Lean project
First step in formalizing Walsh analogue of the Carleson-Hunt theorem

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Math behind it

Dyadic Interval: $I = [2^k n, 2^k(n+1))$ where $k, n \in \mathbb{Z}$.
Walsh Function: For $x \in [0, 1)$

$$W_0(x) \equiv 1,$$
$$W_{2n}(x) = \begin{cases} W_n(2x) & x < 0.5, \\ W_n(2x-1) & x \geq 0.5 \end{cases},$$
$$W_{2n+1}(x) = \begin{cases} W_n(2x) & x < 0.5, \\ -W_n(2x-1) & x \geq 0.5. \end{cases}$$

Math behind it

Walsh Fourier series:
Let $f \in L^2([0,1])$. The series

$$\sum_{l=0}^{\infty} \langle f, W_l \rangle W_l$$

is called the Walsh-Fourier series for f and converges in a sense of L^2, where

$$\langle f, W_l \rangle = \int_0^1 f(x) W_l(x)\, dx.$$

Math behind it

Haar function Define Haar function $h(x)$ as

$$h(x) \equiv \begin{cases} 1, & 0 \leq x < \frac{1}{2}, \\ -1, & \frac{1}{2} < x \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

and for $I = [2^k n, 2^k(n+1))$

$$h_I(x) = 2^{\frac{k}{2}} h\left(2^k x - n\right).$$

Rademacher function

$$r_n(t) = 2^{-n/2} \sum_{I \in \mathcal{I}^n} h_I(t),$$

where $\mathcal{I}_m$ is the set of dyadic intervals contained in $[0, 1)$ of length 2^{-m}.

Motivation

Goal for the future...

Theorem

For $f \in L^p([0,1])$, $1 < p < \infty$, the Walsh-Fourier series converges to $f(x)$ for almost every $x \in [0,1]$.

Where we go with that?

Current focus:

Theorem

For every $N \in \mathbb{N}$ let $\mathcal{M}$ be the unique subset of $\mathbb{N}$ such that

$$N = \sum_{m \in \mathcal{M}} 2^m.$$

Then, for every dyadic test function f and every $x \in [0, \infty)$, we have

$$\sum_{i=0}^{N} \langle f, W_i \rangle W_i(x) = \int_0^1 W_N(x) K_{\mathcal{M}}(x, y) W_N(y) f(y)\, dy$$

with

$$K_{\mathcal{M}}(x, y) = 1 + \sum_{m \in \mathcal{M}} \sum_{I \in \mathcal{I}^m} h_I(x) h_I(y).$$

On the Way to Formalize It

Progress So Far:

- Formalized definition and properties of dyadic intervals and binary representation set.
- Rewritten and given statements:
  - Walsh functions.
  - Walsh series.
  - Haar functions.
  - Rademacher functions.
- Rewritten lemmas used in the proof.

On the Way to Formalize It

What more should be done?

- Finishing sections about
  - Walsh functions.
  - Walsh series.
  - Haar functions.
  - Rademacher functions.
  - Kernel.
- Proving the needed lemmas.
- Proving the main theorem.

On the Way to Formalize It

Highs and lows

- Confusion about properly defining stuff.
- Dealing with Walsh functions.