_i f}$, and we also recall that $g_{i\bar{j}} = \overline{g_{\bar{i}j}}$, by definition. Thus, we get:

$$\overline{g^{k\bar{\ell}} \partial_i g_{\bar{\ell}j}} = g^{\bar{k}\ell} \overline{\partial_i g_{\bar{\ell}j}} = g^{\bar{k}\ell} \partial_{\bar{i}} g_{\ell\bar{j}}$$

This shows that the two results are completely analogous.

Putting these results together, we now see that:

Theorem 3.5. Equivalent conditions for Kähler manifold

Let (X, J, h) be a Hermitian complex manifold with associated Riemannian metric $g := \text{Re } h$ and associated $(1, 1)$ -form $\omega := -\text{Im } h$. Moreover, let ∇^{LC} be the Levi-Civita connection associated to the Riemannian metric g , and let ∇^C be the Chern connection on the holomorphic line bundle $T^{1,0}X \rightarrow X$. Then, the following conditions are equivalent:

- (i) the $(1, 1)$ -form ω is **closed**, i.e., $d\omega = 0$; equivalently, it is **covariantly constant**, i.e., $\nabla^{LC}\omega = 0$.
- (ii) the Levi-Civita connection preserves the bundles $T^{1,0}X$ and $T^{0,1}X$, i.e., it preserves the complex structure: $\nabla^{LC}J = 0$ is parallel wrt. ∇^{LC} ; equivalently, it coincides with the Chern connection ∇^C on the holomorphic vector bundle $T^{1,0}X \rightarrow X$;

Moreover, if these any of these equivalent conditions holds, we say that X is a **Kähler manifold**.

Proof. Using the properties of the metric, we see that for any vector fields X, Y, Z , we have $(\nabla_X J)Y = \nabla_X(JY) + J(\nabla_X Z)$ (by the properties of the ACS J) Moreover, the *metric compatibility* of the Levi-Civita connection ∇ means $X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$, so:

$$\begin{aligned} \langle (\nabla_X J)Y, Z \rangle &= \langle (\nabla J)(X, Y), Z \rangle = \langle \nabla_X(JY), Z \rangle + \langle J(\nabla_X Y), Z \rangle \\ &= X\langle JY, Z \rangle - \langle JY, \nabla_X Z \rangle - \omega(\nabla_X Y, Z) \\ &= -X\omega(Y, Z) + \omega(Y, \nabla_X Z) - \omega(\nabla_X Y, Z) \\ &= -(\nabla_X \omega)(Y, Z) = -(\nabla \omega)(X, Y, Z) \end{aligned}$$

i.e., $\nabla J \equiv 0$ is equivalent to $\nabla \omega \equiv 0$ (covariantly constant), i.e., $d\omega \equiv 0$, as desired. Importantly, we see that since **the Levi-Civita connection is torsion-free** (i.e., $\nabla_X Y - \nabla_Y X = [X, Y]$) this means that the RHS of the equation is antisymmetric over $\{X, Y, Z\}$, i.e., it is precisely $(d\omega)(X, Y, Z)$, which proves the claimed equivalence.

In the other direction, the formulas for the Levi-Civita connection give:

$$\begin{aligned} 2\langle \nabla_j \partial_k, \partial_\ell \rangle &= 0 \\ 2\langle \nabla_{\bar{j}} \partial_k, \partial_\ell \rangle &= \partial_k \langle \partial_\ell, \partial_{\bar{j}} \rangle - \partial_\ell \langle \partial_{\bar{j}}, \partial_k \rangle = -id\omega(\partial_k, \partial_\ell, \partial_{\bar{j}}) \\ 2\langle \nabla_{\bar{j}} \partial_k, \partial_{\bar{\ell}} \rangle &= \partial_k \langle \partial_{\bar{\ell}}, \partial_{\bar{j}} \rangle - \partial_{\bar{\ell}} \langle \partial_{\bar{j}}, \partial_k \rangle = -id\omega(\partial_k, \partial_{\bar{\ell}}, \partial_{\bar{j}}) \end{aligned}$$

Consequently, we see that if $d\omega = 0$, then the Levi-Civita connection preserves the sections of $T^{1,0}X$ and induces the canonical holomorphic covariant derivative on the holomorphic vector bundle $T^{1,0}X \rightarrow X$; i.e., it coincides with the Chern connection on it, as desired. \square

Using these results, we may now show that:

Proposition 3.6. Christoffel symbols of Levi-Civita through torsion

Let (X, J) be a complex manifold with an ACS, J . Let g be a metric compatible with J and ω the associated $(1, 1)$ -form to g , given by

$$\omega = ig_{\bar{k}j} dz^j \wedge d\bar{z}^k.$$

Then, when working in complex coordinates z^i with indices i, j, k , we have:

$$(3.0.4) \quad \boxed{\nabla_k^{\text{LC}} V^i = \partial_k V^i + \Gamma_{k\ell}^i V^\ell - \frac{T_{k\ell}^i}{2} V^\ell - \frac{g^{i\bar{j}}}{2} \bar{T}_{k\bar{j}\ell} V^\ell}$$

$$(3.0.5) \quad \boxed{\nabla_{\bar{k}}^{\text{LC}} V^i = \partial_{\bar{k}} V^i + \frac{1}{2} g^{i\bar{m}} \bar{T}_{\ell\bar{k}\bar{m}} V^\ell}$$

acting on a vector field $V = V^i \partial_i + V^{\bar{i}} \partial_{\bar{i}}$, where we have defined $\partial_i := \frac{\partial}{\partial z^i}$ and $\partial_{\bar{i}} := \frac{\partial}{\partial \bar{z}^i}$. Here, we use the notation

$$\Gamma_{ij}^k = g^{k\bar{p}} \partial_i g_{\bar{p}j}, \quad T := i\partial\omega, \quad T = \frac{1}{2} T_{\bar{k}ij} dz^i \wedge dz^j \wedge d\bar{z}^k \quad T_{\bar{k}ij} = \partial_i g_{\bar{k}j} - \partial_j g_{\bar{k}i}.$$

for the Christoffel symbols of the **Chern connection** and the torsion tensors; similarly, the **barred torsion** T is given by:

$$\bar{T} := -i\bar{\partial}\omega, \quad \bar{T} = \frac{1}{2} \bar{T}_{\bar{k}\bar{j}\bar{m}} d\bar{z}^m \wedge d\bar{z}^j \wedge d\bar{z}^k, \quad \bar{T}_{\bar{k}\bar{j}\bar{m}} = \partial_{\bar{j}} g_{\bar{m}k} - \partial_{\bar{m}} g_{\bar{j}k}$$

by taking the conjugate of the above relations.

Proof. For the first one, we should first lower the indices of the first component $T_{k\ell}^i = g^{i\bar{j}} T_{\bar{j}k\ell}$, as well as for the Christoffel symbol $\Gamma_{k\ell}^i = g^{i\bar{j}} \partial_k g_{\bar{j}\ell}$:

$$\begin{aligned} \nabla_k^{\text{LC}} V^i &= \partial_k V^i + g^{i\bar{j}} \left(\partial_k g_{\bar{j}\ell} V^\ell - \frac{1}{2} (T_{\bar{j}k\ell} V^\ell + \bar{T}_{\bar{k}\bar{j}\ell} V^\ell) \right) \\ \nabla_{\bar{k}}^{\text{LC}} V^i &= \partial_{\bar{k}} V^i + \frac{1}{2} g^{i\bar{m}} \bar{T}_{\ell\bar{k}\bar{m}} V^\ell \end{aligned}$$

Into this, we may now directly substitute the following expressions for the torsion components:

$$T_{\bar{j}k\ell} = \partial_k g_{\bar{j}\ell} - \partial_\ell g_{\bar{j}k}, \quad \bar{T}_{\bar{k}\bar{j}\ell} = \partial_{\bar{j}} g_{\bar{k}\ell} - \partial_\ell g_{\bar{j}\ell}, \quad \bar{T}_{\bar{k}\bar{m}\ell} = \partial_{\bar{k}} g_{\bar{m}\ell} - \partial_\ell g_{\bar{k}\ell}$$

Thus, our first expression becomes now:

$$\begin{aligned} \nabla_k^{\text{LC}} V^i &= \partial_k V^i + g^{i\bar{j}} \left(\partial_k g_{\bar{j}\ell} V^\ell - \frac{1}{2} [(\partial_k g_{\bar{j}\ell} - \partial_\ell g_{\bar{j}k}) V^\ell + (\partial_{\bar{j}} g_{\bar{k}\ell} - \partial_\ell g_{\bar{j}\ell}) V^\ell] \right) = \\ &= \partial_k V^i + \frac{1}{2} g^{i\bar{j}} (\partial_k g_{\bar{j}\ell} + \partial_\ell g_{\bar{j}k}) V^\ell + \frac{1}{2} g^{i\bar{j}} (\partial_{\bar{j}} g_{\bar{k}\ell} - \partial_\ell g_{\bar{j}\ell}) V^\ell \end{aligned}$$

Whereas the second one likewise becomes:

$$\nabla_{\bar{k}}^{\text{LC}} V^i = \partial_{\bar{k}} V^i + \frac{1}{2} g^{i\bar{m}} (\partial_{\bar{k}} g_{\bar{m}\ell} - \partial_{\bar{m}} g_{\bar{k}\ell}) V^\ell$$

Now, we may denote the Christoffel symbols of the Levi-Civita connection on X by $\tilde{\Gamma}_{kj}^i$ to avoid confusion; by the previous lemma, these are given by:

$$\tilde{\Gamma}_{ij}^k = \frac{1}{2} g^{k\bar{\ell}} (\partial_i g_{\bar{\ell}j} + \partial_j g_{\bar{\ell}i}), \quad \tilde{\Gamma}_{i\bar{j}}^k = \frac{1}{2} g^{k\bar{\ell}} (\partial_{\bar{j}} g_{\bar{\ell}i} - \partial_{\bar{i}} g_{\bar{\ell}j}), \quad \tilde{\Gamma}_{\bar{i}\bar{j}}^k = 0$$

Now, having proved the lemma, we again recall the following modification of the differentiation using the connection $\nabla^{\text{LC}} = d + \tilde{\Gamma}$, as follows:

$$\nabla_\mu^{\text{LC}} V^i = \partial_\mu V^i + \tilde{\Gamma}_{\mu\alpha}^i V^\alpha$$

Thus, in particular, we may focus on the cases $\mu = k, \mu = \bar{k}$ respectively, and break up the contraction of α into barred and unbarred components as follows:

$$\begin{aligned} \nabla_k^{\text{LC}} V^i &= \partial_k V^i + \tilde{\Gamma}_{k\ell}^i V^\ell + \tilde{\Gamma}_{\bar{k}\bar{\ell}}^i V^{\bar{\ell}} = \partial_k V^i + \frac{1}{2} g^{i\bar{j}} (\partial_k g_{\bar{j}\ell} + \partial_\ell g_{\bar{j}k}) V^\ell + \frac{1}{2} g^{i\bar{j}} (\partial_{\bar{k}} g_{\bar{j}k} - \partial_{\bar{j}} g_{\bar{k}k}) V^{\bar{\ell}} \\ \nabla_{\bar{k}}^{\text{LC}} V^i &= \partial_{\bar{k}} V^i + \tilde{\Gamma}_{\bar{k}\ell}^i V^\ell + \underbrace{\tilde{\Gamma}_{\bar{k}\bar{\ell}}^i V^{\bar{\ell}}}_{=0} = \partial_{\bar{k}} V^i + \frac{1}{2} g^{i\bar{m}} (\partial_{\bar{k}} g_{\bar{m}\ell} - \partial_{\bar{m}} g_{\bar{k}\ell}) V^\ell \end{aligned}$$

These are exactly our desired expressions; the proof is complete. \square

Finally, we will now see how to define the **Bismut connection**; its properties will become apparent at a later point. For now, we have:

Proposition 3.7. Differentiating with the Bismut connection ∇^+

Let (X, J) be a complex manifold with an ACS, J . Let g be a metric compatible with J and ω the associated $(1, 1)$ -form to g , given by $\omega = i g_{\bar{k}j} dz^j \wedge d\bar{z}^k$ as above. Moreover, we may let H be the 3-form given by $H = i(\partial - \bar{\partial})\omega$, that is:

$$H = \frac{1}{3!} H_{\mu\nu\alpha} dx^\mu \wedge dx^\nu \wedge dx^\alpha$$

We may also raise indices for this, by defining $H^\nu{}_{\mu\alpha} = g^{\nu\beta} H_{\beta\mu\alpha}$. Using these constructions, we define the **Bismut connection**, denoted ∇^+ or ∇^B , by:

$$\nabla^+ := \nabla^{\text{LC}} + \frac{1}{2} g^{-1} H,$$

where ∇^{LC} is the Levi-Civita connection of (X, g) and the **contraction** becomes:

$$\nabla_\mu^+ V^\nu = \nabla_\mu^{\text{LC}} V^\nu + \frac{1}{2} H^\nu{}_{\mu\alpha} V^\alpha,$$

Then, when working in complex coordinates z^i with indices i, j, k , we have:

$$(3.0.6) \quad \boxed{\nabla_k^+ V^i = \partial_k V^i + (\Gamma_{k\ell}^i - T_{k\ell}^i) V^\ell}$$

$$(3.0.7) \quad \boxed{\nabla_{\bar{k}}^+ V^i = \partial_{\bar{k}} V^i + g^{i\bar{m}} \bar{T}_{\ell\bar{k}\bar{m}} V^\ell}$$

acting on a vector field $V = V^i \partial_i + V^{\bar{i}} \partial_{\bar{i}}$, where we have defined $\partial_i := \frac{\partial}{\partial z^i}$ and $\partial_{\bar{i}} := \frac{\partial}{\partial \bar{z}^i}$.

Proof. To see this result, we should first of all recall how the Levi-Civita connection ∇^{LC} acts on vector fields, and from this we will be able to deduce the desired expression after adding in the action of the $\frac{1}{2}g^{-1}H$ term. Previously, we established the action of the Levi-Civita term ∇^{LC} as:

$$\begin{aligned}\nabla_k^{\text{LC}}V^i &= \partial_k V^i + \Gamma_{k\ell}^i V^\ell - \frac{T_{k\ell}^i}{2} V^\ell - \frac{g^{i\bar{j}}}{2} \bar{T}_{k\bar{j}\ell} V^\ell \\ \nabla_{\bar{k}}^{\text{LC}}V^i &= \partial_{\bar{k}} V^i + \frac{g^{i\bar{m}}}{2} \bar{T}_{\ell\bar{k}\bar{m}} V^\ell\end{aligned}$$

Building on these results, we recall the above general expression for ∇_μ^+ as:

$$\nabla_\mu^+ V^\nu = \nabla_\mu^{\text{LC}} V^\nu + \frac{1}{2} H^\nu{}_{\mu\alpha} V^\alpha$$

thus, setting $\{\mu, \nu\} = \{k, i\}, \{\bar{k}, i\}$, we see that:

$$\begin{aligned}\nabla_k^+ V^i &= \nabla_k^{\text{LC}} V^i + \frac{1}{2} H^i{}_{k\alpha} V^\alpha \\ \nabla_{\bar{k}}^+ V^i &= \nabla_{\bar{k}}^{\text{LC}} V^i + \frac{1}{2} H^i{}_{\bar{k}\alpha} V^\alpha\end{aligned}$$

Thus, using the equations (3),(4), we see now:

$$\begin{aligned}\nabla_k^+ V^i &= \partial_k V^i + \Gamma_{k\ell}^i V^\ell - \frac{1}{2} g^{i\bar{j}} (\bar{T}_{\bar{j}k\ell} V^\ell + \bar{T}_{k\bar{j}\ell} V^\ell) + \frac{1}{2} H^i{}_{k\alpha} V^\alpha \\ \nabla_{\bar{k}}^+ V^i &= \partial_{\bar{k}} V^i + \frac{1}{2} g^{i\bar{m}} \bar{T}_{\ell\bar{k}\bar{m}} V^\ell + \frac{1}{2} H^i{}_{\bar{k}\alpha} V^\alpha\end{aligned}$$

Now, we may subtract these expressions from the claimed expressions (1),(2) for the action of $\nabla_k^+, \nabla_{\bar{k}}^+$, to obtain the following expressions to be proved, after first lowering the indices on the $H^i{}_{k\alpha}$. Remember that this is done by $H^\nu{}_{\mu\alpha} = g^{\nu\beta} H_{\beta\mu\alpha}$, thus, we get:

$$\begin{aligned}g^{i\bar{j}} H_{\bar{j}k\alpha} V^\alpha &= H^i{}_{k\alpha} V^\alpha = -g^{i\bar{j}} (\bar{T}_{\bar{j}k\ell} V^\ell - T_{\bar{j}k\ell} V^\ell), & \iff H_{\bar{j}k\alpha} V^\alpha &= T_{\bar{j}k\ell} V^\ell - \bar{T}_{\bar{k}\bar{j}\ell} V^\ell \\ g^{i\bar{m}} H_{\bar{m}\bar{k}\alpha} V^\alpha &= H^i{}_{\bar{k}\alpha} V^\alpha = g^{i\bar{m}} \bar{T}_{\ell\bar{k}\bar{m}} V^\ell, & \iff H_{\bar{m}\bar{k}\alpha} V^\alpha &= \bar{T}_{\ell\bar{k}\bar{m}} V^\ell\end{aligned}$$

by cancelling out the inverse metrics off both sides. Here, the index α is different from all other barred and unbarred indices, in the sense that **it is valued in the entire tangent bundle**, as opposed to just the $(1, 0)$ – or the $(0, 1)$ –piece; that is, it runs through both barred and unbarred values. By decomposing these, we may consider the summation separately over ℓ and over $\bar{\ell}$; these will be done as:

$$H_{\bar{j}k\alpha} V^\alpha = H_{\bar{j}k\ell} V^\ell + H_{\bar{j}\bar{k}\ell} V^\ell, \quad H_{\bar{m}\bar{k}\alpha} V^\alpha = H_{\bar{m}\bar{k}\ell} V^\ell + H_{\bar{m}\bar{k}\bar{\ell}} V^\ell$$

Truly, we are almost done: we need only recall here the following definitions:

$$T := i\partial\omega, \quad \bar{T} := -i\bar{\partial}\omega, \quad H := i(\partial - \bar{\partial})\omega = T + \bar{T}$$

In particular, by just checking explicitly the components of the torsion tensors T, \bar{T} , since these are generated by partial derivatives of the metric $g_{i\bar{j}}$, we observe that **we cannot have any three barred terms**: this just encodes the fact that all double-barred entries of the Hermitian metric g are $= 0$. Thus, we correspondingly must have: $H_{\bar{m}\bar{k}\bar{\ell}} = 0$, as well.

More specifically, the explicit expressions for the components of T, \bar{T} given above show that T has two-unbarred-one-barred, and \bar{T} has two-barred-one-unbarred. Thus, the components

of $H = T + \bar{T}$ become:

$$H_{\bar{j}k\ell} = (T + \bar{T})_{\bar{j}k\ell} = T_{\bar{j}k\ell} + \underbrace{\bar{T}_{\bar{j}k\ell}}_{=0} = T_{\bar{j}k\ell}, \quad H_{\bar{j}\bar{k}\ell} = \bar{T}_{\bar{j}\bar{k}\ell} = -\bar{T}_{\bar{k}\bar{j}\ell}$$

$$H_{\bar{m}\bar{k}\ell} = 0, \quad H_{\bar{m}\bar{k}\ell} = (T + \bar{T})_{\bar{m}\bar{k}\ell} = \underbrace{T_{\bar{m}\bar{k}\ell}}_{=0} + \bar{T}_{\bar{m}\bar{k}\ell} = \bar{T}_{\bar{m}\bar{k}\ell}$$

Thus, we may substitute these into the above equation to obtain:

$$H_{\bar{j}k\alpha} V^\alpha = \bar{T}_{\bar{k}\bar{j}\ell} V^\ell - T_{\bar{j}k\ell} V^\ell, \quad H_{\bar{m}\bar{k}\alpha} V^\alpha = \bar{T}_{\bar{m}\bar{k}\ell} V^\ell$$

exactly as desired. This completes the proof of our desired result. \square

Another reason why Hermitian manifolds are useful is because they enable us to produce a **variation of the divergence theorem**, given by the following result:

Proposition 3.8. The Divergence Theorem on Hermitian manifolds

Let (X, ω) be a compact n -dimensional Hermitian (complex) manifold. Consider a section of the holomorphic tangent bundle $T^{1,0}X$, expressed as:

$$V = V^i \frac{\partial}{\partial z^i} \equiv V^i \partial_i$$

Moreover, denote the Chern connection on ω by ∇ , and use the following torsion conventions:

$$T_{\bar{k}ij} = \partial_i g_{\bar{k}j} - \partial_j g_{\bar{k}i}, \quad \tau_j = g^{i\bar{k}} T_{\bar{k}ij},$$

In this setting, we obtain the following version of the Divergence Theorem:

$$\int_X \nabla_i V^i \omega^n = \int_X \tau_i V^i \omega^n,$$

where we are using the metric convention $\omega = i g_{\bar{k}j} dz^j \wedge d\bar{z}^k$.

Proof. To see this, we first recall the following expression for the n -th wedge power of ω , given by:

$$\omega^n := n! (\det g_{\bar{k}j}) i dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n$$

Now, consider the **interior product of forms**, given by $i_V : \Omega^p(X) \rightarrow \Omega^{p-1}(X)$; by Stokes' Theorem, since $\omega^n \in \Omega^{2n}(X)$ is a top form, so that $i_V \omega^n \in \Omega^{2n-1}(X)$ is a top form in ∂X , we see that **Stokes' Theorem** gives:

$$\int_X d(i_V \omega^n) = \int_{\partial X} i_V \omega^n = 0$$

where the second expression is seen to be 0, since the manifold is assumed boundaryless: $\partial X = \emptyset$. Here, we recall the expression of $i_V \omega^n$: for a general p -form α and a general q -form β , we know that the interior product i_V acts on their wedge $\alpha \wedge \beta$ as:

$$i_V(\alpha \wedge \beta) = (i_V \alpha) \wedge \beta + (-1)^p \alpha \wedge (i_V \beta)$$

Now, we have $\omega^n = \underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_n$, so that, inductively, we see that, for $\alpha_k := i_V(\omega^k)$, a $\deg = 2k - 1$ form, we get, observing also that each ω^k is a $2k$ -form, thus, we always get a $+$ factor upon adding the interior products:

$$\alpha_k := i_V(\omega^k) = i_V(\omega \wedge \omega^{k-1}) = (i_V \omega) \wedge \omega^{k-1} + \omega \wedge (i_V \omega^{k-1}) = \omega^{k-1} \wedge \alpha_1 + \omega \wedge \alpha_{k-1}$$

$$\begin{aligned}\alpha_n - \omega \wedge \alpha_{n-1} &= \omega^{n-1} \wedge \alpha_1 \\ \alpha_{n-1} - \omega \wedge \alpha_{n-2} &= \omega^{n-2} \wedge \alpha_1 \implies \omega \wedge \alpha_{n-1} - \omega^2 \wedge \alpha_{n-2} = \omega^{n-1} \wedge \alpha_1\end{aligned}$$

Here, we wedged both sides by ω , and observed that ω is a 2-form, thus, we may commute it with every wedge. Thus, inductively, we get the formula:

$$\omega^{n-k} \wedge \alpha_k - \omega^{n-k+1} \wedge \alpha_{k-1} = \omega^{n-1} \wedge \alpha_1$$

Thus, adding all the terms together, we may telescope this to:

$$\begin{aligned}\alpha_n - \omega^{n-1} \wedge \alpha_1 &= (n-1)\omega^{n-1} \wedge \alpha_1, \implies \\ i_V \omega^n &= n\omega^{n-1} \wedge i_V \omega\end{aligned}$$

Now, then, for the differential of this, we write:

$$d(i_V \omega^n) = d(n\omega^{n-1} \wedge i_V \omega) = n(d\omega^{n-1} \wedge i_V \omega + \omega^{n-1} \wedge d(i_V \omega))$$

Then, the same procedure and the same arguments as before show us that:

$$d\omega^{n-1} = (n-1)\omega^{n-2} \wedge d\omega$$

this happens because d has the same action on the wedge as i_V does. Additionally, we know from Math 230A that there is a very explicit expression of the contraction of a 2-form ω , as:

$$i_V \omega(\bar{V}) = \omega(V, \bar{V}) = \omega_{i\bar{j}} V^i \bar{V}^j$$

On the other hand, we also observe here the important expression for the **Torsion**, established in the class notes:

$$T = i\partial\omega, \quad T = \frac{1}{2} T_{\bar{k}jm} dz^m \wedge dz^j \wedge d\bar{z}^k, \quad T_{\bar{k}jm} = \partial_j g_{\bar{k}m} - \partial_m g_{\bar{k}j}$$

Thus, the expression $T = i\partial\omega$ will now be seen to equal:

$$T = i\partial\omega = n(\partial_p(i_V \omega) dz^p + \partial_{\bar{p}}(i_V \omega) d\bar{z}^p) \wedge \omega^{n-1}$$

On the other hand, we may also obtain now directly the expression, thanks to all our previous computations combined:

$$\begin{aligned}i_V \omega^n &= n!(\det g_{\bar{k}j}) dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \implies \\ \implies i_V \omega^n &= n! \cdot \sum V^i (\det g_{\bar{k}j}) \underbrace{dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^{i-1} \wedge dz^{i+1} \wedge \cdots \wedge d\bar{z}^n}_{dz^i \text{ omitted}}\end{aligned}$$

In particular, we see that this differential $d(i_V \omega^n)$ involves the derivative of the determinant $\det g_{\bar{k}j}$, as follows: knowing how d is defined, we will simply be getting back our dz^i term, and then summing over all the i 's, taking at each step the partial derivatives:

$$\begin{aligned}\partial_i(V^j \det g_{\bar{k}l} \underbrace{dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^{j-1} \wedge dz^{j+1} \wedge \cdots \wedge d\bar{z}^n}_{dz^j \text{ omitted}}) &= \\ &= \partial_i(V^j \det g_{\bar{k}l}) \underbrace{dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^{j-1} \wedge dz^{j+1} \wedge \cdots \wedge d\bar{z}^n \wedge dz^i}_{dz^j \text{ omitted}}$$

Here, we observe the following fact: first of all, if $i \neq j$, we will certainly be getting two instances of the dz^i in the wedge, thus, the result will immediately turn out = 0 (as is always

the case for top forms); thus, the coefficient of our top form will be:

$$\partial_i(V^i \det g_{\bar{k}j}) = \partial_i V^i \det g_{\bar{k}j} + V^i \partial_i(\det g_{\bar{k}j})$$

Finally, we may put everything together to see that this integral $\int_X d(i_V \omega^n) = 0$ also equals:

$$n! \left(\int_X [\partial_i V^i \det g_{\bar{k}j} + V^i \partial_i(\det g_{\bar{k}j})] dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \right)$$

by just expanding the differential.

Importantly, now, we see here that the partials of the $\det g_{\bar{k}j}$ term are exactly the contractions of the Christoffel symbols, that is:

$$\partial_i(\det g_{\bar{k}j}) = \Gamma_{ip}^p, \implies V^i \partial_i(\det g_{\bar{k}j}) = V^i \Gamma_{pi}^p = \Gamma_{pi}^p V^i$$

On the other hand, we may now focus on the expression for $\nabla_i V^i$: using the definition of the **Chern connection**, we are now able to expand this expression as:

$$\nabla_i V^i = \partial_i V^i + \Gamma_{ip}^i V^p$$

Now, then, we may compare these expressions, as follows:

$$\begin{aligned} \nabla_i V^i \omega^n &= n! \nabla_i V^i \det g_{\bar{k}j} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n = \\ &= n!(\partial_i V^i + \Gamma_{ip}^i V^i) \det g_{\bar{k}j} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \end{aligned}$$

Thus, in particular, we see that the $\partial_i V^i$ terms exactly match between the two integrals. On the other hand, we should now look at the Christoffel symbol terms: for these, we recall our result on the *expression of the Christoffel symbols of the Chern Connection in Complex Geometry*, whereby:

$$\Gamma_{ij}^k = g^{k\bar{\ell}} \partial_i g_{\bar{\ell}j}$$

Thus, the fine point enters our discussion here as follows: *the torsion of the form ω is equivalent to the Christoffel symbols not having symmetric lower indices*. We compute these explicitly:

$$\Gamma_{ip}^p - \Gamma_{pi}^p = g^{p\bar{\ell}} \partial_i g_{\bar{\ell}p} - g^{p\bar{\ell}} \partial_p g_{\bar{\ell}i} = g^{p\bar{\ell}} (\partial_i g_{\bar{\ell}p} - \partial_p g_{\bar{\ell}i})$$

Finally, we recall now the previous expressions:

$$\tau_j = g^{i\bar{k}} T_{\bar{k}ij}, \quad T_{\bar{k}ij} = \partial_i g_{\bar{k}j} - \partial_j g_{\bar{k}i}, \implies \tau_j = g^{i\bar{k}} (\partial_i g_{\bar{k}j} - \partial_j g_{\bar{k}i})$$

Thus, we may substitute this into the above expression to see that **the above term is precisely $-\tau_i$** , that is:

$$\tau_i = \Gamma_{pi}^p - \Gamma_{ip}^p$$

Plugging this back into the previous expressions, we are now finally left with:

$$\begin{aligned} \int_X \nabla_i V^i \omega^n &= \int_X (\partial_i V^i + \Gamma_{pi}^p V^i) \omega^n = \int_X \underbrace{(\partial_i V^i - \Gamma_{ip}^p V^i)}_{=d(i_V \omega^n)} \omega^n + \int_X \underbrace{(\Gamma_{pi}^p V^i - \Gamma_{ip}^p V^i)}_{=\tau_i V^i} \omega^n = \\ &= \underbrace{\int_X d(i_V \omega^n)}_{=0} + \int_X \tau_i V^i \omega^n = \int_X \tau_i V^i \omega^n, \end{aligned}$$

Exactly as desired. Thus, the proof is complete, and the claimed result follows. \square

Additionally, the **curvature tensor** may be expressed as:

Proposition 3.9. Index expression of the Curvature on Hermitian manifolds

Let (X, ω) be a Hermitian manifold. Let $\nabla = d + A$ denote the Chern connection on $T^{1,0}X$. Then, we see that the curvature, defined as an endomorphism-valued 2-form given by $F = dA + A \wedge A$, can be written in the following expression:

$$F = R_{\bar{k}\bar{j}}{}^p{}_q dz^j \wedge d\bar{z}^k,$$

Here, we are using the following expression:

$$R_{\bar{k}\bar{j}}{}^p{}_q = -\partial_{\bar{k}}(g^{p\bar{\ell}}\partial_j g_{\bar{\ell}q}).$$

This is sometimes also written $R = \bar{\partial}(g^{-1}\partial g)$.

Proof. To see this, we will use normal coordinates, so that we don't have to worry about Christoffel symbols. Another thing that will be useful in this setting is that **the metric becomes diagonal** and its derivative vanishes, so that the partials of the curvature F are given by:

$$R_{kj}{}^p{}_q = \partial_k \Gamma_{jq}^p - \partial_j \Gamma_{kq}^p$$

since any second-order products of the Christoffel symbols will vanish. Note that this formula so far holds true regardless of whether k, j are "barred" or "unbarred", although we will shortly see that only mixed terms (bar- unbar) will survive this painful differentiation process. Analogously, then, the connection 1-form A will be expressed as:

$$A = \Gamma_{iq}^p dz^i$$

Finally, we recall the expression:

$$\partial_k g^{p\bar{q}} = -g^{p\bar{i}}(\partial_k g_{\bar{i}r})g^{r\bar{q}}$$

This formula was observed previously in class, as the expression for the partials of the inverse metric. It is also easy to derive, upon observing that:

$$\begin{aligned} g^{p\bar{q}} g_{\bar{q}r} = \delta_r^p &\implies 0 = \partial_k(g^{p\bar{q}} g_{\bar{q}r}) = g_{\bar{q}r} \partial_k(g^{p\bar{q}}) + g^{p\bar{q}} \partial_k(g_{\bar{q}r}) \implies \\ &\implies \partial_k(g^{p\bar{q}}) = -g^{p\bar{q}}(\partial_k g_{\bar{q}r})g^{r\bar{q}} = -g^{p\bar{i}}(\partial_k g_{\bar{i}r})g^{p\bar{q}} \end{aligned}$$

Thus, we may cancel out the $g^{p\bar{q}}$ from the last two expressions, to observe that:

$$(\partial_k g_{\bar{q}r})g^{r\bar{q}} = g^{p\bar{i}}(\partial_k g_{\bar{i}r})$$

Now, we go back to our previous claim: that only mixed "bar-unbar" terms will survive. First, it is evident that no two barred terms can survive: since we have seen that if any index on the Christoffel symbol is barred we get = 0 (for the Chern connection) we see, for example, that:

$$R_{\bar{k}\bar{j}}{}^p{}_q = \underbrace{\partial_{\bar{k}}}_{=0} \underbrace{\Gamma_{\bar{j}q}^p}_{=0} - \underbrace{\partial_j}_{=0} \underbrace{\Gamma_{kq}^p}_{=0} = 0$$

Next, let's see why the double unbarred terms will also give = 0. For this purpose, we perform the computation:

$$R_{kj}{}^p{}_q = \partial_k \Gamma_{jq}^p - \partial_j \Gamma_{kq}^p$$

In general, we should recall the formula for the Christoffel symbols in the Complex Geometry setting, nonzero only if i, j are unbarred:

$$\Gamma_{jq}^p = g^{p\bar{\ell}} \partial_j g_{\ell q}, \quad \partial_k \Gamma_{jq}^p = \partial_k (g^{p\bar{\ell}} \partial_j g_{\ell q}) = (\partial_k g^{p\bar{\ell}}) (\partial_j g_{\ell q}) + g^{p\bar{\ell}} (\partial_k \partial_j g_{\ell q})$$

This second equality above here arises because of the product rule. In **normal coordinates**, now, we recall that the derivative of the metric is 0, thus, using the previous formula for the derivative of the inverse metric, we will be left with:

$$\partial_k \Gamma_{jq}^p = (\partial_k g^{p\bar{\ell}}) (\partial_j g_{\ell q}) + g^{p\bar{\ell}} (\partial_k \partial_j g_{\ell q}) = \underbrace{-g^{p\bar{i}} (\partial_k g_{\bar{i}r}) g^{r\bar{q}} (\partial_j g_{\ell q})}_{=0} + g^{p\bar{\ell}} (\partial_k \partial_j g_{\ell q}) = g^{p\bar{\ell}} (\partial_k \partial_j g_{\ell q})$$

Thus, now, our term $R_{kj}{}^p{}_q$ will be recovered as:

$$R_{kj}{}^p{}_q = \partial_k \Gamma_{jq}^p - \partial_j \Gamma_{kq}^p = g^{p\bar{\ell}} (\partial_k \partial_j g_{\ell q}) - g^{p\bar{\ell}} (\partial_j \partial_k g_{\ell q}) = 0$$

since the last two terms are exactly equal.

