Signals and Systems

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Open Question Points: 1

Setting

{Exercise #from Chapter 3}

Let $x[n]$ be a periodic signal with period $N = 8$ and Fourier series coefficients $a_k = -a_{k-4}$. A signal $y[n] = \left(\frac{1+(-1)^n}{2}\right) x[n-1]$ With period $N = 8$ is generated. Denoting the Fourier series coefficients of $y[n]$ by b_k, find a function $f[k]$ such that $b_k = f[k] a_k$

Open Question is only supported on Version 2.0 or newer.

Answer

Solution:

We have

$$\mathbf{e}^{\boldsymbol{j4(2\pi/8)}n}x[n] = \mathbf{e}^{j\pi n}x[n]$$

$$= (-1)^n x[n] \overset{\boldsymbol{FS}}{\leftrightarrow} a_{k-4}$$

and therefore, $(-1)^{n+\mathbf{1}}x[n] \overset{\boldsymbol{FS}}{\leftrightarrow} -a_{k-4}$

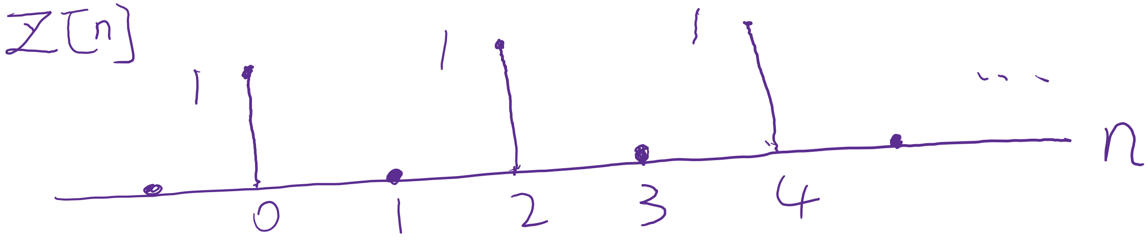
If $\boldsymbol{a_k} = -\boldsymbol{a_{k-4}}$**, then** $x[\mathbf{0}] = x[\pm\mathbf{2}] =$

$x[\pm\mathbf{4}]$**=…=0.**

Now, note that in the signal $\boldsymbol{p[n] = x[n-1]}$,

$\boldsymbol{p[\pm 1] = p[\pm 3] = \cdots = 0}$.

Now let us plot the signal $\boldsymbol{z[n] = \left(\frac{1+(-1)^n}{2}\right)}$,



so $\boldsymbol{z[n] = 0, for\ n = odd}$,

$\boldsymbol{z[n] = 1, for\ n = even}$.

clearly, the signal $y[n] = z[n]p[n] = p[n]$,

because $p[n]$ is zero whenever $z[n]$ is zero.

Therefore, $y[n] = x[n-1]$.

The FS coefficients are $b_k = a_k \, \mathrm{e}^{-jk(2\pi/8)}$

So, $f[k] = \mathrm{e}^{-jk(2\pi/8)}$

Notice:

In this and the following chapters

Notation: $\boldsymbol{\tilde{x}(t)} \rightarrow \boldsymbol{periodic}\ signal$

$\boldsymbol{x(t)} \rightarrow \boldsymbol{aperiodic}\ signal$

4. The Continuous time Fourier Transform

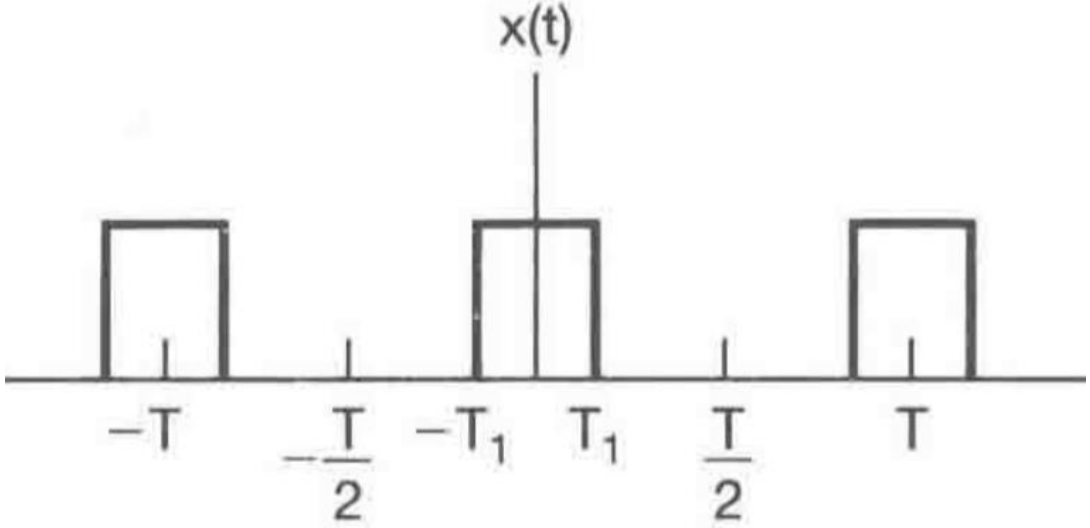
4.1 Representation of Aperiodic signals:

The Continuous time Fourier Transform

4.1.1 Development of the Fourier transform representation of the continuous time Fourier transform

4 The continuous time Fourier transform

(1) Example (From Fourier series to Fourier transform)



$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

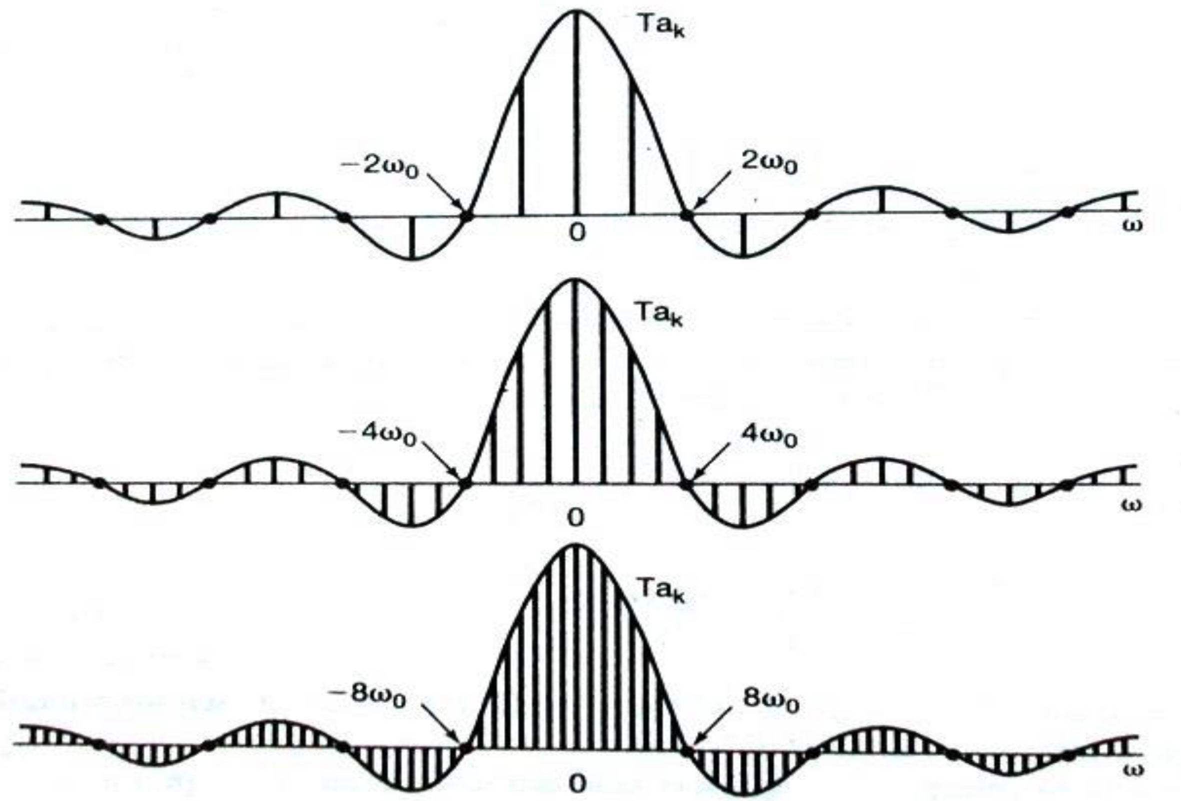
4 The continuous time Fourier transform

(1) Example (From Fourier series to Fourier transform)

$$T = 4T_1$$

$$T = 8T_1$$

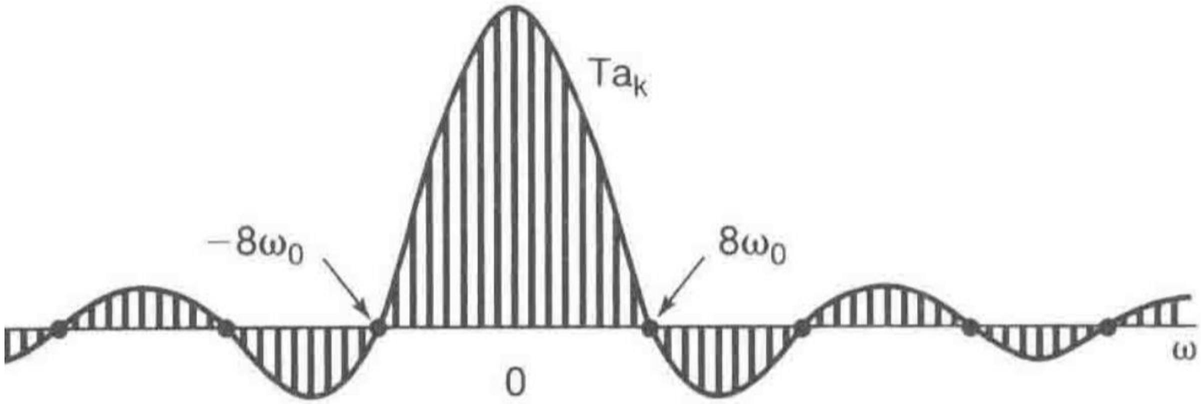
$$T = 16T_1$$

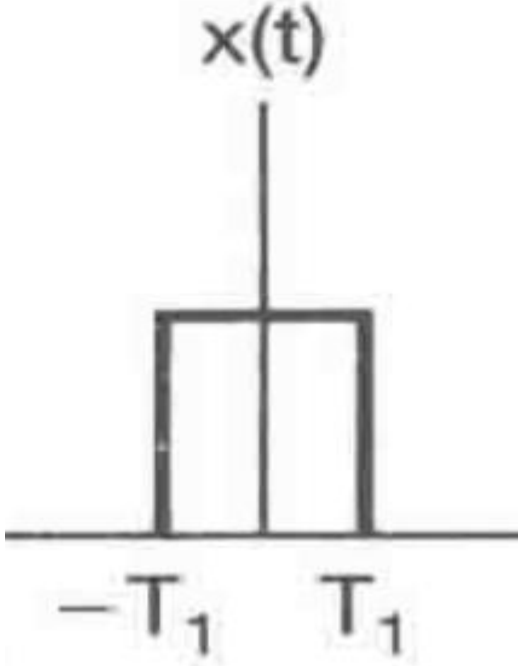


$$Ta_k = \frac{2\sin(\omega T_1)}{\omega}\Big|_{\omega = k\omega_0}$$

4 The continuous time Fourier transform

(1) Example (From Fourier series to Fourier transform)

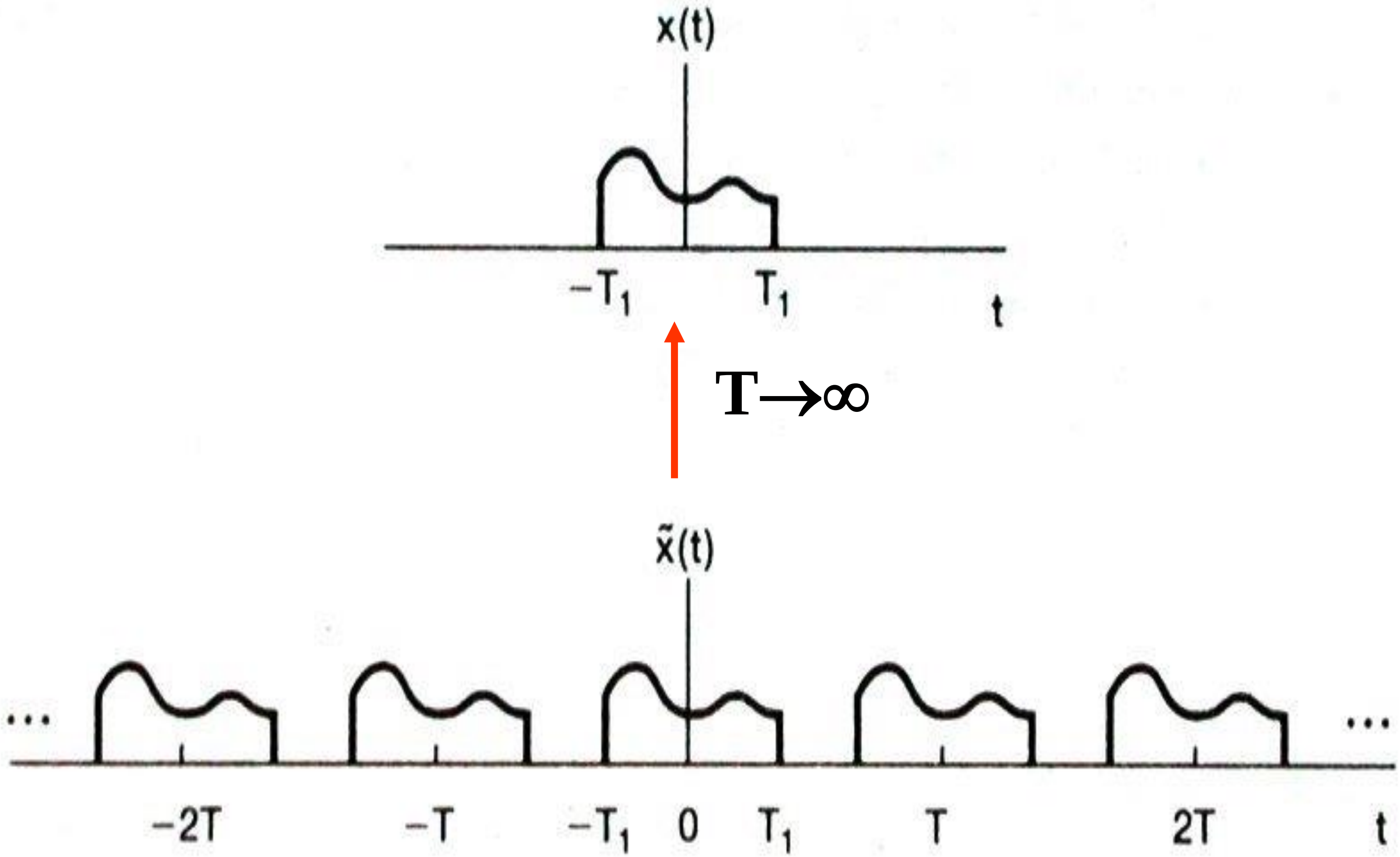




$$Ta_k = \frac{2\sin(\omega T_1)}{\omega}\Big|_{\omega = k\omega_0}$$

$T \to \infty, ?$

4 The continuous time Fourier transform



4 The continuous time Fourier transform

(2) Fourier transform representation of Aperiodic signal

For periodic signal :

$$\tilde{x}(t)\begin{cases}\tilde{x}(t)=\sum_{k=-\infty}^{+\infty}a_k e^{jk\omega_0 t}\\ a_k=\frac{1}{T}\int_T \tilde{x}(t)e^{-jk\omega_0 t}dt\end{cases}$$

For aperiodic signal x(t) :

$$x(t)=\lim_{T\to\infty}\tilde{x}(t)\ ,or\quad \tilde{x}(t)\xrightarrow{T\to\infty}x(t)$$

4 The continuous time Fourier transform

When T→∞ ,

$$\tilde{x}(t) \xrightarrow{T\to\infty} x(t)$$

$$\omega_0 = \frac{2\pi}{T} \xrightarrow{T\to\infty} d\omega$$

$$k\omega_0 \xrightarrow{T\to\infty} \omega$$

So

$$\boldsymbol{a_k T = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt = X(j\omega)}$$

The development is gained as following:

4 The continuous time Fourier transform

$$
\begin{aligned}
x(t) &= \lim_{T\to\infty} \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\
&= \lim_{T\to\infty} \sum_{k=-\infty}^{+\infty} \frac{X(jk\omega_0)}{T} e^{jk\omega_0 t} \\
&= \lim_{\omega_0\to 0} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \frac{\omega_0}{2\pi} \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega
\end{aligned}
$$

4 The continuous time Fourier transform

Fourier transform:

$$\begin{cases} X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \\ x(t) = \dfrac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \end{cases}$$

or $$x(t) \xleftrightarrow{F} X(j\omega)$$

Relation between Fourier series and Fourier transform:

$$\begin{cases} a_k = \dfrac{1}{T} X(j\omega) \big|_{\omega = k\omega_0} \quad \textit{(Periodic signal)} \\ X(j\omega) = T \cdot a_k \big|_{k\omega_0 = \omega} \quad \textit{(Aperiodic signal)} \end{cases}$$

4 The continuous time Fourier transform

In other way, Specifically

$$\tilde{x}(t) \xleftrightarrow{FS} a_k$$

$$a_k = \frac{1}{T}\int_T \tilde{x}(t)e^{-jk\omega_0 t}dt$$

$x(t)$ **is one period of** $\tilde{x}(t)$ **over** $s \le t \le s+T$

$$a_k = \frac{1}{T}\int_T \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_s^{s+T} x(t)e^{-jk\omega_0 t}dt$$

extend to *all t*, $\quad = \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}dt$

Comparing the following two equations:

$$a_k = \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}dt$$

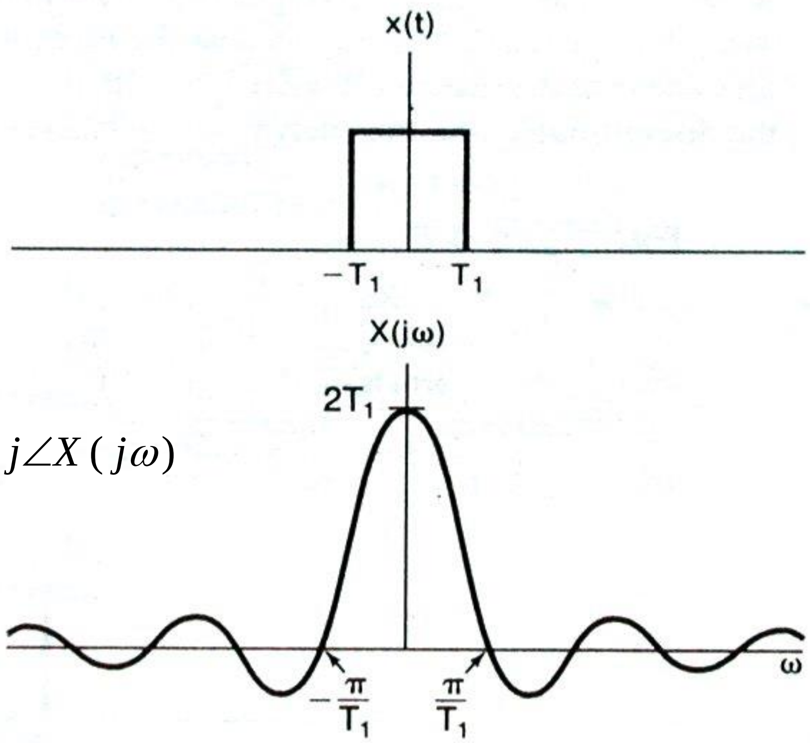
$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

Conclusion:

$$a_k = \frac{1}{T}X(j\omega)|_{\omega=k\omega_0}$$

4 The continuous time Fourier transform

$$x(t) \xleftrightarrow{F} X(j\omega)$$



In polar form:

$$X(j\omega) = |X(j\omega)| e^{j\varphi(\omega)} = |X(j\omega)| e^{j\angle X(j\omega)}$$

In rectangular form:

$$X(j\omega) = R_e[X(j\omega)] + jI_m[X(j\omega)]$$

4 The continuous time Fourier transform

Now we have $X(j\omega)$, we can derive $\hat{x}(t)$ from it

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

Question:: when is $\hat{x}(t)$ a valid representation of the original signal $x(t)$?

