MATH4045GU

IZ TECUM

Contents

1. Basic definitions and first consequences

Definition 1.1. A Riemann Surface (RS) is a topological space X that is 2nd countable and Hausdorff equipped with a complex structure.

$$\mathcal{A} = \{\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}\}.$$

Remark 1.2. Let X be a topological space such that A, B are complex structures so that you have $(X, \mathcal{A})$ and $(X, \mathcal{B})$. If $\mathcal{A} \cup \mathcal{B}$ happens to be a complex structure, then $(X, \mathcal{A} \cup \mathcal{B})$ are the same RS (refer Zoren's lemma).

Proposition 1.3. *All Riemann Surfaces are also orientable* 2*-dimensional real manifolds.*

Definition 1.4. Let X be an n-dimensional real manifold that is orientable. If $\forall\, \phi_{\alpha\beta} : \mathbb{R}^n \to \mathbb{R}^n$ one has

$$d\phi_{\alpha\beta} : \mathbb{R}^n \to \mathbb{R}^n.$$

Corollary 1.5. *Every compact Riemann Surface is a compact orientable* 2*-dimensional real manifold which are classified by* $g \in \mathbb{Z}_{\geq \nvdash}$.

2. Projective line and basic examples

Example 2.1. Consider $\mathbb{P}^1_{\mathbb{C}} = S^2$. Recall $\mathbb{P}^n_{\mathbb{R}} = \mathbb{R}^{n+1} \setminus \{(0, 0, \dots, 0)\}/\sim$, where

$$(x_0, x_1, \dots, x_n) \sim (\lambda x_0, \lambda x_1, \dots, \lambda x_n).$$

Similarly, $\mathbb{P}^n_{\mathbb{C}} = \mathbb{C}^{n+1} \setminus \{(0, 0, \dots, 0)\}/\sim$, where

$$(z_0, z_1, \dots, z_n) \sim (\lambda z_0, \lambda z_1, \dots, \lambda z_n) \qquad (\lambda \in \mathbb{C}_{\neq 0})$$

is called the complex projective space.

Fact 2.2. $\mathbb{P}^1$ *is homeomorphic to* S^2.

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Proof. For a complex structure on $\mathbb{P}^1_{\mathbb{C}} = \{[z,w] : z,w \in \mathbb{C},\ (z,w) \neq (0,0)\}$, set

$$U_1 = \mathbb{P}^1_{\mathbb{C}} \setminus [0,1], \qquad \phi_1 : U_1 \to \mathbb{C}, \qquad [z,w] \mapsto \frac{w}{z}.$$

(And similarly $U_2 = \mathbb{P}^1_{\mathbb{C}} \setminus [1,0]$ with $\phi_2([z,w]) = \frac{z}{w}$.) We claim that ϕ_1, ϕ_2 are compatible, i.e. ϕ_{21} is biholomorphic. Observe,

$$\phi_{21} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, \qquad t \mapsto [1,t] \mapsto \frac{1}{t},$$

so $\phi(t) = \frac{1}{t}$ is a biholomorphic map. □

Example 2.3. Fix $\omega_1, \omega_2 \in \mathbb{C}$ that are $\mathbb{R}$-linearly independent. Set

$$L = \mathbb{Z}\{\omega_1, \omega_2\} = \{n_1\omega_1 + n_2\omega_2 :\ n_1, n_2 \in \mathbb{Z}\}.$$

Check $L \subset \mathbb{C}$.

Example 2.4. Consider $X = \mathbb{C}$ and $X = \{z \in \mathbb{C} : |z| < 1\}$ are examples of non-compact Riemann Surfaces.

3. Affine plane curves and the implicit function theorem

Definition 3.1. An affine plane curve is the zero set of a single polynomial, say $f \in \mathbb{C}[z,w]$. Formally,

$$X = \{(z,w) \in \mathbb{C}^2 : f(z,w) = 0\} \subset \mathbb{C}^2.$$

Theorem 3.2 (Implicit function)**.** *Consider $\rho_0 = (z_0, w_0) \in X = \{f(z,w) = 0\}$. Assume $\frac{\partial f}{\partial w}(\rho_0) \neq 0$. Then there exists $U \subset X$ open neighborhood of ρ_0, there also exists $V \subset \mathbb{C}$ such that $g : V \to \mathbb{C}$ is a holomorphic function such that*

$$U = \{(t, g(t)) \in \mathbb{C}^2 : t \in V\}.$$

Definition 3.3. An affine plane curve $X = (f(z,w) = 0)$ is smooth at $p \in X$ if

$$\left(\frac{\partial f}{\partial z}(p), \frac{\partial f}{\partial w}(p)\right) \neq (0,0).$$

If X is smooth $\forall\, p \in X$, then we say that X is a smooth affine curve.

4. Smooth affine plane curves are Riemann surfaces

Theorem 4.1. *Every smooth affine plane curve is necessarily a non-compact Riemann Surface.*

Proof. We already have a topology on $X \subset \mathbb{C}^2$. For each $p \in X$, either $\frac{\partial f}{\partial z}(p) \neq 0$ or $\frac{\partial f}{\partial w}(p) \neq 0$. Assume the latter.

Using Definition 3.2, there exists an open neighborhood $U_p \ni p$, an open set $V_p \subset \mathbb{C}$, and a holomorphic map $g_p : V_p \to \mathbb{C}$ such that

$$U_p = \{(t, g_p(t)) \in \mathbb{C}^2 :\ t \in V_p\}.$$

Define the chart

$$\phi_p : U_p \to V_p, \qquad (t, g_p(t)) \mapsto t.$$

This defines local charts $\{\phi_p : U_p \to V_p\}$ around each point where $\partial f/\partial w \neq 0$. Similarly, whenever $\frac{\partial f}{\partial z}(p) \neq 0$, we can instead solve for $z = h_p(w)$ and define

$$\psi_p : U_p \to W_p, \qquad (h_p(s), s) \mapsto s,$$

for suitable open $W_p \subset \mathbb{C}$.

It remains to check that on overlaps the transition maps are biholomorphic. Take charts centered at p and q and consider $U_p \cap U_q \neq \varnothing$. There are three cases.

Case 1. $\frac{\partial f}{\partial w}(p) \neq 0$ and $\frac{\partial f}{\partial w}(q) \neq 0$. Then on $U_p \cap U_q$ both charts are of the form

$$\phi_p(z,w) = z, \qquad \phi_q(z,w) = z$$

(because each chart uses the z-coordinate as parameter). Hence the transition map is

$$\phi_q \circ \phi_p^{-1} :\ \phi_p(U_p \cap U_q) \to \phi_q(U_p \cap U_q), \qquad t \mapsto t,$$

which is holomorphic with holomorphic inverse.

Case 2. $\frac{\partial f}{\partial z}(p) \neq 0$ and $\frac{\partial f}{\partial z}(q) \neq 0$. Then on $U_p \cap U_q$ both charts use the w-coordinate as parameter:

$$\psi_p(z,w) = w, \qquad \psi_q(z,w) = w.$$

Thus the transition map is again the identity

$$\psi_q \circ \psi_p^{-1}(s) = s,$$

hence biholomorphic.

