BÀI TẬP LỚN GIẢI TÍCH 2

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MSSV: B20DCCN075

Bài 1: a) $F(x,y,z) = y^2 + \frac{2}{x} - \sqrt{z^2-y^2} = 0$

$F'_x(x,y,z) = -\frac{2}{x^2}$, $F'_y(x,y,z) = 2y + \frac{y}{\sqrt{z^2-y^2}}$, $F'_z = \frac{-z}{\sqrt{z^2-y^2}}$

$$y'_x = -\frac{F'_x}{F'_y} = \frac{\frac{2}{x^2}}{2y+\frac{y}{\sqrt{z^2-y^2}}} \quad , \quad y'_z = -\frac{F'_z}{F'_y} = \frac{\frac{z}{\sqrt{z^2-y^2}}}{2y+\frac{y}{\sqrt{z^2-y^2}}}$$

$$\Rightarrow x^2 . y'_x + \frac{1}{z} y'_z = \frac{1}{y}\left(2+\frac{1}{\sqrt{z^2-y^2}}\right) . \frac{1}{2+\frac{1}{\sqrt{z^2-y^2}}} = \frac{1}{y}$$

$$\Rightarrow A = x^2 y'_x + \frac{1}{z} y'_z - \frac{1}{y} = 0$$

b) $\begin{cases} x+y+z=0 \\ x^2+y^2+z^2=1 \end{cases}$, $x = x(y)$, $z = z(y)$

$$\Rightarrow \begin{cases} x+z = -y \\ x^2+z^2 = 1-y^2 \end{cases} \Rightarrow \begin{cases} x'(y) + z'(y) = -1 \\ x.x'(y) + z.z'(y) = -y \end{cases} \Rightarrow \begin{cases} x'(y) = \frac{y-z}{z-x} \\ z'(y) = \frac{y-x}{x-z} \end{cases}$$

d) $A = \sqrt[4]{14e^{0,02} + \sqrt{1,03} + (0,98)^3}$

$A = f(x,y,z) = \sqrt[4]{14e^x + \sqrt{y} + z^3}$

$$\approx f(x_0,y_0,z_0) + f'_x(x_0,y_0,z_0)\Delta x + f'_y(x_0,y_0,z_0)\Delta y + f'_z(x,y,z).\Delta z$$

với $\begin{cases} x_0 = 0 \\ y_0 = 1 \\ z_0 = 1 \end{cases}$, $\begin{cases} \Delta x = 0,02 \\ \Delta y = 0,03 \\ \Delta z = -0,02 \end{cases}$

$f(x_0,y_0,z_0) = 4$

$f'_x = \frac{1}{4}(14e^x + \sqrt{y} + z^3)^{-\frac{3}{4}} . 14e^x \Rightarrow f'_x(x_0,y_0,z_0) = \cancel{\frac{14}{32}} = \frac{7}{16}$.

$f'_y = \frac{1}{4} . \frac{1}{2\sqrt{y}}(14e^x + \sqrt{y} + z^3)^{-\frac{3}{4}} \Rightarrow f'_y(x_0,y_0,z_0) = \frac{1}{64}$

$f'_z = \frac{1}{4} . 3z^2 . (14e^x + \sqrt{y} + z^3)^{-\frac{3}{4}} \Rightarrow f'_z(x_0,y_0,z_0) = \frac{3}{32}$

$$\Rightarrow A = 4 + \frac{7}{16} . 0,02 + \frac{1}{64} . 0,03 + \frac{3}{32} . (-0,02) = \frac{25647}{6400}$$

1c). $z = \int_u^v e^{t^2} du$, $u = x^2 - 3y$, $v = 2x - \cos y^2$

$dz = z'_u du + z'_v dv$, $du = u'_x dx + u'_y dy$, $dv = v'_x dx + v'_y dy$

$\Rightarrow \cancel{z'_u = z'_v}$

$z'_u = e^{u^2}$, $z'_v = e^{v^2}$

$\Rightarrow dz = e^{u^2}(2x\,dx + (-3)dy) + e^{v^2}(2dx + 2y \sin y^2\, dy)$

$= e^{(x^2-3y)^2}(2x\,dx - 3dy) + e^{(2x-\cos y^2)^2}(2dx + 2y \sin y^2\, dy)$

Bài 2:

