

Maxwell's Equations are

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho_f \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{H} &= \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}
\end{aligned} \tag{2}$$

in a dielectric material. In a linear material, $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$, $\mathbf{M} = \chi_m \mathbf{H}$, $\mathbf{D} = \epsilon \mathbf{E}$, $\mu \mathbf{H} = \mathbf{B}$ where $\epsilon = \epsilon_0(1 + \chi_e)$ and $\mu = \mu_0(1 + \chi_m)$. The boundary conditions are

$$\begin{aligned}
\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp &= \sigma_f \\
B_1^\perp - B_2^\perp &= 0 \\
\mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel &= \mathbf{0} \\
\frac{1}{\mu_1} \mathbf{B}_1^\parallel - \frac{1}{\mu_2} \mathbf{B}_2^\parallel &= \mathbf{K}_f \times \hat{\mathbf{n}}
\end{aligned} \tag{3}$$

The energy per unit volume (J/m^3) in an electromagnetic field is

$$u = \frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \tag{4}$$

The Poynting vector, defined as

$$\mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}) \tag{5}$$

describes the energy per unit time crossing an infinitesimal surface $d\mathbf{a}$ (W/m^2). Poynting's theorem says

$$\frac{dW}{dt} = -\frac{d}{dt} \int_V u d^3x - \oint_S \mathbf{S} \cdot d\mathbf{a} \tag{6}$$

which reduces to

$$\frac{\partial u}{\partial t} = -\nabla \cdot \mathbf{S} \tag{7}$$

in empty space. It can be shown that the force per unit volume \mathbf{f} is

$$\begin{aligned}
\mathbf{f} &= \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \\
&= \nabla \cdot \overleftrightarrow{\mathbf{T}} - \epsilon_0 \mu_0 \frac{\partial \mathbf{S}}{\partial t}
\end{aligned} \tag{8}$$

where the components of $\overleftrightarrow{\mathbf{T}}$, the Maxwell stress tensor, are

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right) \quad (9)$$

Physically, T_{ij} is the force per unit area in the i th direction acting on an element of surface oriented in the j th direction (whatever that means), so

$$\mathbf{F} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \epsilon_0 \mu_0 \frac{d}{dt} \int_V \mathbf{S} d^3x \quad (10)$$

Newton's second law says

$$\mathbf{F} = \frac{d\mathbf{p}_{\text{mech}}}{dt} \quad (11)$$

which naturally leads to the field momentum density being

$$\mathbf{g} = \mu_0 \epsilon_0 \mathbf{S} = \frac{1}{c^2} \mathbf{S} = \epsilon_0 (\mathbf{E} \times \mathbf{B}) \quad (12)$$

The field's angular momentum density is

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{g} \quad (13)$$

The wave equation

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial x^2} &= \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2} \quad (1D) \\ \nabla^2 \Psi &= \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2} \quad (3D) \end{aligned} \quad (14)$$

admits solutions of the form

$$\begin{aligned} \Psi(\mathbf{r}, t) &= \Psi_R(\mathbf{r}, t) + \Psi_L(\mathbf{r}, t) \\ &= A e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + B e^{-i(\mathbf{k} \cdot \mathbf{r} + \omega t)} \end{aligned} \quad (15)$$

The relevant quantities to wave dynamics are given by

$$\begin{aligned} k &= \frac{2\pi}{\lambda}, & \omega &= 2\pi f \\ v &= \lambda f = \frac{\omega}{k}, & \mathbf{k} \cdot \mathbf{v} &= \omega \\ c &= \frac{1}{\sqrt{\epsilon_0 \mu_0}}, & v &= \frac{1}{\sqrt{\epsilon \mu}} = \frac{c}{n} \end{aligned} \quad (16)$$

where

$$n = \sqrt{\frac{\epsilon \mu}{\epsilon_0 \mu_0}} \quad (17)$$

The general solutions to EM waves in a vacuum are

$$\begin{aligned} \tilde{\mathbf{E}}(\mathbf{r}, t) &= \tilde{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \hat{\mathbf{n}} \\ \tilde{\mathbf{B}}(\mathbf{r}, t) &= \frac{1}{c} \hat{\mathbf{k}} \times \tilde{\mathbf{E}} \end{aligned} \quad (18)$$

The average values of densities are

$$\begin{aligned}\langle u \rangle &= \frac{1}{2} \epsilon_0 E_0^2 \\ \langle \mathbf{S} \rangle &= \frac{1}{2} c \epsilon_0 E_0^2 \hat{\mathbf{z}} \\ \langle \mathbf{g} \rangle &= \frac{1}{2c} \epsilon_0 E_0^2 \hat{\mathbf{z}}\end{aligned}\tag{19}$$