Case 3. $\frac{\partial f}{\partial w}(p) \neq 0$ and $\frac{\partial f}{\partial z}(q) \neq 0$ (or vice versa). Then ϕ_p uses z as parameter and ψ_q uses w as parameter. On $U_p \cap U_q$ we may write

$$U_p = \{(t, g_p(t)) : \ t \in V_p\}, \qquad U_q = \{(h_q(s), s) : \ s \in W_q\},$$

with g_p holomorphic on V_p and h_q holomorphic on W_q. On the overlap, points satisfy simultaneously $w = g_p(z)$ and $z = h_q(w)$, so

$$h_q(g_p(t)) = t, \qquad g_p(h_q(s)) = s$$

whenever the expressions are defined on the overlap domains. Therefore the transition map is

$$\psi_q \circ \phi_p^{-1} : \ \phi_p(U_p \cap U_q) \to \psi_q(U_p \cap U_q), \qquad t \mapsto \psi_q(t, g_p(t)) = g_p(t),$$

which is holomorphic. Its inverse is

$$\phi_p \circ \psi_q^{-1} : \ \psi_q(U_p \cap U_q) \to \phi_p(U_p \cap U_q), \qquad s \mapsto \phi_p(h_q(s), s) = h_q(s),$$

also holomorphic. Hence the transition maps are biholomorphic.

Thus the atlas formed by all such local charts is compatible and defines a complex structure on X. Since X is a proper closed subset of $\mathbb{C}^2$ given by a single holomorphic equation, it is non-compact in general (and in particular smooth affine curves are not compact in the induced topology). Therefore X is a non-compact Riemann surface. □

5. Projective Space, Plane Curves, and Smoothness

Example 5.1 (Singularities of an affine plane curve)**.** Solve the following system in $\mathbb{C}^2$:

$$y^2 = x^2(x+1), \qquad f(x,y) = y^2 - x^2(x+1).$$

Proof. The curve is the zero set $V(f) \subset \mathbb{C}^2$ with

$$f(x,y) = y^2 - x^2(x+1) = y^2 - x^3 - x^2.$$

A point $(x_0, y_0) \in V(f)$ is singular if and only if $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$.

Compute partial derivatives:

$$f_x(x,y) = -(3x^2 + 2x), \qquad f_y(x,y) = 2y.$$

Thus $f_y = 0$ forces $y = 0$. Then $f_x = 0$ gives

$$3x^2 + 2x = x(3x+2) = 0 \quad \Longrightarrow \quad x = 0 \text{ or } x = -\frac{2}{3}.$$

Now impose the curve equation $f(x,0) = 0$:

$$f(0,0) = 0 \quad \Rightarrow \quad (0,0) \in V(f).$$

But

$$f\left(-\frac{2}{3}, 0\right) = -\left(-\frac{2}{3}\right)^2 \left(-\frac{2}{3} + 1\right) = -\frac{4}{9} \cdot \frac{1}{3} = -\frac{4}{27} \neq 0,$$

so $\left(-\frac{2}{3}, 0\right) \notin V(f)$.

Therefore the unique singular point is $(0,0)$. □

Proposition 5.2 (Singular points of a plane affine curve)**.** *Let $f \in \mathbb{C}[x, y]$ and let $C = V(f) \subset \mathbb{C}^2$. A point $p = (x_0, y_0) \in C$ is singular if and only if*

$$\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = 0.$$

Proof. This is the Jacobian criterion in the hypersurface case: C is smooth at p iff the differential $df_p \neq 0$, equivalently not both partials vanish at p. □

Example 5.3 (Complex structure on projective space)**.** Is projective space $\mathbb{P}^n$ a complex manifold?

Proof. Let $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ with homogeneous coordinates $[z_0 : \cdots : z_n]$.

For each $i \in \{0, \ldots, n\}$ define the standard chart

$$U_i = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \mid z_i \neq 0\}.$$

Define a map $\phi_i : U_i \to \mathbb{C}^n$ by

$$\phi_i([z_0 : \cdots : z_n]) = \left(\frac{z_0}{z_i}, \ldots, \widehat{\frac{z_i}{z_i}}, \ldots, \frac{z_n}{z_i}\right),$$

where the hat means we omit the ith coordinate, so ϕ_i has n entries.

This map is well-defined on projective classes: if we replace $(z_0, \ldots, z_n)$ by $\lambda(z_0, \ldots, z_n)$ with $\lambda \in \mathbb{C}^*$, then each ratio $\frac{\lambda z_j}{\lambda z_i} = \frac{z_j}{z_i}$ is unchanged.

The inverse map $\psi_i : \mathbb{C}^n \to U_i$ is given by inserting 1 in the ith slot:

$$\psi_i(w_1, \ldots, w_n) = [w_0 : \cdots : w_{i-1} : 1 : w_{i+1} : \cdots : w_n],$$

where the w's are placed in the non-i positions in the obvious order. Hence ϕ_i is a homeomorphism onto its image.

It remains to check the transition maps are holomorphic. On overlaps $U_i \cap U_j$ we have $z_i \neq 0$ and $z_j \neq 0$, and

$$\phi_j \circ \psi_i : \mathbb{C}^n \supset \phi_i(U_i \cap U_j) \to \mathbb{C}^n$$

is given by rational functions in the coordinates with denominator equal to the jth coordinate in the i-chart. Concretely, if $\phi_i([z]) = (w_0, \ldots, \widehat{w_i}, \ldots, w_n)$ with $w_k = z_k/z_i$, then on $U_i \cap U_j$ we have $w_j = z_j/z_i \neq 0$ and

$$\phi_j([z]) = \left(\frac{z_0}{z_j}, \ldots, \widehat{\frac{z_j}{z_j}}, \ldots, \frac{z_n}{z_j}\right) = \left(\frac{w_0}{w_j}, \ldots, \widehat{\frac{w_j}{w_j}}, \ldots, \frac{w_n}{w_j}\right),$$

which is holomorphic on the domain $w_j \neq 0$.

Therefore these charts define a complex manifold structure on $\mathbb{P}^n$. □

Definition 5.4 (Homogeneous polynomial)**.** A polynomial $F \in \mathbb{C}[x_0, \ldots, x_n]$ is *homogeneous of degree d* if

$$F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d F(x_0, \ldots, x_n) \quad \text{for all } \lambda \in \mathbb{C}.$$

Equivalently, every monomial appearing in F has total degree d.

Definition 5.5 (Homogenization)**.** Given $f \in \mathbb{C}[x_1, \ldots, x_n]$ of total degree $\leq d$, its *homogenization* of degree d is

$$F(x_0, x_1, \ldots, x_n) := x_0^d\, f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in \mathbb{C}[x_0, \ldots, x_n],$$

which is homogeneous of degree d and satisfies $F(1, x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$.

Example 5.6 (Homogenizing a plane curve)**.** In the affine chart $x_0 = 1$, the curve $x_2 - x_1^3 = 0$ in $\mathbb{C}^2_{(x_1,x_2)}$ homogenizes to

$$F(x_0, x_1, x_2) = x_0^2 x_2 - x_1^3,$$

so the associated projective curve in $\mathbb{P}^2_{[x_0:x_1:x_2]}$ is $V(F)$.

Definition 5.7 ((Smooth) projective plane curve)**.** A *projective plane curve* is the zero set of a homogeneous polynomial $F \in \mathbb{C}[x_0, x_1, x_2]$ in $\mathbb{P}^2$:

$$X = \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 \mid F(x_0, x_1, x_2) = 0\} \subset \mathbb{P}^2.$$

It is *smooth* if it has no singular points (equivalently, if it is smooth in each affine chart $U_i = \{x_i \neq 0\}$).