2.1 a): $z = e^{x}(x+y)(x-y+4) = e^{x}(x^2-y^2+4x+4y)$

$z'_x = e^{x}(x^2+6x-y^2+4y+4)$, $z'_y = e^{x}(4-2y)$

$\begin{cases} z'_x = 0 \\ z'_y = 0 \end{cases} \Leftrightarrow \begin{cases} x^2-y^2+6x+4y+4=0 \\ 4-2y=0 \end{cases} \Leftrightarrow \begin{cases} y=2 \\ (x+2)(x+4)=0 \end{cases} \Leftrightarrow \begin{cases} y=2 \\ \left[\begin{array}{l} x=-2 \\ x=-4 \end{array}\right. \end{cases}$

$\Rightarrow$ Hàm số có 2 điểm dừng là $M_1(-4,2)$, $M_2(-2,2)$

$A = z''_{x^2} = e^{x}(x^2+8x-y^2+4y+10) = e^{x}[(x+4)^2-(y-2)^2-2]$

$B = z''_{xy} = e^{x}(4-2y)$

$C = z''_{y^2} = -2e^{x}$

$\Rightarrow \Delta = B^2 - AC = \cancel{2e^{2x}(x^2+y^2+8x-4y+18)} =$

$= [e^{x}(4-2y)]^2 + 2e^{2x}(x^2+8x-y^2+4y+10)$

Điểm dừng	Δ	A	Kết luận
$M_1(-4,2)$	$-$	$-$	Điểm cực đại
$M_2(-2,2)$	$+$	$\bullet$	không là cực trị

$\Rightarrow z_{CĐ} = z_{max} = 4e^{-4}$

2.1 b) $z = xy^2 + x^4 + y^4$

$\begin{cases} z'_x = 0 \\ z'_y = 0 \end{cases} \Leftrightarrow \begin{cases} y^2+4x^3=0 \\ 2xy+4y^3=0 \end{cases} \Leftrightarrow \begin{cases} y^2+4x^3=0 \\ 2y(x+2y^2)=0 \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} y=0 \\ x=0 \end{cases} \\ \begin{cases} y^2+4x^3=0 \\ 2y^2+x=0 \end{cases} \quad (1) \end{array}\right.$

Giải (1): $\begin{cases} 2y^2+x=0 \\ 8x^3-x=0 \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x = -\frac{1}{2\sqrt{2}} \\ y = \pm\frac{1}{2\sqrt[4]{2}} \end{cases} \\ \begin{cases} x=0 \\ y=0 \end{cases} \end{array}\right.$

$\Rightarrow$ Hàm số có 3 điểm dừng $M_1(0,0)$, $M_2\left(-\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt[4]{2}}\right)$, $M_3\left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt[4]{2}}\right)$

$A = z''_{x^2} = 12x^2$

$B = z''_{xy} = 2y$

$C = z''_{y^2} = 2x + \cancel{2y^2}\ 12y^2 \Rightarrow \Delta = B^2 - AC = 4y^2 - 12x^2(2x+12y^2)$

Điểm dừng	Δ	A	Kết luận
$M_1(0,0)$	0		M_1 là điểm nghi ngờ
$M_2\left(-\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt[4]{2}}\right)$	−	+	M_2 là điểm cực tiểu
$M_3\left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt[4]{2}}\right)$	−	+	điểm cực tiểu

$z(0,0) = 0$

Với $\cancel{x = y = \frac{1}{n} \text{ thì } z\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{n}}$

