Signals and Systems

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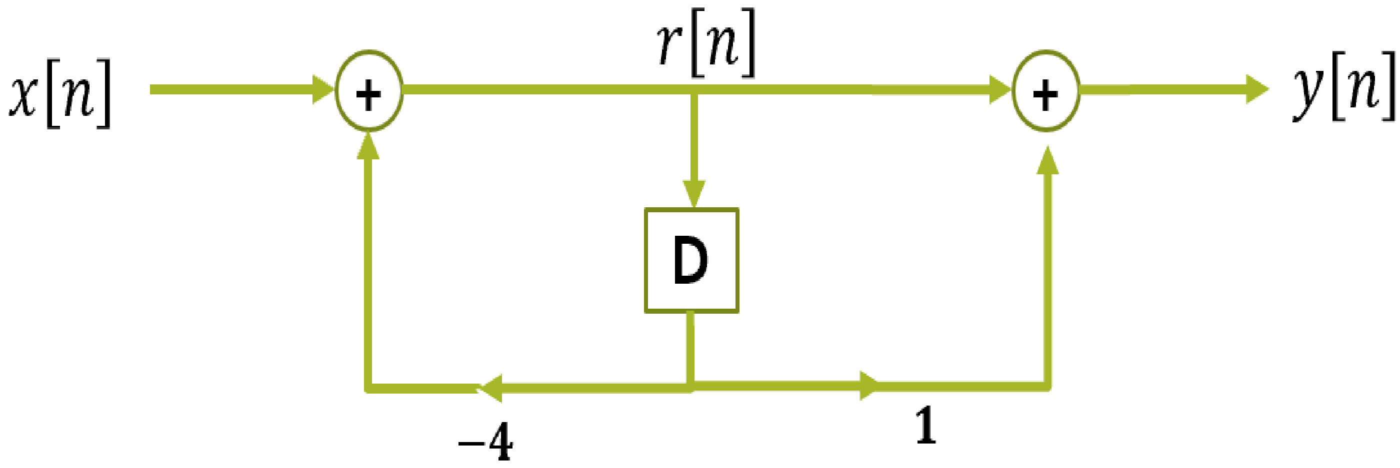
Exercise: Consider the block diagram in the figure. The system is causal and is initially at rest.



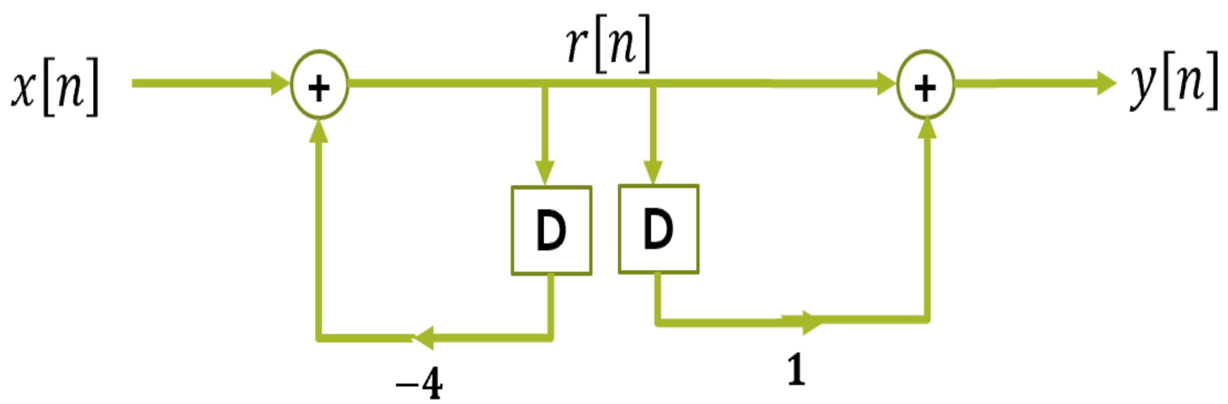
(a) Find the difference equation relating $x[n]$ and $y[n]$.

(b) For $x[n] = \delta[n]$, find $r[n]$ for all n.

(c) Find the system impulse response.

Solution:

(a) In the following figure we convert the block diagram from the original figure to direct form I.



$r[n]$ is given by $r[n] = x[n] - 4r[n-1]$

while $y[n] = r[n] + r[n-1]$

Substituting for $r[n]$ yields

$$y[n] + 4y[n-1] = x[n] + x[n-1]$$

(b) The relation between $x[n]$ and $r[n]$ is $r[n] = -4r[n-1] + x[n]$. For such a simple equation, we solve it recursively when $\delta[n] = x[n]$.

n	$\delta[n]$	$r[n-1]$	$r[n]$
<0	0	0	0
0	1	0	1
1	0	1	-4
2	0	-4	16
3	0	16	-64

We see that $r[n] = (-4)^n u[n]$.

(c) Since $r[n] = (-4)^n u[n]$, and $y[n] = r[n] + r[n-1]$

So $y[n] = (-4)^n u[n] + (-4)^{n-1} u[n-1]$

Now $y[n] = h[n]$,when $x[n] = \delta[n]$,

so $\boldsymbol{h[n] = (-4)^n u[n] + (-4)^{n-1} u[n-1]}$

This expression for $h[n]$ can be **further simplified**:

$$h[n] = (-4)^n u[n] + (-4)^{n-1} u[n-1]$$

$$\text{Or } h[n] = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ -3(-4)^{\mathrm{n}-1}, & \mathrm{n} > 0 \end{cases}$$

Thus, $\boldsymbol{h[n] = \delta[n] - 3(-4)^{\mathrm{n}-1}\, u[n-1]}$

Chapter 2 Review---- idea related to chapter 3

If: $x[n] = a_1\emptyset_1[n] + a_2\emptyset_2[n] + \ldots$

$\emptyset_1[n] \text{----}\!\!\rightarrow \varphi_1[n]$

and system is linear

Then: $y[n] = a_1\varphi_1[n] + a_2\varphi_2[n] + \ldots$

Identical for C-T

For LTI system

D-T:

$\emptyset_k[n] = \boldsymbol{\delta}[n-k]$

$\varphi_k[n] = \boldsymbol{h}[n-k]$

C-T:

$\emptyset_k(t) = \boldsymbol{\delta}(t-k)$

$\varphi_k(t) = \boldsymbol{h}(t-k)$

3.Fourier Series Representation of Periodic Signal

Jean Baptiste Joseph Fourier, born in 1768, in France.

1807, periodic signal could be represented by sinusoidal series.

1829, Dirichlet provided precise conditions.

1960s, Cooley and Tukey discovered fast Fourier transform.

3.2 The Response of LTI Systems to Complex Exponentials

(1) Continuous time LTI system

$$x(t)=e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t)=H(s)e^{st}$$

$$y(t)=x(t)*h(t)=\int_{-\infty}^{+\infty}x(t-\tau)h(\tau)d\tau$$

$$=\int_{-\infty}^{+\infty}e^{s(t-\tau)}h(\tau)d\tau=e^{st}\int_{-\infty}^{+\infty}h(\tau)e^{-s\tau}d\tau$$

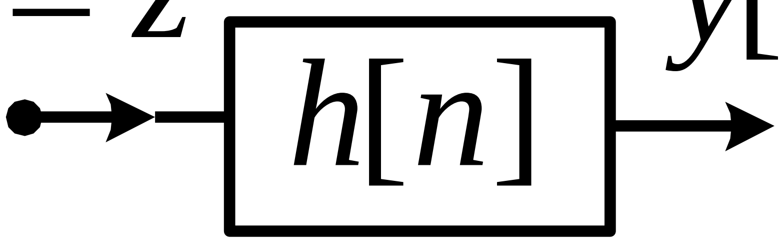
$$=e^{st}H(s)$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

(**system function**)

(2) Discrete time LTI system

$$x[n] = z^n \qquad y[n] = H(z) z^n$$



3 Fourier Series Representation of Periodic Signals

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k]$$

$$= \sum_{k=-\infty}^{+\infty} z^{(n-k)}h[k] = z^n \sum_{k=-\infty}^{+\infty} z^{-k}h[k]$$

$$= z^n H(z)$$

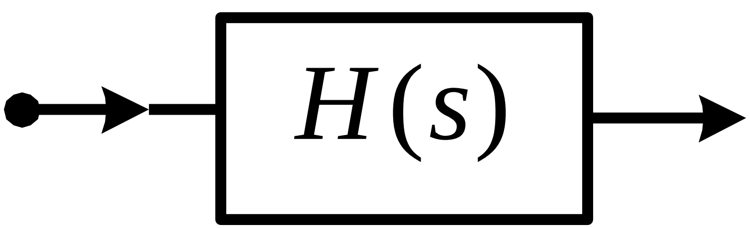
$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

(**system function**)

(3) Input as a combination of Complex Exponentials

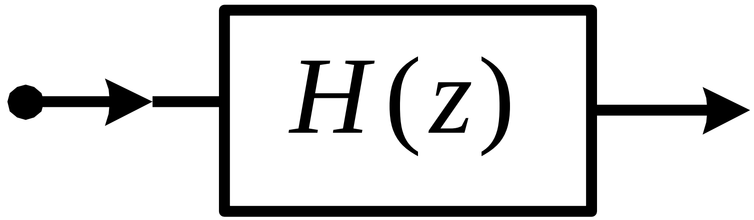
Continuous time LTI system:

$$x(t)=\sum_{k=1}^{N}a_k e^{s_k t} \qquad y(t)=\sum_{k=1}^{N}a_k H(s_k)e^{s_k t}$$



Discrete time LTI system:

$$x[n] = \sum_{k=1}^{N} a_k z_k^{\,n} \qquad y[n] = \sum_{k=1}^{N} a_k H(z_k) z_k^{\,n}$$



Example 3.1 LTI system: $y(t) = x(t-3)$

$$x(t) = e^{j2t}, H(s) = ?$$

- From $y(t) = x(t-3)$,we know

$h(t) = \delta(t-3)$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

$$= \int_{-\infty}^{+\infty} \delta(\tau-3)e^{-s\tau}d\tau = e^{-j6}$$

Example 3.1 $y(t) = x(t-3)$

$$x(t) = e^{j2t}, H(s) = ?$$

- $y(t) = x(t-3) = e^{j2(t-3)} = e^{-j6}e^{j2t} = H(s)x(t)$

- $H(s) = e^{-j6}$

3.3 Fourier Series Representation of Continuous-time Periodic Signals

3.3.1 Linear Combinations of Harmonically Related Complex Exponentials

(1) General Form—general complex exponential

The set of harmonically related complex exponentials:

$$\Phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t},$$

$$k = 0, \pm 1, \pm 2 \cdots$$

Fundamental period: *T* (common period)

3 Fourier Series Representation of Periodic Signals

$e^{j\omega_0 t}, e^{-j\omega_0 t}$: **Fundamental components**

$e^{j2\omega_0 t}, e^{-j2\omega_0 t}$: **Second harmonic components**

$e^{jN\omega_0 t}, e^{-jN\omega_0 t}$: **Nth harmonic components**

So, arbitrary periodic signal can be represented as

(Fourier Series)

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

a_k : FS coefficients

3 Fourier Series Representation of Periodic Signals

Example $$x(t)=\sum_{k=-3}^{3}a_k e^{jk2\pi t}$$ $a_k=?$

$$\Rightarrow x(t)=1+\frac{1}{4}(e^{j2\pi t}+e^{-j2\pi t})+\frac{1}{2}(e^{j4\pi t}+e^{-j4\pi t})$$

$$+\frac{1}{3}(e^{j6\pi t}+e^{-j6\pi t})$$

$$\Rightarrow a_0=1, a_1=a_{-1}=\frac{1}{4}, a_2=a_{-2}=\frac{1}{2}, a_3=a_{-3}=\frac{1}{3}$$

$$\Rightarrow x(t)=1+\frac{1}{2}\cos 2\pi t+\cos 4\pi t+\frac{2}{3}\cos 6\pi t$$

(2) Representation for Real Signal

Real periodic signal: x(t)=x*(t)

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

So $a_{-k} = a^*_k$

$$x(t)=a_0+\sum_{k=1}^{+\infty}[a_k e^{jk\omega_0 t}+a_{-k}e^{-jk\omega_0 t}]$$

$$=a_0+\sum_{k=1}^{+\infty} 2\,\mathrm{Re}[\,a_k e^{jk\omega_0 t}]$$

3 Fourier Series Representation of Periodic Signals

Let $a_k = A_k e^{j\theta_k}$ $\quad A_k = |a_k|$ **--magnitude**

$a_k e^{jk\omega_0 t} = A_k e^{j(k\omega_0 t + \theta_k)}$ $\quad \theta_k$ **------phase**

$$\therefore x(t) = a_0 + \sum_{k=1}^{+\infty} 2A_k \cos(k\omega_0 t + \theta_k)$$