4.1.2 Convergence of Fourier transform

Dirichlet conditions:

(1) $x(t)$ **is absolutely integrable.**

$$\int_{-\infty}^{+\infty} |x(t)|dt < \infty$$

(2) $x(t)$ **have a finite number of maxima and minima within any finite interval.**

(3) $x(t)$ **have a finite number of discontinuity within any finite interval. Furthermore, each of these discontinuities must be finite.**

4.1.2 Convergence of Fourier transform

Dirichlet conditions satisfied

→ signal has Fourier transform

However,

Some periodic signals, neither absolutely integrable, nor square integrable, however, it still has FT

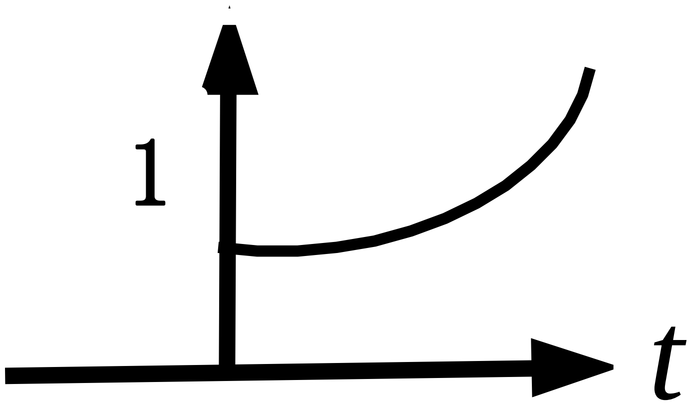
4 The continuous time Fourier transform

For example,

$$x(t) = e^{t}u(t)$$

It has not FT.

$$x(t) = e^{t}u(t)$$



Note: Dirichlet conditions can be extanded to the Singularity Functions.

4 The continuous time Fourier transform

Key point Review:
Fourier Transform pair

$$\begin{cases} X(j\omega)=\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt \\ x(t)=\dfrac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega \end{cases}$$

or $$x(t)\xleftrightarrow{F}X(j\omega)$$

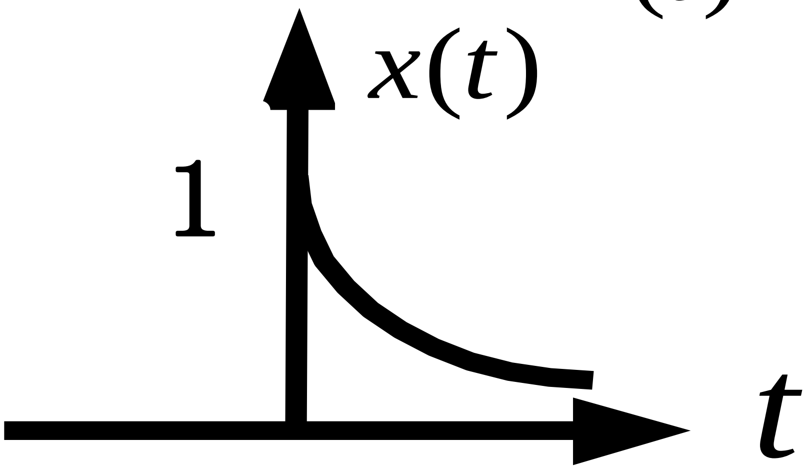
Relation between Fourier series and Fourier transform:

$$\begin{cases} a_k=\dfrac{1}{T}X(j\omega)\big|_{\omega=k\omega_0} \quad \text{(Periodic signal)} \\ X(j\omega)=T\cdot a_k\big|_{k\omega_0=\omega} \quad \text{(Aperiodic signal)} \end{cases}$$

4.1.3 Examples of Continuous time Fourier Transform

Example 4.1

$$x(t) = e^{-at}u(t) = \begin{cases} e^{-at} & (t \geq 0) \\ 0 & (t < 0) \end{cases}$$



$$(a > 0)$$

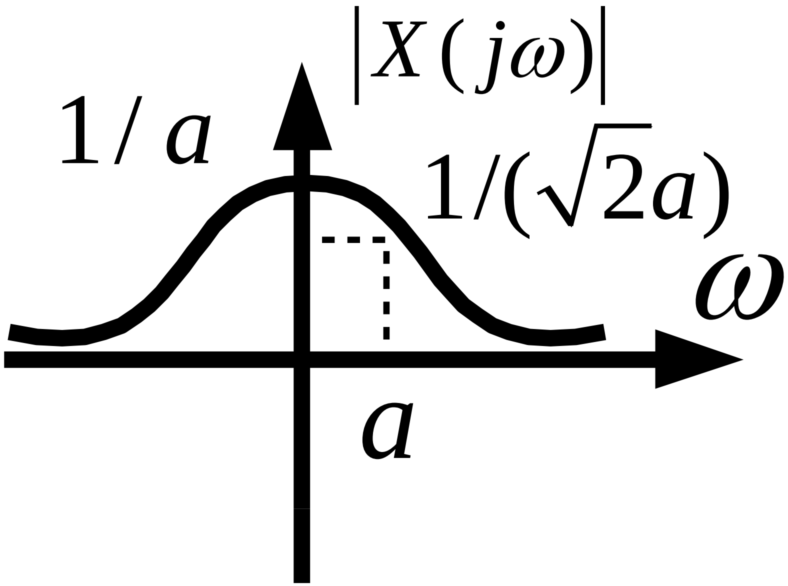
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \frac{1}{a+j\omega}, (a > 0)$$

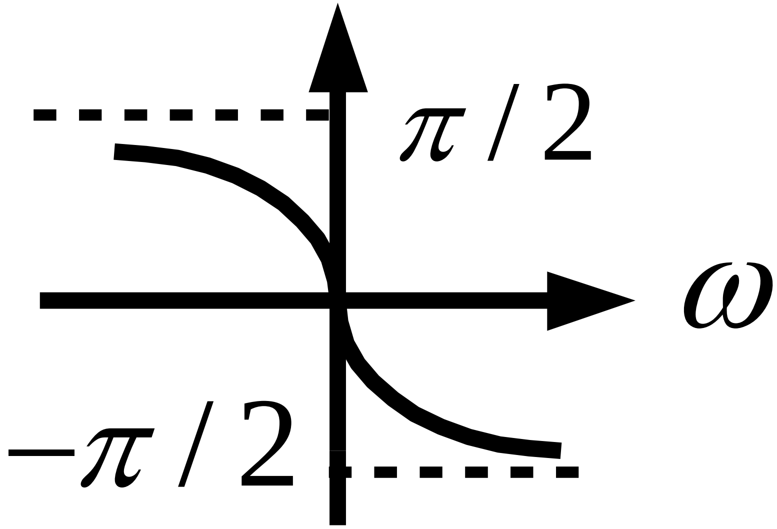
In polar form:

$$X(j\omega) = |X(j\omega)|e^{j\varphi(\omega)} = |X(j\omega)|e^{j\measuredangle X(j\omega)}$$

4 The continuous time Fourier transform

$$\left|X(j\omega)\right| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \varphi(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$





$$\measuredangle X(j\omega) = \varphi(\omega)$$

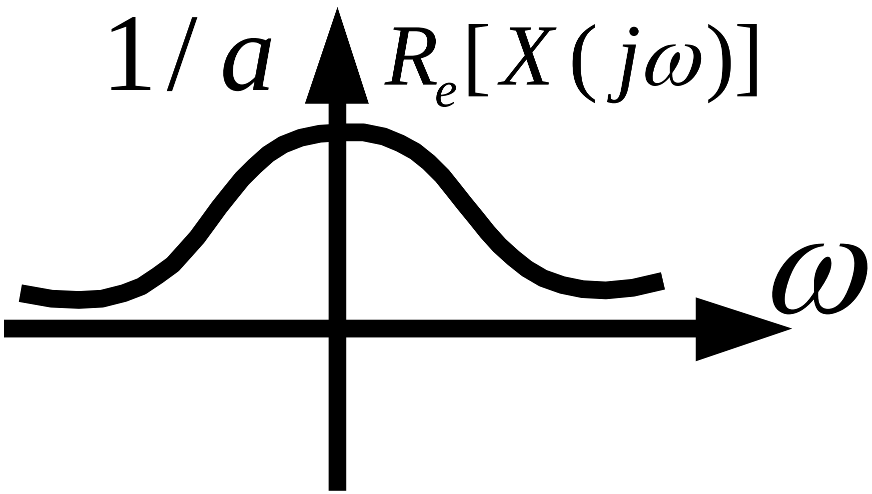
magnitude *phase*

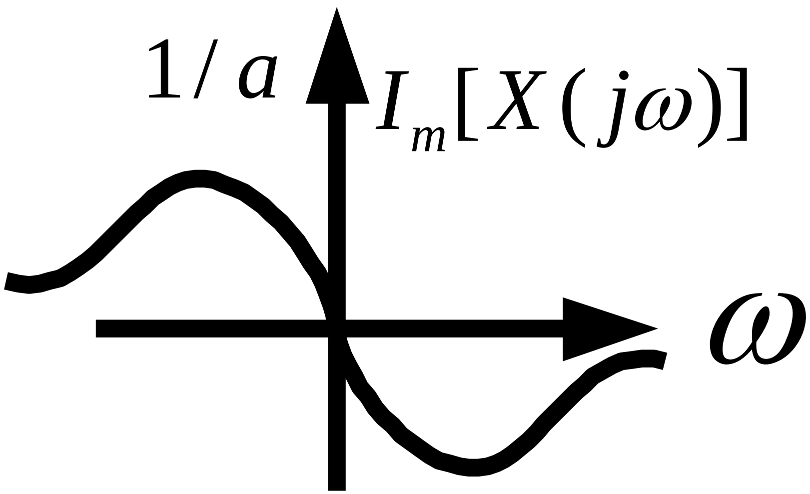
4 The continuous time Fourier transform

In rectangular form:

$$X(j\omega)=\frac{1}{a+j\omega}=\frac{a}{a^2+\omega^2}+\frac{-j\omega}{a^2+\omega^2}$$

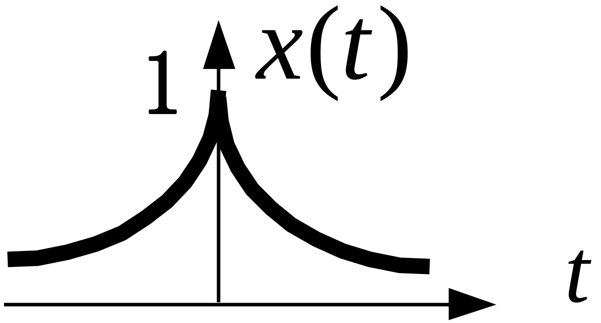
$$X(j\omega)=R_e[X(j\omega)]+jI_m[X(j\omega)]$$





4 The continuous time Fourier transform

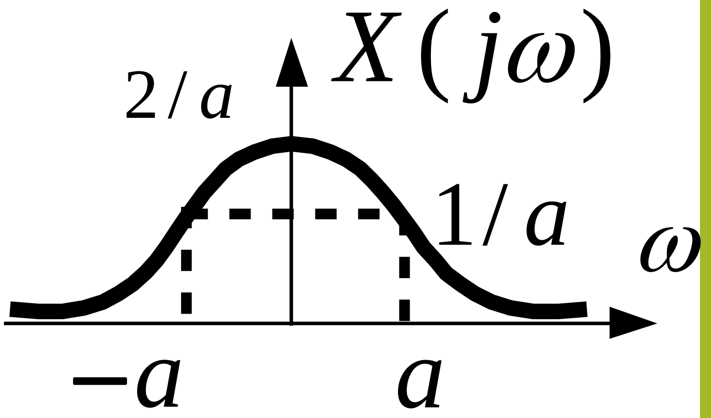
Example 4.2 $\quad x(t) = e^{-a|t|} \quad , a > 0$



$$X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}$$

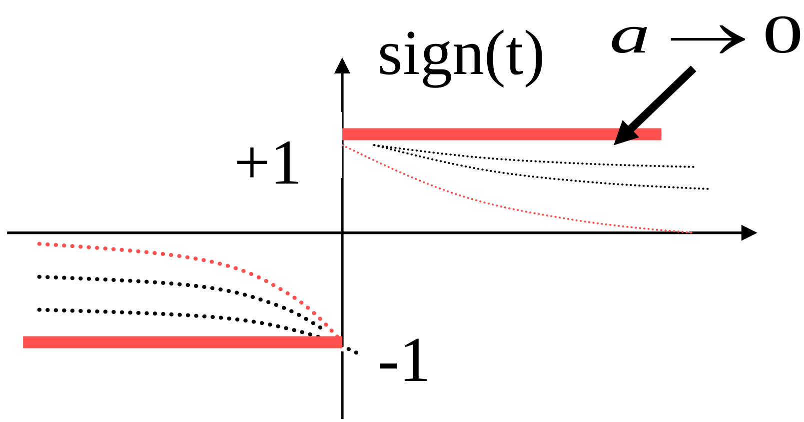


$$\therefore X(j\omega) = \frac{2a}{a^2 + \omega^2}$$

Note:

$$x(t) = sign(t) \cdot e^{-a|t|}$$

$$f(t) = \mathrm{sign}(t) = \begin{cases} +1 & (t > 0) \\ -1 & (t < 0) \end{cases}$$

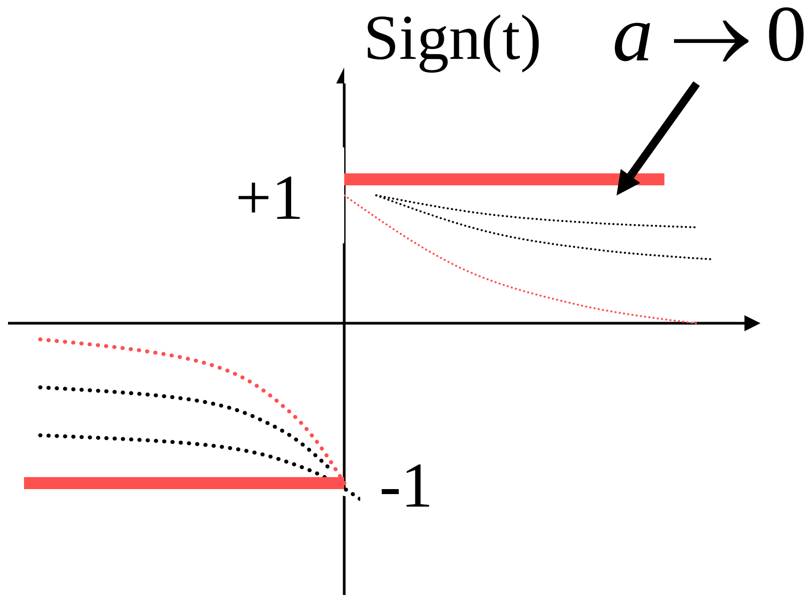


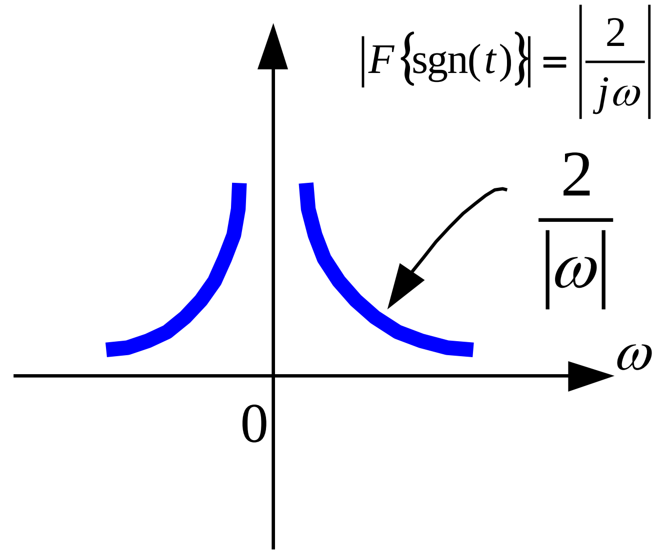
$$f(t) = \lim_{a \to 0} x(t) = \lim_{a \to 0} [sign(t) \cdot e^{-a|t|}]$$

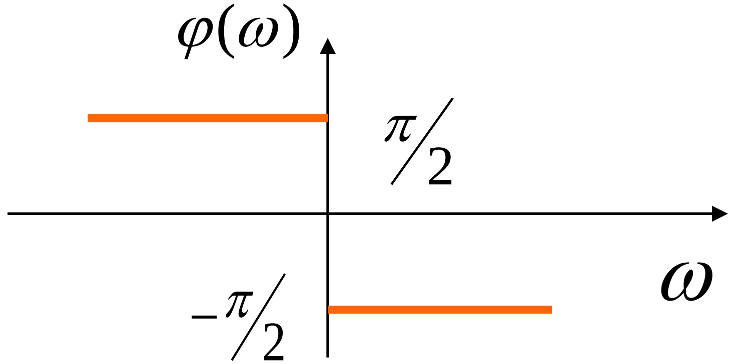
$$F(\omega) = \lim_{a \to 0} X(j\omega) = \lim_{a \to 0} \frac{-2j\omega}{a^2 + \omega^2} = \frac{2}{j\omega}$$

$$|F(\omega)| = \frac{2}{\omega} \qquad \varphi(\omega) = \begin{cases} -\frac{\pi}{2} & (\omega > 0) \\ +\frac{\pi}{2} & (\omega < 0) \end{cases}$$

4 The continuous time Fourier transform







$$\operatorname{sign}(t) = \begin{cases} +1 & (t > 0) \\ -1 & (t < 0) \end{cases} \quad \xleftrightarrow{F} \quad \frac{2}{j\omega}$$

4 The continuous time Fourier transform

Example 4.4

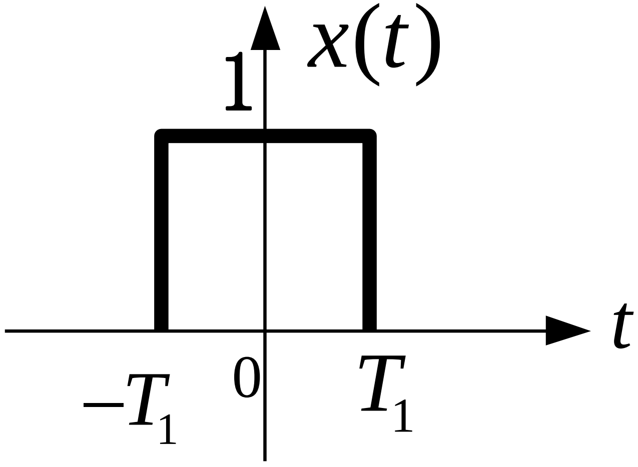
$$x(t)=\begin{cases}1, & |t|<T_1\\ 0, & |t|>T_1\end{cases}$$

