UESTC4004
Digital Communications

Channel Coding

$G = [P | I_k]$

Systematic Linear Block codes

- A systematic (n,k) linear block code is a mapping a k-dimensional message vector to an n-dimensional code word such that part of the sequence has k message digits and remaining (n-k) are parity digits
- A systematic linear code will have a generator matrix

$$G = \begin{bmatrix} P & \vdots & I_k \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1,(n-k)} & 1 & 0 & \cdots & 0 \\ p_{21} & p_{22} & \cdots & p_{2,(n-k)} & 0 & 1 & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ p_{k1} & p_{k2} & \cdots & p_{k,(n-k)} & 0 & 0 & \cdots & 1 \end{bmatrix} \quad (6.27)$$

- Combining (6.26) and (6.27):

$$\mathrm{u}_1, \mathrm{u}_2, \ldots \mathrm{u}_n = [\mathrm{m}_1, \mathrm{m}_2, \ldots \mathrm{m}_k] \times \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1,(n-k)} & 1 & 0 & \cdots & 0 \\ p_{21} & p_{22} & \cdots & p_{2,(n-k)} & 0 & 1 & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ p_{k1} & p_{k2} & \cdots & p_{k,(n-k)} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

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行数x列数(不要写反)

解码.

$$G = \overbrace{[P \mid I_k]}^{n} \}k$$

$k \times n$

Decoding using Parity-Check Matrix

- Let H denote the parity check matrix, that will enable us to decode the received vectors
- For each (k x n) generator matrix G, there exists an (n-k)x n matrix H, such that rows of G are orthogonal to the rows of H: $GH^T=0$
- Fulfilling the orthogonality requirements:

$$H = \left[I_{n-k} \vdots P^T \right] \qquad (6.32)$$

$$H = [I_{n-k} \mid P^T]$$

$(n-k) \times n$

$n-k+k=n$

And

$$H^T = \begin{bmatrix} I_{n-k} \\ \hline P \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \\ \hline p_{11} & p_{12} & \cdots & p_{1,(n-k)} \\ p_{21} & p_{22} & \cdots & p_{2,(n-k)} \\ \vdots & & & \\ p_{k1} & p_{k2} & \cdots & p_{k,(n-k)} \end{bmatrix}}_{n-k} \qquad (6.33)$$

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Syndrome Testing

- Let r be received vector where U vector was transmitted :

$$\boxed{\mathbf{r = U + e}}$$

- The syndrome of r is defined as:

$$\boxed{\mathbf{S = rH^T}}$$

伴随式

- Combining

$$\boxed{\mathbf{S = (U + e)H^T = U\,H^T + e\,H^T >> S = e\,H^T}}$$

H

- **Requirements of the parity-check matrix**
  - No column of H can be all zeros, or else an error in the corresponding codeword position would not affect the syndrome and would be undetectable
  - All columns of H must be unique. If two columns of H were identical, errors in these two corresponding codeword positions would be indistinguishable

$\forall h_k \neq \vec{0}$

$h_i \neq h_j$

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Example

- Codeword U = 1 0 1 1 1 0 , and r = 0 0 1 1 1 0 Find S=rH^T

$$S = rH^T$$

$$= [0\ 0\ 1\ 1\ 1\ 0]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= [1,\ \underline{1+1},\ \underline{1+1}] = [1 \quad 0 \quad 0]$$

$$[1\ 0\ 0\ 0\ 0\ 0]H^T = [1\ 0\ 0]$$

Error Correction

元组

- Arranging 2n n-tuples, representing possible received vectors, in an array is called standard array. Standard array for (n,k) code is:

比特（带脚标）

$$
\begin{array}{cccccc}
U_1 & U_2 & \cdots & U_i & \cdots & U_2^k \\
e_2 & U_2 + e_2 & \cdots & U_i + e_2 & \cdots & U_2^k + e_2 \\
e_3 & U_2 + e_3 & \cdots & U_i + e_3 & \cdots & U_2^k + e_3 \\
\vdots & \vdots & & \vdots & & \\
e_j & U_2 + e_j & \cdots & U_i + e_j & \cdots & U_2^k + e_j \\
\vdots & \vdots & & \vdots & & \\
e_2^{n-k} & U_2 + e_2^{n-k} & \cdots & U_i + e_2^{n-k} & \cdots & U_2^k + e_2^{n-k}
\end{array}
\qquad (6.38)
$$

