Signals and Systems

Chapter 10

The Z-Transform

Previous Knowledge:

Chap.3

$$x(t) = est \rightarrow \boxed{h(t)} \rightarrow y(t) = H(s)e^{st}$$

$$x[n] = e^{j\omega n} \rightarrow \boxed{h[n]} \rightarrow y[n] = H(z)e^{j\omega n}$$

$$\begin{cases} H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt \\ H(z) = \sum_{k=-\infty}^{+\infty} h[k]\, z^{-k} \end{cases}$$

(**system function**)

Previous Knowledge:

$$\underline{z = e^{j\omega}}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]\, e^{-j\omega n}$$

(**Frequency response**)

Chap.5: DT Fourier Transform

Previous Knowledge:

$z = re^{j\omega}$: General complex variable

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]\, z^{-n} \Rightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$\Downarrow$

Chap.10: Z-Transform

10. The Z- Transform

10.1 The Z- Transform

An LTI system of D-T

$$x[n] = z^n \longrightarrow y[n] = H(z)z^n$$

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]z^{-n}$$

(1) Definition

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} \quad \boxed{\text{ROC}}$$

$$x[n] \xleftrightarrow{Z} X(z) \quad \boxed{\text{ROC}}$$

(2) Region of Convergence (ROC)

ROC: Range of |z| for X(z) to converge

Representation:

A. Inequality

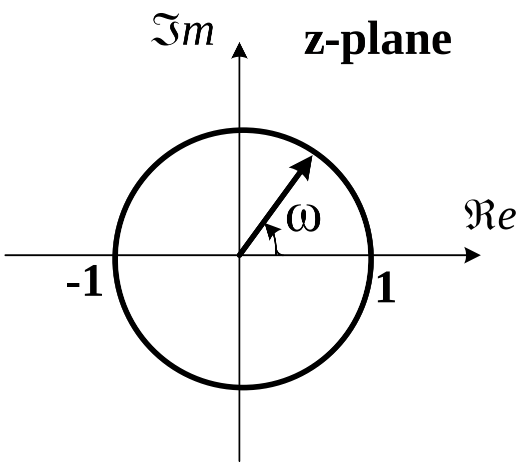
B. Region in Z-plane

Let,
$z = re^{j\omega}$

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n}$$

Then

$$= \sum_{n=-\infty}^{\infty} \{x[n]r^{-n}\}e^{-j\omega n} = \boxed{F\{x[n]r^{-n}\}}$$



FT converges, and Z-transform converges.

*** ROC of X(z):** $|z|$? ?

Example 10.1

$$x[n] = a^n u[n]$$

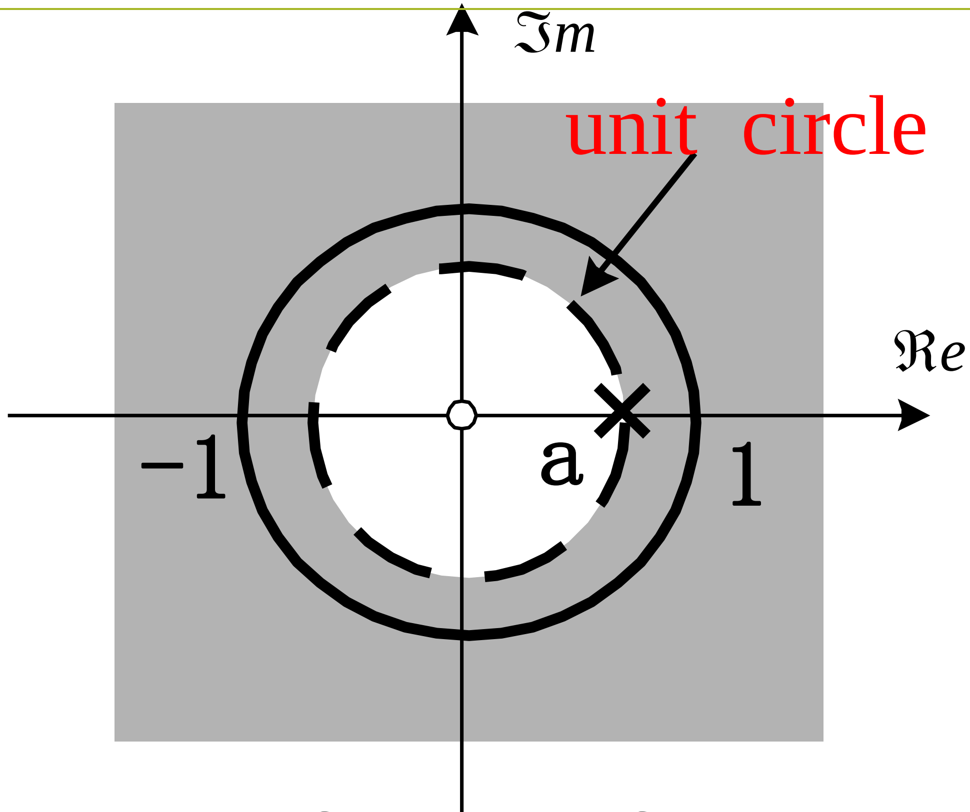
$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

