Project Summary

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1 Introduction

This project aims to take the first steps toward proving the Walsh analogue of the Carleson-Hunt theorem using the Linearized Metric Carleson theorem.

The work includes formal definitions, theorems, and proofs concerning Haar, Rademacher, and Walsh functions, along with the structure of dyadic intervals and binary representation sets. In particular, I have focused on:

- Formalizing the definitions and properties of dyadic intervals, Haar, Rademacher, and Walsh functions, Walsh series and inner products, as well as the binary representation set, which plays a key role in proving the main statement.
- Stating and proving lemmas that lay the groundwork for the main theorem.

2 Main Results and Formalization Details

I have formalized several fundamental objects that are important for this topic:

- **Dyadic Intervals:** I define dyadic intervals and prove their basic properties. These will be useful for simplifying future work.
- **Haar and Rademacher Functions:** I construct Haar and scaled Haar functions, as well as the Rademacher functions, and prove some of their key properties.
- **Walsh Functions:** I define Walsh functions and simplify their representation.
- **Binary Representation Set:** I formalize the binary representation set, which helps describe the functions mentioned above and analyze their behavior in a more structured way.

The ultimate goal of this project was to formalize the following theorem:

For every $N \in \mathbb{N}$, let $\mathcal{M}$ be the unique subset of $\mathbb{N}$ such that

$$N = \sum_{m \in \mathcal{M}} 2^m.$$

Then, for every dyadic test function f and every $x \in [0, \infty)$, we have

$$\sum_{i=0}^{N} \langle f, W_i \rangle W_i(x) = \int_0^1 W_N(x) K_{\mathcal{M}}(x, y) W_N(y) f(y)\, dy$$

with

$$K_{\mathcal{M}}(x, y) = 1 + \sum_{m \in \mathcal{M}} \sum_{I \in \mathcal{I}^m} h_I(x) h_I(y).$$

However, this theorem has not yet been fully formalized. The statement of the theorem, along with the supporting lemmas that would be used to prove it in the future, remains part of the project.

3 Incomplete Proofs and `sorry`'s

The following parts remain unfinished:

- The main theorem and lemmas from the file `Utiles`.
- The theorem about the kernel. Additional statements about its properties are also needed.
- Statements about the product of Walsh functions and the values they take.
- The theorem about the binary representation set, providing an explicit way to write natural numbers and describing other properties.

4 References and Sources

The formalization is based mostly on my own work and the lecture notes of Prof. Christopher Thiele from previous years' Harmonic Analysis course.