$M(x, x) \in \Omega_\delta(M_1)$

$\Delta f(M_1) = f(M) - f(M_1) = x^3 + 2x^4$

$= x^3(1 + 2x)$

$\Delta f(M_1) > 0$ nếu $x > 0$

$\Delta f(M_1) < 0$ nếu $-\frac{1}{2} < x < 0$

$\Rightarrow M_1$ không là cực trị

2.2a. $z = f(x,y) = x + 2y$ (1) với điều kiện $x^2 + y^2 = 5$

Hàm Lagrange $L(x, y, \lambda) = x + 2y + \lambda(x^2 + y^2 - 5)$ (2)

$$\begin{cases} L'_x = 0 \\ L'_y = 0 \\ L'_\lambda = 0 \end{cases} \Leftrightarrow \begin{cases} 1 + 2\lambda x = 0 \\ 2 + 2\lambda y = 0 \\ x^2 + y^2 - 5 = 0 \end{cases} \Leftrightarrow \begin{cases} x = -\frac{1}{2\lambda} \\ y = -\frac{1}{\lambda} \\ x^2 + y^2 = 5 \end{cases}$$

$\Rightarrow \left(-\frac{1}{2\lambda}\right)^2 + \left(-\frac{1}{\lambda}\right)^2 = 5 \Rightarrow \lambda = \pm\frac{1}{2} \Rightarrow$

$$\left[\begin{array}{l} \begin{cases} \lambda = \frac{1}{2} \\ x = -1 \\ y = -2 \end{cases} \\ \begin{cases} x = 1 \\ y = 2 \\ \lambda = -\frac{1}{2} \end{cases} \end{array}\right.$$

$\Rightarrow$ Hàm (2) có 2 điểm dừng $A_1\left(-1, -2, \frac{1}{2}\right)$, $A_2\left(1, 2, -\frac{1}{2}\right)$

$L''_{x^2} = 2\lambda$, $L''_{xy} = 0$, $L''_{y^2} = 2\lambda$

$\Rightarrow d^2L = 2\lambda\,dx^2 + 2\lambda\,dy^2$

Điểm	$d^2L(x_0, y_0, \lambda_0)$	Kết luận
$A_1\left(-1, -2, \frac{1}{2}\right)$	> 0	$\Rightarrow M_1(-1, -2)$ là điểm cực tiểu $\Rightarrow z_{CT} = z_{min} = -5$
$A_2\left(1, 2, -\frac{1}{2}\right)$	< 0	$\Rightarrow M_2(1, 2)$ là điểm cực đại $\Rightarrow z_{CĐ} = z_{max} = 5$

2.2b. $f(x,y,z) = xyz$ thỏa mãn điều kiện $x+y+z=5$, $xy+yz+zx=8$

$xy+yz+zx=8 \Rightarrow xyz + z^2(x+y) = 8z$

$\Rightarrow xyz = 8z - z^2(5-z) = z^3-5z^2+8z = g(z)$

$g'(z) = 3z^2-10z+8 = 0 \Leftrightarrow \left[\begin{array}{l} z=2 \\ z=\frac{4}{3}\end{array}\right.$

Với $z=2$, ta có $\begin{cases} x+y+2=5 \\ xy+2(x+y)=8 \end{cases} \Leftrightarrow \begin{cases} x+y=3 \\ xy=2 \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x=1 \\ y=2 \end{cases} \\ \begin{cases} x=2 \\ y=1 \end{cases} \end{array}\right.$

Với $z=\frac{4}{3}$, ta có $\begin{cases} x+y+\frac{4}{3}=5 \\ xy+\frac{4}{3}(x+y)=8 \end{cases} \Leftrightarrow \begin{cases} x+y=\frac{11}{3} \\ xy=\frac{28}{9} \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x=\frac{7}{3} \\ y=\frac{4}{3} \end{cases} \\ \begin{cases} x=\frac{4}{3} \\ y=\frac{7}{3} \end{cases} \end{array}\right.$

Bảng biến thiên

z	$-\infty$		$\frac{4}{3}$		2		$+\infty$
$g'(z)$		$+$	0	$-$	0	$+$	
$g(z)$	$-\infty$	$\nearrow$	$\frac{112}{27}$	$\searrow$	4	$\nearrow$	$+\infty$