How to determine the ROC of X(z)?

$$\therefore X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, |z| > |a|$$

$$a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, |z| > |a|$$

Z-plane



The ROC of signals for 0<a<1
(right-sided signal)

Specially, some commonly used ZT pairs are in next slide.

$$u[n] \xleftrightarrow{Z} \frac{1}{1-z^{-1}} \quad ,|z|>1$$

$$\cos(\omega_0 n)u[n] \xleftrightarrow{Z} \frac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}},|z|>1$$

$$\sin(\omega_0 n)u[n] \xleftrightarrow{Z} \frac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}},|z|>1$$

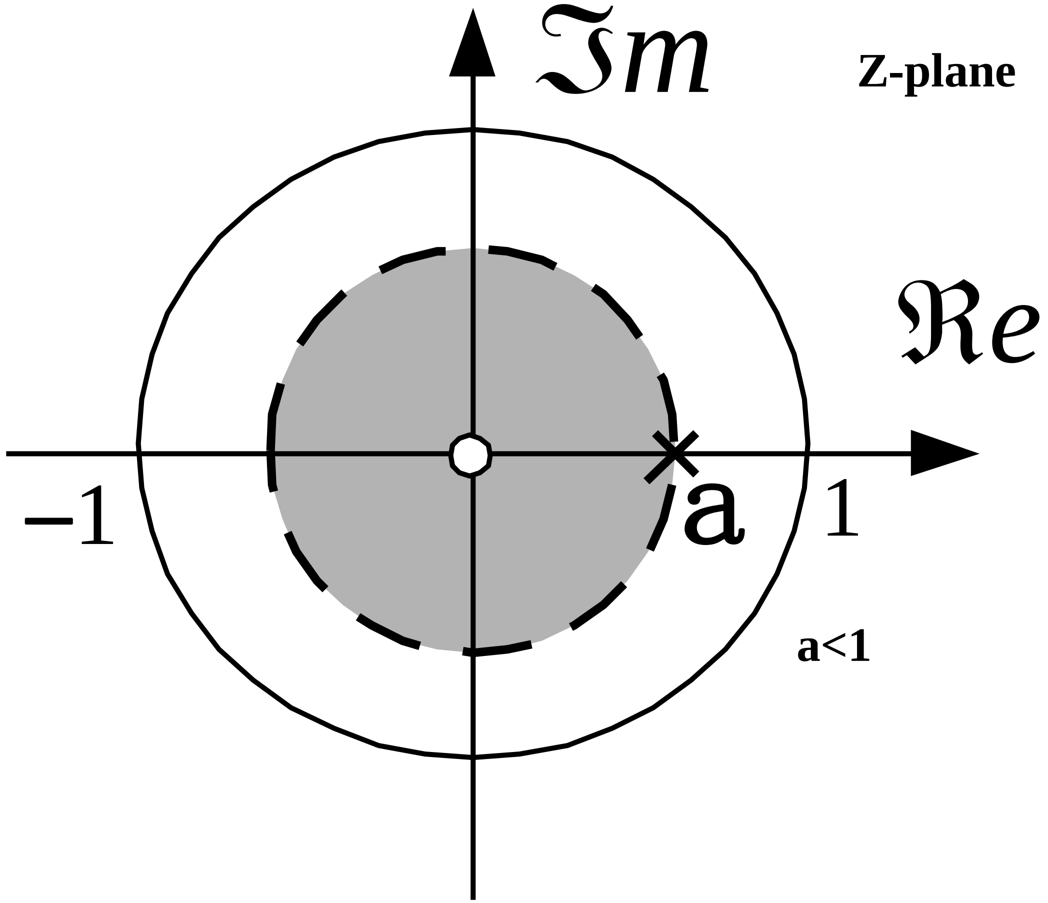
Example 10.2 $x[n] = -a^n u[-n-1]$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\boxed{\sum_{n=-\infty}^{-1}} (az^{-1})^n$$

$$= -\sum_{n=1}^{\infty} (a^{-1}z)^n = -\frac{a^{-1}z}{1-a^{-1}z}, |z| < |a|$$

$$\therefore X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, |z| < |a|$$

$$-a^n u[-n-1] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, |z| < |a|$$



If a>1?

