Predicting future states using Markov Chains

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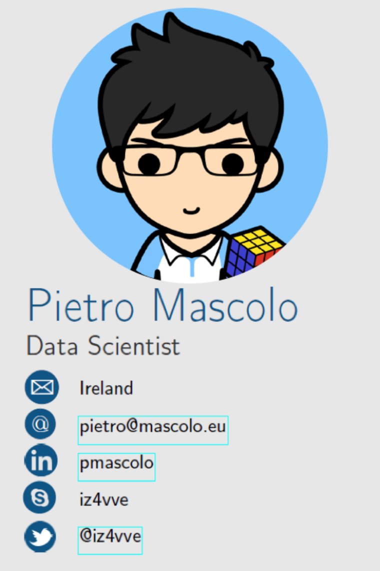
What I'm going to tell you

- Introduction;
- definition and representation of Markov Chains;
- how Markov Chains can be used;
- a bit of maths;
- a bit of code;
- a small demo...

Who I am

- **Mad scientist**;
- **Python enthusiast**;
- **Kotlin**/Scala practitioner;
- **Golang** padawan;
- Ham Radio Operator (EI/IZ4VVE);
- Mountain hiker;
- Karateka;
- Amateur Photographer;
- ...



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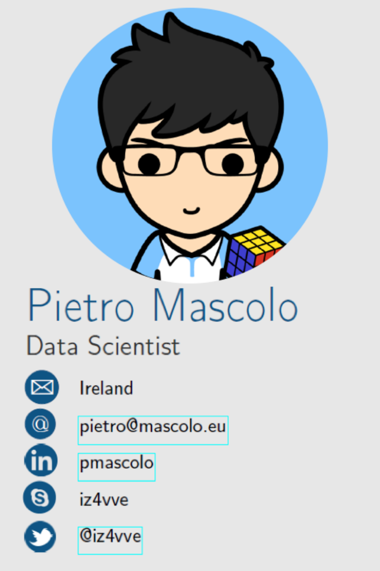

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OUR VALUES

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Why all this?

Problem statement (sorry for having to "anonymise" everything...)

For each user of a system, a sequence of events occurs. Some sequences are good, some are bad. We want to determine driving factors of bad sequences as well as when/where they occur in the system, and for which users.

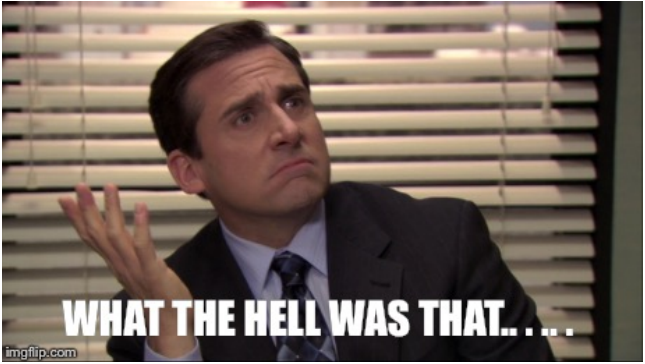
Markov Chains - definition and representation 1/2

Definition

A Markov chain is collection of random variables X_t having the property that, given the present, the future is conditionally independent of the past.

$$P\left(X_t = j | X_0 = i_0, X_1 = i_1, ..., X_{t-1} = i_{t-1}\right) = P\left(X_t = j | X_{t-1} = i_{t-1}\right)$$

Markov Chains - definition and representation 1/2



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Don't panic!

Everything will be clearer with an example...

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Let us imagine we have the following sequence:

$$1, 2, 1, 2, 1, 2, 3, 1, 2$$

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Based on these numbers we can say that:

- If the current state is 1, there is a 100% probability of moving to state 2;
- if the current state is 2, there is 66% probability of evolving to state 1 and 33% of evolving to state 3;
- if the current state is 3, there is a 100% probability of evolving to state 1.

Markov Chains - definition and representation 2/2

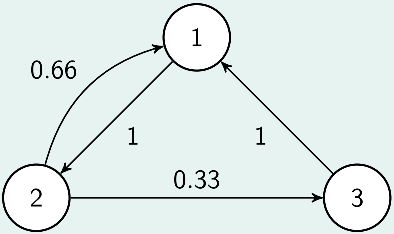
The states we have determined can be represented in a more visual fashion by using a matrix (or - even better - a graph).