Vậy hàm số có 2 điểm cực tiểu là $M_1(1,2,2)$ và $M_2(2,1,2)$, $f_{CT} = 4$

Hàm số có 2 điểm cực đại là $M_3\left(\frac{7}{3},\frac{4}{3},\frac{4}{3}\right)$ và $M_4\left(\frac{4}{3},\frac{7}{3},\frac{4}{3}\right)$, $f_{CĐ} = \frac{112}{27}$

2.3a. GTLN, GTNN của $f(x,y) = x+y$ trong miền $D=\{(x,y) \mid x^2+y^2 \le 1\}$

* Xét trên miền trong của D:

$\text{int } D = U = \{(x,y) \mid x^2+y^2<1\}$

$\begin{cases} f'_x=0 \\ f'_y=0 \end{cases} \Leftrightarrow \begin{cases} 1=0 \\ 1=0 \end{cases}$ (vô lý)

* Xét tại các điểm thuộc biên của D

$\partial D = \{(x,y) \mid x^2+y^2=1\}$

$y^2 = 1-x^2 \Rightarrow \left[\begin{array}{ll} f(x,y)\big|_{y\ge0} = g(x) = x+\sqrt{1-x^2}, & -1\le x\le 1 \\ f(x,y)\big|_{y\le0} = h(x) = x-\sqrt{1-x^2}, & -1\le x\le 1 \end{array}\right.$

• Xét $g(x) = 1 + \frac{-x}{\sqrt{1-x^2}} = 0 \Leftrightarrow x = \frac{1}{\sqrt{2}} \Rightarrow y = \cancel{*}\frac{1}{\sqrt{2}}$.

$\cancel{g(\pm 1)}$ $g(1) = 1, \; g(-1) = -1$

$\cancel{f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} \; f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = g\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$

Bảng biến thiên

x	-1		$\frac{1}{\sqrt{2}}$		1
$g'(x)$	‖	+	0	−	‖
$g(x)$	-1	↗	$\sqrt{2}$	↘	1

$\Rightarrow \max\limits_{\partial D} g(x) = \sqrt{2}$ tại $M_1\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\min\limits_{\partial D} g(x) = -1$ tại $N_1(-1, 0)$ (1)

⊕ Xét $h(x) = 1 + \frac{x}{\sqrt{1-x^2}} = 0 \Leftrightarrow x = -\frac{1}{\sqrt{2}} \Rightarrow y = -\frac{1}{\sqrt{2}}$

$h(-1) = -1, \; h(1) = 1$

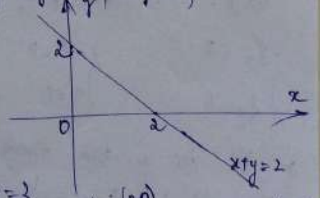
$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = h\left(-\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$ $\Rightarrow \begin{cases} \max\limits_{\partial D} h(x) = \cancel{\sqrt{2}}\ 1 \text{ tại } M_2(1,0) \; (2) \\ \min\limits_{\partial D} h(x) = -\sqrt{2} \text{ tại } N_2\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \end{cases}$

Từ (1), (2) $\Rightarrow \max\limits_{D} f = \sqrt{2}$ tại $M_1\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\min\limits_{D} f = -\sqrt{2}$ tại $N_2\left(-\frac{1}{2}, -\frac{1}{\sqrt{2}}\right)$

<u>2.3 b</u>: $z = x^2 + y^2 - 8x - 6y + 1$ trong miền $D = \{(x,y) : x, y \geq 0 ; \; x + y \leq 2\}$

* Xét trên miền trong của D: $\text{int } D = \{(x,y) \mid x, y > 0 ; \; x + y < 2\}$



$\begin{cases} z'_x = 0 \\ z'_y = 0 \end{cases} \Leftrightarrow \begin{cases} 2x - 8 = 0 \\ 2y - 6 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 4 \\ y = 3 \end{cases} \notin \text{int } D$