Lemma 5.8 (Euler's identity)**.** *If $F \in \mathbb{C}[x_0, x_1, x_2]$ is homogeneous of degree d, then*

$$x_0 \frac{\partial F}{\partial x_0} + x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2} = d\,F.$$

Proof. Write F as a sum of monomials $m = x_0^{a_0} x_1^{a_1} x_2^{a_2}$ with $a_0 + a_1 + a_2 = d$. For such a monomial,

$$x_0 \frac{\partial m}{\partial x_0} + x_1 \frac{\partial m}{\partial x_1} + x_2 \frac{\partial m}{\partial x_2} = (a_0 + a_1 + a_2)m = d\,m.$$

Sum over monomials. □

Theorem 5.9 (Projective Jacobian criterion for plane curves)**.** *Let $X = V(F) \subset \mathbb{P}^2$ be a projective plane curve with $F \in \mathbb{C}[x_0, x_1, x_2]$ homogeneous. A point $p \in X$ is singular if and only if*

$$\frac{\partial F}{\partial x_0}(p) = \frac{\partial F}{\partial x_1}(p) = \frac{\partial F}{\partial x_2}(p) = 0.$$

Proof. Work in an affine chart, say $U_0 = \{x_0 \neq 0\}$ with coordinates

$$u = \frac{x_1}{x_0}, \qquad v = \frac{x_2}{x_0}.$$

Let $d = \deg(F)$ and define the dehomogenization

$$f(u, v) := F(1, u, v).$$

Then $X \cap U_0$ is identified with the affine curve $V(f) \subset \mathbb{C}^2_{(u,v)}$.

By Proposition 5.2, a point $p \in X \cap U_0$ is singular iff $f_u(p) = f_v(p) = 0$. We relate these to the homogeneous derivatives. Using $F(x_0, x_1, x_2) = x_0^d f(x_1/x_0, x_2/x_0)$, a direct chain rule computation yields, at points of U_0,

$$\frac{\partial F}{\partial x_1} = x_0^{d-1}\, f_u, \qquad \frac{\partial F}{\partial x_2} = x_0^{d-1}\, f_v,$$

so $f_u = f_v = 0$ is equivalent to $\partial_{x_1} F = \partial_{x_2} F = 0$ on U_0.

Finally, on X we have $F = 0$, so Euler's identity (Lemma 5.8) gives

$$x_0 \frac{\partial F}{\partial x_0} + x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2} = 0.$$

Thus, at any point with $x_0 \neq 0$, if $\partial_{x_1} F = \partial_{x_2} F = 0$, then automatically $\partial_{x_0} F = 0$. Hence, in U_0, singularity is equivalent to the vanishing of all three partials.

The same argument applies in the other charts U_1 and U_2, so the criterion holds globally on $\mathbb{P}^2$. □

Theorem 5.10 (Smooth projective plane curves are compact Riemann surfaces)**.** *Every smooth projective plane curve $X \subset \mathbb{P}^2$ is a compact Riemann surface.*

Proof. Since $\mathbb{P}^2$ is a compact complex manifold (Example 5.3), any closed subset is compact in the subspace topology. A projective curve $X = V(F)$ is closed, hence compact.

Smoothness means that for every point $p \in X$ the Jacobian criterion (Theorem 5.9) guarantees a nonvanishing gradient, so X is a complex submanifold of codimension 1 in $\mathbb{P}^2$ near each point. Therefore X is a complex 1-manifold, i.e. a Riemann surface. Combining with compactness gives a compact Riemann surface.

$$\therefore f|_U : U \to \mathbb{C} \text{ is a constant}$$

□

Remark 5.11. Euler's identity (Lemma 5.8) is exactly why, on X inside an affine chart, it suffices to check only the "affine" partials: the remaining homogeneous derivative is forced by the others once $F = 0$.

Example 5.12. Choose $F(x,y,z) = x^4 + y^4 + z^4$ to be a homogeneous $X = (-z^4 = x^4 + y^4) \subset \mathbb{P}^2$ is a smooth polynomial. After evaluation, find that

$$F_x = 4x^3 = F_y = F_z = 0$$
$$\implies x = y = z = 0 \implies [0,0,0] \in \mathbb{P}^2$$

Example 5.13. Consider the three-degree polynomial $f(x,y) = y^2 - x(x+1)(x-1)$, where $f \in \mathbb{C}^2$.

$$F(x,y,z) = y^2z - x(x+z)(x-z)$$

After partitioning the above F,

$$F_x = -3x^2 + z^2$$
$$F_y = 2yz$$
$$F_z = y^2 - 2xz$$

Compute when $z = 0$

$$\implies x = y = 0 \implies \emptyset$$

Compute when $y = 0$

$$\implies xz = 0 \implies 3x^2 = z^2$$
$$\implies x = z = 0 \implies \emptyset$$
$$\therefore \text{smooth} \implies \text{ compact riemann surface}$$

6. Meromorphic Functions

Remark 6.1. For the purpose of notation we recognize X as a noncompact Riemann surface

Definition 6.2 (Holomorphic function on a Riemann surface)**.** Let X be a Riemann surface Let $W \subset X$ be open Let $f : W \to \mathbb{C}$ be a function We say that f is holomorphic at $p \in W$ if there exists a chart $\phi : U \to V$ with $p \in U \subset W$ such that the map

$$f \circ \phi^{-1} : V \to \mathbb{C}$$

is holomorphic at the point $\phi(p) \in V$ We say that f is holomorphic on W if it is holomorphic at every $p \in W$

Proposition 6.3 (Chart independence)**.** *Let X be a Riemann surface Let $W \subset X$ be open Let $f : W \to \mathbb{C}$ be a function Then the following are equivalent*
(1) f is holomorphic at $p \in W$
(2) For every chart $\phi : U \to V$ with $p \in U \subset W$ the map $f \circ \phi^{-1} : V \to \mathbb{C}$ is holomorphic at $\phi(p)$

Proof. Fix $p \in W$
Proof that (2) implies (1) Assume (2) Choose any chart $\phi : U \to V$ with $p \in U \subset W$ Then $f \circ \phi^{-1}$ is holomorphic at $\phi(p)$ So by definition f is holomorphic at p This proves (1)
Proof that (1) implies (2) Assume (1) Then there exists a chart $\phi_1 : U_1 \to V_1$ with $p \in U_1 \subset W$ such that

$$f \circ \phi_1^{-1} : V_1 \to \mathbb{C}$$

is holomorphic at $\phi_1(p)$

Let $\phi_2 : U_2 \to V_2$ be any other chart with $p \in U_2 \subset W$ Set

$$U_{12} = U_1 \cap U_2 \qquad V_{12} = \phi_2(U_{12})$$

Then $p \in U_{12}$ and V_{12} is open in $\mathbb{C}$ On V_{12} we have the identity

$$f \circ \phi_2^{-1} = (f \circ \phi_1^{-1}) \circ (\phi_1 \circ \phi_2^{-1}) \tag{1.1}$$

because for any $z \in V_{12}$ we have $\phi_2^{-1}(z) \in U_{12}$ so

$$(f \circ \phi_1^{-1})\big(\phi_1(\phi_2^{-1}(z))\big) = f(\phi_2^{-1}(z))$$

which is exactly (1.1)

The transition map

$$\phi_1 \circ \phi_2^{-1} : V_{12} \to \phi_1(U_{12})$$

is holomorphic because X is a Riemann surface The map $f \circ \phi_1^{-1}$ is holomorphic on a neighborhood of $\phi_1(p)$ by assumption Therefore the composition on the right side of (1.1) is holomorphic on a neighborhood of $\phi_2(p)$ Thus $f \circ \phi_2^{-1}$ is holomorphic at $\phi_2(p)$ Since ϕ_2 was arbitrary this proves (2) $\square$

