Maxwell's Equations are

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{D} &= \rho_f \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\boldsymbol{\nabla} \times \mathbf{H} &= \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}
\end{aligned} \tag{2}$$

in a dielectric material. In a linear material, $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$, $\mathbf{M} = \chi_m \mathbf{H}$, $\mathbf{D} = \epsilon \mathbf{E}$, $\mu \mathbf{H} = \mathbf{B}$ where $\epsilon = \epsilon_0(1 + \chi_e)$ and $\mu = \mu_0(1 + \chi_m)$. The boundary conditions are

$$\begin{aligned}
\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} &= \sigma_f \\
B_1^{\perp} - B_2^{\perp} &= 0 \\
\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} &= \mathbf{0} \\
\frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} &= \mathbf{K}_f \times \hat{\mathbf{n}}
\end{aligned} \tag{3}$$

The energy per unit volume (J/m^3) in an electromagnetic field is

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \tag{4}$$

The Poynting vector, defined as

$$\mathbf{S} = \frac{1}{\mu_0}\left(\mathbf{E} \times \mathbf{B}\right) \tag{5}$$

describes the energy per unit time crossing an infinitesimal surface $d\mathbf{a}$ (W/m^2). Poynting's theorem says

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V u d^3x - \oint\!\!\!\oint_S \mathbf{S} \cdot d\mathbf{a} \tag{6}$$

which reduces to

$$\frac{\partial u}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{S} \tag{7}$$

in empty space. It can be shown that the force per unit volume $\mathbf{f}$ is

$$\begin{aligned}
\mathbf{f} &= \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \\
&= \boldsymbol{\nabla} \cdot \overleftrightarrow{\mathbf{T}} - \epsilon_0 \mu_0 \frac{\partial \mathbf{S}}{\partial t}
\end{aligned} \tag{8}$$

where the components of $\overleftrightarrow{\mathbf{T}}$, the Maxwell stress tensor, are

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right) \tag{9}$$

Physically, T_{ij} is the force per unit area in the ith direction acting on an element of surface oriented in the jth direction (whatever that means), so

$$\mathbf{F} = \oiint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \epsilon_0 \mu_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_V \mathbf{S} d^3 x \tag{10}$$

Newton's second law says

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}_{\mathrm{mech}}}{\mathrm{d}t} \tag{11}$$

which naturally leads to the field momentum density being

$$\mathbf{g} = \mu_0 \epsilon_0 \mathbf{S} = \frac{1}{c^2}\mathbf{S} = \epsilon_0 \left(\mathbf{E} \times \mathbf{B} \right) \tag{12}$$

The field's angular momentum density is

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{g} \tag{13}$$

The wave equation

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial x^2} &= \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} \quad \text{(1D)} \\ \nabla^2 \Psi &= \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} \quad \text{(3D)} \end{aligned} \tag{14}$$

admits solutions of the form

$$\begin{aligned} \Psi\left(\mathbf{r}, t\right) &= \Psi_R\left(\mathbf{r}, t\right) + \Psi_L\left(\mathbf{r}, t\right) \\ &= A e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} + B e^{-i(\mathbf{k}\cdot\mathbf{r} + \omega t)} \end{aligned} \tag{15}$$

The relevant quantities to wave dynamics are given by

$$\begin{aligned} k &= \frac{2\pi}{\lambda}, \qquad \omega = 2\pi f \\ v = \lambda f &= \frac{\omega}{k}, \qquad \mathbf{k}\cdot\mathbf{v} = \omega \\ c &= \frac{1}{\sqrt{\epsilon_0 \mu_0}}, \qquad v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n} \end{aligned} \tag{16}$$

where

$$n = \sqrt{\frac{\epsilon\mu}{\epsilon_0 \mu_0}} \tag{17}$$

The general solutions to EM waves in a vacuum are

$$\begin{aligned} \tilde{\mathbf{E}}(\mathbf{r}, t) &= \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \hat{\mathbf{n}} \\ \tilde{\mathbf{B}}(\mathbf{r}, t) &= \frac{1}{c}\hat{\mathbf{k}} \times \tilde{\mathbf{E}} \end{aligned} \tag{18}$$

The average values of densities are

$$\begin{aligned}
\langle u \rangle &= \frac{1}{2}\epsilon_0 E_0^2 \\
\langle \mathbf{S} \rangle &= \frac{1}{2}c\epsilon_0 E_0^2 \hat{\mathbf{z}} \\
\langle \mathbf{g} \rangle &= \frac{1}{2c}\epsilon_0 E_0^2 \hat{\mathbf{z}}
\end{aligned} \tag{19}$$

The intensity is $I = \langle S \rangle = \frac{1}{2}c\epsilon_0 E_0^2$. In a material, the formulas translate by taking constants and magnitudes to their material versions: $\epsilon_0 \to \epsilon$, $\mu_0 \to \mu$, $c \to v$. When a wave is incident upon a boundary, we get a reflected and transmitted wave. The wave amplitudes are related by

$$E_{0R} = \left| \frac{n_1 - n_2}{n_1 + n_2} \right| E_{0I}, \quad E_{0T} = \left(\frac{2n_1}{n_1 + n_2} \right) E_{0I} \tag{20}$$

