电子科技大学
格拉斯哥学院
Glasgow College, UESTC

Circuit Analysis and Design

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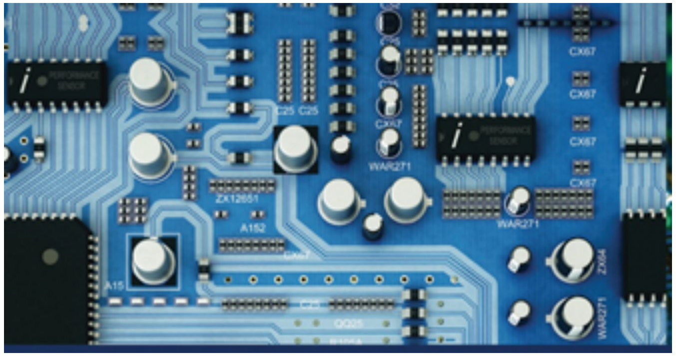
Masood Ur-Rehman, Qammer H. Abbasi, Guodong Zhao

{masood.urrehman, *qammer.abbasi, guodong.zhao*}@glasgow.ac.uk

"A good student never steal or cheat"

Agenda



- Natural response of RC circuit
- Step response of RC circuit
- Natural response of RL circuit
- Step response of RL circuit
- Solving general first-order differential equations

Introduction

- **When a capacitor or an inductor possesses initial energy, the circuit responds to the initial energy until all the energy is spent, even when there is no input signal. The response of a circuit due to initial energy only is called a natural response (also transient response, zero input response, and source-free response).**
- **The response of a circuit to a dc input signal (step input) is called a step response*.* The step response includes the response due to the initial energy stored in the capacitor or inductor.**

Natural Response of RC Circuit

- The switch in Figure 7.1 has been in position *a* for a long time before it is moved to position *b* at $t = 0$. At t = 0, the voltage across the capacitor is equal to the voltage of the source V_S; that is, $v(0) = V_0 = V_S$. For $t \geq 0$, the circuit shown in Figure 7.1 becomes the circuit shown in Figure 7.2, with initial voltage of $v(0) = V_0 = V_S$.

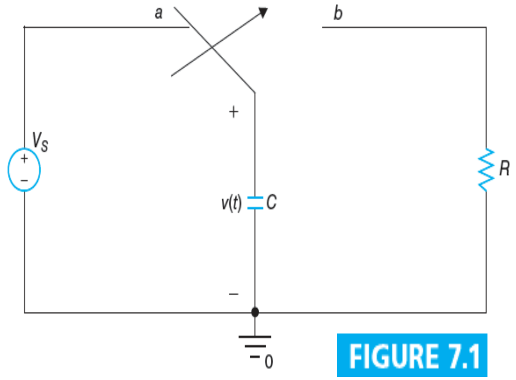


FIGURE 7.1

- Summing the currents leaving node 1, we obtain

$$C\frac{dv(t)}{dt}+\frac{v(t)}{R}=0 \Rightarrow \frac{dv(t)}{dt}=-\frac{v(t)}{RC} \Rightarrow \frac{\frac{dv(t)}{dt}}{v(t)}=-\frac{1}{RC} \Rightarrow \frac{d}{dt}\ln\left[v(t)\right]=-\frac{1}{RC} \quad (1)$$

- Integrating on both sides of the last expression of (1), we get

$$\ln\left[v(t)\right]=-\int_0^t \frac{1}{RC}dt+K \quad (2)$$

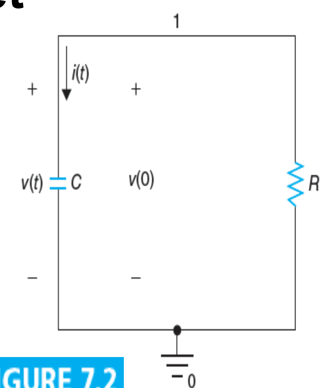


FIGURE 7.2

- Exponentiation on both sides of Equation (2), we obtain

$$e^{\ln[v(t)]}=v(t)=e^{\left(-\frac{t}{RC}+K\right)}=e^{K}e^{-\frac{t}{RC}}=Ae^{-\frac{t}{RC}} \quad (3)$$

Natural Response of RC Circuit (Continued)

- **At $t = 0$, Equation (3) becomes** $v(0) = Ae^{-\frac{0}{RC}} = A$

- **Thus, $A = v(0) = V_0$. The voltage across the capacitor (also resistor) is given by**

$$v(t) = v(0)e^{-\frac{t}{RC}}u(t) = V_0 e^{-\frac{t}{RC}}u(t) \quad (4)$$

where $u(t) = 1$ for $t \geq 0$ and $u(t) = 0$ for $t < 0$. $u(t)$ is called unit step function.

- **The current through the capacitor is given by**

$$i(t) = C\frac{dv(t)}{dt} = C\left(-\frac{1}{RC}\right)v(0)e^{-\frac{t}{RC}} = -\frac{v(0)}{R}e^{-\frac{t}{RC}} = -\frac{V_0}{R}e^{-\frac{t}{RC}}u(t)$$

- **The instantaneous power on the capacitor is given by**

$$p_C(t) = v(t)i(t) = -\frac{v^2(0)}{R}e^{-\frac{2t}{RC}}u(t) \quad \text{(power is released)}$$

- **The energy absorbed by the resistor is given by** $w_R(t) = \int_0^t p_R(\lambda)d\lambda = \frac{Cv^2(0)}{2}\left(1 - e^{-\frac{2t}{RC}}\right)u(t)$
- **At $t = \infty$, $w(\infty) = 0.5CV_0^2$.**

EXAMPLE 7.1

- **Let** $v(0) = V_S = 5$ V, $R = 2$ kΩ, $C = 1$ μF **in the circuit shown in Figure 7.2. Find the expression of** $v(t)$, $i(t)$, $p_C(t)$ **and plot** $v(t)$, $i(t)$, **and** $p_C(t)$ **for** $t \geq 0$. $RC = 2\times10^{-3} = 2$ ms, $1/(RC) = 500$ (1/s)

$$v(t) = V_0 e^{-\frac{t}{RC}} u(t) = 5e^{-500t} u(t)\,\mathrm{V}, i(t) = \frac{-V_0}{R} e^{-\frac{t}{RC}} u(t) = -2.5e^{-500t} u(t)\,\mathrm{mA}, p_C(t) = \frac{-V_0^2}{R} e^{-\frac{2t}{RC}} u(t) = -12.5e^{-1000t} u(t)\,\mathrm{mW}$$

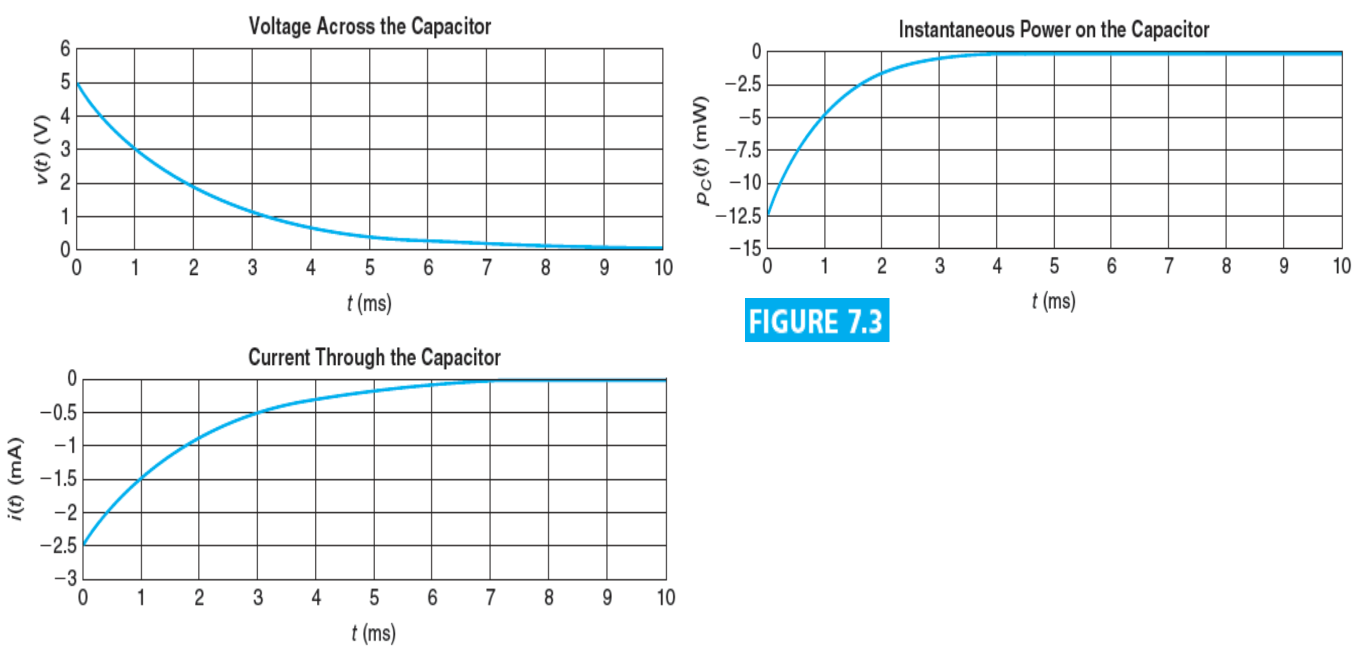
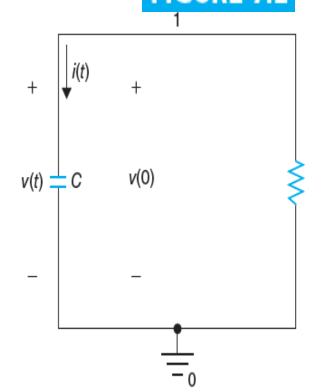


FIGURE 7.3

FIGURE 7.2



Time Constant of RC Circuit

•**The product of R and C, RC, is measured in seconds and is called a time constant, denoted by τ. Thus,**τ = RC

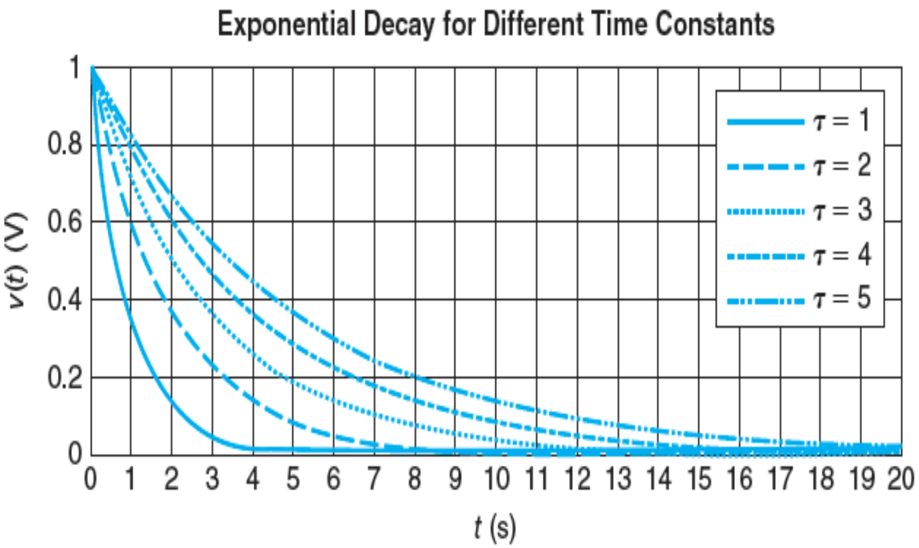
•**In terms of τ,** *v(t), i(t),* **and** $p_C(t)$ **for the circuit shown in Figure 7.2 become, respectively**

$$v(t) = V(0)e^{-\frac{t}{\tau}}u(t), i(t) = \frac{-V(0)}{R}e^{-\frac{t}{\tau}}u(t), p_C(t) = \frac{-V^2(0)}{R}e^{-\frac{2t}{\tau}}u(t)$$