Let $a_k = B_k + jC_k$--rectangular form

$$\therefore x(t) = a_0 + 2\sum_{k=1}^{+\infty} [B_k \cos k\,\omega_0 t - C_k \sin k\,\omega_0 t]$$

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

(Orthogonal function set)

Determining the coefficient by *orthogonality*:

(Multiply two sides by $e^{-jn\omega_0 t}$)

$$x(t)e^{-jn\omega_0 t}=\sum_{k=-\infty}^{+\infty} a_k e^{j(k-n)\omega_0 t}$$

3 Fourier Series Representation of Periodic Signals

$$\int_T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

$$\int_T x(t)e^{-jn\omega_0 t}\,dt = \int_T \sum_{k=-\infty}^{+\infty} a_k e^{j(k-n)\omega_0 t}\,dt$$

$$= a_n T$$

$$\therefore \quad a_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}\,dt$$

Fourier Series Representation:

$$\begin{cases} x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} & \textbf{(Synthesis equation)} \\ a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt & (\textbf{\textit{Analysis equation}}) \end{cases}$$

Or $x(t) \overset{FS}{\leftrightarrow} a_k$

$\{a_k\}$ **are called** ***Fourier Series coefficients*** **or** ***spectral coefficients*** **of** $x(t)$**.**

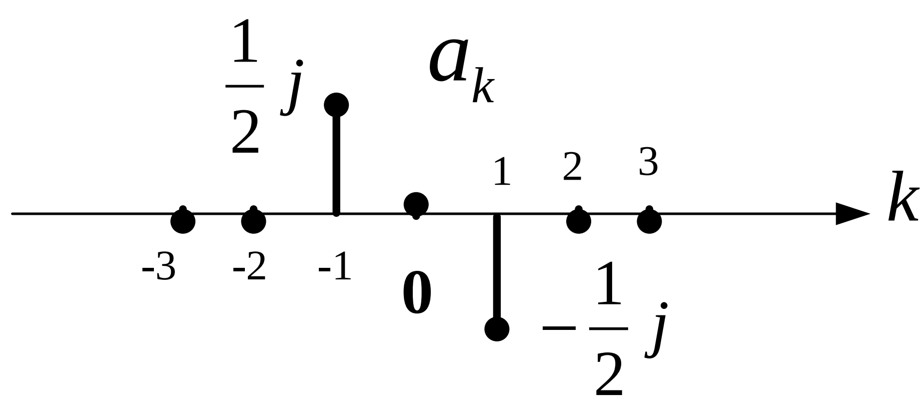
3 Fourier Series Representation of Periodic Signals

Example $x(t) = \sin\omega_0 t$, $a_k = ?$

$$\because x(t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} \qquad \therefore a_k = \begin{cases} 1/2j, k=1 \\ -1/2j, k=-1 \\ 0, k \neq \pm 1 \end{cases}$$

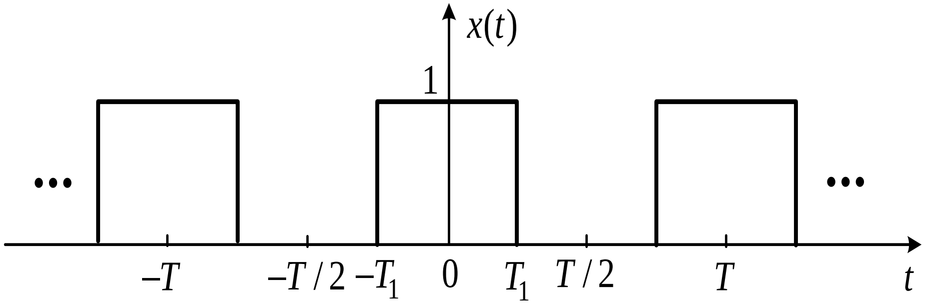
or $a_1 = \frac{1}{2j}$ $\qquad a_{-1} = -\frac{1}{2j}$ $\qquad a_k = 0, k \neq \pm 1$

Spectrum



Example 3.5: **Periodic square wave**, **and defined over one period, determine** a_k**.**

$$x(t)=\begin{cases}1, & |t|<T_1\\ 0, & T_1<|t|<T/2\end{cases}$$



$$a_k=\frac{1}{T}\int_{-T/2}^{T/2}x(t)e^{-jk\omega_0 t}dt=\frac{1}{T}\int_{-T_1}^{T_1}e^{-jk\omega_0 t}dt$$

$$=-\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}=\frac{1}{jk\omega_0 T}[e^{jk\omega_0 T_1}-e^{-jk\omega_0 T_1}]$$

$$a_k = 2\frac{\sin k\,\omega_0 T_1}{k\omega_0 T} = \frac{\sin k\,\omega_0 T_1}{k\pi}$$

$$\omega_0 T = 2\pi \qquad a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

So, Fourier Series Representation of $x(t)$

3 Fourier Series Representation of Periodic Signals

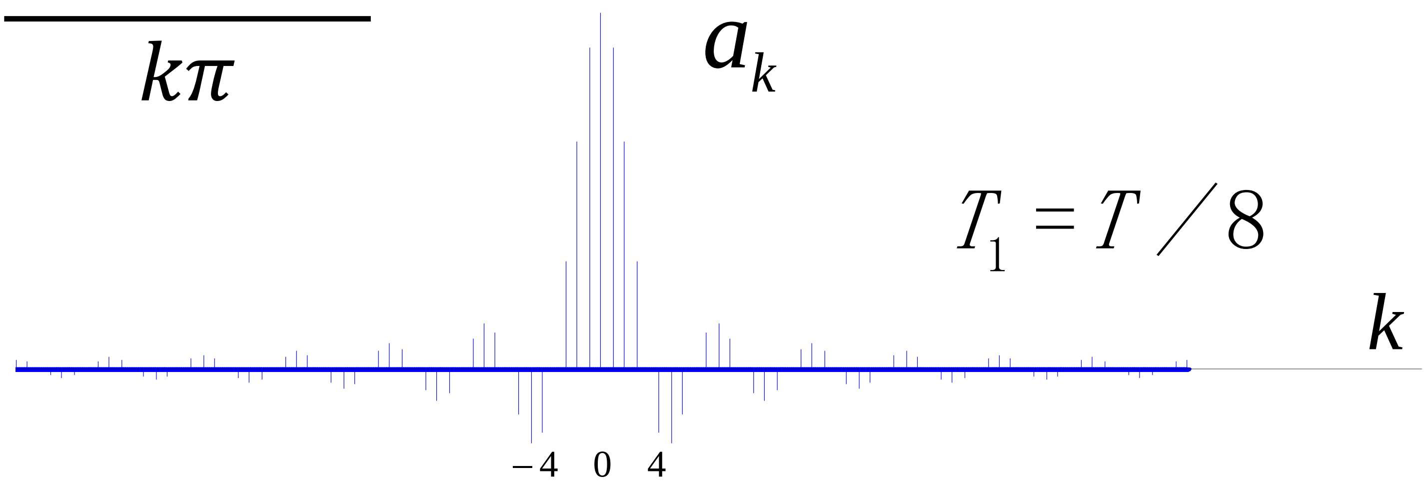
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

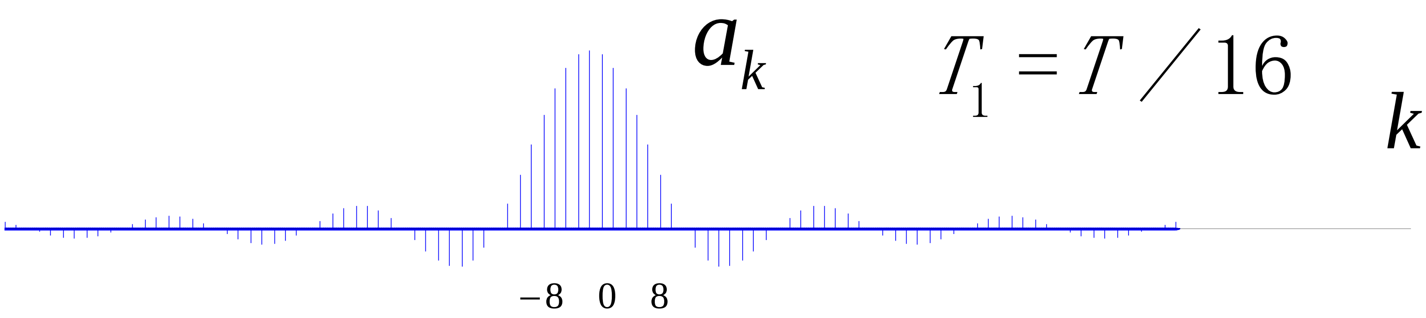
$$= \sum_{k=-\infty}^{+\infty} \frac{\sin k\,\omega_0 T_1}{k\pi} e^{jk\omega_0 t}$$

a_k ------**Spectrum of $x(t)$ (in next slide)**

3 Fourier Series Representation of Periodic Signals

$$a_k = \frac{\sin k\,\omega_0 T_1}{k\pi}$$

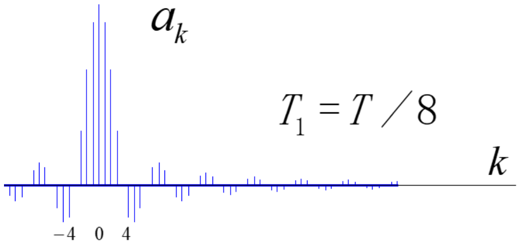




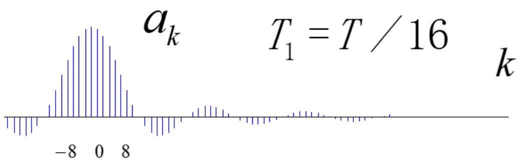
Bandwidth of signal:
defined by application. Normally,
90% power BW, first zero-cross BW are used.

Physically, Bandwidth of square signal become wider when T_1 is decreasing.

From $$a_k = \frac{\sin k\,\omega_0 T_1}{k\pi}$$



first zero-cross at $k\omega_0 T_1 = \pm\pi$

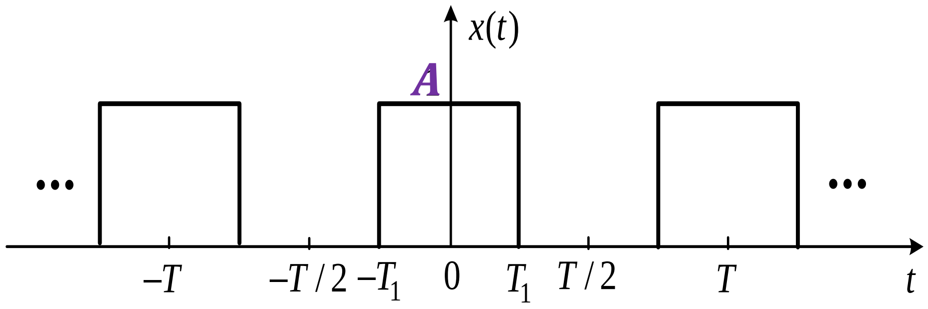


first zero-cross BW= $2\pi/T_1$ **(rad/s).**

3 Fourier Series Representation of Periodic Signals

Extended **Periodic square wave, period T, determine a_k.**

$$x(t) = \begin{cases} A, & |t| < T_1 \\ 0, & T_1 < |t| < \dfrac{T}{2} \end{cases}$$



$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} Ae^{-jk\omega_0 t}dt$$

$$a_k = \frac{A\sin k\,\omega_0 T_1}{k\pi}, \qquad a_0 = \frac{1}{T}\int_{-T_1}^{T_1} A\,dt = \frac{2AT_1}{T}$$

Example

A continuous–time periodic signal $x(t)$ is real valued and has a fundamental period T =8. The nonzero Fourier series coefficients for $x(t)$ are

$$a_1 = a_{-1} = 2, a_3 = a_{-3}^* = 4j$$

Express $\boldsymbol{x(t)}$ in the form

$$x(t) = \sum_{k=0}^{+\infty} A_k \cos(\omega_k t + \theta_k)$$

Solution

Because we have: $a_1 = a_{-1} = 2, a_3 = a_{-3}^{*} = 4j$

Using the Fourier series synthesis equation

$$x(t)$$
$$= a_1 e^{j\left(\frac{2\pi}{T}\right)t} + a_{-1} e^{-j\left(\frac{2\pi}{T}\right)t} + a_3 e^{j3\left(\frac{2\pi}{T}\right)t} + a_{-3} e^{-j3\left(\frac{2\pi}{T}\right)t}$$
$$= 2e^{j\left(\frac{2\pi}{8}\right)t} + 2e^{-j\left(\frac{2\pi}{8}\right)t} + 4je^{j3\left(\frac{2\pi}{8}\right)t} - 4je^{-j3\left(\frac{2\pi}{8}\right)t}$$
$$= 4\cos\left(\frac{\pi}{4}t\right) - 8\sin\left(\frac{3\pi}{4}t\right)$$
$$= 4\cos\left(\frac{\pi}{4}t\right) + 8\sin\left(\frac{3\pi}{4}t + \frac{\pi}{2}\right)$$