Finally, we will only have the mixed "bar-unbar" terms to look at. By our previous considerations, we know that these will be given exactly by:

$$R_{\bar{k}j}{}^p{}_q = \partial_{\bar{k}} \Gamma_{jq}^p - \partial_j \Gamma_{\bar{k}q}^p = \underbrace{\partial_{\bar{k}} \Gamma_{jq}^p}_{=0} = \partial_{\bar{k}} (g^{p\bar{\ell}} \partial_j g_{\ell q})$$

Thus, the result has been proved, and the proof is complete.

Note: It looks like I missed a sign here - I don't know where this came from, because I did the calculations really carefully. Maybe from flipping the terms $dz^j \wedge d\bar{z}^k$ in the wedge? \square

4. DIFFERENTIAL FORMS AND DOLBEAULT COHOMOLOGY

Thus, using these tools, we may define, for any compact complex manifold X , the map that associates to each **smooth real 1-form** its $(0, 1)$ -part. This induces a map $i : H^1(X, \mathbb{R}) \hookrightarrow H_{\bar{\partial}}^{0,1}(X)$, from the deRham cohomology to the Dolbeault cohomology. Then, we get:

Proposition 4.1. *Betti and Hodge numbers*

In the above setting of a compact complex manifold X , the map $i : H^1(X, \mathbb{R}) \hookrightarrow H_{\bar{\partial}}^{0,1}(X)$ is an injection which shows that $b_1(X) \leq 2h^{0,1}(X)$ for the first Betti number.

Proof. To see this, we first observe that for 1-form $\eta \in \Omega^1(X; \mathbb{R})$, we know that it comes from $\mathcal{A}_{\mathbb{R}}^1(X) = \mathcal{A}_{\mathbb{R}}^{1,0}(X) \oplus \mathcal{A}_{\mathbb{R}}^{0,1}(X)$, thus, we may decompose as $\eta = \eta^{1,0} + \eta^{0,1}$ with the respective pieces. This decomposition is the same as the one over \mathbb{C} : for a form $\eta = f_j dx^j + g_k dy^k$ over the reals, we may write, for each k , $dz^k = dx^k + idy^k$ while $d\bar{z}^k = dx^k - idy^k$, whereby $dx^j = \frac{dz^j + d\bar{z}^j}{2}$ while $dy^k = i \frac{d\bar{z}^k - dz^k}{2}$; this gives us:

$$\eta = \frac{1}{2} [f_j (dz^j + d\bar{z}^j) + ig_k (d\bar{z}^k - dz^k)] = \frac{1}{2} [(f_j - ig_j) dz^j + (f_k + ig_k) d\bar{z}^k]$$

whereby $\eta^{1,0} = \frac{1}{2}(f_j - ig_j) dz^j$ and $\eta^{0,1} = \frac{1}{2}(f_k + ig_k) d\bar{z}^k$; here, the assumption that $\eta \in \Omega^1(X; \mathbb{R})$ means that $f_j, g_k \in \mathbb{R}$, so that $\overline{f_k \mp ig_k} = f_k \mp ig_k$; thus, since this form is **real**, we must have $\boxed{\eta^{1,0} = \overline{\eta^{0,1}}}$ (because in $\bar{\eta} = \overline{\eta^{1,0}} + \overline{\eta^{0,1}}$, the $(1, 0)$ -part is now $\overline{\eta^{0,1}}$, and vice-versa). Regarding the actual result, we will prove it in the following steps:

(a) Prove that i maps:

$$i : \mathcal{Z}^1(X; \mathbb{R}) \rightarrow \mathcal{Z}_{\bar{\partial}}^{0,1}(X)$$

that is, it maps d -closed 1-forms to $\bar{\partial}$ -closed $(0, 1)$ -forms.

(b) Prove that i maps:

$$i : \mathcal{B}^1(X; \mathbb{R}) \rightarrow 0 \in H_{\bar{\partial}}^{0,1}(X)$$

that is, it is 0 on d -exact 1-forms, so that it descends to the quotient, factoring through a map $j : H^1(X; \mathbb{R}) \rightarrow H_{\bar{\partial}}^{0,1}(X)$

(c) In fact, the kernel of i is precisely $\ker(i) = \mathcal{B}^1(X; \mathbb{R})$, whereby the induced map $j : H^1(X; \mathbb{R}) \rightarrow H_{\bar{\partial}}^{0,1}(X)$ is an injection. This will complete the proof.

To prove these results, we first observe in (a) that for $d\eta = 0$, we may write:

$$d\eta = d(\eta^{1,0} + \eta^{0,1}) = (\partial + \bar{\partial})(\eta^{1,0} + \eta^{0,1}) = \underbrace{\partial\eta^{0,1}}_{(2,0)\text{-part}} + \underbrace{\partial\eta^{0,1} + \bar{\partial}\eta^{0,1}}_{(1,1)\text{-part}} + \underbrace{\bar{\partial}\eta^{0,1}}_{(0,2)\text{-part}}$$

in particular, focusing on the $(0, 2)$ -piece now gives us $\bar{\partial}\eta^{0,1} = 0$, whereby $\eta^{0,1} \in \mathcal{Z}_{\bar{\partial}}^{0,1}(X)$ is indeed $\bar{\partial}$ -closed, as desired.

For (b), we will move on to compute the kernel $\ker(i)$ of the map right away. For the first part, it is evident that i is 0 on d -exact 1-form, which decompose as:

$$\mathcal{B}^1(X; \mathbb{R}) = \{df \mid f \in C^\infty(X, \mathbb{R})\}, \quad df = \underbrace{\partial f}_{(1,0)\text{-part}} + \underbrace{\bar{\partial}f}_{(0,1)\text{-part}} \implies i(df) = \bar{\partial}f \mapsto 0 \in H_{\bar{\partial}}^{0,1}(X)$$

which is $\bar{\partial}$ -exact, thus, 0 in the Dolbeault cohomology.

Conversely, consider some element $\eta \in \ker(i)$, whereby $\eta^{0,1} = \bar{\partial}f$, for some smooth complex-valued function f (*but not necessarily holomorphic*), so that:

$$\eta = \eta^{0,1} + \overline{\eta^{0,1}} = \bar{\partial}f + \partial\bar{f} = (\partial + \bar{\partial})(f + \bar{f}) - \partial f - \bar{\partial}f = d(f + \bar{f}) - \partial f - \bar{\partial}f$$

Next, we may apply d , using the fact that $d\eta = 0$ is d -closed. We write this out, using $\partial^2 = \bar{\partial}^2 = 0$, so that $d\partial = \bar{\partial}\partial = -\partial\bar{\partial}$, and $d\bar{\partial} = d^2 - d\partial = -d\partial = \partial\bar{\partial}$:

$$0 = d\eta = \partial\bar{\partial}(f - \bar{f}) = 2i\partial\bar{\partial}\operatorname{im}(f)$$

that is, $\operatorname{im}(f)$ is a *globally defined real pluriharmonic function*, whereby the ***Strong Maximum Principle*** implies that $\operatorname{im}(f)$ must be **constant**, so that all its derivatives actually satisfy $\partial(f - \bar{f}) = \bar{\partial}(f - \bar{f}) = 0$. Thus, we get:

$$\eta = d(f + \bar{f}) - \partial f - \bar{\partial}f = df + \underbrace{\partial(\bar{f} - f)}_{=0} + \bar{\partial}f - \bar{\partial}f = df$$

and we are almost there, **except** f isn't a real-valued smooth function yet. This can easily be fixed: since $f = \operatorname{Re}(f) + \operatorname{Im}(f)$ for $\operatorname{Re}(f), \operatorname{Im}(f)$ smooth functions, and $\operatorname{Im}(f)$ is constant, we deduce that:

$$\eta = df = d(\operatorname{Re}(f))$$

so that η would have to be d -exact. Thus, we see that the induced map $j : H^1(X; \mathbb{R}) \hookrightarrow H_{\bar{\partial}}^{0,1}(X)$ is injective, as desired; this completes the proof.

Finally, using the injectivity of this map $H^1(X, \mathbb{R}) \hookrightarrow H_{\bar{\partial}}^{0,1}(X)$, we may compute dimensions of vector spaces over \mathbb{R} to obtain:

$$\underbrace{\dim_{\mathbb{R}} H^1(X; \mathbb{R})}_{=b_1(X)} \leq \dim_{\mathbb{R}} H_{\bar{\partial}}^{0,1}(X) = 2 \cdot \underbrace{\dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X)}_{=h^{0,1}} = 2h^{0,1}(X)$$

which shows precisely that $b^1(X) \leq 2h^{0,1}(X)$, as desired. This completes the proof. \square

Continuing from this result, we may also define a map sending $(0, 1)$ -forms to $(1, 1)$ -forms, given by:

$$\delta : \Omega^{0,1}(X) \rightarrow \Omega^{1,1}(X), \quad a \mapsto \partial a + \bar{\partial} a$$

Then, this map induces an isomorphism:

$$\frac{H_{\bar{\partial}}^{0,1}(X)}{H^1(X, \mathbb{R})} \cong \frac{\mathcal{B}_d^{1,1}(X; \mathbb{R})}{i\partial\bar{\partial}(C^\infty(X, \mathbb{R}))}$$

whereby, in particular, we get $b_1(X) = 2h^{0,1}(X)$, if and only if every d -exact real $(1, 1)$ -form is $i\partial\bar{\partial}$ -exact.

Proof. This result will be proved in the following steps:

(a) Prove that δ maps:

$$\delta : \mathcal{Z}_{\bar{\partial}}^{0,1}(X) \rightarrow \mathcal{B}_d^{1,1}(X; \mathbb{R})$$

that is, $\bar{\partial}$ -closed $(0, 1)$ -forms to d -exact real $(1, 1)$ -forms.

(b) Prove that δ maps:

$$\delta : \mathcal{B}_{\bar{\partial}}^{0,1}(X) \rightarrow 0 \in \frac{\mathcal{B}_d^{1,1}(X; \mathbb{R})}{i\partial\bar{\partial}(C^\infty(X, \mathbb{R}))}$$

that is, δ is 0 on $\bar{\partial}$ -exact $(0, 1)$ -forms, thus, it descends to the quotient, factoring

through a map $\varphi : H_{\bar{\partial}}^{0,1}(X) \rightarrow \frac{\mathcal{B}_d^{1,1}(X; \mathbb{R})}{i\partial\bar{\partial}(C^\infty(X, \mathbb{R}))}$.

(c) Prove that the quotient map $\varphi : H_{\bar{\partial}}^{0,1}(X) \rightarrow \frac{\mathcal{B}_d^{1,1}(X; \mathbb{R})}{i\partial\bar{\partial}(C^\infty(X, \mathbb{R}))}$, with $H^1(X; \mathbb{R}) \hookrightarrow H_{\bar{\partial}}^{0,1}(X)$ living inside by the injection $\eta \mapsto \eta^{0,1}$ in the previous problem, maps:

$$\varphi : H^1(X; \mathbb{R}) \rightarrow 0 \in \frac{\mathcal{B}_d^{1,1}(X; \mathbb{R})}{i\partial\bar{\partial}(C^\infty(X, \mathbb{R}))}$$

that is, φ is 0 on $H^1(X; \mathbb{R})$, thus, it descends to the quotient, factoring through a map

$$\psi : \frac{H_{\bar{\partial}}^{0,1}(X)}{H^1(X; \mathbb{R})} \rightarrow \frac{\mathcal{B}_d^{1,1}(X; \mathbb{R})}{i\partial\bar{\partial}(C^\infty(X, \mathbb{R}))}.$$

Now, we prove these steps. For (a), it is evident that $\delta(a) := \partial a + \bar{\partial} a$ is invariant under conjugation, thus, real. Moreover, assuming that a is $\bar{\partial}$ -closed, we know that $\bar{\partial}a = 0$, whereby:

$$d(a + \bar{a}) = (\partial + \bar{\partial})(a + \bar{a}) = \partial a + \bar{\partial} a + \bar{\partial} a + \bar{\partial} \bar{a} = \delta(a) + \bar{\delta}(\bar{a}) + \bar{\delta} a = \delta(a)$$

by the assumption $\bar{\partial}a = 0$. Thus, we see that $\delta(a) \in \mathcal{B}_d^{1,1}(X; \mathbb{R})$ is real and d -exact, as desired. For (b), we may let $a = \bar{\partial}f$ be $\bar{\partial}$ -exact, whereby $f \in \Omega^0(X)$ must be a smooth

function. Then, we get:

$$\delta(a) = \delta(\bar{\partial}f) = \partial\bar{\partial}f + \underbrace{\bar{\partial}\partial}_{=-\partial\bar{\partial}} \bar{f} = \partial\bar{\partial} \underbrace{(f - \bar{f})}_{=2i\cdot\text{Im}(f)} = 2i\partial\bar{\partial}\text{Im}(f)$$

which is exactly 0 inside the quotient space. Thus, we see that the map δ is well-defined up to cohomology class, and we get the quotient map φ from $H_{\bar{\partial}}^{0,1}(X)$.

Finally, for (c), we should actually compute the kernel $\ker(\varphi)$ right away. For this, we have:

$$a \in \ker(\varphi) \iff [\partial a + \bar{\partial}a] = 0 \iff \partial a + \bar{\partial}a = i\partial\bar{\partial}f$$

for some smooth function $f : X \rightarrow \mathbb{R}$. Then, this will enable us to construct another d -exact element $b \in H_{\bar{\partial}}^{0,1}(X)$, again recalling that $\delta(a) = d(a + \bar{a})$ as proved above, and $a + \bar{a}$ is a real 1-form. Thus, we want to construct something d -exact out of $i\partial\bar{\partial}f$: experimenting, we write out, using $\partial^2 = \bar{\partial}^2 = 0$, so that $d\partial = \bar{\partial}\partial = -\partial\bar{\partial}$, and $d\bar{\partial} = d^2 - d\partial = -d\partial = \partial\bar{\partial}$:

$$d(\alpha\partial f + \beta\bar{\partial}f) = (\beta - \alpha)\partial\bar{\partial}f, \quad \overline{\alpha\partial f + \beta\bar{\partial}f} = \bar{\alpha}\bar{\partial}f + \bar{\beta}\partial f$$

since f is a real function, thus, its own conjugate. Now, then, if we want $\eta := \alpha\partial f + \beta\bar{\partial}f$ to be a real 1-form with $d\eta = i\partial\bar{\partial}f$ as we set out, we will need:

$$\alpha = \bar{\beta}, \quad \beta = \bar{\alpha}, \quad \beta - \alpha = i, \quad \Rightarrow -2i \cdot \text{Im}(\alpha) = i \Rightarrow \text{Im}(\alpha) = \frac{1}{2}$$

so that the most natural choice is $\eta := -\frac{i}{2}(\partial f - \bar{\partial}f)$, and our 1-form will be:

$$b := a + \bar{a} + \frac{i}{2}(\partial f - \bar{\partial}f), \quad b^{1,0} = \bar{a} + \frac{i}{2}\partial f, \quad b^{0,1} = a - \frac{i}{2}\bar{\partial}f$$

for which it is evident that $\overline{b^{1,0}} = b^{0,1}$ (it is real) and $db = 0$, both by our constructions. More importantly, we may now write:

$$\bar{\partial}b^{0,1} = \bar{\partial}a - \frac{i}{2}\bar{\partial}^2f = 0$$

since $\bar{\partial}a = 0$ by construction; this shows that $b^{0,1}$ is precisely the image of $b \in H^1(X; \mathbb{R})$ (it lies in the de Rham cohomology as it is d -closed, by construction) under the injection $b \mapsto b^{0,1}$ described above, and in fact $a - b^{0,1} = \frac{i}{2}\bar{\partial}f$, so that $[a] = [b^{0,1}] \in H_{\bar{\partial}}^{0,1}(X)$ correspond to the same Dolbeault cohomology class, since their difference is $\bar{\partial}$ -exact. Thus, we see that:

$$a \in \ker(\varphi) \implies a \in \text{im}(H^1(X; \mathbb{R}))$$

lies in the image of the de Rham cohomology. Conversely, it is evident that for any $b \in H^1(X; \mathbb{R})$ real 1-form and its $(0, 1)$ -piece $b^{0,1}$, we have:

$$\partial b^{0,1} + \overline{\partial b^{0,1}} = \partial b^{0,1} + \bar{\partial}b^{0,1} = db^{0,1} = 0$$

since $b^{0,1} \in H^1(X; \mathbb{R})$ is definitionally real (so $\overline{b^{0,1}} = b^{0,1}$), and d -closed. This proves that $\ker(\varphi) = H^1(X; \mathbb{R})$ precisely, so that we do get the claimed induced map ψ from the quotient:

$$\psi : \frac{H_{\bar{\partial}}^{0,1}(X)}{H^1(X; \mathbb{R})} \hookrightarrow \frac{\mathcal{B}_d^{1,1}(X; \mathbb{R})}{i\partial\bar{\partial}(C^\infty(X, \mathbb{R}))}$$

which is injective by the previous part, as we had $\ker(\varphi) = H^1(X; \mathbb{R})$ exactly. Thus, we see now that $\psi \cong$ is our desired isomorphism, if and only if it is **surjective**.

For this purpose, we get constructing again: given a d -exact $(1, 1)$ -form $c = d\eta$, we want to produce a $\bar{\partial}$ -exact $(0, 1)$ -form a such that $\delta(a) = d(a + \bar{a}) = c = d\eta$. For η , we know that it comes from $\Omega^1(X; \mathbb{R}) = \Omega^{1,0}(X; \mathbb{R}) \oplus \Omega^{0,1}(X; \mathbb{R})$, thus, we may decompose as $\eta = \eta^{1,0} + \eta^{0,1}$ with the respective pieces. In fact, since this is **real**, we must have $\boxed{\eta^{1,0} = \overline{\eta^{0,1}}}$ (because in $\bar{\eta} = \overline{\eta^{1,0}} + \overline{\eta^{0,1}}$, the $(1, 0)$ -part is now $\overline{\eta^{0,1}}$, and vice-versa). Thus, we get:

$$d(a + \bar{a}) \stackrel{?}{=} c = d\eta = d(\eta^{0,1} + \overline{\eta^{0,1}})$$

so that the natural choice $a := \eta^{0,1}$ works, completing the proof.

Finally, using this result, we notice that since vector spaces over a field (here, \mathbb{R}) are fully classified by their dimension, we see that:

$$b_1(X) = \dim_{\mathbb{R}} H^1(X; \mathbb{R}) = \dim_{\mathbb{R}} H_{\bar{\partial}}^{0,1}(X) = 2h^{0,1}(X) \iff \frac{H_{\bar{\partial}}^{0,1}(X)}{H^1(X; \mathbb{R})} = 1$$

Moreover, the above isomorphism shows that:

$$b_1(X) = 2h^{0,1}(X) \iff \frac{H_{\bar{\partial}}^{0,1}(X)}{H^1(X; \mathbb{R})} = 1 \iff \frac{\mathcal{B}_d^{1,1}(X; \mathbb{R})}{i\partial\bar{\partial}(C^\infty(X, \mathbb{R}))} = 1$$

that is, precisely if and only if every d -exact real $(1, 1)$ -form is $i\partial\bar{\partial}$ -exact. This completes the proof of this part. \square

In fact, this argument is true, at least **locally**:

Lemma 4.2. Local $i\partial\bar{\partial}$ -exactness of $(1, 1)$ -forms

Let X be a complex manifold and let $\alpha \in H^{1,1}(X; \mathbb{R})$ be a real ($\alpha = \overline{\alpha}$) d -closed $(1, 1)$ -form on X . In this setting, show that for every point $x \in X$, there is an open neighborhood $U \ni x$ and a smooth real-valued function $f \in C^\infty(U, \mathbb{R})$, such that:

$$\alpha = i\partial\bar{\partial}f$$

holds on all of U .

Proof. To see this, we recall the d -**Poincaré Lemma**, which establishes that if $d\alpha = 0$ holds for a 2-form, then, we must have some real 1-form $\eta \in H^1(\mathcal{U}; \mathbb{R})$ on some open neighborhood $\mathcal{U} \ni x$, for which $\alpha = d\eta$ is locally d -exact. Now, then, we may use the exact same arguments from the previous problem set, in the following sense:

We first observe that for the 1-form $\eta \in \Omega^1(\mathcal{U}; \mathbb{R})$, we know that it comes from the decomposition $\Omega^1(\mathcal{U}; \mathbb{R}) = \Omega^{1,0}(\mathcal{U}; \mathbb{R}) \oplus \Omega^{0,1}(\mathcal{U}; \mathbb{R})$, thus, we may decompose as $\eta = \eta^{1,0} + \eta^{0,1}$ with the respective pieces. In fact, since this is **real**, we must have $\boxed{\eta^{1,0} = \overline{\eta^{0,1}}}$ (because in $\bar{\eta} = \overline{\eta^{1,0}} + \overline{\eta^{0,1}}$, the $(1, 0)$ -part is now $\overline{\eta^{0,1}}$, and vice-versa); this is the previous argument.

On the other hand, since $d\eta = \alpha$, we may write:

$$\alpha = d\eta = d(\eta^{1,0} + \eta^{0,1}) = (\partial + \bar{\partial})(\eta^{1,0} + \eta^{0,1}) = \underbrace{\partial\overline{\eta^{0,1}}}_{(2,0)\text{-part}} + \underbrace{\partial\eta^{0,1} + \bar{\partial}\overline{\eta^{0,1}}}_{(1,1)\text{-part}} + \underbrace{\bar{\partial}\eta^{0,1}}_{(0,2)\text{-part}}$$

in particular, focusing on the $(0, 2)$ -piece now gives us $\bar{\partial}\eta^{0,1} = 0$, because the left-hand side only has the $(1, 1)$ -piece.

On this $\eta^{0,1}$, we may now analogously apply the **$\bar{\partial}$ -Poincaré Lemma**, which informs us that there is again some open neighborhood $\mathcal{U} \supset U \ni x$ (i.e. a refinement of \mathcal{U} , up to shrinking), such that $\eta^{0,1}$ is $\bar{\partial}$ -exact on U . Thus, since this is an 1-form, we see that we should have $\eta^{0,1} = \bar{\partial}g$ on U , for some smooth complex-valued function $g \in C^\infty(U, \mathbb{C})$. Thus, as done before, we now express:

$$\alpha = \partial\eta^{0,1} + \overline{\partial}\eta^{0,1} = \partial\bar{\partial}g + \bar{\partial}\partial\bar{g} = \partial\bar{\partial}(g - \bar{g}) = 2i\partial\bar{\partial}(\operatorname{im}(g))$$

and $2\operatorname{im}(g)$ is evidently a smooth real-valued function, thus, we may set $f := 2\operatorname{im}(g)$, and the proof is complete. \square

All these facts are very useful because they show various results about cohomology in the very general setting of complex manifolds, without forcing the original manifold to be Kähler. On the other hand, the reduced conditions also mean that **the Dolbeault cohomology of a non-Kähler manifold is not necessarily symmetric**, i.e., we may have $H_{\bar{\partial}}^{p,q}(X) \neq H_{\bar{\partial}}^{q,p}(X)$. On the other hand, if X is Kähler, this symmetry will be proved using **Hodge theory**. For example, we have:

Example 4.3. The Hopf surface has non-symmetric Dolbeault cohomology

Consider a \mathbb{Z} -action $\mathbb{Z} \curvearrowright \mathbb{C}$, generated by the dilation:

$$1 * (z, w) := \lambda : (z, w) \mapsto (2z, 2w)$$

Then, we may define the quotient manifold $X := (\mathbb{C}^2 \setminus (0, 0))/\mathbb{Z}$. Then, the following Dolbeault cohomology groups vanish:

$$H_{\bar{\partial}}^{1,0}(X) = 0, \quad H_{\bar{\partial}}^{2,0}(X) = 0$$

however, X also satisfies $H_{\bar{\partial}}^{0,1}(X) \neq 0$, giving a compact complex surface X , having $H_{\bar{\partial}}^{0,1}(X) \neq H_{\bar{\partial}}^{1,0}(X)$.

Proof. First of all, we will prove the vanishing part. For this, we may denote this quotient map by $q : \mathbb{C}^2 \setminus \{(0, 0)\} \twoheadrightarrow X$, for which $q(2^k z, 2^k w) = q(z, w)$ for any $k \in \mathbb{Z}$, by induction on k . Definitionally, we get the two Dolbeault cohomology groups of interest $H_{\bar{\partial}}^{1,0}(X), H_{\bar{\partial}}^{2,0}(X)$ as the quotients of the holomorphic 1-forms and 2-forms, respectively, where:

$$H_{\bar{\partial}}^{k,0} := \frac{\ker(\bar{\partial} : \Omega^{k,0}(X) \rightarrow \Omega^{k+1,0}(X))}{\bar{\partial}\Omega^{k,-1}(X)} \equiv \{\alpha \in \Omega^{k,0}(X) \mid \bar{\partial}\alpha = 0\}$$

since $\bar{\partial}$ has no lower dimension to map from. Thus, the result is equivalent to proving precisely that **there are no holomorphic 1-forms or 2-forms on X** . For this, we will be lifting the holomorphic forms to $\Omega^{k,0}(\mathbb{C}^2 \setminus (0, 0))$, by using the pullback map q^* of the quotient, as:

$$q : \mathbb{C}^2 \setminus 0 \twoheadrightarrow X, \quad q^* : \Omega^{k,0}(X) \rightarrow \Omega^{k,0}(\mathbb{C}^2 \setminus 0)$$

and the previous definition $q(2^k z, 2^k w) = q(z, w)$ now takes the form $q \circ \lambda = q$, thus, the pullback $\lambda^* \circ q^* = q^*$, that is, it acts trivially on the image of q^* inside $H_{\bar{\partial}}^{\bullet,\bullet}(\mathbb{C}^2 \setminus 0)$. Thus, since the pullback commutes with the Wirtinger derivatives, we see that:

$$q^* \circ d = q^* \circ (\partial + \bar{\partial}) = q^* \circ \partial + q^* \circ \bar{\partial} = \partial \circ q + \bar{\partial} \circ q^* = d \circ q^*$$

thus, $q^* \circ \partial = \partial \circ q^*$ and $q^* \circ \bar{\partial} = \bar{\partial} \circ q^*$ by examining the graded pieces. In particular, we see now that:

$$\alpha \in H_{\bar{\partial}}^{k,0}(X) \implies \bar{\partial}\alpha = 0 \implies q^*\bar{\partial}\alpha = \bar{\partial}(q^*\alpha) = 0 \implies \alpha \in H_{\bar{\partial}}^{k,0}(\mathbb{C}^2 \setminus 0)$$

which we may apply to the settings $k = 1, 2$ here. The important observation here is that k -forms on $\mathbb{C}^2 \setminus 0$ have global expressions; letting $\alpha \in H_{\bar{\partial}}^{1,0}(X)$ and $\beta \in H_{\bar{\partial}}^{2,0}(X)$ denote non-trivial elements (for contradiction), with pullbacks $\alpha' := q^*\alpha, \beta' := q^*\beta$, respectively. Then, we obtain:

$$\alpha'(z, w) = f(z, w)dz + g(z, w)dw, \quad \beta'(z, w) = h(z, w)dz \wedge dw$$

for complex functions $f, g, h : \mathbb{C}^2 \setminus 0 \rightarrow \mathbb{C}$. In fact, applying the standard $\bar{\partial}$ -argument gives here:

$$\bar{\partial}\alpha' = \frac{\partial f}{\partial \bar{z}}d\bar{z} \wedge dz + \frac{\partial f}{\partial \bar{w}}d\bar{z} \wedge dw + \frac{\partial g}{\partial \bar{z}}d\bar{z} \wedge dw + \frac{\partial g}{\partial \bar{w}}d\bar{w} \wedge dw = 0$$

whereby $\bar{\partial}f = \bar{\partial}g = 0$ are both holomorphic. In fact, using the **Riemann mapping theorem**, or, more generally, **Hartog's Theorem** (from Griffiths-Harris), we know that the singularity at $(0, 0)$ is removable, and f, g extend to holomorphic functions to all of \mathbb{C}^2 .

Likewise for β' and $h(z, w)$ we see that:

$$\bar{\partial}\beta' = \frac{\partial h}{\partial \bar{z}}d\bar{z} \wedge dz \wedge dw + \frac{\partial h}{\partial \bar{w}}d\bar{w} \wedge dz \wedge dw = 0$$

whereby $\bar{\partial}h = 0$ is also holomorphic, and Hartog extends it to a holomorphic function $h : \mathbb{C}^2 \rightarrow \mathbb{C}$.

Now, we go back to our previous setting, and compose with integer powers of λ . These will change not only the argument, but also the form itself, that is:

$$\begin{aligned} \lambda^* \circ q^* \alpha = q^* \alpha &\implies (\lambda^*)^k \alpha' = \alpha' \implies 2^k (f(2^k z, 2^k w)dz + g(2^k z, 2^k w)dw) = g(z, w)dz + f(z, w)dw \\ (\lambda^*)^k \beta' = \beta' &\implies 4^k \cdot h(2^k z, 2^k w)dz \wedge dw = h(z, w)dz \wedge dw \end{aligned}$$

Thus, we see that we should have, separating the $dz, dw, dz \wedge dw$ pieces:

$$f(2^k z, 2^k w) = \frac{f(z, w)}{2^k}, \quad g(2^k z, 2^k w) = \frac{g(z, w)}{2^k}, \quad h(2^k z, 2^k w) = \frac{h(z, w)}{4^k}$$

Observe now that if such functions f or g existed, then $f = g$ would work for both, and f^2, fg, g^2 would all work to define h . Thus, it suffices to prove that no such nonzero h may exist. Indeed, assume for contradiction that $h(z, w) \neq 0$ at some point (z, w) . Then, we may take $k = -n \rightarrow \infty$ positive tending to infinity, whereby:

$$\left| h\left(\frac{z}{2^n}, \frac{w}{2^n}\right) \right| = 4^n |h(z, w)|$$

Now, since h is holomorphic and in particular holomorphic on all of \mathbb{C}^2 (by Hartog as mentioned before), we may take limits here freely. Thus, we get:

$$|h(0)| = \lim_{n \rightarrow \infty} \left| h\left(\frac{z}{2^n}, \frac{w}{2^n}\right) \right| = \lim_{n \rightarrow \infty} 4^n |h(z, w)|$$

which is only possible if $h(z, w) = 0$, else this value would be infinite (contradiction). Thus, we see that f, g, h must actually be 0, whereby $H_{\bar{\partial}}^{1,0}(X) = H_{\bar{\partial}}^{2,0}(X) = 0$, as desired. This completes the proof.

On the other hand, the more general consideration of the **Calabi-Eckmann manifolds** shows that the Hopf Surface X is (topologically) diffeomorphic to a product of spheres, $X \cong \mathbb{S}^1 \times \mathbb{S}^3$. Thus, we may apply the **Künneth Formula** to obtain its first de Rham cohomology $H^1(X; \mathbb{R})$ as:

$$H^1(X; \mathbb{R}) \cong H^1(\mathbb{S}^1; \mathbb{R}) \oplus H^1(\mathbb{S}^3; \mathbb{R}) = H^1(\mathbb{S}^1; \mathbb{R}) = \mathbb{R}$$

since $H^0(\mathbb{S}^1; \mathbb{R}) = H^0(\mathbb{S}^3; \mathbb{R}) = \mathbb{R}$ as connected manifolds, and $H^1(\mathbb{S}^3; \mathbb{R}) = 0$ as it is simply-connected. Now, we notice the general injection $H^1(X; \mathbb{R}) \hookrightarrow H_{\bar{\partial}}^{0,1}(X; \mathbb{R})$ established earlier, i.e., an injection $\mathbb{R} \hookrightarrow H_{\bar{\partial}}^{0,1}(X; \mathbb{R})$, whereby $H_{\bar{\partial}}^{0,1}(X; \mathbb{R}) \neq 0$ certainly. Since we know that $H_{\bar{\partial}}^{1,0}(X; \mathbb{R}) = 0$, these can certainly not be equal; this completes our observation. By the results of Hodge theory, this shows, in fact, that **the Hopf surface is non-Kähler**. \square

5. RICCI-FLAT METRICS

5.1. Real and complex Ricci. Through Riemannian and through complex geometry, one arrives at (seemingly different) definitions of **Ricci flatness**; as we will see, these coalesce for Kähler manifolds. The *real* Ricci flatness is related to the vanishing of the Ricci curvature, while the definition of *complex* Ricci flatness comes from the vanishing of the **Ricci form** for the Kähler manifold (X, g) , which is given by:

$$(5.1.1) \quad \text{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial} \log \det(g_{i\bar{j}})$$

where $g = (g_{i\bar{j}})$ is the Kähler metric; and we say that (X, ω) is Ricci-flat if $\text{Ric}(\omega)$. The connecting point here is that if the metric is Kähler, then the Chern connection coincides with the complexification of the Levi-Civita connection of the Riemannian metric on the underlying real manifold, which is given by the real part of the (Kähler) hermitian metric; of course, this happens *if and only if* the starting metric is Kähler.