Example 6.4 (Holomorphicity of a chart)**.** Let X be a Riemann surface Let $\phi : U \to V$ be a chart Consider ϕ as a function $\phi : U \to \mathbb{C}$ Then ϕ is holomorphic on U

Indeed take the chart ϕ itself Then

$$\phi \circ \phi^{-1} = \mathrm{id}_V$$

which is holomorphic on V So ϕ is holomorphic by the definition

Example 6.5 (Restriction of a coordinate projection on an affine curve)**.** Let $X \subset \mathbb{C}^2$ be a one dimensional complex manifold realized as a smooth affine plane curve Fix the map

$$\pi_x : X \to \mathbb{C} \qquad \pi_x(x, y) = x$$

Then π_x is holomorphic on X

Proof Fix $p \in X$ Choose a chart $\phi : U \to V$ for X near p Then the map $\phi^{-1} : V \to U \subset \mathbb{C}^2$ is holomorphic by definition of a complex chart Write

$$\phi^{-1}(z) = \big(\alpha(z)\ ,\ \beta(z)\big)$$

where $\alpha : V \to \mathbb{C}$ and $\beta : V \to \mathbb{C}$ are holomorphic functions Then on V we have

$$\pi_x \circ \phi^{-1}(z) = \alpha(z)$$

which is holomorphic So π_x is holomorphic at p Since p was arbitrary π_x is holomorphic on X $\square$

Proposition 6.6 (Maximum modulus on a compact Riemann surface)**.** *Let X be a compact Riemann surface Then every holomorphic function $f : X \to \mathbb{C}$ is constant*

Proof. Since X is compact and f is continuous the image $f(X)$ is a compact subset of $\mathbb{C}$ Hence the function

$$|f| : X \to \mathbb{R} \qquad |f|(p) = |f(p)|$$

attains a maximum at some point $p_0 \in X$

Choose a chart $\phi : U \to V$ with $p_0 \in U$ Define

$$g = f \circ \phi^{-1} : V \to \mathbb{C}$$

Then g is holomorphic on V and $|g|$ has a local maximum at $\phi(p_0)$ By the maximum modulus principle on planar domains g is constant on the connected component of V containing $\phi(p_0)$ Therefore f is constant on U

Now let

$$A = \{p \in X \mid f \text{ is constant on a neighborhood of } p\}$$

We have shown A is nonempty By definition A is open To see A is closed take a sequence $p_k \in A$ with $p_k \to p$ Choose a chart neighborhood U of p For large k we have $p_k \in U$ On U the function f agrees on overlaps with constant values near p_k By connectedness of a small neighborhood in U these constants coincide So f is constant near p Hence $p \in A$ so A is closed

Because X is connected and A is nonempty open and closed we have $A = X$ Thus f is locally constant everywhere and therefore constant on X $\square$

Theorem 6.7 (Classification of isolated singularities in the plane)**.** *Let $V \subset \mathbb{C}$ be open and let $z_0 \in V$ Let $f : V \setminus \{z_0\} \to \mathbb{C}$ be holomorphic Then exactly one of the following holds*

(1) Removable singularity There exists a holomorphic function $\tilde{f} : V \to \mathbb{C}$ such that $\tilde{f} = f$ on $V \setminus \{z_0\}$

(2) Pole There exists an integer $n \geq 1$ and a holomorphic function $g : V \to \mathbb{C}$ with $g(z_0) \neq 0$ such that

$$f(z) = \frac{g(z)}{(z - z_0)^n} \qquad \textit{for } z \in V \setminus \{z_0\}$$

(3) Essential singularity The singularity at z_0 is neither removable nor a pole Equivalently f has a Laurent expansion

$$f(z) = \sum_{k \in \mathbb{Z}} a_k (z - z_0)^k$$

with infinitely many negative index coefficients $a_k \neq 0$

Definition 6.8 (Meromorphic at a point in the plane)**.** Let $V \subset \mathbb{C}$ be open and let $z_0 \in V$ Let $f : V \setminus \{z_0\} \to \mathbb{C}$ be holomorphic We say that f is meromorphic at z_0 if the singularity at z_0 is either removable or a pole

Definition 6.9 (Type of singularity on a Riemann surface)**.** Let X be a Riemann surface Let $W \subset X$ be open and let $p \in W$ Let $f : W \setminus \{p\} \to \mathbb{C}$ be holomorphic We say that f has a removable singularity at p or a pole at p or an essential singularity at p if there exists a chart $\phi : U \to V$ with $p \in U \subset W$ such that the function

$$f \circ \phi^{-1} : V \setminus \{\phi(p)\} \to \mathbb{C}$$

has the corresponding type of isolated singularity at the point $\phi(p)$ in the usual complex analytic sense

Definition 6.10 (Meromorphic function on a Riemann surface)**.** Let X be a Riemann surface and let $W \subset X$ be open A function $f : W \to \mathbb{C} \cup \{\infty\}$ is meromorphic on W if for every $p \in W$ there exists a chart $\phi : U \to V$ with $p \in U \subset W$ such that the coordinate representative

$$f \circ \phi^{-1} : V \to \mathbb{C} \cup \{\infty\}$$

is meromorphic at $\phi(p)$ in the usual one variable sense Equivalently there exists a discrete set $D \subset W$ such that f is holomorphic on $W \setminus D$ and every point of D is a pole or removable singularity of f

7. Meromorphic Functions and Orders

Proposition 7.1. *If $f : W \dashrightarrow \mathbb{C}$ is meromorphic on a Riemann surface (or complex manifold of dimension 1), then for every $p \in W$ there exists an open neighborhood $U \subset W$ of p and holomorphic functions $g, h \in \mathcal{O}(U)$ with $h \not\equiv 0$ such that*

$$f = \frac{g}{h} \qquad \textit{on } U.$$

Proof. Fix $p \in W$.

By definition of meromorphic, there exists a chart (U, ϕ) with $p \in U$ and $\phi : U \to \phi(U) \subset \mathbb{C}$ biholomorphic such that the coordinate expression

$$F := f \circ \phi^{-1} : \phi(U) \dashrightarrow \mathbb{C}$$

is meromorphic in the complex-analytic sense on the open set $\phi(U) \subset \mathbb{C}$.

Let $z_0 = \phi(p)$. Since F is meromorphic at z_0, there exists an integer $n \geq 0$ and a holomorphic function G on a smaller neighborhood $\phi(U_0) \subset \phi(U)$ of z_0 such that

$$F(z) = \frac{G(z)}{(z - z_0)^n} \qquad \text{on } \phi(U_0),$$

and $G(z_0) \neq 0$ if $n > 0$ (after absorbing maximal power of $(z - z_0)$ into n).

Define

$$g := G \circ \phi \in \mathcal{O}(U_0), \qquad h := (z - z_0)^n \circ \phi \in \mathcal{O}(U_0),$$

where $U_0 := \phi^{-1}(\phi(U_0)) \subset U$. Then for every $q \in U_0$,

$$f(q) = F(\phi(q)) = \frac{G(\phi(q))}{(\phi(q) - z_0)^n} = \frac{(G \circ \phi)(q)}{\big((z - z_0)^n \circ \phi\big)(q)} = \frac{g(q)}{h(q)}.$$

Hence $f = g/h$ on U_0. Since $p \in U_0$ and U_0 is open, the claim holds. ∎

Fact 7.2. *Let $f : W \dashrightarrow \mathbb{C}$ be meromorphic and let (U, ϕ) be a chart with $p \in U$, $z = \phi(\cdot)$, $z_0 = \phi(p)$. Then there exist $n \in \mathbb{Z}_{\geq 0}$ and holomorphic G near z_0 such that*

$$(f \circ \phi^{-1})(z) = \frac{G(z)}{(z - z_0)^n}.$$

Equivalently, with $g := G \circ \phi$ and $h := (z - z_0)^n \circ \phi$,

$$f = \frac{g}{h} \qquad \textit{on a neighborhood of } p.$$

Remark 7.3 (Stalks and sheaf maps). Let X be a Riemann surface.