•**Figure 7.4 shows v(t) for** τ = *1, 2, 3, 4*, **and** *5* **[(***v(0)* = *1 V***].**

•**As the time constant τ increases, it takes longer time for the voltage across the capacitor to decay. The speed at which the charges stored on the capacitor plates are discharged is controlled by the time constant τ.**

FIGURE 7.4



Time Constant of RC Circuit (Continued)

- **At time zero (t = 0), the voltage is at its peak value v(0). At time t = τ, the voltage is**

$$v(\tau) = v(0)e^{-\frac{\tau}{\tau}} = v(0)e^{-1} = 0.3678794412\,v(0)$$

- **At time** t = τ, **the voltage across the capacitor drops to** *36.788%* **of the initial value at** *t = 0*. **For** *t = 2τ, 3τ, 4τ, 5τ, . . . , 10τ*, **we have the values shown in Table 7.1.**

TABLE 7.1 Voltage Across the Capacitor Normalized to V_0 at $t = n\tau$.

n	exp(−n)
0	1.000000000000000
1.000000000000000	0.367879441171442
2.000000000000000	0.135335283236613
3.000000000000000	0.049787068367864
4.000000000000000	0.018315638888734
5.000000000000000	0.006737946999085
6.000000000000000	0.002478752176666
7.000000000000000	0.000911881965555
8.000000000000000	0.000335462627903
9.000000000000000	0.000123409804087
10.000000000000000	0.000045399929762

Time Constant of RC Circuit (Continued)

- **At five times the time constant, the voltage across the capacitor due to initial energy on the capacitor is less than *1%* of the initial voltage (*0.6738%*). For all practical purposes, the transient response can be ignored after about five times the time constant.**
- **The time derivative of the voltage across the capacitor is given by**

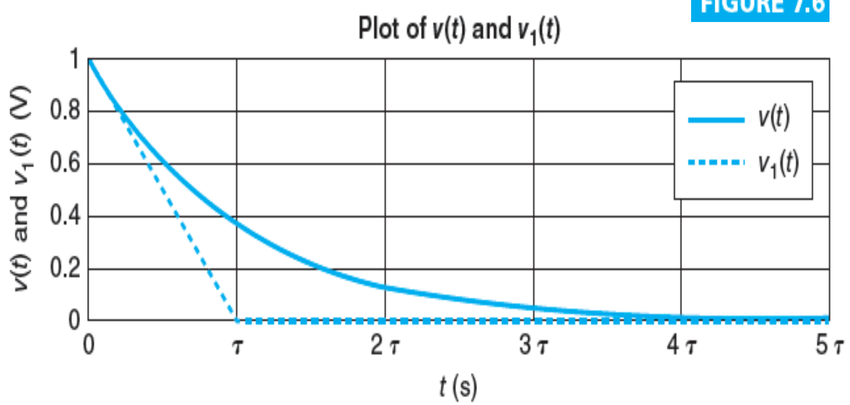
$$\frac{dv(t)}{dt} = v(0)\frac{d}{dt}e^{-\frac{t}{\tau}} = -\frac{v(0)}{\tau}e^{-\frac{t}{\tau}}$$

- **The rate of decay of the voltage across the capacitor is at its maximum at t = 0 and slows down as time progresses. The rate of decay at *t = 0* is – *v(0)/τ*. If the voltage decreases at this rate, *v(t)* is given by**

$$v_1(t) - v(0) = -\frac{v(0)}{\tau}(t-0)$$

- **Figure 7.6 shows *v(t)* and $v_1(t)$.**

FIGURE 7.6



Finding the Time Constant

- **If there is one resistor with resistance** R **and one capacitor with capacitance** C**, as in the circuit shown in Figure 7.2, the time constant is the product of the** R **and** C**; that is,** $\tau = RC$**.**
- **If there is one capacitor with capacitance** C **and more than one resistor in the circuit, find the equivalent resistance** R_{eq} **of all the resistors in the circuit that connects in parallel to the capacitor. Then, the circuit reduces to one capacitor with capacitance** C **and one resistor with resistance** R_{eq} **connected in parallel. The time constant is given by** $\tau = R_{eq}C$**.**
- **The voltage across the capacitor is given by**

$$v(t) = v(0)e^{-\frac{t}{\tau}}u(t) \text{ V}$$

- **Finding the voltage** $v(t)$ **across the capacitor for the given circuit involves finding the initial voltage** $v(0)$ **across the capacitor, finding the equivalent resistance** R_{eq}**, and finding the time constant** $\tau = R_{eq}C$**.**

EXAMPLE 7.2

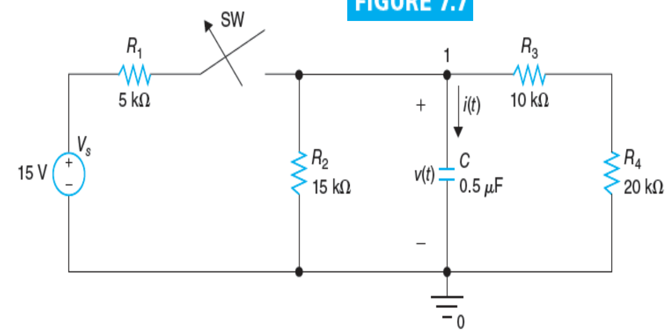
- **In the circuit shown in Figure 7.7, the switch has been closed for a long time before it is opened at** $t = 0$**. Find the voltage** $v(t)$ **across the capacitor and the current** $i(t)$ **through the capacitor for** $t \geq 0$ **and plot** $v(t)$ **and** $i(t)$ **for** $t \geq 0$**. Before the switch is opened, the capacitor can be treated as an open circuit.**
- $R_2 \parallel (R_3 + R_4) = 15 \times 30/(15 + 30)\ k\Omega = 10\ k\Omega$
- **Voltage divider rule:** $v(0) = V_0 = 15\ V \times 10\ k\Omega / 15\ k\Omega = 10\ V$
- $R_{eq} = R_2 \parallel (R_3 + R_4) = 15 \times 30/(15 + 30)\ k\Omega = 10\ k\Omega$
- $\tau = R_{eq}C = 10^4 \times 5 \times 10^{-7} = 5 \times 10^{-3} = 5\ ms,\ 1/\tau = 200\ (1/s)$
- **The voltage across the capacitor is given by**

$$v(t) = V(0)e^{-\frac{t}{\tau}}u(t) = 10e^{-200t}u(t)\ \text{V}$$

- **The current through the capacitor is**

$$i(t) = C\frac{dv(t)}{dt} = 5\times10^{-7}\times10\times(-200)e^{-200t}u(t) = -e^{-200t}u(t)\ \text{mA}$$

FIGURE 7.7



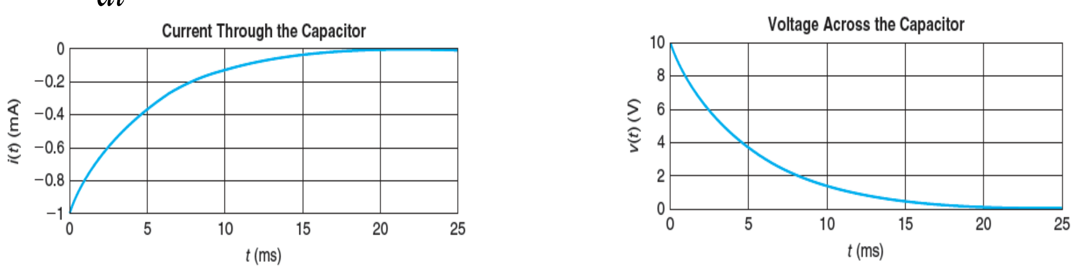


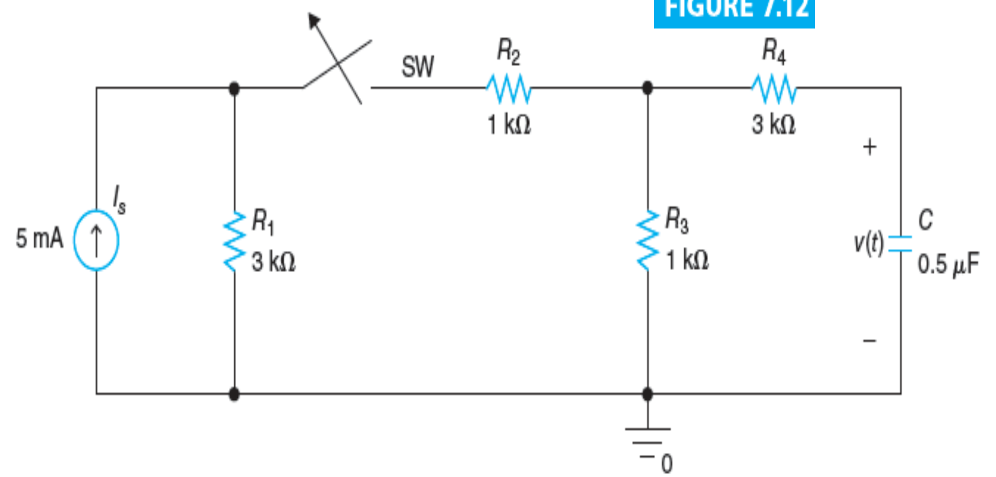
FIGURE 7.10

EXAMPLE 7.3

- **In the circuit shown in Figure 7.12, the switch has been closed for a long time before it is opened at** $t = 0$**. Find the voltage v(t) across the capacitor for** $t \geq 0$**. Before the switch is opened, the capacitor can be treated as an open circuit. The current through** R_4 **is zero.**
- **From the current divider rule, the current through** R_3 **is given by** $5\ mA \times 3\ k\Omega / 5\ k\Omega = 3\ mA$
- **Ohm's law:** $v(0) = V_0 = 1\ k\Omega \times 3\ mA = 3\ V$
- $R_{eq} = R_3 + R_4 = 4\ k\Omega$
- $\tau = R_{eq}C = 4 \times 10^3 \times 5 \times 10^{-7} = 2 \times 10^{-3} = 2\ ms,\ 1/\tau = 500\ (1/s)$
- **The voltage across the capacitor is given by**

$$v(t) = V(0)e^{-\frac{t}{\tau}}u(t) = 3e^{-500t}u(t)\ \mathrm{V}$$

FIGURE 7.12



Step Response of RC Circuit

- The switch in the circuit shown in Figure 7.15 is closed at $t = 0$. At $t = 0$, the voltage across the capacitor is $v(0) = V_0$.
- Collecting the voltage drops around the mesh in the clockwise direction for $t \geq 0$, we have