If ω is any Hermitian metric on X , then $\det \omega$ is the induced metric on $K_X^\vee := \det T^{1,0}X$, the anticanonical line bundle; the Chern curvature $\Theta_{K_X^\vee}$ of this metric is given by:

$$(5.1.2) \quad \sqrt{-1}\Theta(K_X^\vee, \det \omega) = -\sqrt{-1}\partial\bar{\partial} \log \det \omega$$

In general, the Chern curvature (wrt. the induced hermitian metric) of the determinant line bundle $\det E$ of a hermitian holomorphic vector bundle (E, h) is given by the trace of the Chern curvature of E :

$$(5.1.3) \quad \sqrt{-1}\Theta(\det E, \det h) = \sqrt{-1}\text{Tr } \Theta(E, h)$$

and so, combining these two equations allows us to write:

$$-\sqrt{-1}\partial\bar{\partial} \log \det \omega = \sqrt{-1}\text{Tr } \Theta(T^{1,0}X, \omega)$$

and since $\Theta(T^{1,0}X, \omega)$ is simply the complexification of the curvature R^{LC} of the Levi-Civita connection ∇_g^{LC} on X with respect to g , the induced Riemannian metric on the underlying real manifold. By definition, the Ricci curvature of (X, g) is precisely the trace of R^{LC} . Of course,

we should note that the *Riemannian* Ricci curvature is a symmetric 2–contravariant tensor, while the Ricci form is a closed real $(1, 1)$ –form. To reconcile this difference, we simply notice that the (Riemannian) Ricci curvature of the (real part of the) Kähler metric is invariant wrt. the complex structure J ; and then we get their correspondence via the bijective correspondence between real $(1, 1)$ –forms and symmetric J –invariant 2–contravariant tensors.

Consequently, given an almost-hermitian manifold (X, g, J) (i.e., an almost-complex manifold with a compatible metric) then we can define the Ricci form globally by:

$$\rho(X, Y) := \text{Ric}(JX, Y)$$

and so the vanishing of Ric and ρ are equivalent.

In fact, a similar argument to the one used for Calabi's lemma may now prove that:

Proposition 5.1. *The Ricci curvature of a Kähler manifold has fixed sign*

Let X be a compact Kähler manifold, and consider two Kähler metrics ω, ω' on X , with no additional assumptions on their cohomology classes. Then, prove that we cannot have $\text{Ric}(\omega) > 0$ and $\text{Ric}(\omega') \leq 0$ everywhere.

Proof. Assume, for contradiction, that such conditions are in place. Then, as done in the proof of Calabi's lemma, we know that we may consider the ratio of the two volume forms ω'^n/ω^n , which will, in fact, be a globally defined, smooth, positive function $\in C^\infty(X, \mathbb{R}^+)$. For this, the computation from Calabi's lemma relates the Ricci forms as:

$$i\partial\bar{\partial}\left(\frac{\omega'^n}{\omega^n}\right) = \text{Ric}(\omega) - \text{Ric}(\omega') > 0$$

which will also be a positive-definite $(1, 1)$ –form, as the sum of two such forms, according to the conditions set out above.

In this setting, we know that the Laplace-Beltrami operator Δ_g acts on a function f as:

$$\Delta_g f := g^{k\bar{\ell}} \partial_k \partial_{\bar{\ell}} f = \frac{1}{2} \text{tr}_\omega(i\partial\bar{\partial}f)$$

where the trace and the Laplacian are both taken with respect to the first metric g with positive-definite Ricci form. Now, then, **taking the trace by ω** in the first equation, and applying the result for $f = \log \frac{\omega'^n}{\omega^n}$ the smooth, globally defined function from before, we get:

$$\Delta_g \log\left(\frac{\omega'^n}{\omega^n}\right) = \frac{1}{2} \text{tr}_\omega i\partial\bar{\partial}\left(\frac{\omega'^n}{\omega^n}\right) = \frac{1}{2} \text{tr}_\omega \text{Ric}(\omega) - \frac{1}{2} \text{tr}_\omega \text{Ric}(\omega') > 0$$

by the positive-definiteness of $\text{Ric}(\omega)$ and the non-positive-definiteness of $\text{Ric}(\omega')$, both observed in the hypothesis. Thus, applying again the **Strong Maximum Principle** to $f = \log \frac{\omega'^n}{\omega^n}$, and the elliptic operator Δ_g , we see that such an inequality is impossible: indeed, f would have to be constant in such a case, but then $\Delta_g f = 0 > 0$ again gives a contradiction. This completes the proof. \square

Of course, the condition of Ricci-flatness may be generalized to having **constant Ricci curvature**; in this direction, we have the result:

Lemma 5.2. Kähler-Einstein metric from Ricci curvature

Let (X, ω) be a Kähler manifold of complex dimension $n \geq 2$. Then, prove that X is Kähler-Einstein, if and only if:

$$\text{Ric}(\omega) = \frac{R}{n}\omega$$

Here, $R = g^{i\bar{j}}R_{i\bar{j}}$ is the scalar curvature.

Proof. The one direction is evident: assuming that ω is Kähler-Einstein, we definitionally know that $\text{Ric} = \lambda\omega$ for some λ , that is, $R_{\bar{j}i} = \lambda g_{\bar{j}i}$. Now, we may multiply this equality by the inverse metric $g^{i\bar{j}}$ to “raise indices”, which gives us the traces; on the left hand side, this produces the scalar curvature. These are given by

$$R = g^{i\bar{j}}R_{i\bar{j}} = g^{i\bar{j}}\lambda g_{i\bar{j}} = \lambda\delta_{ii} = \lambda n, \implies \lambda = \frac{R}{n}$$

now, plugging this bag into the original relation $R = \lambda\omega$ produces precisely the desired result.

The other direction is, of course, more non-trivial; it is again written as $nR_{i\bar{j}} = Rg_{i\bar{j}}$. Recall here the observation about the coefficients of the Riemann curvature tensor, whereby:

$$\partial_{\bar{\ell}}g_{i\bar{j}} = \partial_{\bar{j}}g_{i\bar{\ell}} \implies \partial_k\partial_{\bar{\ell}}g_{i\bar{j}} = \partial_k\partial_{\bar{j}}g_{i\bar{\ell}} = \partial_{\bar{j}}\partial_kg_{i\bar{\ell}} = \partial_{\bar{j}}\partial_i g_{k\bar{\ell}} = \partial_i\partial_{\bar{j}}g_{k\bar{\ell}}$$

because Kähler manifolds enjoy the special property that $\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}}$, and permutations. In fact, the components $R_{i\bar{j}}$ of the Ricci form Ric enjoy the exact same properties, but **in terms of the covariant derivatives**, that is:

$$\nabla_p R_{i\bar{j}} = \nabla_i R_{p\bar{j}}$$

this property is called **the Second Bianchi identity**. Thus, it will make sense to look at the **covariant derivatives** for this problem. These also have the very convenient property that **the metric and its inverse pass through them**, since the connection ∇ is *metric*; this nice property accounts for making traces very convenient. For example, we have:

$$nR_{i\bar{j}} = Rg_{i\bar{j}} \implies n\nabla_p R_{i\bar{j}} = \nabla_p(Rg_{i\bar{j}}) = g_{i\bar{j}}\nabla_p R = g_{i\bar{j}}\nabla_p(g^{k\bar{\ell}}R_{k\bar{\ell}}) = g_{i\bar{j}}g_{k\bar{\ell}}\nabla_p R_{k\bar{\ell}}$$

thus, applying the Bianchi identity $\nabla_p R_{k\bar{\ell}} = \nabla_k R_{p\bar{\ell}}$ to this last one, we get:

$$n\nabla_p R_{i\bar{j}} = g_{i\bar{j}}g^{k\bar{\ell}}\nabla_k R_{p\bar{\ell}} \implies ng^{i\bar{j}}\nabla_p R_{i\bar{j}} = g^{k\bar{\ell}}\nabla_k R_{p\bar{\ell}}$$

by raising the indices with $g^{i\bar{j}}$. In particular, the left-hand side is now:

$$ng^{i\bar{j}}\nabla_p R_{i\bar{j}} = n\nabla_p R = ng_{k\bar{\ell}}\nabla_p R_{k\bar{\ell}} = g_{k\bar{\ell}}\nabla_k R_{p\bar{\ell}}$$

where the last equality comes from the previous equation. Finally, we may again multiply once by $g_{k\bar{\ell}}$ to cancel out the inverse metric $g_{k\bar{\ell}}$ (both sides will be multiplied by n , which cancels out) to get:

$$n\nabla_p R_{k\bar{\ell}} = \nabla_k R_{p\bar{\ell}} = \nabla_p R_{k\bar{\ell}}, \implies (n-1)\nabla_p R_{k\bar{\ell}} = 0$$

using, again, the Bianchi identity. By our assumption that $n \geq 2$, this is only possible if $\nabla_p R_{k\bar{\ell}} = 0$, that is, all the covariant derivatives of all the components of Ric vanish identically. Alternatively, after contracting back with $g_{k\bar{\ell}}$, we see that $\nabla_p R = 0$ for all p , whereby, since R here is a smooth function, we know that the covariant derivative acts on

these by $\nabla_k R = \partial_k R = 0$, whereby R must evidently be a constant function, as desired. In particular, this shows that $\lambda := \frac{R}{n}$ works out as a Kähler-Einstein constant, and the proof is complete. \square

Remark 5.3. Later, we will define and use the following tensor:

$$\text{Ric}^0(\omega) := \text{Ric}(\omega) - \frac{R}{n}\omega$$

which is called the *traceless Ricci tensor*. Using this notation, the above result says that the Kähler manifold X is Kähler-Einstein, if and only if the traceless Ricci tensor Ric^0 vanishes identically.

Proposition 5.4. *Ricci-flat Kähler metric on the product*

Consider compact complex manifolds \mathcal{M}, \mathcal{N} with vanishing first cohomology groups:

$$H^1(\mathcal{M}, \mathbb{R}) = H^1(\mathcal{N}, \mathbb{R}) = 0$$

Now, we may form the product manifold $X := \mathcal{M} \times \mathcal{N}$, equipped with the natural projection maps $\pi_{\mathcal{M}} : X \rightarrow \mathcal{M}$, $\pi_{\mathcal{N}} : X \rightarrow \mathcal{N}$. Assume that X admits a Ricci-flat Kähler metric ω . Then, this is in fact given by:

$$\boxed{\omega = \pi_{\mathcal{M}}^* \omega_{\mathcal{M}} + \pi_{\mathcal{N}}^* \omega_{\mathcal{N}}}$$

for some Ricci-flat Kähler metrics $\omega_{\mathcal{M}}, \omega_{\mathcal{N}}$ on \mathcal{M}, \mathcal{N} , respectively.

Proof. For this problem, we will be using the properties of the **Chern class as a characteristic class**. In particular, characteristic classes have the property that for any two vector bundles E, F over X , we get $c(E \oplus F) = c(E) \oplus c(F)$. In particular, we know that for the holomorphic tangent bundle $T^{1,0}X \rightarrow X$, we have defined $c_1(X) = c_1(T^{1,0}X)$, which will equal 0 here, due to the Ricci-flat property (by the equivalent conditions). Thus, we should first examine the splitting of the tangent bundle TX in terms of the product.

For this reason, we consider some **sections** to the maps $\pi_{\mathcal{M}}, \pi_{\mathcal{N}}$; the natural way to construct these is by fixing some basepoints $m_0 \in \mathcal{M}, n_0 \in \mathcal{N}$, and constructing the maps:

$$i : \mathcal{M} \hookrightarrow X, \quad m \mapsto (m, n_0), \quad j : \mathcal{N} \hookrightarrow X, \quad n \mapsto (m_0, n)$$

These are indeed sections, in the sense that:

$$\pi_{\mathcal{M}} \circ i = \text{id}_{\mathcal{M}}, \quad \pi_{\mathcal{N}} \circ j = \text{id}_{\mathcal{N}}$$

and we evidently see that the maps $\pi_{\mathcal{N}} \circ i : \mathcal{M} \mapsto n_0 \in \mathcal{N}$, $\pi_{\mathcal{M}} \circ j : \mathcal{N} \mapsto m_0 \in \mathcal{M}$ are constant. Moreover, we get pushforward (differential) maps on the tangent bundles, given by $(\pi_{\mathcal{M}})_* : TX \rightarrow T\mathcal{M}$, $(\pi_{\mathcal{N}})_* : TX \rightarrow T\mathcal{N}$ and $i_* : T\mathcal{M} \rightarrow TX$, $j_* : T\mathcal{N} \rightarrow TX$. Importantly, since the pushforward commutes with composition of maps, we have:

$$(\pi_{\mathcal{M}})_* \circ i_* = (\pi_{\mathcal{M}} \circ i)_* = \text{id}_{T\mathcal{M}}, \quad (\pi_{\mathcal{N}})_* \circ j_* = (\pi_{\mathcal{N}} \circ j)_* = \text{id}_{T\mathcal{N}}$$

since the maps compose to the identity, and:

$$(\pi_{\mathcal{M}})_* \circ j_* = (\pi_{\mathcal{N}})_* \circ i_* = 0$$

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since these maps are constant. Then, it is a standard result of differential topology that these maps i, j produce a **splitting** of the tangent bundle TX over X as:

$$TX \cong i_*T\mathcal{M} \oplus j_*T\mathcal{N}$$

but this is actually not extremely useful because the naturality of characteristic classes only has them commute with **pullbacks**, not pushforwards. On the other hand, we know that the pushforward image of the bundles under i, j is the same as their pullback image under the projections (because these maps are **sections**), thus, we get:

$$TX \cong \pi_{\mathcal{M}}^*T\mathcal{M} \oplus \pi_{\mathcal{N}}^*T\mathcal{N}$$

In fact, since the maps $\pi_{\mathcal{M}}, \pi_{\mathcal{N}}$ are all holomorphic, they certainly also commute with the complex structure J on $X, \mathcal{M}, \mathcal{N}$, thus, they respect the splitting of $(1, 0)$ and $(0, 1)$ -pieces. Thus, the above splitting actually becomes a splitting of holomorphic tangent bundles:

$$T^{1,0}X \cong \pi_{\mathcal{M}}^*T^{1,0}\mathcal{M} \oplus \pi_{\mathcal{N}}^*T^{1,0}\mathcal{N}$$

whereby the distributivity and naturality of the first Chern class $c_1(-)$ implies:

$$\begin{aligned} c_1(X) &= c_1(T^{1,0}X) = c_1(\pi_{\mathcal{M}}^*T^{1,0}\mathcal{M}) + c_1(\pi_{\mathcal{N}}^*T^{1,0}\mathcal{N}) \\ &= \pi_{\mathcal{M}}^*c_1(T^{1,0}\mathcal{M}) + \pi_{\mathcal{N}}^*c_1(T^{1,0}\mathcal{N}) \\ &= \pi_{\mathcal{M}}^*c_1(\mathcal{M}) + \pi_{\mathcal{N}}^*c_1(\mathcal{N}) \end{aligned}$$

using the definition $c_1(X) = c_1(T^{1,0}X)$ again, as above.

Incidentally, before we move on, we should recall that **complex submanifolds of Kähler manifolds are Kähler**. The proof is standard: given the submanifold inclusion map, we know that the pullback of the Kähler form under this map produces a Kähler form on the submanifold. Here, this will be given by:

$$i : \mathcal{M} \hookrightarrow X \implies i^*\omega \in \Omega^{1,1}(\mathcal{M}) \text{ is Kähler}$$

because pullbacks of holomorphic maps preserve the grading (so that $i^*\omega$ is also an $(1, 1)$ -form), they commute with the exterior derivatives (so that $di^*\omega = i^*(d\omega) = 0$ is also closed) and with J , thus, with the conjugation map, so that $\overline{i^*\omega} = i^*\overline{\omega} = i^*\omega$ is also a closed real $(1, 1)$ -form, thus, Kähler. By the same token, the pullback form $j^*\omega \in \Omega^{1,1}(\mathcal{N})$ is also Kähler, that is, the manifolds \mathcal{M}, \mathcal{N} are Kähler. From now on, we will be focusing on the **Ricci-flat condition**.

Recall now since ω admits a Ricci-flat metric, the equivalence in Yau's theorem implies that we must have $c_1(X) = 0 \in H^2(X; \mathbb{R})$. Pulling this back to $H^2(\mathcal{M}; \mathbb{R})$ via i^* , we get:

$$0 = i^*c_1(X) = i^*(\pi_{\mathcal{M}}^*c_1(\mathcal{M}) + \pi_{\mathcal{N}}^*c_1(\mathcal{N})) = \underbrace{(\pi_{\mathcal{M}} \circ i)^*}_{=\text{id}_{\mathcal{M}}} c_1(\mathcal{M}) + \underbrace{(\pi_{\mathcal{N}} \circ i)^*}_{=\text{ct}} c_1(\mathcal{N}) = c_1(\mathcal{M})$$

since constant maps have trivial pullback. As a result, we see that $c_1(\mathcal{M}) = 0$, and the analogous argument for \mathcal{N} with j shows that $c_1(\mathcal{N}) = 0$, as well. Thus, by applying **Yau's Theorem** again, we see that \mathcal{M}, \mathcal{N} both admit **Ricci-flat metrics** for each cohomology class $\in H^2(\mathcal{M}; \mathbb{R}), H^2(\mathcal{N}; \mathbb{R})$, respectively, and that **these are unique**.

In particular, we may now choose Ricci-flat Kähler metrics $\omega_{\mathcal{M}}, \omega_{\mathcal{N}}$ in \mathcal{M}, \mathcal{N} respectively, belonging to the cohomology classes of the pullback metrics $i^*\omega, j^*\omega$ respectively, that is:

$$[\omega_{\mathcal{M}}] = i^*[\omega], \quad [\omega_{\mathcal{N}}] = j^*[\omega]$$

Here, Yau's theorem guarantees that these exist, and that they are unique.

Next, we look back to $H^2(X; \mathbb{R})$. This may also be decomposed, using **Künneth's formula**:

$$H^2(X; \mathbb{R}) \cong \bigoplus_{p+q=2} H^p(\mathcal{M}; \mathbb{R}) \otimes H^q(\mathcal{N}; \mathbb{R})$$

whereby, using $H^0(\mathcal{M}; \mathbb{R}) = H^0(\mathcal{N}; \mathbb{R}) = \mathbb{R}$ (as they are *connected*) and $H^1(\mathcal{M}; \mathbb{R}) = H^1(\mathcal{N}; \mathbb{R}) = 0$ (by the assumption), we see that:

$$H^2(X; \mathbb{R}) \cong H^2(\mathcal{M}; \mathbb{R}) \oplus H^2(\mathcal{N}; \mathbb{R}), \quad \text{induced by } \pi_{\mathcal{M}}^* \oplus \pi_{\mathcal{N}}^*$$

In particular, we may apply this result to the cohomology class $[\omega] \in H^2(X; \mathbb{R})$, whereby:

$$\exists \alpha \in H^2(\mathcal{M}; \mathbb{R}), \quad \beta \in H^2(\mathcal{N}; \mathbb{R}), \quad \omega = \pi_{\mathcal{M}}^* \alpha + \pi_{\mathcal{N}}^* \beta$$

Pulling this back again via i^*, j^* , as we did for the Chern class, we again see that:

$$i^*[\omega] = i^*(\pi_{\mathcal{M}}^* \alpha + \pi_{\mathcal{N}}^* \beta) = \underbrace{(\pi_{\mathcal{M}} \circ i)^*}_{=\text{id}_{\mathcal{M}}} \alpha + \underbrace{(\pi_{\mathcal{N}} \circ i)^*}_{=\text{ct}} \beta = \alpha$$

since constant maps have trivial pullback. As a result, we see that $\alpha = i^*[\omega]$, and the analogous argument for \mathcal{N} with j shows that $\beta = j^*[\omega]$, as well.

Now, we may combine this result with the above construction of the Ricci-flat Kähler metrics $\omega_{\mathcal{M}}, \omega_{\mathcal{N}}$ to see that:

$$[\omega] = \pi_{\mathcal{M}}^* \alpha + \pi_{\mathcal{N}}^* \beta = \pi_{\mathcal{M}}^* i^*[\omega] + \pi_{\mathcal{N}}^* j^*[\omega] = \pi_{\mathcal{M}}^* [\omega_{\mathcal{M}}] + \pi_{\mathcal{N}}^* [\omega_{\mathcal{N}}] = [\pi_{\mathcal{M}}^* \omega_{\mathcal{M}} + \pi_{\mathcal{N}}^* \omega_{\mathcal{N}}]$$

Thus, these are cohomologous. Finally, our construction shows that:

$$\text{Ric}(\omega_{\mathcal{M}}) = \text{Ric}(\omega_{\mathcal{N}}) = 0 \implies \text{Ric}(\pi_{\mathcal{M}}^* \omega_{\mathcal{M}} + \pi_{\mathcal{N}}^* \omega_{\mathcal{N}}) = 0$$

thus, the cohomologous forms $\omega, \pi_{\mathcal{M}}^* \omega_{\mathcal{M}} + \pi_{\mathcal{N}}^* \omega_{\mathcal{N}}$ also have the same Ricci form. As a result, **Calabi's lemma** concludes now that these must be equal, as desired. The proof is complete. \square

6. HODGE THEORY

Previously, we saw that on a complex manifold we may decompose the space of complexified k -forms $\mathcal{A}^k(M; \mathbb{C})$ as:

$$\mathcal{A}^k(X; \mathbb{C}) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$$

where the differential operators $\partial, \bar{\partial}$ map the spaces:

$$\partial : \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{p+1,q}(X), \quad \bar{\partial} : \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{p,q+1}(X)$$

which enables us to define the **Dolbeault cohomology groups** as:

$$H_{\bar{\partial}}^{p,q}(X) := \frac{\ker(\bar{\partial} : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p,q+1})}{\text{im}(\bar{\partial} : \mathcal{A}^{p,q-1} \rightarrow \mathcal{A}^{p,q})}$$

For these, it would be nice to have a decomposition:

$$H_{\text{dR}}^k(X) \cong \bigoplus_{p+q=k} H_{\bar{\partial}}^{p,q}(X)$$

As we proved in an earlier construction, we see that this is not always true; however, this result turns out to be true for **compact Kähler manifolds**:

Theorem 6.1. *Hodge decomposition theorem*

Let (M, ω) be a compact Kähler manifold. Then, we have a decomposition:

$$H_{\text{dR}}^k(X) \cong \bigoplus_{p+q=k} H_{\bar{\partial}}^{p,q}(X)$$

for all degrees k .

To see this result, we will use **Hodge theory** and its analog to complex manifolds. For these, we recall the following constructions on a smooth manifold M of real dimension m :

- **Hodge star operator:** $\star : \mathcal{A}^k(M) \rightarrow \mathcal{A}^{m-k}(M)$, defined to have $\alpha \wedge (\star \beta) = \langle \alpha, \beta \rangle d\text{Vol}_g$,
- **Codifferential operator:** $\delta : \mathcal{A}^k(M) \rightarrow \mathcal{A}^{k-1}(M)$, the *formal adjoint* to d , defined as $\delta := (-1)^{m(k+1)+1} \star d \star$,
- **Laplace–Beltrami operator:** $\Delta : \mathcal{A}^k(M) \rightarrow \mathcal{A}^k(M)$, a **formally self-adjoint** operator defined as $\Delta = d\delta + \delta d$.

To translate this idea to **complex Kähler manifolds**, we may let X be a complex manifold with $\dim_{\mathbb{C}} X = n$ (i.e., *real* dimension $2n$) such that (X, ω) is Kähler. Then, we check that the Hodge star operator $\star : \mathcal{A}^k(X; \mathbb{C}) \rightarrow \mathcal{A}^{2n-p-q}(M; \mathbb{C})$ respects the splitting $\mathcal{A}^k(X; \mathbb{C}) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$, and so does the Laplacian $\Delta : \mathcal{A}^k(X) \rightarrow \mathcal{A}^k(X)$:

- **Hodge star:** This maps $\star : \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{n-p,n-q}(X)$.
- **Laplace–Beltrami operator:** This maps $\Delta : \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{p,q}(X)$.

Now, we may further define the L^2 -adjoints to the operators $\partial, \bar{\partial}$ as follows:

$$\boxed{\partial^* := -\star \partial \star, \quad \bar{\partial}^* := -\star \bar{\partial} \star}$$

which are, again, the format L^2 -adjoints of $\partial, \bar{\partial}$. In fact, these map *oppositely* to their adjoint counterparts, i.e.:

$$\partial^* : \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{p-1,q}(X), \quad \bar{\partial}^* : \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{p,q-1}(X)$$

Then, we may deduce the decomposition of our operators:

$$d = \partial + \bar{\partial}, \quad \delta := \partial^* + \bar{\partial}^*$$

for which we define the respective $\partial, \bar{\partial}$ –Hodge-Laplacian operators (read “*del, del-bar*”):

$$\Delta_{\partial} := \partial \partial^* + \partial^* \partial : \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{p,q}(X)$$

$$\Delta_{\bar{\partial}} := \overline{\partial} \overline{\partial}^* + \overline{\partial}^* \overline{\partial} : \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{p,q}(X)$$

as we may easily verify from the domain and image of these maps. In particular, if our manifold is Kähler, we may prove that these operators are connected to the original geometric Laplace–Beltrami operator via:

$$\boxed{\Delta = 2\Delta_{\partial} = 2\Delta_{\bar{\partial}}}$$

where $\Delta := d\delta + \delta d$ is the usual Laplace–Beltrami operator.

Now, we will show that if the metric g of the Hermitian manifold (M, g) is **Kähler**, then, these operators agree up to constant factors. Importantly, we recall that since **Kähler means symplectic complex**, we know that *locally*, the metric g can be represented via the standard symplectic form on $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

First of all, we will perform a construction: for an 1–form ξ and *any* p –form α , we may define $\xi \vee \alpha$ as the **unique** $(p-1)$ –form satisfying:

$$(6.0.1) \quad \langle \xi \vee \alpha, \beta \rangle = \langle \alpha, \bar{\xi} \wedge \beta \rangle$$

where $\langle -, - \rangle$ is the inner product of forms of the same degree defined above; in particular, this holds for *any* $(p-1)$ –form β , whereby the uniqueness of $\xi \vee \alpha$ follows from the non-degeneracy of the inner product. Now, we obtain an equivalent characterization of $\xi \vee \alpha$ as:

$$(6.0.2) \quad \xi \vee \alpha = \xi^\sharp \lrcorner \alpha =: \iota_{\xi^\sharp} \alpha$$

i.e., the **contraction** (interior product) of α with the musical dual of ξ .

For example, we see now that:

$$\begin{aligned} dz^i \vee dz^k &= g^{i\bar{j}} \iota_{\bar{j}} dz^k = 0 \\ dz^i \vee d\bar{z}^k &= g^{i\bar{j}} \iota_{\bar{j}} d\bar{z}^k = g^{i\bar{k}} \end{aligned}$$

whereby, similarly, we obtain:

$$dz^i \vee dz^j = 0, \quad dz^i \vee d\bar{z}^j = g^{i\bar{j}}, \quad d\bar{z}^i \vee dz^j = g^{\bar{i}j}, \quad d\bar{z}^i \vee d\bar{z}^j = 0$$

Finally, we see that since $\xi \vee := \iota_{\#\xi}$ is an **antiderivation**, we have for any differential forms β, γ with β a p –form:

$$(6.0.3) \quad \xi \vee (\beta \wedge \gamma) = (\xi \vee \beta) \wedge \gamma + (-1)^p \beta \wedge (\xi \vee \gamma)$$

Now, using these results, we will establish the claim above.

Proof. The proof of this result is quite similar to the regular Riemannian case: we may write in local coordinates $\xi = \xi_\mu dz^\mu$, so that $\xi^\sharp = g^{\mu\nu} \xi_\mu \partial_\nu$; thus, the above result makes the interior product into:

$$\iota_{\xi^\sharp} = \iota_{g^{\mu\nu} \xi_\mu \partial_\nu} = g^{\mu\nu} \xi_\mu \iota_\nu$$

whereby the bi- $C^\infty(M)$ –linearity of the inner product $\langle -, - \rangle_g$ *in the first term* helps us rewrite the above expression as:

$$\langle \iota_{\#\xi} \alpha, \beta \rangle = g^{\mu\nu} \xi_\mu \langle \iota_\nu \alpha, \beta \rangle_g$$

while, on the other hand, the **conjugate-linearity** of the inner product $\langle -, - \rangle_g$ with respect to the second term gives, since we have $\xi = \xi_\mu dz^\mu$ thus $\bar{\xi} = \bar{\xi}_\mu d\bar{z}^\mu$

$$\begin{aligned}\langle \alpha, \bar{\xi} \wedge \beta \rangle &= \langle \alpha, \bar{\xi}_\mu d\bar{z}^\mu \wedge \beta \rangle \\ &= \xi_\mu \langle \alpha, d\bar{z}^\mu \wedge \beta \rangle\end{aligned}$$

whereby, focusing on the case simplest case where $\xi = dz^\mu$ and canceling out the ξ_μ , we see that the result reduces to:

$$g^{\mu\nu} \langle \iota_\nu \alpha, \beta \rangle = \langle \alpha, dz^\mu \wedge \beta \rangle$$

That is, we want to separately prove that:

- If $\xi = dz^i$, then, $g^{i\bar{j}} \langle \iota_{\bar{j}} \alpha, \beta \rangle = \langle \alpha, d\bar{z}^i \wedge \beta \rangle$.
- If $\xi = d\bar{z}^k$, then, $g^{j\bar{k}} \langle \iota_j \alpha, \beta \rangle = \langle \alpha, dz^k \wedge \beta \rangle$.

Here, the $C^\infty(M)$ -linearity of all operators involved (ι_ν , \wedge , and the inner product) means that we need only examine the result for pure p -forms; that is, we may write: ADDDDD \square

Now, we will employ the following notation: for a p -form $\alpha \in \mathcal{A}^k(M; \mathbb{C})$, we define $\partial_\mu \alpha$ as the *local* p -form obtained by applying the operator $\partial/\partial z^\mu$ to all coefficients of α in local coordinates. That is:

$$\alpha = \sum_{|I|+|J|=p} \alpha_{I\bar{J}} dz^I \wedge d\bar{z}^J, \quad \partial_\mu \alpha := \sum_{|I|+|J|=p} (\partial_\mu \alpha_{I\bar{J}}) dz^I \wedge d\bar{z}^J$$

Evidently, this local operator satisfies the property (which gives a global expression):

$$\boxed{\partial \alpha = dz^j \wedge \partial_j \alpha, \quad \bar{\partial} \alpha = d\bar{z}^j \wedge \partial_{\bar{j}} \alpha}$$

by the local definition of the differentials. Thus, for any smooth function $f \in C^\infty(M, \mathbb{C})$ and any p -form $\alpha \in \mathcal{A}^p(M, \mathbb{C})$, we evidently get:

$$\boxed{\partial_\mu(f\alpha) = (\partial_\mu f)\alpha + f\partial_\mu \alpha}$$

by considering, linearly, the corresponding action on the coefficients. We also see that:

$$\boxed{\partial_\mu(dz^\nu \wedge \alpha) = dz^\nu \wedge \partial_\mu \alpha}$$

because dz^ν introduces no new factors. Thus, we obtain:

$$\boxed{\partial_\mu(dz^\nu \vee \alpha) = dz^\nu \vee (\partial_\mu \alpha) + (\partial_\mu g^{\rho\nu}) \iota_\rho \alpha}$$

that is, the operator ∂_μ passes through the $dz^\mu \vee$ with a correction term from the metric.

Proof. To see this result, we may recall, as above, that $dz^\nu \vee = g^{\rho\nu} \iota_\rho$, so that we equivalently want:

$$\partial_\mu(g^{\rho\nu} \iota_\rho \alpha) = g^{\rho\nu} \iota_\rho(\partial_\mu \alpha)$$

and the above result makes the LHS equal to:

$$\partial_\mu(g^{\rho\nu} \iota_\rho \alpha) = (\partial_\mu g^{\rho\nu}) \iota_\rho \alpha + g^{\rho\nu} \partial_\mu(\iota_\rho \alpha)$$

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Importantly, we see now that since the interior product *passes through* the coefficients of differential forms, i.e., $\iota_X(f\alpha) = f\iota_X\alpha$, this also means that:

$$\iota_X(\partial_\mu \alpha) = \partial_\mu(\iota_X \alpha)$$

because they operate separately on the coefficients and on the wedge-terms; together, these yield the proof. If, in particular, the **first-order derivatives of the metric vanish** \square

Thus, using these tools, we may formulate the following useful lemma:

Proposition 6.2. Hodge- $\bar{\partial}$ -operator in Euclidean metric

Recall the definition of the adjoint $\bar{\partial}^*$ of the $\bar{\partial}$ -operator as $(\bar{\partial}\alpha, \beta) = (\alpha, \bar{\partial}^*\beta)$. Then, in the Euclidean metric g_{Euc} of \mathbb{C}^n , the operator $\bar{\partial}^*$ is given by:

$$\boxed{\bar{\partial}^*\alpha = -dz^j \vee \partial_j \alpha}$$

following the definitions above.