* Xét trên biên $x = 0$, $0 \leq y \leq 2$

$z = g(y) = y^2 - 6y + 1$, $g'(y) = 2y - 6 = 0 \Leftrightarrow y = 3$

BBT

y	0		2
$g'(y)$		−	
$g(y)$	1	↘	-7

$\Rightarrow \max\limits_{\substack{x=0 \\ y\in[0,2]}} z = 1$ tại $(0,0)$, $\min\limits_{\substack{x=0 \\ y\in[0,2]}} z = -7$ tại $(0,2)$ (1)

+ Xét trên biên $y=0$, $0 \le x \le 2$

$z = g(x) = x^2 - 8x + 1$, $g'(x) = 0 \Leftrightarrow 2x - 8 = 0 \Leftrightarrow x = 4$

$g(0) = 1$, $g(2) = -11$

$\Rightarrow \max\limits_{\substack{y=0\\ x\in[0,2]}} z = 1$ tại $(0,0)$, $\min\limits_{\substack{y=0\\ x\in[0,2]}} z = -11$, tại $(2,0)$ (2)

+ Xét trên biên $x+y=2$, $0 \le x \le 2$ thì:

$z = x^2 + (2-x)^2 - 8x - 6(2-x) + 1$

$= 2x^2 - 6x - 7 = g(x)$

$g'(x) = 4x - 6 = 0 \Leftrightarrow x = \frac{3}{2} \Rightarrow y = \frac{1}{2}$

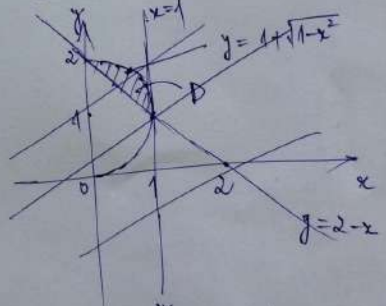
x	0		$\frac{3}{2}$		2
z'		$-$	0	$+$	
z	-7	$\searrow$	$-\frac{23}{2}$	$\nearrow$	-11

$\Rightarrow \max\limits_{\substack{x+y=2\\ x\in[0,2]}} z = -7$ tại $(0,2)$, $\min\limits_{\substack{x+y=2\\ x\in[0,2]}} z = -\frac{23}{2}$ tại $\left(\frac{3}{2}, \frac{1}{2}\right)$ (3)

Từ (1), (2), (3) $\Rightarrow \max\limits_{D} z = 1$ tại $O(0,0)$, $\min\limits_{D} z = -\frac{23}{2}$ tại $M\left(\frac{3}{2}, \frac{1}{2}\right)$

Bài 3:

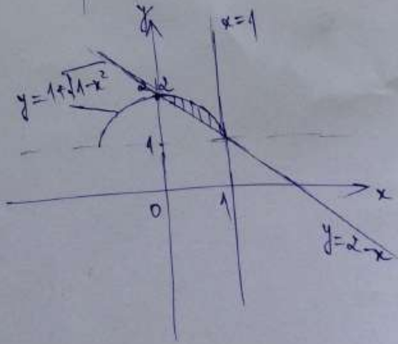
a) $I = \int_0^1 dx \int_{2-x}^{1+\sqrt{1-x^2}} f(x,y)\, dy$



$\begin{cases} 0 \le x \le 1 \\ 2-x \le y \le 1+\sqrt{1-x^2} \end{cases} \Rightarrow \begin{cases} 1 \le y \le 2 \\ 2-y \le x \le \sqrt{2y-y^2} \end{cases}$

$\Rightarrow I = \int_1^2 dy \int_{2-y}^{\sqrt{2y-y^2}} f(x,y)\, dx$

b) $W = \underbrace{\int_0^{\sqrt{2}} dy \int_0^y f(x,y)\, dx}_{W_1} + \underbrace{\int_{\sqrt{2}}^2 dy \int_0^{\sqrt{4-y^2}} f(x,y)\, dx}_{W_2}$