Define the presheaves (indeed sheaves) of holomorphic and meromorphic functions:

$$\mathcal{O}_X(U) := \mathcal{O}(U), \qquad \mathcal{M}_X(U) := \{\text{meromorphic } U \dashrightarrow \mathbb{C}\}.$$

For each point $p \in X$, define stalks

$$\mathcal{O}_{X,p} := \varinjlim_{p \in U} \mathcal{O}_X(U), \qquad \mathcal{M}_{X,p} := \varinjlim_{p \in U} \mathcal{M}_X(U).$$

There is a canonical morphism of sheaves (restriction-compatible maps on each open set)

$$\iota : \mathcal{O}_X \longrightarrow \mathcal{M}_X, \qquad \iota_U : \mathcal{O}_X(U) \to \mathcal{M}_X(U), \qquad g \mapsto g,$$

and hence induced morphisms on stalks

$$\iota_p : \mathcal{O}_{X,p} \to \mathcal{M}_{X,p}.$$

Moreover, $\mathcal{M}_X$ is the sheaf of total quotient rings of $\mathcal{O}_X$ in the sense that on stalks

$$\mathcal{M}_{X,p} \cong \mathrm{Frac}(\mathcal{O}_{X,p}),$$

and under this identification, every germ of a meromorphic function at p is a quotient of germs of holomorphic functions:

$$[f]_p = \frac{[g]_p}{[h]_p} \qquad \text{with } [g]_p, [h]_p \in \mathcal{O}_{X,p},\ [h]_p \neq 0.$$

Categorical diagram (stalk as a colimit over neighborhoods):

$$[columnsep = large]\{\mathcal{O}_X(U)\}_{p\in U}[r, "\iota_U"][d]\{\mathcal{M}_X(U)\}_{p\in U}[d]\mathcal{O}_{X,p}[r, "\iota_p"]\mathcal{M}_{X,p}.$$

Example 7.4. Let X be a compact Riemann surface and take $W = X$. A meromorphic function is defined by local data: for each point $p \in X$ one requires the coordinate expression to be meromorphic in a neighborhood. One cannot demand a single global holomorphic chart on all of X (no global coordinate in general), so the locality in the definition is essential.

Example 7.5. Let $X = \mathbb{P}^1_{[x:y]}$. Consider

$$U_0 := \{[x : y] \in \mathbb{P}^1 :\ y \neq 0\}, \qquad \phi_0 : U_0 \to \mathbb{C}, \qquad \phi_0([x : y]) = \frac{x}{y}.$$

Define

$$f : U_0 \to \mathbb{C}, \qquad f([x : y]) = \frac{x}{y}.$$

Then f is holomorphic on U_0.

Let $\infty := [1 : 0]$ and $U_\infty := \{[x : y] : x \neq 0\}$. Define the chart

$$\phi_\infty : U_\infty \to \mathbb{C}, \qquad \phi_\infty([x : y]) = \frac{y}{x}.$$

On $U_0 \cap U_\infty$ we have

$$(\phi_0 \circ \phi_\infty^{-1})(t) = \frac{1}{t}.$$

Compute the coordinate expression near ∞: for $t \in \mathbb{C}$,

$$\phi_\infty^{-1}(t) = [1 : t],$$

so

$$(f \circ \phi_\infty^{-1})(t) = f([1 : t]) = \frac{1}{t}.$$

Thus f has a pole of order 1 at ∞, so f extends to a meromorphic function on all of $\mathbb{P}^1$ with at worst a pole at $[1:0]$.

Definition 7.6 (Order of a meromorphic function)**.** Let $f : W \dashrightarrow \mathbb{C}$ be meromorphic, fix $p \in W$, choose a chart (U, ϕ) with $p \in U$, and write $z = \phi(\cdot)$ and $z_0 = \phi(p)$.

There exists a unique integer $r \in \mathbb{Z}$ and a holomorphic function H near z_0 with $H(z_0) \neq 0$ such that

$$(f \circ \phi^{-1})(z) = (z - z_0)^r \, H(z).$$

Define the order of f at p to be

$$\operatorname{ord}_p(f) := r \in \mathbb{Z}.$$

Remark 7.7. Let $\operatorname{ord}_p(f) = r$.

(1) $r > 0 \iff f$ has a zero of order r at p.
(2) $r = 0 \iff f$ is holomorphic at p and $f(p) \neq 0$.
(3) $r < 0 \iff f$ has a pole of order $|r|$ at p.

Lemma 7.8. *Definition 7.6 is well-defined: if (U, ϕ) and (U', ψ) are charts around p with $z = \phi(\cdot)$ and $w = \psi(\cdot)$, and*

$$(f \circ \phi^{-1})(z) = (z - z_0)^r H(z), \qquad H(z_0) \neq 0,$$
$$(f \circ \psi^{-1})(w) = (w - w_0)^s K(w), \qquad K(w_0) \neq 0,$$

then $r = s$.

Proof. Let $z_0 = \phi(p)$ and $w_0 = \psi(p)$. Consider the transition map

$$T := \psi \circ \phi^{-1} : \phi(U \cap U') \to \psi(U \cap U'), \qquad w = T(z),$$

which is biholomorphic with $T(z_0) = w_0$.

Write the power series of T at z_0:

$$T(z) - w_0 = a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots, \qquad a_1 \neq 0.$$

Factor

$$T(z) - w_0 = (z - z_0)\,\alpha(z), \qquad \alpha(z) := a_1 + a_2(z - z_0) + a_3(z - z_0)^2 + \cdots, \qquad \alpha(z_0) = a_1 \neq 0.$$

Now compare the two expressions for f on $U \cap U'$:

$$(f \circ \phi^{-1})(z) = (f \circ \psi^{-1})(T(z)).$$

Substitute:

$$(z - z_0)^r H(z) = \big(T(z) - w_0\big)^s K(T(z)) = \big((z - z_0)\alpha(z)\big)^s K(T(z)) = (z - z_0)^s \alpha(z)^s K(T(z)).$$

Rearrange:

$$(z - z_0)^{r-s} = \frac{\alpha(z)^s K(T(z))}{H(z)}.$$

The right-hand side is holomorphic and nonvanishing at z_0 because

$$\alpha(z_0) \neq 0, \qquad K(w_0) \neq 0, \qquad H(z_0) \neq 0,$$

so $\alpha(z)^s K(T(z))/H(z)$ has a holomorphic inverse near z_0.