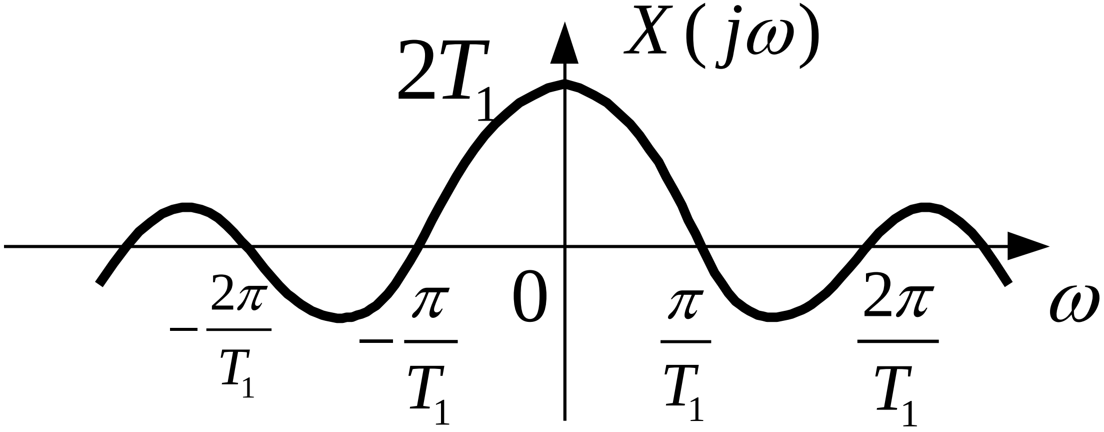
$$X(j\omega)=\int_{-T_1}^{T_1} e^{-j\omega t}\,dt=\frac{2\sin(\omega T_1)}{\omega}$$

$$X(j\omega)=2T_1\left(\frac{\sin(\omega T_1)}{\omega T_1}\right)=2T_1 Sa(\omega T_1)$$

$$Sa(\theta)=\frac{\sin(\theta)}{\theta}$$

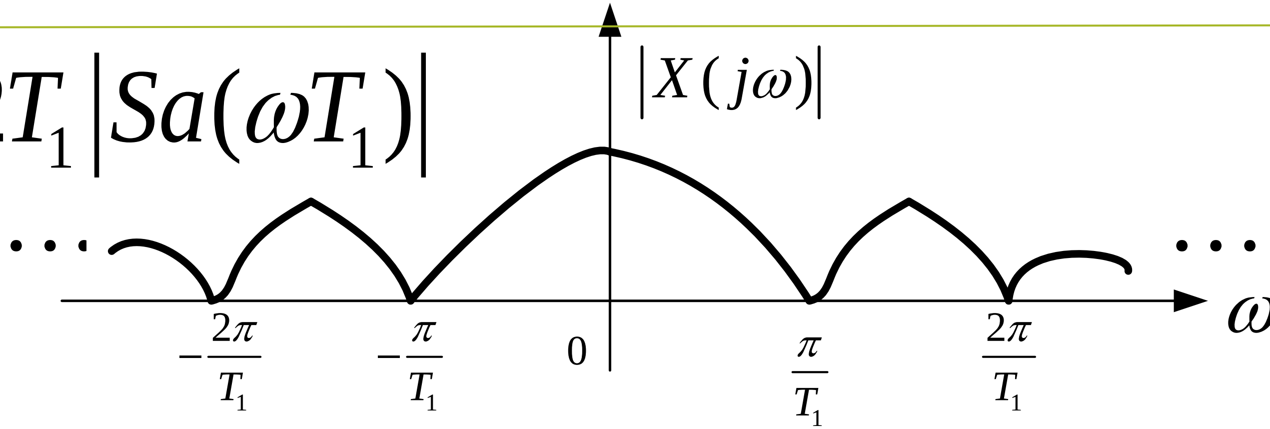
Sample function

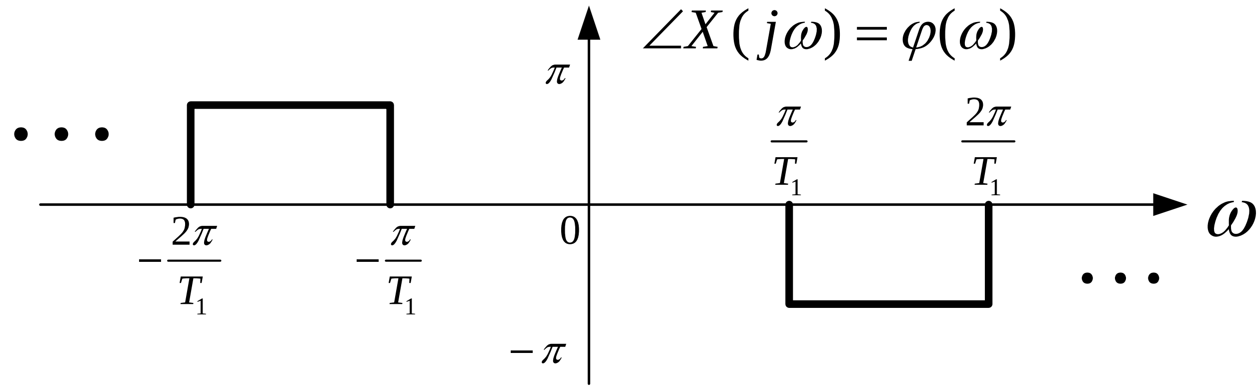




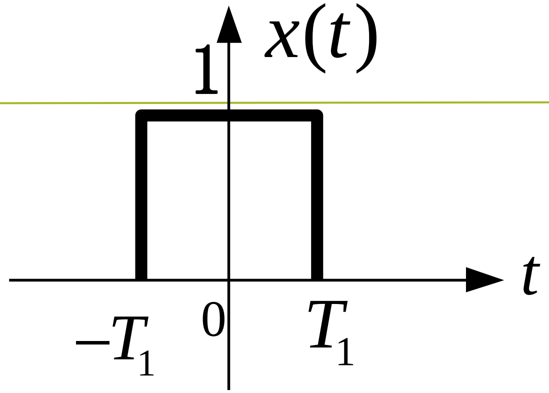
4 The continuous time Fourier transform

$$\left|X(j\omega)\right| = 2T_1\left|Sa(\omega T_1)\right|$$





$$\varphi(\omega) = \begin{cases} 0, & \left(\frac{2n\pi}{T_1} < \left|\omega\right| < \frac{(2n+1)\pi}{T_1}\right) \\ \mp\pi, & \left(\frac{(2n+1)\pi}{T_1} < \left|\omega\right| < \frac{2(n+1)\pi}{T_1}\right) \end{cases}$$



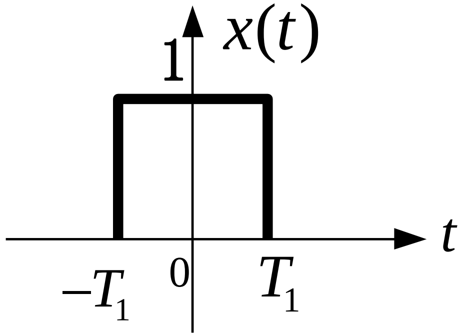
Inverse Fourier transform

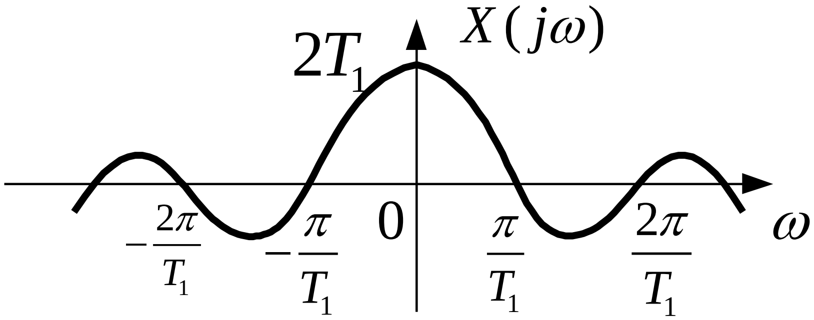
$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\,\frac{sin\omega T_1}{\omega} e^{j\omega t} d\omega$$

As $\omega \to \infty$, this signal converges to $x(t)$ everywhere, except at the discontinuities.

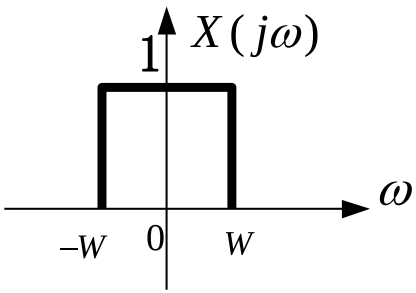
$t = \pm T_1$, $\hat{x}(t)$ converges to ½ , which is the average of the values of $x(t)$ on both sides of the discontinuity.

4 The continuous time Fourier transform





Example 4.5



$$X(j\omega)=\begin{cases}1, & |\omega|<W\\ 0, & |\omega|>W\end{cases}$$

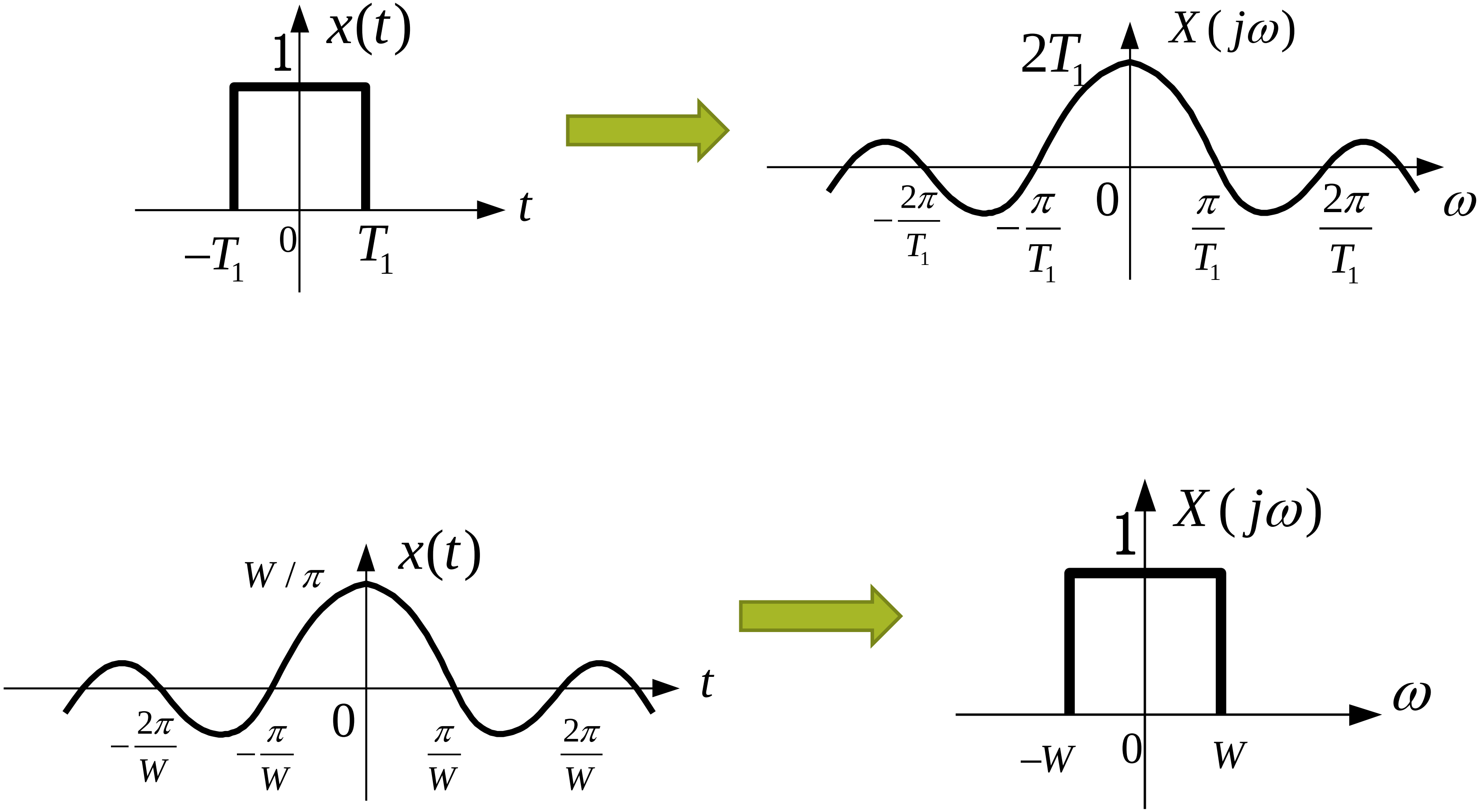
$$x(t)=\frac{1}{2\pi}\int_{-W}^{W}e^{j\omega t}d\omega=\frac{\sin(Wt)}{\pi t}$$

$$\frac{\sin(\pi\theta)}{\pi\theta}=\sin c(\theta)$$

$$\frac{\sin(Wt)}{\pi t}=\frac{W}{\pi}\sin c\left(\frac{Wt}{\pi}\right)$$

------------Sinc function

4 The continuous time Fourier transform



4 The continuous time Fourier transform

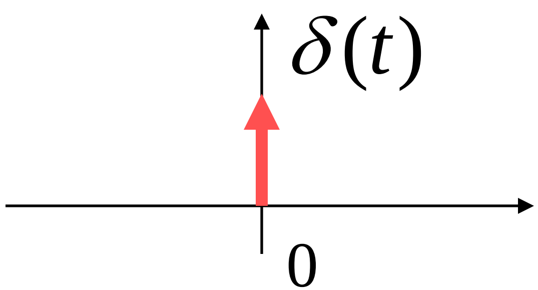
Example 4.3

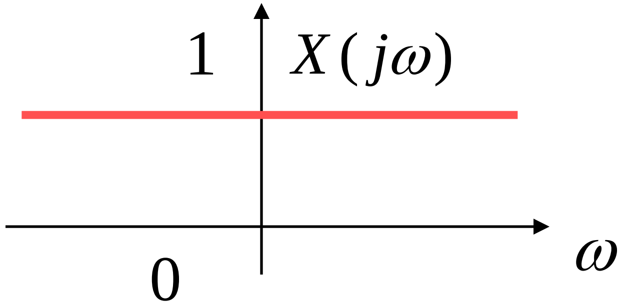
$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$x(t) = \delta(t) \xleftrightarrow{F} X(j\omega) = 1$$

$$\therefore \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 \cdot e^{j\omega t} d\omega$$



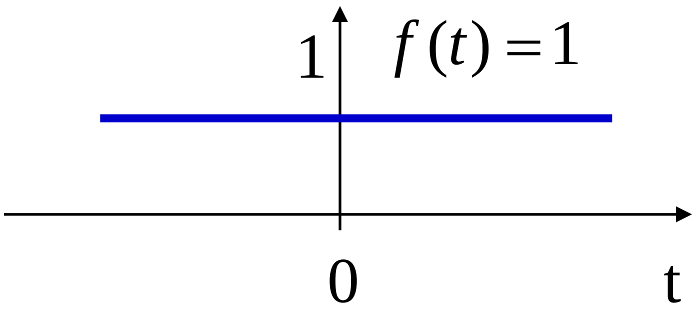


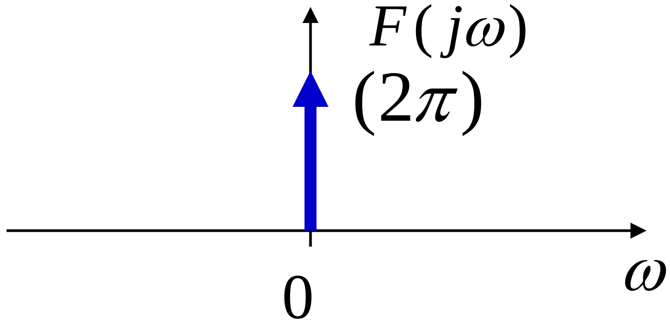
4 The continuous time Fourier transform

Vise versa $f(t)=1 \xleftrightarrow{F} F(j\omega)=2\pi\delta(\omega)$

$$FT^{-1}[2\pi\delta(\omega)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}2\pi\delta(\omega)e^{j\omega t}d\omega=1$$

$$\therefore\ \delta(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}1\cdot e^{-j\omega t}dt$$





$$\delta(y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\pm jy\xi}d\xi$$

4.2 The Fourier Transform for Periodic Signal

Periodic signal:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$e^{jk\omega_0 t} \xleftrightarrow{F} 2\pi\delta(\omega - k\omega_0)$$

thus

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \xleftrightarrow{F} X(j\omega) = \sum_{k=-\infty}^{+\infty} a_k 2\pi\delta(\omega - k\omega_0)$$

4 The continuous time Fourier transform

Example (1)

$$e^{j\omega_0 t} \xleftrightarrow{F} 2\pi\delta(\omega-\omega_0) \qquad (4.21)$$

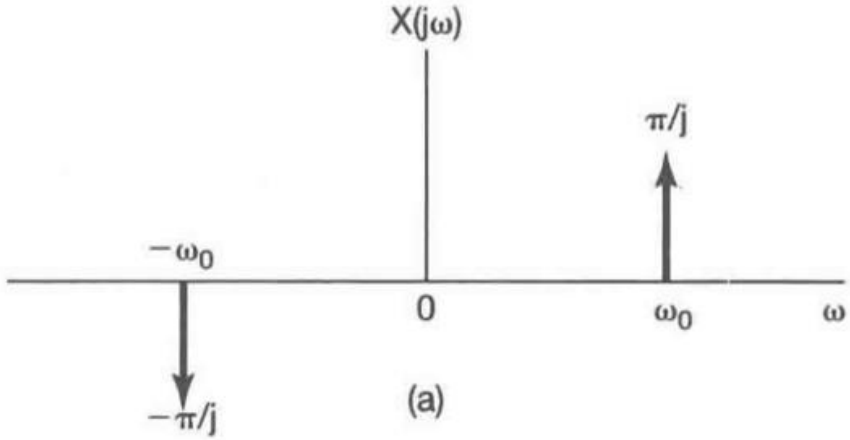
$$Solution: x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}d\omega$$

$$e^{j\omega_0 t}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}2\pi\delta(\omega-\omega_0)e^{j\omega t}d\omega$$

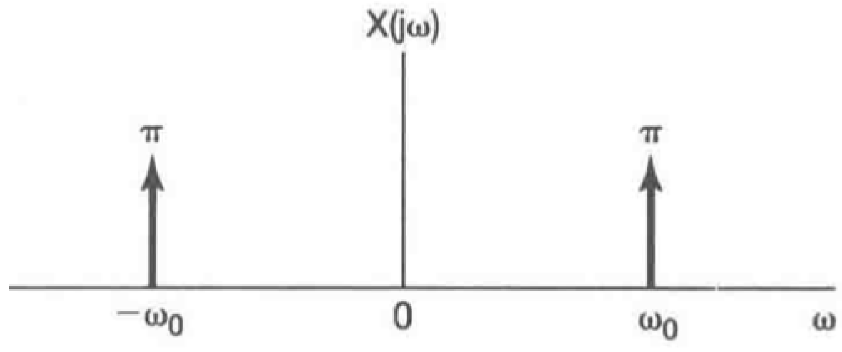
4 The continuous time Fourier transform

Example 4.7

$$x(t) = \sin\omega_0 t \stackrel{F}{\longleftrightarrow} X(j\omega) = j\pi\delta(\omega+\omega_0) - j\pi\delta(\omega-\omega_0)$$



$$x(t) = \cos\omega_0 t \stackrel{F}{\longleftrightarrow} X(j\omega) = \pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)$$



4 The continuous time Fourier transform

Compare these results from two examples

$$x(t) = \sin\omega_0 t \xleftrightarrow{F} X(j\omega) = j\pi\delta(\omega+\omega_0) - j\pi\delta(\omega-\omega_0)$$

$$x(t) = \cos\omega_0 t \xleftrightarrow{F} X(j\omega) = \pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)$$

$$e^{j\omega_0 t} \xleftrightarrow{F} 2\pi\delta(\omega-\omega_0)$$

Example 4.6 The FS of periodic signal

$$x(t) = \sum_{k=-\infty}^{+\infty} \frac{\sin k\,\omega_0 T_1}{k\pi} e^{jk\omega_0 t}$$

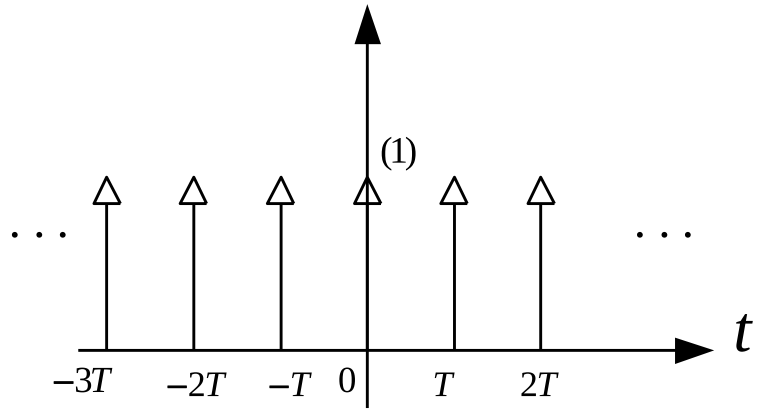
The FT of $x(t)$ is

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$$

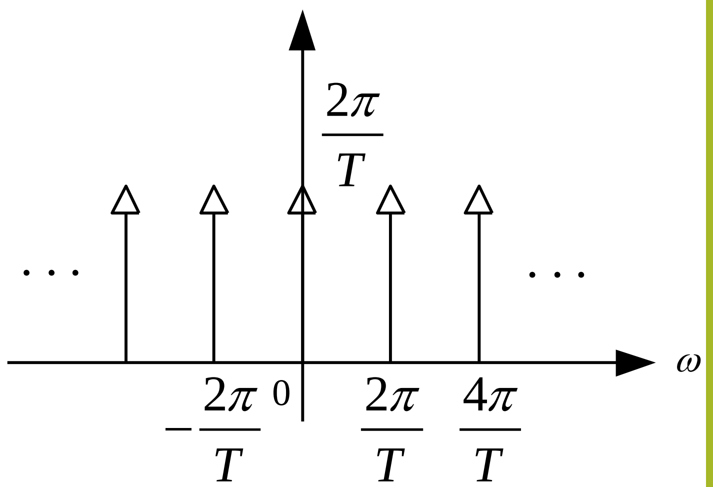
4 The continuous time Fourier transform

Example 4.8

$$x(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT) \xleftrightarrow{F} X(j\omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-k\frac{2\pi}{T}\right)$$



$$\xleftrightarrow{F}$$



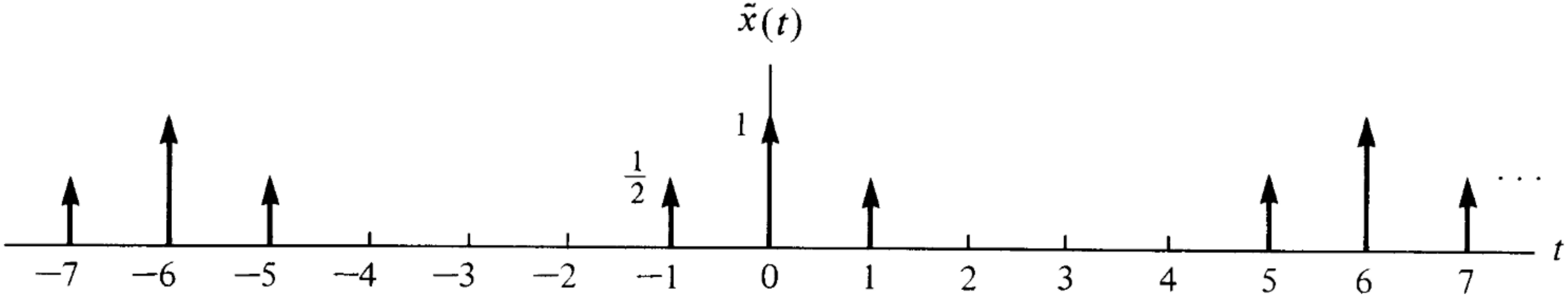
The FS of

$$x(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT)=\sum_{k=-\infty}^{+\infty}\frac{1}{T}e^{jk\omega_0 t}$$

By using (4.21),it is easy to get the FT pairs.