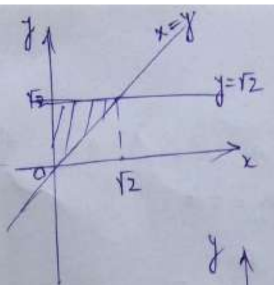


$\Rightarrow W = W_1 + W_2$

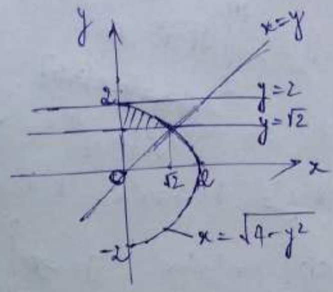
$$W_1 = \int_0^{\sqrt2} dy \int_0^{y} f(x,y)\,dx$$

$$\begin{cases} 0 \le y \le \sqrt2 \\ 0 \le x \le y \end{cases} \Rightarrow \begin{cases} 0 \le x \le \sqrt2 \\ x \le y \le \sqrt2 \end{cases}$$

$$\Rightarrow W_1 = \int_0^{\sqrt2} dx \int_x^{\sqrt2} f(x,y)\,dy \quad (1)$$



$$W_2 = \int_{\sqrt2}^{2} dy \int_0^{\sqrt{4-y^2}} f(x,y)\,dx \quad (2)$$



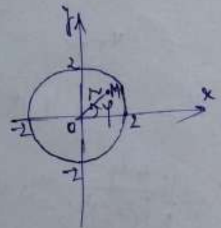
$$\begin{cases} \sqrt2 \le y \le 2 \\ 0 \le x \le \sqrt{4-y^2} \end{cases} \Rightarrow \begin{cases} \sqrt2 \le x \le 2 \\ \sqrt2 \le y \le \sqrt{4-x^2} \end{cases}$$

$$\Rightarrow W_2 = \int_{\sqrt2}^{2} dx \int_{\sqrt2}^{\sqrt{4-x^2}} f(x,y)\,dy \quad (2)$$

Từ (1), (2) $\Rightarrow W = \int_0^{\sqrt2} dx \int_x^{\sqrt2} f(x,y)\,dy + \int_{\sqrt2}^{2} dx \int_{\sqrt2}^{\sqrt{4-x^2}} f(x,y)\,dy = \int_0^{2} dx \int_y^{\sqrt{4-x^2}} f(x,y)\,dy$

Bài 4.

4.1).a $J = \iint_D \sqrt{\frac{4-x^2-y^2}{4+x^2+y^2}}\,dx\,dy$ với miền D là $x^2+y^2 \le 4$



Đặt $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \Rightarrow J = r, \quad x^2+y^2 = r^2$

$D \to \Delta = \left\{(r,\varphi) \;\middle|\; \begin{array}{l} 0 \le \varphi \le 2\pi \\ 0 \le r \le 2 \end{array}\right\}$

$$I = \iint_\Delta \sqrt{\frac{4-r^2}{4+r^2}}\, r\,dr\,d\varphi = \frac12 \int_0^{2\pi} d\varphi \int_0^{2} \sqrt{\frac{4-r^2}{4+r^2}}\, d(r^2)$$

Xét $U = \int_0^2 \sqrt{\frac{4-r^2}{4+r^2}}\, r\,dr$

$\cancel{J = \int \sqrt{\frac{4+x^2}{4+x^2}}\, r\,dr}$

Đặt $\sqrt{4+x} = t \Rightarrow \cancel{dx = 2tdt \Rightarrow J = \int \sqrt{\frac{8-t}{t}}\,dt = \int \sqrt{\frac{8}{t}-1}\,dt}$

$\Rightarrow \cancel{dx = 2tdt}$

Đặt $\sqrt{4+r^2} = t \Rightarrow 2r\,dr = 2t\,dt, \quad r^2 = t^2 - 4 \Rightarrow 4 - r^2 = 8 - t^2$