Hence $(z - z_0)^{r-s}$ is holomorphic and nonvanishing at z_0. This is possible only if $r - s = 0$. Therefore $r = s$. □

Example 7.9. Let

$$f(z) = \frac{z(z-1)^3(z-i)^5}{(z+2)^4(z-\sqrt{2})^6} \qquad (z \in \mathbb{C}).$$

Then

$$\operatorname{ord}_0(f) = 1, \qquad \operatorname{ord}_1(f) = 3, \qquad \operatorname{ord}_i(f) = 5, \qquad \operatorname{ord}_{-2}(f) = -4, \qquad \operatorname{ord}_{\sqrt{2}}(f) = -6,$$

and for $p \in \mathbb{C} \setminus \{0, 1, i, -2, \sqrt{2}\}$,

$$\operatorname{ord}_p(f) = 0.$$

Proposition 7.10. *Let f, g be nonzero meromorphic functions on an open set W and let $p \in W$. Then*

(1) $\mathrm{ord}_p(fg) = \mathrm{ord}_p(f) + \mathrm{ord}_p(g)$.
(2) $\mathrm{ord}_p\left(\frac{f}{g}\right) = \mathrm{ord}_p(f) - \mathrm{ord}_p(g)$.
(3) $\mathrm{ord}_p(f+g) \geq \min\{\mathrm{ord}_p(f), \mathrm{ord}_p(g)\}$, *with equality if* $\mathrm{ord}_p(f) \neq \mathrm{ord}_p(g)$.

Proof. Fix $p \in W$. Choose a chart (U, ϕ) around p with coordinate z, $z_0 = \phi(p)$, and write

$$f(z) = (z - z_0)^r H(z), \qquad H(z_0) \neq 0,$$
$$g(z) = (z - z_0)^s K(z), \qquad K(z_0) \neq 0,$$

where $r = \mathrm{ord}_p(f)$ and $s = \mathrm{ord}_p(g)$.

(1)

$$(fg)(z) = (z - z_0)^{r+s}\, H(z)K(z), \qquad (HK)(z_0) = H(z_0)K(z_0) \neq 0,$$

so $\mathrm{ord}_p(fg) = r + s$.

(2)

$$\frac{f}{g}(z) = (z - z_0)^{r-s}\, \frac{H(z)}{K(z)}, \qquad \frac{H(z_0)}{K(z_0)} \neq 0,$$

so $\mathrm{ord}_p(f/g) = r - s$.

(3)

$$f(z) + g(z) = (z - z_0)^{\min\{r,s\}}\Big((z - z_0)^{r-\min\{r,s\}} H(z) + (z - z_0)^{s-\min\{r,s\}} K(z)\Big).$$

The bracketed term is holomorphic near z_0. Hence

$$\mathrm{ord}_p(f+g) \geq \min\{r, s\}.$$

If $r < s$, then

$$f(z) + g(z) = (z - z_0)^r \Big(H(z) + (z - z_0)^{s-r} K(z)\Big),$$

and the bracket satisfies

$$H(z_0) + (0) \cdot K(z_0) = H(z_0) \neq 0,$$

so $\mathrm{ord}_p(f+g) = r = \min\{r, s\}$. The case $s < r$ is identical. This proves the equality statement when $r \neq s$. □

Remark 7.11. For a fixed point $p \in X$, the map

$$\mathrm{ord}_p : \mathcal{M}_X(W)^\times \to \mathbb{Z}, \qquad f \mapsto \mathrm{ord}_p(f)$$

is a group homomorphism (multiplicative group to additive group) by Definition 7.10(1):

$$\mathrm{ord}_p(fg) = \mathrm{ord}_p(f) + \mathrm{ord}_p(g).$$

It is a discrete valuation in the sense that the inequality in Definition 7.10(3) holds:

$$\mathrm{ord}_p(f+g) \geq \min\{\mathrm{ord}_p(f), \mathrm{ord}_p(g)\}.$$

8. Meromorphic Functions on $\mathbb{P}^1$

Proposition 8.1. *Let X be a compact Riemann surface and let $f : X \dashrightarrow \mathbb{C}$ be meromorphic. Then f has only finitely many zeros and poles on X.*

Proof. Define the zero set and pole set

$$Z := \{p \in X : \mathrm{ord}_p(f) > 0\}, \qquad P := \{p \in X : \mathrm{ord}_p(f) < 0\}.$$

We show Z and P are discrete subsets of X.

Fix $p \in X$. Choose a chart (U, ϕ) around p with coordinate z and $z_0 = \phi(p)$. Write

$$(f \circ \phi^{-1})(z) = (z - z_0)^r H(z), \qquad H(z_0) \neq 0, \qquad r = \mathrm{ord}_p(f).$$

If $r > 0$ then z_0 is an isolated zero of $f \circ \phi^{-1}$ because H is nonvanishing near z_0. Hence there is a neighborhood of p in which p is the only zero of f. Thus zeros are isolated. If $r < 0$ then f has a pole at p, equivalently $1/f$ has a zero at p, and poles are isolated by the same argument. Therefore Z and P are discrete.

A discrete subset of a compact space is finite. Indeed, if Z were infinite, choose distinct points $p_n \in Z$; by compactness, (p_n) has an accumulation point $p \in X$. But Z is discrete, so p has a neighborhood containing at most one point of Z, contradicting accumulation. Hence Z is finite. The same argument shows P is finite. □

Theorem 8.2. *Every meromorphic function $f : \mathbb{P}^1 \dashrightarrow \mathbb{C}$ is of the form*

$$f([x:y]) = \frac{F(x,y)}{G(x,y)}$$

where $F, G \in \mathbb{C}[x,y]$ are homogeneous polynomials of the same degree and $G \not\equiv 0$. Conversely, any such quotient defines a meromorphic function on $\mathbb{P}^1$.

Proof. **(1)** Assume F, G are homogeneous of the same degree d and $G \not\equiv 0$. On $U_0 = \{y \neq 0\}$, set $t = x/y$ and write

$$F(x,y) = y^d F(t,1), \qquad G(x,y) = y^d G(t,1),$$

so

$$\frac{F(x,y)}{G(x,y)} = \frac{F(t,1)}{G(t,1)},$$

a meromorphic function of $t \in \mathbb{C}$. On $U_\infty = \{x \neq 0\}$, set $s = y/x$ and similarly obtain

$$\frac{F(x,y)}{G(x,y)} = \frac{F(1,s)}{G(1,s)},$$

a meromorphic function of $s \in \mathbb{C}$. On $U_0 \cap U_\infty$ we have $t = 1/s$, and the two expressions agree because both equal F/G on that overlap. Hence the quotient defines a meromorphic function on $\mathbb{P}^1$.

(2) Let $f : \mathbb{P}^1 \dashrightarrow \mathbb{C}$ be meromorphic. Restrict to the affine chart

$$U_0 = \{y \neq 0\}, \qquad \phi_0([x:y]) = t = \frac{x}{y}.$$

Then

$$f_0(t) := (f \circ \phi_0^{-1})(t)$$

is a meromorphic function on $\mathbb{C}$. By Definition 8.1 applied to $\mathbb{P}^1$, f has finitely many poles on $\mathbb{P}^1$; in particular f_0 has finitely many poles in $\mathbb{C}$. Let these poles be

$$t = a_1, \dots, a_m \in \mathbb{C}$$

with orders

$$\mathrm{ord}_{a_i}(f_0) = -n_i, \qquad n_i \in \mathbb{Z}_{\geq 1}.$$

Define the polynomial

$$Q(t) := \prod_{i=1}^{m} (t - a_i)^{n_i} \in \mathbb{C}[t].$$

Then $Q(t) f_0(t)$ is meromorphic on $\mathbb{C}$ and has no poles in $\mathbb{C}$ by construction, hence $Q f_0$ is entire on $\mathbb{C}$.