The reflection and transmission coefficients are

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2, \quad T = \frac{4n_1 n_2}{(n_1 + n_2)^2} \tag{21}$$

and $R + T = 1$. The **First Law** of Optics says the incident, reflected, and transmitted wave vectors form a plane of incidence which also includes the plane of incidence. The **Second Law** says $\theta_I = \theta_R$. The **Third Law**, **Snell's law**, says

$$\frac{\sin \theta_T}{\sin \theta_I} = \frac{n_1}{n_2} \tag{22}$$

where n_1 is the incident side and n_2 is the transmitted side. **Fresnel's equations** say

$$\tilde{E}_{0R} = \left(\frac{\alpha - \beta}{\alpha + \beta} \right) \tilde{E}_{0I}, \quad \tilde{E}_{0T} = \left(\frac{2}{\alpha + \beta} \right) \tilde{E}_{0I} \tag{23}$$

where

$$\beta = \frac{\mu_1 n_2}{\mu_2 n_1}, \quad \alpha = \frac{\cos \theta_T}{\cos \theta_I} \tag{24}$$

Brewster's angle is the angle at which the reflected angle is completely extinguished. It is

$$\sin^2 \theta_B = \frac{1 - \beta^2}{(n_1/n_2)^2 - \beta^2} \tag{25}$$

When $\mu_1 \approx \mu_2$, $\beta \approx n_2/n_1$, $sin^2\theta_B \approx \beta^2/(1+\beta^2)$,

$$\tan \theta_B \approx \frac{n_2}{n_1} \tag{26}$$

In a conductor, $\mathbf{J}_f = \sigma \mathbf{E}$. The wave equations become

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t}, \quad \nabla^2 \mathbf{B} = \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{B}}{\partial t} \tag{27}$$

The solutions now admit a complex wave number $\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$. The real and imaginary components are

$$\begin{aligned} k &= \operatorname{Re}\tilde{k} = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2} \\ \kappa &= \operatorname{Im}\tilde{k} = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2} \end{aligned} \tag{28}$$

The solutions become

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{-\kappa z} e^{i(kz-\omega t)} \tag{29}$$

In a linear dielectric, the polarization and complex susceptibility are

$$\begin{aligned} \mathbf{P}(t) &= \epsilon_0 \tilde{\chi}(\omega)\mathbf{E}(t) \\ \chi(\omega) &= \frac{q^2 n}{\epsilon_0 m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \end{aligned} \tag{30}$$

where n is the number of molecules per unit volume, m is the mass of each molecule, ω is the wave's frequency, ω_0 is the resonant frequency, and γ is the damping of each molecule. The solutions to the wave equation become

$$\tilde{E}(z,t) = E_t e^{i(k \operatorname{Re}\sqrt{\kappa} z - \omega t)} e^{-k \operatorname{Im}\sqrt{\kappa} z} \tag{31}$$

where

$$\kappa = 1 + \chi, \quad k = \frac{\omega}{c} \tag{32}$$

Defining

$$\operatorname{Re}\sqrt{\kappa} + i \operatorname{Im}\sqrt{\kappa} = n(\omega) + iK(\omega) \tag{33}$$

we see the $n(\omega)$ term acts as a frequency dependent modifier on wave speed and wavelength while $K(\omega)$ is a frequency dependent attenuation factor. Their formulas are

$$\begin{aligned} n(\omega) &= 1 + \frac{1}{2}\frac{\Omega_p^2\left(\omega_0^2 - \omega^2\right)}{\left(\omega_0^2-\omega^2\right)^2 + \omega^2\gamma^2} \\ K(\omega) &= \frac{1}{2}\frac{\Omega_p^2 \omega\gamma}{\left(\omega_0^2-\omega^2\right)^2 + \omega^2\gamma^2} \end{aligned} \tag{34}$$

where

$$\Omega_p^2 = \frac{q^2 n}{\epsilon_0 m} \tag{35}$$

Far away from the resonant frequency ω_0, $\chi \approx$ real so attenuation is minimal and can be ignored. The velocity of a sinusoidal component is $v = \omega/k$ while the group velocity of the wave envelope is $v_g = d\omega/dk$. In a linear dielectric, the group and phase velocities are related by

$$\frac{1}{v_g} = \frac{1}{v_{ph}} + \frac{\omega}{c}\frac{\partial n}{\partial \omega} \tag{36}$$

A Fourier series gives a series representation of a function in terms of sinusoidal functions. The important identities are

$$\begin{aligned}
\int_0^a \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)dx &= \frac{a}{2}\delta_{mn} \\
\int_0^a \cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}x\right)dx &= \frac{a}{2}\delta_{mn} \\
\int_0^a \sin\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}x\right)dx &= 0
\end{aligned} \tag{37}$$