The intensity is $I = \langle S \rangle = \frac{1}{2} c \epsilon_0 E_0^2$. In a material, the formulas translate by taking constants and magnitudes to their material versions: $\epsilon_0 \rightarrow \epsilon$, $\mu_0 \rightarrow \mu$, $c \rightarrow v$. When a wave is incident upon a boundary, we get a reflected and transmitted wave. The wave amplitudes are related by

$$E_{0R} = \left| \frac{n_1 - n_2}{n_1 + n_2} \right| E_{0I}, \quad E_{0T} = \left(\frac{2n_1}{n_1 + n_2} \right) E_{0I}\tag{20}$$

The reflection and transmission coefficients are

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2, \quad T = \frac{4n_1 n_2}{(n_1 + n_2)^2}\tag{21}$$

and $R + T = 1$. The **First Law** of Optics says the incident, reflected, and transmitted wave vectors form a plane of incidence which also includes the plane of incidence. The **Second Law** says $\theta_I = \theta_R$. The **Third Law**, **Snell's law**, says

$$\frac{\sin \theta_T}{\sin \theta_I} = \frac{n_1}{n_2}\tag{22}$$

where n_1 is the incident side and n_2 is the transmitted side. **Fresnel's equations** say

$$\tilde{E}_{0R} = \left(\frac{\alpha - \beta}{\alpha + \beta} \right) \tilde{E}_{0I}, \quad \tilde{E}_{0T} = \left(\frac{2}{\alpha + \beta} \right) \tilde{E}_{0I}\tag{23}$$

where

$$\beta = \frac{\mu_1 n_2}{\mu_2 n_1}, \quad \alpha = \frac{\cos \theta_T}{\cos \theta_I}\tag{24}$$

Brewster's angle is the angle at which the reflected angle is completely extinguished. It is

$$\sin^2 \theta_B = \frac{1 - \beta^2}{(n_1/n_2)^2 - \beta^2}\tag{25}$$

When $\mu_1 \approx \mu_2$, $\beta \approx n_2/n_1$, $\sin^2 \theta_B \approx \beta^2/(1 + \beta^2)$,

$$\tan \theta_B \approx \frac{n_2}{n_1}\tag{26}$$

In a conductor, $\mathbf{J}_f = \sigma \mathbf{E}$. The wave equations become

$$\nabla^2 \mathbf{E} = \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu \sigma \frac{\partial \mathbf{E}}{\partial t}, \quad \nabla^2 \mathbf{B} = \mu \epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu \sigma \frac{\partial \mathbf{B}}{\partial t}\tag{27}$$

The solutions now admit a complex wave number $\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$. The real and imaginary components are

$$\begin{aligned} k = \text{Re } \tilde{k} &= \omega \sqrt{\frac{\epsilon\mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1 \right]^{1/2} \\ \kappa = \text{Im } \tilde{k} &= \omega \sqrt{\frac{\epsilon\mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1 \right]^{1/2} \end{aligned} \quad (28)$$

The solutions become

$$\tilde{\mathbf{E}}(z, t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz - \omega t)}, \quad \tilde{\mathbf{B}}(z, t) = \tilde{\mathbf{B}}_0 e^{-\kappa z} e^{i(kz - \omega t)} \quad (29)$$

The velocity of a sinusoidal component is $v = \omega/k$ while the group velocity of the wave envelope is $v_g = d\omega/dk$. In a linear dielectric, the polarization and complex susceptibility are

$$\begin{aligned} \mathbf{P}(t) &= \epsilon_0 \tilde{\chi}(\omega) \mathbf{E}(t) \\ \chi(\omega) &= \frac{q^2 n}{\epsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \end{aligned} \quad (30)$$

where n is the number of molecules per unit volume, m is the mass of each molecule, ω is the wave's frequency, ω_0 is the resonant frequency, and γ is the damping of each molecule. The solutions to the wave equation become

$$\tilde{E}(z, t) = E_t e^{i(k \text{Re } \sqrt{\kappa} z - \omega t)} e^{-k \text{Im } \sqrt{\kappa} z} \quad (31)$$

where

$$\kappa = 1 + \chi, \quad k = \frac{\omega}{c} \quad (32)$$

Defining

$$\text{Re } \sqrt{\kappa} + i \text{Im } \sqrt{\kappa} = n(\omega) + iK(\omega) \quad (33)$$

we see the $n(\omega)$ term acts as a frequency dependent modifier on wave speed and wavelength while $K(\omega)$ is a frequency dependent attenuation factor. Their formulas are

$$\begin{aligned} n(\omega) &= 1 + \frac{1}{2} \frac{\Omega_p^2 (\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2} \\ K(\omega) &= \frac{1}{2} \frac{\Omega_p^2 \omega \gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2} \end{aligned} \quad (34)$$

where

$$\Omega_p^2 = \frac{q^2 n}{\epsilon_0 m} \quad (35)$$

In a linear dielectric, the group and phase velocities are related by

$$\frac{1}{v_g} = \frac{1}{v_{ph}} + \frac{\omega}{c} \frac{\partial n}{\partial \omega} \quad (36)$$