- Each row, called a coset consists of an error pattern in the first column called coset leader
- If error pattern is not a coset leader, erroneous decoding will result

coset leader

Locating the Error Pattern

- Example of a standard array for a (6,3) code is shown:

000000	110100	011010	101110	101001	011101	110011	000111
000001	110101	011011	101111	101000	011100	110010	000110
000010	110110	011000	101100	101011	011111	110001	000101
000100	110000	011110	101010	101101	011001	110111	000011
001000	111100	010010	100110	100001	010101	111011	001111
010000	100100	001010	111110	111001	001101	100011	010111
100000	010100	111010	001110	001001	111101	010011	100111
010001	100101	001011	111111	111000	001100	100010	010110

The syndrome of a Coset

- Coset is a short for “a set of numbers having a common feature”
- If e_j is the coset leader then U_i+e_j is an n-tuple in this coset.
- Syndrome of this n-tuple is:

$$\mathbf{S = (U_i + e_j)\, H^T = e_j\, H^T}$$

- The syndrome must be unique to estimate the error pattern

Error Correction Decoding

$e_j H^T$

- Calculate the syndrome of R using $S=rH^T$
- Locate the coset leader (error pattern) e_j, whose syndrome equals $\mathbf{rH}^T$
- This error pattern is assumed to be the corruption caused by the channel
- The corrected received vector, or code word, is identified as $U=r+e_j$. We retrieve the valid codeword by subtracting(adding) the identified error

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- The results are:

pattern
error 的伴随式

Error Pattern	Syndrome
000000	000
000001	101
000010	011
000100	110
001000	001
010000	010
100000	100
010001	111

Table: Syndrome lookup Table

$rH^T = \hat{e}H^T$

Error Correction Example

- Error pattern is an estimate of error, the decoder addes the estimated error to received signal to obtain an estimate of transmitted code word as:

$$\hat{U} = r + \hat{e} = (U + e) + \hat{e} = U + (e + \hat{e})$$

- Example: let U=101110, and r=001110, then show how the decoder can correct the error using syndrome look-up table

$$S = [0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0]H^T = [1 \quad 0 \quad 0]$$

rH^T

$$\text{estimated error}:$$

$$\hat{e} = 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$$

if $\hat{e}H^T = rH^T$

$$\text{The corrected vector is estimated by}:$$

$$\hat{U} = r + \hat{e}$$

$$= 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 + 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$$

$$= 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0$$

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Error Detection and Correcting Capability

Weight and Distance of Binary Vector

能力判别的指标.

- Hamming distance between two codewords is the number of elements in which they differ
- Hamming weight is the number of nonzero elements

汉明距离：2个码的区别

汉明权重：1的个数

Example:

$$U = 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1$$

$$V = 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0$$

$$w(U) = 5$$

$$d(U,V) = w(U+V) = 6$$

$$d(u,v) = w(u+v)$$

$$u+v = 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1$$ 异或

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dmin

Minimum Distance of a Linear Code: The minimum distance among all the distances between each pair of codes in the code set

校正能力

- **The error-correcting capability t of a code:** the maximum number of guaranteed correctable errors per codeword defined in terms of d_{min}

最多能被

$$\boxed{t = \left\lfloor \frac{d_{min}-1}{2} \right\rfloor}$$

correct 的个数 t

- Error detecting capability defined in terms of d_{min}

$$\boxed{e = d_{\min} - 1}$$

最多能被 测出来的error数 e

Message Vector	Codeword
000	000000
100	110100
010	011010
110	101110
001	101001
101	011101
011	110011
111	000111

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Question

- Consider a (7, 4) code whose generator matrix is

$G = [P \mid I_k]$

$$G = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$GH^T = 0 \qquad H = [I_{n-k} \mid P^T]$

1) Find **H**, the parity check matrix of the code.
2) What is the code rate and redundancy rate for the given coder?
3) Find the code word corresponding to message 1110.
4) Compute the syndrome for the received vector 0001110.