$-V_S + R\,i(t) + v(t) = 0 \quad (1)$

- The current through the capacitor is given by $$i(t) = C\frac{dv(t)}{dt} \quad (2)$$

- Substitution of Equation (2) into Equation (1) yields

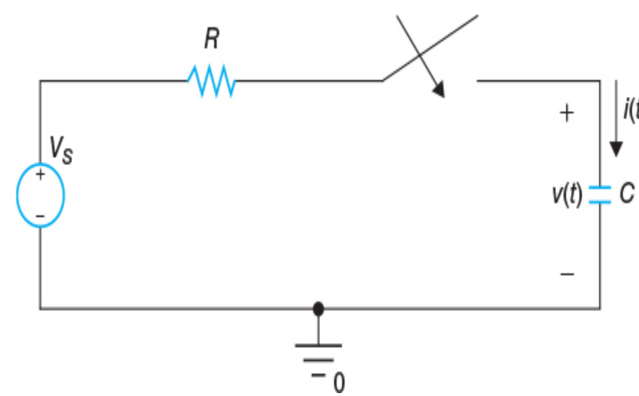
$$-V_S + RC\frac{dv(t)}{dt} + v(t) = 0 \Rightarrow \frac{dv(t)}{dt} = \frac{-1}{RC}\left[v(t) - V_S\right] \quad (3)$$

- Dividing by $v(t) - V_S$ on both sides, we obtain

$$\frac{\frac{dv(t)}{dt}}{v(t) - V_S} = \frac{-1}{RC} \Rightarrow \frac{d}{dt}\left[\ln\left|v(t) - V_S\right|\right] = \frac{-1}{RC} \quad (4)$$

FIGURE 7.15

RC circuit with step input.



Step Response of RC Circuit (Continued)

- **Integrating on both sides of Equation (4), we obtain**

$$\ln\left|v(t)-V_S\right| = \int_0^t \frac{-1}{RC}dt + K = \frac{-t}{RC} + K \quad (5)$$

- **Exponentiation on both sides of Equation (5) yields**

$$e^{\ln\left|v(t)-V_S\right|} = \left|v(t)-V_S\right| = e^K e^{-\frac{t}{RC}} \Rightarrow v(t)-V_S = \pm e^K e^{-\frac{t}{RC}} \quad (6)$$

- **Equation (6) can be rewritten as**

$$v(t) = V_S + Ae^{-\frac{t}{RC}} \quad (7)$$

- **The constant A can be found by applying the initial condition:**

$v(0) = V_0 = V_S + A \Rightarrow A = V_0 - V_S$

Step Response of RC Circuit (Continued)

- **The voltage across the capacitor can be written as**

$$v(t) = V_S + \left(V_0 - V_S\right)e^{-\frac{t}{RC}} \quad (8)$$

- **This solution is valid for $t \geq 0$. At t = 0, the voltage is $v(0) = V_0$, and at $t = \infty$, the voltage is $v(\infty) = V_S$. The voltage across the capacitor changes from the initial value of $v(0) = V_0$ at $t = 0$ to the final value of $v(\infty) = V_S$ at $t = \infty$. The final value of $v(\infty) = V_S$ can be obtained from the circuit shown in Figure 7.15. At $t = \infty$, the capacitor can be treated as an open circuit. The current through the resistor is zero and the voltage drop across the resistor is zero. The voltage across the capacitor is V_S.**
- **Equation (8) can be rewritten as ($\tau = RC$)**

$$v(t) = v(\infty) + \left[v(0) - v(\infty)\right]e^{-\frac{t}{\tau}} = (\text{Final Value}) + \left[(\text{Initial Value}) - (\text{Final Value})\right]e^{-\frac{t}{(\text{Time Constant})}} \quad (9)$$

- **If there is a time delay t_d, replace t by $t - t_d$.**

Step Response of RC Circuit (Continued)

- **Equation (9) is the solution to a differential equation given by the first equation in Equation (3):**

$$-V_S + RC\frac{dv(t)}{dt} + v(t) = 0 \Rightarrow \frac{dv(t)}{dt} + \frac{1}{RC}v(t) = \frac{1}{RC}V_S \Rightarrow \frac{dv(t)}{dt} + \frac{1}{\tau}v(t) = \frac{1}{\tau}V_S \quad (10)$$

- **In the steady state at** $t = \infty$**, since** $dv(t)/dt = 0$**, Equation (10) becomes**

  $$\frac{1}{\tau}v(\infty) = \frac{1}{\tau}V_S$$

  Thus, $v(\infty) = V_S$**.**
- **If there is a time delay** t_d**, replace** t **by** $t - t_d$ **in Equation (9).**

Time Constant

•For RC circuits with one capacitor in the circuit, the time constant is given by $\tau = R_{eq}C$ where R_{eq} is the equivalent resistance seen from the capacitor. The equivalent resistance R_{eq} is the Thévenin equivalent resistance when the rest of the circuit (excluding the capacitor) is converted to the Thévenin equivalent circuit. In general, R_{eq} can be found by deactivating independent sources (short-circuit voltage sources and open-circuit current sources) and finding the equivalent resistance seen from the capacitor. Other methods, such as test voltage and test current, can also be used.

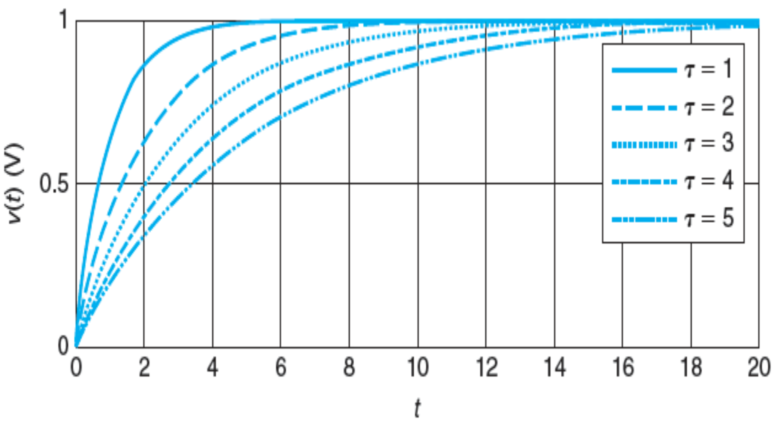
•Figure 7.16 shows *v(t)* given by Equation (9) for $V_S = 1\ V$, $V_0 = 0\ V$, and five different values of τ

•At $t = \tau$, $v(\tau) = 0.63212\ V_S$. At $t = \tau$, the voltage reaches *63.212%* of the final value.

•At $t = 5\tau$, the voltage reaches *99.3262%* of the final value. Refer to Table 7.2 in the text.

FIGURE 7.16

Step Response of RC Circuit for Different Time Constants



EXAMPLE 7.4

- **Let** *V_S = 1 V, R = 4 kΩ, C = 1 μF, v(0) = 0 V* **for the circuit shown in Figure 7.15. Find the voltage** *v(t)* **across the capacitor and the current i(t) through the capacitor for** *t ≥ 0*. **Plot** *v(t)* **and i(t) for** *t ≥ 0*.
- *τ = RC = 4000×1×10^{-6} = 4×10^{-3} = 4 ms, 1/ τ = 250 (1/s), v(∞) = V_S = 1 V*.
- **The voltage across the capacitor is given by**

$$v(t) = V_S + \left(V_0 - V_S\right)e^{-\frac{t}{RC}} = 1 + (0-1)e^{-250t} = \left(1 - e^{-250t}\right)u(t)\ \mathrm{V}$$

- **The current through the capacitor is given by**

$$i(t) = C\frac{dv(t)}{dt} = CV_S\frac{1}{RC}e^{-\frac{t}{RC}} = \frac{V_S}{R}e^{-\frac{t}{RC}} = 250e^{-250t}u(t)\ \mu\mathrm{A}$$

- **Figure 7.19 shows** *v(t)* **and** *i(t).*

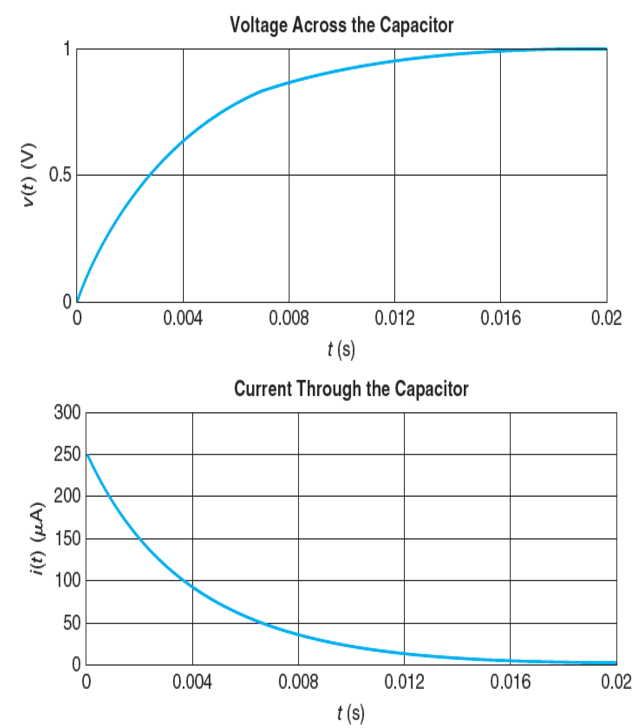


FIGURE 7.19

EXAMPLE 7.5

- **The switch in the circuit shown in Figure 7.20 is closed at *$t = 0$*. At *$t = 0$*, the voltage across the capacitor is *$v(0) = V_0$*. Find the voltage *$v(t)$* across the capacitor and the current *$i(t)$* through the capacitor.**
- **Applying the source transformation to the circuit shown in Figure 7.20, we obtain the circuit shown in Figure 7.21.**
- *Final value = $v(\infty) = I_S R$*, *time constant = $\tau = RC$*
- **The voltage across the capacitor is given by**

$$v(t) = I_S R + \left(V_0 - I_S R\right) e^{-\frac{t}{RC}}$$

- **The current through the capacitor is given by**

$$i(t) = C\frac{dv(t)}{dt} = \frac{-C}{RC}\left(V_0 - I_s R\right) e^{-\frac{t}{RC}} = \left(I_s - \frac{V_0}{R}\right) e^{-\frac{t}{RC}} u(t) \text{ A}$$

FIGURE 7.20

An *RC* circuit.

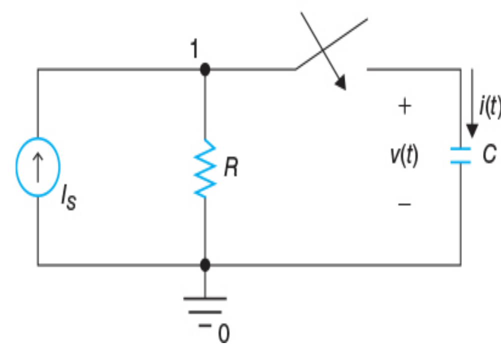
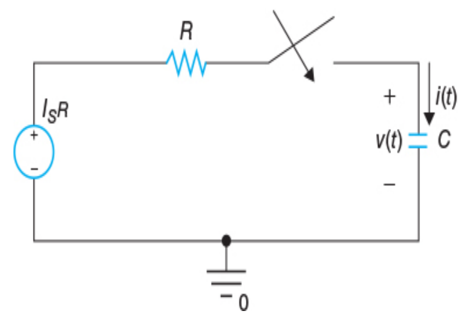


FIGURE 7.21

Thévenin equivalent circuit.