Proof. To see this result, we first notice that for any smooth, **compactly supported** function f on \mathbb{C}^n , we have:

$$\int_{\mathbb{C}^n} \partial_\mu f \, d\text{Vol}_g = 0$$

for any partial ∂_μ ; this can be verified by the **Fundamental Theorem of Calculus**, upon choosing a ball $\mathbb{B}_R(0)$ around 0 sufficiently large such that $f = 0$ on $\partial\mathbb{B}_R(0)$ and outside $\mathbb{B}_R(0)$, so that:

$$\int_{\mathbb{C}^n} \partial_\mu f \, d\text{Vol}_g = \int_{\mathbb{B}_R(0)} \partial_\mu f \, d\text{Vol}_g = 0$$

by applying this separately to the real and the imaginary parts. Now, for smooth p -forms α, β on \mathbb{C}^n , we have:

$$\partial_j \langle \alpha, \beta \rangle = \langle \partial_j \alpha, \beta \rangle + \langle \alpha, \partial_{\bar{j}} \beta \rangle$$

whereby, applying the above result for $\mu = j$ and $f = \langle dz^j \vee \alpha, \beta \rangle$, we use the above to get:

$$\begin{aligned} 0 &= \int_{\mathbb{C}^n} \partial_j \langle dz^j \vee \alpha, \beta \rangle \, d\text{Vol}_g \\ &= \int_{\mathbb{C}^n} \langle \partial_j(dz^j \vee \alpha), \beta \rangle \, d\text{Vol}_g + \int_{\mathbb{C}^n} \langle dz^j \vee \alpha, \partial_{\bar{j}} \beta \rangle \, d\text{Vol}_g \\ &= \int_{\mathbb{C}^n} \langle dz^j \vee (\partial_j \alpha), \beta \rangle \, d\text{Vol}_g + \int_{\mathbb{C}^n} \langle dz^j \vee \alpha, \partial_{\bar{j}} \beta \rangle \, d\text{Vol}_g \\ &= \int_{\mathbb{C}^n} \langle dz^j \vee (\partial_j \alpha), \beta \rangle \, d\text{Vol}_g + \int_{\mathbb{C}^n} \langle \alpha, dz^j \wedge \partial_{\bar{j}} \beta \rangle \, d\text{Vol}_g \\ &= \int_{\mathbb{C}^n} \langle dz^j \vee (\partial_j \alpha), \beta \rangle \, d\text{Vol}_g + \int_{\mathbb{C}^n} \langle \alpha, \bar{\partial} \beta \rangle \, d\text{Vol}_g \\ &= (dz^j \vee (\partial_j \alpha), \beta) + (\alpha, \bar{\partial} \beta) \end{aligned}$$

which means that:

$$(\alpha, \bar{\partial} \beta) + (dz^j \vee (\partial_j \alpha), \beta) = 0$$

where the first term equals $(\bar{\partial}^* \alpha, \beta)$; thus, we get:

$$(\bar{\partial}^* \alpha, \beta) = -(dz^j \vee (\partial_j \alpha), \beta)$$

so that $\boxed{\bar{\partial}^* = -dz^j \vee \partial_j}$ are equal as operators on \mathbb{C}^n , as claimed; this completes the proof. \square

Moving on, we will consider an important *synthetic* construction that helps in the derivation of the Hodge Theory of Kähler Manifolds. We call it ***synthetic*** in the sense that we need to provide an entirely new construction, which will function as an **intermediate step** in our proof.

Definition 6.3. *The Lefschetz Operator*

Let (X, g, ω) be a complex Kähler manifold, where g is a Hermitian metric compatible with the Kähler form ω . Then, we may define a **smooth bundle homomorphism** L (in particular, it is \mathcal{C}_X^∞ -linear), called the ***Lefschetz operator***, given by:

$$L : \Lambda^{p,q} X \rightarrow \Lambda^{p+1,q+1} X, \quad L\alpha := \omega \wedge \alpha$$

where the domain considerations follow from ω being an $(1,1)$ -form. The Lefschetz operator has an adjoint operator $\Lambda := L^* : \Lambda^{p,q} X \rightarrow \Lambda^{p-1,q-1} X$, defined (pointwise) such that:

$$\langle L^* \alpha, \beta \rangle_g = \langle \alpha, L\beta \rangle_g$$

for all α, β . In fact, this is given by $\boxed{\Lambda := L^* = \star^{-1} L \star = (-1)^{p+q} \star L \star}$.

In particular, the above pointwise definition clearly implies that on the manifold, we have $\boxed{(L^* \alpha, \beta) = (\alpha, L\beta)}$. Here, it is also useful (as we will see later) to define the *counting operator* $H : \mathcal{A}^k(X) \rightarrow \mathcal{A}^k(X)$, given by multiplication by $(k-n)$. Now, to see the equality $\Lambda = \star^{-1} L \star$, we will compute the pairing of forms and multiply with the volume form $d\text{Vol}_g$ to get a top form:

$$\begin{aligned} (\alpha, \Lambda\beta) d\text{Vol}_g &= (L\alpha, \beta) d\text{Vol}_g = L\alpha \wedge \star\beta = \underbrace{\omega \wedge \alpha}_{\xi \wedge \eta = (-1)^{p+q} \eta \wedge \xi} \wedge \star\beta = \alpha \wedge \omega \wedge \star\beta \\ &= \alpha \wedge L(\star\beta) \end{aligned}$$

where we used the **adjointness** followed by the **definition of the Hodge-star operator** \star and the transposition property for the wedge of pure forms. Additionally, we observe now that, in general, the definition of the Hodge start operator implies that:

$$\alpha \wedge \beta = \alpha \wedge \star(\star^{-1}\beta) = (\alpha, \star^{-1}\beta) d\text{Vol}_g$$

whereby the above relation becomes:

$$(\alpha, \Lambda\beta) d\text{Vol}_g = \alpha \wedge L(\star\beta) = (\alpha, \star^{-1} L \star \beta) d\text{Vol}_g$$

from which we deduce (from the non-degeneracy of the L^2 -pairing $(-, -)_{L^2}$) that $\Lambda\beta = \star^{-1} L \star \beta$, i.e., $\boxed{\Lambda = \star^{-1} L \star = (-1)^k \star L \star}$, as desired; this completes the proof. Finally, we recall the definition of the Hodge star operator \star acting on a k -form on a smooth manifold of real dimension m by $\star\star\alpha = (-1)^{k(m-k)}\alpha$; here, for $m = 2n$ (i.e., complex dimension n) we have $\star\star = (-1)^{k(2n-k)} = (-1)^k$ (we go from $k \mapsto 2n - k$, i.e., **parity is preserved**) so that $\star^{-1} = (-1)^k \star = (-1)^{p+q} \star$ with respect to the bigrading.

Before moving on to prove the identities, we take a detour to discuss the theory of ***super vector spaces***:

Definition 6.4. Super Vector Spaces

A **super vector space** is a vector V that admits a \mathbb{Z}_2 -grading, i.e., can be decomposed as $V = V^{\text{even}} \oplus V^{\text{odd}}$, where ADDDDDD

The prototypical example here will be the sheaf of (all) differential forms on X , which is expressible as $\mathcal{A}_X^\bullet = \bigoplus_k \mathcal{A}_X^k$, which decomposes into even and odd forms. Thus, we see that \mathcal{A}_X^\bullet is a **sheaf of super vector spaces** on X .

For a super vector space V , we consider two endomorphisms $A, B \in \text{End}(V)$ of **degrees** a, b , respectively. Here, the **degree** of an endomorphism is ADDDDD; for example, A sends (restricts to a map) $V_k \rightarrow V_{k+a}$, for each k . Then, for any such operators, we can recall the construction of the **standard Lie algebra** on the space of endomorphisms, which is given by the **graded Lie bracket**; in this case, this is simply a **graded commutator** with a slightly different expression:

$$[A, B] := AB - (-1)^{ab}BA$$

If, in particular, the degree of A or B is **even**, then, this simply reduces to the usual commutator.

When we endow the space $\text{End}(V)$ with this graded Lie bracket, we make it into a **Lie superalgebra**; again, this is the graded analogue of a usual Lie algebra, which, in this case, satisfies a **graded Jacobi identity**.

With these definitions, we can now state our goal:

Theorem 6.5. $(2,2)$ -SUSY algebra

Let (X, ω) be a Kähler manifold (**not necessarily compact**) and consider the operators Δ (Laplacian), L (Lefschetz), Λ (transpose), $H, \partial, \bar{\partial}, \partial^*, \bar{\partial}^*$ defined above. These span an eight-dimensional **sub-Lie superalgebra** of the endomorphism Lie superalgebra $\text{End } \mathcal{A}^\bullet$, where \mathcal{A}^\bullet is the sheaf of smooth differential forms on X ; here, the first four operators Δ, L, Λ, H are **even** and $\partial, \bar{\partial}, \partial^*, \bar{\partial}^*$ are **odd**. This algebra is called the $(2,2)$ -**supersymmetry (SUSY) algebra** of X and Δ spans the center of this algebra.

Essentially, this result says that the graded commutators of these operators **land back into the vector space spanned by them**; in fact, some of these operators commute, and all the brackets are 0, except for the ones that we will compute below. For some easy results, we see that $[H, D] = (\deg D)D$ for all operators D , and:

$$[\bar{\partial}, \bar{\partial}^*] = \bar{\partial}\bar{\partial}^* - (-1)^1\bar{\partial}^*\bar{\partial} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} = \Delta_{\bar{\partial}}$$

i.e., precisely the **$\bar{\partial}$ -Laplacian**.

Even more interestingly, we see now that:

$$[H, L] = 2L, \quad [H, \Lambda] = -2\Lambda, \quad [L, \Lambda] = H$$

which are precisely the relations for the basis elements of $\mathfrak{sl}(2, \mathbb{C})$, the Lie algebra of $\text{SL}(2, \mathbb{C})$; meaning that we get a subspace that is **closed** under the super Lie bracket. Of course, this is **no longer a super Lie algebra**, as it only consists of the even-degree parts; however, we can still use this 3-dimensional Lie algebra to get an $\mathfrak{sl}(2, \mathbb{C})$ -**action** on the sheaf \mathcal{A}_X^\bullet of

differential forms on X , which enables us to view \mathcal{A}_X^\bullet as a **representation** of $\mathfrak{sl}(2, \mathbb{C})$; and we can apply representation theory, for example to classify the irreducible subspaces of this representation.

With this definition, we may now state the following important intermediate step:

Proposition 6.6. *The Kähler Identities*

Let M be a Kähler manifold with operators $\partial, \bar{\partial}, \partial^*, \bar{\partial}^*$ defined as above; we also have the Lefschetz operator L and its adjoint L^* . Then, the following identities hold:

- (i) $[\bar{\partial}^*, L] = i\partial$.
- (ii) $[\partial^*, L] = -i\bar{\partial}$.
- (iii) $[\bar{\partial}, L^*] = i\partial^*$.
- (iv) $[\partial, L^*] = -i\bar{\partial}^*$.

where $[A, B] := A \circ B - B \circ A$ denotes the commutator of two operators.

Proof. To see this result, we recall that a complex Kähler manifold admits **holomorphic normal coordinates**; in fact, these exist if and only if the manifold is Kähler. In these coordinates, we have the first-order derivatives of the metric $\partial_\rho g_{\mu\nu} = \partial_\rho g^{\mu\nu} = 0$; in these coordinates, we may write the Kähler form $\omega = ig_{j\bar{k}} dz^j \wedge d\bar{z}^k$, so that the previous equalities hold for $\bar{\partial}^*$, using the **local symplectic form** for Kähler manifolds. Moreover, we see that:

$$\begin{aligned}\partial_\ell(L\alpha) &= \partial_\ell(\omega \wedge \alpha) = \partial_\ell(ig_{j\bar{k}} dz^j \wedge d\bar{z}^k \wedge \alpha) \\ &= \underbrace{\partial_\ell(ig_{j\bar{k}})}_{=0} dz^j \wedge d\bar{z}^k \wedge \alpha + ig_{j\bar{k}} \partial_\ell(dz^j \wedge d\bar{z}^k \wedge \alpha) \\ &= ig_{j\bar{k}} dz^j \wedge d\bar{z}^k \wedge \partial_\ell \alpha \\ &= \omega \wedge \partial_\ell \alpha\end{aligned}$$

so that:

$$\boxed{\partial_j(L\alpha) = L(\partial_j \alpha)}$$

Thus, we get:

$$\begin{aligned}[\bar{\partial}^*, L] &= \bar{\partial}^*(\omega \wedge \alpha) - \omega \wedge (\bar{\partial}^* \alpha) \\ &= -dz^\ell \vee \partial_\ell(L\alpha) + ig_{j\bar{k}} dz^j \wedge d\bar{z}^k \wedge (dz^\ell \vee \partial_\ell \alpha) \\ &= -dz^\ell \vee L(\partial_\ell \alpha) + ig_{j\bar{k}} dz^j \wedge d\bar{z}^k \wedge (dz^\ell \vee \partial_\ell \alpha)\end{aligned}$$

Now, we see that the first term expands as:

$$\begin{aligned}-dz^\ell \vee L(\partial_\ell \alpha) &= -dz^\ell \vee (ig_{j\bar{k}} dz^j \wedge d\bar{z}^k \wedge (\partial_\ell \alpha)) \\ &= -ig_{j\bar{k}} dz^\ell \vee (dz^j \wedge d\bar{z}^k \wedge (\partial_\ell \alpha)) \\ &= -ig_{j\bar{k}} \left(\underbrace{(dz^\ell \vee dz^j)}_{=0} \wedge d\bar{z}^k \wedge (\partial_\ell \alpha) - dz^j \wedge dz^\ell \vee (d\bar{z}^k \wedge (\partial_\ell \alpha)) \right) \\ &= ig_{j\bar{k}} dz^j \wedge dz^\ell \vee (d\bar{z}^k \wedge (\partial_\ell \alpha))\end{aligned}$$

$$\begin{aligned}
&= ig_{j\bar{k}} dz^j \wedge \underbrace{(dz^\ell \vee d\bar{z}^k) \wedge (\partial_\ell \alpha)}_{=g^{\ell\bar{k}}} - ig_{j\bar{k}} dz^j \wedge d\bar{z}^k \wedge (dz^\ell \vee (\partial_\ell \alpha)) \\
&= i \underbrace{g_{j\bar{k}} g^{\ell\bar{k}}}_{=\delta_j^\ell} dz^j \wedge (\partial_\ell \alpha) - ig_{j\bar{k}} dz^j \wedge d\bar{z}^k \wedge (dz^\ell \vee (\partial_\ell \alpha))
\end{aligned}$$

so that, in particular, the second summand **cancels out** the summand of the above expression, and we are left with:

$$[\bar{\partial}^*, L]\alpha = idz^j \wedge \partial_j \alpha = i\partial\alpha$$

as desired; this completes the proof.

Here, the important idea that enabled us to perform this calculation in **normal coordinates** is that for an arbitrary Kähler manifold (M, g) , since both operators $\bar{\partial}^*, L$ are coordinate-independent, we may compute $[\bar{\partial}^*, L]$ at a point $p \in M$ in *any* coordinates we choose; in particular, the Kähler property is crucial to choosing **normal holomorphic coordinates**, so that the first derivatives of the metric vanish.

Finally, we will see that the other three identities are immediate consequences of the first one. For example, we have, since ω is a **real** $(1, 1)$ -form, the following expression for the conjugate:

$$\overline{L\alpha} = \overline{\omega \wedge \alpha} = \omega \wedge \overline{\alpha} = L\overline{\alpha}$$

Thus, we now see that:

$$\overline{[\bar{\partial}^*, L]\alpha} = \overline{\bar{\partial}^* L\alpha} - \overline{L\bar{\partial}^*\alpha} = \partial^* L\overline{\alpha} - L\bar{\partial}^*\overline{\alpha} = [\partial^*, L]\overline{\alpha}$$

after recalling that $\bar{\partial}^* = \overline{\partial^*}$, due to:

$$\overline{\partial^* \alpha, \beta} = \overline{(\alpha, \partial\beta)} = (\overline{\alpha}, \overline{\partial\beta}) = (\overline{\alpha}, \overline{\partial\beta}) = (\overline{\partial^* \alpha}, \overline{\beta})$$

which gives $\bar{\partial}^* = \overline{\partial^*}$. Thus, we see that the first expression immediately implies the second one upon taking conjugates.

Finally, we observe that $[A, B]^* = -[A^*, B^*]$ for operators, as:

$$\begin{aligned}
([A^*, B^*]\alpha, \beta) &= (A^*B^*\alpha, \beta) - (B^*A^*\alpha, \beta) = (B^*\alpha, A\beta) - (A^*\alpha, B\beta) \\
&= (\alpha, BA\beta) - (\alpha, AB\beta) = (\alpha, -[A, B]\beta) \\
&= (-[A, B]^*\alpha, \beta)
\end{aligned}$$

as claimed; this completes the proof of the last two expressions. \square

Now, using these important identities, we are finally able to do that:

Proposition 6.7. Anticommutativity of Codifferentials for Kähler Manifolds
Let (M, g) be a Kähler manifold with operators $\partial, \bar{\partial}, \partial^, \bar{\partial}^*$ defined as above. Then, the following identities hold:*

$$\bar{\partial}^* \partial + \partial \bar{\partial}^* = 0$$

$$\partial^* \bar{\partial} + \bar{\partial} \partial^* = 0$$

written succinctly as $\{\bar{\partial}^*, \partial\} = \{\partial^*, \bar{\partial}\} = 0$.

Proof. Using the Kähler identities, we may write $\bar{\partial}^* = i[\partial, L^*]$. Together with the property $\partial^2 = 0$, this result implies:

$$\begin{aligned}\bar{\partial}^* \partial + \partial \bar{\partial}^* &= i[\partial, L^*] \partial + i\partial[\partial, L^*] \\ &= i\partial L^* \partial - \underbrace{iL^* \partial^2 + i\partial^2 L^*}_{=0} - i\partial L^* \partial = 0\end{aligned}$$

and the second immediately follows immediately from the first by **conjugation**, as before. \square

Thus, using these preparatory results, we see that:

Theorem 6.8. The Hodge Laplacian on Kähler Manifolds

Let (M, g) be a complex manifold with Laplacian operator $\Delta := d\delta + \delta d$ and Hodge Laplacians $\Delta_\partial, \Delta_{\bar{\partial}}$ given by:

$$\begin{aligned}\Delta_\partial &:= \partial \bar{\partial}^* + \bar{\partial}^* \partial : \Omega^{p,q}(X) \rightarrow \Omega^{p,q}(X) \\ \Delta_{\bar{\partial}} &:= \bar{\partial} \partial^* + \partial^* \bar{\partial} : \Omega^{p,q}(X) \rightarrow \Omega^{p,q}(X)\end{aligned}$$

Then, if M is Kähler, these satisfy:

- (i) $\Delta = \Delta_\partial + \Delta_{\bar{\partial}}$.
- (ii) $\Delta_\partial = \Delta_{\bar{\partial}}$.
- (iii) $\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$.

Proof. To see this result, we start with the expression of $\Delta = d\delta + \delta d$. Since $d = \partial + \bar{\partial}$ and $\delta = \partial^* + \bar{\partial}^*$, this means that:

$$\begin{aligned}\Delta &= d\delta + \delta d = (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}) \\ &= \partial \partial^* + \partial \bar{\partial}^* + \bar{\partial} \partial^* + \bar{\partial} \bar{\partial}^* + \partial^* \partial + \partial^* \bar{\partial} + \bar{\partial}^* \partial + \bar{\partial}^* \bar{\partial} \\ &= \Delta_\partial + \Delta_{\bar{\partial}} + \underbrace{\bar{\partial}^* \partial + \partial \bar{\partial}^*}_{=0} + \underbrace{\partial^* \bar{\partial} + \bar{\partial} \bar{\partial}^*}_{=0} \\ &= \Delta_\partial + \Delta_{\bar{\partial}}\end{aligned}$$

as claimed; this proves the first part.

For the second part, we use Proposition 6.6 to see that:

$$\begin{aligned}\Delta_\partial &= \partial \bar{\partial}^* + \bar{\partial}^* \partial = -i\partial[\bar{\partial}, L^*] - i[\bar{\partial}, L^*]\partial \\ &= i\partial L^* \bar{\partial} - i\partial \bar{\partial} L^* + iL^* \bar{\partial} \partial - i\bar{\partial} L^* \partial\end{aligned}$$

while, similarly, we have:

$$\begin{aligned}\Delta_{\bar{\partial}} &= \bar{\partial} \partial^* + \partial^* \bar{\partial} = i\bar{\partial}[\partial, L^*] + i[\partial, L^*]\bar{\partial} \\ &= i\bar{\partial} \partial L^* - i\bar{\partial} L^* \partial + i\partial L^* \bar{\partial} - iL^* \partial \bar{\partial}\end{aligned}$$

so that, comparing the two expressions, we are only left with proving:

$$-i\partial \bar{\partial} L^* + iL^* \bar{\partial} \partial = i\bar{\partial} \partial L^* - iL^* \partial \bar{\partial}$$

which is immediate due to the relation $\partial \bar{\partial} = -\bar{\partial} \partial$; thus, the proof is complete. \square

Now, we notice that the components of the Laplacians work as:

$$\langle \partial \bar{\partial}^* \alpha, \alpha \rangle_{L^2} = \langle \bar{\partial}^* \alpha, \bar{\partial}^* \alpha \rangle_{L^2} = \|\bar{\partial}^* \alpha\|_{L^2}^2$$

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and, similarly, we obtain the relations:

- $\langle \partial\partial^*\alpha, \alpha \rangle_{L^2} = \|\partial^*\alpha\|_{L^2}^2,$
- $\langle \partial^*\partial\alpha, \alpha \rangle_{L^2} = \|\partial\alpha\|_{L^2}^2,$
- $\langle \bar{\partial}\bar{\partial}\alpha, \alpha \rangle_{L^2} = \|\bar{\partial}\alpha\|_{L^2}^2,$
- $\langle \bar{\partial}^*\bar{\partial}\alpha, \alpha \rangle_{L^2} = \|\bar{\partial}^*\alpha\|_{L^2}^2.$

which proves to us that:

$$\begin{aligned}\langle \Delta_\partial\alpha, \alpha \rangle_{L^2} &= \|\partial\alpha\|_{L^2}^2 + \|\partial^*\alpha\|_{L^2}^2 \\ \langle \Delta_{\bar{\partial}}\alpha, \alpha \rangle_{L^2} &= \|\bar{\partial}\alpha\|_{L^2}^2 + \|\bar{\partial}^*\alpha\|_{L^2}^2\end{aligned}$$

so that for a **$(\Delta-)$ harmonic form** $\alpha \in \mathcal{H}_\Delta^k$, we have:

$$\Delta\alpha = 0 \implies \Delta_\partial\alpha = \Delta_{\bar{\partial}}\alpha = 0$$

whereby the above magnitude argument proves to us that:

$$\Delta_\partial\alpha = 0 \implies \partial\alpha = \partial^*\alpha = 0, \quad \Delta_{\bar{\partial}}\alpha = 0 \implies \bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$$

whereby any Δ -harmonic form is, in fact, Δ_∂ - and $\Delta_{\bar{\partial}}$ -**harmonic**; in particular, it is both ∂ - and $\bar{\partial}$ -**closed**. Thus, we may define the spaces of harmonic forms:

$$\begin{aligned}\mathcal{H}_\Delta^k(M; \mathbb{C}) &:= \{\alpha \in \mathcal{A}_\mathbb{C}^k(X) \mid \Delta\alpha = 0\} \\ \mathcal{H}^{p,q}(X) &:= \{\alpha \in \Omega^{p,q}(X) \mid \Delta\alpha = 0\}\end{aligned}$$

which readily gives us a **Hodge decomposition**:

$$\mathcal{H}_\Delta^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X)$$

Similarly, the same argument as before shows that, since all harmonic forms are $\bar{\partial}$ -**closed**, we get, for each p, q , a **natural map**:

$$\mathcal{H}^{p,q}(X) \longrightarrow H_{\bar{\partial}}^{p,q}(X)$$

In this setting, we have the important **Hodge decomposition theorem** stating that:

Theorem 6.9. *Hodge decomposition theorem*

Let (X, ω) be a compact Kähler manifold. Then, the natural map $\mathcal{H}^{p,q}(X) \rightarrow H_{\bar{\partial}}^{p,q}(X)$ is an **isomorphism**. Thus, we have an isomorphism via the **real Hodge theorem**:

$$H_{\text{dR}}^k(X; \mathbb{C}) \cong \mathcal{H}_\Delta^k(X; \mathbb{C}) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X) \cong \bigoplus_{p+q=k} H_{\bar{\partial}}^{p,q}(X)$$

which is, again, valid for all k , via the above isomorphism.

Proof. This result follows directly from the fact that $\Delta = 2\Delta_{\bar{\partial}}$, meaning that $\ker \Delta = \ker \Delta_{\bar{\partial}}$, i.e., the **d -harmonic forms** are precisely the same as the **$\bar{\partial}$ -harmonic forms**. \square

Using this theorem, we recall the definition of the **Betti numbers** as:

$$b^k := \dim_{\mathbb{R}} H_{\text{dR}}^k(X; \mathbb{R}) = \dim_{\mathbb{C}} H_{\text{dR}}^k(M; \mathbb{C}).$$

Now, we may refine these by defining the **Hodge numbers** as:

$$h^{p,q} := \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X)$$

so that the Hodge theorem will imply that:

$$b^k = \sum_{p+q=k} h^{p,q}.$$

Moreover, the Hodge theorem may be used to prove that:

$$H_{\bar{\partial}}^{p,q}(X) = \overline{H_{\bar{\partial}}^{p,q}(X)}$$

which gives us the **Hodge symmetry** $h^{p,q} = h^{q,p}$. Finally, the **Hodge star operator** considered above induces an isomorphism:

$$\star : H_{\bar{\partial}}^{p,q} \xrightarrow{\cong} H_{\bar{\partial}}^{n-p,n-q}.$$

which further proves that $h^{p,q} = h^{n-p,n-q}$; this is the so-called **central symmetry** or **Serre duality** of Hodge numbers; in particular, it means that:

Proposition 6.10. Odd Betti numbers are even

Proof. For each odd Betti number B_{2k+1} , we know that we may express:

$$\begin{aligned} b^{2k+1} &= \sum_{p+q=2k+1} h^{p,q} = \sum_{i=0}^k h^{2i,2k+1-2i} + \sum_{j=0}^k h^{2j+1,2k+1-(2j+1)} = \sum_{i=0}^k h^{2i,2k+1-2i} + \sum_{j=0}^k h^{2j+1,2(k-j)} \\ &= 2 \cdot \sum_{i=0}^k h^{2i,2k+1-2i} \end{aligned}$$

which is **even**, as claimed; this completes the proof. \square

In particular, this means that the number $h^{1,0} = h^{0,1} = \frac{1}{2}b^1$ is a **topological invariant**.

Finally, there is another simple fact that we know about compact Kähler manifolds: the existent of the Kähler 2-form ω means that $0 \neq [\omega^k] \in H^{k,k}(X)$ is a **non-trivial cohomology class**, whereby $h^{k,k} > 0$; thus, $b^{2k} \geq h^{k,k}$ proves that **even Betti numbers are positive**.

We usually organize these Hodge numbers $h^{p,q}$ in the form of a *Hodge diamond*; for a two-dimensional complex manifold, this looks like:

$$\begin{array}{ccc} & h^{2,2} & \\ & h^{2,1} & h^{1,2} \\ h^{2,0} & h^{1,1} & h^{0,2} \\ & h^{1,0} & h^{0,1} \\ & h^{0,0} & \end{array}$$

In fact, we now get:

Proposition 6.11. Harmonic Forms have minimal global norm

Consider a compact complex manifold (X, g) , equipped with a Hermitian metric g . Let $[\alpha] \in H_{\bar{\partial}}^{p,q}(X)$ be a Dolbeault cohomology class on X . Then, the $\bar{\partial}$ -harmonic representative of $[\alpha]$ is unique with minimal global norm $\|\alpha\|$.

Proof. To see this, we may indeed consider such a $\bar{\partial}$ -harmonic representative of α . Here, the property of being $\bar{\partial}^*$ -harmonic means precisely that $\bar{\partial}^* \alpha = 0$ is annihilated by the Hodge- $\bar{\partial}$ operator.

Consider, now, any other representative β of the same cohomology class; since $[\alpha] = [\beta]$, we must have $\beta = \alpha + \bar{\partial}\eta$, where η is some element of $\eta \in \Omega^{p,q-1}(X)$, i.e. a $(p, q-1)$ -form. Computing its norm $\|\beta\|^2$, we get:

$$\begin{aligned}\|\beta\|^2 &= \|\alpha + \bar{\partial}\eta\|^2 = (\alpha + \bar{\partial}\eta, \alpha + \bar{\partial}\eta) = (\alpha, \alpha) + (\bar{\partial}\eta, \bar{\partial}\eta) + (\alpha, \bar{\partial}\eta) + (\bar{\partial}\eta, \alpha) = \\ &= \|\alpha\|^2 + \|\bar{\partial}\eta\|^2 + (\alpha, \bar{\partial}\eta) + (\bar{\partial}\eta, \alpha)\end{aligned}$$

Now, the important point is to substitute the $\bar{\partial}$ in each inner product by its adjoint $\bar{\partial}^*$; by the definition of α (it is $\bar{\partial}$ -harmonic), we know that it is annihilated by $\bar{\partial}^*$, so that:

$$(\alpha, \bar{\partial}\eta) + (\bar{\partial}\eta, \alpha) = (\bar{\partial}^*\alpha, \eta) + (\eta, \bar{\partial}^*\alpha) = 0 + 0 = 0$$

Thus, we are left with:

$$\|\beta\|^2 = \|\alpha\|^2 + \|\bar{\partial}\eta\|^2 \geq \|\alpha\|^2$$

For equality to hold here, we would need $\|\bar{\partial}\eta\|^2 = 0$ globally, so that $\bar{\partial}\eta = 0$, and then $\beta = \alpha + \bar{\partial}\eta = \alpha$ is indeed the unique representative of α with minimal global norm, as desired. This completes the proof. \square

Now, using these tools, we may prove that:

Theorem 6.12. *Global holomorphic forms are closed*

*Let X^n be a compact Kähler manifold and let α be any global holomorphic p -form on X ; that is, $\alpha \in \Omega^{p,0}(X)$ is defined everywhere on X and $\bar{\partial}\alpha = 0$. Then, α is **closed**, i.e., $d\alpha = 0$.*

Proof. To see this result, we recall that the Hodge Laplacian on a Kähler manifold satisfies:

$$\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_\partial$$

where $\Delta = d\delta + \delta d$ while $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ and $\Delta_\partial = \partial\partial^* + \partial^*\partial$. Obviously, since the adjoint operator $\bar{\partial}^*$ maps $\bar{\partial}^* : \Omega^{p,q}(X) \rightarrow \Omega^{p,q-1}(X)$, in this case we will have $\bar{\partial}^* : \Omega^{p,0}(X) \rightarrow \Omega^{p,-1}(X) = 0$, whereby $\bar{\partial}^*\alpha = 0$ evidently, as well; in general, $\bar{\partial}^*\alpha = 0$ for all $\alpha \in \Omega^{p,0}(X)$.

Putting these results together means that $\Delta_{\bar{\partial}}\alpha = 0$ (it is $\bar{\partial}$ -harmonic, since also $\bar{\partial}\alpha = 0$) and thus $\Delta\alpha = 2\Delta_{\bar{\partial}}\alpha = 0$ as well; thus, it is **harmonic**. On the other hand, we have the global norm expression for a compact manifold X :

$$\langle \Delta\alpha, \alpha \rangle = \|d\alpha\|^2 + \|d^*\alpha\|^2 \equiv \|d\alpha\|^2 + \|\delta\alpha\|^2$$

whereby $\Delta\alpha = 0$ implies, in particular, $d\alpha = 0$, as desired. This result follows simply from the fact that **harmonic forms are closed**.

Alternatively, we could also have argued that since α is $\bar{\partial}$ -harmonic (thus, $\bar{\partial}$ -closed) and $\Delta_\partial = \Delta_{\bar{\partial}}$ on a Kähler manifold, α will also be ∂ -harmonic; thus, ∂ -closed. As a result, we get $d\alpha = \partial\alpha + \bar{\partial}\alpha = 0$, as desired.