1) $H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$

2). $\text{code rate} = \dfrac{k}{n} = \dfrac{4}{7} \qquad \text{rerate} = \dfrac{n-k}{k} = \dfrac{3}{4} = 0.75$

3). $u = [1110] \begin{bmatrix} V1 \\ V2 \\ V3 \\ V4 \end{bmatrix} = V_1 + V_2 + V_3 = 0101110$

4). $S = rH^T = [0001110] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = [0 \; 1 \; 0]$

CONVOLUTIONAL ENCODING

- A convolutional code is described by three integers, n, k, and K where the ratio k/n is called the rate of the code
- The integer K is constraint length; it represents number of k-tuple stages in the encoding shift register.
- Encoder has memory—the n-tuple emitted by the convolutional encoding procedure is not only a function of an input k-tuple, but is also a function of the previous K-1 input k-tuples

Block Diagram of a Typical Communication Link

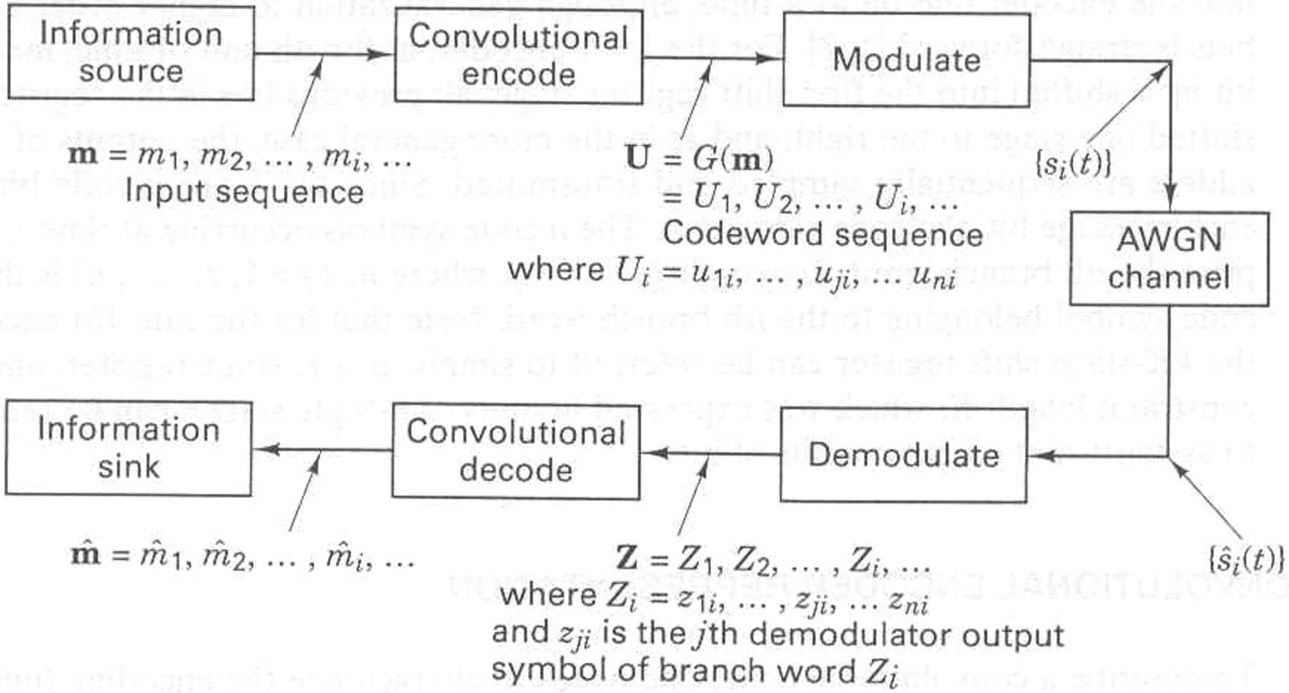

Figure: Encode/decode and modulate/demodulate portions of a communication link

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Connection Representation

length

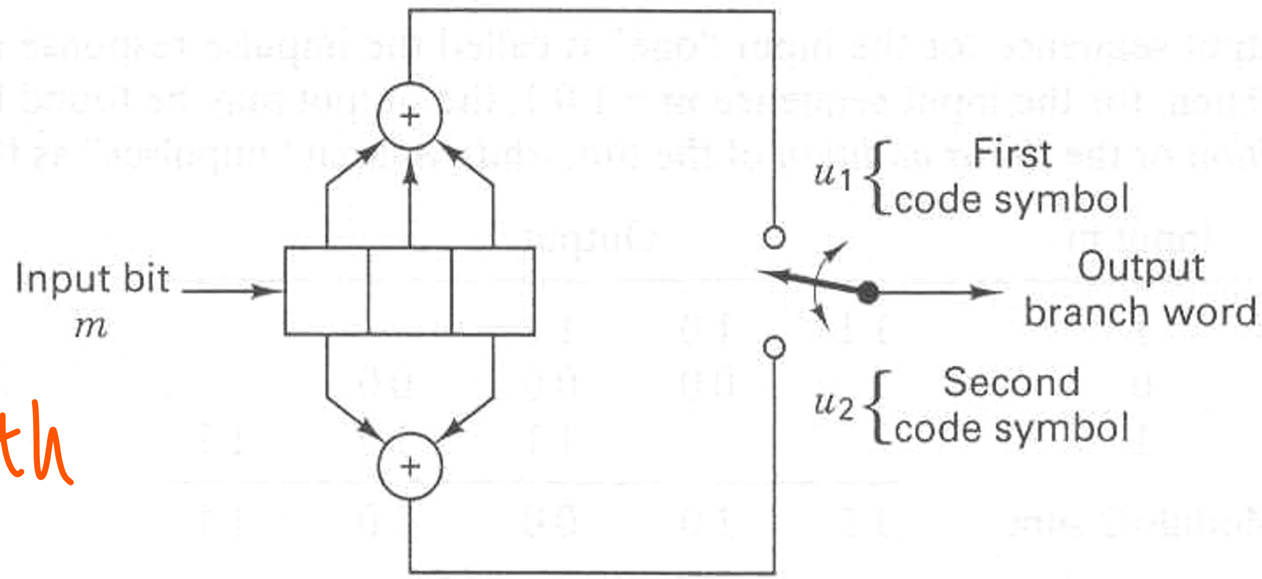

Figure: Convolutional Encoder (rate ½, K=3)