Two representations

Matrix representation

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0.66 & 0 & 0.33 \\ 1 & 0 & 0 \end{pmatrix}$$

Graph representation



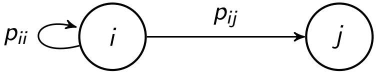
A bit of Maths 1/3 - description

Now we need a bit of maths to show what we can do with Markov Chains...

A bit of Maths 1/3 - description

We can describe a Markov Chain as follows:

Given a set of possible states: $S = \{s_1, s_2, ..., s_n\}$, at each step, a generic state i can move to a new state j or remain in the same initial state with certain *transition probabilities*:



A bit of Maths 2/3 - transition probabilities and next state

All the transition probabilities can be collected in a transition matrix:

$$T = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \\ p_{n1} & & p_{nn} \end{pmatrix}$$

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And given an initial state, the next state transition probabilities are given by:

$$\mathbf{S_{i+1}} = \mathbf{S_i} \times \mathbf{T}$$

and both S_i and S_{i+1} can be either a definite state (vector of all zeros and a single 1) or a probabilistic state (vector of decimals summing up to 1).

A bit of Maths 3/3 - High order Markov Chains

So far we only treated first order Markov Chains: now we can expand on that! What will the next state be, given that the current state is j **AND** the previous state was i?

The transition matrix can be computed the same way as before, only considering all subsequent pairs in the sequence of states.

A bit of Maths 3/3 - High order Markov Chains

In case of a second order chain, the transition matrix will be of size $N(N-1) \times N(N-1)$, where N is the cardinality of the set of possible states, and it will relate pairs of states instead of single states.

$$T = \begin{pmatrix} p_{(11)(11)} & \cdots & p_{(11)(1n)} \\ \vdots & \ddots & \\ p_{(n1)(11)} & & p_{(nn)(nn)} \end{pmatrix}$$

Caveat

WARNING!

For clarity, I will omit a lot of stuff from the code (error handling, correct matrix handling, logging, dosctrings, ...).
You should NOT run the code as is: IT WON'T WORK!

Markov Chain object implementation

Let's start simple, shall we?

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def __init__(self, n_states, order=1):
    self.number_of_states = n_states
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    self.possible_states = {
        j: i for i, j in
        enumerate(
                itertools.product(range(n_states),
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    self.transition_matrix = sparse.dok_matrix((
        (len(self.possible_states), len(self.possible_states))
    ), dtype=np.float64)
```

Transition update

Having initialised our matrix T, we need to update it when we examine a sequence:

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```
def update_transition_matrix(self, states_sequence):

    visited_states = [
        states_sequence[i: i + self.order]
        for i in range(len(states_sequence) - self.order + 1)
    ]
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Transition update

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    ]

    for state_index, i in enumerate(visited_states):
        self.transition_matrix[
            self.possible_states[tuple(i)],
            self.possible_states[tuple(visited_states[
                state_index + self.order
            ])]
        ] += 1
```

Something is mising though...

Something is missing though...

Something is still missing though...

Normalisation

This matrix represents probabilities: **ROWS MUST SUM TO 1**!

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```
def normalise_transitions(self):
    self.transition_matrix = preprocessing.normalize(
        self.transition_matrix, norm="l1"
    )
```

Fit

Calling update_transition_matrix on a sequence of sequences:

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```
def fit(self, state_sequences):
    for index, sequence in enumerate(state_sequences):
        self.update_transition_matrix(sequence)
    self.normalize_transitions()
```

Predicting next state

Now we need the predict method...

Predicting next state

Now we need the predict method... How do we implement that?

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Remember:

$$S_{i+1} = S_i \times T$$

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Predicting next state

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We have to take this all the way down to S_i.
It looks a bit unwieldy to implement...
Oh, wait!
If we **DO** take it all the way to S_i we're only left with T multiplied by itself N times!

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Oh, wait!
If we **DO** take it all the way to S_i we're only left with T multiplied by itself N times!

$$S_{i+N} = S_i \times T^N$$

Now, **this** we can implement :D

Predicting next state

```python
def predict_state(self, current_state, num_steps=1):
    _next_state = sparse.csr_matrix(current_state).dot(
        np.power(self.transition_matrix, num_steps)
    )

    return _next_state
```

There...

There we go!

And that's it for the (simplified) implementation... We're good to go!

Code and demo!! :D

Demo!

Q/A

You've made it through!!

Thanks for your attention!

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Slides and code: http://bit.ly/2H8giAW