EXAMPLE 7.6

- **Switch 1 is opened at** $t = 0$ **and switch 2 is closed at** $t = 0.5\ ms$ **in the circuit shown in Figure 7.22. Find the voltage** $v(t)$ **across the capacitor for** $t \geq 0$**.**
- **The initial voltage at** $t = 0$ **across the capacitor is** $v(0) = 10\ V \times 4\ k\Omega / 10\ k\Omega = 4\ V$**.**
- **For** $0 \leq t \leq 0.5\ ms$, $\tau = R_2C = 4000 \times 0.25 \times 10^{-6} = 1\ ms$, $1/\tau = 1000$. $v(t) = 4\ \exp(-1000t)\ u(t)\ V$**.**
- **At** $t = 0.5\ ms$, $V_1 = v(0.5\ ms) = 4\ \exp(-0.5) = 2.4261\ V$
- **For** $0.5\ ms \leq t$**, the final value is** $v(\infty) = 20\ V \times 4\ k\Omega / 10\ k\Omega = 8\ V$. $R_{eq} = R_2 \parallel R_3 = 2.4\ k\Omega$
- **For** $0.5\ ms \leq t$, $\tau = R_{eq}C = 2400 \times 0.25 \times 10^{-6} = 0.6\ ms$, $1/\tau = 1666.6667\ (1/s)$.
- **For** $0.5\ ms \leq t$, $v(t) = [8 + (2.4261 - 8)\exp(-1666.6667(t - 0.0005))]$
  $= [8 - 5.5739\exp(-1666.6667(t - 0.0005))]\ u(t)\ V$

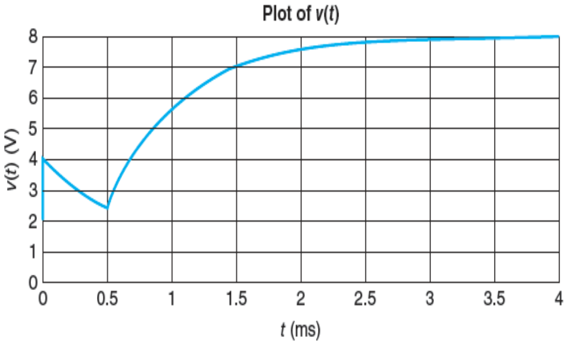
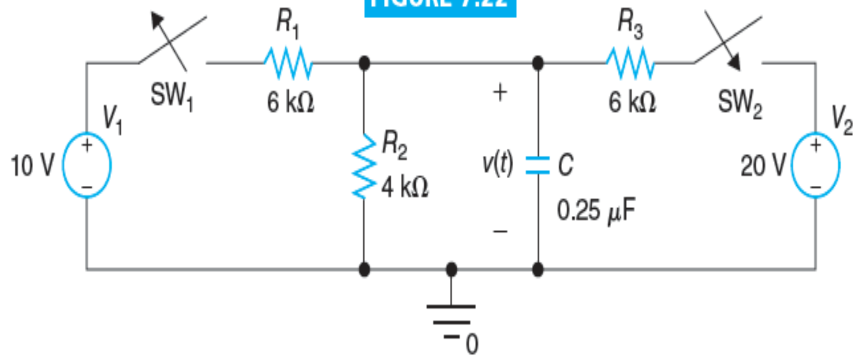


FIGURE 7.22

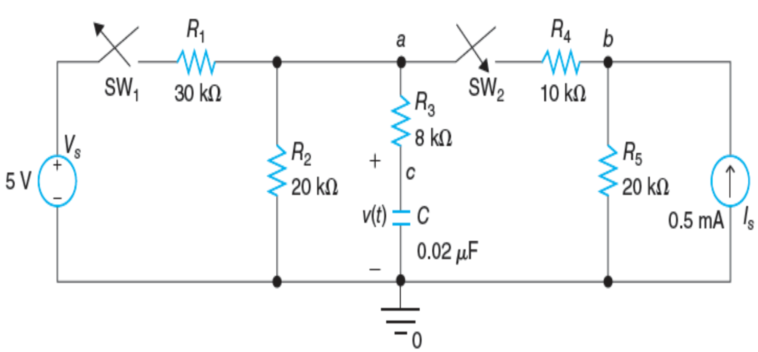


EXAMPLE 7.7

- **Switch 1 is opened at** $t = 0$ **and switch 2 is closed at** $t = 0$ **in the circuit shown in Figure 7.25. Find the voltage** $v(t)$ **across the capacitor and voltage** $v_a(t)$ **at node *a* for** $t \geq 0$**.**
- **For** $t \leq 0$**, the current through** R_3**-C path is zero. Voltage divider rule:** $v(0) = 5\ V \times 20/50 = 2\ V$**.**
- **At** $t = \infty$**, the current through** R_3**-C path is zero. Current through** R_4**-**R_2 **path is** $i_2 = 0.2\ mA$**: Current divider rule:** $0.5\ mA \times 20/(20 + 10 + 20) = 0.2\ mA$, $v(\infty) = R_2 i_2 = 20\ k\Omega \times 0.2\ mA = 4\ V$
- **For** $0 \leq t$, $R_{eq} = R_3 + [R_2 \,||\, (R4 + R5)] = 8\ k\Omega + [20\ k\Omega \,||\, 30\ k\Omega] = 20\ k\Omega$
- **For** $0 \leq t$, $\tau = R_{eq}C = 20{,}000 \times 2 \times 10^{-8} = 0.4\ ms$, $1/\tau = 2500\ (1/s)$
- **For** $0 \leq t$, $v(t) = [4 + (2 - 4)\exp(-2500t)]\ u(t)\ V = [4 - 2\exp(-2500t)]\ u(t)\ V$
- **For** $0 \leq t$, $i(t) = C\ dv(t)/dt = 0.1\exp(-2500t)]\ u(t)\ mA$

  $v_a(t) = R_3 i + v(t) = [4 - 1.2\exp(-2500t)]\ u(t)\ V$

FIGURE 7.25



EXAMPLE 7.8

- **Find the voltage *v(t)* across the capacitor in the circuit shown in Figure 7.27. The initial voltage across the capacitor is *v(0) = 0 V*.**
- **The voltage v_a can be obtained by applying the voltage divider rule:**

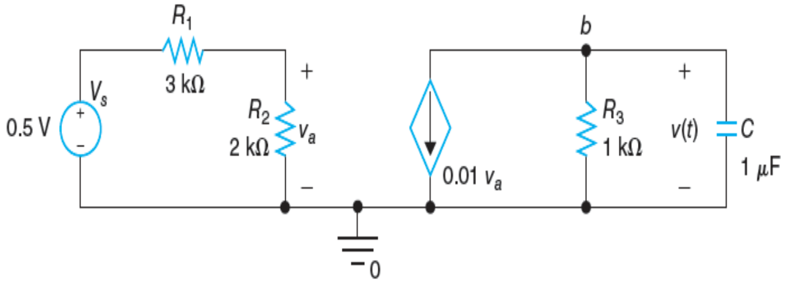
  $v_a = V_s \times R_2/(R_1 + R_2) = 0.5\ V \times 2/5 = 0.2\ V$
- **Summing the currents leaving node *b*, we obtain**

$$0.01\times 0.2+\frac{v(t)}{1000}+1\times 10^{-6}\frac{dv(t)}{dt}=0\Rightarrow\frac{dv(t)}{dt}+1000v(t)=-2000$$

- **At** $t = \infty$, $dv(t)/dt = 0$, $1000v(\infty) = -2000 \Rightarrow v(\infty) = -2\ V$
- $1/\tau = 1000\ (1/s)$, $\tau = 1/1000 = 1\ ms$
- **The voltage across the capacitor is**

$$v(t)=\left[v(\infty)+\left(v(0)-v(\infty)\right)e^{-\frac{t}{\tau}}\right]u(t)=\left[-2+2e^{-1000t}\right]u(t)\ \mathrm{V}$$

FIGURE 7.27



Natural Response of RL Circuit

- The switch in Figure 7.29 has been closed for a long time before it is opened at $t = 0$. At $t = 0$, the current through the inductor is equal to the current from the source I_S; that is, $i(0) = I_0 = I_S$. For $t \geq 0$, the circuit shown in Figure 7.29 becomes the circuit shown in Figure 7.30, with initial current of $i(0) = I_0 = I_S$.
- Summing the currents leaving node 1, we obtain

$$i(t) + \frac{L}{R}\frac{di(t)}{dt} = 0 \Rightarrow \frac{di(t)}{dt} = -\frac{R}{L}i(t) \Rightarrow \frac{\frac{di(t)}{dt}}{i(t)} = -\frac{R}{L} \Rightarrow \frac{d}{dt}\ln\left[i(t)\right] = -\frac{R}{L} \quad (1)$$

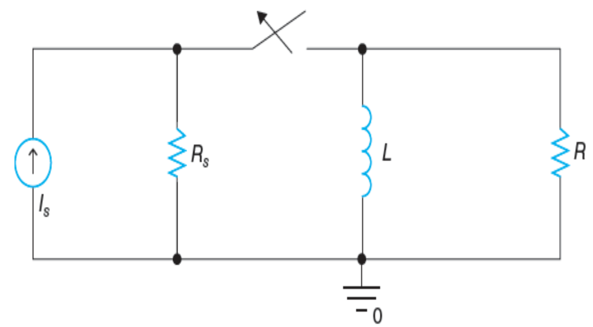


FIGURE 7.29

- Integrating on both sides of the last expression of (1), we get

$$\ln\left[i(t)\right] = -\int_0^t \frac{R}{L}dt + K \quad (2)$$

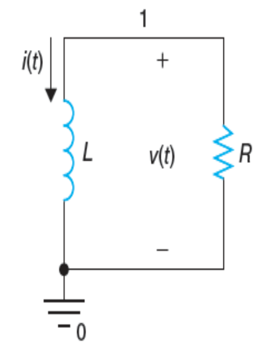


FIGURE 7.30

- Exponentiation on both sides of Equation (2), we obtain

$$e^{\ln\left[i(t)\right]} = i(t) = e^{\left(-\frac{R}{L}t + K\right)} = e^K e^{-\frac{R}{L}t} = Ae^{-\frac{R}{L}t} \quad (3)$$

Natural Response of RL Circuit (Continued)

- **At t = 0, Equation (3) becomes** $i(0) = Ae^{-\frac{R}{L}0} = A$

- **Thus,** $A = i(0) = I_0$**. The current through the inductor (also resistor) is given by**

$$i(t) = i(0)e^{-\frac{t}{L/R}}u(t) = I_0 e^{-\frac{t}{L/R}}u(t) \quad (4)$$

  where $u(t) = 1$ **for** $t \geq 0$ **and** $u(t) = 0$ **for** $t < 0$**.** $u(t)$ **is called unit step function.**