Example:

Consider the periodic signal $\tilde{x}(t)$ in the Figure, which is composed solely of impulses.

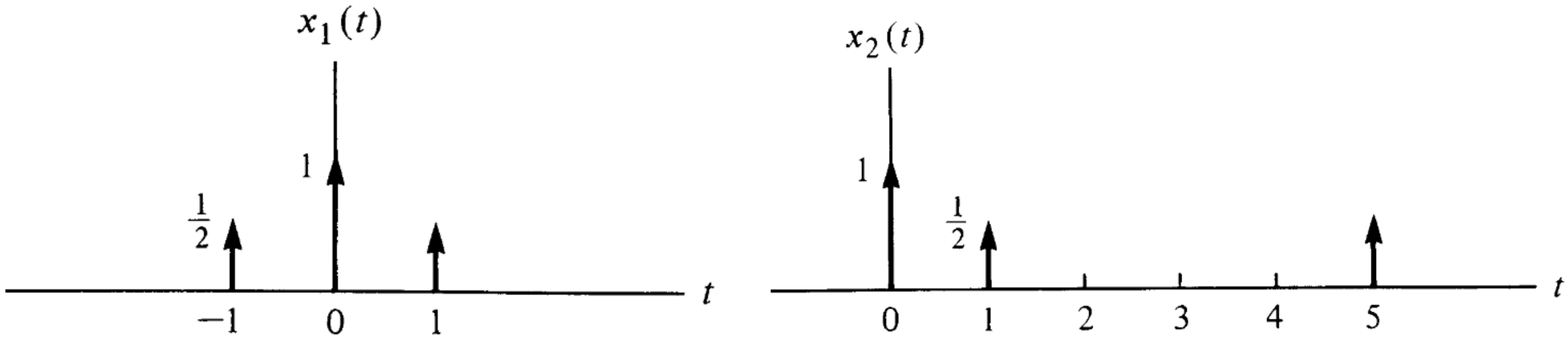


(a) What is the fundamental period T_0 ?

(b) Find the Fourier series of $\tilde{x}(t)$.

Example:

(c) Find the Fourier transform of the signal in the following figures.



Solution:

(a) By inspection, T_0=6.

(b) $a_k = \frac{1}{T_0}\int_{T_0} \tilde{x}(t) e^{-jk\left(\frac{2\pi}{T_0}\right)t} dt$

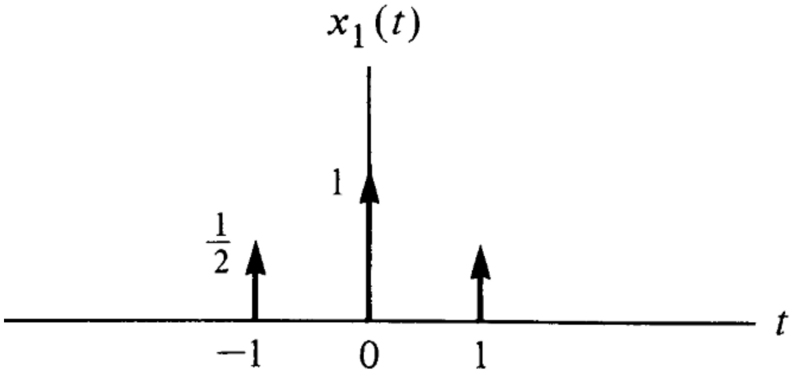
we integrate from -3 to 3:

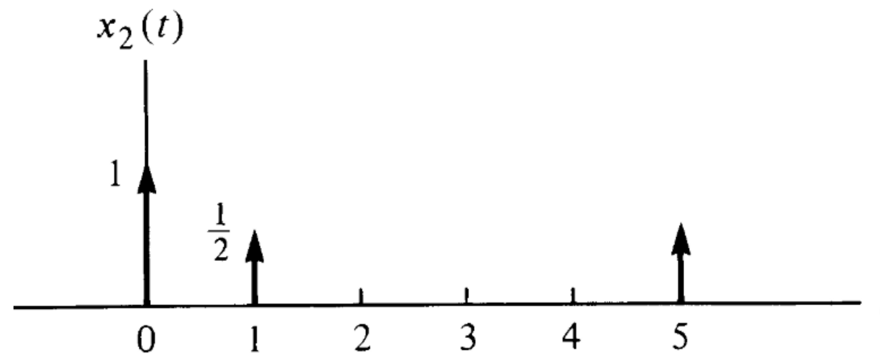
$$a_k = \frac{1}{6}\int_{-3}^{3}[\frac{1}{2}\delta(t+1) + \delta(t) + \frac{1}{2}\delta(t-1)]e^{-jk\left(\frac{2\pi}{6}\right)t} dt$$

$$= \frac{1}{6}\left(\frac{1}{2}e^{jk\left(\frac{2\pi}{6}\right)} + 1 + \frac{1}{2}e^{-jk\left(\frac{2\pi}{6}\right)}\right)$$

$$= \frac{1}{6}\left(1 + cos\frac{2\pi k}{6}\right)$$

(c) Find the Fourier transform of the signal in the following figures.





$$X_1(j\omega) = \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} [\tfrac{1}{2}\delta(t+1) + \delta(t) + \tfrac{1}{2}\delta(t-1)] e^{-j\omega t}\, dt$$

$$= \tfrac{1}{2}e^{j\omega} + 1 + \tfrac{1}{2}e^{-j\omega} = 1 + \cos\omega$$

$$X_2(j\omega) = \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} [\delta(t) + \tfrac{1}{2}\delta(t-1) + \tfrac{1}{2}\delta(t-5)] e^{-j\omega t}\, dt$$

$$= 1 + \tfrac{1}{2}e^{-j\omega} + \tfrac{1}{2}e^{-j5\omega}$$

4 The continuous time Fourier transform

4.3 Properties of the Continuous time Fourier Transform

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
4.3.2	Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega)=X^*(-j\omega) \\ \Re e\{X(j\omega)\}=\Re e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\}=-\mathcal{I}m\{X(-j\omega)\} \\ \|X(j\omega)\|=\|X(-j\omega)\| \\ \sphericalangle X(j\omega)=-\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t)=\mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t)=\mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals		

$$\int_{-\infty}^{+\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega$$

4.3 Properties of the Continuous time Fourier Transform

$$\begin{cases} x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega & eq.4.8/4.24 \\ X(j\omega) = \displaystyle\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt & eq.4.9\ /4.25 \end{cases}$$

$$\begin{cases} x(t) \dashrightarrow \boldsymbol{F}^{-1}\{X(j\omega)\} \\ X(j\omega) \dashrightarrow \boldsymbol{F}\{x(t)\} \end{cases}$$

4.3 Properties of the Continuous time Fourier Transform

For example

$$\begin{cases} \dfrac{1}{a+j\omega} = \boldsymbol{F}\{e^{-at}u(t)\} \\ e^{-at}u(t) = \boldsymbol{F^{-1}}\{\dfrac{1}{a+j\omega}\} \end{cases}$$

$$e^{-at}u(t) \overset{F}{\leftrightarrow} \frac{1}{a+j\omega}$$

4.3.1 Linearity

If

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$y(t) \xleftrightarrow{F} Y(j\omega)$$

then

$$ax(t) + by(t) \xleftrightarrow{F} aX(j\omega) + bY(j\omega)$$

4.3.2 Time Shifting

If $x(t) \xleftrightarrow{F} X(j\omega)$

then $x(t-t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(j\omega)$

$$\Pr oof: x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

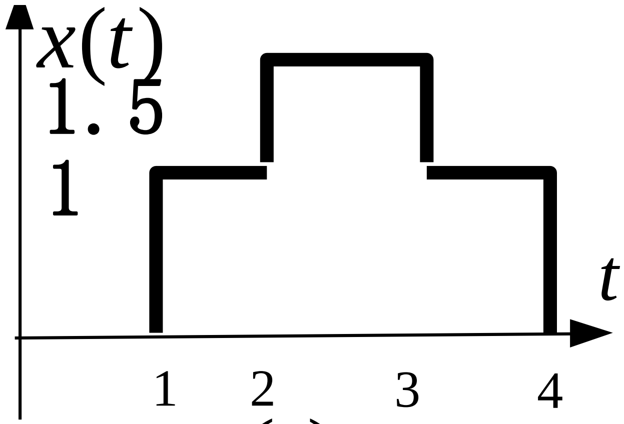
$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega(t-t_0)} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{-j\omega t_0} e^{j\omega t} d\omega$$

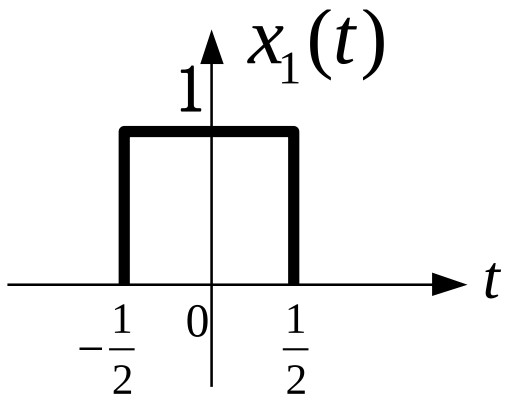
That is $x(t-t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(j\omega)$

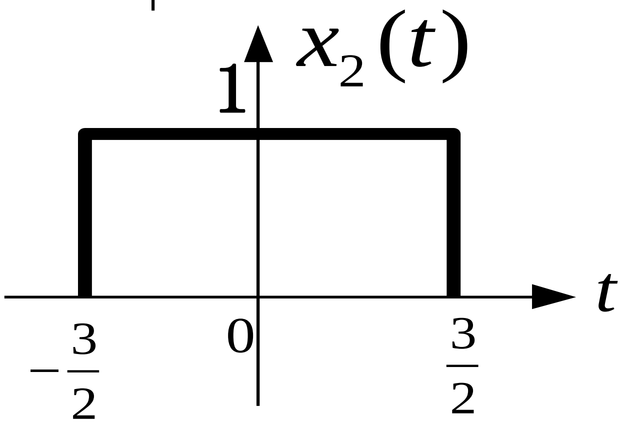
4 The continuous time Fourier transform

Example 4.9



$$x(t)=\frac{1}{2}x_1(t-2.5)+x_2(t-2.5)$$





$$X_1(j\omega)=\frac{2\sin(\omega/2)}{\omega} \qquad X_2(j\omega)=\frac{2\sin(3\omega/2)}{\omega}$$

$$X(j\omega)=e^{-j5\omega/2}\left\{\frac{\sin(\omega/2)+2\sin(3\omega/2)}{\omega}\right\}$$

4.3.3 Conjugation and Conjugate Symmetry

(1) If $$x(t) \stackrel{F}{\longleftrightarrow} X(j\omega)$$

then $$x^*(t) \stackrel{F}{\longleftrightarrow} X^*(-j\omega)$$

$$\mathrm{Pr}oof: x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)e^{-j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(-j\omega)e^{j\omega t}d\omega$$

4.3.3 Conjugation and Conjugate Symmetry

(2) If $x(t) = x^*(t)$ (real)

then $X(j\omega) = X^*(-j\omega)$

Ex. $x(t) = e^{-at}u(t)$

$$X(j\omega) = \frac{1}{a+j\omega}$$

$$X(-j\omega) = \frac{1}{a-j\omega} = X^*(-j\omega)$$

(3) If $x(t) = x_e(t) + x_o(t) = x^*(t)$ **(real)**

then

$$X(j\omega) = X_R(j\omega) + jX_I(j\omega) = X_e(j\omega) + X_o(j\omega)$$

and

$$x_e(t) \xleftrightarrow{F} X_R(j\omega) = X_e(j\omega)$$

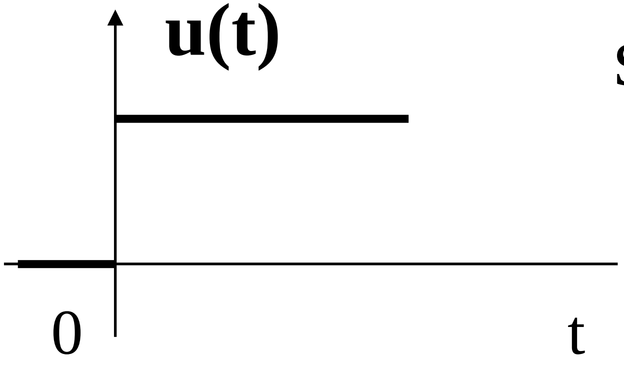
$$x_o(t) \xleftrightarrow{F} jX_I(j\omega) = X_o(j\omega)$$

Prove by yourself!

4 The continuous time Fourier transform

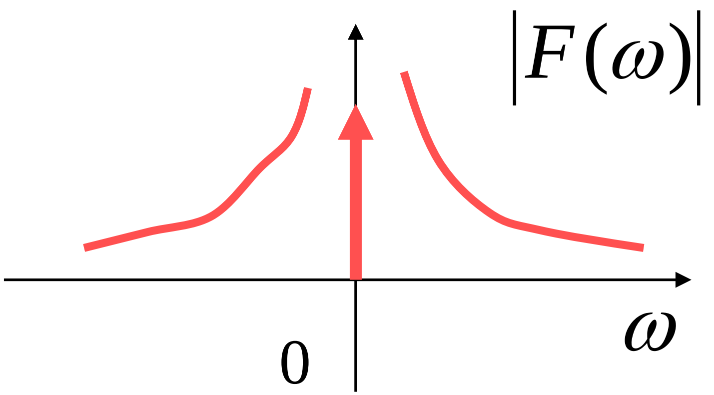
Example

$$u(t) = \frac{1}{2} + \frac{1}{2} sign(t)$$



$$\mathrm{sign}(t) = \begin{cases} +1 & (t > 0) \\ -1 & (t < 0) \end{cases} \overset{F}{\longleftrightarrow} \frac{2}{j\omega}$$

$$FT[u(t)] = \cdots = \pi\delta(\omega) + \frac{1}{j\omega}$$



4.3.4 Differentiation and Integration

(1) If $$x(t) \xleftrightarrow{F} X(j\omega)$$

then $$x'(t) \xleftrightarrow{F} j\omega X(j\omega)$$

$$\begin{cases} x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega & eq.4.24 \\ X(j\omega) = \displaystyle\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt & eq.4.25 \end{cases}$$

4.3.4 Differentiation and Integration

(1) If $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega)$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} j\omega X(j\omega)e^{j\omega t}d\omega = j\omega\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

So $$\frac{dx(t)}{dt} \overset{F}{\longleftrightarrow} j\omega X(j\omega)$$

4.3.4 Differentiation and Integration

(2) If $$x(t) \xleftrightarrow{F} X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

Proof:

$$\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t)$$

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega)$$

$$u(t) \overset{F}{\longleftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega)$$

$$x(t) * u(t) \overset{F}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

4 The continuous time Fourier transform

Example $$\frac{d}{dt}[\delta(t)] \stackrel{FT}{\leftrightarrow} ??$$

From $\delta(t) \stackrel{FT}{\leftrightarrow} 1$ $$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} d\omega$$

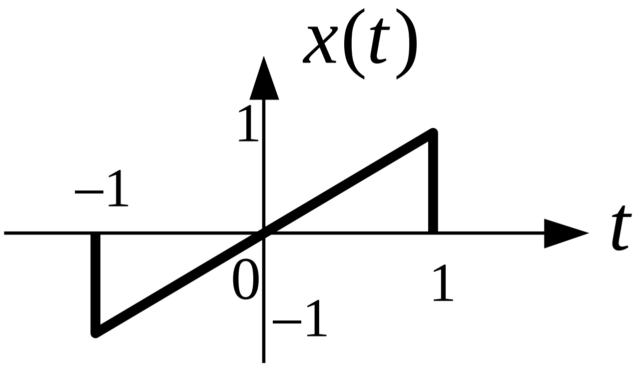
$$\frac{d}{dt}[\delta(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega) e^{j\omega t} d\omega$$

So $FT\left[\frac{d}{dt}\delta(t)\right] = j\omega$ **or** $\frac{d}{dt}\delta(t) = \delta^{'}(t) \stackrel{FT}{\leftrightarrow} j\omega$

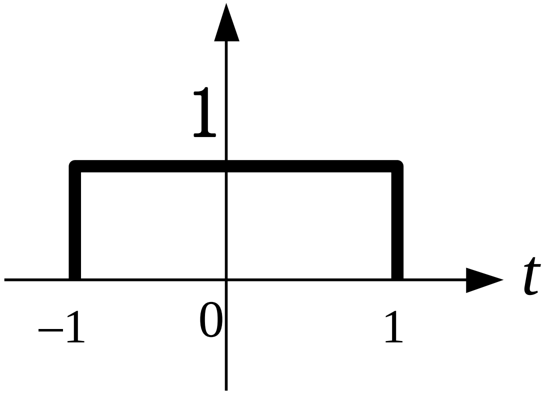
$$\frac{d^n}{dt^n}\delta(t) \stackrel{FT}{\longleftrightarrow} (j\omega)^n$$

Example 4.12

Determine the FT of $x(t)$



$$g(t)=\frac{d}{dt}x(t)=$$



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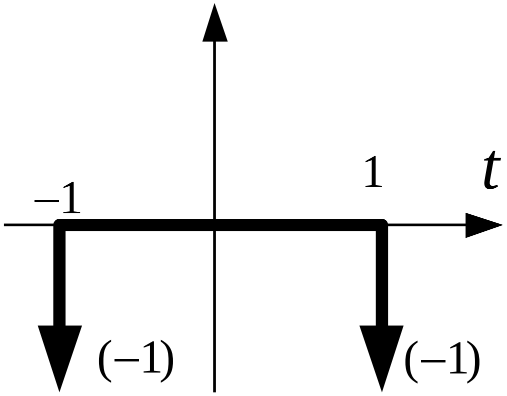
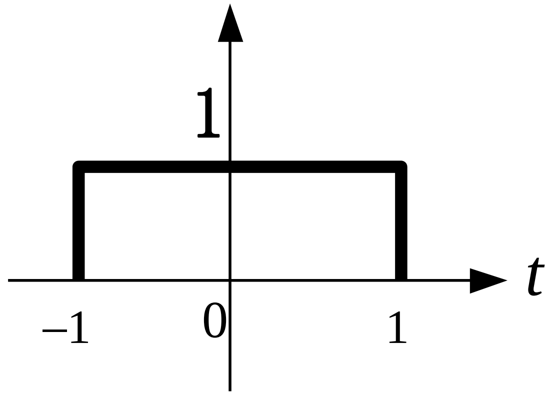


4 The continuous time Fourier transform

$$g(t) = \frac{d}{dt}x(t) =$$



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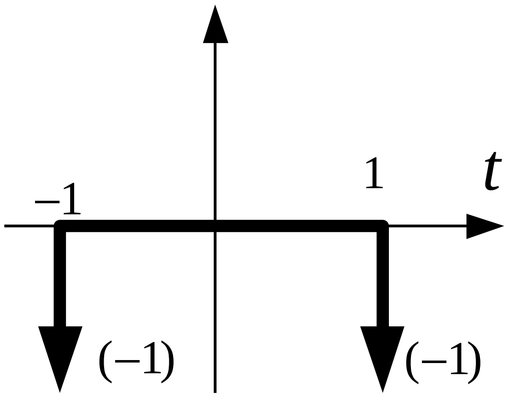


$$G(j\omega) = \frac{2\sin(\omega)}{\omega} - e^{j\omega} - e^{-j\omega}$$

$$X(j\omega) = \frac{1}{j\omega}G(j\omega) + \pi G(0)\delta(\omega)$$

$$X(j\omega) = \frac{2\sin(\omega)}{j\omega^2} - \frac{2\cos(\omega)}{j\omega}$$

4.3.5 Time and Frequency Scaling

If $x(t) \xleftrightarrow{F} X(j\omega)$

then $x(at) \xleftrightarrow{F} \frac{1}{|a|} X(\frac{j\omega}{a})$

$$\text{Proof}: x(at) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega at}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{|a|} X(j\omega/a)e^{j\omega t}d\omega$$

$$\therefore x(at) \xleftrightarrow{F} \frac{1}{|a|} X(j\omega/a)$$

4 The continuous time Fourier transform

Especially, $x(-t) \xleftrightarrow{F} X(-j\omega)$

$$x\left(\frac{t-b}{a}\right) \xleftrightarrow{F} |a|\, X(ja\omega) e^{-jb\omega}$$

Prove it by yourself!