Now inspect the behavior at ∞. Use the chart

$$U_\infty = \{x \neq 0\}, \qquad \phi_\infty([x:y]) = s = \frac{y}{x}.$$

On $U_0 \cap U_\infty$ we have $t = 1/s$. Define

$$f_\infty(s) := (f \circ \phi_\infty^{-1})(s), \qquad \phi_\infty^{-1}(s) = [1:s].$$

Then

$$f_0\left(\frac{1}{s}\right) = f_\infty(s), \qquad (s \neq 0).$$

Also

$$Q\left(\frac{1}{s}\right) f_0\left(\frac{1}{s}\right) = Q\left(\frac{1}{s}\right) f_\infty(s)$$

extends meromorphically across $s = 0$ (since f_∞ is meromorphic at 0 and $Q(1/s)$ has a pole of finite order at 0). Thus the entire function $Q f_0$ has at most polynomial growth at ∞ (equivalently, $Q(1/s) f_\infty(s)$ has at most a pole at $s = 0$).

Concretely, there exists $N \geq 0$ such that

$$s^N Q\left(\frac{1}{s}\right) f_\infty(s)$$

is holomorphic at $s = 0$. Since $s = 1/t$, this means there exists N such that

$$\frac{Q(t) f_0(t)}{t^N}$$

is bounded for $|t|$ large, hence $Q(t)f_0(t)$ is a polynomial in t (entire function with at most polynomial growth). Therefore there exists $P(t) \in \mathbb{C}[t]$ such that

$$Q(t) f_0(t) = P(t), \qquad f_0(t) = \frac{P(t)}{Q(t)}.$$

(3) Let

$$\deg P \leq d, \qquad \deg Q \leq d, \qquad d := \max\{\deg P, \deg Q\}.$$

Define homogeneous polynomials of degree d by

$$F(x,y) := y^d P\left(\frac{x}{y}\right), \qquad G(x,y) : y^d Q\left(\frac{x}{y}\right)$$

(on the level of polynomials this is the standard homogenization: $P(t) = \sum_{k=0}^{d} p_k t^k$ gives $F(x,y) = \sum_{k=0}^{d} p_k x^k y^{d-k}$, similarly for G). Then on U_0,

$$\frac{F(x,y)}{G(x,y)} = \frac{P(x/y)}{Q(x/y)} = f_0(x/y) = f([x:y]).$$

By analytic continuation this equality holds as meromorphic functions on $\mathbb{P}^1$. Thus

$$f([x:y]) = \frac{F(x,y)}{G(x,y)}$$

with F, G homogeneous of the same degree. □

Example 8.3 (Projective plane curve)**.** Let $X = (F = 0) \subset \mathbb{P}^2$ for $F \in \mathbb{C}[x,y,z]$ homogeneous.
Say X is a smooth projective plane curve and, consequently, a connected Riemann Surface. Take two homogeneous polynomials, $G, H \in \mathbb{C}[x,y,z]$ of same degree such that

$$\frac{G}{H} : \mathbb{P}^2 \dashrightarrow \mathbb{C}$$

We define the following linear mapping:

$$\frac{G}{H} : X \dashrightarrow \mathbb{C} \qquad [x,y,z] \mapsto \frac{G(x,y,z)}{H(x,y,z)}$$

Take an open affine subset, and say

$$X_0 = \{[x,y,z] \in X | x \neq 0\} = X \cap U_0$$

$$X_1 = X \cap U_1$$

$$X_2 = X \cap U_2$$

We take an example, X_0

$$X_0 := (f(1,y,z) = 0) \subset \mathbb{C}^2_{(y,z)}$$

On X_0,

$$\frac{G(1,y,z)}{H(1,y,z)}$$

is meromorphic because of the prior example on X_0. Unless $F \mid H$, this is a smooth projective plane curve.

9. Holomorphic maps between two Riemann Surfaces

Intuitively, we are now extending the codomain from $\mathbb{C}^2 \to Y$.

Definition 9.1. Let X, Y be Riemann Surfaces. A map $f : X \to Y$ is holomorphic at $p \in X$ if

$$\exists p \in U, \exists F(p) \in U' \to V'$$

such that

$$\psi \circ F \circ \psi^{-1} : V \to V'$$

is holomorphic at $\phi(p)$.

Lemma 9.2. *Let $F : X \to Y$ be a continuous map between two Riemann Surfaces, then F is holomorphic at $p \in X$ iff*

$$\forall p \in U \to V \qquad F(p) \in U \to U' \qquad F(p) \in V \to V'$$

so

$$\psi \circ F \circ \psi^{-1}$$

is holomorphic at $\phi(p)$.

Proposition 9.3. *A holomorphic map has the following properties,*

1. *Every holomorphic map $f : X \to Y$ where X, Y are Riemann Surfaces is continuous.*
2. *If you have $X \to Y \to Z$ are holomorphic maps between Riemann Surfaces, then $g \circ f : X \to Z$ is holomorphic.*
3. *If $X \to Y$ is a holomorphic map, and $Y \dashrightarrow \mathbb{C}$ is a meromorphic map, then $g \circ f : X \xrightarrow{f} Y \dashrightarrow \mathbb{C}$ is a meromorphic function*

Remark 9.4. If $X \xrightarrow{f} \mathbb{C}$ is a holomorphic function and $X \xrightarrow{f} Y$ is a holomorphic function, then you can do operations on the two observe holomorphic functions.

$$f, g : X \to \mathbb{C} \qquad f + g : X \to \mathbb{C} \qquad p \mapsto f(p) + g(p)$$

Definition 9.5. Let X, Y be two Riemann Surfaces. If $X \xrightarrow{F} Y$, and $Y \xrightarrow{G} X$ are holomorphic maps such that

$$X \xrightarrow{F} Y \xrightarrow{G} X = \mathrm{id}_X \qquad F \circ G : Y \xrightarrow{G} X \xrightarrow{F} Y = \mathrm{id}_Y$$

Then we say F, G are isomorphisms (or biholomorphism) between X, Y, and X, Y are also isomorphisms. We denote isomorphisms by $X \cong Y$

If X, Y are isomorphisms, where $X \cong Y$, then this means X, Y belong the same way as a Riemann Surface.

Lemma 9.6. *Let $F : X \to Y$ be a holomorphic map between Riemann Surfaces. If F is bijective (as sets), then F is biholomorphic.*

Proof. WTS $F^{-1} : Y \to X$ is holomorphic.

Fix $q \in Y$.

Since F is bijective as a set map, there exists a unique $p \in X$ such that

$$F(p) = q.$$

Choose complex charts

$$\phi : U \to V \subset \mathbb{C} \qquad \text{with } p \in U \subset X,$$
$$\psi : U' \to V' \subset \mathbb{C} \qquad \text{with } q \in U' \subset Y.$$

Since F is continuous, $F^{-1}(U')$ is open in X and contains p because $F(p) = q \in U'$. Hence

$$p \in U \cap F^{-1}(U') \neq \emptyset.$$

Replace U by $U \cap F^{-1}(U')$ so that

$$U \subset F^{-1}(U'), \qquad \text{equivalently } F(U) \subset U'.$$

Now consider the coordinate representative of F on U:

$$f := \psi \circ F \circ \phi^{-1} : \ \phi(U) \to \psi(U').$$

Because F is holomorphic, f is holomorphic at $\phi(p)$ by the definition of holomorphic map between Riemann surfaces.

Since F is bijective as a set map, the restriction

$$F|_U : U \to F(U)$$

is bijective, hence its inverse

$$(F|_U)^{-1} : F(U) \to U$$

is well-defined.

Compute the inverse of f explicitly. First write f again:

$$f = \psi \circ F \circ \phi^{-1}.$$

We claim that on the set $\psi(F(U)) \subset \psi(U')$ the inverse is

$$f^{-1} = \phi \circ F^{-1} \circ \psi^{-1}.$$

Verify by direct computation.

(1) Compute $f \circ (\phi \circ F^{-1} \circ \psi^{-1})$.