3.4 Convergence of the Fourier Series

(1) Finite series

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

Approximation error:

$$e_N(t) = x(t) - x_N(t)$$

$$= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t} = \sum_{|k|>N} a_k e^{jk\omega_0 t}$$

$$\lim_{N\to\infty} \frac{1}{T}\int_T |e_N(t)|^2\, dt = 0$$

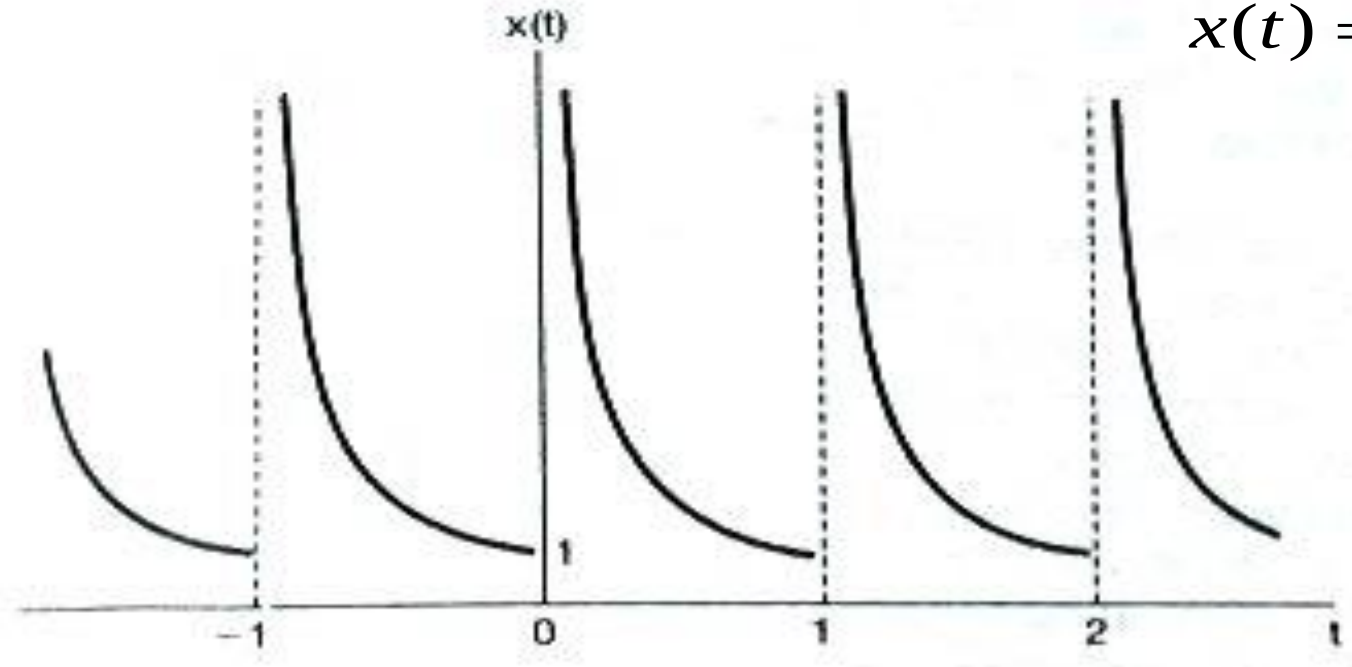
If convergent. ($x_N(t) \to x(t)$)

(2) Dirichlet condition

Condition 1: Absolutely Integrable

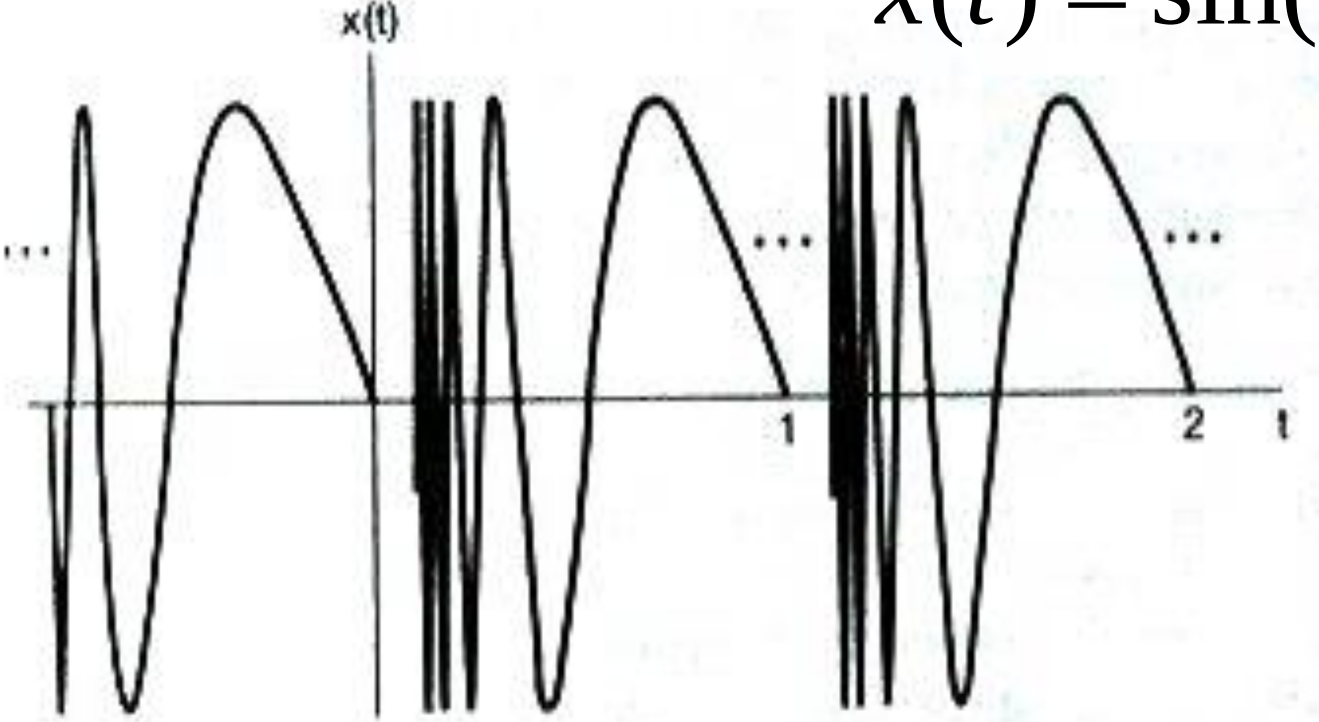
$$\int_T |x(t)|\,dt < \infty$$

$$x(t) = \frac{1}{t}$$

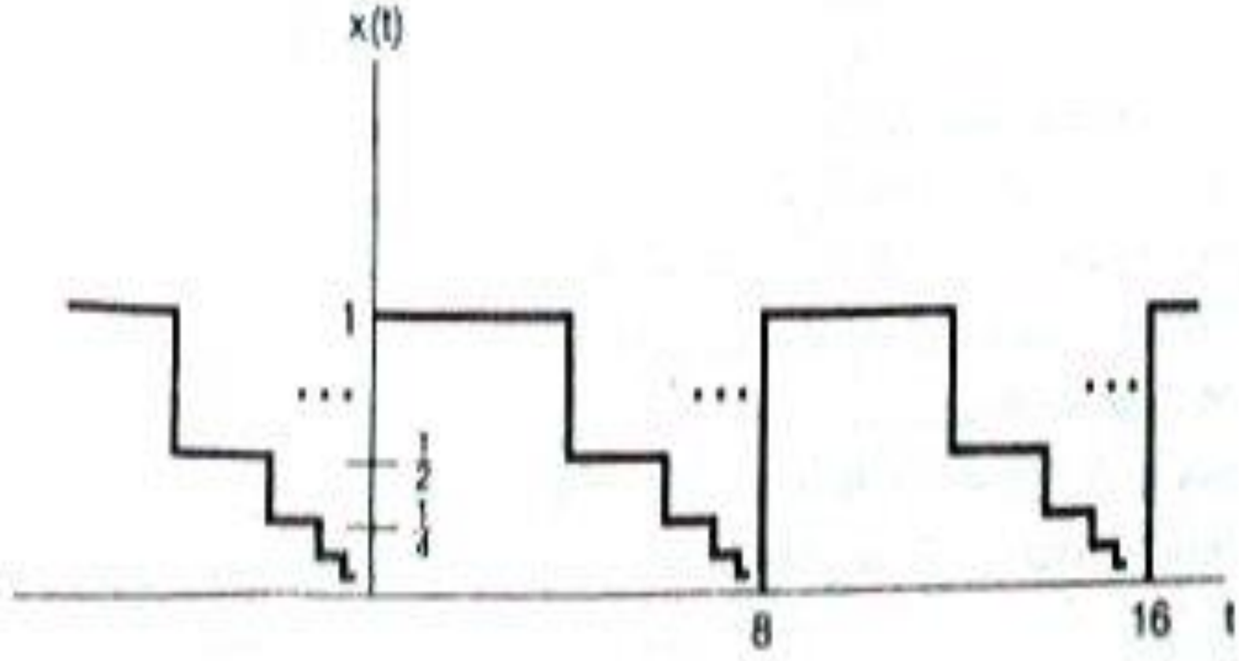


Condition 2: Finite number of maxima and minima during a single period

$$x(t) = \sin(\frac{2\pi}{t})$$



Condition 3: Finite number of discontinuity



(3) Gibbs phenomenon (read by yourself)

1898, Albert Michelson , An American physicist

Constructed a harmonic analyzer

Observed truncated Fourier series

$x_N(t)$ [Eq(3.52)] looked very much like $x(t)$

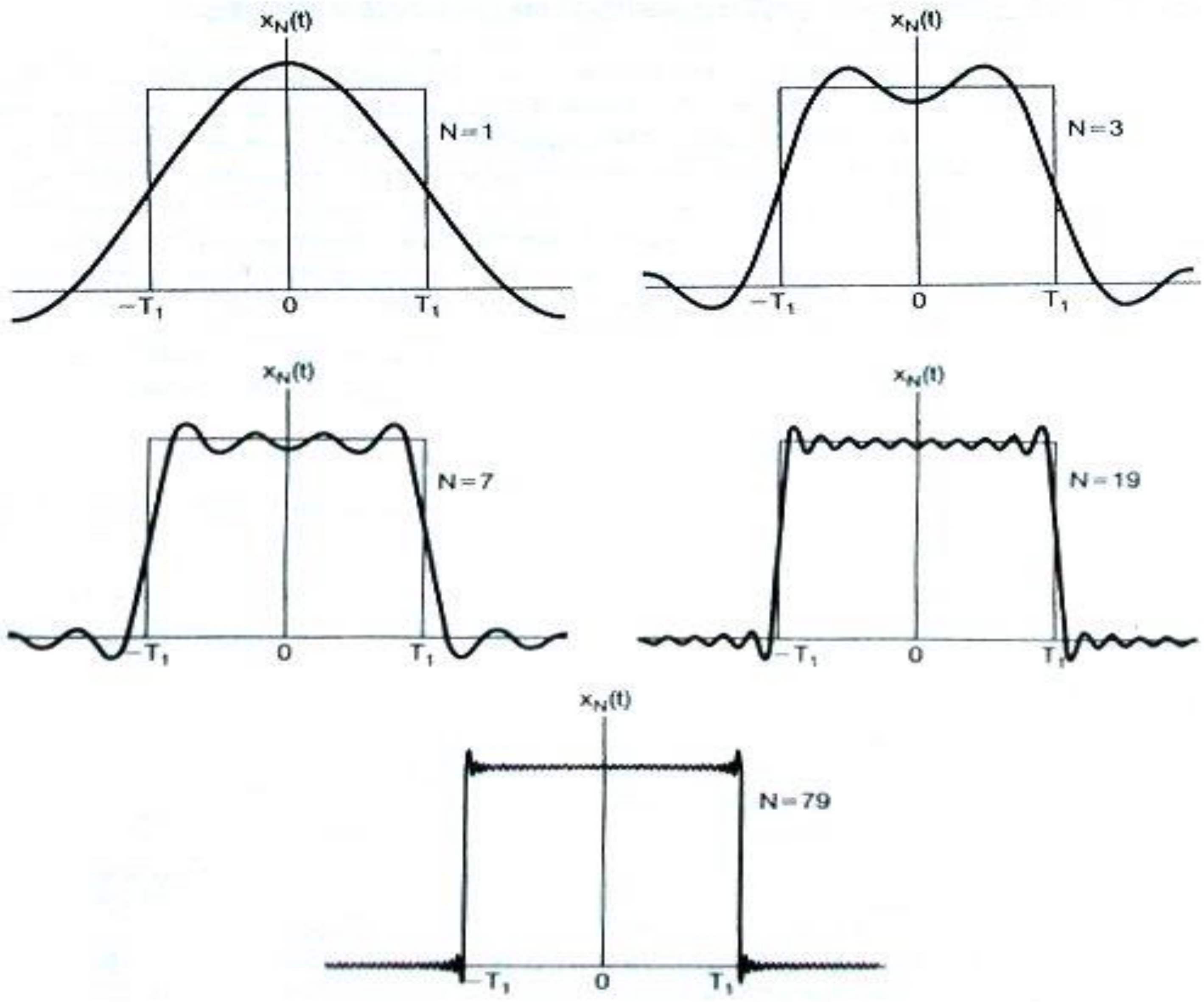
Found a strange phenomenon

Josiah Gibbs, An American mathematical physicist

Gave out a mathematical Explanation

We can get the illustration of Gibbs phenomenon from following figure.