4.3.6 Duality

If $$x(t) \xleftrightarrow{F} X(j\omega)$$

then $$X(jt) \xleftrightarrow{F} 2\pi x(-\omega)$$

$$\text{Proof}: x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

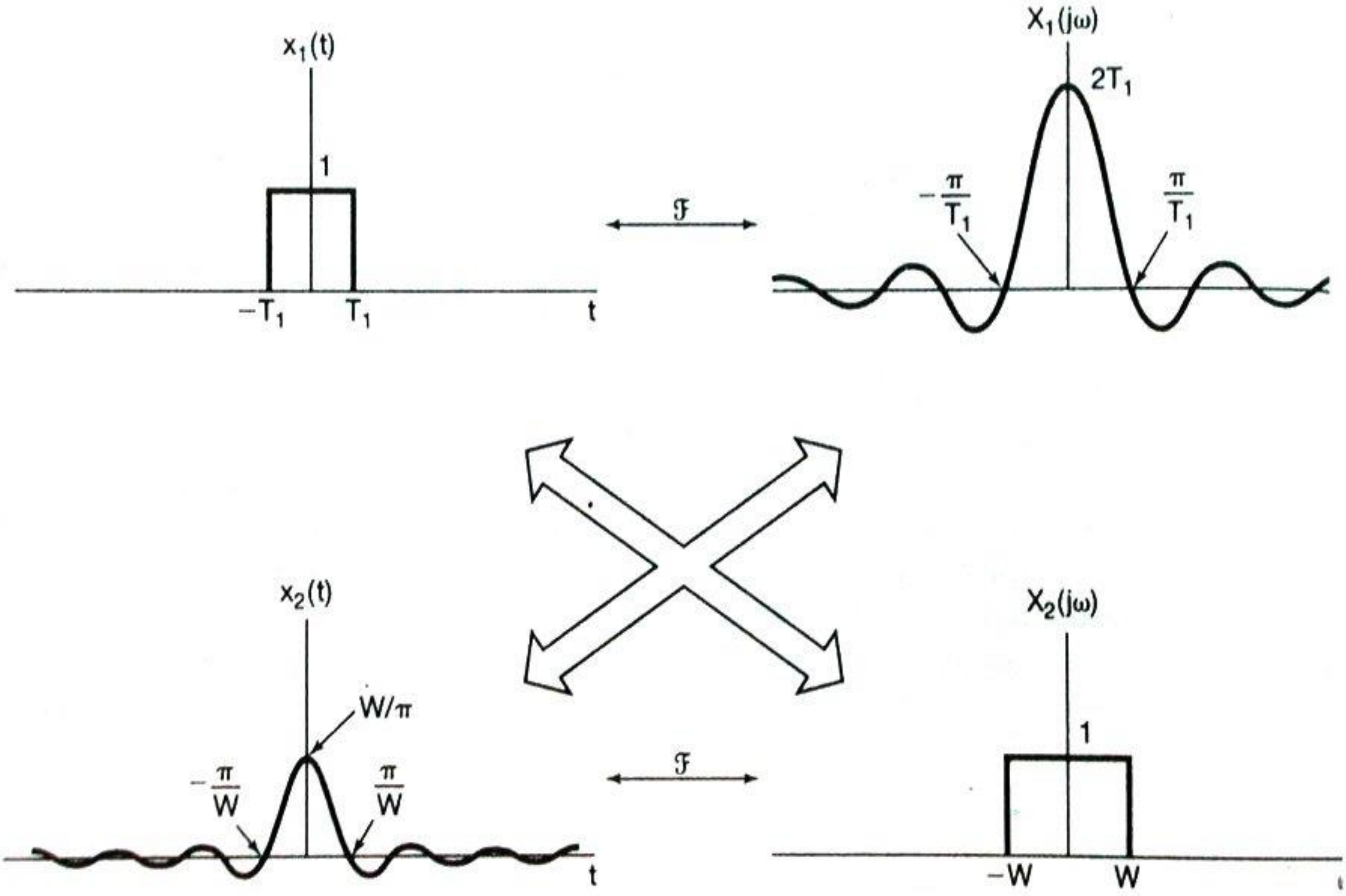
exchange t *and* ω:

$$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jt)e^{j\omega t}dt$$

$$2\pi \quad x(-\omega) = \int_{-\infty}^{+\infty} X(jt)e^{-j\omega t}dt$$

$$\therefore\ X(jt) \xleftrightarrow{F} 2\pi \quad x(-\omega)$$

4 The continuous time Fourier transform



Example 4.13

$$g(t) = \frac{2}{1+t^2}$$

From FT pair

$$\because x(t) = e^{-a|t|} \overset{F}{\longleftrightarrow} X(j\omega) = \frac{2a}{a^2+\omega^2}$$

And Duality,we get

$$\therefore g(t) = \frac{2}{1+t^2} \overset{F}{\longleftrightarrow} G(j\omega) = 2\pi e^{-|\omega|}$$

Example **From FT pair**

$$\frac{d^n}{dt^n}\delta(t) \overset{FT}{\longleftrightarrow} (j\omega)^n$$

And Duality,we get

$$t^n \overset{FT}{\longleftrightarrow} 2\pi(j)^n \frac{d^n}{d\omega^n}[\delta(\omega)]$$

4 The continuous time Fourier transform

Example $$x'(t) = \frac{d}{dt}x(t) \xleftrightarrow{F} j\omega X(j\omega)$$

Using Duality,we get

$$-jtx(t) \xleftrightarrow{F} \frac{d}{d\omega}X(j\omega) \qquad \textbf{(4.40)}$$

Example $$te^{-at}u(t) \xleftrightarrow{F} \frac{1}{(a+j\omega)^2}, (a>0)$$

$$x(t-t_0) \xleftrightarrow{F} e^{-j\omega t_0}X(j\omega)$$

Using Duality,we get

$$e^{j\omega_0 t}x(t) \xleftrightarrow{F} X(j(\omega-\omega_0)) \qquad \textbf{(4.41)}$$

4 The continuous time Fourier transform

Example

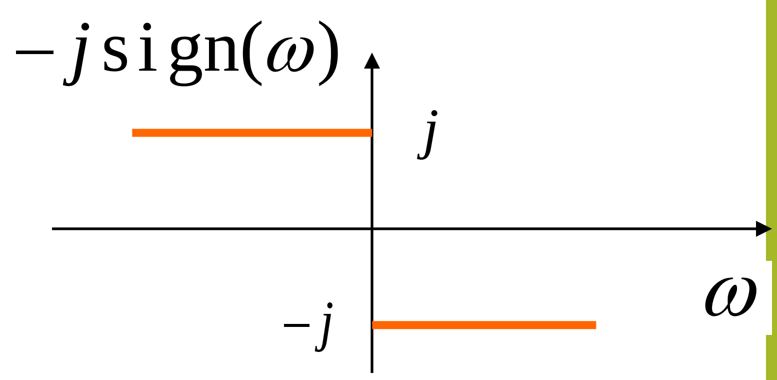
$$\frac{1}{\pi t} \xleftrightarrow{F} -j\,\mathrm{sign}(\omega) = \begin{cases} -j & (\omega > 0) \\ +j & (\omega < 0) \end{cases}$$

From FT pair

$$\mathrm{sign}(t) = \begin{cases} +1 & (t > 0) \\ -1 & (t < 0) \end{cases} \xleftrightarrow{F} \frac{2}{j\omega}$$

Using Duality,we can get it easily.

$$\because \frac{2}{jt} \xleftrightarrow{F} 2\pi\,\mathrm{sign}(-\omega)$$



4.3.7 Parseval's Relation

If $x(t) \xleftrightarrow{F} X(j\omega)$

then $\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 \, d\omega$

$$\mathrm{Pr}\,oof: \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} x(t) x^*(t) dt$$

$$= \int_{-\infty}^{+\infty} x(t) [\frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) e^{-j\omega t} d\omega] dt$$

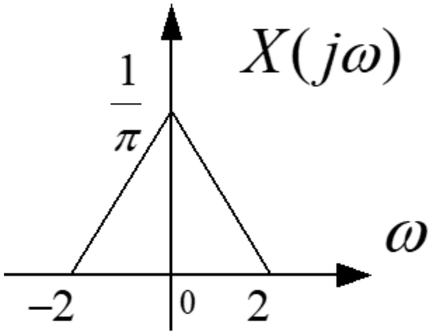
$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) [\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) X(j\omega) d\omega \quad = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 \, d\omega$$

Example 4.14 read by yourself!

4 The continuous time Fourier transform

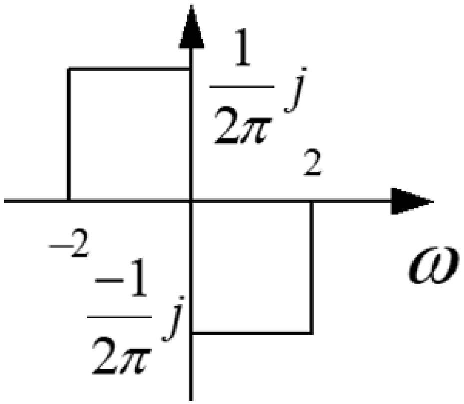
Example $$x(t) = \left(\frac{\sin t}{\pi t}\right)^2 \overset{F}{\longleftrightarrow}$$



$$\int_{-\infty}^{+\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt = ?$$

Solution:

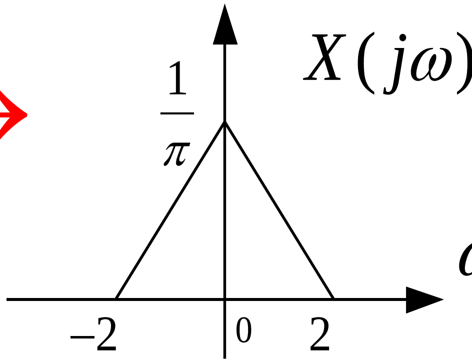
$$\because\ g(t) = t x(t) = t\left(\frac{\sin t}{\pi t}\right)^2 \overset{F}{\longleftrightarrow}$$



$$G(j\omega) = -j\frac{d}{d\omega}X(j\omega)$$

4 The continuous time Fourier transform

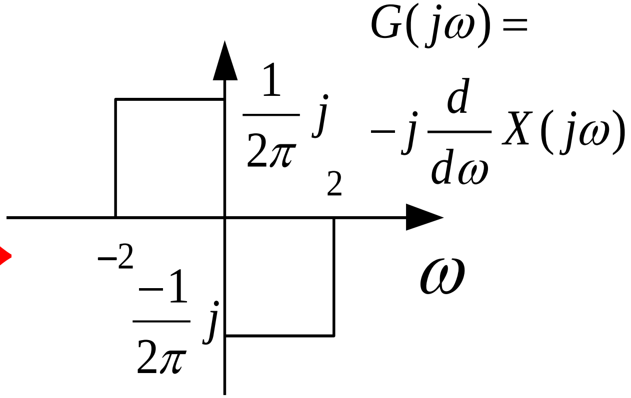
Example $$x(t) = \left(\frac{\sin t}{\pi t}\right)^2 \xleftrightarrow{F}$$



$$\int_{-\infty}^{+\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt = ?$$

Solution:

$$\because\ g(t) = t x(t) = t\left(\frac{\sin t}{\pi t}\right)^2 \xleftrightarrow{F}$$



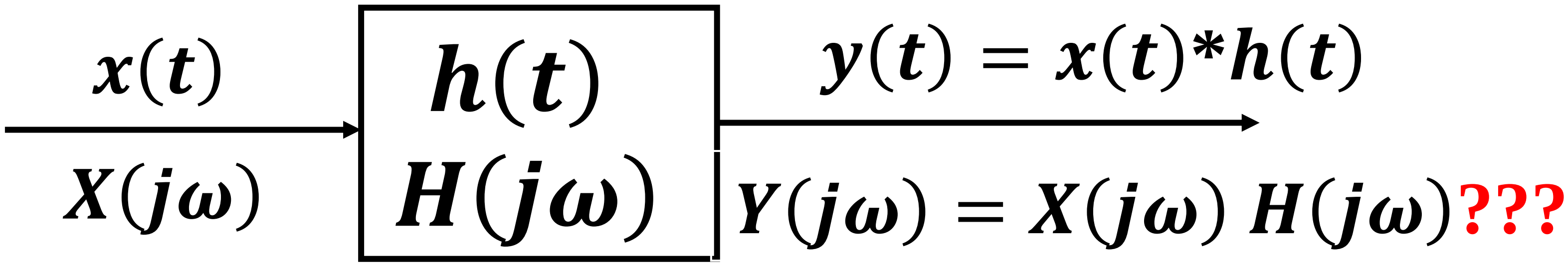
$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-2}^{2} \frac{1}{4\pi^2} d\omega = \frac{1}{2\pi^3}$$

$$\therefore \int_{-\infty}^{+\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt = \frac{1}{2\pi^3}$$

4.4 The Convolution Property

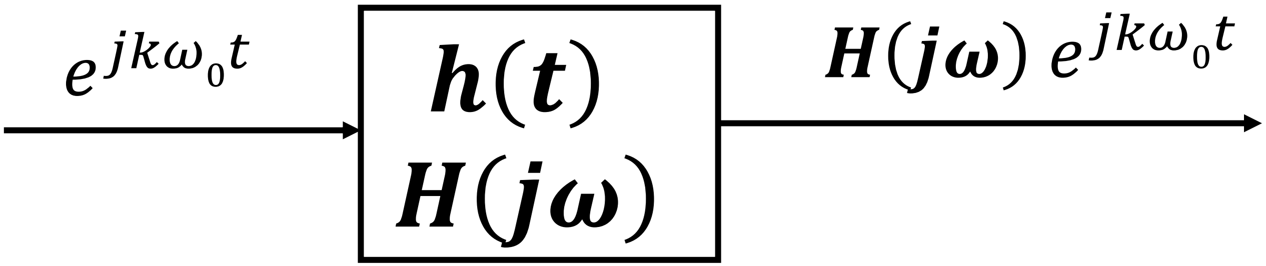
Consider an LTI system:



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

$$= \lim_{n\to\infty}\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

4 The continuous time Fourier transform



$$H(jk\omega_0) = \int_{-\infty}^{+\infty} h(t) e^{-jk\omega_0 t} dt$$

For LTI system,

$$\frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

$$\dashrightarrow \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) H(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

4 The continuous time Fourier transform

$$y(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) H(jk\omega_0) e^{jk\omega_0 t} \omega_0 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) H(j\omega) e^{j\omega t} d\omega,$$

from **inverse Fourier transform**

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Y(j\omega) e^{j\omega t} d\omega$$

So $Y(j\omega) = X(j\omega) H(j\omega)$

4 The continuous time Fourier transform

Another way to derive the result:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$Y(j\omega) = F\{y(t)\}$$

$$= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right] e^{-j\omega t} dt$$

$$interchange\ the\ \ order$$

$$= \int_{-\infty}^{+\infty} x(\tau) \left[\int_{-\infty}^{+\infty} h(t-\tau)e^{-j\omega t} dt\right] d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)[H(j\omega)]\, e^{-j\omega\tau} d\tau = H(j\omega)X(j\omega)$$

4.4.1 Examples

Example 4.15 $h(t)=\delta(t-t_0)\quad H(j\omega)=e^{-j\omega t_0}$

$$\because\ y(t)=x(t)*h(t)=x(t)*\delta(t-t_0)=x(t-t_0)$$

$$\therefore Y(j\omega)=X(j\omega)H(j\omega)$$

So, $y(t)=x(t-t_0)\xleftrightarrow{F}e^{-j\omega t_0}X(j\omega)=Y(j\omega)$

4.4.1 Examples

Example 4.16: a differentiator $y(t) = \frac{dx(t)}{dt}$

$$y(t) = \frac{dx(t)}{dt} \xleftrightarrow{F} Y(j\omega) = j\omega X(j\omega)$$

From previous knowledge:

$$h(t) = \delta'(t) \qquad H(j\omega) = j\omega$$

$$y(t) = x(t) * h(t), \;\; Y(j\omega) = j\omega X(j\omega)$$

4 The continuous time Fourier transform

Example 4.17: integrator

$$y(t)=\int_{-\infty}^{t}x(\tau)d\tau$$

$$\because\ y(t)=x(t)*u(t)$$

$$\therefore h(t)=u(t)\overset{FT}{\leftrightarrow}H(j\omega)=\pi\delta(\omega)+\frac{1}{j\omega}$$

$$y(t)=x(t)*u(t)\xleftrightarrow{F}Y(j\omega)=\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega$$

Example 4.19

$$h(t) = e^{-at}u(t), a > 0$$

$$x(t) = e^{-bt}u(t), b > 0$$

$$X(j\omega) = \frac{1}{b + j\omega} \qquad H(j\omega) = \frac{1}{a + j\omega}$$

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}$$

4 The continuous time Fourier transform

$$Y(j\omega)=X(j\omega)H(j\omega)=\frac{1}{(a+j\omega)(b+j\omega)}$$

By ***partial-fraction expansions***

$$Y(j\omega)=\frac{A}{a+j\omega}+\frac{B}{b+j\omega}$$ **and** $$A=\frac{1}{b-a}=-B$$

$$\therefore Y(j\omega)=\frac{1}{b-a}[\frac{1}{a+j\omega}-\frac{1}{b+j\omega}]$$

Inverse FT

$$y(t)=\frac{1}{b-a}[e^{-at}u(t)-e^{-bt}u(t)]$$

If a=b,

$$Y(j\omega)=\frac{1}{(a+j\omega)^2}$$ $$y(t)=te^{-at}u(t)$$

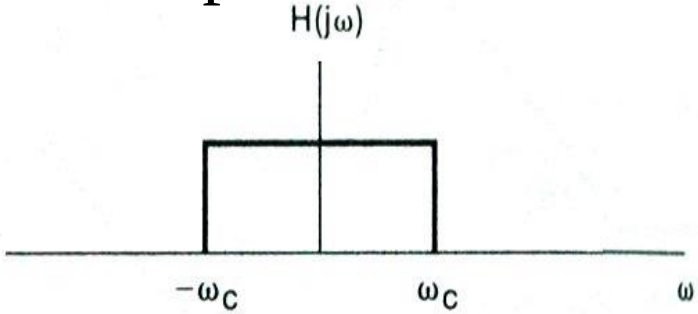
Deduce it by yourself!

