IMAGE TRANSFORMS – KARHUNEN-LOÈVE TRANSFORM

- Diagonalizing the image covariance

$$\begin{bmatrix} 0.35 & 0 & 0 & 0 \\ 0 & 0.37 & 0 & 0 \\ 0 & 0 & 0.22 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}$$

$$X \xrightarrow{P} Y$$

- Linear, orthogonal

IMAGE TRANSFORMS – KARHUNEN-LOÈVE TRANSFORM

- Diagonalizing the image covariance

$$\begin{bmatrix} 0.35 & 0 & 0 & 0 \\ 0 & 0.37 & 0 & 0 \\ 0 & 0 & 0.22 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}$$

$$X \xrightarrow{P} Y$$

- Linear, orthogonal

$$Y = PX \qquad Y = \begin{bmatrix} P_1 \\ \dots \\ P_i \\ \dots \end{bmatrix} X \qquad y_i = \langle P_i, X \rangle$$

IMAGE TRANSFORMS – KARHUNEN-LOÈVE TRANSFORM

- Linear, orthogonal
- The covariance in transform domain

$$C_Y = \frac{1}{n-1} YY^T \qquad Y = PX \qquad C_Y = \frac{1}{n-1} YY^T$$

$$= \frac{1}{n-1}(PX)(PX)^T \qquad P = E^T \qquad = \frac{1}{n-1}(E^T X)(E^T X)^T$$

$$= \frac{1}{n-1} PXX^T P^T \qquad A = XX^T \qquad = \frac{1}{n-1} E^T XX^T E$$

$$= \frac{1}{n-1} PAP^T \qquad A = EDE^T \quad E^{-1} = E^T \qquad = \frac{1}{n-1} E^T (EDE^T) E$$

D:Diagonal Matrix

E:Orthogonal Matrix

$$= \frac{1}{n-1} D$$

IMAGE TRANSFORMS – KARHUNEN-LOÈVE TRANSFORM

- Karhunen-Loève Transform

$$\boldsymbol{C}_x = \frac{1}{L}\sum_{i=1}^{L}(\boldsymbol{X}_i - \boldsymbol{m}_x)(\boldsymbol{X}_i - \boldsymbol{m}_x)^{\mathrm{T}} = \frac{1}{L}\left[\sum_{i=1}^{L}\boldsymbol{X}_i \boldsymbol{X}_i^{\mathrm{T}}\right] - \boldsymbol{m}_x \boldsymbol{m}_x^{\mathrm{T}}$$

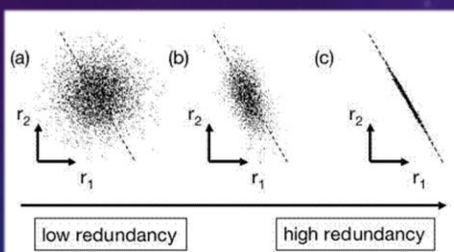
$$C_x \to \{\lambda_i, \vec{u}_i\} \qquad P = [\vec{u}_1, \ldots, \vec{u}_i, \ldots]^T$$

$$Y = P(X - m_x)$$

$$X = P^T Y + m_x$$

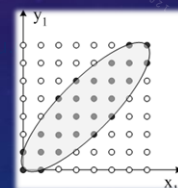
PRINCIPAL COMPONENT ANALYSIS

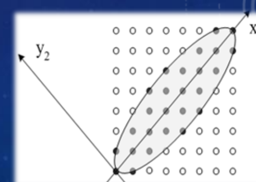
- Arrange the eigenvalues in descending order，the KL approximation is the one that minimizes the total mean square error(KL transform optimally compacts the energy)



$$X = P^T Y + m_x$$

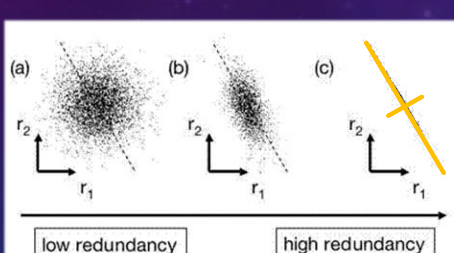
$$\hat{X} = P_K^T Y_K + m_x$$





PRINCIPAL COMPONENT ANALYSIS

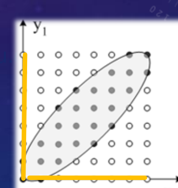
- Arrange the eigenvalues in descending order，the KL approximation is the one that minimizes the total mean square error(KL transform optimally compacts the energy)