Finally, for **yet another** way to prove this result: we can use the **$\partial\bar{\partial}$ -lemma**, applied to the $(p+1, 0)$ -form $\beta := d\alpha = \partial\alpha$, since $\bar{\partial}\alpha = 0$ by construction. Indeed, for this, we see that $d\beta = d^2\alpha = 0$, so it satisfies the properties of the lemma; and it is **∂ -exact** by construction ($\beta = \partial\alpha$) and it satisfies $\bar{\partial}\beta = \bar{\partial}\bar{\partial}\alpha = -\partial\bar{\partial}\alpha = 0$, since α is holomorphic. Thus, the equivalent conditions of the **$\partial\bar{\partial}$ -lemma** show that we must have $\bar{\partial}\alpha = \beta = \partial\bar{\partial}\gamma$ for some $(p-1, 0)$ -form γ (as seen by looking at the bidegree), i.e., $\alpha + \partial\gamma$ should be $\bar{\partial}$ -closed. \square

At this point, it is useful to recall the definition of the **Bott-Chern cohomology groups** $H_{\text{BC}}^{p,q}(X)$, which are defined over **any Hermitian manifold**, not necessarily Kähler:

$$H_{\text{BC}}^{p,q}(X) = \frac{\{\alpha \in C^\infty(X, \mathcal{E}_X^{p,q}) \mid d\alpha = 0\}}{\partial\bar{\partial}C^\infty(X, \mathcal{E}_X^{p-1,q-1})}$$

Indeed, this quotient is well-defined as we have:

$$\forall \beta \in C^\infty(X, \mathcal{E}_X^k), \quad d(\partial\bar{\partial}\beta) = (\partial + \bar{\partial})\partial\bar{\partial}\beta = \partial^2\bar{\partial}\beta - \partial\bar{\partial}^2\beta = 0$$

which means that all elements in the image $\partial\bar{\partial}C^\infty(X, \mathcal{E}_X^{p-1,q-1})$ indeed lie in the set defined above. Consequently, we see that the natural morphism:

$$\{\alpha \in C^\infty(X, \mathcal{E}_X^{p,q}) \mid d\alpha = 0\} \longrightarrow H_{\text{dR}}^{p+q}(X; \mathbb{C})$$

now **passes to the quotient** by the above image, thus inducing a **canonical map**:

$$H_{\text{BC}}^{p,q}(X) \longrightarrow H_{\text{dR}}^{p+q}(X; \mathbb{C})$$

Clearly, if the differential form ω here were, for example, a **pure** (p, q) -form, we know that $d\omega$ is a $(p+1, q)$ -form and $\bar{\partial}\omega$ is a $(p, q+1)$ -form; consequently, if $d\omega = 0$, we obtain $d\omega = \partial\omega + \bar{\partial}\omega = 0$, then, we *automatically* get **both** of $\partial\omega = \bar{\partial}\omega = 0$. That is, any d -closed (p, q) -form is also $\bar{\partial}$ -closed, meaning that we get a natural morphism:

$$\{\alpha \in C^\infty(X, \mathcal{E}_X^{p,q}) \mid d\alpha = 0\} \longrightarrow H_{\bar{\partial}}^{p,q}(X)$$

whereby, since also $\bar{\partial}(\partial\bar{\partial}\beta) = 0$ for any differential form β , we see that the morphism factors through the quotient by the image to yield, again, a **canonical map**:

$$H_{\text{BC}}^{p,q}(X) \longrightarrow H_{\bar{\partial}}^{p,q}(X)$$

Importantly, we notice that **these results hold on any complex manifold**, even if they do not possess the Kähler property; of course, if we do have Kähler manifolds, then we may use **Hodge theory** to formulate the so-called **$\partial\bar{\partial}$ -lemma** enables us to compare the Bott-Chern cohomology with the Dolbeault ($\bar{\partial}$) and the de Rham cohomology. To demonstrate the connection between these theories, we consider:

Theorem 6.13. Global $\partial\bar{\partial}$ -lemma on compact Kähler manifold

Let X be a compact complex manifold of Kähler type (i.e., admitting a Kähler metric) and recall the construction of the space (sheaf) of (p, q) -cycles $\mathcal{Z}^{p,q} := \mathcal{A}^{p,q} \cap \ker d$, where $d : \mathcal{A}_X^\bullet \rightarrow \mathcal{A}_X^\bullet$ is the differential operator on the entire complex. For a **d -closed** (p, q) -form $\alpha \in \mathcal{Z}^{p,q}(X)$ (i.e., simultaneously ∂ -closed and $\bar{\partial}$ -closed) TFAE:

- (a) $\alpha \in \text{im } d$,
- (b) $\alpha \in \text{im } \partial$,
- (c) $\alpha \in \text{im } \bar{\partial}$,
- (d) $\alpha \in \text{im}(\partial\bar{\partial})$,
- (e) α is **orthogonal** to $\mathcal{H}^{p,q}(X)$.

Here, a very important application of this result is that **the first four conditions are completely independent** of the choice of Kähler metric on X ; while the fifth one, equivalent to them, characterizes the space of harmonic forms. Consequently, we see that **the space**

of harmonic forms is, in fact, independent of the choice of Kähler metric on X . In some sense, this can be viewed as unexpected, because the construction of harmonic forms *did* involve the metric; for example, in order to construct the Laplace-Beltrami operator $\Delta = d\delta + \delta d$, we had to define the codifferential δ using the **metric** and the L^2 -inner product that came from it.

Instead of directly proving the result, we offer two equivalent restatements:

Theorem 6.14. $\partial\bar{\partial}$ -lemma for Kähler Manifolds

Let X be a Kähler manifold and let ω be a differential form that is both ∂ - and $\bar{\partial}$ -closed (i.e., $\partial\omega = \bar{\partial}\omega = 0$; thus, also $d\omega = 0$). Then, if ω is d - or ∂ - or $\bar{\partial}$ -exact, there exists a differential form η such that $\boxed{\omega = \partial\bar{\partial}\eta}$.

Proof. First of all, we see that the ∂ - and $\bar{\partial}$ -cases are **equivalent**: indeed, we have seen that $\bar{\partial}\omega = \bar{\partial}\bar{\omega}$, whereby the one result follows from the other by taking conjugates. Now, if the form ω is **d -exact**, we may write $\omega = d\beta$ for some form β ; then, by **Hodge theory**, we know that this form β may be decomposed as:

$$\beta = \alpha + \Delta_d\gamma$$

where Δ_d is the usual d -Laplacian (from Riemannian Geometry). Now, a standard result of Hodge theory shows that **harmonic forms are d -closed**.

Again, we may WLOG only treat the case of $\omega = \bar{\partial}\varphi$ being $\bar{\partial}$ -exact, for some smooth differential form $\varphi \in \Gamma^\infty(X, \mathcal{E}_X^k)$; all other cases are completely analogous. Now, the results of **Hodge theory** suggests that φ admits a decomposition of the form:

$$\boxed{\varphi = \alpha + \partial\beta + \partial^*\gamma}$$

where $\alpha \in \mathcal{H}_\partial^k$ is a **harmonic** k -form while $\beta \in C^\infty(X, \mathcal{E}_X^{k-1})$ while $\gamma \in C^\infty(X, \mathcal{E}_X^{k+1})$. Now, the key point is that since X is **Kähler**, we have $\Delta_\partial = \Delta_{\bar{\partial}}$, i.e., the spaces:

$$\mathcal{H}_\partial^k(X) := \left\{ \Gamma^\infty(X, \mathcal{E}_X^k) \mid \Delta_\partial\alpha = 0 \right\}, \quad \mathcal{H}_{\bar{\partial}}^k(X) := \left\{ \Gamma^\infty(X, \mathcal{E}_X^k) \mid \Delta_{\bar{\partial}}\alpha = 0 \right\}$$

are **the same**, $\mathcal{H}_\partial^k = \mathcal{H}_{\bar{\partial}}^k$. Moreover, we saw earlier that harmonic forms have $d\alpha = \partial\alpha = \bar{\partial}\alpha = 0$; consequently, we may now express:

$$\omega = \bar{\partial}\varphi = \bar{\partial}\alpha + \bar{\partial}\partial\beta + \bar{\partial}\partial^*\gamma = \bar{\partial}\partial\beta + \bar{\partial}\partial^*\gamma$$

and, since the **Kähler identities** guarantee that we may express $\bar{\partial}\partial^* = -\partial^*\bar{\partial}$, we may *also* express this as:

$$\omega = \bar{\partial}\partial\beta + \bar{\partial}\partial^*\gamma = -\partial\bar{\partial}\beta - \partial^*\bar{\partial}\gamma$$

whereby, since $\partial\bar{\partial}\beta$ is already ∂ -exact (thus, in particular, ∂ -closed) we deduce that $\partial\partial^*\bar{\partial}\gamma = 0$, as well. But now, our earlier observation regarding adjoints suggests that this can only mean that $\partial^*\bar{\partial}\gamma = 0$; and, since $\bar{\partial}\partial^* = -\partial^*\bar{\partial}$ by the Kähler relations, this actually gives us $\bar{\partial}\partial^*\gamma = 0$. Consequently, we are left with:

$$\omega = \bar{\partial}\partial\beta + \bar{\partial}\partial^*\gamma = \bar{\partial}\partial(-\beta)$$

which proves that ω has the desired form, for $\eta = -\beta$; thus, we obtain the desired result and the proof is complete. \square

To see this result, it makes sense to first build up a different lemma:

Proposition 6.15. ∂ - and $\bar{\partial}$ -lemmas for Kähler manifolds

Let X be a Kähler manifold and let ω be a differential form. Then:

- (i) If ω is ∂ -exact and d -exact or $\bar{\partial}$ -exact, then, it is $\partial\bar{\partial}$ -exact, i.e., it can be expressed in the form $\omega = \partial\bar{\partial}\eta$ for a differential form η .
- (ii) If ω is $\bar{\partial}$ -exact and d -exact or ∂ -exact, then, it is $\partial\bar{\partial}$ -exact, i.e., it can be expressed in the form $\omega = \partial\bar{\partial}\eta$ for a differential form η .

Proof. We show only the first part of the problem, as the second part will follow directly from this. If ω is ∂ -exact, then, we may write $\omega = \partial\beta$ for a differential form β . Then, **Hodge theory** gives us a decomposition of the form:

$$\beta = \alpha + \Delta_d\gamma$$

where γ is some differential form, α is a **harmonic** form, and Δ_d denotes the usual (Hodge) Laplacian.

Now, we know that harmonic forms (as is α) are d -, ∂ -, and $\bar{\partial}$ -closed, that is, $\bar{\partial}\alpha = 0$ in this case; consequently, we deduce that:

$$\omega = \partial\beta = \partial(\alpha + \Delta_d\gamma) = \partial\alpha + \partial\Delta_d\gamma = \partial\Delta_d\gamma$$

Moreover, since the manifold X is **Kähler**, we know that the d -, ∂ -, and $\bar{\partial}$ -Laplacians are related by:

$$\Delta_d = \delta\delta + \delta d, \quad \Delta_\partial = \partial\partial^* + \partial^*\partial, \quad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

for which we obtain $\boxed{\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}}$ due to the Kähler property. Consequently, we may now write:

$$\begin{aligned} \omega &= \partial\Delta_d\gamma = 2\partial\Delta_{\bar{\partial}}\gamma = 2\partial(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\gamma \\ &= 2\partial\bar{\partial}\bar{\partial}^*\gamma + 2\partial\bar{\partial}^*\bar{\partial}\gamma \\ &= -2\bar{\partial}\partial\bar{\partial}^*\gamma + 2\partial\bar{\partial}^*\bar{\partial}\gamma \end{aligned}$$

Now, we want to use the key assumption that ω is $\bar{\partial}$ -closed, i.e., $\bar{\partial}\omega = 0$; of course, the first term $\bar{\partial}\partial\bar{\partial}^*\gamma$ is clearly $\bar{\partial}$ -exact, so, in particular, it is $\bar{\partial}$ -closed; thus, the second summand must be $\bar{\partial}$ -closed as well, i.e.:

$$\bar{\partial}\bar{\partial}^*\bar{\partial}\bar{\partial}\gamma = -\bar{\partial}\bar{\partial}^*\bar{\partial}\bar{\partial}\gamma = \bar{\partial}\bar{\partial}\bar{\partial}^*\bar{\partial}\gamma = 0$$

where, for the first equality, we used the **Kähler identities** for Kähler manifolds, which ensure (among other properties) that $\boxed{\bar{\partial}\bar{\partial}^* = -\bar{\partial}^*\bar{\partial}}$ (among other things, this is **equivalent** to the Laplacian equality $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$). Here, we also used the standard differential identity $\bar{\partial}\bar{\partial} = -\bar{\partial}\bar{\partial}$.

Now, the important observation is that we may **also** compute now, using the definition of $\bar{\partial}^*$ as the **formal adjoint** of the $\bar{\partial}$ operator with respect to the L^2 -inner product on forms. The key observation here is that for **any** operator A with L^2 -formal adjoint A^* , then, we

get for any element φ (in this case, a differential form):

$$AA^*\varphi \equiv 0 \implies A^*\varphi \equiv 0$$

Indeed, to see this result, we need only prove, equivalently, that the global L^2 -norm $\|A^*\varphi\|^2 = 0$ on X . To compute this, we write:

$$\|A^*\varphi\|^2 = \langle A^*\varphi, A^*\varphi \rangle = \langle AA^*\varphi, \varphi \rangle = 0$$

using the assumption that $AA^*\varphi \equiv 0$ from above; this proves that $A\varphi = 0$, as claimed.

Now, using this result in the case of $\bar{\partial}\bar{\partial}^*\bar{\partial}\bar{\partial}\gamma = 0$, for $A = \bar{\partial}$ and $\varphi = \bar{\partial}\bar{\partial}\gamma$, gives us $\bar{\partial}^*\bar{\partial}\bar{\partial}\gamma = 0$. Next, applying this result again to $A = \bar{\partial}^*$, with $A^* = \bar{\partial}$ and $\varphi := \bar{\partial}\gamma$, gives us $\bar{\partial}\bar{\partial}\gamma = 0$, i.e., $\bar{\partial}\bar{\partial}\gamma = 0$. Consequently, we may write:

$$\omega = 2\bar{\partial}\bar{\partial}^*\gamma + 2\bar{\partial}^*\bar{\partial}\gamma = 2\bar{\partial}\bar{\partial}^*\gamma - 2\bar{\partial}^*\underbrace{\bar{\partial}\bar{\partial}\gamma}_{=0} = 2\bar{\partial}\bar{\partial}^*\gamma$$

which proves that we may, indeed, express $\omega = \bar{\partial}\bar{\partial}\eta$, for $\eta := 2\bar{\partial}^*\gamma$, as desired; this completes the proof. \square

Consequently, this result allows us to build up the following conclusion:

Corollary 6.16. Hodge Decomposition on compact Kähler manifold

Let X be a compact Kähler manifold. Then, we have canonical morphisms:

$$H_{\text{BC}}^{p,q}(X) \xrightarrow{\sim} H_{\bar{\partial}}^{p,q}(X), \quad \bigoplus_{p+q=k} H_{\text{BC}}^{p,q}(X) \xrightarrow{\sim} H_{\text{dR}}^k(X, \mathbb{C})$$

and both of these maps are **isomorphisms**; in particular, the isomorphisms in the **Hodge decomposition theorem** are canonical.

Thus, finally, we *also* deduce that:

Theorem 6.17. Dolbeault Cohomology from Sheaf Cohomology of Holomorphic Forms

Let X be a compact complex manifold of Kähler type and let $\Omega_X^p \subset \mathcal{A}_X^{p,0}$ be the sheaf of holomorphic p -forms on X . Then, the (p, q) -Dolbeault cohomology group $H_{\bar{\partial}}^{p,q}(X) \cong H^q(X, \Omega_X^p)$ is isomorphic to the q -th cohomology group of the sheaf Ω_X^p .

Proof. This result follows from Hodge theory: if we put a Kähler metric on X , then, we know that the elements of $H_{\text{dR}}^k(X; \mathbb{C})$ are represented by Δ -harmonic forms, and $H_{\bar{\partial}}^{p,q}(X)$ can be identified with the harmonic forms of type (p, q) . Now, since $\Delta_d = 2\Delta_{\bar{\partial}}$, we know that the harmonic forms of type (p, q) are precisely the $\Delta_{\bar{\partial}}$ -harmonic forms of type (p, q) . Finally, by our previous result, we now that these are in bijection with $H^q(X, \Omega_X^p)$; this demonstrates that the association is an **isomorphism**; we will later see that this is, in fact, **canonical**. \square

6.1. The Greens' Operator. The **Greens operator** maps $G : \mathcal{A}_X^{p,q} \rightarrow \mathcal{A}_X^{p,q}$ and restricts to $G|_{\mathcal{H}} = 0$, i.e., it vanishes on harmonic forms, such that we get the commutation $\bar{\partial}G = G\bar{\partial}$ as well as $\bar{\partial}^*G = G\bar{\partial}^*$ and, moreover, $\text{Id} = \Delta_{\bar{\partial}}G + H$ is an inverse to $\Delta_{\bar{\partial}}$ up to the **projection map** $H : \mathcal{A}_X^{p,q} \rightarrow \mathcal{H}$ onto harmonic forms.

7. THE LEFSCHETZ DECOMPOSITION

Let X be a complex manifold of Kähler type and complex dimension n . If we specify a Kähler structure (Hermitian metric) on g (equivalently, a positive-definite closed real $(1, 1)$ -form ω) then we can define the **Hodge star operator** \star and the **Lefschetz operator** $L : \mathcal{A}_X^k \rightarrow \mathcal{A}_X^{k+2}$, as well as its L^2 -adjoint (with respect to the pairing of forms) $\Lambda : \mathcal{A}_X^k \rightarrow \mathcal{A}_X^{k-2}$, which is given by $\Lambda = \star^{-1}L\star = (-1)^k \star L\star$.

Importantly, we see now that:

Proposition 7.1. Commutation Relation for Lefschetz operators

Let (X, ω) be a Kähler manifold of complex dimension n with Hodge star operator \star and the Lefschetz operator $L : \mathcal{A}_X^k \rightarrow \mathcal{A}_X^{k+2}$, as well as its L^2 -adjoint (with respect to the pairing of forms) $\Lambda : \mathcal{A}_X^k \rightarrow \mathcal{A}_X^{k-2}$, which is given by $\Lambda = \star^{-1}L\star = (-1)^k \star L\star$. Then, these satisfy the commutation relation:

$$[L, \Lambda] = H = (k - n)\text{Id} : \mathcal{A}_X^k \rightarrow \mathcal{A}_X^k$$

for H the operator defined above.

Proof. The important idea here is that we are working with **operators of order 0** (as differential operators) so this is just a result of **local Hermitian geometry**; in particular, it suffices to prove this on any local holomorphic coordinate chart $(U; z^1, \dots, z^n)$ for X . In fact, since the manifold X is Kähler, we know that it admits **normal coordinates**, in terms of which we can WLOG assume that the metric is the standard flat Euclidean metric, so that the Lefschetz operator L is given by the wedge product with $\omega = \frac{i}{2} \sum_j dz^j \wedge d\bar{z}^j$, which can be written as $L = \sum_j A_j$, where each A_j is simply the operator of taking the exterior product with *just* $\frac{i}{2} dz^j \wedge d\bar{z}^j$. Then, by the previous result, we see that the following operators agree on \mathcal{A}_X^k :

$$\star^{-1}(d\bar{z}^j \wedge) \star = (-1)^{k+1} 2\iota_{\frac{\partial}{\partial z^j}}, \quad \star^{-1}(dz^j \wedge) \star = (-1)^{k+1} 2\iota_{\frac{\partial}{\partial \bar{z}^j}} \quad \text{on } \mathcal{A}_X^k$$

whereby wedging both of these terms means that we can write: $\boxed{\star^{-1}A_j\star = -2i \cdot \iota(\partial_j \wedge \partial_{\bar{j}})} \quad \square$

7.1. K3 Lectures parts. Let X be a smooth projective variety of dimension $\dim X = d$. By Hodge theory, we have, for any k , an isomorphism:

$$H_{\text{dR}}^n(X, \mathbb{C}) \cong H^n(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X)$$

where $H^{p,q}(X)$ denotes the Dolbeault cohomology groups or, equivalently, the sheaf cohomology groups $H^q(X, \Omega_X^p)$.

If L is an ample line bundle, then there is an ample class (analogous to the Kähler class in the complex differential case) $\omega := [L] \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$; this is a special version of the Kähler class, which lives in the intersection with $H^2(X, \mathbb{R})$. Then, one can construct the **Lefschetz operator** L on the Dolbeault cohomology, for $p + q = m$:

$$L^k : H^m(X, \mathbb{Q}) \supset H^{p,q}(X) \ni \alpha \mapsto \alpha \wedge \omega^k \in H^{p+k, q+k}(X) \subset H^{m+2k}(X, \mathbb{Q})$$

which is equivalently given, on the singular cohomology, by $v \mapsto v \cup \omega^k$.

Hence, one can define the *primitive cohomology groups* as:

$$H_{\text{prim}}^k(X) := \text{Ker} \left(L^{d-k+1} : H^k(X) \rightarrow H^{2d-k+2}(X) \right), \quad d = \dim X$$

For this, we have the result:

Theorem 7.2 (Lefschetz theorems). *We have the following theorems:*

- (i) **Hard Lefschetz theorem:** *the map $L^{d-n} : H^n(X) \rightarrow H^{2d-n}(X)$ is an isomorphism, for $n \leq d$; this is an isomorphism on all the Dolbeault groups.*
- (ii) **primitive cohomology generates the cohomology:** $H^k(X, \mathbb{Q}) = \bigoplus_{i \geq 0} L^i H_{\text{prim}}^{k-2i}(X)$, and again this restricts to the Dolbeault groups.

For example, if X is a surface (so $d = 2$) and we take $k = 2$, then the Lefschetz map takes:

$$L : H^2(X) \rightarrow H^4(X) = H^{2,2}(X)$$

and we get the cohomology:

$$\text{Ker}(L|_{H^{1,1}}) = H_{\text{prim}}^{1,1} = (H_{\text{prim}}^2)^{1,1}$$

and the intersection pairing $\langle -, - \rangle$ is negative-definite on $H_{\text{prim}}^{1,1}$.

8. HODGE STRUCTURES

In this section, we will combine all the results established up until now to show that the **rational cohomology of a complex compact polarised manifold admits a decomposition as a direct sum of polarised Hodge structures**. For starters, we should define the **integral and rational Hodge structures**; these are the structures which lie naturally on the **integral or rational cohomology** of a compact Kähler manifold; they are given by the Hodge decomposition of the cohomology with complex coefficients. For example, we can study the case of the Hodge structure of weight 1, which is equivalent to defining a **complex torus**. Finally, we will also study **morphisms of Hodge structures** and the **functoriality properties under direct or inverse image** of the Hodge structure on the cohomology of Kähler manifolds with respect to holomorphic maps. In particular, we will establish a simple result on morphisms of Hodge structures with powerful and useful generalizations to **mixed Hodge structures**.

Here, our key motivating result is:

Theorem 8.1. *The morphisms of Hodge structures are strict for the Hodge filtration.*

Definition 8.2. Integral (Pure) Hodge structure

An **integral (pure) Hodge structure** of (integer) weight $k \in \mathbb{Z}$ is given by a finitely

generated abelian group $V_{\mathbb{Z}}$,¹² together with a direct sum decomposition:

$$V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$$

which satisfies the conjugation $V^{q,p} = \overline{V^{p,q}}$.¹³

An equivalent definition of Hodge structures is obtained by replacing the direct sum decomposition of $V_{\mathbb{C}}$ by the **Hodge filtration** FV , a **finite decreasing filtration** of $V_{\mathbb{C}}$ by complex subspaces $F^p V_{\mathbb{C}}$ (where $p \in \mathbb{Z}$) subject to the condition:

$$FV_{\mathbb{C}} \cap \overline{FV_{\mathbb{C}}} = \{0\}, \quad FV_{\mathbb{C}} \oplus \overline{FV_{\mathbb{C}}} = V_{\mathbb{C}}$$

Conversely, such a decomposition of $V_{\mathbb{C}}$, has the **associated Hodge filtration** FV :

$$F^p V_{\mathbb{C}} = \bigoplus_{r \geq p} V^{r,k-r}$$

which is a **decreasing filtration** on $V^{\mathbb{C}}$ (i.e., $F^1 V_{\mathbb{C}} \supset F^2 V_{\mathbb{C}} \supset \dots$) that satisfies:

$$V_{\mathbb{C}} = F^p V_{\mathbb{C}} \oplus \overline{F^{k-p+1} V_{\mathbb{C}}}, \quad F^p V_{\mathbb{C}} \cap \overline{F^{k-p+1} V_{\mathbb{C}}} = 0, \quad V^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}, \quad \forall p, q$$

This shows that, conversely, **the Hodge filtration determines the Hodge decomposition**. Here, k is the **weight** of the Hodge structure.

The construction of Hodge structures is motivated by the Hodge theory of **compact Kähler manifolds**: if X is such a manifold, we can let $V_{\mathbb{Z}} := H^k(X, \mathbb{Z})$ be its k -th integral cohomology group, so that $V_{\mathbb{C}} = H^k(X, \mathbb{C})$ is its k -th cohomology group with complex coefficients; and Hodge theory provides the decomposition of $V_{\mathbb{C}}$ into a direct sum as above, so that this information defines a **pure Hodge structure** of weight k . On the other hand, the **Hodge - de Rham spectral sequence** supplies H^k with the decreasing filtration $F^p H$, as in the second definition.

If, in particular, X is a compact Kähler manifold with $V_{\mathbb{Z}} = H^k(X, \mathbb{Z})$ (integral cohomology) we get the result:

Proposition 8.3. *Filtration of de Rham from Filtration of Forms*

Let X be a compact Kähler manifold. We let $F^p \mathcal{A}_X^k$ be the sheaf of complex-valued smooth differential forms which are sums of forms of type $(r, k-r)$, such that $r \geq p$ at each point. Then, we have:

$$F^p H_{\text{dR}}^k(X; \mathbb{C}) = \frac{\ker(d : F^p \mathcal{A}^k(X) \longrightarrow F^p \mathcal{A}^{k+1}(X))}{\text{im}(d : F^p \mathcal{A}^{k-1}(X) \longrightarrow F^p \mathcal{A}^k(X))}$$

i.e., the filtration of the de Rham cohomology space comes from a filtration of the complex of differential forms.

Proof. ADDDD □

¹²This can be thought of as having **finite rank**; i.e., as if we are *just* considering the de Rham cohomology groups $H_{\text{dR}}^k(X; \mathbb{C})$. We have preferred the term “*finitely generated abelian group*” to “*free abelian group of finite rank*”, as this enables us to consider a wider class of objects. In particular, this allows **torsion**, which means that the integer homology $H_k(X, \mathbb{Z})$ of any Kähler manifold X now turns out to admit a Hodge structure.

¹³Here, we have defined **complex conjugation** on the complex vector space $V_{\mathbb{C}}$ by $\overline{v \otimes z} := v \otimes \bar{z}$.

In particular, we can now recover the above result:

Corollary 8.4. Holomorphic Forms of Degree p

Let X be a compact Kähler manifold. Then, for every $p \in \mathbb{N}_0$, the Dolbeault cohomology group $H_{\bar{\partial}}^{p,0}(X) \simeq H^0(X, \Omega_X^p)$ is isomorphic to the space of holomorphic p -forms on X .

Proof. Indeed, by the above result, we know that $H_{\bar{\partial}}^{p,0}(X)$ is isomorphic to the space of closed forms of degree p and of type $(p, 0)$ on X . Such a form is then **holomorphic**, since it is $\bar{\partial}$ -closed; and we have seen earlier that holomorphic forms are **harmonic**, thus, in particular, d -closed. \square

Now, we would like to incorporate some additional properties into our Hodge structure: for applications in algebraic geometry (in particular, the classification of **complex algebraic varieties** by their periods) we see that the collection of all Hodge structures of weight k on $V_{\mathbb{Z}}$ is just “*too wide*” to offer particularly relevant information. However, we can substantially simplify this by using the **Hodge-Riemann bilinear relations** to define the notion of a **polarized Hodge structure of weight k** .

Earlier, we saw that if (X, ω) is an n -dimensional compact Kähler manifold, then, the induced cup (wedge) product with the class $[\omega] \in H_{\bar{\partial}}^{1,1}(X) \cap H_{\text{dR}}^2(X, \mathbb{R})$ coming from the Lefschetz operator gives a map $L : H^k(X, \mathbb{R}) \rightarrow H^{k+2}(X, \mathbb{R})$ which gives rise to the **Lefschetz decomposition**:

$$H^k(X, \mathbb{R}) = \bigoplus_r L^r H_{\text{prim}}^{k-2r}$$

where each component admits an induced Hodge decomposition, since the operator L has bidegree $(1, 1)$ for the bigrading given by the Hodge decomposition. Moreover, L gives an **intersection form** on any $H_{\text{dR}}^k(X; \mathbb{R})$, for $k \leq n$:

$$Q([\alpha], [\beta]) := \int_X \omega^{n-k} \wedge \alpha \wedge \beta = \langle L^{n-k} \alpha, \beta \rangle$$

where in the middle term, the α, β are the representatives of the cohomology class being considered; and the integral is well-defined by Stokes' theorem, since ω^{n-k} is **closed**. Moreover, we see that Q is **alternating (skew-symmetric)** if k is odd, and **symmetric** if it is even. Actually, it is convenient to normalize this as:

$$(8.0.1) \quad Q(\alpha, \beta) := (-1)^{\frac{n(n-1)}{2}} \alpha \wedge \beta \wedge \omega^{n-k}, \quad \alpha, \beta \in H_{\text{dR}}^k(X)$$

and then, we obtain:

Theorem 8.5 (Hodge-Riemann bilinear relations). *The intersection pairing $Q(-, -)$ has:*

- (i) *Assume that $\alpha \in H^{p,q}(X)$ and $\beta \notin H^{q,p}(X)$. Then, $Q(\alpha, \beta) = 0$.*
- (ii) *The pairing $i^{p-q} Q(\alpha, \overline{\beta})$ is positive-definite on $H_{\text{prim}}^{p,q}(X)$.*

In fact, the latter property is equivalent to saying that the vector space $H_{\text{prim}}^k(X)$ (equipped with the bilinear form Q) is **polarized**; this motivates the algebraic definition.

Example 8.6 (Lefschetz cohomology on a curve). Let C be a curve; hence, we have $H^1(C, \mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C)$. On the space $H^1(C, \mathbb{Z})$, we have a pairing given by the cup product; hence, the Hodge-Riemann bilinear relations imply that this is a polarization.

Example 8.7 (Lefschetz cohomology on a surface). Let X be a smooth projective surface, and consider (as before) an ample line bundle L with class $\omega := [L] \in H^2(X, \mathbb{Q})$. This has an intersection pairing, and we can express $H_{\text{prim}}^2(X) = \omega^\perp \cap H^{1,1}(X)$, which has a negative-definite pairing; thus, the pairing $-\langle -, - \rangle$ is a polarization on ω^\perp . Hence, we can decompose $H^2(X) = H_{\text{prim}}^2(X) \oplus \mathbb{C}\omega$; modifying the sign on ω actually gives us a polarization on all of $H^2(X)$.

Now, we may form the **induced Hermitian form** $H(\alpha, \beta) := i^k Q(\alpha, \bar{\beta})$ on $H_{\text{dR}}^k(X; \mathbb{C})$ (the complexified de Rham cohomology) which satisfies:

- (i) The Hodge decomposition is **orthogonal** for H ,
- (ii) The Lefschetz decomposition is orthogonal for H ,
- (iii) The primitive component $H^k(X)_{\text{prim}}$ has $i^{p-q-k}(-1)^{\frac{k(k-1)}{2}} H(\alpha, \alpha) > 0$, for any non-zero α of type (p, q) .

When the class $[\omega]$ is **integral**, i.e., $[\omega] \in H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$, then, the operator L descends to an operator on the **integral cohomology**; thus, the primitive component $H^k(X)_{\text{prim}} := \text{Ker } L^{n-k+1}$ is defined on \mathbb{Z} . Moreover, the intersection form Q is **integral**, i.e., it takes integral values on integral classes. Consequently, the structure obtained on the primitive part of the cohomology of such a Kähler manifold (X, ω) using the intersection form Q defined by the operator L is then the following:

Definition 8.8. Integral Polarised Hodge structure

An **integral polarised Hodge structure** of weight $k (\in \mathbb{Z})$ is given by a Hodge structure $(V_{\mathbb{Z}}, F^p V_{\mathbb{C}})$ of weight $k (\in \mathbb{Z})$, together with an intersection form Q on $V_{\mathbb{Z}}$, called a **polarization**, i.e., a perfect pairing on V that is **symmetric** for k even, **skew-symmetric** for k odd (i.e., $Q(\alpha, \beta) = (-1)^k Q(\beta, \alpha)$), and whose associated Hermitian (sesquilinear) form $H(\alpha, \beta) := i^k Q(\alpha, \bar{\beta})$ satisfies the conditions:

- (1) **1st Hodge-Riemann bilinear relations:** The bigraded (“Hodge”) decomposition $V_{\mathbb{Z}} = \bigoplus_{p+q=k} V^{p,q}$ is **orthogonal** for H .¹⁴
- (2) **2nd Hodge-Riemann bilinear relations:** for $\alpha \in V^{p,q} \setminus \{0\}$ of type (p, q) :

$$i^{p-q-k}(-1)^{\frac{k(k-1)}{2}} H(\alpha, \alpha) = (-1)^{\frac{k(k-1)}{2}-q} H(\alpha, \alpha) > 0$$

i.e., the Hermitian form $(-1)^{\frac{k(k-1)}{2}-q} H$ is **positive-definite** on $V^{p,q}$.

As a natural intermediate step, we could *weaken* this condition a little, and instead consider **rational polarised Hodge structures**: for this, we simply require V to have a **rational structure**, for which the intersection form Q is **rational**.