4 The continuous time Fourier transform

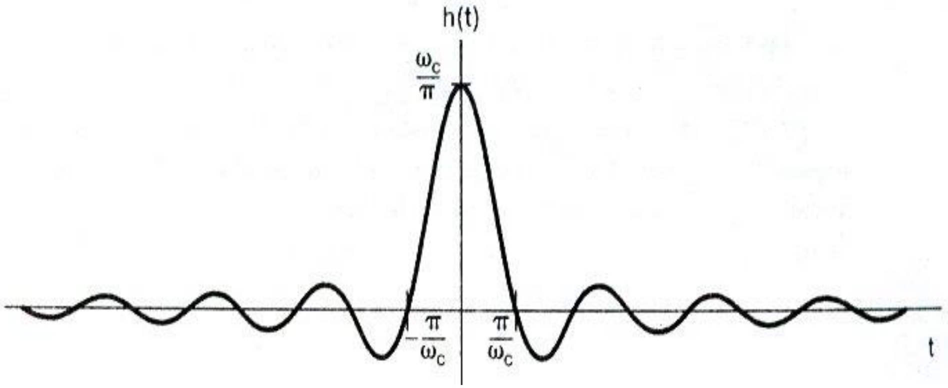
Example 4.18 and 4.20

Ideal Lowpass filter

$$H(j\omega)=\begin{cases}1, & |\omega|\le\omega_c\\ 0, & |\omega|>\omega_c\end{cases}$$

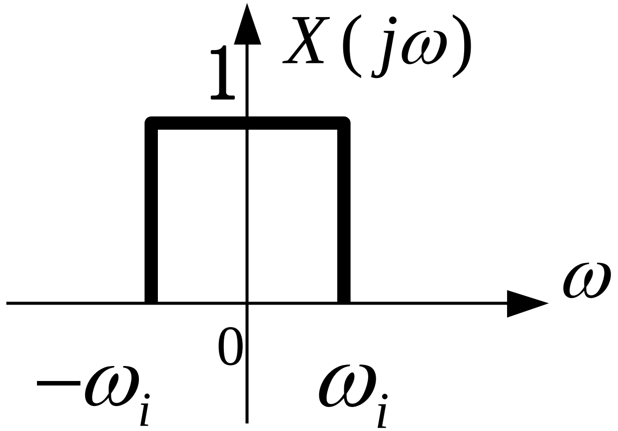


$$h(t)=\frac{\sin\omega_c t}{\pi t}$$



$$x(t)=\frac{\sin\omega_i t}{\pi t}$$

$$X(j\omega)=\begin{cases}1, & |\omega|\le\omega_i\\ 0, & |\omega|>\omega_i\end{cases}$$



4 The continuous time Fourier transform

$$\because Y(j\omega) = H(j\omega)X(j\omega)$$

$$\therefore Y(j\omega) = \begin{cases} 1, & |\omega| \le \omega_0 \\ 0, & |\omega| > \omega_0 \end{cases}$$

$$\omega_0 = \min(\omega_c, \omega_i)$$

$$\therefore y(t) = \begin{cases} \dfrac{\sin \omega_c t}{\pi t}, & if \qquad \omega_c \le \omega_i \\ \dfrac{\sin \omega_i t}{\pi t}, & if \qquad \omega_c > \omega_i \end{cases}$$

4 The continuous time Fourier transform

Example **Hilbert transform**(Extension Problems 4.48)

$$y(t)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{x(\tau)}{t-\tau}\,d\tau$$

$$\underset{X(j\omega)}{\xrightarrow{x(t)}}\ \boxed{\begin{matrix}h(t)\\ H(j\omega)\end{matrix}}\ \underset{Y(j\omega)=X(j\omega)\,H(j\omega)}{\xrightarrow{y(t)=x(t)*h(t)}}$$

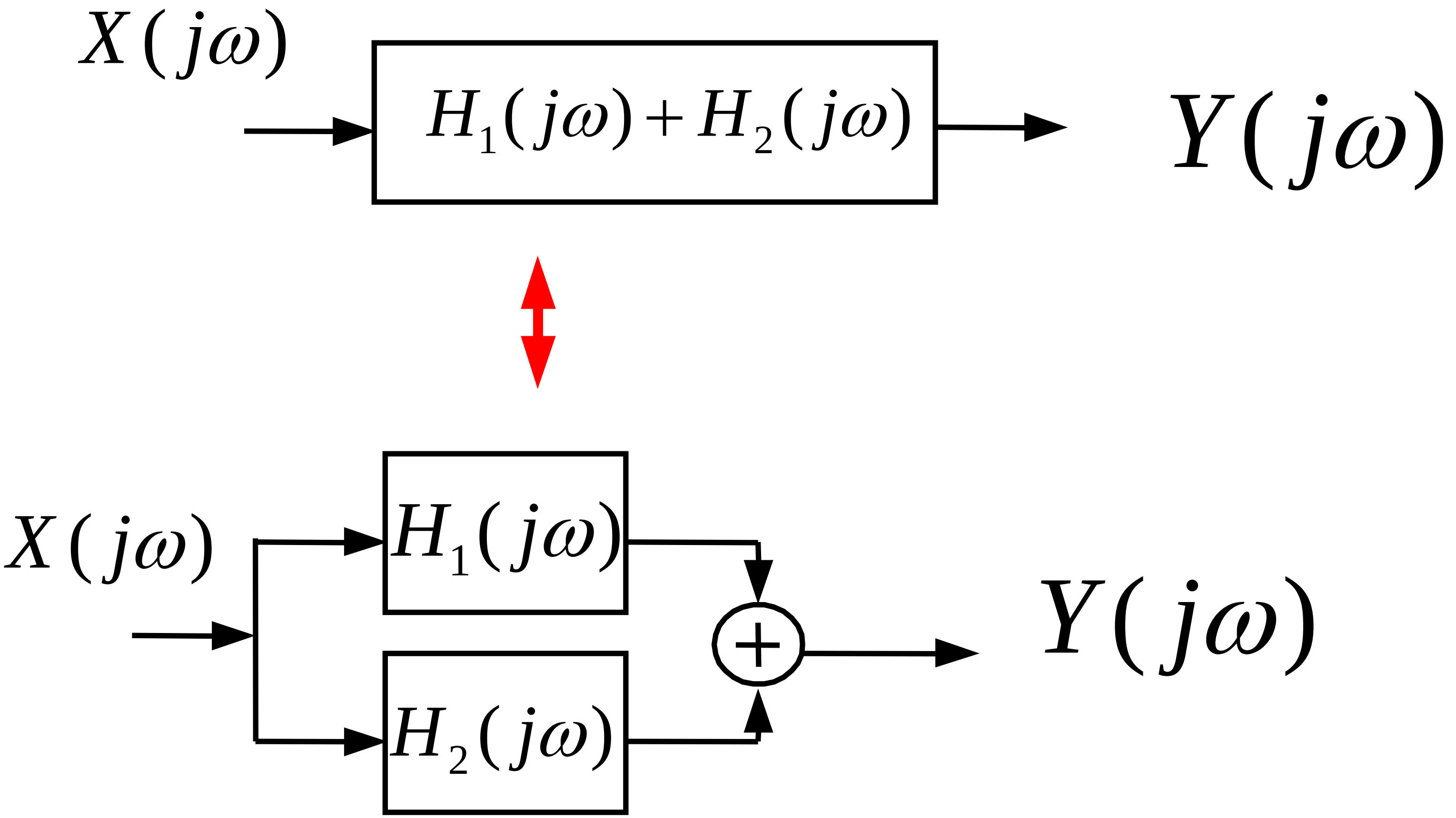
$$h(t)=\frac{1}{\pi t}$$

$$H(j\omega)=-j\,\mathrm{sign}(\omega)=\begin{cases}-j & (\omega>0)\\ +j & (\omega<0)\end{cases}$$

When $x(t)=\cos\omega_0 t,\ y(t)=sin\omega_0 t.$ **Prove it by yourself!**

4 The continuous time Fourier transform

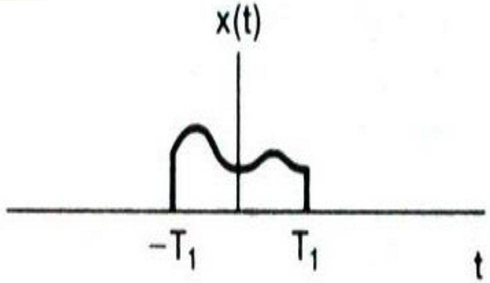
Example



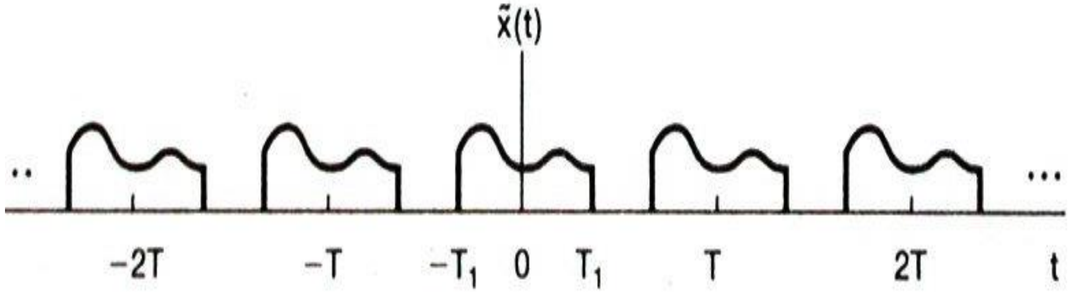
$$Y(j\omega)=[H_1(j\omega)+H_2(j\omega)]X(j\omega)$$

4 The continuous time Fourier transform

Example



$$x(t) \xleftrightarrow{FT} X(j\omega)$$



$$\tilde{x}(t) \xleftrightarrow{FT} \tilde{X}(j\omega)$$

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} x(t-nT) = x(t) * \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

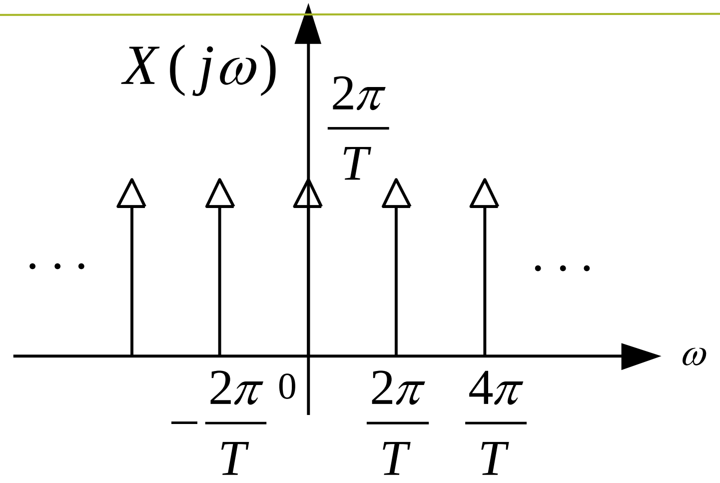
$$\therefore \tilde{X}(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} X(jk\frac{2\pi}{T})\delta(\omega - k\frac{2\pi}{T})$$

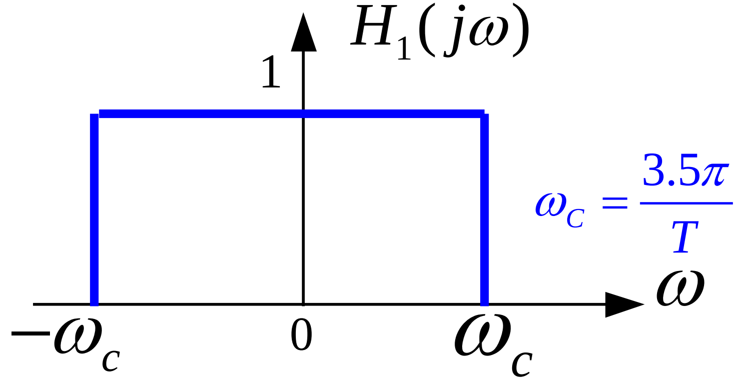
4 The continuous time Fourier transform

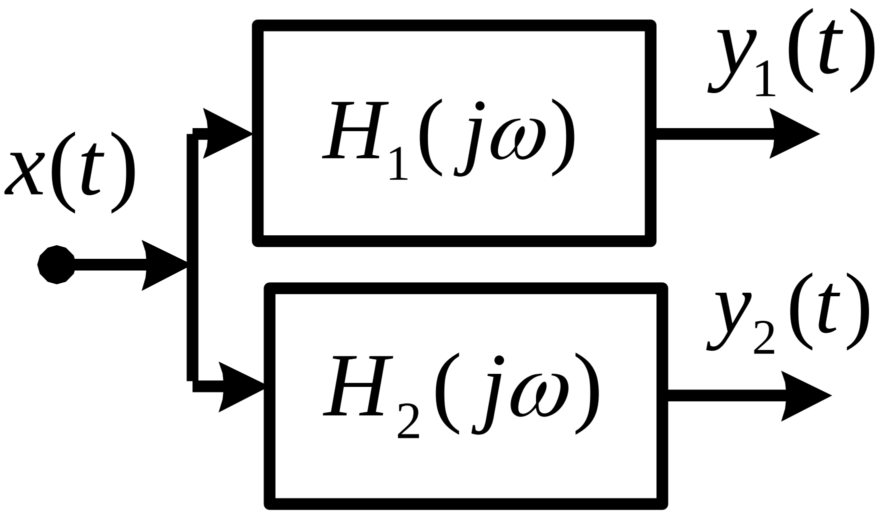
Example

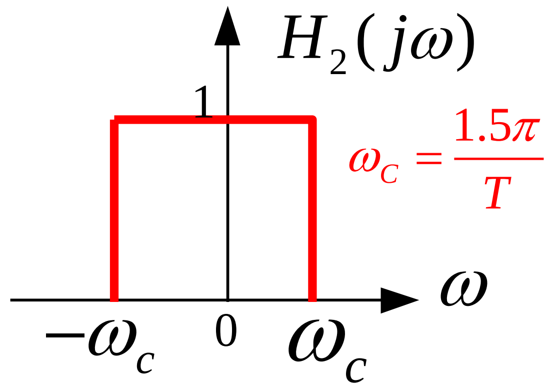
$$x(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT)$$

$$X(j\omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-k\frac{2\pi}{T}\right)$$





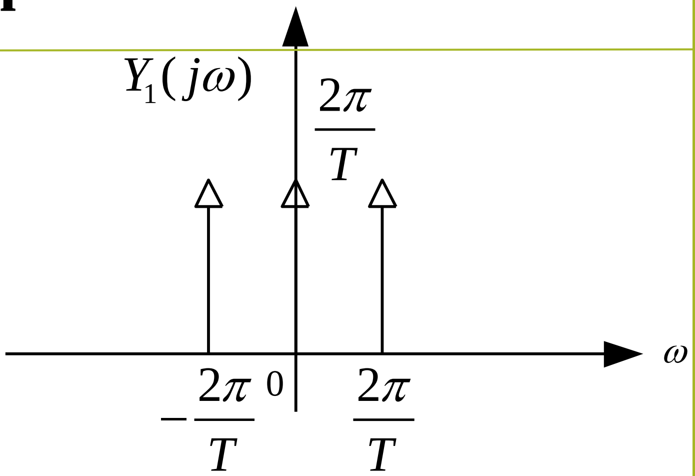




4 The continuous time Fourier transform

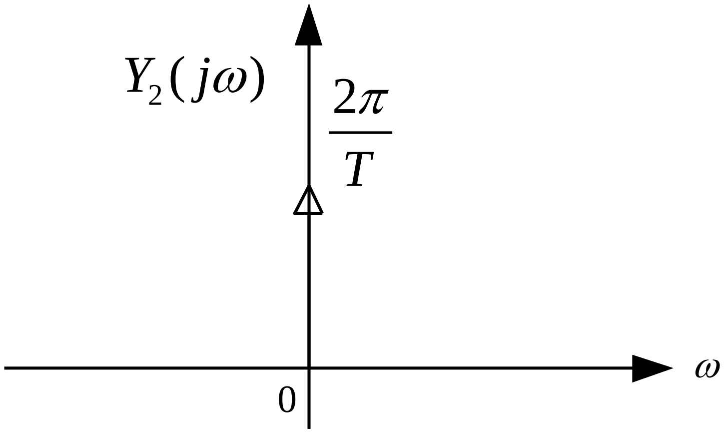
$$Y(j\omega) = X(j\omega)H(j\omega)$$

$$= \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} H(jk\frac{2\pi}{T})\delta(\omega - k\frac{2\pi}{T})$$



$$Y_1(j\omega) = \frac{2\pi}{T}\sum_{k=-1}^{+1}\delta(\omega - k\frac{2\pi}{T}) = \frac{2\pi}{T}[\delta(\omega + \frac{2\pi}{T}) + \delta(\omega) + \delta(\omega - \frac{2\pi}{T})]$$

$$Y_2(j\omega) = \frac{2\pi}{T}\delta(\omega)$$



$$\therefore y_1(t) = \frac{1}{T}[1 + 2\cos(\frac{2\pi}{T}t)]$$

$$\therefore y_2(t) = \frac{1}{T}$$

Waveform in time domain?

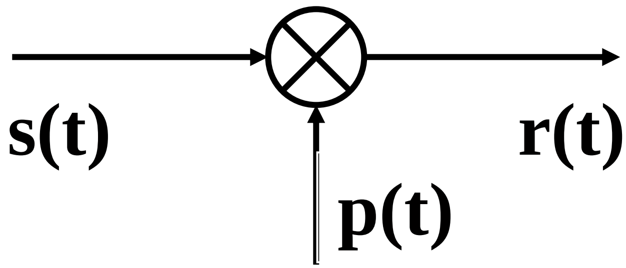
4 The continuous time Fourier transform

$$y(t) = x(t) * h(t) \overset{F}{\longleftrightarrow} Y(j\omega) = X(j\omega)H(j\omega)$$

$$y(t) = x(t)h(t) \overset{F}{\longleftrightarrow} Y(j\omega) = ??????$$

4.5 The Multiplication Property

The multiplication(modulation) property:



$$r(t) = s(t)\,p(t) \xleftrightarrow{F} R(j\omega) = \frac{1}{2\pi} S(j\omega) * P(j\omega)$$

Proof:

In the way Similar to the Convolution Property,
You can do it by yourself !

4 The continuous time Fourier transform

Example 4.21 **(Modulation)**

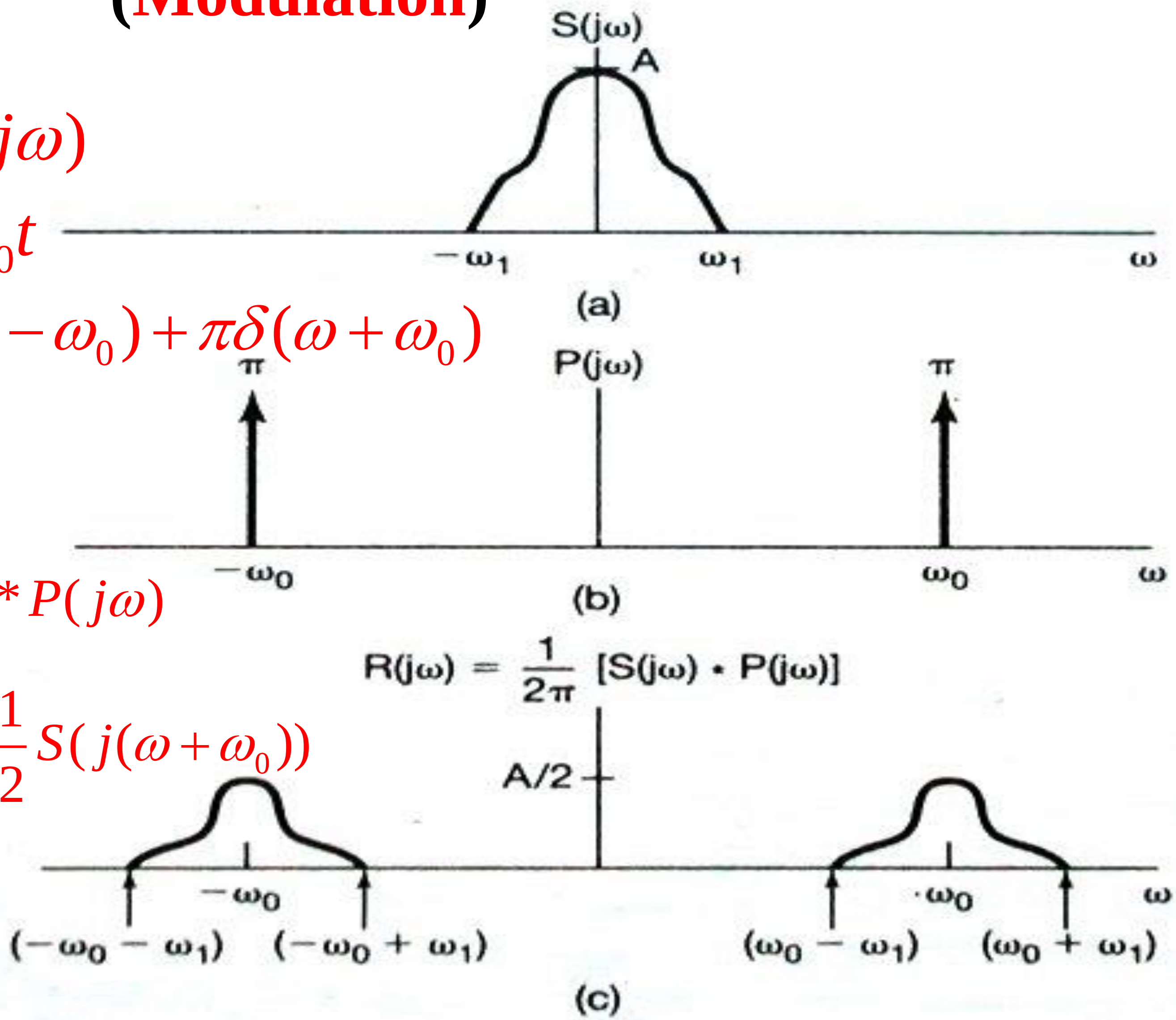
$$s(t) \xleftrightarrow{F} S(j\omega)$$

$$p(t) = \cos\omega_0 t$$

$$P(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$

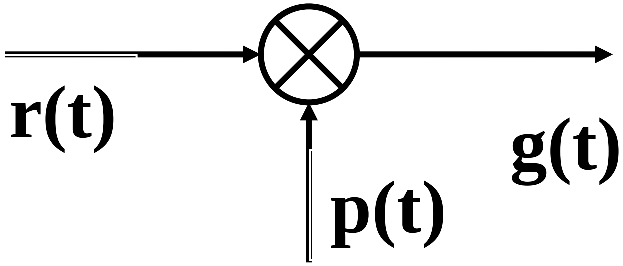
$$R(j\omega) = \frac{1}{2\pi} S(j\omega) * P(j\omega)$$

$$= \frac{1}{2} S(j(\omega-\omega_0)) + \frac{1}{2} S(j(\omega+\omega_0))$$



4 The continuous time Fourier transform

Example 4.22 **(Demodulation)**



$$p(t) = \cos \omega_0 t$$

$$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$G(j\omega) = \frac{1}{2\pi} R(j\omega) * P(j\omega)$$

$$= \frac{1}{2} S(j\omega) + \frac{1}{4} S(j(\omega - 2\omega_0)) + \frac{1}{4} S(j(\omega + 2\omega_0))$$

As shown in Figure 4.24.It will be discussed in Ch.8.