- **The voltage across the inductor is given by**

$$v(t) = L\frac{di(t)}{dt} = L\left(-\frac{R}{L}\right)i(0)e^{-\frac{t}{L/R}} = -Ri(0)e^{-\frac{t}{L/R}} = -RI_0 e^{-\frac{t}{L/R}}u(t)$$

- **The instantaneous power on the inductor is given by**

$$p(t) = v(t)i(t) = -I_0^2 R e^{-\frac{2t}{L/R}}u(t) \quad \text{(power is released)}$$

- **The energy on the resistor is given by** $w(t) = \int_0^t p(\lambda)d\lambda = \frac{1}{2}LI_0^2\left(1 - e^{-\frac{2t}{L/R}}\right)u(t)$
- **At** $t = \infty$, $w(\infty) = 0.5LI_0^2$**.**

Time Constant of an RL Circuit

- **The ratio of *L* over *R*, *L/R*, has a unit of seconds and is called a time constant of the RL circuit. The time constant is denoted by τ. Thus, $\tau = L/R$.**
- **In terms of τ, *i(t), v(t),* and $p_L(t)$ for the circuit shown in Figure 7.30 become, respectively**

$$i(t) = I_0 e^{-\frac{t}{\tau}} u(t), v(t) = -RI_0 e^{-\frac{t}{\tau}} u(t), p_L(t) = -I_0^2 R e^{-\frac{2t}{\tau}} u(t)$$

- **Figure 7.31 shows *i(t)* for τ = *1, 2, 3, 4*, and *5* [(*i(0)* = *1 A*].**
- **As the time constant τ increases, it takes longer time for the current through the inductor to decay.**

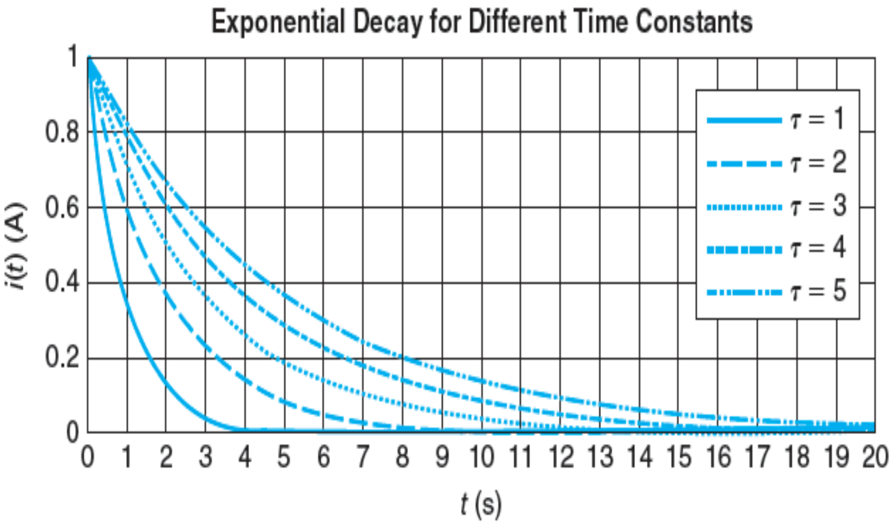


FIGURE 7.31

Time Constant of an RL Circuit (Continued)

•**At** t **=** 0**, the current is at its peak value** $i(0)$**. At time** $t = \tau$**, the current is**

$$i(\tau) = i(0)e^{-\frac{\tau}{\tau}} = i(0)e^{-1} = 0.3678794412\, i(0)$$

•**At** $t = \tau$**, the current through the inductor drops to 36.788% of the initial value at** $t = 0$**. For** $t = 2\tau, 3\tau, 4\tau, 5\tau, \ldots, 10\tau$**, we have the values shown in Table 7.3.**

TABLE 7.3 Current Through the Inductor Normalized to I_0 at $t = n\tau$.

n	$\exp(-n)$
0	1.000000000000000
1.000000000000000	0.367879441171442
2.000000000000000	0.135335283236613
3.000000000000000	0.049787068367864
4.000000000000000	0.018315638888734
5.000000000000000	0.006737946999085
6.000000000000000	0.002478752176666
7.000000000000000	0.000911881965555
8.000000000000000	0.000335462627903
9.000000000000000	0.000123409804087
10.000000000000000	0.000045399929762

Time Constant of an RL Circuit (Continued)

- **At five times the time constant, the current through the inductor due to initial energy on the inductor is less than *1%* of the initial voltage (*0.6738%*). For all practical purposes, the transient response can be ignored after about five times the time constant.**
- **The time derivative of the current through the inductor is given by**

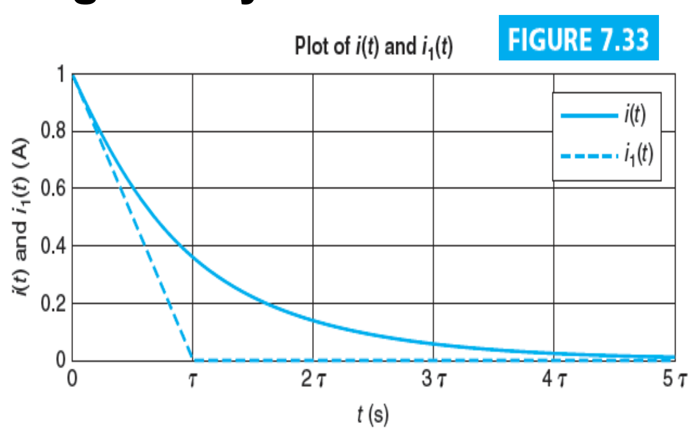
$$\frac{di(t)}{dt} = I_0 \frac{d}{dt} e^{-\frac{t}{\tau}} = -\frac{I_0}{\tau} e^{-\frac{t}{\tau}} = -\frac{RI_0}{L} e^{-\frac{t}{\tau}}$$

- **The rate of decay of the current through the inductor is at its maximum at $t = 0$ and slows down as time progresses. The rate of decay at $t = 0$ is $-I_0/\tau$.**

  If the current decreases at this rate, $i(t)$ is given by

$$i_1(t) - I_0 = -\frac{I_0}{\tau}(t - 0)$$

- **Figure 7.33 shows $i(t)$ and $i_1(t)$.**



Finding the Time Constant

- **If there is one resistor with resistance** R **and one inductor with inductance** L**, as in the circuit shown in Figure 7.30, the time constant is the ratio of** L **over** R**; that is,** $\tau = L/R$**.**
- **If there is one inductor with inductance** L **and more than one resistor in the circuit, find the equivalent resistance** R_{eq} **of all the resistors in the circuit seen from the inductor (Thévenin equivalent resistance). Then, the circuit reduces to one inductor with inductance** L **and one resistor with resistance** R_{eq}**. The time constant is given by** $\tau = L/R_{eq}$**.**
- **The current through the inductor is given by**

$$i(t) = I_0 e^{-\frac{t}{\tau}} u(t)\ \mathrm{A}$$

- **Finding the current** $i(t)$ **through the inductor for the given circuit involves finding the initial current** $i(0)$ **through the inductor, finding the equivalent resistance** R_{eq}**, and finding the time constant** $\tau = L/R_{eq}$**.**

EXAMPLE 7.9

- The switch in the circuit shown in Figure 7.34 has been closed for a long time before it is opened at $t = 0$. Find the current i(t) through the inductor and voltage v(t) across the inductor for $t \geq 0$. Also, plot $i(t)$ and $v(t)$ for $t \geq 0$.
- For $t \leq 0$, the inductor can be treated as a short circuit. $i(0) = I_0 = V_s/R_1 = 12\ V/4\ k\Omega = 3\ mA$
- For $t \geq 0$, the time constant is $\tau = L/R = 0.5/100 = 0.005\ s = 5\ ms$. $1/\tau = 200\ (1/s)$
- For $t \geq 0$, $i(t) = I_0 exp(- t/\tau) = 3\ exp(-200t)\ u(t)\ mA$.
- For $t \geq 0$, $v(t) = L\ di(t)/dt = 0.5 \times 3 \times 10^{-3} \times (-200)\ exp(-200t)\ u(t) = -0.3\ exp(-200t)\ u(t) V$

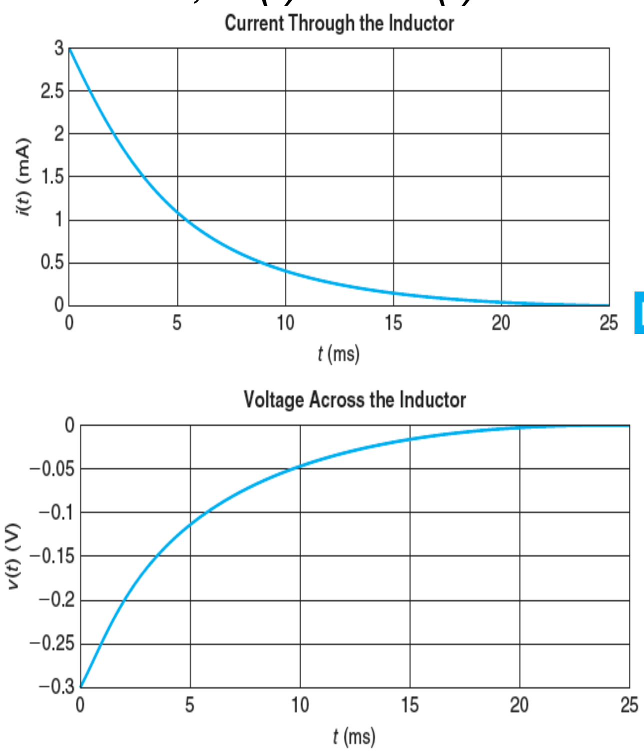
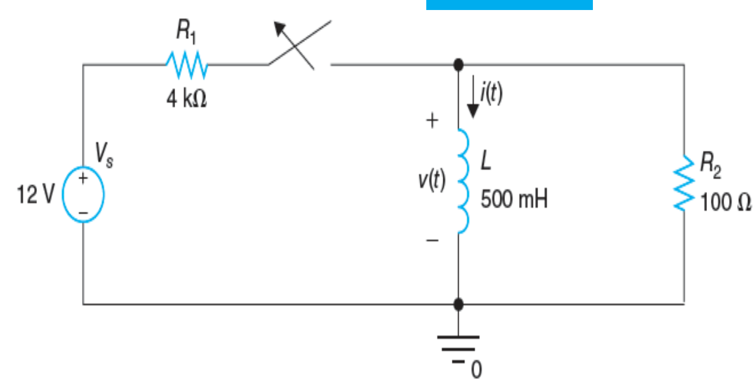