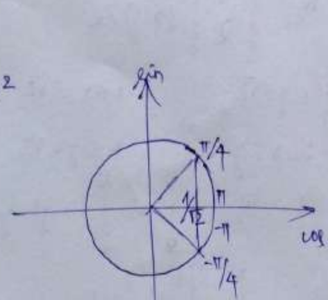
$$\Rightarrow J = \int_2^{2\sqrt2} \frac{\sqrt{8-t^2}}{t}\cdot t\,dt = \int_2^{2\sqrt2} \sqrt{8-t^2}\,dt = \int_2^{2\sqrt2} \sqrt{8-r^2}\,dr$$

$$\Rightarrow I = \int_0^{2\pi} d\varphi \int_2^{2\sqrt2} \sqrt{8-r^2}\,dr$$

$$J = \int_0^2 \sqrt{\frac{4-x^2}{4+x^2}}\, x\,dx$$

Đặt $\sqrt{4+x^2} = t \Rightarrow x^2 = t^2 - 4 \Rightarrow 4 - x^2 = 8 - t^2$

$x^2 = t^2 - 4 \Rightarrow x\,dx = t\,dt$



$$\Rightarrow J = \int_2^{2\sqrt{2}} \sqrt{8-t^2}\, dt$$

Đặt $t = 2\sqrt{2}\cos\alpha,\ \alpha \in [-\pi, \pi] \Rightarrow dt = -2\sqrt{2}\sin\alpha\, d\alpha$

$8 - t^2 = 8 - 8\cos^2\alpha = 8(1-\cos^2\alpha) = 8\sin^2\alpha \Rightarrow \sqrt{8-t^2} = 2\sqrt{2}|\sin\alpha|$

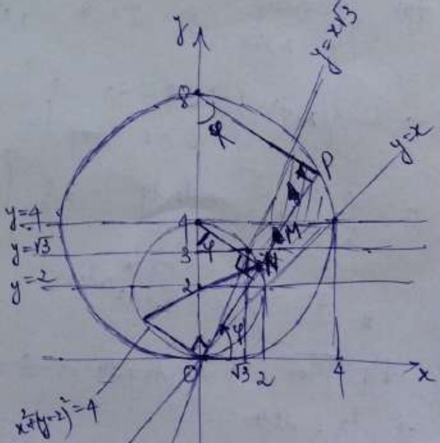
$\cancel{J = 2\sqrt{2}\int_{\frac{\pi}{4}}^{2\pi} |\sin\alpha| \sin\alpha}$ $\quad J = \int_{-\frac{\pi}{4}}^{0} 2\sqrt{2}|\sin\alpha|(-2\sqrt{2})\sin\alpha\, d\alpha \quad \left(|\sin\alpha| = -\sin\alpha,\ -\frac{\pi}{4} \le \alpha \le 0\right)$

$$= 8\int_{-\frac{\pi}{4}}^{0} \sin^2\alpha\, d\alpha = 4\int_{-\frac{\pi}{4}}^{0} (1-\cos 2\alpha)\, d\alpha = 4\left(\alpha - \frac{1}{2}\sin 2\alpha\right)\Big|_{-\frac{\pi}{4}}^{0}$$

$$= \pi - 2$$

$$\Rightarrow I = \int_0^{2\pi} d\varphi\, [\pi - 2] = (\pi-2)\,\varphi\Big|_0^{2\pi} = 2\pi^2 - 4\pi$$

4.1b) $I = \iint_D \frac{dx\,dy}{(x^2+y^2)^2}$, $\begin{cases} 4y \le x^2+y^2 \le 8y \\ x \le y \le \sqrt{3}x \end{cases}$



$\cancel{D = D_1 \cup D_2 \cup D_3,\ D_1 \cap D_2 \cap D_3 = \emptyset}$

$\cancel{\frac{1}{(x^2+y^2)^2} = f(x,y)}$

$\cancel{D_1: \begin{cases} 2 \le y \le 3 \\ \sqrt{4-(y-2)^2} \le x \le y \end{cases}}$

Đặt: $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \Rightarrow J = r$

$$D \to \Delta = \left\{(r,\varphi) \,\middle|\, \begin{array}{l} \frac{\pi}{4} \le \varphi \le \frac{\pi}{3} \\ 4\sin\varphi \le r \le 8\sin\varphi \end{array}\right\} \Rightarrow I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_{4\sin\varphi}^{8\sin\varphi} \frac{1}{r^4}\, r\,dr = \frac{-1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \left[\frac{1}{r^2}\right]\Bigg|_{4\sin\varphi}^{8\sin\varphi}$$