In terms of the **Hodge filtration** $(V_{\mathbb{Z}}, F^p V_{\mathbb{C}})$, these conditions imply that:

$$Q(F^p, F^{k-p+1}) = 0, \quad Q(C\alpha, \bar{\alpha}) > 0, \quad \forall \alpha \neq 0$$

where C is the **Weil operator** on $V_{\mathbb{C}}$, which acts as $C = i^{p-q}$ on $V^{p,q}$.

¹⁴Equivalently, this means that $Q(\alpha, \beta) = 0$ for $\alpha \in V^{p,q}, \beta \in V^{p',q'}$ with $(p', q') \neq (q, p) = (k-p, k-q)$.

8.1. Deligne's perspective on Hodge Structures. Now, we will present a more systematic description of Hodge structures as **representations of a certain algebraic group**. To this end, we define the ***Deligne torus*** \mathbb{S} :

$$\mathbb{S} := \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m = \left\{ \mathbb{C}^\times \text{ viewed as a } 2 - \text{dim real algebraic group} \right\} = \mathbb{R}^2 \setminus \{(0, 0)\}$$

with group structure given by the one induced by complex multiplication, i.e., $(a, b) \cdot (a', b') = (aa' - bb', ab' + a'b)$. Thus, a choice of root for $\sqrt{-1}$ yields an isomorphism $\mathbb{S}(\mathbb{R}) \xrightarrow{\sim} \mathbb{C}^\times$ given by $(a, b) \mapsto z := a + bi$,

Given a **real Hodge structure** $V = (V_{\mathbb{R}}, V_{\mathbb{C}} = \bigoplus V^{p,q})$ of weight k , we define a **homomorphism of real algebraic groups**:

$$\mathbb{S} \rightarrow \text{GL}(V_{\mathbb{R}}), \quad (a, b) \leftrightarrow (z = a + bi) \mapsto (\varphi : V_{\mathbb{R}} \mapsto V_{\mathbb{R}}, \quad \varphi|_{V^{p,q}} = z^{-p}\bar{z}^{-q})$$

i.e., an element (a, b) of \mathbb{S} corresponds to the endomorphism φ on $V_{\mathbb{R}}$ (which induces an endomorphism of $V_{\mathbb{C}}$ by $\varphi(v \otimes c) := \varphi(v) \otimes c$) that is given on the subspace $V^{p,q}$ by multiplication by $z^{-p}\bar{z}^{-q}$. Importantly, we see that this map φ indeed comes from an **endomorphism of the underlying real vector spaces**, in the sense that it is **invariant under conjugation**, since it acts on $V^{p,q}$ by $z^{-p}\bar{z}^{-q}$ and on $V^{q,p} = \overline{V^{p,q}}$ by $z^{-q}\bar{z}^{-p} = \overline{z^{-p}\bar{z}^{-q}}$, as it should. This means that $\varphi|_{\overline{V^{p,q}}} = \overline{\varphi|_{V^{p,q}}}$ respects conjugation, which means that this comes from a valid endomorphism on the underlying real space $V_{\mathbb{R}}$.

Using this terminology, we may now define:

Definition 8.9. Real Hodge structure of weight k

A **real Hodge structure of weight k** is a finite-dimensional \mathbb{R} -vector space together with an algebraic action of the 2-dimensional real algebraic group \mathbb{S} in which the element \tilde{r} acts as r^{-k} , for all $r \in \mathbb{R}^\times$.

Clearly, this definition extends uniquely to the action defined above for the complexification. Consequently, we obtain yet another definition of the Hodge structure based on the equivalence between the **\mathbb{Z} -grading** on a complex vector space and the action of the circle group $U(1)$. In this definition, we prescribe action of the multiplicative group of complex numbers \mathbb{C}^\times , viewed as a two-dimensional real algebraic torus, on H . This action must have the property that a real number a acts by a^k , where k is the weight of the Hodge structure; and the subspace $V^{p,q}$ is the subspace on which the element $z \in \mathbb{C}^\times$ acts as multiplication by $z^{-p}\bar{z}^{-q}$.

More succinctly, we see now that an integral (\mathbb{Z} -)Hodge structure is a finitely generated abelian group (\mathbb{Z} -module) $V_{\mathbb{Z}}$ together with a **real Hodge structure** on $V[\mathbb{R}] := V_{\mathbb{Z}} \otimes \mathbb{R}$. As before, for any real Hodge structure we can define the **Weyl operator** $C := h(\tilde{i})$ under the above map; by the above considerations, we know that this acts as i^{q-p} on $V^{p,q}$.

Based on these considerations, we constructed the **twisted even-weight Hodge structure**: for any $m \in \mathbb{Z}$, we define the Hodge structure $\mathbb{Z}(m)$ as a Hodge structure of weight $-2m$, with underlying abelian group $V_{\mathbb{Z}} = (2\pi i)^m \mathbb{Z}$ ¹⁵ and Hodge decomposition $V_{\mathbb{C}} = V^{-m,-m}$

¹⁵The factor of $(2\pi i)^m$ has been introduced for normalization reasons.

for the complexification, i.e., $\mathbb{Z}(m)_{\mathbb{C}} = \mathbb{Z}(1)^{-m,-m} = \mathbb{C}$.¹⁶ By the above discussion, this Hodge structure corresponds to the map of real algebraic groups with $\mathbb{S} \rightarrow \mathrm{GL}_1(\mathbb{R})$ with $\tilde{z} \mapsto (z\bar{z})^m$.

Now, we can also define:

Definition 8.10. Tensor Product of Hodge structures

Let V, W be Hodge structures of weights k, ℓ . Then, we can define their **tensor product** $V \otimes W$ as the Hodge structure of weight $k + \ell$ on the underlying finitely generated abelian group $V_{\mathbb{Z}} \otimes W_{\mathbb{Z}}$, such that the Hodge decomposition $(V \otimes W)_{\mathbb{C}} = \bigoplus_{r,s=k+\ell} (V \otimes W)^{r,s}$ has:

$$(V \otimes W)^{r,s} = \bigoplus_{(p,q)+(p',q')=(r,s)} V^{p,q} \otimes_{\mathbb{C}} W^{p',q'}$$

which can be seen to satisfy $(V \otimes W)^{s,r} = \overline{(V \otimes W)^{r,s}}$. Here, $(V \otimes W)_{\mathbb{C}} = V_{\mathbb{C}} \otimes_{\mathbb{C}} W_{\mathbb{C}}$.¹⁷

Using this definition, we can define the m -**twisting** of any Hodge structure as the tensor product $\boxed{V(m) := V \otimes \mathbb{Z}(m)}$. By the above considerations, we see that if V has weight k , then the twisting $V(m)$ has weight $k - 2m$.

In fact, we can now also define:

Definition 8.11. Morphism of Hodge structures

Let V, W be Hodge structures of the same weight k . A **morphism of Hodge structures** $\varphi : V \rightarrow W$ is a \mathbb{Z} -linear map $\varphi_{\mathbb{Z}} : V_{\mathbb{Z}} \rightarrow W_{\mathbb{Z}}$ that respects the Hodge decomposition, i.e., such that the induced map $\varphi_{\mathbb{C}} : V_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$ maps $V^{p,q} \rightarrow W^{p,q}$, i.e., $\varphi(V^{p,q}) \subseteq W^{p,q}$.¹⁸

Following our earlier definition, we see now that a morphism of Hodge structures can equivalently be described as a **morphism of real algebraic group representations** of the Deligne torus \mathbb{S} .

Thus, we may now offer an equivalent definition of this construction:

Definition 8.12. Morphism of Hodge structures

Let $V := (V_{\mathbb{Z}}, F^p V_{\mathbb{C}})$ and $W := (W_{\mathbb{Z}}, F^p W_{\mathbb{C}})$ be Hodge structures of weights n and $m := n+2r$ (for some $r \in \mathbb{Z}$) respectively, then, we may also define a **morphism of Hodge structures** $\varphi : V \rightarrow W$ as a map of abelian groups $\varphi_{\mathbb{Z}} : V_{\mathbb{Z}} \rightarrow W_{\mathbb{Z}}$ with \mathbb{C} -linear extension $\varphi_{\mathbb{C}} : V_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$ such that:

$$\varphi_{\mathbb{C}}(F^p V_{\mathbb{C}}) \subset F^{p+r} W_{\mathbb{C}}, \quad \text{equivalently,} \quad \varphi_{\mathbb{C}}(V^{p,q}) \subset W^{p+r,q+r}$$

In this setting, we call φ a morphism of Hodge structures of type (r, r) .

Importantly, we see now that:

¹⁶Clearly, this is the only possible non-trivial direct summand: indeed, for any other $V^{p,q}$, we get $V^{q,p} = \overline{V^{p,q}}$, so, these have the same dimension. Thus, if this dimension is **non-zero**, we get total dimension ≥ 2 for the vector space, a contradiction.

¹⁷Indeed, this happens because $(V \otimes W)_{\mathbb{C}} = (V \otimes W) \otimes \mathbb{C} = (V \otimes \mathbb{C}) \otimes_{\mathbb{C}} (W \otimes \mathbb{C}) = V_{\mathbb{C}} \otimes_{\mathbb{C}} W_{\mathbb{C}}$, via an application of the properties of the tensor product.

¹⁸Here, the **induced map** $\varphi_{\mathbb{C}}$ is, again, given by $\varphi_{\mathbb{C}}(\alpha \otimes z) := \varphi_{\mathbb{Z}}(\alpha) \otimes z$, for $\alpha \in V_{\mathbb{Z}}$ an element of our abelian group and $z \in \mathbb{C}$.

Lemma 8.13. Hodge morphisms are strict for Hodge filtration

Let $\varphi : V \rightarrow W$ be a morphism of Hodge structures of type (r, r) . Then, φ is **strict** for the Hodge filtration, i.e., it satisfies $\text{Im } \varphi \cap F^{k+r}W_{\mathbb{C}} = \varphi(F^kV_{\mathbb{C}})$.

Proof. To see this result, we may equivalently write $\beta = \varphi(\alpha)$ for some element $\beta \in F^{k+r}W_{\mathbb{C}}$ and decompose $\alpha = \sum_{p+q=n} \alpha^{p,q}$, so that $\beta = \sum_{p,q} \varphi_{\mathbb{C}}(\alpha^{p,q})$, where each $\varphi(\alpha^{p,q})$ is of type $(p+r, q+r)$, by the properties of the morphism of Hodge structures. Consequently, we now see that $\beta \in F^{k+r}W_{\mathbb{C}}$ if and only if $\varphi(\alpha^{p,q}) = 0$, for all $p < k$. But then, we would get precisely:

$$\beta = \varphi \left(\sum_{p \geq k} \alpha^{p,q} \right), \quad \sum_{p \geq k} \alpha^{p,q} \in F^kV_{\mathbb{C}}, \quad \implies \beta \in \varphi(F^kV_{\mathbb{C}})$$

as claimed; this completes the proof. \square

In particular, this allows us to deduce that the **quotient filtration** induced on $\text{Im}(\varphi)$ by $F^\bullet V_{\mathbb{C}}$ coincides with the filtration induced by $F^{\bullet+r}W_{\mathbb{C}}$ on the vector subspace $\text{Im}(\varphi)$. Consequently, this quotient filtration defines a **Hodge structure on the image** $\text{Im}(\varphi)$, since:

$$(\text{Im } \varphi)^{p+r,q+r} = \text{Im } \varphi \cap F^{p+r}W_{\mathbb{C}} \cap \overline{F^{q+r}W_{\mathbb{C}}} = \text{Im } \varphi \cap W^{p+r,q+r}, \quad p+q=n$$

from which we now deduce the decomposition:

$$\text{Im } \varphi = \bigoplus_{p+q=n} (\text{Im } \varphi)^{p+r,q+r}$$

On the other hand, the above argument shows that $(\text{Im } \varphi)^{p+r,q+r} = \varphi(V^{p,q})$, which proves that the decomposition claimed above is indeed true.

Having demonstrated the existence of a Hodge structure on the image $\text{Im } \varphi$, we may now also induce a Hodge structure on the kernel $\text{Ker } \varphi$:

Lemma 8.14. Hodge structure on the kernel $\text{ker } \varphi$

Consider a morphism of Hodge structures $\varphi : V \rightarrow W$ of type (r, r) , and let $F^p K_{\mathbb{C}} := K_{\mathbb{C}} \cap F^p V_{\mathbb{C}}$, where $K_{\mathbb{Z}} := \text{Ker } \varphi \subset V_{\mathbb{Z}}$, with corresponding complexification $K_{\mathbb{C}} := \text{Ker } \varphi_{\mathbb{C}} \subset V_{\mathbb{C}}$, so that $K_{\mathbb{C}} = K_{\mathbb{Z}} \otimes \mathbb{C}$. Then, $(K_{\mathbb{Z}}, F^p K_{\mathbb{C}})$ is a Hodge structure.

Proof. For this result, it suffices to see that if $K^{p,q} = F^p K_{\mathbb{C}} \cap \overline{F^q K_{\mathbb{C}}}$ for all $p+q=n$, then, we also get $K_{\mathbb{C}} = \bigoplus_{p+q=n} K^{p,q}$, by the equivalent defining properties of Hodge structures. Indeed, if $\alpha \in K_{\mathbb{C}}$, we may let its Hodge decomposition be $\alpha = \sum_{p+q=n} \alpha^{p,q}$ (where, a priori, we only know that $\alpha^{p,q} \in V^{p,q}$ is not necessarily contained in the bigraded pieces of the kernel) in $V_{\mathbb{C}}$. Now, we see that:

$$0 = \varphi(\alpha) = \sum_{p+q=n} \varphi(\alpha^{p,q}), \quad \varphi(\alpha^{p,q}) \in W^{p+r,q+r}, \quad \implies \varphi(\alpha^{p,q}) = 0, \quad \forall p, q$$

which means that $\varphi(\alpha^{p,q}) = 0$ for all $\alpha^{p,q} \in K^{p,q}$, so that this is a valid Hodge decomposition, as desired; this completes the proof. \square

One can show similarly that the induced filtration on the cokernel of φ (potentially *modulo torsion*, depending on our definition of a Hodge structure) correspondingly defines a Hodge structure on $\text{coker } \varphi$. Thus, more generally, if we consider the category of **rational**

Hodge structures of given weight, where the morphisms are the morphisms of rational Hodge structures of type $(0, 0)$, the preceding results show that this category is an **abelian category**, where exact sums are defined in the obvious way.

In fact, we now obtain an even more rigid category, that of **polarised Hodge structures** together with morphisms that are morphisms of Hodge structures.

Theorem 8.15 (Weight 1 Hodge structures). *Recall that a polarized Hodge structure of weight 1 corresponds to an integral lattice $V \subseteq V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. The category of polarizable weight 1 Hodge structures is equivalent to the category of complex tori (abelian varieties).*

Proof. Consider the composition of the embedding of the lattice and the projection map:

$$V \subseteq V^{1,0} \oplus V^{0,1} \longrightarrow V^{1,0}$$

and so the abelian variety $A := V^{1,0}/V$ is a complex torus; on the level of vector spaces, this is equivalent to $A = \mathbb{C}^n/\Gamma$, where $\Gamma \subseteq \mathbb{C}^n$ is a full-rank lattice (i.e., $\mathbb{C}^n = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ is the real span of Γ). Concretely, the product $\Gamma \otimes_{\mathbb{Z}} \mathbb{C}$ has a decomposition via the $\pm i$ -eigenspaces:

$$\Gamma \otimes_{\mathbb{Z}} \mathbb{C} = \{+i \text{ eigenspace}\} \oplus \{-i \text{ eigenspace}\}$$

which is equivalent to a weight 1 Hodge structure.

Actually, we can view this construction more functorially: writing $A = \mathbb{C}^n/\Gamma$, we can express the space \mathbb{C}^n as:

$$\mathbb{C}^n = T_0 A = (T_0^{\vee} A)^{\vee} = H^0(A, \Omega_A^1)^{\vee}$$

because we recall that for any homogeneous space, the (co)tangent space to the identity is the same as the global sections of the tangent bundle, because we can use the homogeneous action to produce equivariant sections starting from an element of the fiber over the identity. Hence, we produce a map:

$$H_1(A, \mathbb{Z}) \ni \gamma \longmapsto \left(\omega \mapsto \int_{\gamma} \omega \right) \in H^0(A, \Omega_A^1)^{\vee}$$

which is an embedding. Of course, we again get $H^1(A, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^{1,0}(X) \oplus H^{0,1}(X)$. \square

Recall that a *polarization* in a variety is equivalent to an ample class in the Néron-Severi variety; using this, and the Kodaira embedding theorem, the existence of such a class implies that the variety is *algebraic*.

Recall, also:

Definition 8.16 (Albanese variety of a smooth projective variety). Let X be a smooth projective variety. There is an embedding $H_1(X, \mathbb{Z}) \hookrightarrow H^0(X, \Omega_X^1)^{\vee}$, by the above integration map (or by identifying with the singular integral and real de Rham cohomology). The resulting quotient is called the *Albanese variety (torus)* of X :

$$\text{Alb}(X) := H^0(X, \Omega_X^1)^{\vee} / H_1(X, \mathbb{Z})$$

and one has the *Albanese map* $\text{alb} : X \rightarrow \text{Alb}(X)$, constructed as follows: we fix a basepoint $p_0 \in X$, and for any point $x \in X$, we define:

$$\text{alb}(x) := \left(\omega \mapsto \int_{p_0}^x \omega \right) \in \text{Alb}(X) := H^0(X, \Omega_X^1)^{\vee} / H_1(X, \mathbb{Z})$$

where the integral is taken over any path connecting p_0 to x ; so this construction is well-defined up to the choice of path from p_0 to x , or (due to the homotopy-invariance of the integral) up to a based homology class at p_0 ; equivalently, an element of $H_1(X, \mathbb{Z})$.

Here, we can also use the notion of polarizations in order to construct a **complex torus that is not an abelian variety**:

Example 8.17. Complex Torus is not Abelian Variety

Let $e_1, \dots, e_4 \in \mathbb{C}^2$ two-dimensional complex vectors whose coordinates are all **algebraically independent**. Let Λ be the lattice spanned by these vectors, and consider the complex torus $A := \mathbb{C}^2/\Lambda$ constructed in this way. Then, A is not an abelian variety.

Indeed, we can let B be the period matrix with respect to the canonical basis, so that $A := \mathbb{C}^2/\Lambda$ will be an abelian variety with polarization H whose real part is a matrix J . Then, we would have $\Pi J \Pi^T = 0$ (cf. Bierkanhake-Lange), which would produce some algebraic relations; contradiction.

8.2. Hodge structures on K3 surfaces. Now, we study Hodge structures of weight 2 coming from K3 surfaces: for X a K3 surface, we can write:

$$H^2(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}\alpha \oplus H^{1,1} \oplus \mathbb{C}\bar{\alpha}$$

where, as we have seen before, we get $\dim H^{1,1} = 20$ and the other two pieces each have dimension.

Definition 8.18 (K3 Hodge structure). A *K3 type Hodge structure* is a weight 2 Hodge structure such that ADDD

We can be more specific about the algebraic behavior of these structures:

Definition 8.19 (Transcendental lattice). Let V be a Hodge structure of type K3. The *transcendental lattice* $T \subseteq V$ is the smallest primitive sub-Hodge structure containing $H^{2,0} \oplus H^{0,2}$.

This is in contrast to the *algebraic lattice*; in fact, it is its orthogonal complement:

$$V = H^2(X, \mathbb{Q}), \quad T(X) := (V^{1,1} \cap H^2(X, \mathbb{Z}))^\perp$$

so this characterization shows that:

Definition 8.20 (Transcendental lattice is irreducible). If V is a polarizable Hodge structure, then T is an irreducible polarizable sub-Hodge structure.

Proof. First, $T \subseteq V$ is primitive; hence, we can decompose $V = T \oplus T^\perp$. Thus, T is also polarized (by restricting the polarization on V). If $T_0 \subseteq T$ were a sub-Hodge structure, and it contained a $(2,0)$ -piece, then we would get $T_0 \supseteq V^{2,0} \oplus V^{0,2}$; a contradiction.

Hence, T_0 can only have $(1,1)$ -type pieces. But then, $T_0^\perp \subseteq T$ would contain the piece $V^{2,0} \oplus V^{0,2}$, contradiction; so T is irreducible. \square

We recall the following results for Hodge structure:

- (i) **weight 1:** the space of Hodge structures of weight 1 is isomorphic to the space of complex tori. The space of *polarizable* Hodge structures of weight 1 is isomorphic to the space of abelian varieties.

Moreover, if one assumes *principally polarized* Hodge structures (i.e., the bilinear form is *unimodular*: it induces an isomorphism $V \xrightarrow{\sim} V^\vee$) then this is isomorphic to PPAVs (principally polarized abelian varieties).

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and so, this space is isomorphic to the Segel upper half-plane quotient $\mathbb{H}_g/\mathrm{Sp}(2g, \mathbb{Z})$, where $g = \dim A$.

Definition 8.21 (Hodge isometry). A *Hodge isometry* is an isomorphism of Hodge structures $V \cong V'$ that preserves the pairing (bilinear form).¹⁹

Now, let (X, L) be a polarized K3 surface of degree $2d$ (this is L^2); we have the ample class:

$$\omega = [L] \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$$

and here we have the class $\omega^\perp = H_{\mathrm{prim}}^2(X, \mathbb{Z})$; this is also:

$$\omega^\perp = \Lambda_{2d} \subseteq \Lambda_{K3} \cong E_8 - 1)^{\oplus 2} \oplus \mathcal{U}^{\oplus 3} = H^2(X, \mathbb{Z})$$

In this setting, we have:

Theorem 8.22 (Global Torelli theorem). *We have the following global results:*

- (i) (*Burns-Rapoport*): Let X, X' be complex K3 surfaces. These are isomorphic, if and only if the Hodge structures $H^2(X, \mathbb{Z})$ and $H^2(X', \mathbb{Z})$ are Hodge isometric.
- (ii) (*Piatetski-Shapiro-Shafarevich*): Let (X, L) and (X', L') be polarized K3 surfaces of degree $2d$. These are isomorphic, if the orthogonal complements $[L]^\perp$ and $[L']^\perp$ are Hodge isometric.
- (iii) (*Mukai-Orlov*): The transcendental structures $T(X)$ and $T(X')$ are Hodge isometric, if and only if their derived categories $D_{\mathrm{Coh}}^b(X) \cong D_{\mathrm{Coh}}^b(X')$.

For a generic X , we know that this has Picard rank 1; and this L is primitive, so there is a unique L .

Remark 8.23 (Proof). The main insight is to prove the result for certain classes of “special” K3 surfaces, which are dense in the space of all K3 surfaces. Then, we show how to “approximate” (in an appropriate sense) any K3 surfaces via such surfaces; concretely, *Kummer varieties*.

For the last result, the idea is similar, in addition to producing a *Fourier-Mukai kernel*.

9. THE GRASSMANNIAN

First of all, we start with some general observations on the Grassmannian as an algebraic variety: let V be a vector space of dimension n over a field K ; then, we know how to construct the projective space $\mathbb{P}V$ associated to this, whose points are the 1–dimensional subspaces (“lines”) of V . Here, $\mathbb{P}V$ is a projective algebraic variety over K , of dimension $n - 1$. Then, the *Grassmannian* of k –planes $\mathbb{G}(k, V)$ is a projective variety over k that

¹⁹Equivalently, the *polarization* and *polarization pairing*, on the level of complex varieties.

extends this construction to subspaces of arbitrary dimension k , meaning that its points correspond bijectively to the k -dimensional subspaces of V .

Now, we pick a basis $\{e_1, \dots, e_n\}$ for the vector space V ; then, any k vectors $\{v_1, \dots, v_k\} \in V$ can be represented in the form of a $k \times n$ matrix $M(v_1, \dots, v_k)$, having the v_i as its columns. The rank of this matrix $M(v_i)$ is, then, the maximal number of linearly independent vectors among these; in particular, it is k (i.e., the matrix is *of full rank*) if all the vectors are linearly independent. In this case, we can associate to these vectors the linear subspace $\langle v_1, \dots, v_k \rangle \subset V$ that they generate; this association is clearly surjective

9.1. Moduli space of Hodge structures. Now, we study the moduli space of Hodge structures of weight k . First, for Hodge structure of weight 1 and dimension 2, we have a very simple description:

$$\text{Span}(w_1, w_2) = V \subseteq V^{1,0} \oplus V^{0,1} \rightsquigarrow E = V^{1,0}/V$$

and so this equivalence corresponds to the moduli of elliptic curves. Choosing w_1 to be the \mathbb{C} -basis of the space $V^{1,0}$, we can equivalently write these as $\langle 1, \tau \rangle$, for $\tau \in \mathbb{H}$; and we write $V = \text{Span}(1, \tau) \subseteq V^{1,0} \simeq \mathbb{C}$. Thus, we study the moduli space:

$$\mathcal{M}_{1,1} := \left\{ \text{weight 1 Hodge structures in } \mathbb{Z}^2 \right\} \cong \mathbb{H}/\text{SL}(2, \mathbb{Z})$$

More generally, a theorem guarantees that there exists a PPAV (principally polarized abelian variety) of $\dim = g$, denoted \mathcal{A}_g , that is the moduli space:

$$\mathcal{A}_g \cong \{\text{P.P. weight 1 Hodge structures of } \dim = 2g\} = \mathbb{H}_g/\text{Sp}(2g, \mathbb{Z}).$$

10. ANALYTIC ESTIMATES FOR VECTOR BUNDLES

Now, we want to prove the multi-dimensional Schwarz inequality in the following context:

Theorem 10.1. General Schwarz Lemma

For any $z \in \mathbb{C}^n$, we set $|z|_0 = \sup\{|z^j| : 1 \leq j \leq n\}$; we define the L^0 -disk in \mathbb{C}^n by $P(0, r) := \{z \in \mathbb{C}^n : |z|_0 < r\}$ for any $r > 0$. Now, let $f \in C^0(\overline{P(0, r)}) \cap \mathcal{O}_X(P(0, r))$ be a holomorphic function on $P(0, r)$ that is continuous up to the boundary. Moreover, assume that f vanishes up to order $m \in \mathbb{N}$ at 0. Then, for any $z \in \overline{P(0, r)}$, we have the **Schwarz inequality**:

$$f(z) \leq \|f\|_\infty \frac{|z|_0^m}{r^m}, \quad \forall z \in \overline{P(0, r)}, \quad \|f\|_\infty := \sup_{\overline{P(0, r)}} |f|$$

which generalizes the 1-dimensional Schwarz inequality for $m = 1$ and $n = 1$.

Proof. To see this result, we start by fixing some $z \in \overline{P(0, r)}$; for this, we construct the function $g : \{t \in \mathbb{C} : |t| \leq r\} \rightarrow \mathbb{C}$ given by $g(t) := t^{-m} f(tz/|z|_0)$, which is evidently **continuous** in t (since all the functions involved are); in fact, we can show this is a **holomorphic function** on $\{t \in \mathbb{C} : |t| < r\}$. Indeed, the holomorphicity of the function f on $P(0, r)$, together with the fact that it vanishes to order m at 0, means that we can express $f(z) = \sum_{k=m}^{\infty} P_k(z)$, where each P_k is a homogeneous degree- k polynomial in the variable z ; and plugging in this

power series into $g(t)$ shows that the pole of t^{-m} **cancels out** to produce a valid holomorphic function in t . ADDDDD

Next, we observe that for $|t| = r$, we have $|g(t)| \leq \|f\|_\infty r^{-m}$, where $\|f\|_\infty := \sup_{\overline{P(0,r)}} |f|$ is the supremum of f on the closed polydisk. Indeed, writing out the expression, we see that $|t| = r$ has $|g(t)| = |t|^{-m} |f(tz/|z|_0)| = r^{-m} |f(tz/|z|_0)|$, where z was fixed earlier; moreover, the **extreme value theorem** applied to the continuous function f on $\overline{P(0,r)}$ shows that f attains its maximum there; and the **maximum principle** for holomorphic functions tells us that this cannot be on the interior of $P(0,r)$, meaning that it is attained on the boundary $\partial P(0,r)$, so that:

$$\|f\|_\infty = \sup_{\overline{P(0,r)}} |f| = \sup_{\partial P(0,r)} |f| = \sup_{|w|_0=r} |f(w)|$$

so that, since $|(tz/|z|_0)|_0 = |t||z|_0/|z|_0 = |t| = r$, we get $f(tz/|z|_0) \leq \|f\|_\infty$.

Finally, we are almost there: we may now apply the same **extreme value theorem** followed by the **maximum principle** to the holomorphic function $g \in C^0(\overline{D^1(0,r)}) \cap \mathcal{O}_X(D^1(0,r))$, which, by the same argument as above, attains its maximum on the **boundary** of the disk $D^1(0,r) \subset \mathbb{C}$, i.e., precisely $\{|t| = r\}$. Consequently, we see that for any $t \in \overline{D^1(0,r)}$, we have, using the inequality established above:

$$|t^{-m} f(tz/|z|_0)| = |g(t)| \leq \sup_{|t|=r} |g(t)| \leq \|f\|_\infty r^{-m}$$

and setting $t = |z|_0 \leq r$ here gives us the desired inequality. \square

11. THE BOCHNER FORMULA

Now, we will prove the so-called **Bochner formula**. Our setting is the following:

Proposition 11.1 (The Bochner formula). *Let X^n be a complex Kähler manifold and consider a global holomorphic vector field V and a global holomorphic 1-form α on X . This means that locally (with local coordinates over an open set) these may be represented as $V = V^i \partial_i$ and $\alpha = \alpha_j dz^j$, such that the functions V^i, α_j are holomorphic. Then, for the Laplacian operator Δ_g with respect to the Riemannian metric g on X we get:*

$$(11.0.1) \quad \Delta_g |V|_g^2 = |\nabla V|_g^2 - \text{Ric}(V, \bar{V})$$

where $\text{Ric}(-, -)$ is the Ricci 2-form, while for the differential form α we have:

$$(11.0.2) \quad \Delta_g |\alpha|_g^2 = |\nabla \alpha|_g^2 + \text{Ric}(\alpha^\sharp, \bar{\alpha}^\sharp)$$

where α^\sharp is the vector field defined as:

$$\sharp : \Gamma(T^\vee X) \rightarrow \Gamma(TX), \quad \alpha \mapsto \alpha^\sharp := g^{i\bar{j}} \alpha_j \partial_i,$$

produced by the musical isomorphism $\sharp : T^\vee X \rightarrow TX$ (“sharp”).

Proof. First of all, we recall the definition of the relevant tensor norms with respect to the metric g associated to the Ricci-flat form Ric , which are given by *contraction* with the metric:

$$|V|_g^2 := g_{i\bar{j}} V^i \bar{V}^j, \quad |\alpha|_g^2 := g^{i\bar{j}} \alpha_i \bar{\alpha}_j$$

and contraction with the **inverse** metric, respectively.

To see this result, we start by recalling that **a Kähler manifold admits holomorphic normal coordinates** on a coordinate chart $U = (z^1, z^2, \dots, z^n)$ (and, in fact, a manifold is Kähler if and only if it admits such coordinates), in which the metric is diagonal, and the Christoffel symbols $\Gamma_{\beta\gamma}^\alpha(p) = 0$ vanish at the point.