$\frac{k}{n} = \frac{1}{2}$

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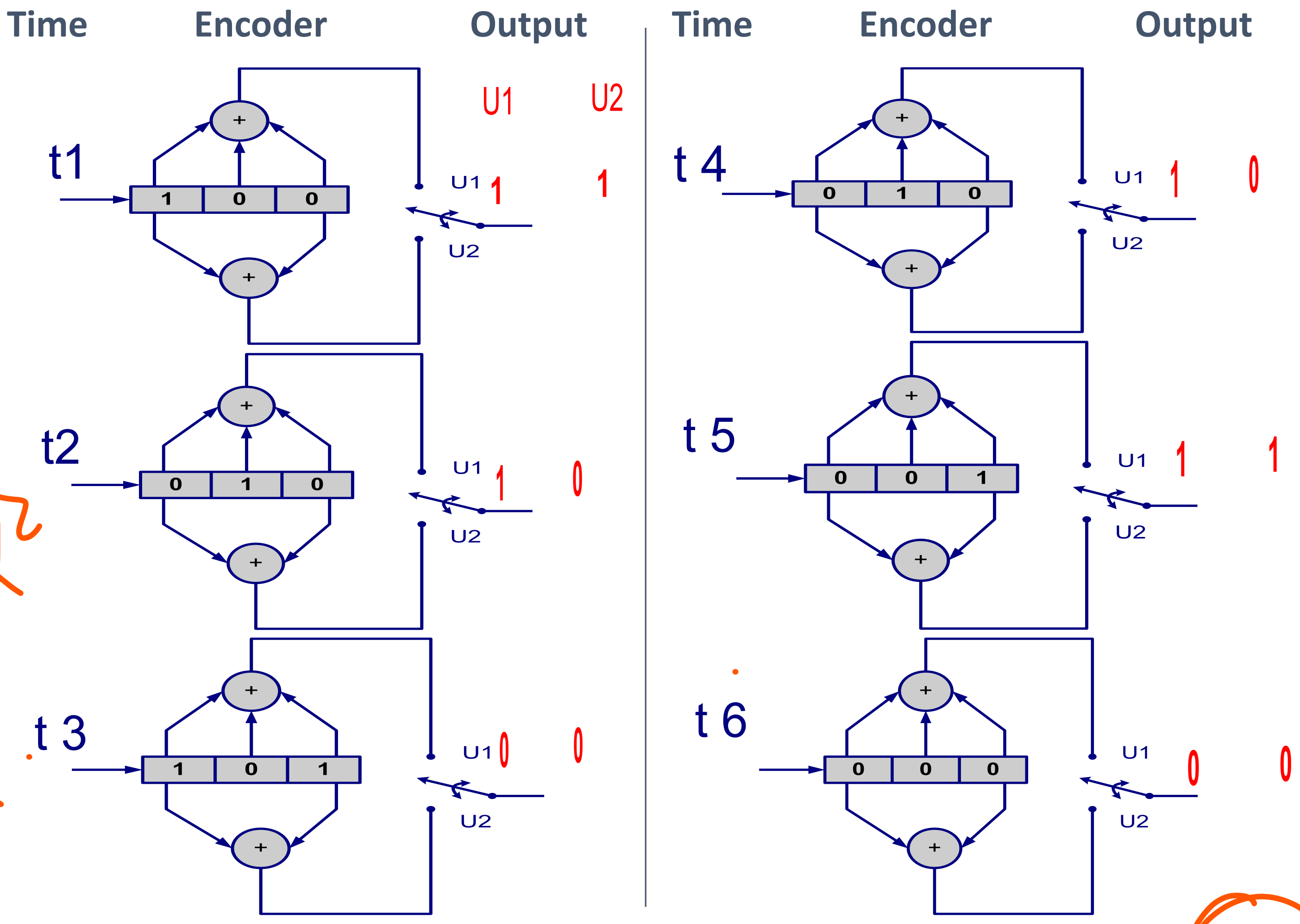
g_1: 3个

g_2: 2个

$\Downarrow$

$g_1: 1+x+x^2$

$g_2: 1+x^2$



Output Sequence: 11 10 00 10 11 00 for input 101

t_0 000 → t_1 100 → t_2 010 → t_3 101 → . . .

进栈

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Polynomial Representation

- Convolutional encoder maybe represented with a set of n generator polynomials, one for each modulo-2 adders
- Continuing with the same example, we can write the generator polynomial for upper connections $g_1(X)$ and $g_2(X)$ for lower connections:

$$g_1(X) = 1 + X + X^2$$

$$g_2(X) = 1 + X^2$$

- U(X) is the output sequence

$$\boxed{U(X) = m(X)g_1(X) \quad \text{interlaced} \quad \text{with} \quad m(X)g_2(X)}$$

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X的系数

Where m= 101, encoding can be done as:

$$m(X)g_1(X) = (1+X^2)(1+X+X^2) = 1+X+X^3+X^4$$

$$m(x)g_2(X) = (1+X^2)(1+X^2) = 1+X^4$$

$$m(X)g_1(X) = 1+ \quad X + 0X^2 + \quad X^3 + X^4$$

$$m(X)g_2(X) = 1 + 0X + 0X^2 + 0X^3 + X^4$$

$$U(X) = (1,1) + (1,0)X + (0,0)X^2 + (1,0)X^3 + (1,1)X^4$$

$$U = 1\ 1 \quad 1\ 0 \quad 0\ 0 \quad 1\ 0 \quad 1\ 1$$

系数√