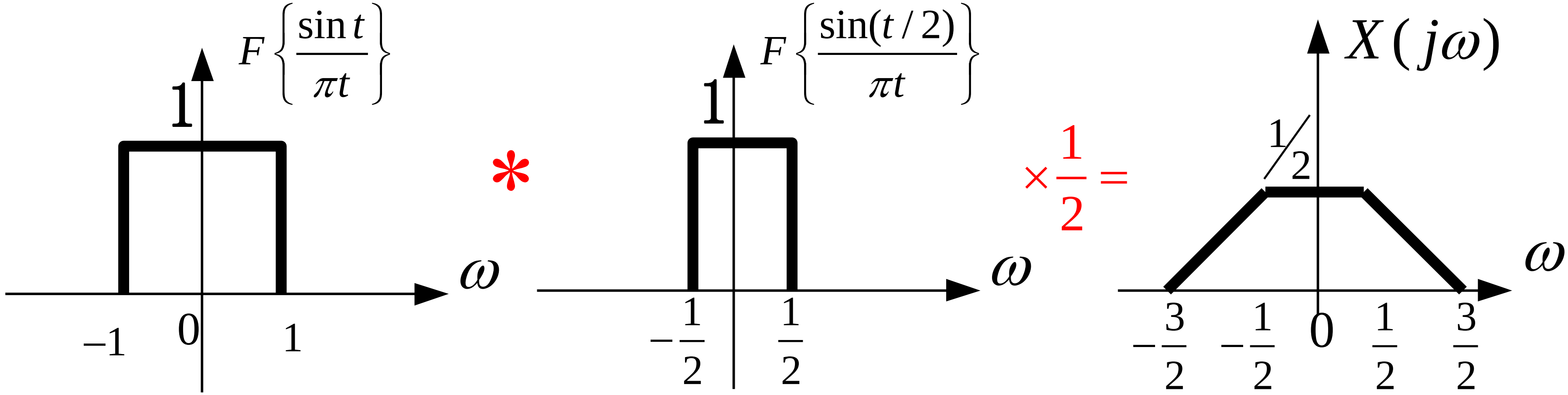
4 The continuous time Fourier transform

Example 4.23

$$x(t) = \frac{(\sin t)\sin(t/2)}{\pi t^2}$$

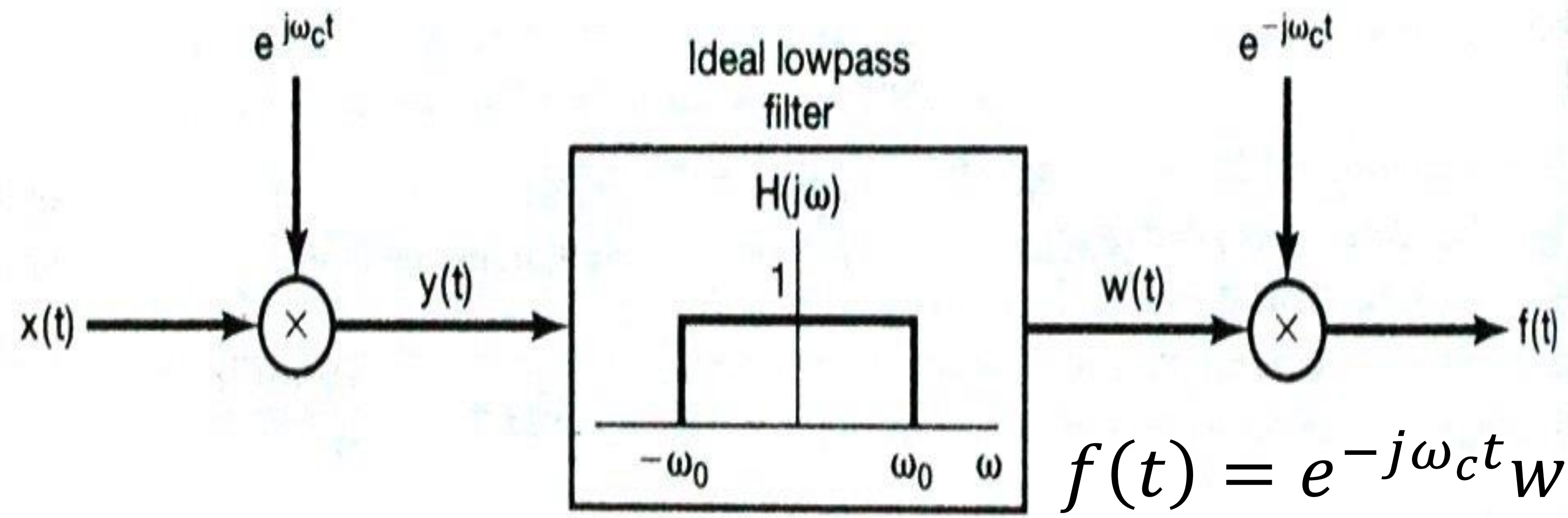
$$x(t) = \pi\left(\frac{\sin t}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right)$$

$$X(j\omega) = \frac{1}{2} F\left\{\frac{\sin t}{\pi t}\right\} * F\left\{\frac{\sin(t/2)}{\pi t}\right\}$$



4.5.1 Frequency-Selective Filtering with Variable Center Frequency

A Bandpass Filter :

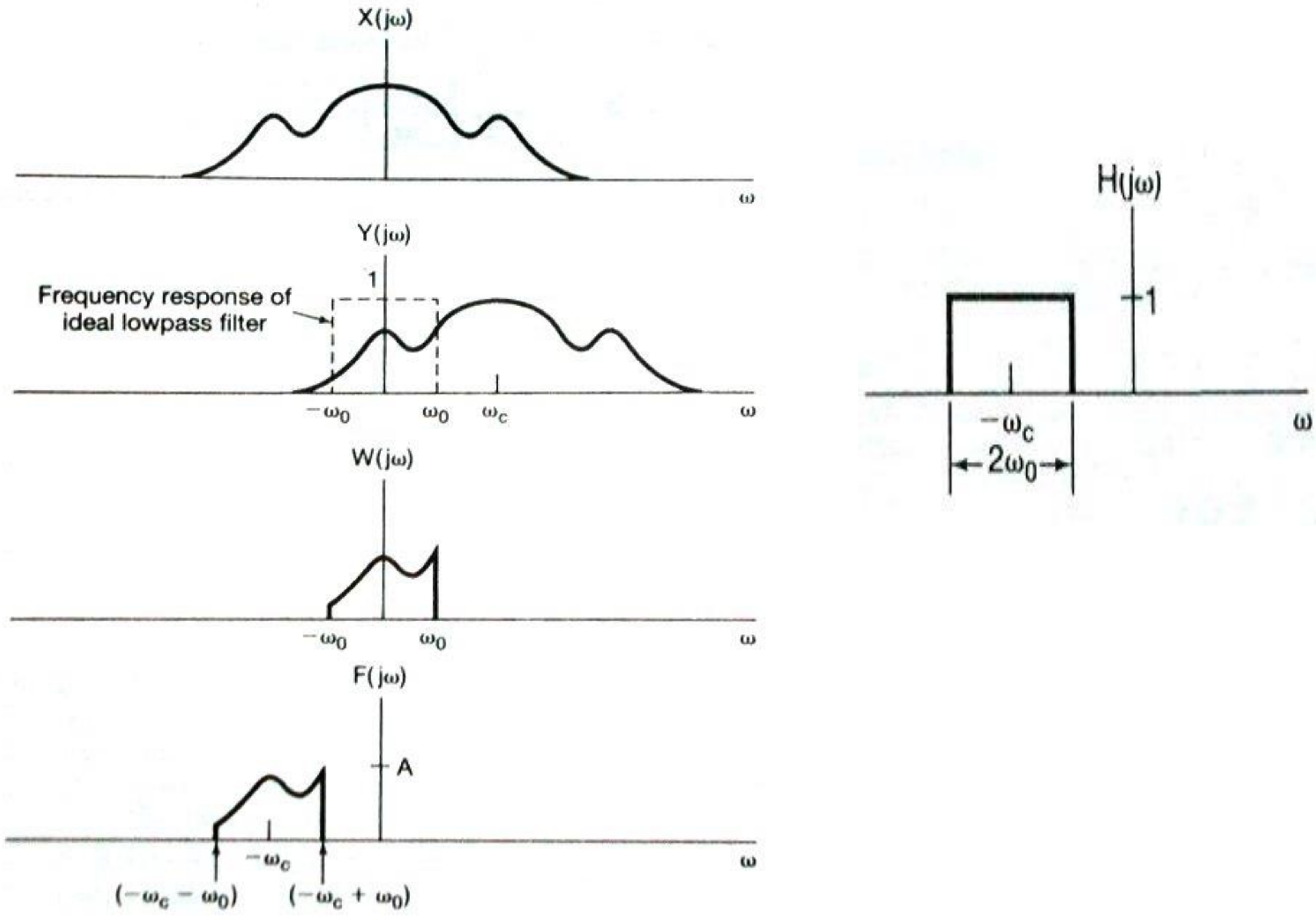


$$f(t) = e^{-j\omega_c t} w(t)$$

$$y(t) = e^{j\omega_c t} x(t)$$

$$F(j\omega) = W(j(\omega + \omega_c))$$

4 The continuous time Fourier transform



4.6 Tables of Fourier Properties and of Basic Fourier Transform Pairs

Table 4.1

Table 4.2

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals $\int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lvert X(j\omega)\rvert^2 d\omega$		

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi}\operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

4.7 System Characterized by Linear Constant-Coefficient Differential Equation

Constant-coefficient differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{dx^k(t)}{dt^k}$$

Fourier transform:

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

Define:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} \quad (\textit{frequency} \quad \textit{response})$$

Example 4.24

$$\frac{dy(t)}{dt} + ay(t) = x(t) \qquad (a > 0)$$

$$H(j\omega) = \frac{1}{a + j\omega}$$

$$h(t) = e^{-at}u(t)$$

4 The continuous time Fourier transform

Example 4.25 $$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=\frac{dx(t)}{dt}+2x(t)$$

$$H(j\omega)=\frac{(j\omega)+2}{(j\omega)^2+4(j\omega)+3}=\frac{j\omega+2}{(j\omega+1)(j\omega+3)}$$

By *partial-fraction expansions* $\therefore H(j\omega)=\dfrac{1/2}{j\omega+1}+\dfrac{1/2}{j\omega+3}$

$$h(t)=\tfrac{1}{2}[e^{-t}u(t)+e^{-3t}u(t)]$$

Example 4.26 If $x(t)=e^{-t}u(t)$ in Example 4.25

$$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=\frac{dx(t)}{dt}+2x(t)$$

$$(j\omega)^2Y(j\omega)+4j\omega Y(j\omega)+3Y(j\omega)$$
$$=j\omega X(j\omega)+2X(j\omega)$$

$$\boxed{H(j\omega)=\frac{1/2}{j\omega+1}+\frac{1/2}{j\omega+3} \qquad X(j\omega)=\frac{1}{j\omega+1}}$$

$$Y(j\omega)=H(j\omega)X(j\omega)=\frac{1/2}{(j\omega+1)^2}+\frac{1/2}{(j\omega+1)\,(j\omega+3)}$$

4 The continuous time Fourier transform

$$Y(j\omega) = \frac{1/2}{(j\omega+1)^2} + \frac{1/2}{(j\omega+1)\,(j\omega+3)}$$

$$= \frac{\frac{1}{2}(j\omega+3) + \frac{1}{2}(j\omega+1)}{(j\omega+1)^2(j\omega+3)}$$

$$= \frac{j\omega+2}{(j\omega+1)^2(j\omega+3)}$$

4 The continuous time Fourier transform

By ***partial-fraction expansions***

$$Y(j\omega) = \frac{A_1}{(j\omega+1)^2} + \frac{A_2}{j\omega+1} + \frac{A_3}{j\omega+3}$$

$$A_1 = \frac{1}{2}, A_2 = \frac{1}{4}, A_3 = -\frac{1}{4}$$

$$y(t) = [\frac{1}{2}te^{-t} + \frac{1}{4}e^{-t} - \frac{1}{4}e^{-3t}]u(t)$$

Example:

Consider the unit impulse response of an LTI system is $\boldsymbol{h(t)} = \frac{\boldsymbol{\sin \pi t \sin 2\pi t}}{\boldsymbol{\pi t^2}}$.

(a) Determine and sketch $\boldsymbol{H(j\omega)}$.

(b) Given the input

$$\boldsymbol{x(t) = 1 + \sin \frac{\pi t}{2} + \cos 2\,\pi t + \sin 5\,\pi t}$$,

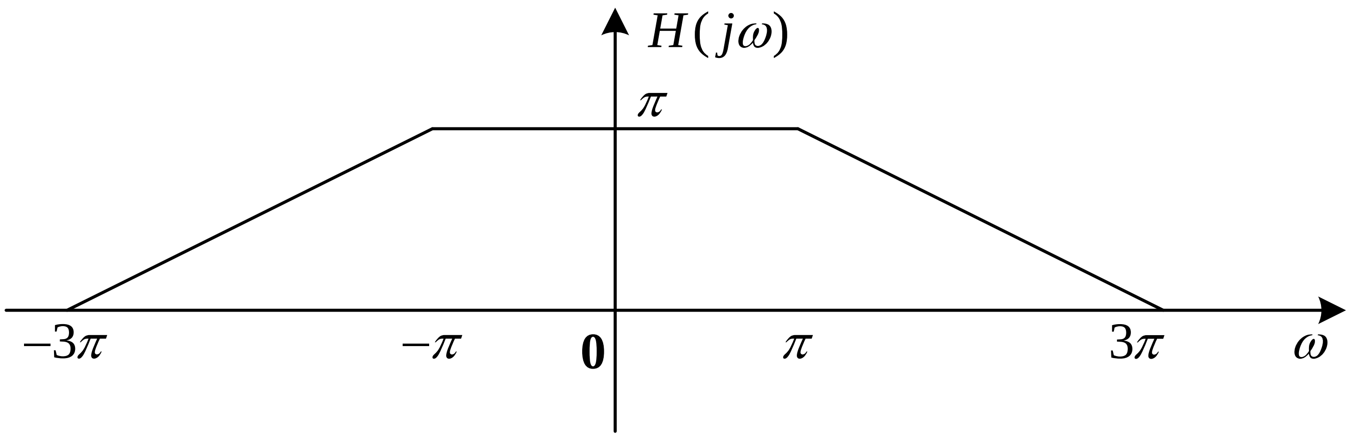
determine the output **y(t)**.

(a) $h(t) = \frac{\sin \pi t \sin 2\pi t}{\pi t^2} = \pi \cdot \frac{\sin \pi t}{\pi t} \cdot \frac{\sin 2\pi t}{\pi t} = \pi \cdot h_1(t) \cdot h_2(t)$

$$h_1(t) \overset{F}{\leftrightarrow} H_1(j\omega)$$

$$h_2(t) \overset{F}{\leftrightarrow} H_2(j\omega)$$

So $H(j\omega) = \pi \cdot \frac{1}{2\pi} H_1(j\omega) * H_2(j\omega) = \frac{1}{2} H_1(j\omega) * H_2(j\omega)$



(b) Since $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$,

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k \boldsymbol{H(jk\omega_0)} e^{jk\omega_0 t}$$

And we have obtained $H(j\omega) = \begin{cases} \pi & |\omega| < \pi \\ -\frac{1}{2}(\omega - 3\pi) & \pi < \omega < 3\pi \\ \frac{1}{2}(\omega + 3\pi) & -3\pi < \omega < -\pi \\ 0 & others \end{cases}$

From $x(t) = 1 + sin\frac{\pi t}{2} + cos\,2\,\pi t + sin\,5\,\pi t$, we need determine

the values of : $\boldsymbol{H(j0) = \pi, H\left(j\frac{\pi}{2}\right) = \pi, H(j2\pi) = \frac{\pi}{2}, H(j5\pi) = 0}$

So $y(t) = \pi + \pi\, sin\frac{\pi t}{2} + \frac{\pi}{2} cos\,2\,\pi t$

Exercise:

Consider the following linear constant-coefficient differential equation:

$$\frac{dy(t)}{dt} + 2y(t) = A\cos\omega_0 t.$$

Find the value of ω_0 such that $y(t)$ will have a maximum amplitude of A/3. Assume that the resulting system is linear and time-invariant.

Solution:

Let $x(t) = Acos\omega_0 t$ so

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

The frequency response of the system is

$$H(j\omega) = \frac{1}{2 + j\omega}$$

We know that the resulting system is linear and time-invariant

So, **t**he output of the system is

$$y(t) = H(j\omega)x(t)$$
$$= |H(j\omega_0)|A\cos(\omega_0 t + \emptyset)$$

$$where\,|H(j\omega_0)| = \frac{1}{\sqrt{4+{\omega_0}^2}}$$

$$\emptyset = \measuredangle H(jw_0)$$

For the maximum value of y(t) to be A/3, we require $\frac{1}{4+{\omega_0}^2} = \frac{1}{9}$

So $\omega_0 = \pm\sqrt{5}$

Resume of Chapter 4

Key points of analysis:

$$x(t) \;\; X(j\omega) \longrightarrow \boxed{\begin{array}{c} h(t) \\ H(j\omega) \end{array}} \longrightarrow y(t) = x(t)*h(t) \;\; Y(j\omega) = X(j\omega)\,H(j\omega)$$

Key points of calculation:

Fourier Properties and

Basic Fourier Transform Pairs

Two important properties:

Convolution Property

Multiplication Property

Homework for Chapter 4:

4.3 4.4(a) **4.10** 4.11 4.14 **4.15**

4.24 **4.25** 4.32(a)(b) **4.35** 4.36

4.37 4.43

Signals and Systems

Chapter 5

The <u>D</u>iscrete <u>T</u>ime <u>F</u>ourier <u>T</u>ransform

5. The Discrete Time Fourier Transform

5.1 Representation of aperiodic signal:DTFT

5.1.1 Development of DTFT

(1) Fourier series (periodic signal)

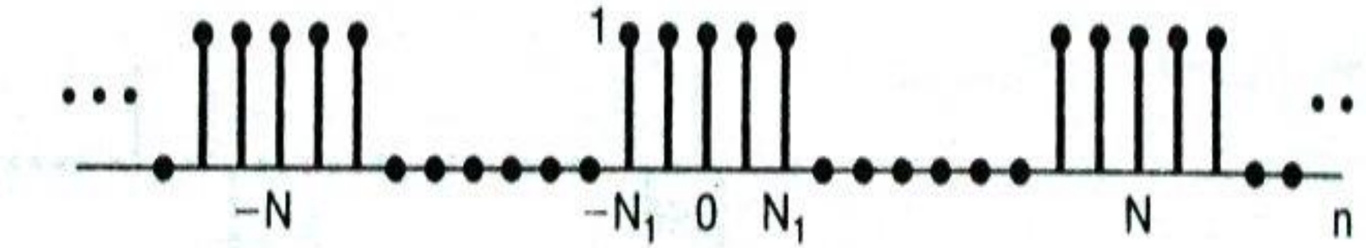
$$\begin{cases} x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n} \\ a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n} \end{cases}$$

(2) Fourier transform (aperiodic signal)

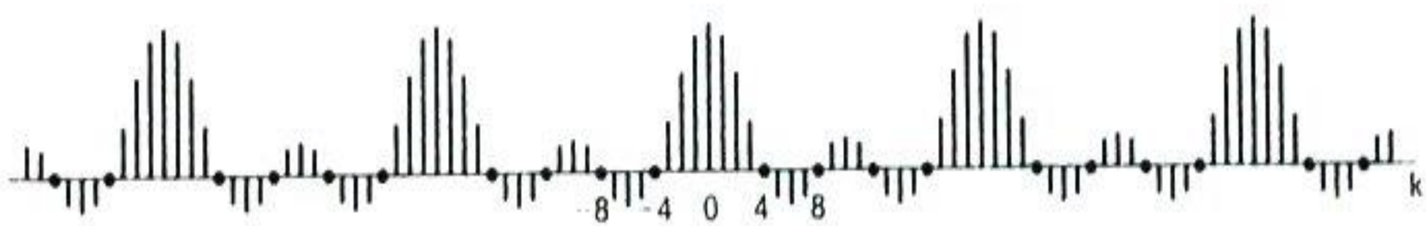
$$\begin{cases} x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \\ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \end{cases}$$

Fourier series (periodic signal)

x[n]