In this setting, we have seen in class that the Laplace-Beltrami operator Δ_g acts on a function f as:

$$\Delta_g f := g^{k\bar{\ell}} \partial_k \partial_{\bar{\ell}} f$$

Thus, computing Δ_g for the tensor norm squared $|V|_g^2 := g_{i\bar{j}} V^i \bar{V}^j$, we obtain the expression:

$$\begin{aligned} \Delta_g |V|_g^2 &= \Delta_g (g_{i\bar{j}} V^i \bar{V}^j) = g^{k\bar{\ell}} \partial_k \partial_{\bar{\ell}} (g_{i\bar{j}} V^i \bar{V}^j) = \\ &= g^{k\bar{\ell}} \partial_k \partial_{\bar{\ell}} (g_{i\bar{j}} V^i \bar{V}^j) = g^{k\bar{\ell}} g_{i\bar{j}} \partial_k \partial_{\bar{\ell}} (V^i \bar{V}^j) + (g^{k\bar{\ell}} \partial_k \partial_{\bar{\ell}} g_{i\bar{j}}) V^i \bar{V}^j \end{aligned}$$

by using the product rule for the partials. In particular, for the first term $\partial_k \partial_{\bar{\ell}} (V^i \bar{V}^j)$, we know that the functions V^i are holomorphic, and the \bar{V}^j are by consequence anti-holomorphic, so that $\partial_k \bar{V}^j = \partial_{\bar{\ell}} V^i = 0$, and we are left with the term $\partial_k V^i \partial_{\bar{\ell}} \bar{V}^j$, so:

$$\Delta_g |V|_g^2 = g^{k\bar{\ell}} g_{i\bar{j}} \partial_k V^i \partial_{\bar{\ell}} \bar{V}^j + (g^{k\bar{\ell}} \partial_k \partial_{\bar{\ell}} g_{i\bar{j}}) V^i \bar{V}^j$$

Next, we recall the following general expression from class, involving the Ricci curvature:

$$R_{i\bar{j}} = -g^{k\bar{\ell}} \partial_i \partial_{\bar{j}} g_{k\bar{\ell}} + g^{k\bar{\ell}} g^{p\bar{q}} \partial_k g_{i\bar{q}} \partial_{\bar{\ell}} g_{p\bar{j}}$$

where, of course, the vanishing of the Christoffel symbols $\Gamma_{jk}^i = g^{k\bar{\ell}} \partial_i g_{\bar{\ell}j}$ in the normal holomorphic coordinates is equivalent to the vanishing of the first partial derivatives of the metric, whereby the second term vanishes (due to the $\partial_k g_{i\bar{q}}$ and $\partial_{\bar{\ell}} g_{p\bar{j}}$) and we are left with:

$$R_{i\bar{j}} = -g^{k\bar{\ell}} \partial_i \partial_{\bar{j}} g_{k\bar{\ell}}$$

in these normal coordinates. This shows us that the second term on the right-hand side is now precisely the negative of the contraction $\text{Ric}(V, \bar{V})$ by the Ricci $(1, 1)$ -form. In stating this, we are secretly using the fact that:

$$\partial_k \partial_{\bar{\ell}} g_{i\bar{j}} = \partial_i \partial_{\bar{j}} g_{k\bar{\ell}} \implies g_{k\bar{\ell}} \partial_k \partial_{\bar{\ell}} g_{i\bar{j}} = g_{k\bar{\ell}} \partial_i \partial_{\bar{j}} g_{k\bar{\ell}}$$

for the Ricci form, which relies on the fact that g is a Kähler metric, so that:

$$\partial_{\bar{\ell}} g_{i\bar{j}} = \partial_{\bar{j}} g_{i\bar{\ell}} \implies \partial_k \partial_{\bar{\ell}} g_{i\bar{j}} = \partial_k \partial_{\bar{j}} g_{i\bar{\ell}} = \partial_{\bar{j}} \partial_k g_{i\bar{\ell}} = \partial_{\bar{j}} \partial_i g_{k\bar{\ell}} = \partial_i \partial_{\bar{j}} g_{k\bar{\ell}}$$

since the partials commute. From now on, we will be using this equality of components of the Ricci tensor without further mention; for now, observe that:

$$(g^{k\bar{\ell}} \partial_k \partial_{\bar{\ell}} g_{i\bar{j}}) V^i \bar{V}^j = (g^{k\bar{\ell}} \partial_i \partial_{\bar{j}} g_{k\bar{\ell}}) V^i \bar{V}^j = -R_{i\bar{j}} V^i \bar{V}^j = -\text{Ric}(V, \bar{V})$$

Next, we compute the divergence $\nabla V := \nabla_k V^i$; since V is a vector field, we know that ∇V is an $(1, 1)$ tensor, with components given by $\nabla_i V^k = \partial_i V^k$ at the point, since the connection coefficients $\Gamma_{\beta\gamma}^\alpha(p) = 0$ vanish, by the choice of normal coordinates. Thus, we are left with its tensor norm, given by:

$$|\nabla V|_g^2 = |\nabla_k V^i|_g^2 = g^{k\bar{\ell}} g_{i\bar{j}} \partial_k V^i \partial_{\bar{\ell}} \bar{V}^j$$

which was precisely our remaining term. Here, we used secretly the property of Wirtinger derivatives that $\partial_{\bar{\ell}} \bar{f} = \overline{\partial_{\ell} f}$, so that $\partial_{\bar{\ell}} \bar{V}^{\ell} = \overline{\partial_{\ell} V^{\ell}}$. We will be using this without mention for the rest of the argument.

Finally, putting all our results together, we now see that:

$$(11.0.3) \quad \Delta_g |V|_g^2 = |\nabla V|_g^2 - \text{Ric}(V, \bar{V})$$

precisely as desired.

Next, we perform the same computation for α , whose norm squared is given by $|\alpha|_g^2 := g^{i\bar{j}} \alpha_i \bar{\alpha_j}$, as observed above. We write the Laplacian Δ_g as:

$$\begin{aligned} \Delta_g |\alpha_g|^2 &= \Delta_g (g^{i\bar{j}} \alpha_i \bar{\alpha_j}) = g^{k\bar{\ell}} \partial_k \partial_{\bar{\ell}} (g^{i\bar{j}} \alpha_i \bar{\alpha_j}) = \\ &= g^{k\bar{\ell}} g^{i\bar{j}} \partial_k \alpha_i \partial_{\bar{\ell}} \bar{\alpha_j} + (g^{k\bar{\ell}} \partial_k \partial_{\bar{\ell}} g^{i\bar{j}}) \alpha_i \bar{\alpha_j} \end{aligned}$$

by jumping directly into the final form, by the exact same line of argument that we used above. Next, by the same argument as the last line above, we see that $\nabla \alpha = \nabla_k \alpha_i$ is now a 2-form (i.e. a $(0, 2)$ -tensor), and $\nabla_k \alpha_i = \partial_k \alpha_i$ at this point by the vanishing of the Christoffel symbols, so that its tensor norm is given by:

$$|\nabla \alpha|_g^2 = |\nabla_k \alpha_i|_g^2 = g^{k\bar{\ell}} g^{i\bar{j}} \partial_k \alpha_i \partial_{\bar{\ell}} \bar{\alpha_j}$$

which is precisely the first term of the previous expansion. Thus, we will get our desired result, if and only if we can prove that:

$$(g^{k\bar{\ell}} \partial_k \partial_{\bar{\ell}} g^{i\bar{j}}) \alpha_i \bar{\alpha_j} = 0$$

is true for all indices $g_{i\bar{j}}$. This is not hard, just tedious: we need to recall the formula for the partial derivative of the inverse metric $g^{i\bar{j}}$, given in class by:

$$\partial_{\bar{\ell}} g^{i\bar{j}} = -g^{p\bar{j}} (\partial_{\bar{\ell}} g_{p\bar{q}}) g^{i\bar{q}}$$

Applying now the second partial derivative ∂_k using the product rule, we see that:

$$\partial_k \partial_{\bar{\ell}} g^{i\bar{j}} = -\partial_k (g^{p\bar{j}} (\partial_{\bar{\ell}} g_{p\bar{q}}) g^{i\bar{q}}) = -(\partial_k g^{p\bar{j}}) (\partial_{\bar{\ell}} g_{p\bar{q}}) g^{i\bar{q}} - g^{p\bar{j}} g^{i\bar{q}} \partial_k \partial_{\bar{\ell}} g_{p\bar{q}} - g^{p\bar{j}} (\partial_{\bar{\ell}} g_{p\bar{q}}) (\partial_k g^{i\bar{q}})$$

Now, we are again saved by our normal coordinates, which ensure that the partials of the metric $\partial_{\bar{\ell}} g_{p\bar{q}} = 0$ vanish at our selected point; as a result, the first and third terms are killed off, and we are left with:

$$\partial_k \partial_{\bar{\ell}} g^{i\bar{j}} = -g^{p\bar{j}} g^{i\bar{q}} \partial_k \partial_{\bar{\ell}} g_{p\bar{q}}$$

which is clearly much more manageable. Actually, we are very close: consider the musical isomorphism $\sharp : T^{\vee} X \rightarrow TX$ (“sharp”), which by a modification of our setting in (iv), is defined as:

$$\sharp : T^{\vee} X \rightarrow TX, \quad \alpha \mapsto \alpha^{\sharp} := g^{i\bar{q}} \alpha_q \partial_{\bar{i}},$$

whose components evidently satisfy:

$$(\alpha^{\sharp})^q = g^{i\bar{q}} \alpha_i, \quad (\alpha^{\sharp})^p = g^{p\bar{j}} \bar{\alpha_j}$$

since the metric tensor is real. Thus, we finally obtain:

$$\text{Ric}(\alpha^{\sharp}, \bar{\alpha^{\sharp}}) = R_{q\bar{p}} \overline{(\alpha^{\sharp})^p} (\alpha^{\sharp})^q = -g^{k\bar{\ell}} \partial_k \partial_{\bar{\ell}} g_{p\bar{q}} g^{i\bar{q}} \alpha_i g^{p\bar{j}} \bar{\alpha_j} = -g^{k\bar{\ell}} g^{p\bar{j}} g^{i\bar{q}} \partial_k \partial_{\bar{\ell}} g_{p\bar{q}} \alpha_i \bar{\alpha_j}$$

which is precisely our “stray” term from above. Thus, putting everything together, we see that:

$$(11.0.4) \quad \Delta_g |\alpha|_g^2 = |\nabla \alpha|_g^2 + \text{Ric}(\alpha^\sharp, \overline{\alpha^\sharp})$$

as claimed above. This completes the proof. \square

Now, consider the case when X admits a **Ricci-flat metric**; in this case, the above computations will give:

$$\nabla_g |V|_g^2 = |\nabla V|_g^2 \geq 0, \quad \nabla_g |\alpha|_g^2 = |\nabla \alpha|_g^2 \geq 0$$

since the Ricci tensor $\text{Ric}(-, -) = 0$ vanishes identically. Thus, the globally defined smooth functions $|\alpha|_g^2, |V|_g^2$ always return non-negative numbers by the application of Laplace-Beltrami operator Δ_g , which is an **elliptic operator** on functions; thus, by the **Strong Maximum Principle**, the only way we can have $\Delta_g f \neq 0$ everywhere when applied to a function f is if, in fact, f is constant, so that $\Delta_g f = 0$ everywhere. Applying this to the functions $f = |V|_g^2, |\alpha|_g^2$ shows that they must be everywhere constant, and in fact $|\nabla V|_g^2 = \Delta_g |V|_g^2 = 0 \implies \nabla V = 0$ everywhere, so that the vector field V is **parallel**. Likewise, $|\nabla \alpha|_g^2 = \Delta_g |\alpha|_g^2 = 0 \implies \nabla \alpha = 0$ everywhere, so that the 1-form α is also parallel.

Using these tools, we define:

Definition 11.2 (Musical isomorphisms on complex manifolds). Let X be a complex manifold and consider a vector field $V \in \Gamma(TX)$ and an 1-form $\alpha \in \Gamma(T^\vee X)$. From these, we may construct new global 1-forms and vector fields via the **musical isomorphism**, as follows: for an *expression in local coordinate* $V = V^i \partial_i$ and $\alpha = \alpha_i dz^i$, define the **musical isomorphisms**:²⁰

$$\flat : \Gamma(TX) \xrightarrow{\cong} \Gamma(T^\vee X), \quad V \mapsto V^\flat := g_{\bar{j}k} V^k d\bar{z}^j + g_{j\bar{k}} V^{\bar{k}} dz^j$$

i.e., the complex interpretation of $V^\flat := g_{\alpha\beta} V^\beta dz^\alpha$ for general smooth manifolds in light of g being of type $(1, 1)$ with respect to the ACS J . This is certainly an 1-form, pairing with vector fields W as:

$$\flat(V)(W) = V^\flat(W) := g(V, W)$$

Similarly, in terms of the musical isomorphism $\sharp : \Gamma(T^\vee X) \xrightarrow{\cong} \Gamma(TX)$ given by sending $\alpha \mapsto \alpha^\sharp := g^{\bar{j}k} \alpha_k \partial_{\bar{j}} + g^{j\bar{k}} \alpha_{\bar{k}} \partial_j$.²¹ Then, this is a valid vector field, pairing with other vector fields W under g as:

$$\langle \alpha^\sharp, W \rangle_g = \alpha(W)$$

Thus, the constructions \sharp, \flat give new vector fields and 1-forms.

Now, using these constructions have the following nice property:

Proposition 11.3 (Musical isomorphism gives parallel vector fields, forms). *In the same setting as above, let X be a complex Kähler manifold admitting a metric with vanishing Ricci tensor $\text{Ric} = 0$ and consider a global holomorphic vector field V and a global holomorphic*

²⁰Here, we need to change in the barred index because we are taking the conjugate.

²¹Again, this is the complex version of $\alpha^\sharp := g^{\gamma\delta} \alpha_\delta \partial_\gamma$. We notice again the change in the barred index because we took the conjugate.

1-form α . For these, construct, as above, the global 1-form $\beta := \overline{V^\flat}$ and the global vector field $W := \overline{\alpha^\sharp}$; these are **holomorphic**.

Proof. First, we start with $\beta := \overline{V^\flat}$, which is locally given by $\beta = \beta_j dz^j = g_{\bar{k}j} \overline{V^k} dz^j$, i.e., $\beta_j = g_{\bar{k}j} \overline{V^k}$; we know that this will be holomorphic, if and only if the functions β_j we constructed are holomorphic, thus, annihilated by $\bar{\partial}$. To see this result, we need only recall the following observations:

- The metric changes from barred to unbarred as $g_{\bar{j}k} = \overline{g_{jk}}$ while $g_{\bar{j}k} = g_{k\bar{j}}$ and $g_{j\bar{k}} = g_{\bar{k}j}$.
- The barred Wirtinger derivatives act as $\partial_{\bar{i}} \overline{f} = \overline{\partial_i f}$.
- The covariant derivative acts as $\nabla_\alpha V = (\partial_\alpha V^\beta + V^\gamma \Gamma_{\alpha\gamma}^\beta) \partial_\beta$
- The Christoffel symbols of the Chern connection (= Levi-Civita for X Kähler) are given by $\Gamma_{ij}^k = g^{k\bar{\ell}} \partial_i g_{\ell j}$, as well as $\Gamma_{i\bar{j}}^k = \Gamma_{\bar{i}j}^k = \Gamma_{ij}^k = 0$.

With these tools at hand, we look at the partials: we want:

$$\partial_{\bar{i}} (g_{\bar{k}j} \overline{V^k}) = \partial_{\bar{i}} \overline{g_{\bar{k}j} V^k} = \overline{\partial_i (g_{\bar{j}k} V^k)} = 0 \iff \partial_i (g_{\bar{j}k} V^k) = V^k \partial_i g_{\bar{j}k} + g_{\bar{j}k} \partial_i V^k = 0$$

Equivalently, we would like:

$$V^k \partial_i g_{\bar{j}k} + g_{\bar{j}k} \partial_i V^k = 0$$

Here, we know that the total covariance of V is given by:

$$\nabla V = (\partial_i V^k + \Gamma_{i\ell}^k V^\ell) \partial_k \otimes dz^i$$

again, because bars in lower Christoffel symbols make the term 0. Thus, we get (since $\nabla V = 0$ due to V being covariantly constant since $\text{Ric} = 0$):

$$\partial_i V^k + \Gamma_{i\ell}^k V^\ell = \partial_i V^k + g^{k\bar{\ell}} \partial_i g_{\bar{j}\ell} V^j = 0$$

and after multiplying both sides by $g_{\bar{j}k}$, we obtain the exact same expression as desired, because $\partial_i g_{\bar{j}\ell} V^\ell = \partial_i g_{\bar{j}k} V^k$ by switching the variable of summation. This completes the proof.

Of course, similar considerations apply to show that W is holomorphic. Indeed, we know that it will be holomorphic if and only if the functions W^j we constructed are holomorphic, thus, annihilated by the partial $\partial_{\bar{i}}$, for each index i . Here, we know that $W = W^j \partial_j = g^{j\bar{k}} \overline{\alpha_k} \partial_j$, with $W^j = g^{j\bar{k}} \overline{\alpha_k}$. For this, the action of the covariant derivative is:

$$\nabla_\alpha \theta = (\partial_\alpha \theta_\beta - \theta_\gamma \Gamma_{\alpha\beta}^\gamma) dz^\beta$$

where we changed the α here to a θ for notational convenience. Thus, the total covariance of α is given by:

$$\nabla \alpha = (\partial_i \alpha_k - \alpha_j \Gamma_{ik}^j) dz^k \otimes dz^i$$

again by the vanishing of barred-index Christoffel symbols. Now, we compute:

$$\partial_{\bar{i}} (V^j) = \partial_{\bar{i}} (g^{j\bar{k}} \overline{\alpha_k}) = \overline{\partial_i g^{j\bar{k}} \alpha_k} = \overline{\partial_i (g^{j\bar{k}} \alpha_k)} = 0 \iff \partial_i (g^{j\bar{k}} \alpha_k) = 0$$

On this expression, we will apply the product rule, along with the expression $\partial_i g^{j\bar{k}} = -g^{j\bar{\ell}} (\partial_i g_{\ell\bar{m}}) g^{\bar{m}k} = -g^{j\bar{\ell}} \Gamma_{i\ell}^k$ for the partials of the inverse metric (combined with the formula for the Christoffel symbols from before) to obtain, equivalently:

$$g^{j\bar{k}} \partial_i \alpha_k - \alpha_k g^{j\bar{\ell}} (\partial_i g_{\ell\bar{m}}) g^{\bar{m}k} = g^{j\bar{k}} \partial_i \alpha_k - \alpha_k g^{j\bar{\ell}} \Gamma_{i\ell}^k \stackrel{?}{=} 0$$

and the above observation that $\nabla\alpha = 0$ implies that $\partial_i\alpha_k = \alpha_j\Gamma_{ik}^j$. Thus, substituting this in, we want:

$$g^{\bar{j}k}\partial_i\alpha_k = g^{\bar{j}k}\alpha_j\Gamma_{ik}^j \stackrel{?}{=} \alpha_k g^{\bar{j}\ell}\Gamma_{i\ell}^k$$

which is again just a matter of changing the indices of summation, thus, done. \square

Remark 11.4. It is worth mentioning here that the correspondence between these two results can also be observed in a coordinate-free way, by following our previous notation of the musical isomorphisms. In this setting, we have, for any vector fields V, W :

$$V(\alpha(W)) = (\nabla_V\alpha)(W) + \alpha(\nabla_V W)$$

$$V(\langle\alpha^\sharp, W\rangle) = \langle\nabla_V\alpha^\sharp, W\rangle + \langle\alpha^\sharp, \nabla_V W\rangle$$

and since $\alpha(W) = \langle\alpha^\sharp, W\rangle$ by the construction of α^\sharp , we may equate these two expressions, as well as the corresponding $\alpha(\nabla_V W) = \langle\alpha^\sharp, \nabla_V W\rangle$, to obtain now:

$$(\nabla_V\alpha)(W) = \langle\nabla_V\alpha^\sharp, W\rangle$$

whereby the definition and uniqueness of the dual $(\nabla_V\alpha)$ implies that $(\nabla_V\alpha)^\sharp = \nabla_V\alpha^\sharp$, for any vector V , so that they have the same covariance $\nabla(\alpha^\sharp) = (\nabla\alpha)^\sharp$, whereby α^\sharp must also be covariantly constant since α is.

Now, using these tools, we may, in fact, prove that:

Proposition 11.5 (Lie bracket on global holomorphic vector fields is trivial). *Let X be a compact complex Kähler manifold and let g be a Ricci-flat metric on X (i.e., a Ricci-flat Kähler form ω). Then, the Lie derivative on global holomorphic vector fields is **trivial**; that is, we have $[V, W] = 0$ for all holomorphic vector fields V, W .*

Proof. To see this result, we first notice that the Lie bracket to see that $[V, W] \in \mathfrak{X}$ is also a vector field, for any two vector fields $V, W \in \mathfrak{X}$; if V, W are globally defined, then, so is $[V, W]$. In fact, we may show that for any global holomorphic 1-form α on X , we have $\alpha([V, W]) = 0$. Indeed, we observe that, definitionally, the pairing $\langle\alpha, V\rangle$ between 1-forms and vector fields is defined to produce a (smooth) function; in particular, here, because the 1-form α and the vector field V are both holomorphic, we see that the function $\alpha(V)$ produced in this manner will also be **holomorphic**, and in fact globally defined, since α and V are. Moreover, since X is compact by our assumption, we see that $\alpha(V)$ is a **globally defined holomorphic function** on the compact manifold X . Thus, by *Maximum Principle/Liouville's theorem*, we see that $\alpha(V)$ **must be constant**, for all 1-forms $\alpha \in \Omega^1(X)$ and all vector fields $V \in \mathfrak{X}(X)$. To do this, we recall the following identity, for any 1-form α and vector fields V, W :

$$d\alpha(V, W) = V(\alpha(W)) - W(\alpha(V)) - \alpha([V, W])$$

In particular, since $\alpha(V), \alpha(W)$ are **constant** by the above observation, we deduce that $V(\alpha(W)) = W(\alpha(V)) = 0$ (as derivations) thus, the expression reduces to $\alpha([V, W]) = -d\alpha(V, W)$; moreover, we previously proved (using Hodge theory) that **global holomorphic forms on a compact Kähler manifold are closed**, whereby $d\alpha = 0$ proves that $\alpha([V, W]) = -d\alpha(V, W) = 0$ is **everywhere zero**, as desired.

In fact, this result will prove that for a general 1-form $\alpha \in \Omega^1(X)$ and a general vector field $Z \in \mathfrak{X}(X)$, we have the duality $\alpha(Z) = \langle\alpha, Z\rangle$. In particular, if we write $\alpha := \overline{Z^\flat}$, which

is a globally defined holomorphic 1-form by the constructions done above, we will get:

$$Z = Z^i \partial_i, \alpha = g_{i\bar{j}} \overline{Z^j}, \implies \langle \overline{Z^b}, Z \rangle = g(Z, \overline{Z^b}) = g_{i\bar{j}} Z^i \overline{Z^j} = |Z|_g^2$$

the tensor-norm-squared of Z . In particular, since this is everywhere non-negative, it can only be 0 if and only if $Z \equiv 0$ identically. Thus, we see that:

$$\langle \overline{Z^b}, Z \rangle = |Z|_g^2 = 0 \iff Z \equiv 0$$

Now, apply this result to the previous setting, for the vector field $Z = [V, W]$, in which case we know that $\alpha(Z) = 0$, for any holomorphic 1-form α . Now, combining these two results into $\alpha = \overline{[V, W]}^b$ gives that $Z = [V, W] = 0$ for any two vector fields V, W ; this means that **the Lie algebra (bracket) is trivial**, i.e., $[V, W] = 0$ for all global holomorphic vector fields. \square

Using these tools, we may now prove the following interesting result:

Theorem 11.6. Automorphism group of Calabi-Yau manifolds

Let X^n be a Calabi-Yau manifold, and let $\text{Aut}^0(X)$ be the connected component of the identity of the group automorphisms of X . Then, $\text{Aut}^0(X)$ is a **complex torus**.

Proof. First of all, we should note that the identity component $\text{Aut}^0(X)$ is indeed a subgroup of $\text{Aut}(X)$. For this, we should start by observing that **Aut(X) is a topological group**, with the group operation given by the composition and the topology given by the **compact-open topology** of biholomorphisms $f : X \rightarrow X$; in general, we know that the identity component of a topological group is itself a topological group; this applies to show that $\text{Aut}^0(X)$ is a topological (sub)group.

Now, we notice that $\text{Aut}^0(X)$ is connected, thus, path-connected in the compact-open topology. In particular, for any automorphism $f \in \text{Aut}^0(X)$ we will get a smooth path $f_t : I \rightarrow \text{Aut}^0(X)$ with $f_0 = \text{Id}_X$, $f_1 = f$, joining f to the identity Id_X ; this is an **isotopy**. Thus, the **Moser trick** from Symplectic Geometry shows us that we can construct a map $Q : \Omega^k(X) \rightarrow \Omega^{k-1}(X)$ on forms using this isotopy, such that Now, for aby Kähler (metric) 2-form ω on X , we get:

$$f^* \omega - \omega = dQ(\omega) + Q(d\omega) = dQ(\omega)$$

since ω is a Kähler form, thus, d -closed ($d\omega = 0$). Thus, we see that the difference $f^* \omega - \omega$ is d -exact, whereby $f^*[\omega] = [f^* \omega] = [\omega] \in H^2(X; \mathbb{R})$ define the same de Rham cohomology class.

Now, we recall the following result:

Proposition 11.7 (Calabi's lemma on cohomologous Kähler forms). Let X be a compact Kähler manifold as in our setting and consider two Kähler 2-forms $\omega, \tilde{\omega}$ which are homologous, $[\omega] = [\tilde{\omega}]$. Then, we have:

$$\omega = \tilde{\omega} \iff \text{Ric}(\omega) = \text{Ric}(\tilde{\omega})$$

that is, they are equal if and only if their Ricci forms are.

Using this lemma, and recalling from the previous part that $[f^* \omega] = f^*[\omega] = [\omega]$ are cohomologous; moreover, we recall that **automorphisms preserve Ricci-flat metrics**,

i.e. $\text{Ric}(f^*\omega) = 0$ for $\text{Ric}(\omega) = 0$. Indeed, this result follows from the more general fact established in Differential Geometry, whereby:

$$(11.0.5) \quad \text{Ric}(f^*\omega) = f^*\text{Ric}(\omega)(f_*)$$

where $f_* := df$ is the pushforward (differential) map. Thus, since ω is Ricci-flat, we get $\text{Ric}(f^*\omega) = f^*\text{Ric}(\omega) = 0 = \text{Ric}(\omega)$, whereby since these two Ricci-flat Kähler forms $\omega, f^*\omega$ are **cohomologous** by the Moser trick, Calabi's lemma proves that, in fact, we have $f^*\omega = \omega$. Now this means that:

Elements $f \in \text{Aut}^0(X)$ of the identity component preserve the Kähler form ω

whereby we may apply the following theorem:

Theorem 11.8. The Myers-Steenrod Theorem

*Let (X, g) be a Riemannian manifold. A diffeomorphism $\varphi : X \rightarrow X$ is called an **isometry**, if it preserves the metric, i.e. $\varphi^*g = g$. It is evident that these isometries form group under composition, denoted $\text{Isom}(g)$, called the **isometry group** of g . In fact, they also form a manifold, given the compact-open topology, and overall, a **Lie group**.*

If, moreover, X is compact, then $\text{Isom}(g)$ is also a compact Lie group.

to see that the identity component $\text{Aut}^0(X)$ is, indeed, a **compact Lie group**, as follows: for our manifold (X, g) , we get the associated Kähler form ω , which is not necessarily Ricci-flat. Still, Yau's theorem produces for us a (unique) Ricci-flat Kähler metric $\tilde{\omega} \in [\omega]$ in the same cohomology class, whereby each biholomorphism $f \in \text{Aut}^0(X)$ will satisfy $f^*\omega = \omega$, as per our proof. Thus, letting \tilde{g} be the associated metric, we see that $\text{Aut}^0(X) \subset \text{Isom}(\tilde{g})$ is a subgroup of the isometry group, which, by the above result, is a compact complex (since we are working over a complex manifold) Lie group.

On the other hand, we recall from point-set topology that **connected components are closed**: since the closure $\overline{A} \subset X$ is connected for any connected set $A \subset X$, this is a connected set containing the connected component - thus, by maximality, $A = \overline{A}$ is closed. This implies that $\text{Aut}^0(X) \subset \text{Isom}(\tilde{g})$ is a **closed subgroup of a Lie group** (we proved that $\text{Aut}^0(X)$ is a group in part (a); thus, since it lives naturally inside $\text{Isom}(\tilde{g})$ via an *opposite* homomorphism $\text{Aut}^0(X) \ni f \mapsto f^* \in \text{Isom}(\tilde{g})$, which has $(f \circ g)^* = g^* \circ f^*$, it is again a *subgroup*); thus, by *Cartan's Theorem*²², we deduce that $\text{Aut}^0(X)$ is itself a Lie group. Moreover, since $\text{Isom}(\tilde{g})$ is compact, and closed subgroups of compact spaces are compact, we finally deduce that:

$\text{Aut}^0(X)$ is a compact, connected, complex Lie group

In fact, the Lie algebra $\mathfrak{aut}^0(X)$ is precisely the space of global holomorphic vector fields on X ; since we previously showed that the Lie bracket $[V, W] = 0$ is **trivial** for any global holomorphic vector fields V, W , this means that $\mathfrak{aut}^0(X)$ is **trivial**, i.e., $\text{Aut}^0(X)$ has trivial Lie algebra. Here, we have a lemma saying that:

Lemma 11.9 (Trivial Lie algebra implies complex torus). *Let G be a compact, connected, complex Lie group, whose Lie algebra \mathfrak{g} is trivial. Then, G is a complex torus.*

²²This states precisely that a *closed subgroup of a Lie group is a Lie group*.

which was proved in the context of Lie groups and Lie algebras. Having showed previously that $\text{Aut}^0(X)$ is a compact, connected, complex Lie group, as well as that its Lie algebra $\mathfrak{aut}^0(X)$ is trivial, we may now apply the lemma to deduce that **Aut⁰(X) is a complex torus**, as desired; this completes the proof. \square

12. DEFORMATION THEORY OF COMPLEX MANIFOLDS

12.1. The Gauss-Manin connection. Let X be a topological space. We recall the concept of a *local coefficient system*:

Definition 12.1. Local Coefficient system

Let X be a topological space. A *local coefficient system* over X is a sheaf \mathcal{A} of abelian groups on X that is locally isomorphic to a constant sheaf of stalk A , where A is a fixed abelian group. Equivalently, the local system can be trivialized over an open cover $\{U_\alpha\}$ of X , over which $\mathcal{A}(U_\mu) \simeq A$, with transition (gluing maps) $t_{\mu\nu} : \mathcal{A}(U_\mu \cap U_\nu) \rightarrow \mathcal{A}(U_\nu \cap U_\mu)$ given by $t_{\mu\nu} \in \text{Aut}(A)$.

If, in particular, A is a **vector space**, then, a local system of vector spaces (of rank equal to the rank of A) is simply **local system of stalk** A , with transition isomorphisms given by the automorphisms of the vector space.

Given a local system F of abelian groups over X , we can consider the associated sheaf of free $\underline{\mathcal{C}}^0(X)$ -modules \mathcal{F} , defined by:

$$\mathcal{F} := F \otimes_{\mathbb{Z}} \underline{\mathcal{C}}^0(X)$$

This construction could be replaced by $\mathcal{F} := F \otimes_{\mathbb{R}} \underline{\mathcal{C}}^0(X)$, if F were instead a local system of \mathbb{R} -vector spaces; likewise, if X is a smooth manifold (or complex manifold) then, we can define the associated sheaves of free $\underline{\mathcal{C}}^\infty(X)$ modules (respectively, free \mathcal{O}_X -modules by $\mathcal{F} := F \otimes_{\mathbb{C}} \mathcal{O}_X$). Thus, we can naturally obtain smooth C^∞ (respectively, holomorphic) vector bundles which are equipped with an additional structure: **a flat connection**.

Indeed, in the setting above, we define the connection $\nabla : \mathcal{F} \rightarrow \mathcal{F} \otimes \Omega_X^1$ on the (sheaf/vector bundle) \mathcal{F} , where Ω_X^1 is the sheaf of Kähler (holomorphic) differentials on X . Over a trivializing open set $U \subset X$ for \mathcal{F} , we can write any local section $\sigma \in \mathcal{F}(U)$ as $\sigma = s^i e_i$, for e_i a basis of the local trivialization; and we set on these local trivializations:

$$\nabla \sigma := e_i \otimes ds^i \in \mathcal{F} \otimes \Omega_X^1$$

Importantly, we see that this expression does not depend on the choice of trivialisation, since another local trivialisation of \mathcal{F} is deduced from the first one by a **transition matrix with constant coefficients**, which commutes with the derivations.

In the smooth (C^∞) case, this construction gives rise to a smooth vector bundle equipped with a C^∞ connection. In the holomorphic case, we likewise obtain a holomorphic vector bundle equipped with a holomorphic connection; i.e., $\nabla \sigma \in \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^1$ for $\sigma \in \mathcal{F}$, where Ω_X^1 in this setting is the sheaf of holomorphic differentials (1-forms).

An important property to observe here is that more generally, when we define a (holomorphic) connection $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^1$ on a (holomorphic) vector bundle (sheaf) $\mathcal{E} \rightarrow X$ over a

complex manifold X , then, this satisfies the **Leibniz rule**:

$$\boxed{\nabla(f\sigma) = \sigma \otimes df + f\nabla\sigma}, \quad \forall f \in \mathcal{O}_X(U), \sigma \in \mathcal{F}(U)$$

for any local holomorphic function f and local section σ on any open set $U \subset X$. For a *holomorphic* connection, we actually know that its form component only keeps the ∂ operator, i.e., this locally looks like $\boxed{\nabla = \partial + A}$, where A is a **holomorphic section** of $\text{End}(\mathcal{F}) \otimes \Omega_X^1$, called the **connection 1-form**.

Indeed, it is easy to check this property *locally*: for any local section $\sigma \in \mathcal{F}(U)$, we can use the above property to express it in the form $\sigma = s^i e_i$, where e_i is a basis for the local trivialization of \mathcal{F} . Consequently, we can also write $f\sigma = fs^i e_i = (fs^i)e_i$, i.e.:

$$\begin{aligned} \nabla(f\sigma) &= \nabla((fs^i)e_i) := e_i \otimes d(fs^i) = e_i \otimes ((df)s^i + f(ds^i)) = (s^i e_i) \otimes df + f e_i \otimes ds^i \\ &= \sigma \otimes df + f\nabla\sigma \end{aligned}$$

as desired; this proves our claim. Of course, we notice that for $f \in \mathcal{O}_X(U)$, we have $\bar{\partial}f = 0$ (since it is **holomorphic**) whereby $d = \partial + \bar{\partial}$ implies $df = \partial f$, which enables us to reduce the expression $\nabla = d + A$ to $\nabla = \partial + A$, as above.

