Maxwell's Equations are

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{E} &= \frac{\rho}{\epsilon_0}\\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0\\ \boldsymbol{\nabla}\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}\\ \boldsymbol{\nabla}\times\mathbf{B} &= \mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\end{aligned}\tag{1}$$

and

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{D} &= \rho_f\\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0\\ \boldsymbol{\nabla}\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}\\ \boldsymbol{\nabla}\times\mathbf{H} &= \mathbf{J}_f+\frac{\partial\mathbf{D}}{\partial t}\end{aligned}\tag{2}$$

in a dielectric material. In a linear material, $\mathbf{P}=\epsilon_0\chi_e\mathbf{E}$, $\mathbf{M}=\chi_m\mathbf{H}$, $\mathbf{D}=\epsilon\mathbf{E}$, $\mu\mathbf{H}=\mathbf{B}$ where $\epsilon=\epsilon_0(1+\chi_e)$ and $\mu=\mu_0(1+\chi_m)$. The boundary conditions are

$$\begin{aligned}\epsilon_1E_1^{\perp}-\epsilon_2E_2^{\perp} &= \sigma_f\\ B_1^{\perp}-B_2^{\perp} &= 0\\ \mathbf{E}_1^{\parallel}-\mathbf{E}_2^{\parallel} &= \mathbf{0}\\ \frac{1}{\mu_1}\mathbf{B}_1^{\parallel}-\frac{1}{\mu_2}\mathbf{B}_2^{\parallel} &= \mathbf{K}_f\times\hat{\mathbf{n}}\end{aligned}\tag{3}$$

The energy per unit volume (J/m^3) in an electromagnetic field is

$$u=\frac{1}{2}\left(\epsilon_0E^2+\frac{1}{\mu_0}B^2\right)\tag{4}$$

The Poynting vector, defined as

$$\mathbf{S}=\frac{1}{\mu_0}\left(\mathbf{E}\times\mathbf{B}\right)\tag{5}$$

describes the energy per unit time crossing an infinitesimal surface $d\mathbf{a}$ (W/m^2). Poynting's theorem says

$$\frac{\mathrm{d}W}{\mathrm{d}t}=-\frac{\mathrm{d}}{\mathrm{d}t}\int_V u\,d^3x-\oiint_S\mathbf{S}\cdot d\mathbf{a}\tag{6}$$

which reduces to

$$\frac{\partial u}{\partial t}=-\boldsymbol{\nabla}\cdot\mathbf{S}\tag{7}$$

in empty space. It can be shown that the force per unit volume $\mathbf{f}$ is

$$\begin{aligned}\mathbf{f} &= \rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\\ &= \boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}}-\epsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t}\end{aligned}\tag{8}$$

where the components of $\overleftrightarrow{\mathbf{T}}$, the Maxwell stress tensor, are

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right) \tag{9}$$

Physically, T_{ij} is the force per unit area in the ith direction acting on an element of surface oriented in the jth direction (whatever that means), so

$$\mathbf{F} = \oiint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \epsilon_0 \mu_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_V \mathbf{S} d^3 x \tag{10}$$

Newton's second law says

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}_{\text{mech}}}{\mathrm{d}t} \tag{11}$$

which naturally leads to the field momentum density being

$$\mathbf{g} = \mu_0 \epsilon_0 \mathbf{S} = \frac{1}{c^2}\mathbf{S} = \epsilon_0 \left(\mathbf{E} \times \mathbf{B} \right) \tag{12}$$

The field's angular momentum density is

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{g} \tag{13}$$

The wave equation

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial x^2} &= \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} \quad \text{(1D)} \\ \nabla^2 \Psi &= \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} \quad \text{(3D)} \end{aligned} \tag{14}$$

admits solutions of the form

$$\begin{aligned} \Psi\left(\mathbf{r}, t\right) &= \Psi_R\left(\mathbf{r}, t\right) + \Psi_L\left(\mathbf{r}, t\right) \\ &= A e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} + B e^{-i(\mathbf{k}\cdot\mathbf{r} + \omega t)} \end{aligned} \tag{15}$$

The relevant quantities to wave dynamics are given by

$$\begin{aligned} k &= \frac{2\pi}{\lambda}, & \omega &= 2\pi f \\ v = \lambda f &= \frac{\omega}{k}, & \mathbf{k} \cdot \mathbf{v} &= \omega \\ c &= \frac{1}{\sqrt{\epsilon_0 \mu_0}}, & v &= \frac{1}{\sqrt{\epsilon \mu}} = \frac{c}{n} \end{aligned} \tag{16}$$

where

$$n = \sqrt{\frac{\epsilon \mu}{\epsilon_0 \mu_0}} \tag{17}$$

The general solutions to EM waves in a vacuum are

$$\begin{aligned} \tilde{\mathbf{E}}(\mathbf{r}, t) &= \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \hat{\mathbf{n}} \\ \tilde{\mathbf{B}}(\mathbf{r}, t) &= \frac{1}{c} \hat{\mathbf{k}} \times \tilde{\mathbf{E}} \end{aligned} \tag{18}$$

The average values of densities are

$$\begin{aligned}\langle u\rangle &= \frac{1}{2}\epsilon_0 E_0^2\\ \langle \mathbf{S}\rangle &= \frac{1}{2}c\epsilon_0 E_0^2\hat{\mathbf{z}}\\ \langle \mathbf{g}\rangle &= \frac{1}{2c}\epsilon_0 E_0^2\hat{\mathbf{z}}\end{aligned} \tag{19}$$