FIGURE 7.35

FIGURE 7.34

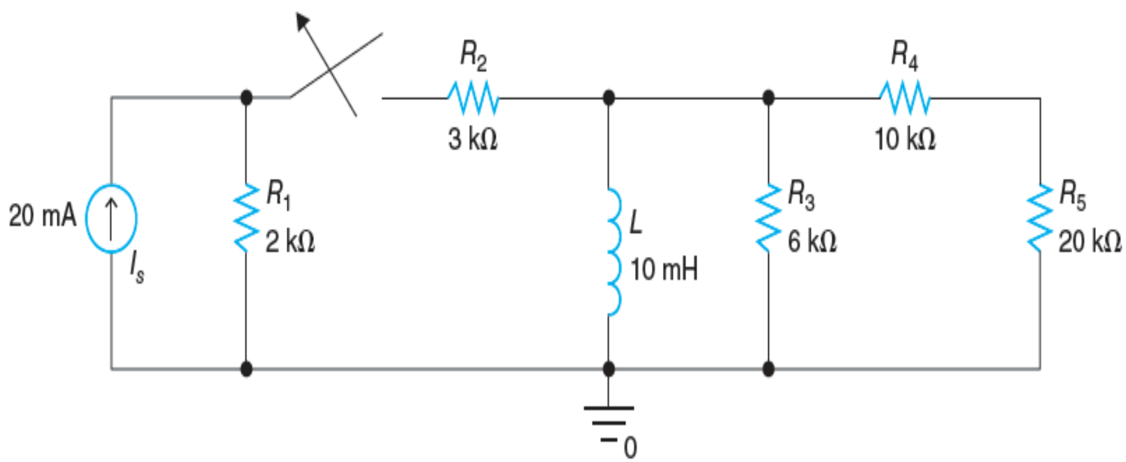


EXAMPLE 7.10

- **The switch in the circuit shown in Figure 7.36 has been closed for a long time before it is opened at** $t = 0$**. Find the current** $i(t)$ **through the inductor for** $t \geq 0$**.**
- **For** $t < 0$**, the inductor can be treated as a short circuit. From the current divider rule, we get**

  $i(0) = I_0 = I_s \times R_1/(R_1 + R_2) = 20\ mA \times 2\ k\Omega/(2\ k\Omega + 3\ k\Omega) = 8\ mA$
- **For** $t \geq 0$**,** $R_{eq} = R_3\ ||\ (R_4 + R_5) = 6 \times 30/(6 + 30)\ k\Omega = 5\ k\Omega$
- **For** $t \geq 0$**, the time constant is** $\tau = L/R_{eq} = 0.01/5000 = 2\ \mu s.\ 1/\tau = 500{,}000\ (1/s)$
- **For** $t \geq 0$**,** $i(t) = I_0 exp(-t/\tau) = 8\ exp(-500{,}000t)\ u(t)\ mA$**.**

FIGURE 7.36



EXAMPLE 7.11

- **The switch in the circuit shown in Figure 7.39 has been closed for a long time before it is opened at $t = 0$. Find the current $i(t)$ through the inductor for $t \geq 0$.**
- **For $t < 0$, the inductor can be treated as a short circuit.**

Let $R_a = R_3 \,||\, R_4$**. Then,** $R_a = 12 \times 4/(12 + 4)\ k\Omega = 48/16\ k\Omega = 3\ k\Omega$**.**

Let $R_b = R_a \,||\, R_5$**. Then,** $R_b = 3 \times 3/(3 + 3)\ k\Omega = 9/6\ k\Omega = 1.5\ k\Omega$**.**

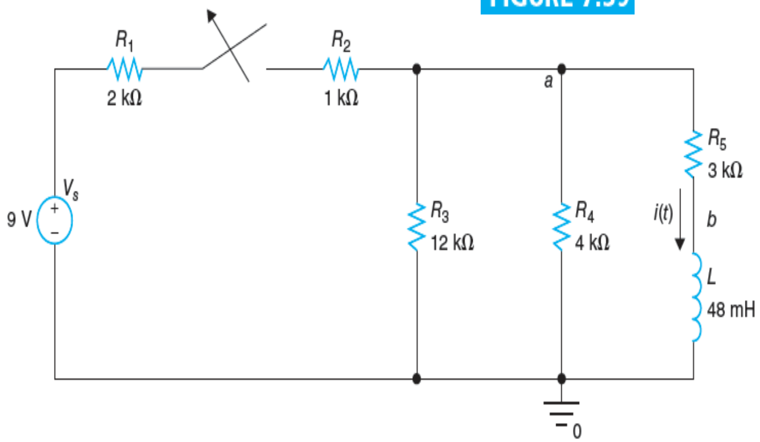
Let $R_c = R_1 + R_2 + R_b$**. Then,** $R_c = 4.5\ k\Omega$. $I_{R1} = V_s/R_c = 2\ mA$, $V_a = V_s - 3\ k\Omega \times 2\ mA = 3\ V$

$i(0) = I_0 = V_a/R_5 = 3\ V/3\ k\Omega = 1\ mA$

- **For** $t \geq 0$, $R_{eq} = R_5 + (R_3 \,||\, R_4) = 3\ k\Omega + 3\ k\Omega = 6\ k\Omega$
- $\tau = L/R_{eq} = 0.048/6000 = 8\ \mu s$. $1/\tau = 125{,}000\ (1/s)$
- **For** $t \geq 0$**,**

$i(t) = I_0 \exp(-t/\tau) = \exp(-125{,}000t)\, u(t)\ mA$**.**

FIGURE 7.39



EXAMPLE 7.12

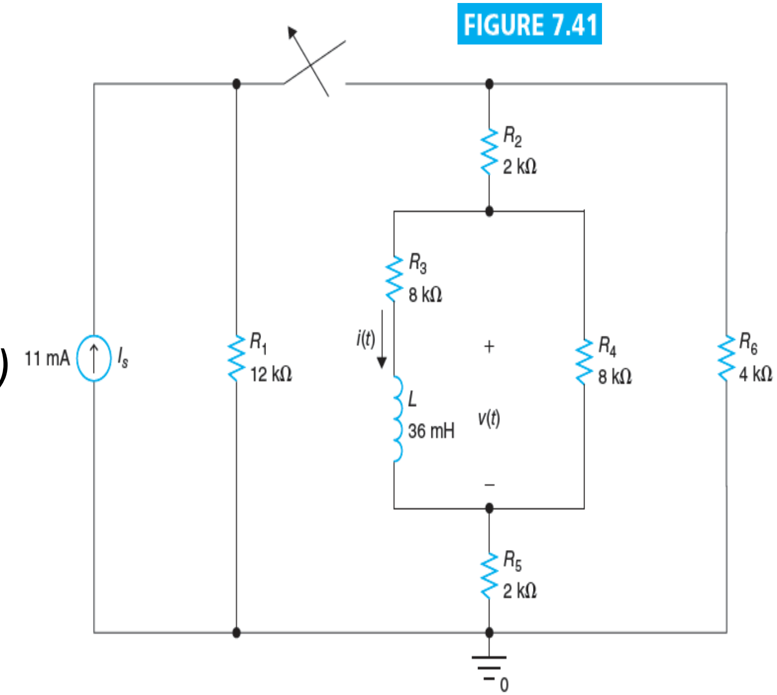
- The switch in the circuit shown in Figure 7.41 has been closed for a long time before it is opened at $t = 0$. Find the current $i(t)$ through the inductor and the voltage $v(t)$ across the inductor for t ≥ 0. $R_a = R_2 + (R_3 \| R_4) + R_5 = 8\ k\Omega$
- From the current divider rule, the current through R_a is

$$I_{R_a} = I_s \times \frac{\frac{1}{R_a}}{\frac{1}{R_1} + \frac{1}{R_a} + \frac{1}{R_6}} = 11\,\text{mA} \times \frac{\frac{1}{8}}{\frac{1}{12} + \frac{1}{8} + \frac{1}{4}} = 11\,\text{mA} \times \frac{3}{11} = 3\,\text{mA}$$

$i(0) = I_0 = I_{Ra}/2 = 1.5\ mA$

- $R_{eq} = R_3 + [R_4 \| (R_2 + R_6 + R_5)] = 12\ k\Omega$
- $\tau = L/R_{eq} = 0.036/12000 = 3\ \mu s.\ 1/\tau = 333{,}333\ (1/s)$
- $i(t) = I_0 \exp(-t/\tau) = 1.5 \exp(-333{,}333t)\ u(t)\ mA.$
- $v(t) = L\ di(t)/dt = -18 \exp(-333{,}333t)\ u(t)\ V.$

FIGURE 7.41



Step Response of RL Circuit

- **The switch in the circuit shown in Figure 7.43 is closed at *t = 0*. At *t = 0*, the current through the inductor is *i(0) = I_0*. For *t ≥ 0*, summing the currents leaving node 1, we obtain**

$$-I_S + \frac{L}{R}\frac{di(t)}{dt} + i(t) = 0 \Rightarrow \frac{di(t)}{dt} = -\frac{R}{L}\left[i(t) - I_S\right] \Rightarrow \frac{\frac{di(t)}{dt}}{i(t) - I_S} = -\frac{R}{L} \Rightarrow \frac{d}{dt}\ln\left[\left|i(t) - I_S\right|\right] = -\frac{R}{L} \quad (1)$$

- **Integrating on both sides of Equation (1), we obtain**

$$\ln\left|i(t) - I_S\right| = \int_0^t \frac{-R}{L}dt = \frac{-Rt}{L} + K \quad (2)$$

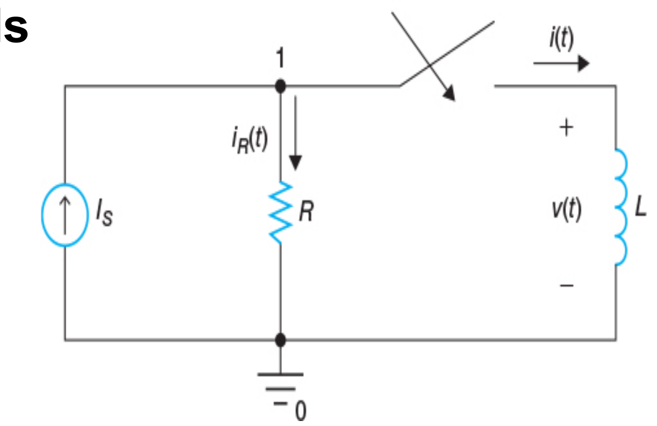
- **Exponentiation on both sides of Equation (2) yields**

$$e^{\ln\left|i(t)-I_S\right|} = \left|i(t) - I_S\right| = e^K e^{-\frac{Rt}{L}} \Rightarrow i(t) - I_S = \pm e^K e^{-\frac{Rt}{L}} \quad (3)$$

- **Let** $A = \pm e^K$

FIGURE 7.43

RL circuit with dc input.