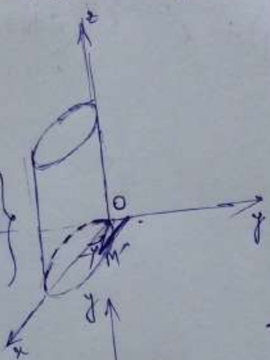
$$\Rightarrow I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_{4\sin\varphi}^{8\sin\varphi} \frac{1}{r^3}\,dr = -\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \left[\frac{1}{r^2}\right]\Bigg|_{4\sin\varphi}^{8\sin\varphi}$$

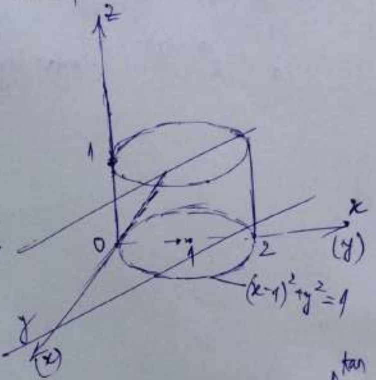
$$= \frac{3}{128}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sin^2\varphi}\,d\varphi = -\frac{3}{128}\cot\varphi\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = -\frac{3}{128}\left(\frac{\sqrt3}{3}-1\right) = \frac{3}{128} - \frac{\sqrt3}{128}$$

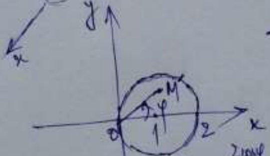
4.2. $I = \iiint\limits_V \frac{dx\,dy\,dz}{(x^2+y^2+1)^2}$, V giới hạn bởi $x^2+y^2 \le 2x$, $0 \le z \le 1$

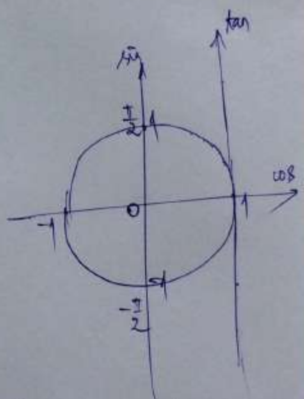
Đặt $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases} \Rightarrow J = r$

$V \to \Omega = \left\{(r,\varphi,z) \;\middle|\; \begin{array}{l} -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le 2\cos\varphi \\ 0 \le z \le 1 \end{array}\right\}$









$$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} \frac{1}{(r^2+1)^2}\,r\,dr \int_0^1 dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} \frac{1}{(r^2+1)^2}\,r\,dr$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} \frac{d(r^2+1)}{(r^2+1)^2} = \cancel{\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \left[\frac{-1}{r^2+1}\right]}$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \left[\frac{1}{r^2+1}\right]\Bigg|_{2\cos\varphi}^{0} = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 - \frac{1}{4\cos^2\varphi+1}\right) d\varphi = \frac{\pi}{2} - \frac{1}{2}J$$

Xét $J = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{1+4\cos^2\varphi}\,d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\frac{1}{\cos^2\varphi}}{\frac{1}{\cos^2\varphi}+4}\,d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d(\tan\varphi)}{5+\tan^2\varphi}$

$$= \cancel{\int_{-1}^{1} \frac{dx}{5+x^2} = \frac{1}{\sqrt5}\arctan\frac{x}{\sqrt5}\Bigg|_{-1}^{1}} = \int_{-\infty}^{+\infty} \frac{dx}{5+x^2} = \frac{1}{\sqrt5}\arctan\frac{x}{\sqrt5}\Bigg|_{-\infty}^{+\infty}$$

$$= \frac{1}{\sqrt5}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{\sqrt5} \Rightarrow I = \cancel{\frac{\pi}{2\sqrt5}}\ \frac{\pi}{2} - \frac{\pi}{2\sqrt5}$$