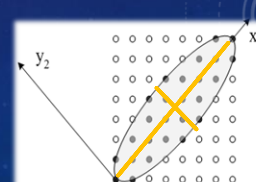


$$X = P^T Y + m_x$$

$$\hat{X} = P_K^T Y_K + m_x$$

$$\varepsilon = \sum_{i=K+1}^{N \times N} \lambda_i$$



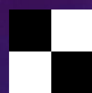


$$X_i = [\, f_i(0,0),\, f_i(0,1), \cdots, f_i(0, N-1),\, f_i(1,0),\, f_i(r, N-1), \cdots, f_i(N-1, N-1)]^{\mathrm{T}}$$

IMAGE TRANSFORMS – KLT

$$C_x = \frac{1}{4}\begin{bmatrix} \frac{1}{4} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & 1 & 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 & 1 & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

- Question: For 2 images of size 2×2, calculate its KLT.

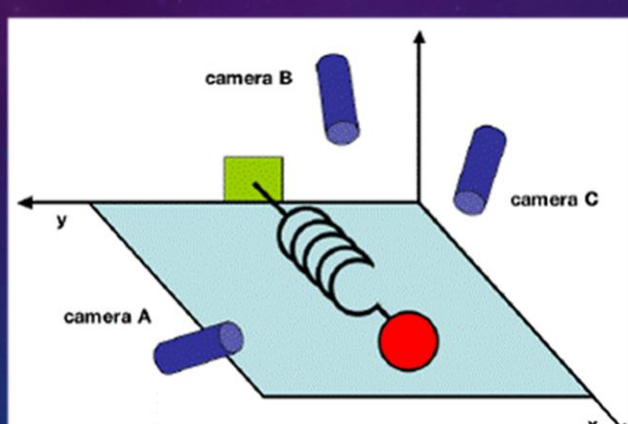
$$f_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0.5 \end{bmatrix}$$  $$f_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}$$ 

$$X_1 = [0,1,1,0.5]^{\mathrm{T}} \qquad X_2 = [0.5,0,0,0]^{\mathrm{T}}$$

$$m_x = E\{X\} = \frac{1}{L}\sum_{i=1}^{L} X_i = [0.25, 0.5, 0.5, 0.25]^T \quad C_x = \frac{1}{L}\sum_{i=1}^{L}(X_i - m_x)(X_i - m_x)^{\mathrm{T}} = \frac{1}{L}\left[\sum_{i=1}^{L} X_i X_i^{\mathrm{T}}\right] - m_x m_x^{\mathrm{T}}$$

$$C_x = \frac{1}{2}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0.5 \end{bmatrix}[0 \quad 1 \quad 1 \quad 0.5] + \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix}[0.5 \quad 0 \quad 0 \quad 0]\right\} - \begin{bmatrix} 0.25 \\ 0.5 \\ 0.5 \\ 0.25 \end{bmatrix}[0.25 \quad 0.5 \quad 0.5 \quad 0.25]$$

PRINCIPAL COMPONENT ANALYSIS



PRINCIPAL COMPONENT ANALYSIS

- PCA Face

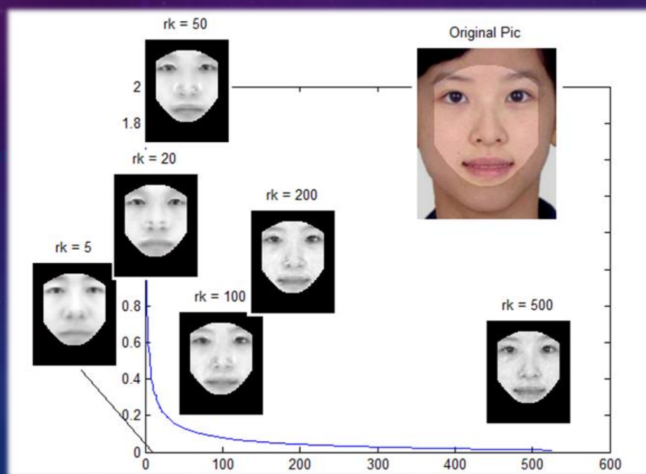
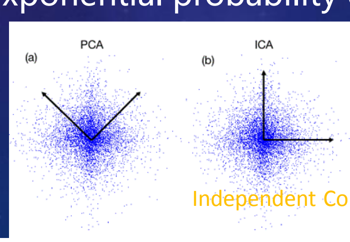


IMAGE TRANSFORMS – KARHUNEN-LOÈVE TRANSFORM

- The importance of the Karhunen–Loève theorem is that it yields the best basis in the sense that it minimizes the total mean squared error
- Pros： Completely decorrelates the original signal，data driven
- Cons：High computational cost, not suitable for data with a non-Gaussian/non-exponential probability distribution