3 Fourier Series Representation of Periodic Signals



Gibbs's conclusion:

Any continuity:

$$x_N(t_1) \rightarrow x(t_1)$$

Vicinity of discontinuity:

ripples

peak amplitude does not seem to decrease

Discontinuity: overshoot 9%

3.5 Properties of Continuous-Time Fourier Series

$$x(t) \overset{FS}{\leftrightarrow} a_k$$

$$y(t) \overset{FS}{\leftrightarrow} b_k$$

They have same Fundamental period.

3.5 Properties of Continuous-Time Fourier Series

3.5.1 Linearity

$$x(t) \overset{FS}{\longleftrightarrow} a_k \qquad y(t) \overset{FS}{\longleftrightarrow} b_k$$

$$z(t) = Ax(t) + By(t) \overset{FS}{\longleftrightarrow} c_k$$

Where $c_k = Aa_k + Bb_k$

3.5.2 Time Shift

$$x(t - t_0) \overset{FS}{\longleftrightarrow} a_k e^{-jk\omega_0 t_0}$$

Where $$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$$

Note: magnitude $$\left| a_k e^{-jk\omega_0 t_0} \right| = \left| a_k \right|$$

Solve{Extended version of **Test #3-1**} by using the time shift property

Use the Fourier series analysis equation to calculate the coefficients a_k for the continuous-time periodic signal

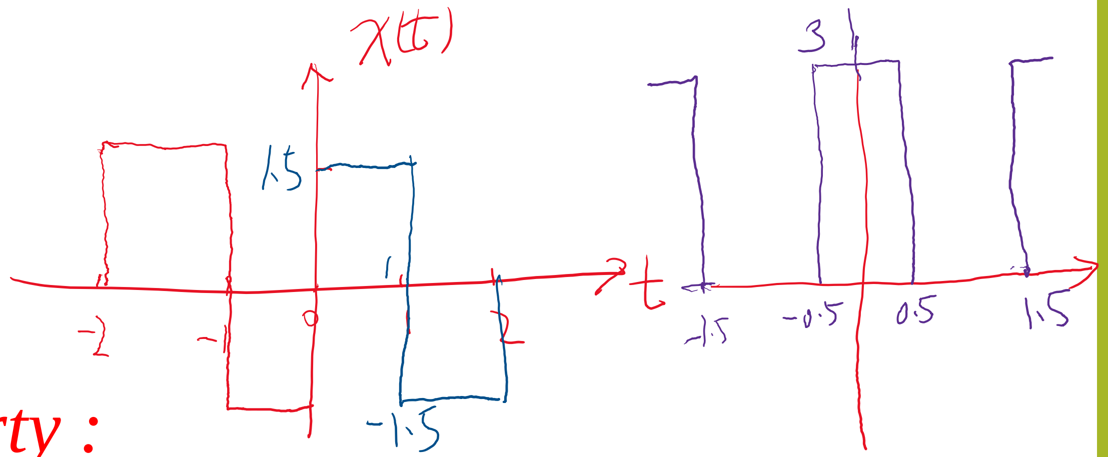
$$x(t) = \begin{cases} 1.5, & 0 \le t \le 1 \\ -1.5, & 1 \le t < 2 \end{cases}$$

with fundamental frequency $\omega_0 = \pi$.

3 Fourier Series Representation of Periodic Signals

Solution 2:

Since $T = \dfrac{2\pi}{\omega_0} = 2.$



Use time shift property :

let $\boldsymbol{r(t) = x(t+0.5)+1.5}$ →even square wave

and square wave results: coefficients for r(t):

$$\boxed{a_k = \frac{\boldsymbol{A}\sin k\,\omega_0 T_1}{k\pi}, \qquad a_0 = \frac{1}{T}\int_{-T_1}^{T_1} \boldsymbol{A}dt = \frac{2\boldsymbol{A}T_1}{T}}$$

$$b_k = \frac{\mathbf{3}\sin[k\pi(0.5)]}{k\pi}, \quad b_0 = \frac{2\times 3\times 0.5}{2} = 1.5$$

Solution 2

$$b_k = \frac{3}{k\pi}\sin(\frac{k\pi}{2}), \quad b_0 = 1.5$$

$$\boldsymbol{r(t) = x(t+0.5) + 1.5}$$

$$\boldsymbol{\rightarrow x(t) = r(t-0.5) - 1.5}$$

Then $a_k = b_k e^{-jk\pi 0.5} \boldsymbol{-1.5}$

So, $a_0 = b_0 - 1.5 = \mathbf{0}$

and for $k \neq 0, a_k = \frac{3}{\pi k} e^{-jk\pi/2} \sin(\frac{k\pi}{2})$

3.5.3 Time Reversal

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0 t}$$

$$let\ k = -n, \qquad x(-t) = \sum_{n=-\infty}^{\infty} a_{-n} e^{jn\omega_0 t}$$

$$x(-t) \overset{FS}{\longleftrightarrow} a_{-k}$$

3.5.3 Time Reversal

$$x(-t) \overset{FS}{\longleftrightarrow} a_{-k}$$

$$if\ x(t) is\ \textbf{even} \rightarrow x(-t) = x(t)$$

$$SO, \quad a_{-k} = a_k$$

$$if\ x(t) is\ \textbf{odd} \rightarrow x(-t) = -x(t)$$

$$SO, \quad a_{-k} = -a_k$$

3.5.4 Time Scaling

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\Rightarrow \quad x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\alpha\omega_0 t}$$

$$x(\alpha t) \overset{FS}{\leftrightarrow} a_k$$

3.5.5 Time Multiplication

$$x(t) \overset{FS}{\leftrightarrow} a_k$$

$$y(t) \overset{FS}{\leftrightarrow} b_k$$

$$x(t)y(t) \overset{FS}{\leftrightarrow} \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

3.5.6 Conjugation and Conjugate Symmetry

$$x(t) \overset{FS}{\leftrightarrow} a_k$$

$$x^*(t) \overset{FS}{\leftrightarrow} a_{-k}^*$$

If $x(t)$ is real and even: $a_k = a_{-k}$, $a_k^* = a_{-k} = a_k$

If $x(t)$ is real and odd: $a_k = -a_{-k}$, $a_k^* = a_{-k} = -a_k$

Some consequences of the property:

If $x(t)$ **is real, then** $a_{-k} = a_k^*$

(Conjugate Symmetry)

If $x(t)$ **is real and even, then** $a_{-k} = a_k$

It implies that a_k **is real and even.**

If $x(t)$ **is real and odd, then**

$$a_{-k} = -a_k$$

It implies that a_k **is purely imaginary and odd.**

3.5.7 Paseval's Relation

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Proof:

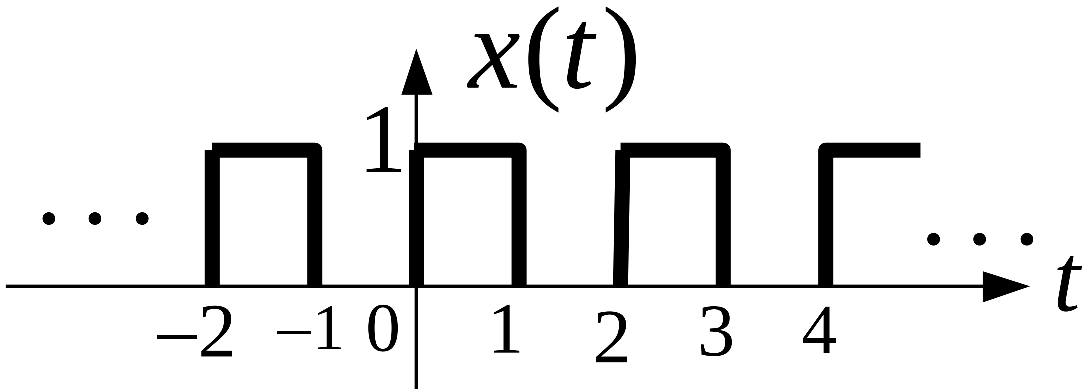
$$\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_T x(t)x(t)^* dt$$

$$= \frac{1}{T}\int_T \{\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}\}\{\sum_{m=-\infty}^{+\infty} a_m^* e^{-jm\omega_0 t}\}dt$$

$$= \frac{1}{T}\int_T \sum_{k=-\infty}^{+\infty} |a_k|^2\, dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

3 Fourier Series Representation of Periodic Signals

Example



$$a_0 = \frac{1}{2} \qquad a_k = j\frac{[(-1)^k - 1]}{2k\pi}, k \neq 0$$

$$\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{2} \qquad \textbf{(A)}$$

$$\sum_{k=-\infty}^{+\infty} |a_k|^2 = a_0{}^2 + 2\sum_{k=1}^{+\infty} |a_k|^2 = (\frac{1}{2})^2 + 2\sum_{k=1,odd}^{+\infty} \frac{1}{(k\pi)^2} \qquad \textbf{(B)}$$

From Parseval's Relation,(A)=(B).We can get:

$$\frac{1}{2}=(\frac{1}{2})^2+2\times\frac{1}{\pi^2}[1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots]$$

So that

$$\frac{\pi^2}{8}=[1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots]$$

You can get this formula in a mathematic manual.

3.5.8 Differentiation

$$\frac{d}{dt}x(t) \stackrel{FS}{\longleftrightarrow} jk\omega_0 a_k$$

Proof:

$$\because x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$\therefore \frac{d}{dt}x(t) = \sum_{k=-\infty}^{+\infty} [a_k jk\omega_0] e^{jk\omega_0 t}$$

3 Fourier Series Representation of Periodic Signals

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau) y(t-\tau) d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right) a_k = \left(\frac{1}{jk(2\pi/T)}\right) a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

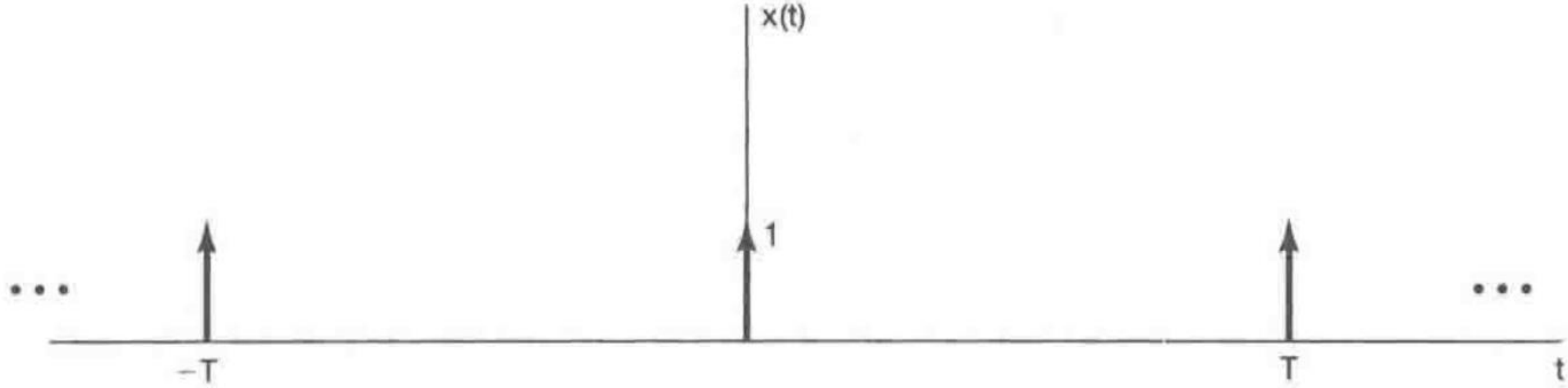
Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Example 3.8