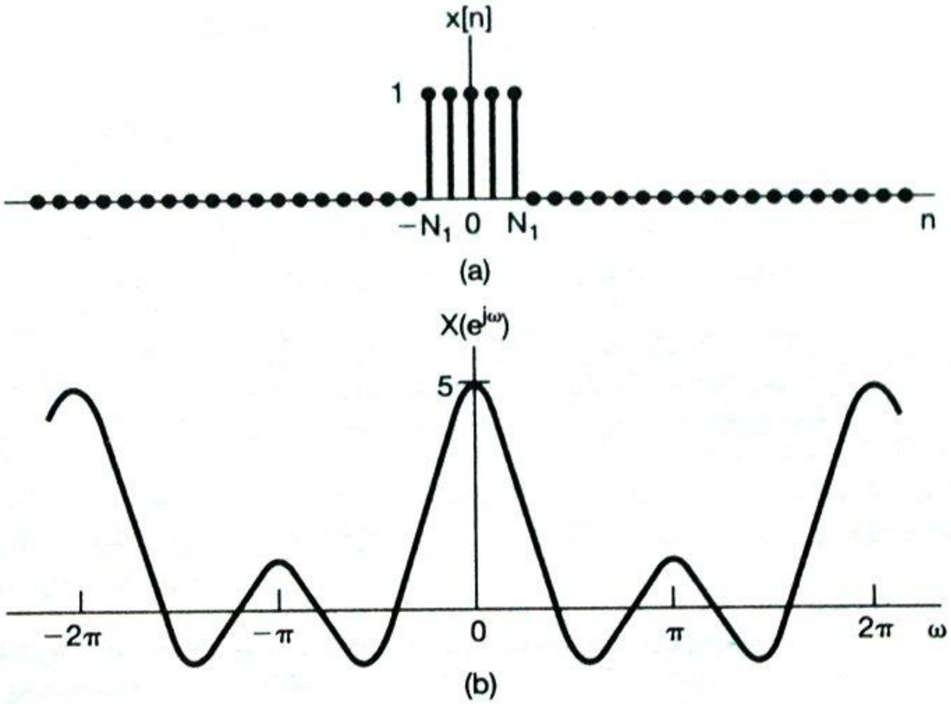


a_k



$$\begin{cases} x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \\ a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} \end{cases}$$

Fourier transform (aperiodic signal)



$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(2\pi/N)n}$$

$$= \frac{1}{N}\sum_{-N_1}^{N_1} x[n]e^{-jk(2\pi/N)n}$$

$$= \frac{1}{N}\underbrace{\sum_{-\infty}^{\infty} x[n]e^{-jk(2\pi/N)n}}_{X(e^{j\omega})}$$

5 The discrete-time Fourier transform

Definition of DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] \overset{DTFT}{\longleftrightarrow} X(e^{j\omega}) \quad \textbf{or} \quad x[n] \overset{FT}{\leftrightarrow} X(e^{j\omega})$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\angle X(e^{j\omega})}$$

$\left|X(e^{j\omega})\right|$ **----Magnitude** $\angle X(e^{j\omega})$ **---phase**

5 The discrete-time Fourier transform

$$x[n] \overset{FT}{\longleftrightarrow} X(e^{j\omega})$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n} d\omega$$

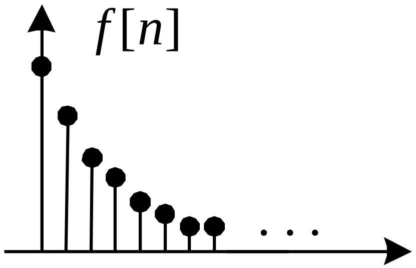
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

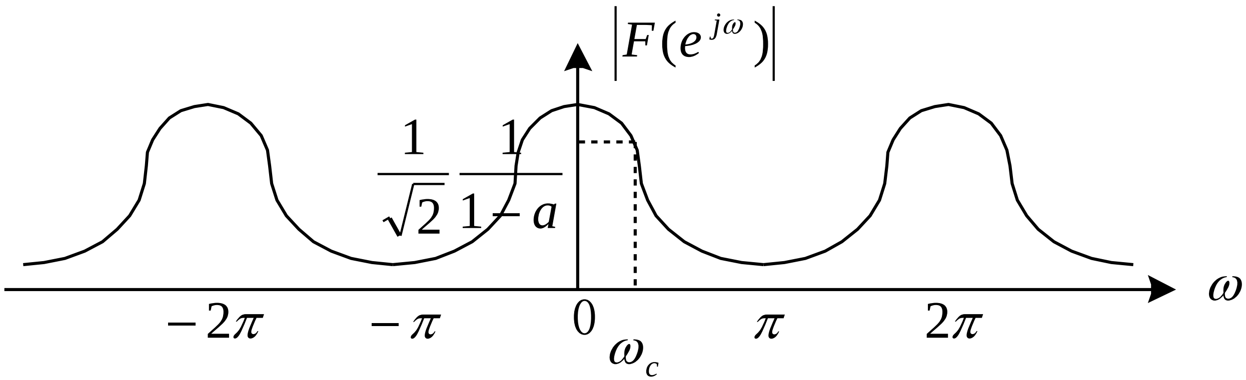
5.1.2 Examples of DTFT

Example $f[n] = a^n u[n]$ $\qquad |a| < 1$

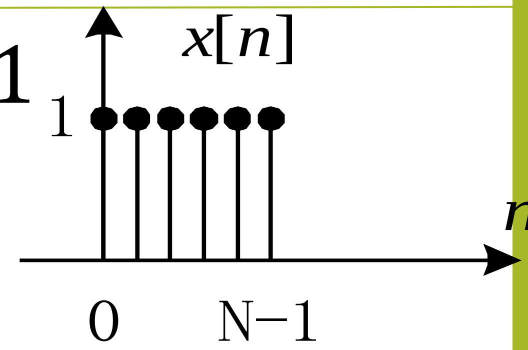
$$F(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-jn\omega} = \frac{1}{1 - ae^{-j\omega}}$$

$$(= \frac{1}{(1 - a\cos\omega) + ja\sin\omega})$$





Example $x[n] = \begin{cases} 1, 0 \le n \le N-1 \\ 0, other\ n \end{cases}$

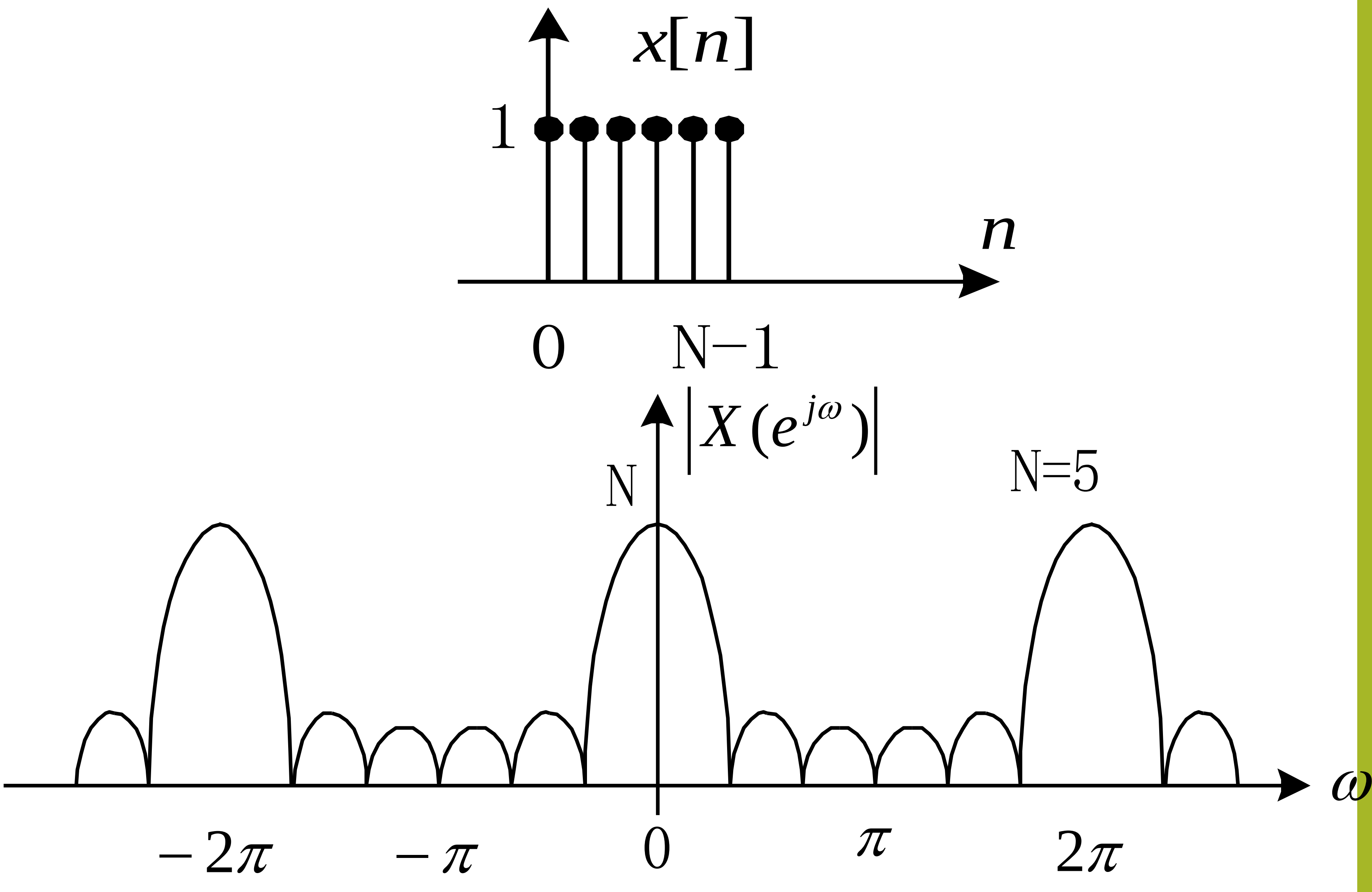


$$X(e^{j\omega}) = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1-e^{-jN\omega}}{1-e^{-j\omega}}$$

Magnitude $$\left|X(e^{j\omega})\right| = \left|\frac{\sin(N\omega/2)}{\sin(\omega/2)}\right|$$

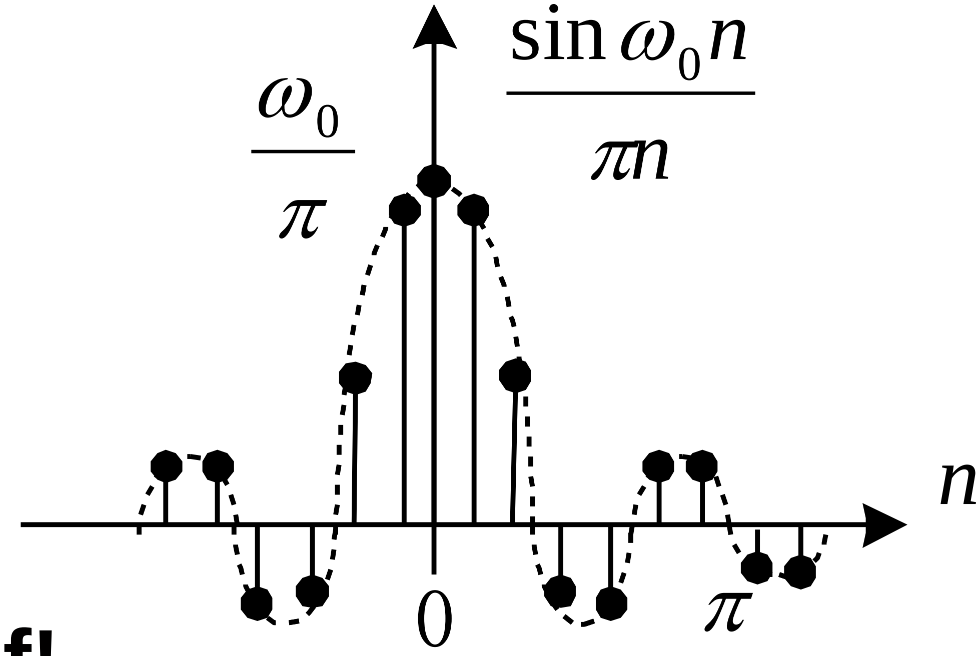
Phase $$\measuredangle X(e^{j\omega}) = -(N-1)\omega/2$$

5 The discrete-time Fourier transform

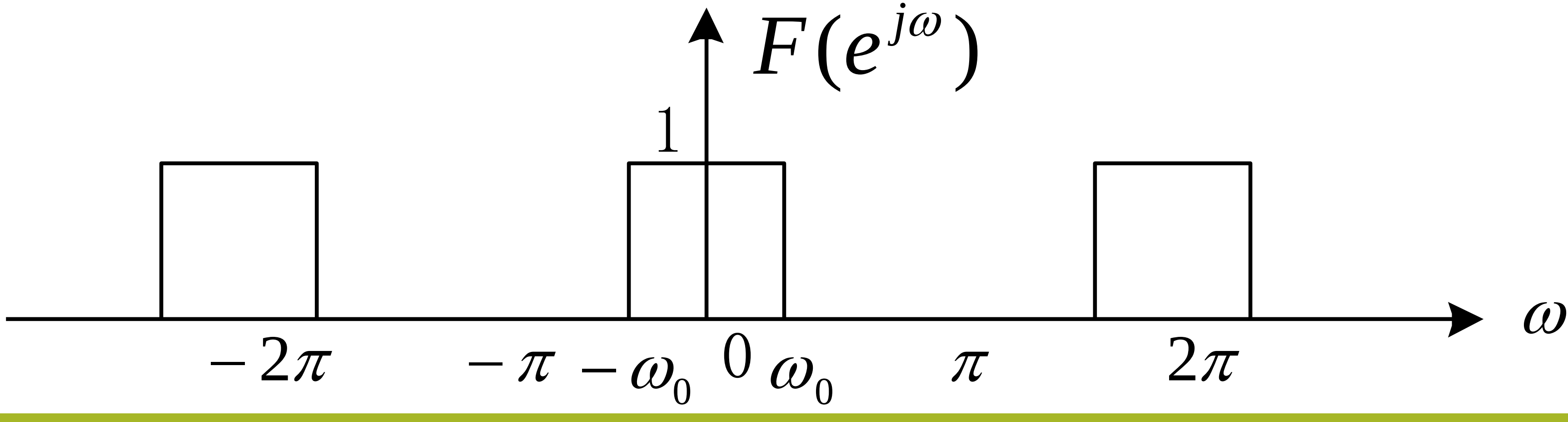


Example

$$f[n] = \frac{\sin \omega_0 n}{\pi n}$$



Prove it by yourself!



Commonly Used DTFT Pairs

Sequence **DTFT**

$$\delta[n] \leftrightarrow 1$$

$$1 \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$$

$$e^{j\omega_o n} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_o+2\pi k)$$

$$u[n] \leftrightarrow \frac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{\infty} \pi \delta(\omega+2\pi k)$$

$$\alpha^n u[n], \quad (|\alpha|<1) \leftrightarrow \frac{1}{1-\alpha e^{-j\omega}}$$

5.2 The Properties of DTFT(5.3~5.6)

The properties of DTFT are similar to the properties of FT in continuous-time.

Table 5.1

Type of Property	Sequence	DTFT
	$x[n]$	$X(e^{j\omega})$
	$h[n]$	$H(e^{j\omega})$
	$y[n]$	$Y(e^{j\omega})$
Linearity	$ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
Time-shifting	$x[n-n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$

5 The discrete-time Fourier transform

Frequency-shifting $e^{j\omega_0 n}x[n]$ $X(e^{j(\omega-\omega_0)})$

Differentiation $nx[n]$ $j\frac{dX(e^{j\omega})}{d\omega}$

Convolution $y[n] = x[n] * h[n]$ $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

Modulation $x[n]y[n]$ $\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's relation

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

Example If it is known that $a^n u[n] \Leftrightarrow \dfrac{1}{1-ae^{-j\omega}}$ **,**
please give the value of

$$\mathbf{I=}\quad \int_{-\pi}^{\pi} \frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})} d\omega$$

Solution **Because**

$$a^n u[n] \Leftrightarrow \frac{1}{1-ae^{-j\omega}}; b^n u[n] \Leftrightarrow \frac{1}{1-be^{-j\omega}}$$

From the convolution in time domain, the following equation can be obtained

5 The discrete-time Fourier transform

$$a^n u[n] * b^n u[n]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})} e^{jn\omega} d\omega$$

So , Let n=0,we can get

I= $2\pi\, \mathrm{a}^n\, u[\mathrm{n}] * \mathrm{b}^n\, u[\mathrm{n}]\,|_{\mathrm{n}=0} = 2\pi \sum_{\mathrm{m}=0}^{\mathrm{n}} \mathrm{a}^{\mathrm{m}} \mathrm{b}^{\mathrm{n}-\mathrm{m}}\,|_{\mathrm{n}=0} = 2\pi$

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{array}{l} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{array}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0} X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}} X(e^{j\omega})$ $+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \Re e\{X(e^{j\omega})\} = \Re e\{X(e^{-j\omega})\} \\ \Im m\{X(e^{j\omega})\} = -\Im m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real and even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n] \text{ real}]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n] \text{ real}]$	$\Re e\{X(e^{j\omega})\}$ $j\Im m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals		

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

5.8 Finite-dimensional LTI Systems

Linear constant coefficients difference equation :

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

Example: Causal LTI system

$$y[n]-ay[n-1]=x[n], |a|<1$$

$$\boldsymbol{So}\ \ Y\left(e^{j\omega}\right)-ae^{-j\omega}Y\left(e^{j\omega}\right)=X(e^{j\omega})$$

$$H\left(e^{j\omega}\right)=\frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}=\frac{1}{1-ae^{-j\omega}}$$

$$h[n]=a^{n}u[n]$$

Example: Causal LTI system

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n]$$

$$So\ \ Y\left(e^{j\omega}\right)-\frac{3}{4}e^{-j\omega}Y\left(e^{j\omega}\right)+\frac{1}{8}e^{-2j\omega}Y\left(e^{j\omega}\right)=2X(e^{j\omega})$$

$$H\left(e^{j\omega}\right)=\frac{Y\left(e^{j\omega}\right)}{X(e^{j\omega})}=\frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}}$$

5 The discrete-time Fourier transform

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}$$

$$= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} = \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}$$

$$h[n] = 4(\frac{1}{2})^n u[n] - 2(\frac{1}{4})^n u[n]$$

Example:

A particular discrete-time system has input $x[n]$ and output $y[n]$. The Fourier transforms of these signals are related by the following equation:

$$Y(e^{j\omega}) = 2X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega}) - \frac{dX(e^{j\omega})}{d\omega}$$

(a) Is this system linear? Clearly justify your answer.

(b) Is the system time-invariant? Clearly justify your answer.

(c) What is $y[n]$ if $x[n] = \delta[n]$?

Solution:

(a) Here $Y(e^{j\omega}) = 2X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega}) - \frac{dX(e^{j\omega})}{d\omega}$

The system is linear because if

$$\boldsymbol{x}[n] = \boldsymbol{a}\boldsymbol{x}_{\mathbf{1}}[n] + \boldsymbol{b}\boldsymbol{x}_{\mathbf{2}}[n]$$

Then $\boldsymbol{y}[n] = \boldsymbol{a}\boldsymbol{y}_{\mathbf{1}}[n] + \boldsymbol{b}\boldsymbol{y}_{\mathbf{2}}[n]$, where $\boldsymbol{y}_{\mathbf{1}}[n]$ is obtained from $\boldsymbol{x}_{\mathbf{1}}[n]$ via the given transfer function. The similar result applies for $\boldsymbol{y}_{\mathbf{2}}[n]$.

(b) The system is time-varying by the following argument.

If $\boldsymbol{x}[n] \rightarrow y[n]$, does $\boldsymbol{x}[n-\mathbf{1}] \rightarrow y[n-\mathbf{1}]$?

$$\boldsymbol{x}[n-\mathbf{1}] \overset{\boldsymbol{F}}{\leftrightarrow} \mathbf{e}^{-j\omega} X\left(\mathbf{e}^{j\omega}\right)$$

The corresponding $Y\left(\mathbf{e}^{j\omega}\right)$ is

$$\mathbf{2e}^{-j\omega} X\left(\mathbf{e}^{j\omega}\right) + \mathbf{e}^{-j\omega} X\left(\mathbf{e}^{j\omega}\right) \mathbf{e}^{-j\omega} + \boldsymbol{j}\mathbf{e}^{-j\omega} X\left(\mathbf{e}^{j\omega}\right) - \mathbf{e}^{-j\omega} \frac{\boldsymbol{dX}\left(\mathbf{e}^{j\omega}\right)}{\boldsymbol{d\omega}} \neq \mathbf{e}^{-j\omega}\left[2X\left(\mathbf{e}^{j\omega}\right) + \mathbf{e}^{-j\omega} X\left(\mathbf{e}^{j\omega}\right) - \frac{\boldsymbol{dX}\left(\mathbf{e}^{j\omega}\right)}{\boldsymbol{d\omega}}\right]$$

(c) If $\boldsymbol{x}[n] = \boldsymbol{\delta}[n]$,

$$X\left(\mathbf{e}^{j\omega}\right) = \mathbf{1}$$

$$\boldsymbol{then} \quad Y\left(\mathbf{e}^{j\omega}\right) = \mathbf{2} + \mathbf{e}^{-j\omega}$$

$$\boldsymbol{so} \quad \boldsymbol{y}[\boldsymbol{n}] = \mathbf{2}\,\boldsymbol{\delta}[n] + \boldsymbol{\delta}[n - \mathbf{1}]$$

No Class Test and Homework for Chapter 5