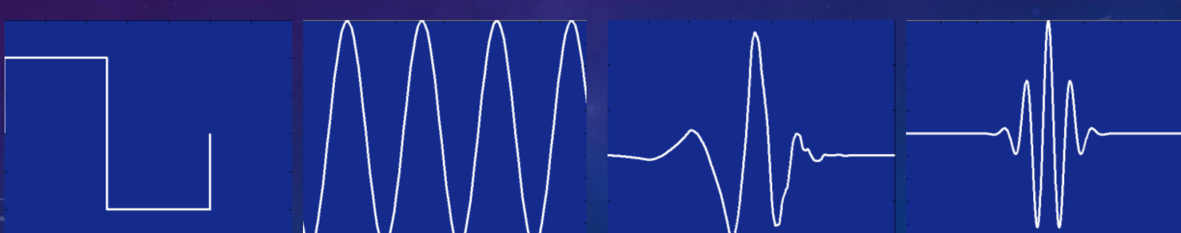
Independent Component Analysis

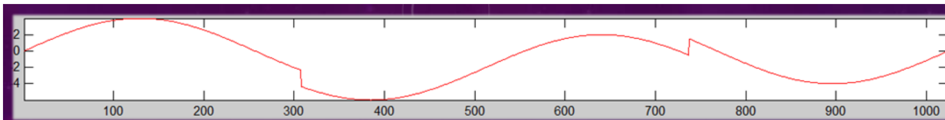
IMAGE TRANSFORMS – DWT

- Wavelets and wavelet transforms are relatively new imaging tools that are being rapidly applied to a wide variety of image processing problems. It is now often replacing the conventional Fourier transform
- Wavelet transforms are broadly divided into three classes: continuous, discrete and multiresolution-based

IMAGE TRANSFORMS – DWT

- Wavelets are small waves of varying frequencies and limited duration.
- Q: Which of the following waves is not a wavelet function?





compressed sensing

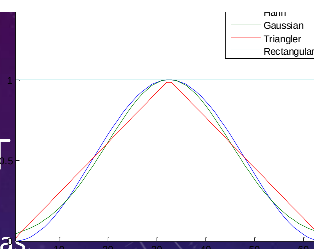
Short-Time Fourier Transform

Gibbs phenomenon

IMAGE TRANSFORMS – DWT

- Wavelets and wavelet transforms are relatively new imaging tools that are being rapidly applied to a wide variety of image processing problems. It is now often replacing the conventional Fourier transform
- Wavelet transforms are broadly divided into three classes: continuous, discrete and multiresolution-based
- Both Fourier and wavelet transforms are frequency-localized, but wavelets have an additional time-localization property.

IMAGE TRANSFORMS – DWT STFT



- The Short-time Fourier transform of a signal $f(x)$ is defined as:

$$\mathrm{WFT}_f(b,\omega)=\frac{1}{\sqrt{2\pi}}\int_R f(x)W^*(x-b)\mathrm{e}^{-\mathrm{j}\omega x}\,\mathrm{d}x$$

W(x) is the windowing function, commonly a Hann window, Gaussian window centered around 0. STFT/WFT

Inverse STFT:

$$f(x)=\frac{1}{\sqrt{2\pi}}\iint_{R^2}\mathrm{WFT}_f(b,\omega)W(x-b)\mathrm{e}^{\mathrm{j}\omega x}\,\mathrm{d}\omega\,\mathrm{d}b$$

IMAGE TRANSFORMS – DWT STFT

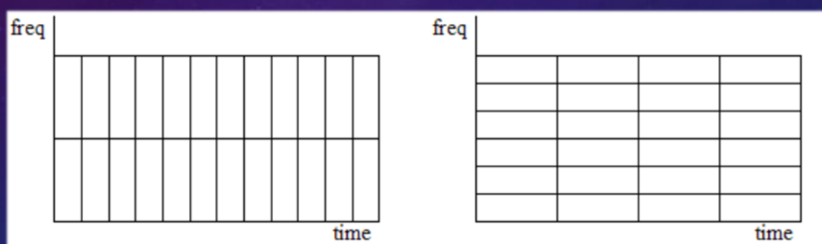
Heisenberg Box

Gabor transform

Gabor limit

$$\Delta t \Delta \omega \geq 0.5$$

- STFT has a fixed resolution
- The width of the windowing function relates to how the signal is represented



- Wavelet transform and multiresolution analysis can give good time resolution for high-frequency events and good frequency resolution for low-frequency events, the combination best suited for many real signals

IMAGE TRANSFORMS – DWT

Time-frequency tilings for the basis functions associated with

- (a)Sampled data
- (b)Fourier transform
- (c)STFT
- (d)Wavelet transform