Determine the Fourier series coefficients a_k of $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(n - kT)$$



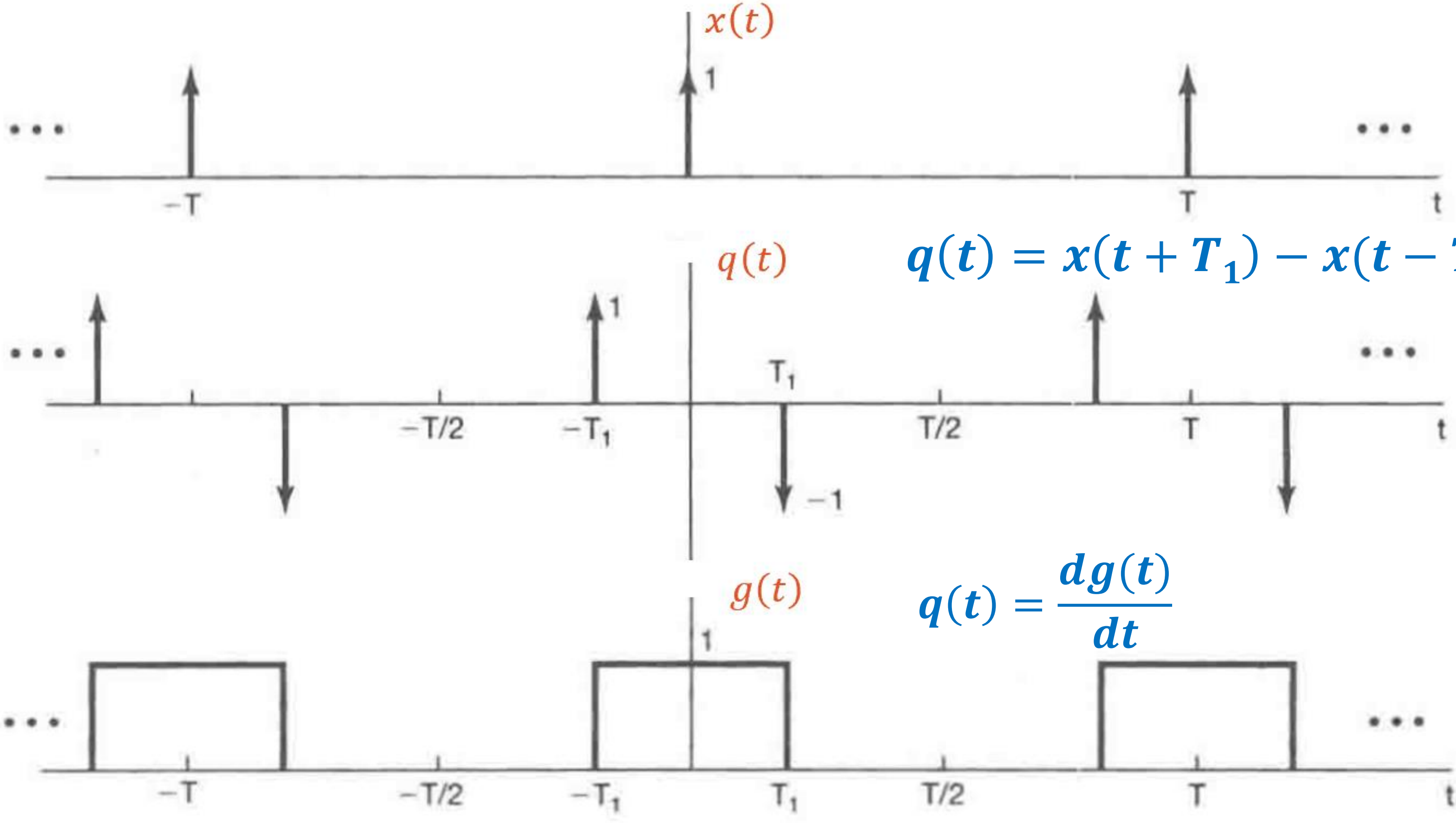
Use equation:

$$a_k = \frac{1}{T}\int x(t)e^{-jk\omega_0 t}dt$$
$$= \frac{1}{T}\int x(t)e^{-jk2\pi/Tt}dt$$

Choose integration interval: $-\frac{\mathrm{T}}{2}$ to $\frac{\mathrm{T}}{2}$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-jk2\pi/Tt}dt = \frac{1}{T}$$

Use this result to solve Square – wave signal



$$q(t) = x(t + T_1) - x(t - T_1)$$

$$q(t) = \frac{dg(t)}{dt}$$

Use this result to solve Square − wave signal

$$q(t) = x(t+T_1) - x(t-T_1)$$

$$\rightarrow \quad b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k$$

$$= \frac{1}{T}\left(e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right) = \frac{2j\sin(k\omega_0 T_1)}{T}$$

$$q(t) = \frac{dg(t)}{dt} \qquad \rightarrow b_k = jk\omega_0 c_k$$

$$g(t) \overset{FS}{\Leftrightarrow} c_k$$

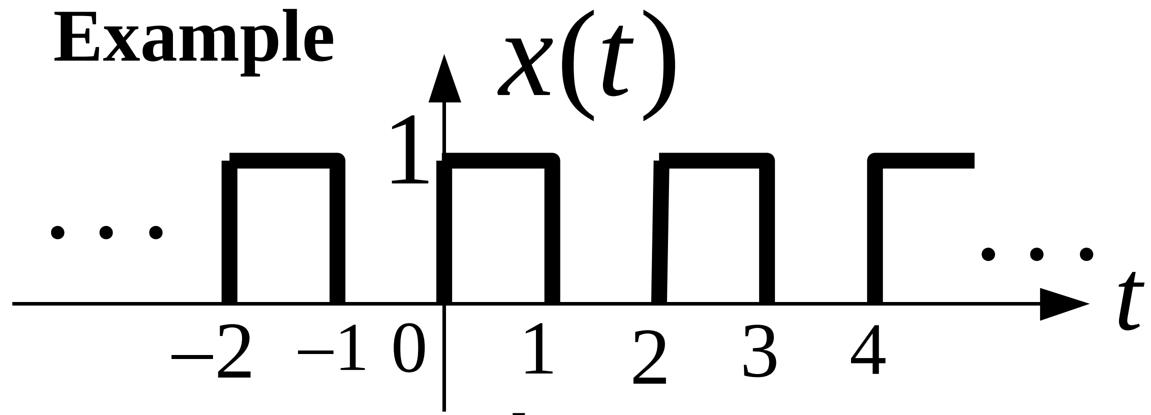
$$c_k = \frac{b_k}{jk\omega_0} = \frac{sin(k\omega_0 T_1)}{k\pi}, k \neq 0$$

$$c_0 = \frac{2T_1}{T}$$

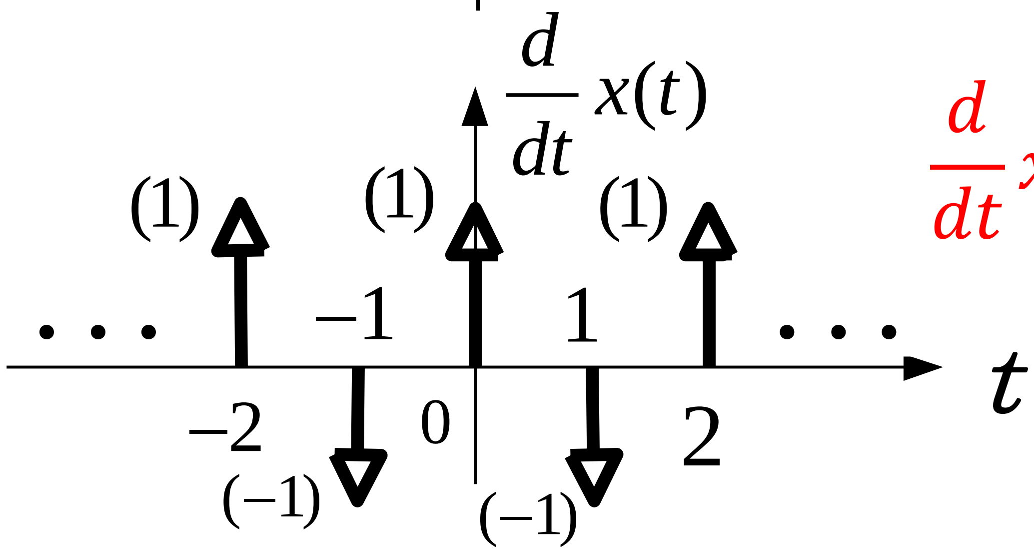
Consistent with the result we got before

3 Fourier Series Representation of Periodic Signals

Example



$$x(t) \overset{FS}{\longleftrightarrow} a_k = ?$$



$$\frac{d}{dt}x(t) = \delta_T(t) - \delta_T(t-1)$$

$$T = 2, \omega_0 = 2\pi/T = \pi$$

$$\frac{d}{dt}x(t) \overset{FS}{\leftrightarrow} jk\omega_0 a_k = b_k$$

3 Fourier Series Representation of Periodic Signals

$$\delta(t) \overset{FS}{\leftrightarrow} \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_0 t}dt = \frac{1}{T}$$

$$\delta(t-1) \overset{FS}{\leftrightarrow} \frac{1}{T}e^{-jk\omega_0}$$

$$\therefore\ b_k = jk\omega_0 a_k = \frac{1}{T} - \frac{1}{T}e^{-jk\omega_0}$$

3 Fourier Series Representation of Periodic Signals

$$\therefore a_k = \frac{1}{jk\omega_0} \times \frac{1}{T}\left[1 - e^{-jk\omega_0}\right]$$

$$= j\frac{\left[(-1)^k - 1\right]}{2k\pi}, k \neq 0$$

$$a_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{2}\int_0^1 1dt = \frac{1}{2}$$

Example:

Let $x(t)$ be a periodic signal whose Fourier series coefficients are $a_k = \begin{cases} 2, & k = 0 \\ j(\frac{1}{2})^{|k|}, & otherwise \end{cases}$

Use **Fourier series properties** to answer the following questions:

(a) Is $x(t)$ real?
(b) Is $x(t)$ even?
(c) Is $dx(t)/dt$ even?

Solution:

(a) If $x(t)$ is real, then $x(t) = x^*(t)$. This implies that for $x(t)$ real $a_k = a^*_{-k}$. Since this is **not true** in this case problem, $x(t)$ is not real.

(b) If $x(t)$ is even, then $x(t)= x(-t)$ and $a_k = a_{-k}$. Since this is true for this case, $x(t)$ is even.

Solution:

(c) We have $g(t) = \frac{dx(t)}{dt} \overset{FS}{\leftrightarrow} b_k = jk\frac{2\pi}{T_0}a_k$.

Therefore, $b_k = \begin{cases} 0, & k = 0 \\ -k(\frac{1}{2})^{|k|}\frac{2\pi}{T_0}, & otherwise \end{cases}$

Since b_k is not even, $g(t)$ is not even.