The intensity is $I = \langle S\rangle = \frac{1}{2}c\epsilon_0 E_0^2$. In a material, the formulas translate by taking constants and magnitudes to their material versions: $\epsilon_0 \to \epsilon$, $\mu_0 \to \mu$, $c \to v$. When a wave is incident upon a boundary normally (so $\theta = 0$, we get a reflected and transmitted wave. The wave amplitudes are related by

$$E_{0R} = \left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{0I}, \quad E_{0T} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{0I} \tag{20}$$

The reflection and transmission coefficients are

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2, \quad T = \frac{4n_1 n_2}{(n_1 + n_2)^2} \tag{21}$$

and $R + T = 1$. See Fresnel's equations for the more general case of a wave incident at an angle. The **First Law** of Optics says the incident, reflected, and transmitted wave vectors form a plane of incidence which also includes the plane of incidence. The **Second Law** says $\theta_I = \theta_R$. The **Third Law**, **Snell's law**, says

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2} \tag{22}$$

where n_1 is the incident side and n_2 is the transmitted side. **Fresnel's equations** say

$$\tilde{E}_{0R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)\tilde{E}_{0I}, \quad \tilde{E}_{0T} = \left(\frac{2}{\alpha + \beta}\right)\tilde{E}_{0I} \tag{23}$$

where

$$\beta = \frac{\mu_1 n_2}{\mu_2 n_1}, \quad \alpha = \frac{\cos\theta_T}{\cos\theta_I} \tag{24}$$

Brewster's angle is the angle at which the reflected angle is completely extinguished. It is

$$\sin^2\theta_B = \frac{1 - \beta^2}{(n_1/n_2)^2 - \beta^2} \tag{25}$$

When $\mu_1 \approx \mu_2$, $\beta \approx n_2/n_1$, $sin^2\theta_B \approx \beta^2/(1 + \beta^2)$,

$$\tan\theta_B \approx \frac{n_2}{n_1} \tag{26}$$

In a conductor, $\mathbf{J}_f = \sigma\mathbf{E}$. The wave equations become

$$\nabla^2\mathbf{E} = \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{E}}{\partial t}, \quad \nabla^2\mathbf{B} = \mu\epsilon\frac{\partial^2\mathbf{B}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{B}}{\partial t} \tag{27}$$

The solutions now admit a complex wave number $\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$. The real and imaginary components are

$$\begin{aligned} k = \mathrm{Re}\,\tilde{k} = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2} \\ \kappa = \mathrm{Im}\,\tilde{k} = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2} \end{aligned} \tag{28}$$

The solutions become

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{-\kappa z} e^{i(kz-\omega t)} \tag{29}$$

In a linear dielectric, the polarization and complex susceptibility are

$$\begin{aligned} \mathbf{P}(t) &= \epsilon_0 \tilde{\chi}(\omega)\mathbf{E}(t) \\ \chi(\omega) &= \frac{q^2 n}{\epsilon_0 m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \end{aligned} \tag{30}$$

where n is the number of molecules per unit volume, m is the mass of each molecule, ω is the wave's frequency, ω_0 is the resonant frequency, and γ is the damping of each molecule. The solutions to the wave equation become

$$\tilde{E}(z,t) = E_t e^{i(k\,\mathrm{Re}\,\sqrt{\kappa} z - \omega t)} e^{-k\,\mathrm{Im}\,\sqrt{\kappa} z} \tag{31}$$

where

$$\kappa = 1 + \chi, \quad k = \frac{\omega}{c} \tag{32}$$

Defining

$$\mathrm{Re}\,\sqrt{\kappa} + i\,\mathrm{Im}\,\sqrt{\kappa} = n(\omega) + iK(\omega) \tag{33}$$

we see the $n(\omega)$ term acts as a frequency dependent modifier on wave speed and wavelength while $K(\omega)$ is a frequency dependent attenuation factor. Their formulas are

$$\begin{aligned} n(\omega) &= 1 + \frac{1}{2}\frac{\Omega_p^2\left(\omega_0^2 - \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2} \\ K(\omega) &= \frac{1}{2}\frac{\Omega_p^2 \omega\gamma}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2} \end{aligned} \tag{34}$$

where

$$\Omega_p^2 = \frac{q^2 n}{\epsilon_0 m} \tag{35}$$

Far away from the resonant frequency ω_0, $\chi \approx$ real so attenuation is minimal and can be ignored. The velocity of a sinusoidal component is $v = \omega/k$ while the group velocity of the wave envelope is $v_g = d\omega/dk$. In a linear dielectric, the group and phase velocities are related by

$$\frac{1}{v_g} = \frac{1}{v_{ph}} + \frac{\omega}{c}\frac{\partial n}{\partial\omega} \tag{36}$$

A Fourier series gives a series representation of a function in terms of sinusoidal functions. The important identities are

$$\begin{aligned}
\int_0^a \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)dx &= \frac{a}{2}\delta_{mn} \\
\int_0^a \cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}x\right)dx &= \frac{a}{2}\delta_{mn} \\
\int_0^a \sin\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}x\right)dx &= 0
\end{aligned} \tag{37}$$