Step Response of RL Circuit (Continued)

- **Then, Equation (3) can be rewritten as**

$$i(t) = I_S + Ae^{-\frac{t}{L/R}} \quad (4)$$

- **The constant A can be found by applying the initial condition:**

  $i(0) = I_0 = I_S + A \Rightarrow A = I_0 - I_S$

- **The current through the inductor can be written as (**$\tau = L/R$**)**

$$i(t) = I_S + \left(I_0 - I_S\right)e^{-\frac{t}{L/R}} = I_S + \left(I_0 - I_S\right)e^{-\frac{t}{\tau}} \quad (5)$$

- **This solution is valid for** $t \geq 0$**. At** $t = 0$**, the current is** $i(0) = I_0$**, and at** $t = \infty$**, the current is** $i(\infty) = I_S$**. The current through the inductor changes from the initial value of** $i(0) = I_0$ **at** $t = 0$ **to the final value of** $i(\infty) = I_S$ **at** $t = \infty$**.**

Step Response of RL Circuit (Continued)

- **The final value of $i(\infty) = I_S$ can be obtained from the circuit shown in Figure 7.43. At $t = \infty$, the inductor can be treated as a short circuit. The current through the inductor is I_S.**
- **Equation (5) can be rewritten as ($\tau = L/R$)**

$$i(t) = i(\infty) + \left[i(0) - i(\infty)\right]e^{-\frac{t}{\tau}} = (\text{Final Value}) + \left[(\text{Initial Value}) - (\text{Final Value})\right]e^{-\frac{t}{(\text{Time Constant})}} \quad (6)$$

- **Equation (6) is the solution to a differential equation given by the first equation in Equation (1):**

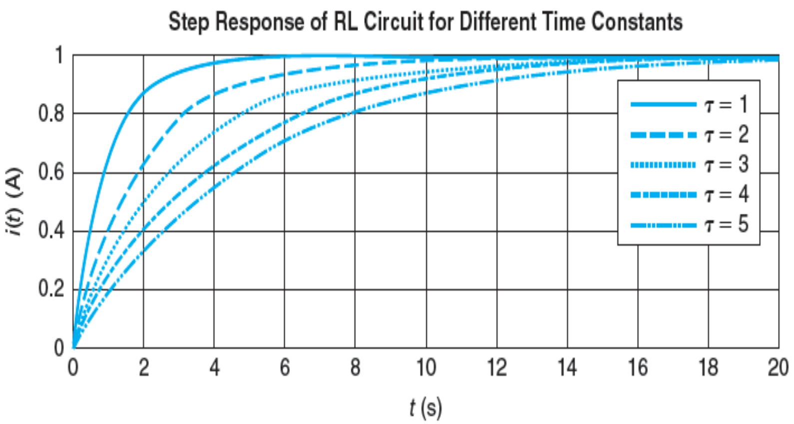
$$\frac{di(t)}{dt} + \frac{1}{L/R}i(t) = \frac{1}{L/R}I_S \Rightarrow \frac{di(t)}{dt} + \frac{1}{\tau}i(t) = \frac{1}{\tau}I_S \quad (7)$$

- **In the steady state at $t = \infty$, since $di(t)/dt = 0$, Equation (7) becomes** $\frac{1}{\tau}i(\infty) = \frac{1}{\tau}I_S$ **Thus, $i(\infty) = I_S$.**
- **If there is a time delay t_d, replace t by $t - t_d$ in Equation (6).**

Time Constant

- **For RL circuits with one inductor in the circuit, the time constant is given** *by* $\tau = L/R_{eq}$ **where** R_{eq} **is the equivalent resistance seen from the inductor. The equivalent resistance** R_{eq} **is the Thévenin equivalent resistance when the rest of the circuit (excluding the inductor) is converted to the Thévenin equivalent circuit. In general,** R_{eq} **can be found by deactivating independent sources (short-circuit current sources and open-circuit voltage sources) and finding the equivalent resistance seen from the inductor. Other methods, such as test voltage and test current, can also be used.**
- **Figure 7.44 shows** *i(t)* **given by Equation (5) for** $I_S = 1\ A$, $I_0 = 0\ A$, **and five different values of** τ.
- **At** $t = \tau$, $i(\tau) = 0.63212\ I_S$. **At** $t = \tau$, **the current reaches** *63.212%* **of the final value.**
- **At** $t = 5\tau$, **the current reaches** *99.3262%* **of the final value.**

FIGURE 7.44



EXAMPLE 7.13

- **Let** *L = 700 mH, R = 200 Ω, I_S = 5 mA*, **and** *I_0 = 0 A* **in the circuit shown in Figure 7.43. Find the current** *$i(t)$* **through the inductor and voltage** *$v(t)$* **across the inductor for** *$t \geq 0$*, **and plot** *$i(t)$* **and** *$v(t)$*.
- **Final value:** $i(\infty) = I_S = 5\ mA$
- **Time constant:** $\tau = L/R = 0.7/200 = 3.5\ ms,\ 1/\tau = 285.7143\ (1/s)$
- $i(t) = [I_S + (I_0 - I_S)exp(-285.7143t)]\ u(t) = 5[1 - exp(-285.7143t)]\ u(t)\ mA$
- $v(t) = L\ di(t)/dt = 0.7\times(-0.005)\times(-285.7143)\ exp(-285.7143t)\ u(t) =$

$exp(-285.7143t)]\ u(t)\ \ V$

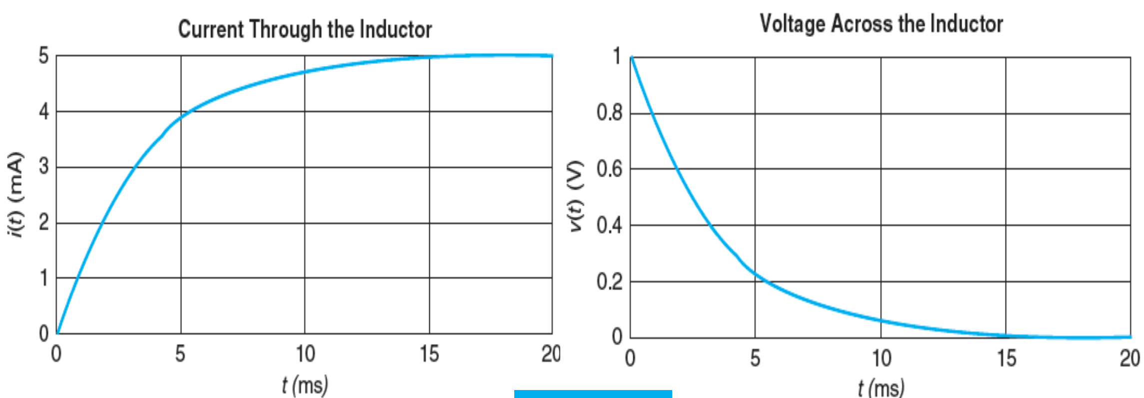
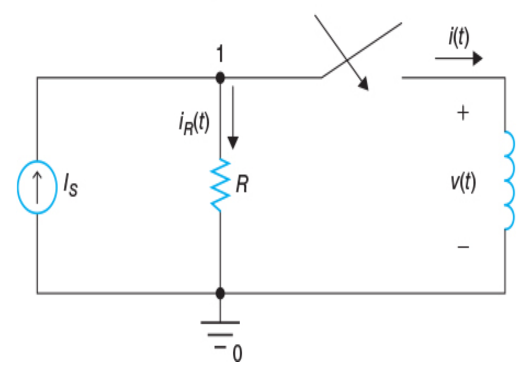


FIGURE 7.45

FIGURE 7.43

RL circuit with dc input.



EXAMPLE 7.14

- The switch in the circuit shown in Figure 7.46 is closed at $t = 0$. The initial current through the inductor is $i(0) = I_0$. Find the current $i(t)$ through the inductor and the voltage $v(t)$ across the inductor.

- Sum the voltage drops around the mesh: $-V_s + Ri(t) + L\frac{di(t)}{dt} = 0 \Rightarrow \frac{di(t)}{dt} + \frac{R}{L}i(t) = \frac{1}{L}V_s$

- Final value: $\frac{R}{L}i(t) = \frac{1}{L}V_s \Rightarrow i(\infty) = \frac{V_s}{R}$

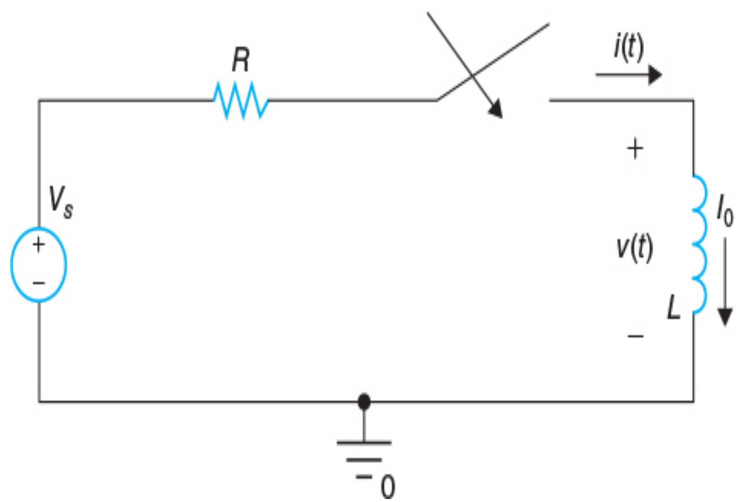
- Time constant: $\tau = L/R$

- Current: $i(t) = \left[\frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-\frac{t}{L/R}}\right]u(t)\ \text{A}$

- Voltage: $v(t) = \left(V_s - RI_0\right)e^{-\frac{t}{L/R}}u(t)\ \text{V}$

FIGURE 7.46

An *RL* circuit for EXAMPLE 7.14.



EXAMPLE 7.15

- **In the circuit shown in Figure 7.48, switch 1 has been closed for a long time before it is opened at t = 0. Switch 2 is closed at** $t = 12\ \mu s$. **Find the current** $i(t)$ **through the inductor for** $t \geq 0$. **Initial value: from the current divider rule,** $i(0) = 9\ mA \times 2/(2 + 1) = 6\ mA$
- **For** $0 \leq t \leq 12\ \mu s$, $R_{eq} = R_3 + R_4 = 4\ k\Omega$, $\tau = L/R = 6\ \mu s$, $1/\tau = 166{,}667\ (1/s)$
- **For** $0 \leq t \leq 12\ \mu s$, $i(t) = 6\ \exp(-166{,}667t)[u(t) - u(t - 12\times10^{-6})]\ mA$, $i(12\times10^{-6}) = 0.8120\ mA$
- **At** $t = \infty$, $V_{R4} = 18\ V \times 0.75/(6 + 0.75) = 2\ V$, $i(\infty) = V_{R4}/R_3 = 2\ mA$
- **For** $12\ \mu s \leq t$, $R_{eq} = R_3 + (R_4||R_5) = 3\ k\Omega$, $\tau = L/R_{eq} = 8\ \mu s$, $1/\tau = 125{,}000\ (1/s)$
- $i(t) = [2 + (0.8120 - 2)\exp(-125{,}000(t - 12\times10^{-6})]\ u(t - 12\times10^{-6})\ mA$
- $= [2 - 1.188\ \exp(-125{,}000(t - 12\times10^{-6}))]\ u(t - 12\times10^{-6})\ mA$

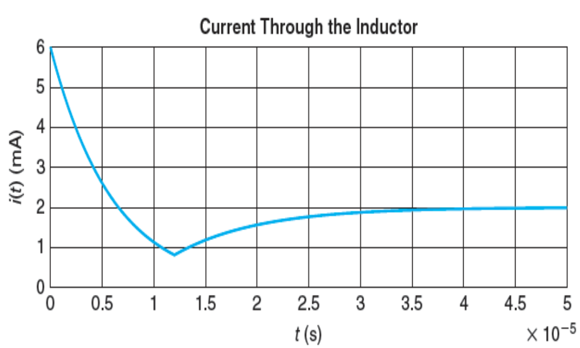
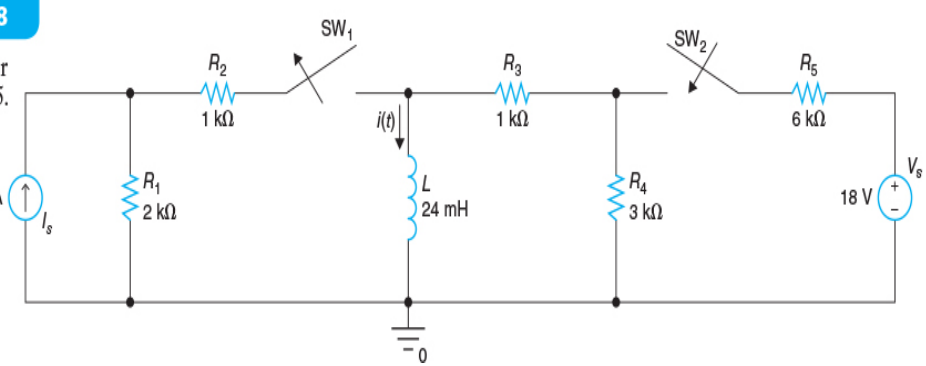


FIGURE 7.50

FIGURE 7.48

Circuit for EXAMPLE 7.15.