3.6 Fourier Series Representation of Discrete-time Periodic Signals

Some important differences

with C-T periodic signal

$$\begin{cases} x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\ a_k = \dfrac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt \end{cases}$$

C-T

$$\begin{cases} x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \\ a_k = \dfrac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} \end{cases}$$

D-T

3.6 Fourier Series Representation of Discrete-time Periodic Signals

3.6.1 Linear Combination of Harmonically Related Complex Exponentials

Periodic signal $x[n]$ with period N :

$$x[n] = x[n+N]$$

Discrete-time complex exponential orthogonal signal set:

$$\Phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, \quad k = 0, \pm 1, \pm 2 \cdots$$

Property of orthogonal signal set:

$$(1)\ \Phi_k[n] = \Phi_{k+rN}[n]$$

$$(2)\ \sum_{n=<N>} \Phi_k[n]\Phi_r{}^*[n] = \sum_{n=<N>} e^{j(k-r)\omega_0 n}$$

$$= \begin{cases} 0, & k \neq r \\ N, & k = r \end{cases}$$

3.6.2 Determination of the Fourier Series Representation of Periodic Signals

Fourier series of periodic signal x[n]:

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n} = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$\omega_0 = 2\pi / N$$

3.6.2 Determination of the Fourier Series Representation of Periodic Signals

When x[n] is already known, then

$$x[0] = \sum_{k=<N>} a_k \qquad x[1] = \sum_{k=<N>} a_k e^{jk\omega_0}$$

$$\ldots \qquad x[N-1] = \sum_{k=<N>} a_k e^{jk(N-1)\omega_0}$$

From which, we can solve the coefficients a_k

3.6.2 Determination of the Fourier Series Representation of Periodic Signals

Other strategy:

Determine the coefficients a_k by orthogonality

$$x[n]e^{-jr(2\pi/N)n}$$

$$= \sum_{k=<N>} a_k e^{j(k-r)(2\pi/N)n}$$

3 Fourier Series Representation of Periodic Signals

$$\sum_{n=<N>} x[n]e^{-jr(2\pi/N)n} = \sum_{n=<N>} \sum_{k=<N>} a_k e^{j(k-r)(2\pi/N)n}$$

$$= \sum_{k=<N>} a_k \sum_{n=<N>} e^{j(k-r)(2\pi/N)n}$$

$$= a_r \cdot N$$

$$\therefore \quad a_k = \frac{1}{N} \sum_{n=<N>} x[n]e^{-jk(2\pi/N)n}$$

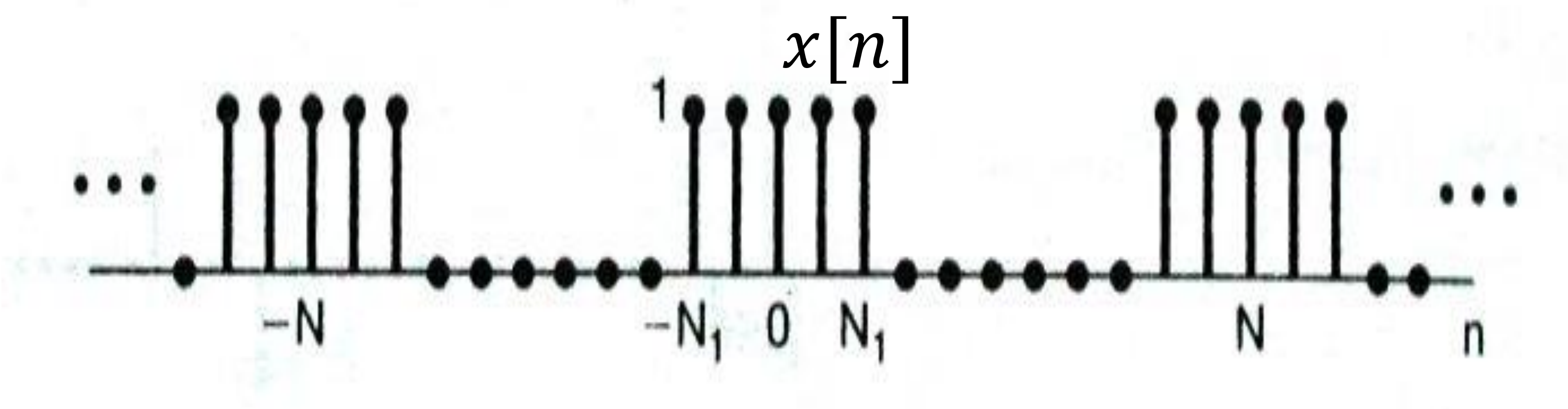
The equations of Fourier series:

$$
\begin{cases}
x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \\
a_k = \dfrac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}
\end{cases}
$$

Note:(a_k is periodic)

Example 3.12

The discrete-time periodic square wave shown in the figure with period N



Evaluate the Fourier series $\boldsymbol{a_k}$ for this signal

3 Fourier Series Representation of Periodic Signals

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}$$

let $m = n + N_1$

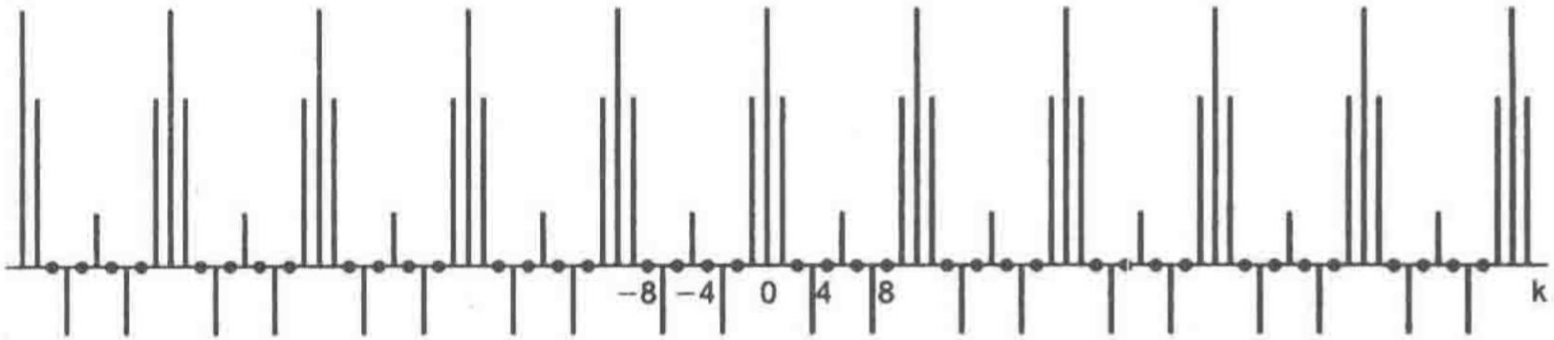
$$a_k = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)}$$

$$= \frac{1}{N} e^{jk(2\pi/N)N_1} \sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)m}$$

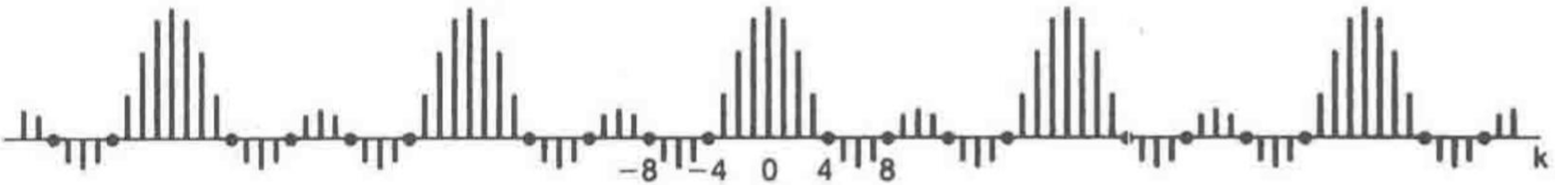
$$= \frac{1}{N} \frac{\sin[2\pi k(N_1+1/2)/N]}{sin(\pi k/N)}, \mathrm{k} \neq 0, \pm N, \pm 2N, \ldots$$

3 Fourier Series Representation of Periodic Signals

$$a_k = \frac{1}{N}\frac{\sin[2\pi k(N_1+1/2)/N]}{sin(\pi k/N)}, \mathrm{k} \neq 0, \pm N, \pm 2N, \ldots$$



(a)



(b)

Example:

Suppose we are given the following information about a periodic signal $x[n]$ with period **8** and Fourier series coefficients a_k :

(a) $a_k = -a_{k-4}$.

(b) $x[2n+1] = (-1)^n$

Sketch one period of $x[n]$.

Solution:

From Table 3.2, we know that if

$$x[n] \overset{FS}{\leftrightarrow} a_k,$$

then $\quad (-1)^n = e^{j\pi n} = e^{j(2\pi/N)(N/2)n}$

So $\quad (-1)^n x[n] \overset{FS}{\leftrightarrow} a_{k-N/2}$

In this case, N=8, therefore, $(-1)^n x[n] \overset{FS}{\leftrightarrow} a_{k-4}$,

since it is given that $a_k = -a_{k-4}$, we have

$$x[n] = -(-1)^n x[n]$$

Solution to be continued:

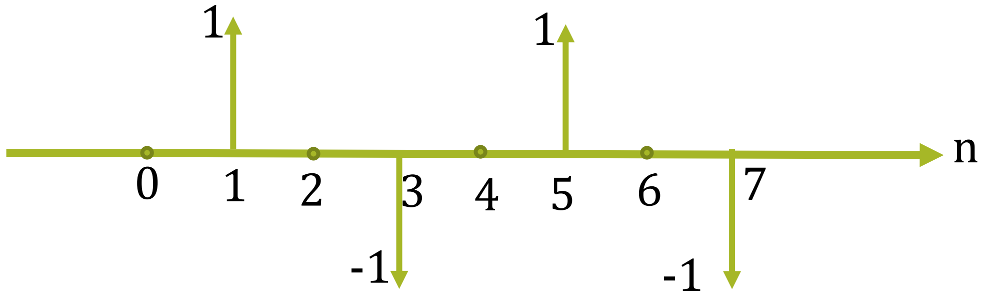
$$x[n] = -(-1)^n x[n]$$

This implies that $x[0] = x[\pm 2] = x[\pm 4] = \cdots = 0$

We are also given that $x[2n+1] = (-1)^n$

So $x[1] = x[5] = 1, and\ x[3] = x[7] = -1.$

Therefore, one period of $x[n]$ is as shown in the figure.



Example:

Let $\boldsymbol{x}[\boldsymbol{n}] = \begin{cases} \mathbf{1}, & \mathbf{0 \leq n \leq 7} \\ \mathbf{0}, & \mathbf{8 \leq \boldsymbol{n} \leq 9} \end{cases}$

be a periodic signal with fundamental period $\boldsymbol{N} = 10$ and Fourier series coefficients $\boldsymbol{a_k}$. Also, let

$$\mathbf{g[n] = x[n] - x[n-1]}.$$

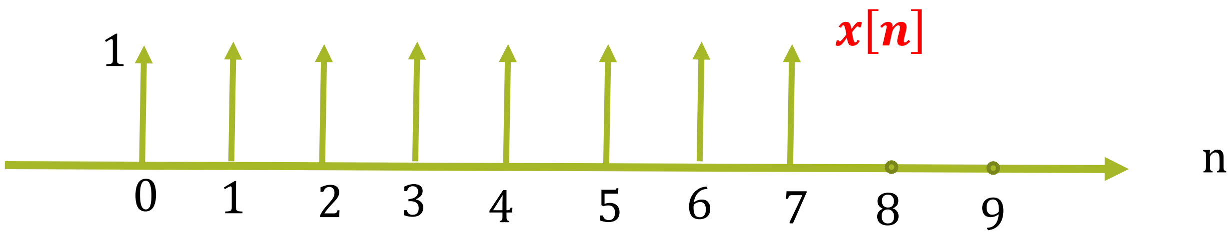
(a) Show that $\boldsymbol{g}[\boldsymbol{n}]$has a fundamental period of 10.

(b) Determine the **Fourier series coefficients** of $g[n]$.

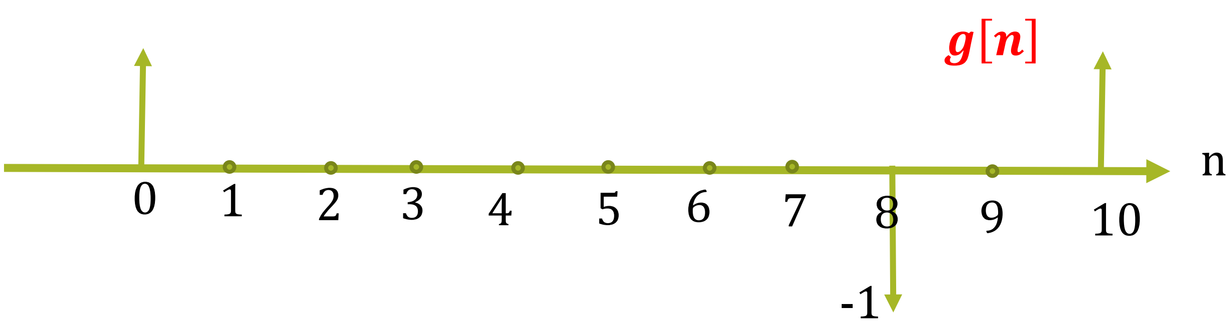
(c) Using the Fourier series coefficients of $g[n]$ and the **first-Difference property** in Table 3.2, determine a_k for $k \neq 0$.