Take any $w \in \psi(F(U))$. Then there exists $u \in U$ such that

$$w = \psi(F(u)).$$

Compute

$$(f \circ (\phi \circ F^{-1} \circ \psi^{-1}))(w) = f\big(\phi(F^{-1}(\psi^{-1}(w)))\big).$$

Substitute $f = \psi \circ F \circ \phi^{-1}$:

$$f\big(\phi(F^{-1}(\psi^{-1}(w)))\big) = (\psi \circ F \circ \phi^{-1})\big(\phi(F^{-1}(\psi^{-1}(w)))\big).$$

Now apply $\phi^{-1} \circ \phi = \mathrm{id}$ on U:

$$\phi^{-1}\big(\phi(F^{-1}(\psi^{-1}(w)))\big) = F^{-1}(\psi^{-1}(w)).$$

Hence

$$(\psi \circ F \circ \phi^{-1})\big(\phi(F^{-1}(\psi^{-1}(w)))\big) = \psi\Big(F\big(F^{-1}(\psi^{-1}(w))\big)\Big).$$

Since $F \circ F^{-1} = \mathrm{id}$ on $F(U)$ and $\psi^{-1}(w) \in F(U)$, we get

$$F\big(F^{-1}(\psi^{-1}(w))\big) = \psi^{-1}(w).$$

Therefore

$$\psi\Big(F\big(F^{-1}(\psi^{-1}(w))\big)\Big) = \psi(\psi^{-1}(w)) = w.$$

So

$$f \circ (\phi \circ F^{-1} \circ \psi^{-1}) = \mathrm{id}_{\psi(F(U))}.$$

(2) Compute $(\phi \circ F^{-1} \circ \psi^{-1}) \circ f$.

Take any $z \in \phi(U)$. Then there exists $u \in U$ such that

$$z = \phi(u).$$

Compute

$$((\phi \circ F^{-1} \circ \psi^{-1}) \circ f)(z) = \phi\Big(F^{-1}\big(\psi^{-1}(f(z))\big)\Big).$$

Substitute $f(z) = (\psi \circ F \circ \phi^{-1})(z)$:

$$\psi^{-1}(f(z)) = \psi^{-1}\big((\psi \circ F \circ \phi^{-1})(z)\big) = F(\phi^{-1}(z)).$$

Thus

$$\phi\Big(F^{-1}\big(\psi^{-1}(f(z))\big)\Big) = \phi\Big(F^{-1}\big(F(\phi^{-1}(z))\big)\Big).$$

Since $\phi^{-1}(z) \in U$ and $F^{-1} \circ F = \mathrm{id}$ on U, we get

$$F^{-1}\big(F(\phi^{-1}(z))\big) = \phi^{-1}(z).$$

Therefore
$$\phi\Big(F^{-1}\big(F(\phi^{-1}(z))\big)\Big) = \phi(\phi^{-1}(z)) = z.$$
So
$$(\phi \circ F^{-1} \circ \psi^{-1}) \circ f = \mathrm{id}_{\phi(U)}.$$
Hence f is bijective with inverse
$$f^{-1} = \phi \circ F^{-1} \circ \psi^{-1}.$$
Now use the chart-compatibility principle already used in the notes when checking overlaps in the proof of *Smooth affine plane curves are Riemann surfaces*: if a map between charts is holomorphic and bijective, then its inverse is the corresponding reverse transition map (and is holomorphic). Therefore f^{-1} is holomorphic at $\psi(q)$.

Finally, observe that f^{-1} is exactly the coordinate representative of F^{-1}:
$$f^{-1} = \phi \circ F^{-1} \circ \psi^{-1}.$$
Since this is holomorphic at $\psi(q)$, by the definition of holomorphic map between two Riemann surfaces, F^{-1} is holomorphic at q.

Since $q \in Y$ was arbitrary, F^{-1} is holomorphic on Y. Therefore F is biholomorphic. □

Theorem 9.7 (Open mapping theorem)**.** *Every non-constant holomorphic map $F : X \to Y$ is an open map (i.e. if $U \subset X$, then $f(U) \subset Y$) is open).*

Example 9.8. Let
$$f : \mathbb{R} \to \mathbb{R} \qquad x \mapsto x^2 \qquad f((-1,1)) = [0,1)$$

Theorem 9.9 (Identity)**.** *Let $F, G : X \to Y$ be holomorphic maps between Riemann Surfaces. If there exists $S \subset X$ such that S has a limit point and $F(p) = G(p)$ for $\forall p \in S$, then $F = G$ on entire X.*

Proposition 9.10. *Let X, Y be two compact, connected Riemann Surfaces, then every non-constant holomorphic map*
$$F : X \to Y$$
is surjective.

Proof. By open mapping theorem, $F(X)$ is an open subset of Y.

Since X is compact and F is continuous, $F(X)$ is compact.

Since Y is a Hausdorff topological space (because Y is a Riemann surface), compact subsets of Y are closed. Hence $F(X)$ is closed in Y.

Therefore $F(X)$ is both open and closed in Y.

Since X is nonempty, $F(X)$ is nonempty. Since Y is connected, the only subsets of Y that are both open and closed are $\emptyset$ and Y. Hence
$$F(X) = Y.$$
Therefore F is surjective. □

Proposition 9.11. *Let X, Y be two Riemann Surfaces, and $F : X \to Y$ is a non-constant holomorphic map between two Riemann Surfaces, then*
$$\forall y \in Y, F^{-1}(y) \subset X$$
is discrete.

Proof. Fix $y \in Y$ and set
$$S := F^{-1}(y) = \{p \in X : \ F(p) = y\}.$$
Suppose S is not discrete. Then there exists an accumulation point $p_0 \in X$ of S.

Define the constant map
$$G : X \to Y \qquad p \mapsto y.$$
Then G is holomorphic.

For every $p \in S$ we have
$$F(p) = y = G(p).$$

Thus

$$F(p) = G(p) \qquad \text{for all } p \in S.$$

The set S has a limit point p_0 by assumption.

By the Identity Theorem, since F and G are holomorphic and agree on a subset with a limit point, we conclude

$$F = G \qquad \text{on all of } X.$$

But G is constant, so F is constant. This contradicts the assumption that F is non-constant.

Therefore our assumption was false, and $S = F^{-1}(y)$ must be discrete. □

Corollary 9.12. *Let X, Y be compact, connected Riemann Surfaces, with $F : X \to Y$ is a nonconstant holomorphic map, then*

(1) *F is surjective.*
(2) *For every y, $F^{-1}(y)$ is finite.*

Proof. We already proved part 1.

Fix $y \in Y$.

By the previous proposition, the set

$$F^{-1}(y) \subset X$$

is discrete.

Since X is compact, every infinite subset of X has an accumulation point. Equivalently, a discrete subset of a compact space must be finite.

To show this directly, assume for contradiction that $F^{-1}(y)$ is infinite. Then we can choose a sequence of distinct points

$$p_1, p_2, p_3, \dots \in F^{-1}(y).$$

Since X is compact, the sequence (p_n) has a convergent subsequence (p_{n_k}) with limit $p_* \in X$:

$$p_{n_k} \to p_* \in X.$$

Thus p_* is an accumulation point of $F^{-1}(y)$.

This contradicts that $F^{-1}(y)$ is discrete.

Therefore $F^{-1}(y)$ must be finite. □

Fact 9.13. *There exists a finite subset $\{z_1, \dots, z_n\} \subset Y$ such that*

$$F : X \setminus F^{-1}(z_i) \to Y \setminus \{z_i\}$$

is a covering topological space.