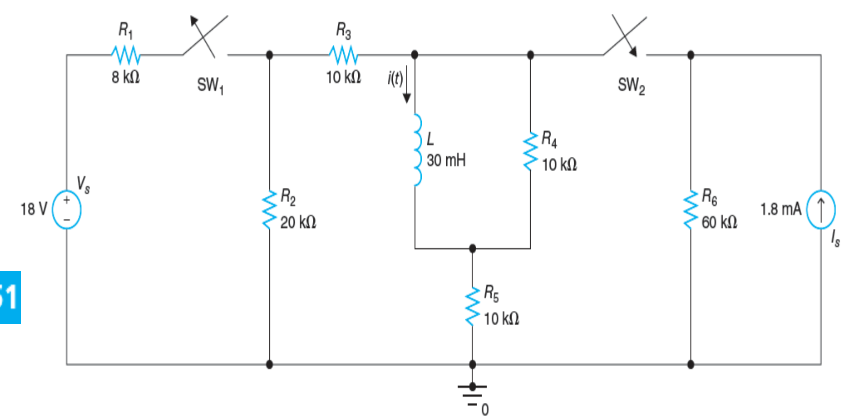
EXAMPLE 7.16

- **In the circuit shown in Figure 7.51, switch 1 has been closed for a long time before it is opened at t = 0. Switch 2 is closed at** $t = 2\ \mu s$**. Find the current** $i(t)$ **through the inductor for** $t \geq 0$**. Initial value:** $R_2 \parallel (R_3 + R_5) = 10\ k\Omega$, $I_{R1} = 18\ V/18\ k\Omega = 1\ mA$, $i(0) = 1\ mA/2 = 0.5\ mA$
- **For** $0 \leq t \leq 2\ \mu s$, $R_{eq1} = R_4 \parallel (R_3 + R_2 + R_5) = 8\ k\Omega$, $\tau = L/R_{eq1} = 3.75\ \mu s$, $1/\tau = 266{,}667\ (1/s)$
- **For** $0 \leq t \leq 2\ \mu s$, $i(t) = 0.5\exp(-266{,}667t)[u(t) - u(t - 2\times10^{-6})]\ mA$, $i(2\times10^{-6}) = 0.2933\ mA$
- **At** $t = \infty$, $R_a = R_6 \parallel (R_3 + R_2) = 20\ k\Omega$, $i(\infty) = I_s \times R_a/(R_5 + R_a) = 1.2\ mA$
- **For** $2\ \mu s \leq t$, $R_{eq2} = R_4 \parallel (R_a + R_5) = 7.5\ k\Omega$, $\tau = L/R_{eq2} = 4\ \mu s$, $1/\tau = 250{,}000\ (1/s)$
- $i(t) = [1.2 + (0.2933 - 1.2)\exp(-250{,}000(t - 2\times10^{-6})]\ u(t - 2\times10^{-6})\ mA$

$= [1.2 - 0.9067\exp(-250{,}000(t - 2\times10^{-6}))]$

$\times\ u(t - 2\times10^{-6})\ mA$

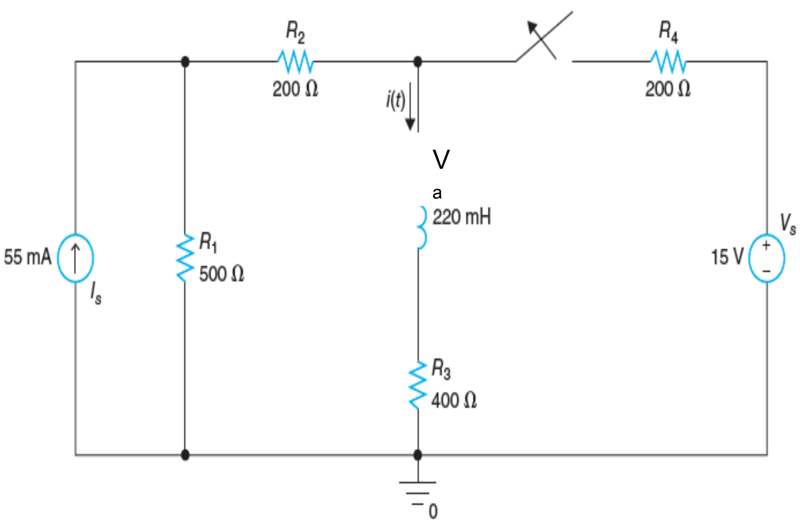
FIGURE 7.51



EXAMPLE 7.17

- **In the circuit shown in Figure 7.55, the switch has been closed for a long time before it is opened at** $t = 0$. **Find the current i(t) through the inductor for** $t \geq 0$.
- $i(0)$ **from** V_s: $R_a = (R_1 + R_2)||R_3 = 254.5455\ \Omega$, $V_{a1} = V_s \times R_a/(R_a + R_4) = 8.4$ V, $i_1(0) = V_{a1}/R_3 = 0.021$ A
- $i(0)$ **from** I_s: $R_b = R_3||R_4 = 133.3333\ \Omega$, $R_c = R_1||(R_2 + R_b) = 200\ \Omega$, $V_{R1} = R_c \times I_s = 11$ V
- $V_{a2} = V_{R1} \times R_b/(R_b + R_2) = 4.4$ V, $i_2(0) = V_{a2}/R_3 = 0.011$ A
- $i(0) = i_1(0) + i_2(0) = 32$ mA
- $i(\infty) = I_s \times R_1/(R_1 + R_2 + R_3) = 25$ mA
- $R_{eq} = R_2 + R_1 + R_3 = 1100\ \Omega$
- $\tau = L/R_{eq} = 2 \times 10^{-4}$ s $= 0.2$ ms, $1/\tau = 5000$ (1/s)
- $i(t) = [25 + (32 - 25)\exp(-5000t)]\, u(t)$ mA
- $i(t) = [25 + 7\exp(-5000t)]\, u(t)$ mA

FIGURE 7.55

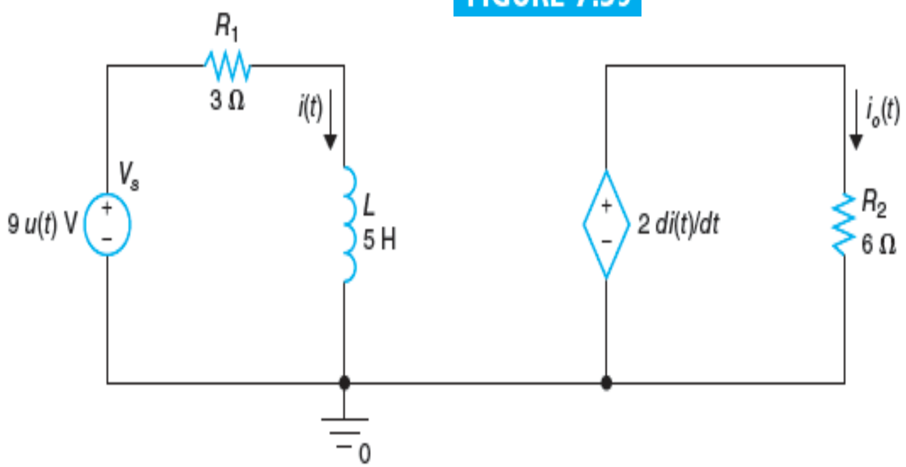


EXAMPLE 7.18

- **The initial current through the inductor is *$i(0) = 1$ A* in the circuit shown in Figure 7.59. Find the current *$i_o(t)$* through *R_2* for *$t \geq 0$*.**
- **Sum the voltage drops around the mesh in the left side:**

$$-9 + 3i(t) + 5\frac{di(t)}{dt} = 0 \Rightarrow \frac{di(t)}{dt} + 0.6i(t) = 1.8$$

FIGURE 7.59



- $0.6\ i(\infty) = 1.8 \Rightarrow i(\infty) = 3\ A$
- $1/\tau = 0.6 \Rightarrow \tau = 1/0.6 = 5/3 = 1.6667$
- $i(t) = [3 + (1 - 3)\exp(-0.6t)]\ u(t)\ A$
- $i(t) = [3 - 2\exp(-0.6t)]\ u(t)\ A$
- $di(t)/dt = 1.2\ \exp(-0.6t)\ u(t)\ A$
- $I_o(t) = (2/6)\times di(t)/dt = 0.4\ \exp(-0.6t)\ u(t)$ A

Summary

- **Figure 7.2 shows a circuit with one capacitor with capacitance** *C* **and one resistor with resistance** *R* **connected in parallel. Let the initial voltage across the capacitor at** $t = 0$ **be** $v(0)$. **Then, the voltage** $v(t)$ **across the capacitor for** $t \geq 0$ **is given by**

$$v(t) = v(0)e^{-\frac{t}{\tau}}u(t) \text{ V}$$

where the time constant τ **is given by** $\tau = RC$**.**

- **Figure 7.30 shows a circuit with one inductor with inductance** *L* **and one resistor with resistance** *R* **connected in parallel. Let the initial current through the inductor at** $t = 0$ **be** $i(0)$. **Then, the current** $i(t)$ **through the inductor for** $t \geq 0$ **is given by**

$$i(t) = i(0)e^{-\frac{t}{\tau}}u(t) \text{ A}$$

where the time constant τ **is given by** $\tau = L/R$**.**

Summary (Continued)

- **Figure 7.15 shows a circuit with a voltage source with voltage V_S, a resistor with resistance R, and a capacitor with capacitance C connected in series. Let the initial voltage across the capacitor at t = 0 be $v(0) = V_0$. Then, the voltage $v(t)$ across the capacitor for t ≥ 0 is given by**

$$v(t) = V_S + \left(V_0 - V_S\right)e^{-\frac{t}{\tau}}$$

where the time constant τ is given by $\tau = RC$.

- **Figure 7.43 shows a circuit with a current source with current I_s, a resistor with resistance R, and an inductor with inductance L connected in parallel. Let the initial current through the inductor at $t = 0$ be $i(0) = I_0$. Then, the current $i(t)$ through the inductor for $t \geq 0$ is given by**

$$i(t) = I_S + \left(I_0 - I_S\right)e^{-\frac{t}{\tau}}$$

where the time constant τ is given by $\tau = L/R$.