Solution:

(a) $g[n]$ is as shown in the figure, clearly it has a fundamental period of 10.



$$\mathbf{g[n] = x[n] - x[n-1]}$$



Solution:

(b) From the last step

$$\boldsymbol{g[n] = \delta_T[n] - \delta_T[n-8]}$$

The Fourier series coefficients of $g[n]$ are

$$\boldsymbol{b_k = \left(\frac{1}{10}\right)\left[1 - e^{-j(2\pi/10)8k}\right]}$$

(c) Since $\boldsymbol{g[n] = x[n] - x[n-1]}$, the FS coefficients a_k and b_k must related as

$$\boldsymbol{b_k = a_k - e^{-j(2\pi/10)k}\, a_k}$$

Therefore, $\boldsymbol{a_k = \frac{b_k}{1-e^{-j(2\pi/10)k}} = \frac{\left(\frac{1}{10}\right)\left[1-e^{-j(2\pi/10)8k}\right]}{1-e^{-j(2\pi/10)k}}}$

3.8 Fourier Series and LTI System

(1) System function

Continuous time system:

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

Discrete-time system:

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]\, z^{-k}$$

(2) Frequency response

Continuous-time system:

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt$$

Discrete-time system:

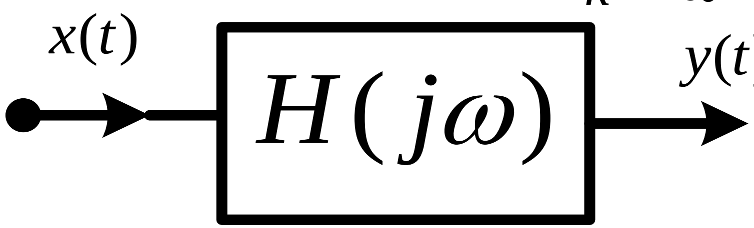
$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]\, e^{-j\omega n}$$

3 Fourier Series Representation of Periodic Signals

(3) System response

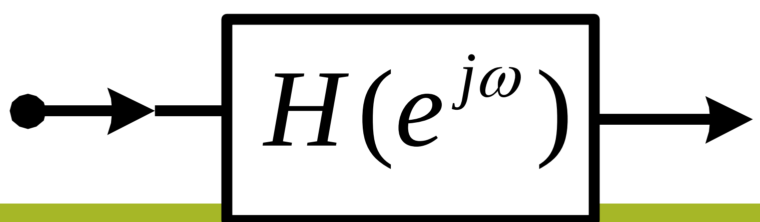
Continuous time system:

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad y(t)=\sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}$$



Discrete-time system:

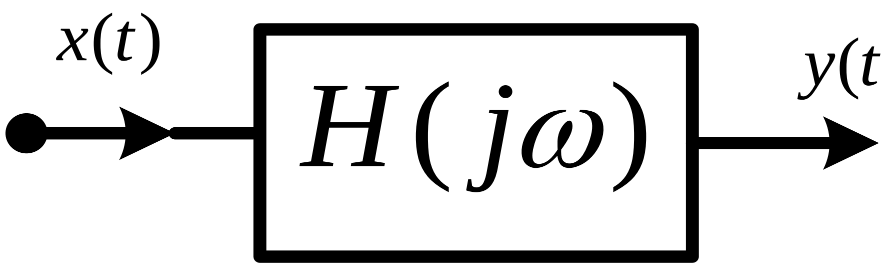
$$x[n]=\sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad y[n]=\sum_{k=<N>} a_k H(e^{jk\omega_0})e^{jk\omega_0 n}$$



3 Fourier Series Representation of Periodic Signals

General Example:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$



$$y(t) = \sum_{k=-\infty}^{+\infty} a_k \boldsymbol{H(jk\omega_0)} e^{jk\omega_0 t}$$

3 Fourier Series Representation of Periodic Signals

Example 3.16

$$x(t)=\sum_{k=-3}^{3} a_k e^{jk2\pi t} \quad h(t)=e^{-t}u(t),$$

$$a_0=1, a_1=a_{-1}=\frac{1}{4},\ a_2=a_{-2}=\frac{1}{2}, a_3=a_{-3}=\frac{1}{3}$$

$$x(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t)$$

$$H(j\omega)=\int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-j\omega\tau}d\tau=\int_{0}^{\infty} e^{-\tau}e^{-j\omega\tau}d\tau$$

$$=-\frac{1}{1+j\omega}e^{-\tau}e^{-j\omega\tau}\Big|_0^{\infty}=\frac{1}{1+j\omega}$$

3 Fourier Series Representation of Periodic Signals

$$y(t) = \sum_{k=-3}^{3} b_k e^{jk2\pi t} \qquad b_k = a_k H(jk2\pi)$$

$$b_0 = 1 \qquad b_1 = \frac{1}{4}\frac{1}{1+j2\pi} \qquad b_{-1} = \frac{1}{4}\frac{1}{1-j2\pi}$$

$$b_2 = \frac{1}{2}\frac{1}{1+j4\pi} \qquad b_{-2} = \frac{1}{2}\frac{1}{1-j4\pi}$$

$$b_3 = \frac{1}{3}\frac{1}{1+j6\pi} \qquad b_{-3} = \frac{1}{3}\frac{1}{1-j6\pi}$$

Read Example 3.17 by yourself.

Example:

Consider a linear, time-invariant system with impulse response

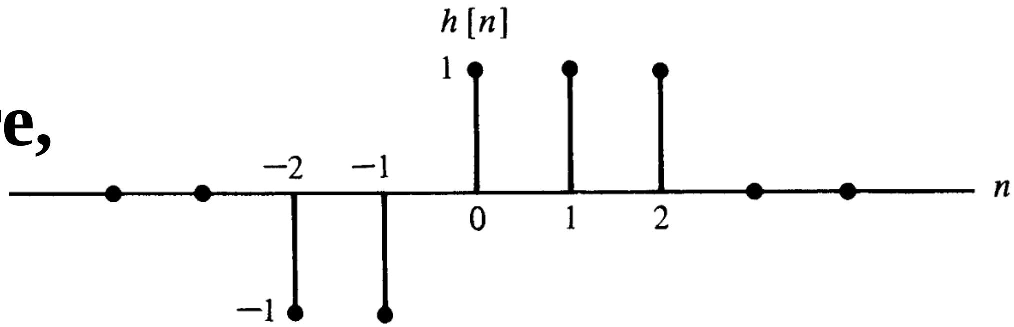
$$h[n] = \begin{cases} 1, & 0 \le n \le 2 \\ -1, & -2 \le n \le -1 \\ 0, & otherwise \end{cases}$$

Find the Fourier series representation of the output $y[n]$ for the giving input.

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - 4k]$$

Solution:

h[n] is sketched in Figure,



The frequency response of the system may be evaluated as $H(e^{j\omega}) = -e^{j2\omega} - e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega}$.

For $x[n]$, **N=4 and** $\omega_0 = {}^{\pi}/_{2}$, **the FS coefficients of the input** $x[n]$ **are** $a_k = \frac{1}{4}$, **for all n.**

Therefore, the FS coefficients of the output are

$$\boldsymbol{b}_k = a_k \boldsymbol{H}\left(e^{jk\omega_0}\right) = \frac{1}{4}[\mathbf{1} - e^{jk\pi/2} + e^{-jk\pi/2}]$$

Example:

Consider a discrete-time LTI system whose **frequency response** is

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \dfrac{\pi}{8} \\ 0, & \dfrac{\pi}{8} < |\omega| < \pi \end{cases}$$

Show that if the input $x[n]$ to this system has a period N=3, the output $y[n]$ has **only one nonzero Fourier series coefficient per period**.

Solution:

Let the FS coefficients of the input be $\boldsymbol{a_k}$. The FS coefficients of the output are of the form

$$\boldsymbol{b}_k = a_k \boldsymbol{H}\left(e^{jk\omega_0}\right)$$

Where $\omega_0 = \frac{2\pi}{3}$. Note that in the range $-1 \leq k \leq 1$, $\boldsymbol{H}\left(e^{jk\omega_0}\right) = \boldsymbol{0}\ \boldsymbol{for}\ \boldsymbol{k} = -\boldsymbol{1} \boldsymbol{and}\ \boldsymbol{k} = \boldsymbol{1}$.

Therefore, only b_0 has a nonzero value at $\boldsymbol{k} = \boldsymbol{0}$ among $\boldsymbol{b}_k$ in the range $-1 \leq k \leq 1$.

3.9 Filtering

- frequency-shaping filters
- frequency-selective filters

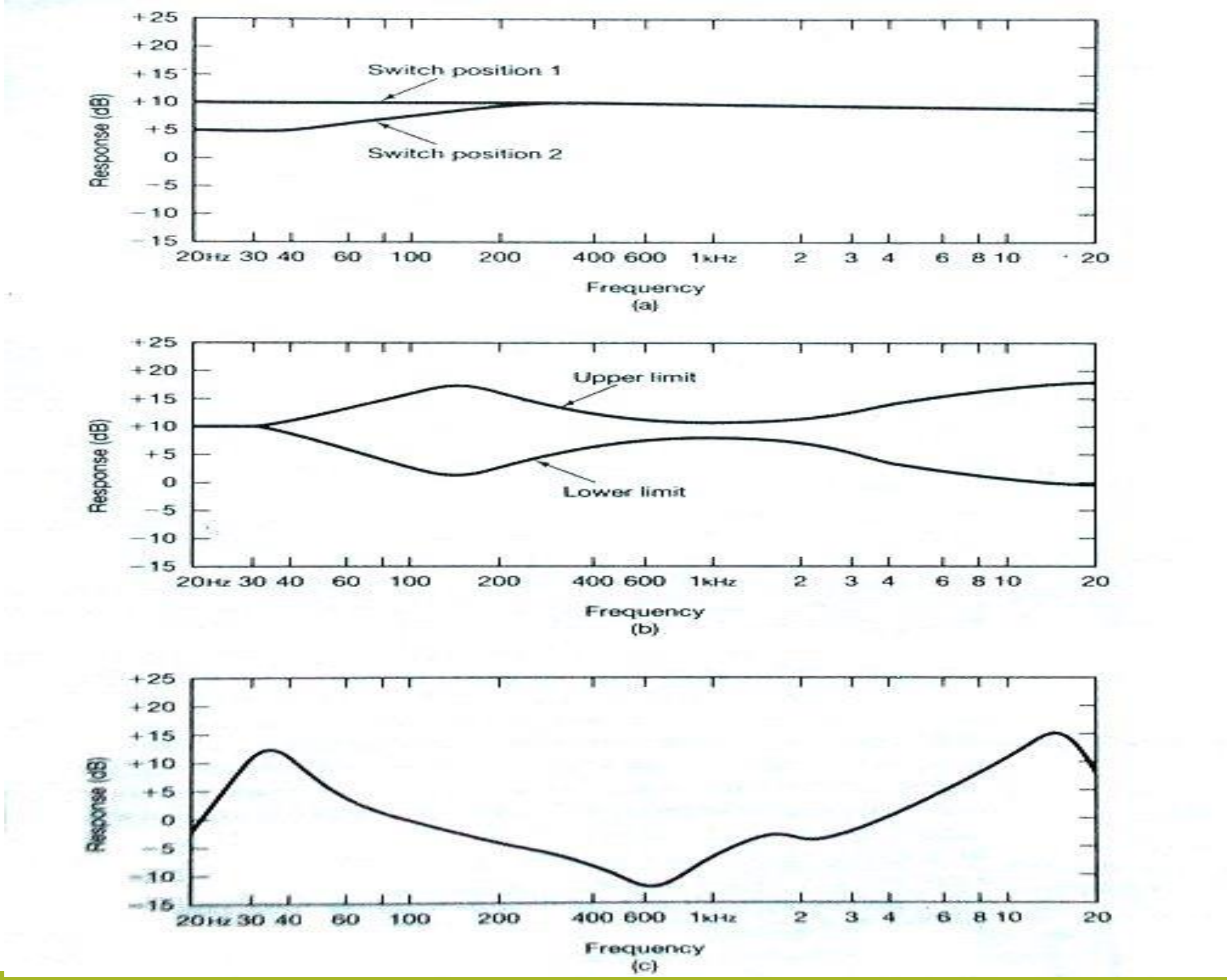
3.9 Filtering

3.9.1 Frequency-shaping filters

Example1:

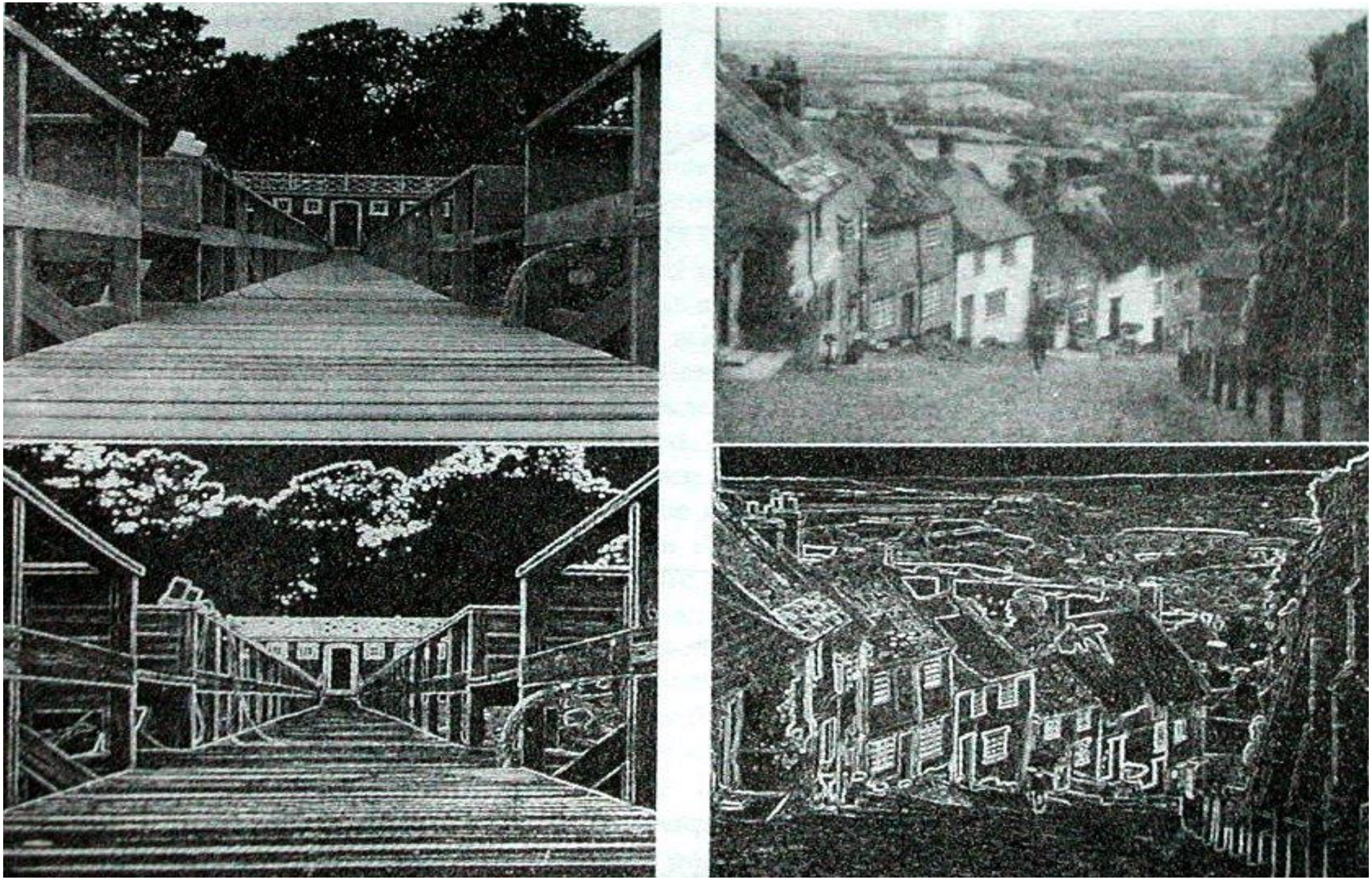
Equalizer

(Designed for audio speaker)



3 Fourier Series Representation of Periodic Signals

Example2: Image Filtering (Edge enhancement)



3.9.2 Frequency-selective filters

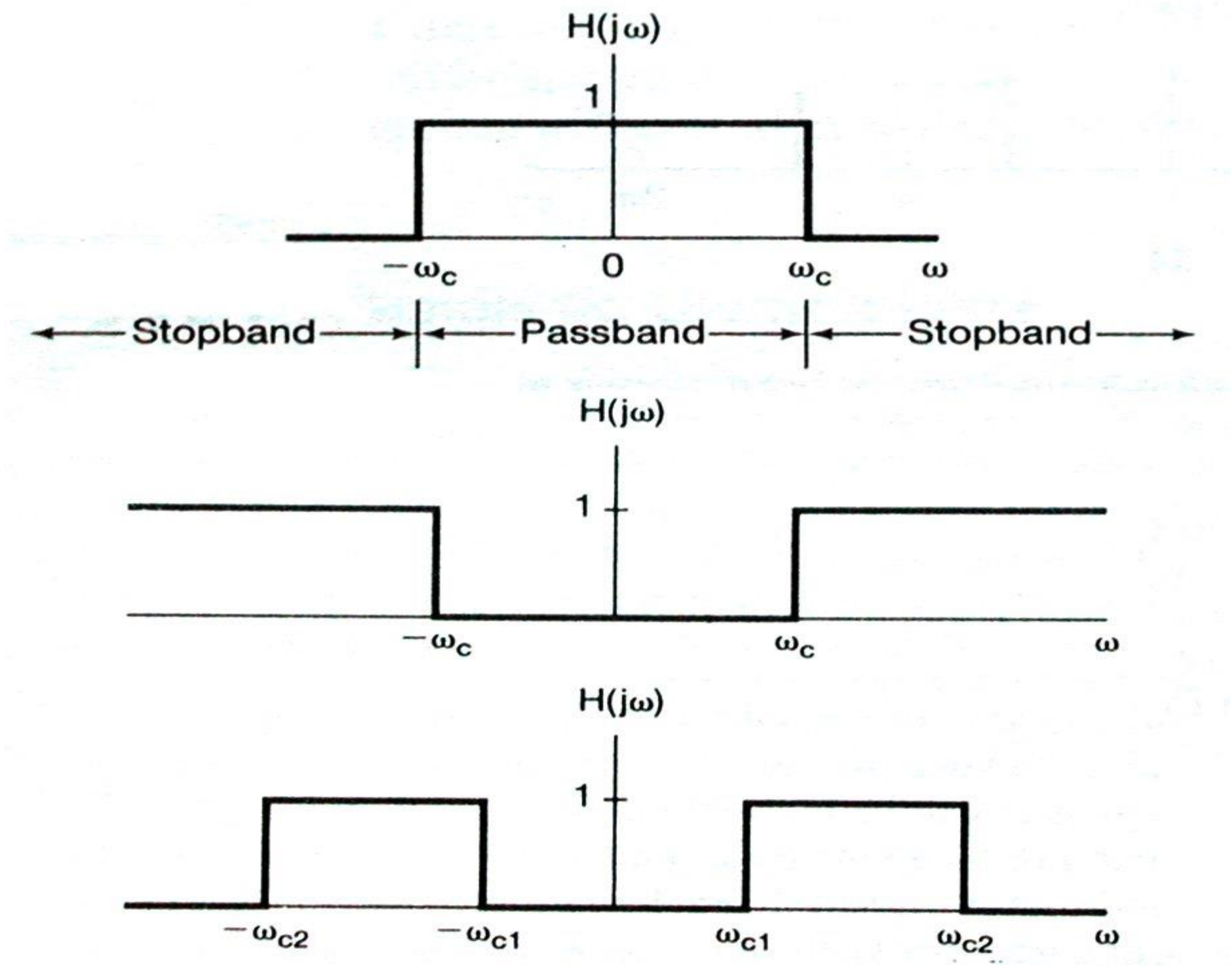
Several type of filter :

(1) Lowpass filter

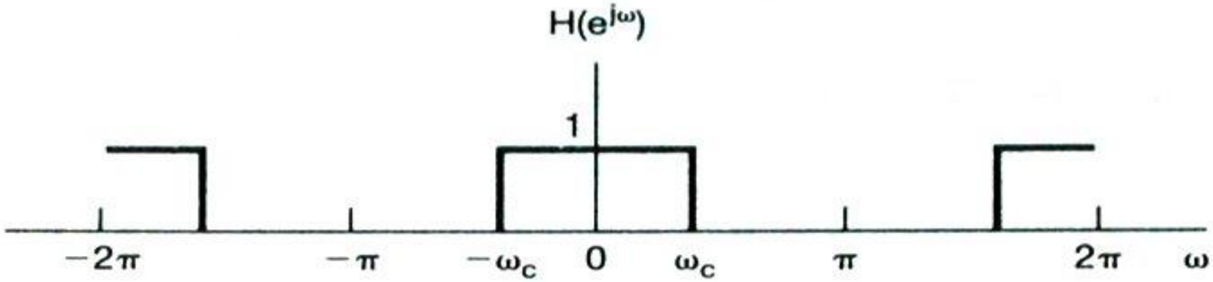
(2) Highpass filter

(3) Bandpass filter

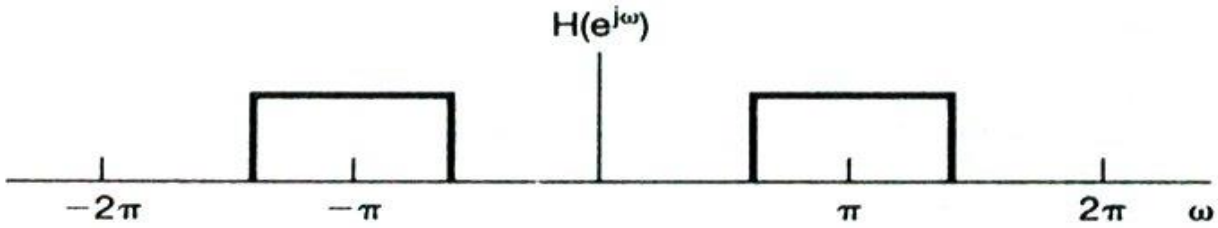
3 types of filter (Continuous time)



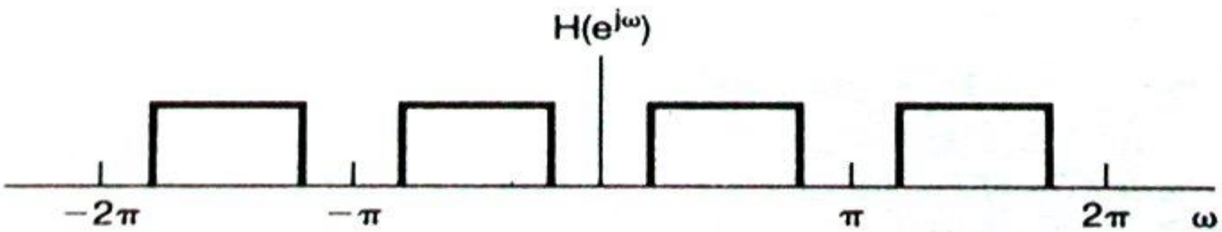
3 types of filter (Discrete time)



(a)



(b)



(c)

Example

Consider a lowpass filter with real frequency response $H(\omega)$ as shown in the figure. Which of the following properties does the filter *impulse response* have?

(a) Real-valued

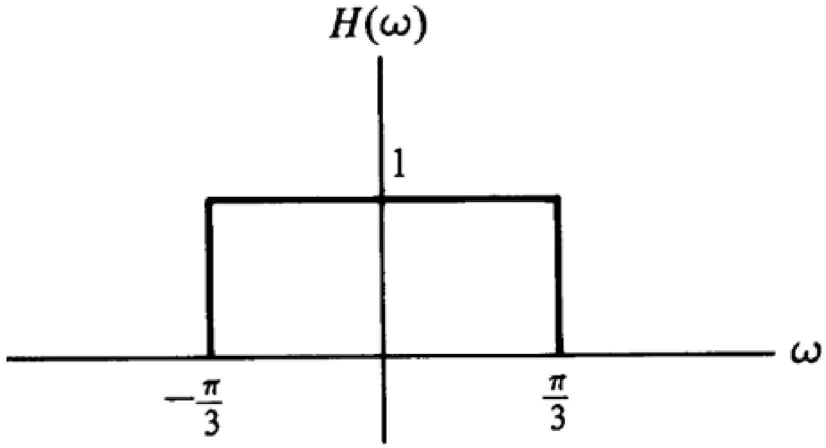
(b) Complex-valued

(c) Even

(d) Odd

(e) Causal

(f) Noncausal



Solution: The *impulse response* is real because

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\, d\omega,$$

$$h^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H^*(\omega)e^{-j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{-j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(-\omega)e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\, d\omega = h(t)$$

Where we used the fact that $H(\omega) = H^*(\omega) = H(-\omega)$

The *impulse response* is even because

$$H(-\omega) = H(\omega),$$

$$h(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\, d\omega$$

$$= h(t)$$

The impulse response is noncausal because

$$\boldsymbol{h(t) = h(-t) \neq 0}$$

3 Fourier Series Representation of Periodic Signals

Resume of Chapter 3

$$x(t) \longrightarrow \boxed{\begin{matrix} h(t) \\ H(j\omega) \end{matrix}} \longrightarrow y(t) = x(t)*h(t)$$

Key points of analysis:

Signals decomposition

A periodic signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Response synthesis

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

Similar as D-T

Homework list for Chapter 3:

1, **13**, 15, 34, 35, **43**

Upload your homework by a **PDF file**