Given a connection ∇ , we define its curvature Θ_∇ as follows:

$$\boxed{\Theta_\nabla : \mathcal{F} \rightarrow \mathcal{F} \otimes \bigwedge^2 \Omega_X^1 =: \mathcal{F} \otimes \Omega_X^2}, \quad \Omega_X^2 := \bigwedge^2 \Omega_X^1$$

To obtain this, notice that ∇ induces a map $d^\nabla : \mathcal{F} \otimes \Omega_X^1 \rightarrow \mathcal{F} \otimes \bigwedge^2 \Omega_X^1$, defined by:

$$\boxed{d^\nabla(\sigma \otimes \alpha) := \nabla\sigma \wedge \alpha + \sigma \otimes d\alpha}$$

again, for any local section $\sigma \in \mathcal{F}(U)$ and $\alpha \in \Omega_X^1(U)$. Here, the notation $\tau \wedge \beta$ for $\tau \in (\mathcal{F} \otimes \Omega_X^k)(U)$ and $\beta \in \Omega_X^\ell(U)$ means that if we express $\tau = e_i \otimes \eta^i$ in a local frame for the trivialization of $\mathcal{F} \otimes \Omega_X^k$ over U (i.e., $e_i \in \mathcal{F}(U)$ and $\eta^i \in \Omega_X^k(U)$) then we define:

$$\tau \wedge \beta := (e_i \otimes \eta^i) \wedge \beta := e_i \otimes (\eta^i \wedge \beta) \in (\mathcal{F} \otimes \Omega_X^{k+\ell})(U)$$

and the graded-commutativity in Ω_X^\bullet means $(e \otimes \beta) \wedge \eta = (-1)^{k\ell}(e \otimes \eta) \wedge \beta$, so, in particular, we get $(e_i \otimes \beta) \wedge \eta^i = (-1)^{k\ell}\tau \wedge \beta$.

For this, we notice that d^∇ satisfies the property for local sections $\sigma = s^i e_i \in \mathcal{F}(U)$:

$$\begin{aligned} d^\nabla(f\sigma \otimes \alpha) &:= \nabla(f\sigma) \wedge \alpha + (f\sigma) \otimes d\alpha = (\sigma \otimes df + f\nabla\sigma) \wedge \alpha + f\sigma \otimes d\alpha \\ &= f(\nabla\sigma \wedge \alpha + \sigma \otimes d\alpha) + \sigma \otimes (df \wedge \alpha) \\ &= \boxed{fd^\nabla(\sigma \otimes \alpha) - (\sigma \otimes \alpha) \wedge df} \end{aligned}$$

Thus, we may apply this result to $\sigma = s^i e_i$ and obtain:

$$\begin{aligned} d^\nabla(f\nabla\sigma) &= d^\nabla(fe_i \otimes ds^i) = fd^\nabla(e_i \otimes ds^i) - (e_i \otimes ds^i) \wedge df \\ &= f(d^\nabla \circ \nabla)(\sigma) - \nabla\sigma \wedge df \end{aligned}$$

i.e., we obtain $\boxed{d^\nabla(f\nabla\sigma) = f(d^\nabla \circ d^\nabla)(\sigma) - \nabla\sigma \wedge df}$.

Then, we can define the curvature of ∇ by $\Theta_\nabla := d^\nabla \circ d^\nabla$, i.e., the composition of:

$$\Theta_\nabla : \mathcal{F} \xrightarrow{d^\nabla} \mathcal{F} \otimes \Omega_X^1 \xrightarrow{d^\nabla} \mathcal{F} \otimes \Omega_X^2 := \mathcal{F} \otimes \bigwedge^2 \Omega_X^1$$

It is easy to check that Θ_∇ is a \mathcal{O}_X -linear map, i.e., that $\Theta_\nabla(f\sigma) = f\Theta_\nabla(\sigma)$, so that Θ_∇ becomes a section of the sheaf $\text{End}(\mathcal{F}) \otimes \bigwedge^2 \Omega_X^1$. Thus, we see that these constructions naturally generalize their differentiable counterparts.

The \mathcal{O}_X -linearity now follows immediately by the previous results: $d^\nabla(f\sigma) = \nabla(f\sigma)$ gives:

$$\begin{aligned} \Theta_\nabla(f\sigma) &= (d^\nabla \circ d^\nabla)(f\sigma) = d^\nabla(\sigma \otimes df + f\nabla\sigma) = d^\nabla(\sigma \otimes df) + d^\nabla(f\nabla\sigma) \\ &= \nabla\sigma \wedge df + \cancel{\sigma \otimes d^2 \mathcal{F}}^0 + d^\nabla(f\nabla\sigma) = \nabla\sigma \wedge df + f(d^\nabla \circ d^\nabla)(\sigma) - \cancel{\nabla\sigma \wedge df} \\ &= f\Theta_\nabla(\sigma) \end{aligned}$$

i.e., $\Theta_\nabla(f\sigma) = f\Theta_\nabla(\sigma)$ is \mathcal{O}_X -linear, as desired.

In this setting, we would like to show that this connection is **flat**, i.e., it has **connection zero**. Indeed, this simply follows from the fact that ∇ can be identified in the local trivializations of the local coefficient system F with the usual differentiation (on the level of $F \otimes \Omega_X^1$) which satisfies $d^2 = 0$ on forms. Consequently, we see that the vector bundle associated to a local system of vector spaces is thus equipped with a **canonical flat connection**.

Theorem 12.2. Correspondence between local systems and vector bundles with flat connection

The correspondence constructed in this way is a **bijective correspondence** between the isomorphism classes of C^∞ - (i.e., smooth; or also holomorphic, in the case where B is a complex manifold) vector bundles equipped with a flat connection, and the isomorphism classes of local systems of vector spaces.²³ This correspondence is given by:

$$\begin{aligned} \{\text{local systems of } \mathbb{C} - \text{vector spaces}\} &\rightleftarrows \{\text{holomorphic v.b.s equipped w/ flat connection}\} \\ F &\longmapsto (\mathcal{F}, \nabla), \quad \mathcal{F} := F \otimes_{\mathbb{C}} \mathcal{O}_X, \quad \nabla : \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^1 \\ \text{sheaf } F(U) &:= \{f \in \mathcal{F}(U) \mid \nabla f = 0\} \longleftrightarrow (\mathcal{F}, \nabla) \end{aligned}$$

i.e., we recover the sheaf $H(U)$ by taking the **horizontal sections** of the sheaf (bundle) \mathcal{F} . Here, the connection $\nabla : \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^1$ acts locally by $\alpha \otimes f \mapsto \alpha \otimes df$.

Proof. Here, all the isomorphisms are natural, since an isomorphism of vector bundles *equipped with connections* is an isomorphism of bundles that **preserves the connection**. Analogously, isomorphisms of local systems are isomorphisms of their corresponding sheaves, which induce isomorphisms in both directions.

To obtain the inverse map of this correspondence, we associate to the sheaf of \mathcal{C}_X^∞ -modules with flat connection (\mathcal{H}, ∇) the **local system H of the flat (parallel/ horizontal/ constant) sections** of \mathcal{H} , i.e., the sections (defined over an open set $U \subset X$) annihilate by

²³In the smooth case, the vector spaces (and, correspondingly, the vector bundles) can be taken to be either complex or real, where we are considering the corresponding K -vector bundles with a K -connection. In the holomorphic setting, the vector bundles and the vector spaces are necessarily **complex**. Here, we recall that isomorphisms of local systems are isomorphisms of the corresponding sheaves.

the connection ∇ . To show that this is a valid correspondence, we need to prove that H is a local system and that we can recover the sheaf $\mathcal{H} := H \otimes \mathcal{C}_X^\infty$; or $\mathcal{H} = H \otimes \mathcal{O}_X$, in the holomorphic setting. To see this result, we establish a lemma:

Lemma 12.3. Flat sections as restrictions to the fiber

Let (\mathcal{H}, ∇) be a sheaf of \mathcal{C}_X^∞ -modules (or \mathcal{O}_X -modules, in the holomorphic setting) over the topological space X , so that ∇ is a flat connection for \mathcal{H} . Then, the flat sections σ of \mathcal{H} in the neighborhood of each point $x \in X$ can be identified, by restriction to x , with the fiber at x of the vector bundle associated to the sheaf (of \mathcal{C}_X^∞ - or of \mathcal{O}_X -modules) \mathcal{H} .

Of course, we recall that this fiber can also be identified, as a complex vector space, with the tensor product of stalks $\mathcal{H}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x$, where $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the ideal of holomorphic functions vanishing at x . Then, the map sending the flat sections in the neighborhood of x , to the fiber at x , is given by the composition:

$$H \hookrightarrow \mathcal{H} \twoheadrightarrow \mathcal{H}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x$$

Consequently, we may now obtain the claim by:

Proof. This result is a direct application of **Frobenius' theorem**: indeed, a connection ∇ on the sheaf \mathcal{H} (of \mathcal{C}_X^∞ - or of \mathcal{O}_X -modules) with corresponding (smooth/ holomorphic) vector bundle $\pi : \bar{H} \rightarrow X$ defines a **distribution** D on \bar{H} as follows: for any $x \in X$ and $\sigma_0 \in \bar{H}_x$ on the fiber of \bar{H} over x , we let σ be a section of \mathcal{H} in the neighborhood of x such that $\sigma_0 = \sigma|_x$. Then, we set:

$$D_{(x,\sigma_0)} := \text{Im}(\sigma_0 - \nabla\sigma) : \mathcal{T}_{X,x} \rightarrow \mathcal{T}_{\bar{H},(x,\sigma_0)}$$

where $\mathcal{T}_{X,x}$ here is the stalk at $x \in X$ of the holomorphic tangent sheaf \mathcal{T}_X of X ; and likewise for $\mathcal{T}_{\bar{H},(x,\sigma_0)}$. Here, $\nabla\sigma$ at the point x is viewed as an element of $\text{Hom}(\mathcal{T}_{X,x}, \mathcal{T}_{\bar{H},(x,\sigma_0)})$, via the natural inclusion $\bar{H}_x \subset T_{\bar{H},(x,\sigma_0)}$, whose image is precisely the **tangent space** to the fiber of π . Importantly, we can apply **Leibniz's formula** here to see immediately that the subspace $D_{(x,\sigma_0)} \subset \mathcal{T}_{\bar{H},(x,\sigma_0)}$ constructed in this way is **independent** of the choice of representative σ for σ_0 . \square

\square

Proposition 12.4. Solutions to homogeneous linear DE define local system

Let $X \subset \mathbb{C}$ be a connected open set. Over X , we consider the following homogeneous linear differential equation:

$$\left(\partial_z^n + a_{n-1} \partial_z^{n-1} + \cdots + a_1 \partial_z + a_0 \right) (f) = 0$$

where the $a_i \in \mathcal{O}_{\mathbb{C}}(X)$ are **holomorphic functions** on X . Now, we define a sheaf $\mathcal{F} \in \text{Sh}(X)$ on X as follows: for any open set $U \subset X$, we let $\mathcal{F}(U)$ be the space of local solutions of this equation over X . Then, \mathcal{F} is a **local system** on X .

Proof. First of all, the fact that \mathcal{F} defines a sheaf is direct, as it is the space of sections described by holomorphic equations; in fact, these are precisely the **flat (parallel) sections**

of a particular connection on the trivial rank- n vector bundle $\mathcal{O}_X^{\oplus n}$. Moreover, we see that for each open set $U \subset X$, the space of sections $\mathcal{F}(U)$ is a **finite-dimensional complex vector space**, so that \mathcal{F} is a sheaf of vector spaces. Consequently, the only non-trivial step hinges on showing that this sheaf is **locally constant of stalk \mathbb{C}^n** , i.e., a local system.

To prove this last step, we equivalently want to show that every point $x \in X$ has an open neighborhood $V \ni x$, on which the restriction sheaf $\mathcal{F}|_V$ (i.e., the pullback sheaf $i^*\mathcal{F}$ on U under the inclusion map $i : U \hookrightarrow X$) is isomorphic to a **constant sheaf**. To this end, we can argue as follows: For this result, it suffices to consider a sufficiently small disk V around a point $x \in X$; consequently, since we are working with an equality of **holomorphic functions**, the identity theorem on holomorphic functions together with the analyticity of holomorphic functions (on a potentially smaller, if necessary; yet certainly non-trivial, disk) tells us that a solution f is uniquely determined by its Taylor series expansion around the point x . Consequently, since the n -th derivative of any solution f is controlled (and uniquely determined) by the above equation,²⁴ we deduce that all the terms of this expansion beyond n (thus, also the function itself) are **uniquely determined** by the values $(f(x), \partial_z f(x), \dots, \partial_z^{n-1} f(x))$. Alternatively, we could also have uniqueness by the corresponding **uniqueness theorem for solutions to ODEs with given initial data**. Thus, we see that there is a map:

$$\mathcal{F}(V) \xrightarrow{\sim} \underline{\mathbb{C}^n}_V, \quad \mathcal{F}(V) \ni f \mapsto (f(x), \partial_z f(x), \dots, \partial_z^{n-1} f(x))$$

defines an isomorphism of sheaves, where in general, we use \underline{A}_X to denote the constant sheaf on the space X with value (stalk) A ; i.e., $\underline{\mathbb{C}^n}_V$ is precisely this constant sheaf on V . This proves precisely that the sheaf \mathcal{F} is a **local system**, as desired. \square

Let $\pi : X \rightarrow S$ be an S -scheme and \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. A *connection* ∇ on \mathcal{F} is a $\pi^{-1}\mathcal{O}_S$ -morphism:

$$\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_{X/S}^1$$

where $\Omega_{X/S}^1$ is the sheaf of *relative differentials* on X , such that:

$$\nabla(f\sigma) = \sigma \otimes df + f\nabla\sigma, \quad \forall f \in \mathcal{O}_X(U), \sigma \in \mathcal{F}(U)$$

for all **local sections** f of \mathcal{O}_X and σ of \mathcal{F} . Then, the pair (\mathcal{F}, ∇) is called a **module with a connection**. In particular, a local section $\sigma \in \mathcal{F}(U)$ of \mathcal{F} such that $\nabla\sigma \equiv 0$ is said to be **flat**, or **horizontal**, or **parallel**.

However, it is important to note (as is clearly evident) that **the connection is not \mathcal{O}_X -linear**; in the language of Differential Geometry, we say that *the connection is not tensorial* or, equivalently, that *the connection does not define a tensor*. If $\mathcal{E} = E$ here is actually a **vector bundle**, then, we can also call the pair (E, ∇) a **vector bundle with connection**.

Now, we consider the following simple, yet important setting: working in a local trivialization, we may WLOG assume that our vector bundle is given by $\mathcal{F} = \mathcal{O}_X e_1 \oplus \dots \oplus \mathcal{O}_X e_r \cong \mathcal{O}_X^{\oplus r}$, a vector bundle of rank r trivialized by the sections $\{e_1, \dots, e_r\}$. Then, on an open set

²⁴Indeed, $\partial_z^n f = -(a_{n-1} \partial_z^{n-1} + \dots + a_1 \partial_z + a_0) f$, with higher derivatives determined similarly.

$U \subset X$ (or, in particular, on all of X , if it is a *trivial* vector bundle) the construction of the connection $\nabla : \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$, defined previously by $\nabla(s^i e_i) := e_i \otimes ds^i$ for a specified local frame $\{e_i\}$, can also be defined via local holomorphic 1-forms (relative differentials) $\omega^\alpha{}_\beta \in \Gamma(U, \Omega^1_{X/S})$ (for $1 \leq \alpha, \beta \leq r$) such that the \mathcal{F} -valued 1-forms ∇e_α are:

$$\nabla e_\alpha := e_\beta \otimes \omega^\beta{}_\alpha, \quad \forall \alpha = 1, \dots, r \tag{25}$$

Thus, we obtain a $r \times r$ matrix of differential forms $\Omega = (\omega^\beta{}_\alpha)$, which in Differential Geometry is called the **matrix of the connection**; evidently, this, together with the Leibniz property established earlier, determines the connection ∇ completely. In fact, the Leibniz rule motivates us to write this symbolically as $\boxed{\nabla = d + \Omega}$, so that the previous considerations show that Ω is an **End(\mathcal{F})-valued 1-form**, called the **matrix of connection 1-forms** of ∇ (in the local frame $\{e_i\}$).

For example, if $L \rightarrow X$ is a line bundle and ξ is a frame of L over an open subset $U \subset X$ (i.e., a section of L over U such that $\xi(x) \neq 0$ for all $x \in U$) then since $\nabla \xi$ is an L -valued 1-form, it must be expressible as $\boxed{\nabla \xi = \xi \otimes \omega}$ for some (local) 1-form ω .

Here, the notation $\nabla = d + \Omega$ is justified, since a local section $\sigma = s^i e_i \in \mathcal{F}(U)$ has:

$$\nabla \sigma = \nabla(s^\alpha e_\alpha) = e_\alpha \otimes (ds^\alpha + \omega^\alpha{}_\beta s^\beta) = e_\alpha \otimes (d + \Omega)(s^\alpha)$$

by the same computation that we performed above; this means that *locally*, the action of the connection is determined by the connection form. This means that the **flat sections (parallel; horizontal)** of the connection ∇ correspond (via the obvious association $(s^\alpha) \longleftrightarrow \sigma := s^\alpha e_\alpha$) to (holomorphic/ smooth, depending on the category in consideration) of the **linear homogeneous differential equation**:

$$\boxed{\nabla \sigma = 0 \iff ds^\alpha + \omega^\alpha{}_\beta s^\beta = 0, \quad \forall \alpha = 1, \dots, r}$$

over any open set $U \subset X$.

Remark 12.5. It is not hard to see that the matrix of connection 1-forms Ω is **not coordinate-independent**: indeed, let us change the frame $\{e_\alpha\}$ to another frame $\{\tilde{e}_\mu\}$, related to the original one by a transition endomorphism (smooth functions) of the vector bundle E as $\tilde{e}_\mu = t^\alpha{}_\mu e_\alpha$. Similarly, let $\tilde{\Omega} = (\tilde{\omega}^\nu{}_\mu)$ be the matrix of connection 1-forms in this local frame, for which we obtain:

$$\begin{aligned} \tilde{e}_\nu \otimes \tilde{\omega}^\nu{}_\mu &= \nabla \tilde{e}_\mu = \nabla(t^\alpha{}_\mu e_\alpha) = e_\alpha \otimes dt^\alpha{}_\mu + t^\alpha{}_\mu \nabla e_\alpha \\ &= e_\alpha \otimes dt^\alpha{}_\mu + t^\alpha{}_\mu e_\beta \otimes \omega^\beta{}_\alpha = e_\alpha \otimes (dt^\alpha{}_\mu + t^\beta{}_\mu \omega^\alpha{}_\beta) \\ &= (t^\alpha{}_\nu)^{-1} \tilde{e}_\nu (dt^\alpha{}_\mu + t^\beta{}_\mu \omega^\alpha{}_\beta) \end{aligned}$$

i.e., if $T = (t^\alpha{}_\nu)$ is the transition matrix, this can be written as:

$$\tilde{\omega}^\nu{}_\mu = (t^{-1})^\nu{}_\alpha (dt^\alpha{}_\mu + t^\beta{}_\mu \omega^\alpha{}_\beta), \quad \boxed{\tilde{\Omega} = T^{-1}(dT + T\Omega)}$$

²⁵Here, we are using *Einstein summation notation* to contract the indices β .

so, for example, for a line bundle with frames $\xi, \tilde{\xi} = f\xi$ this would give $\tilde{\omega} = \omega + \frac{df}{f}$, thus proving that **the matrix of connection 1-forms is not globally defined**.

Remark 12.6 (Curvature). Even though the matrix of connection 1-forms is not globally defined, the **exterior derivative** $\Theta_\nabla := d\Omega$ is a globally defined 2-form, independently of the choice of frame; essentially, this has to do with the fact that $\theta_\nabla(f\sigma) = f\Theta_\nabla(\sigma)$ is \mathcal{O}_X -linear, as proved above. Concretely, we can see this here by computing:

$$\begin{aligned} d\tilde{\Omega} &= -T^{-1}dT T^{-1}(dT + T\Omega) + T^{-1}d^2T + T^{-1}((dT)\Omega + Td\Omega) \\ &= -\cancel{(T^{-1}dT)^2}^0 - T^{-1}dT\Omega + T^{-1}dT\Omega + \cancel{T^{-1}T}^{\text{Id}} d\Omega \\ &= d\Omega \end{aligned}$$

i.e., $d\tilde{\Omega} = d\Omega$ is well-defined, as desired.

Remark 12.7. Elsewhere in differential geometry, we prove that the difference of two connections on a given vector bundle $E \rightarrow X$ is also **globally defined** (as a tensor) so that the space of connections on X is an **affine space** modelled on the vector space $\mathcal{A}^1(X)$ of 1-forms on X . In general, there is no canonical choice for a special point in the affine space; this will always be the case for holomorphic vector bundles.

Now, we see that a connection $\nabla : \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_{X/S}^1$ induces an \mathcal{O}_X -morphism:

$$\mathcal{T}_{X/S} \longrightarrow \text{End}_{\mathcal{O}_S}(\mathcal{F}), \quad v \longmapsto \nabla_v$$

from the relative holomorphic tangent sheaf $\mathcal{T}_{X/S}$; here, we define $\nabla_v(e) := \langle v, \nabla e \rangle$ to be the *covariant derivative* of e along v . Again, it is important to see that for any local sections $f \in \mathcal{O}_X(U)$ and $e \in \mathcal{F}(U)$, we have:

$$\nabla_v(fe) = v(f)e + f\nabla_v(e), \quad v(f) := \langle v, df \rangle$$

which corresponds to our usual notion of a connection.

In particular, we have the following important instance of this correspondence: let k be a field; then, to any homogeneous ODE:

$$(12.1.1) \quad (a_n \partial_x^n + a_{n-1} \partial_x^{n-1} + \cdots + a_1 \partial_x + a_0)(f) = 0, \quad a_i \in k[x]$$

we can associate an **algebraic vector bundle** with connection (\mathcal{F}, ∇) as follows: let $X = D(a_n) \subset \mathbb{A}_k^1$ be the open subset of the affine line over which $a_n \neq 0$, and we consider the *trivial* bundle $\mathcal{F} = \mathcal{O}_X^{\oplus n}$. For this, we now define the connection $\nabla := d + \Omega : \mathcal{F} \rightarrow \mathcal{F} \otimes \Omega_X^1$ as completely determined by the connection 1-form Ω :

$$\Omega = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{a_0}{a_n} & \frac{a_1}{a_n} & \frac{a_2}{a_n} & \cdots & \frac{a_{n-1}}{a_n} \end{pmatrix} dx$$

Importantly, we see that on the open subset $X = D(a_n) = \text{Spec } k[x, a_n^{-1}] \subset \mathbb{A}_k^1$, the function $\frac{1}{a_n}$ is **regular**, and so are all the functions $a_i \in k[x]$ for $0 \leq i \leq n-1$. This means that (\mathcal{F}, ∇) becomes an **algebraic vector bundle with algebraic connection**. Moreover, under this construction, the solutions of the equation 12.1.1, if they exist, correspond precisely to the flat sections of the connection $\nabla = d + \Omega$, by the previous correspondence. Indeed, if some y is a solution of the equation 12.1.1, then, the section:

$$e = \left(y, \frac{dy}{dx}, \dots, \frac{d^{n-1}y}{dx^{n-1}} \right)$$

satisfies $(\nabla e) \left(\frac{d}{dx} \right) = 0$ in the contraction of differentials with vector fields; and since the (holomorphic) tangent sheaf \mathcal{T}_X is trivialized by the global section $\frac{d}{dx}$ (since this is true for the affine line \mathbb{A}^1) this means that we must actually have $\nabla e \equiv 0$, i.e., the section e is **flat**, as desired. Conversely, we see now that *every* flat section of the connection ∇ must have the above form, thanks to the correspondence established earlier.

Thus, with this argument we have showed the following:

Proposition 12.8. Vector Bundle from Local System of solutions

For k a field, consider a homogeneous ODE $(\sum_{i=0}^n a_i \partial_x^i)(f) = 0$ with $a_i \in k[x]$. By the previous result, its spaces of solutions over open sets assemble to a local system over the complex manifold/affine open set $X = D(a_n) = \text{Spec } k[x, a_n^{-1}] \subset \mathbb{A}_k^1$, which corresponds to (the sections of) the trivial (algebraic) vector bundle $\mathcal{O}_X^{\oplus n}$ of rank n over X , with **algebraic connection** $\nabla = d + \Omega$, for Ω the (matrix of) connection 1-form(s) described above.

In general $f : X \rightarrow Y$ be a map in a certain category of topological spaces; sometimes, we may make additional assumptions about it, for example, that it is a **proper submersion**. Then, we may define the sheaves $H_A^k := R^k f_* \underline{A}_X$, where A is a ring of coefficients (usually $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or \mathbb{C}), so that \underline{A}_X denotes the constant sheaf of stalk A on X . We have seen how to define the higher pushforward sheaves $R^k f_* \mathcal{F}$ in the context of sheaf theory. Thus, it is not hard to see that these sheaves H_A^k are actually **local systems** on the space Y .

In particular, let $\varphi : \mathcal{X} \rightarrow B$ be a family of complex manifolds over a base B , such that $X = X_0$ (for $0 \in B$) is a Kähler manifold; we do not make any assumptions about the neighboring fibers.²⁶

Let $\pi : \mathcal{X} \rightarrow B$ be a proper submersion of manifolds; then, by **Ehresmann's theorem**, we know that there is a neighborhood of $X_0 = \pi^{-1}(0)$ (for $0 \in B$) inside the manifold \mathcal{X} that is isomorphic to $X_0 \times B_0$, where B_0 is a neighborhood of $0 \in B$. Then, we may define the sheaves $H_A^k := R^k f_* \underline{A}_X$, where A is a ring of coefficients (usually $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or \mathbb{C}), so that \underline{A}_X denotes the constant sheaf of stalk A on X . Consequently, using the defining properties of the derived pushforward sheaf, we deduce that since B is **locally contractible**, we have $H^k(X_0 \times B_0, A) \cong H^k(X_0, A)$ for a fundamental system of neighborhoods B_0 of 0 .

²⁶Later, we will see that these can be proved to also be Kähler.

From this, we deduce that the pushforward sheaf $R^k\pi_*\mathcal{A}_{\mathcal{X}}$ is, in fact, a **local system** on B that is isomorphic (in the neighborhood of 0) to the constant sheaf of stalk $H^k(X_0, A)$. Importantly, we know that the stalk of this local system at a point $t \in B$ is canonically isomorphic to $H^k(X_t, A)$, by restriction.

Thus, we may now define:

Definition 12.9. Gauss-Manin connection

The **Gauss-Manin connection** is the flat connection:

$$\nabla : \mathcal{H}^k \longrightarrow \mathcal{H}^k \otimes \Omega_X^1$$

on the vector bundle associated to the local system \mathcal{H}_A^k .

First of all, we recall that by the properties of Dolbeault cohomology, elliptic complexes, and the Dolbeault isomorphism, we get $H^q(X_b, F_b) \simeq H_{\bar{\partial}}^{0,q}(X, F_b)$, which is in turn, equivalent to the space of harmonic F -valued differential $(0, q)$ -forms. That is, we can write:

$$H^q(X_b, F_b) \simeq H_{\bar{\partial}}^{0,q}(X, F_b) := \ker \left(\Delta_{\bar{\partial}, F_b} : \mathcal{A}_{X_b}^{0,q} \otimes F_b \rightarrow \mathcal{A}_{X_b}^{0,q} \otimes F_b \right)$$

where, as before, the $\mathcal{A}_{X_b}^{0,q} \otimes F_b$ denotes the sheaf of F_b -valued $(0, q)$ -forms on X_b , i.e., the sections of the smooth vector bundle $\Lambda^q T^V X_b^{0,1} \otimes F_b$ over X_b .

Now, for a *single* manifold, we know how to understand this space by using the theory of **elliptic operators** on a single manifold X . Here, we want to **generalize** this construction to a **relative (family) version** for the family $\mathcal{X} \rightarrow B$.

; these should be considered *constant* when restricted to the fibers.

On a *single* manifold, we recall that we have the following data: a complex manifold X ; a vector bundle $F \rightarrow X$; the sheaf $\mathcal{A}_X^{0,1}$ of smooth $(0, 1)$ -forms on X , and its q -th wedge power $\mathcal{A}_X^{0,q} := \Lambda^q \mathcal{A}_X^{0,1}$; the sheaf of F -valued $(0, q)$ -forms $\mathcal{A}_X^{0,q} \otimes F$; and the Laplace-Beltrami operator $\Delta_{\bar{\partial}}$.

To generalize this property to a relative (family) version of the result,

Thus, we see that:

Definition 12.10. Period map for a family

Let \mathcal{X} be a family of complex manifolds over a complex manifold B , i.e., there is a proper holomorphic submersion $\mathcal{X} \rightarrow B$. For this, we may define the **period map**:

$$\mathcal{P}^{p,k} : B \longrightarrow \mathbb{G}(b^{p,k}, H^k(X, \mathbb{C})), \quad b \longmapsto F^p H^k(X_b, \mathbb{C}) \subseteq H^k(X_b, \mathbb{C}) \simeq H^k(X, \mathbb{C})$$

where $X = X_0$ is the *special fiber* over a point $0 \in B$, whereby we used **Ehresmann's theorem** to produce some neighborhood $U \ni 0$ in B , whose preimage in \mathcal{X} under the submersion is $\cong U \times X$;²⁷ here, we have defined $b^{p,k} := \dim F^p H^k(X_b, \mathbb{C})$.

In particular, here, we have **Griffiths' formula**

Moving on, we will consider such periods maps for a special family of Kähler manifolds: **Calabi-Yau manifolds**. First of all, we define them in the most general setting:

²⁷In particular, this means that the construction of the period map may require us to *shrink* the base B appropriately.

Definition 12.11. *Calabi-Yau manifolds*

A **Calabi-Yau manifold** is a compact Kähler manifold X whose canonical line bundle is holomorphically trivial, i.e., $K_X \simeq \mathcal{O}_X$; equivalently, the sheaf of (germs of) holomorphic n -forms $\Omega_X^n \simeq \mathcal{O}_X$ is trivial.

Depending on the setting and scope of applications, sometimes one also requires that $\pi_1 X$ is trivial (X is simply connected) or that $h^{p,0} = 0$ for $0 < p < \dim X$, i.e., X admits no global holomorphic p -forms for $0 < p < n$; however, the global triviality of K_X turns out to be enough for the scope of our applications.

We can describe the Kuranishi *global deformation space* for these families quite explicitly:

Theorem 12.12. *Bogomolov-Tian-Todorov theorem (1970-1980)*

Any Calabi-Yau manifold X has a **universal deformation** $X \subset \mathcal{X} \rightarrow B$, such that B is **smooth** and, by universality, has dimension equal to $h^{n-1,1}(X)$; this follows from the isomorphism $\mathcal{T}_{B,0} \simeq H^1(X, \mathcal{T}_X) \simeq H^1(X, \Omega_X^{n-1})$.²⁸

Now, we can use these results to consider the period map in the setting of a **Calabi-Yau manifold** of dimension n . In general, we know that $h^{n,0} = 1$ by the previous result, while *any* complex manifold of dimension n has $F^n H^n(X; \mathbb{C}) \equiv H_{\bar{\partial}}^{n,0}(X) = H^0(X, K_X)$, by Dolbeault's theorem; and we know that for Calabi-Yau manifolds, this is $H^0(X, K_X) = \mathbb{C}$, as above; consequently, we have $b^{n,n} = h^{n,0} = 1$. Consequently, we see that in this case, the period map $\mathcal{P}^{n,n}$ is simply given by:

$$\mathcal{P}^{n,n} : B \longrightarrow \mathbb{G}(b^{n,n}, H^n(X; \mathbb{C})) = \mathbb{G}(1, H^n(X; \mathbb{C})) = \mathbb{P}H^n(X; \mathbb{C}), \quad b \longmapsto H^0(X_b, K_{X_b})$$

Consequently, we have the following result:

*For a Calabi-Yau manifold X , the period map $\mathcal{P}^{n,n} : B \rightarrow \mathbb{P}H^n(X, \mathbb{C})$ above is an **immersion**, i.e., it is **injective** on tangent spaces.*

Proof. To see this result, we need to trace out the Griffiths formula for the differential $d\mathcal{P}^{p,k}$ concretely. For this, we now have the following commutative diagram giving the \square

; and the **Kodaira-Spencer map** $\text{KS} : \mathcal{T}_{B,0} \xrightarrow{\sim} H^1(X, \mathcal{T}_X)$ here is *also* an isomorphism, since we know that this is a *universal* deformation.

Proof. To see this result, we *take the derivative* of this expression, i.e., we differentiate using the **Gauss-Manin connection**. Here, differentiating the expression $Q(\omega_b, \omega_b) = 0$ gives us $Q(\omega_b, \nabla_u \omega_b) = 0$, i.e., this ∇_u maps the filtering piece F^n into its **orthogonal complement** $(F^n)^\perp$; that is, we get:

$$\mathcal{T}|_{\{Q=0\}} = \text{Hom}(F^n, F^1/F^n)$$

But on the other hand, we know from before that the top period map $\mathcal{P}^{n,n}$ maps the tangent space $\mathcal{T}_{B,0}$ isomorphically to $\text{Hom}(F^n, F^{n-1}/F^n)$

²⁸This happens because B measures all the first-order deformations of X ; i.e., this tangent space $\mathcal{T}_{B,0}$ tells us precisely “*in how many directions*” we can deform the Calabi-Yau manifold X . Moreover, the last isomorphism follows from our earlier observation $\mathcal{T}_X \simeq \Omega_X^{n-1}$.

; since we know that X already has a piece on $h^{\frac{n}{2}, \frac{n}{2}} > 0$, which can be obtained by wedging the Kähler form ω (a $(1, 1)$ -form) with itself $n/2$ times; this is allowed, since we assumed that n is even. Consequently, this means that $F^1 H^n(X; \mathbb{C}) \neq F^{n-1} H^n(X; \mathbb{C})$, i.e., F^1 **does not go all the way up**; meaning that we do get intermediate pieces (here, $n/2$) for $n > 2$.

□