The ROC of X(z) when 0<a<1
(a left-sided signal)

Example

$$\delta[n] \stackrel{Z}{\longleftrightarrow} 1, \quad 0 \le |z| \le \infty$$

$$\because \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1$$

The ROC is the entire z-plane .

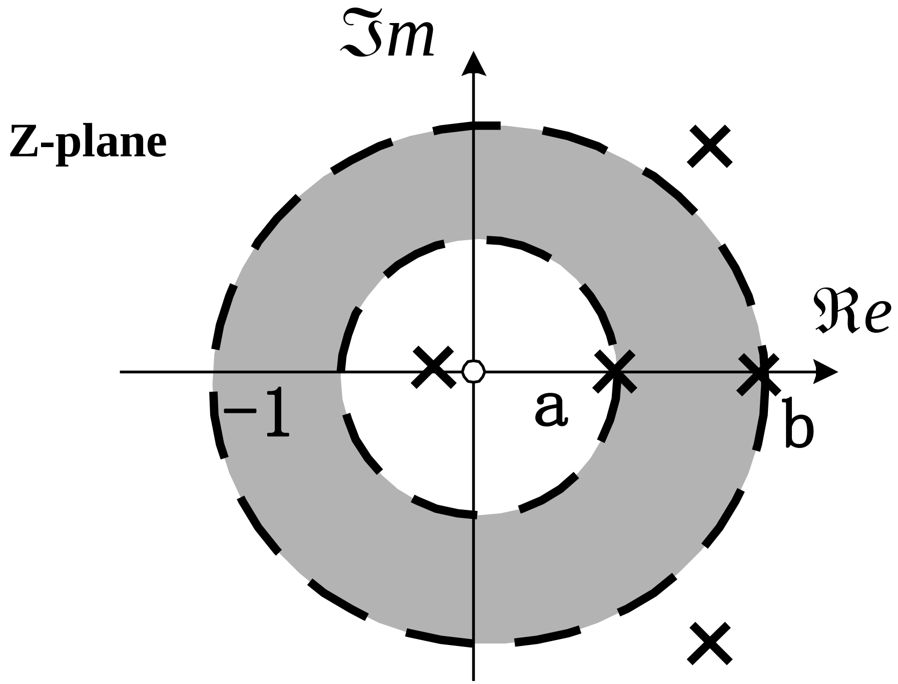
Read example 10.3~4 by yourself!

Normally, X(z) is rational in z (or z^{-1}) form.

10.2 The Region of Convergence for *Z* Transform

Property1: The ROC of X(z) consists of a ring in the z-plane centered about the origin.

Property2: the ROC does not contain any poles.



Property3: If x[n] is of finite duration and is absolutely summable, then the ROC is the entire z-plane , except possibly z=0 and/or z=∞.

Example:

$$\delta[n-1] \xleftrightarrow{Z} z^{-1}, 0 < |z| \leq \infty$$

$$\because \sum_{n=-\infty}^{\infty} \delta[n-1] z^{-n} = z^{-1}$$

$$\delta[n+1] \xleftrightarrow{Z} z, 0 \leq |z| < \infty$$

Example 10.6

$$x[n]=\begin{cases} a^n, 0\le n\le N-1 \\ 0, otherwise \end{cases}$$

finite duration

Length: N

$$0<a<1$$

$$X(z)=\sum_{n=0}^{N-1} a^n z^{-n} = \frac{1-a^N z^{-N}}{1-az^{-1}}$$

or $$=\frac{1}{z^{N-1}}\frac{z^N-a^N}{z-a}$$

$$0<|z|\le\infty$$

ROC:

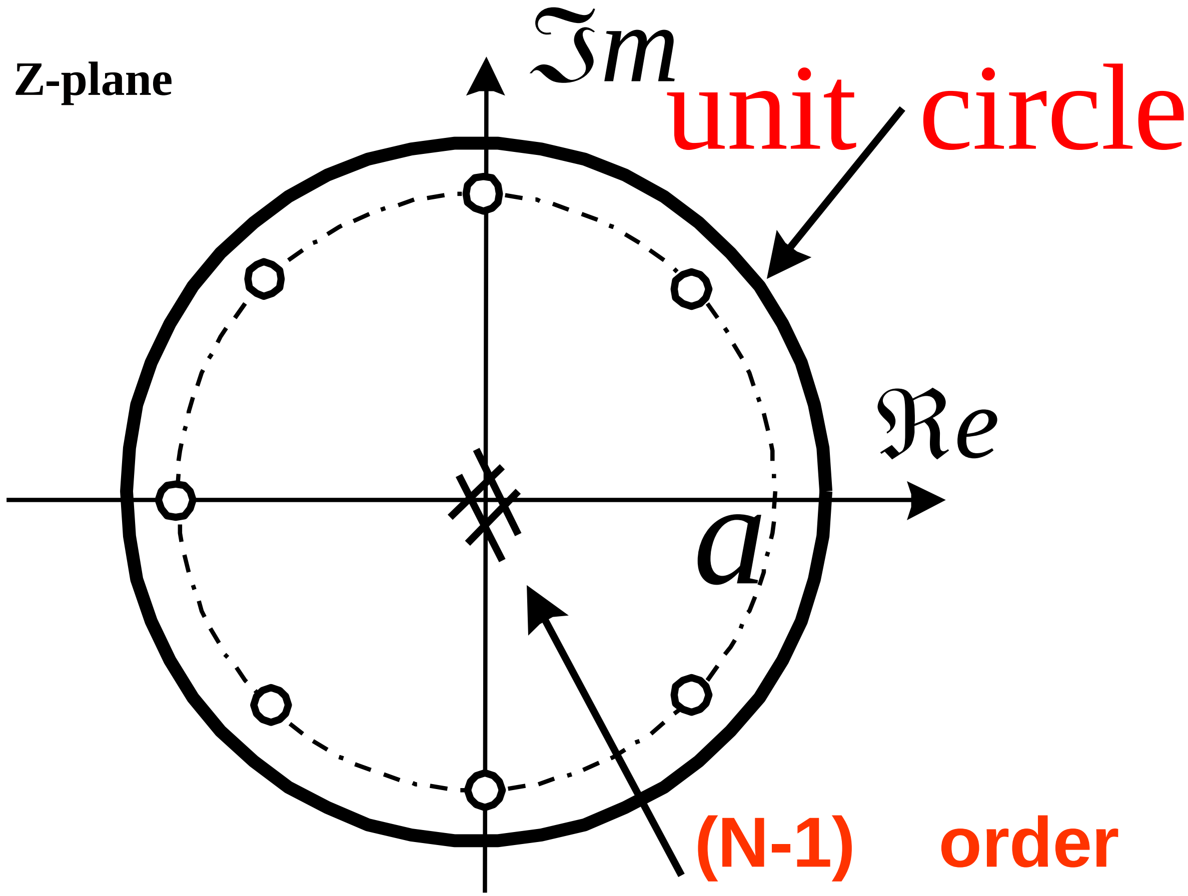
zeros:

$$z_k = ae^{j(\frac{2\pi}{N})k}$$

$$k = 1,2,\ldots, N-1$$

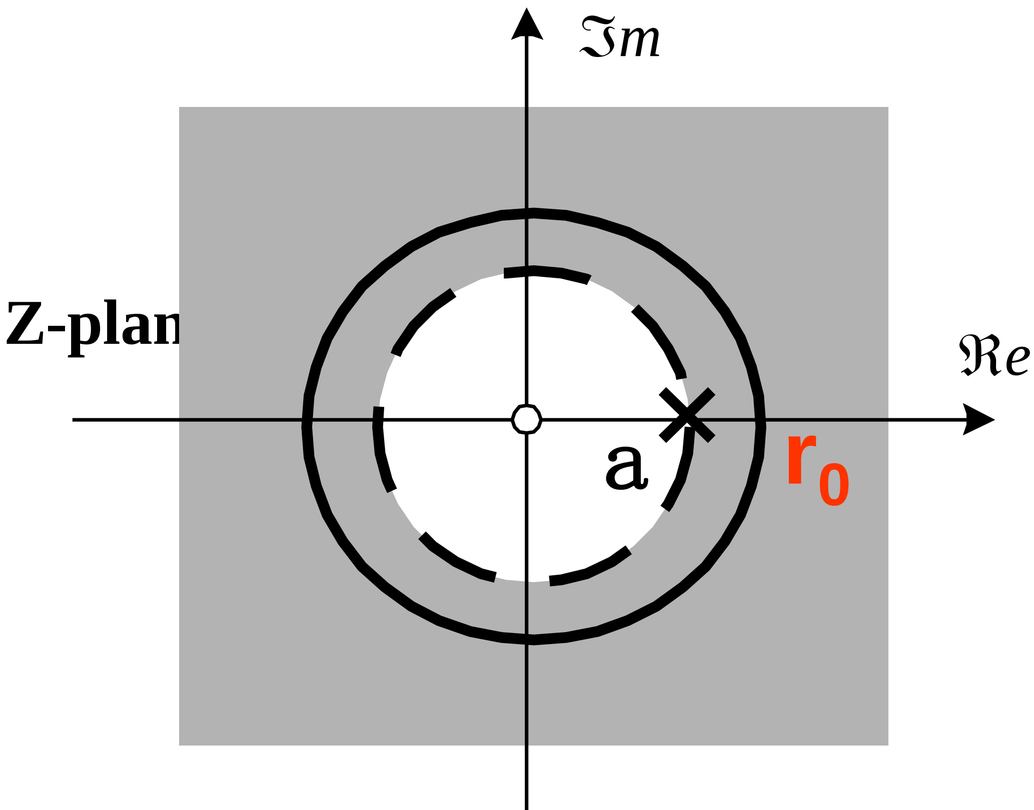
poles: z=0, (N-1)th order, and z=a

Zero-pole plot is shown in the figure of next slide.



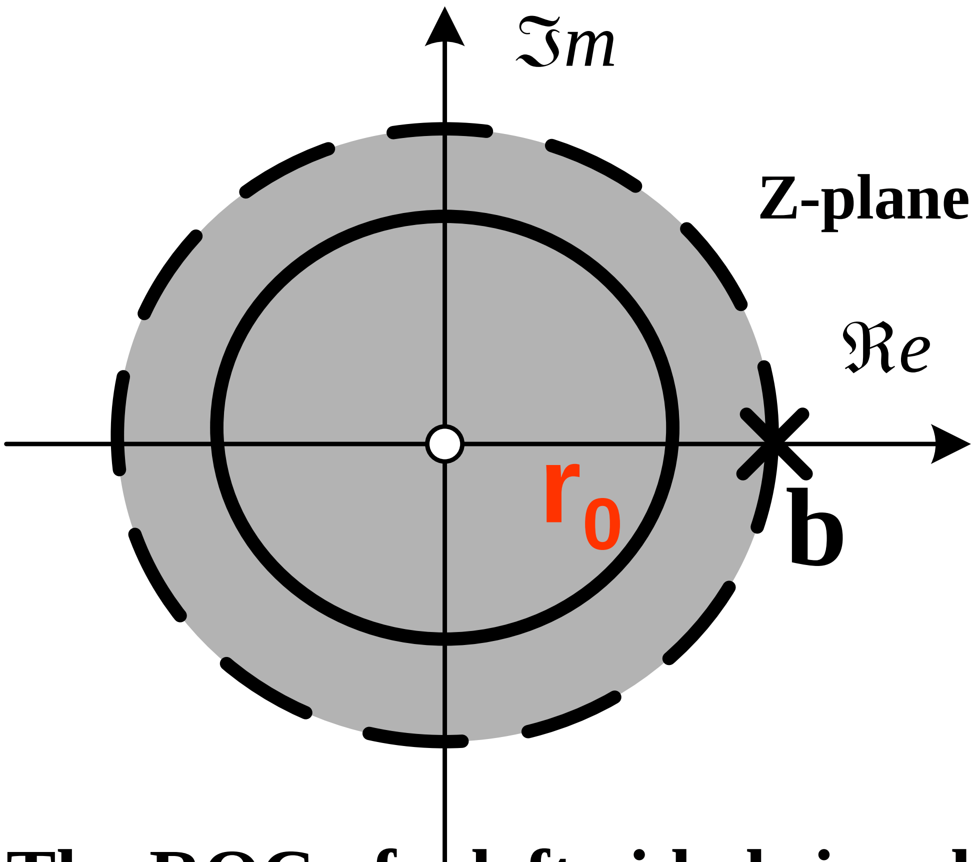
Note: z=a is cancelled.

Property4: If x[n] is right sided, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also in the ROC.



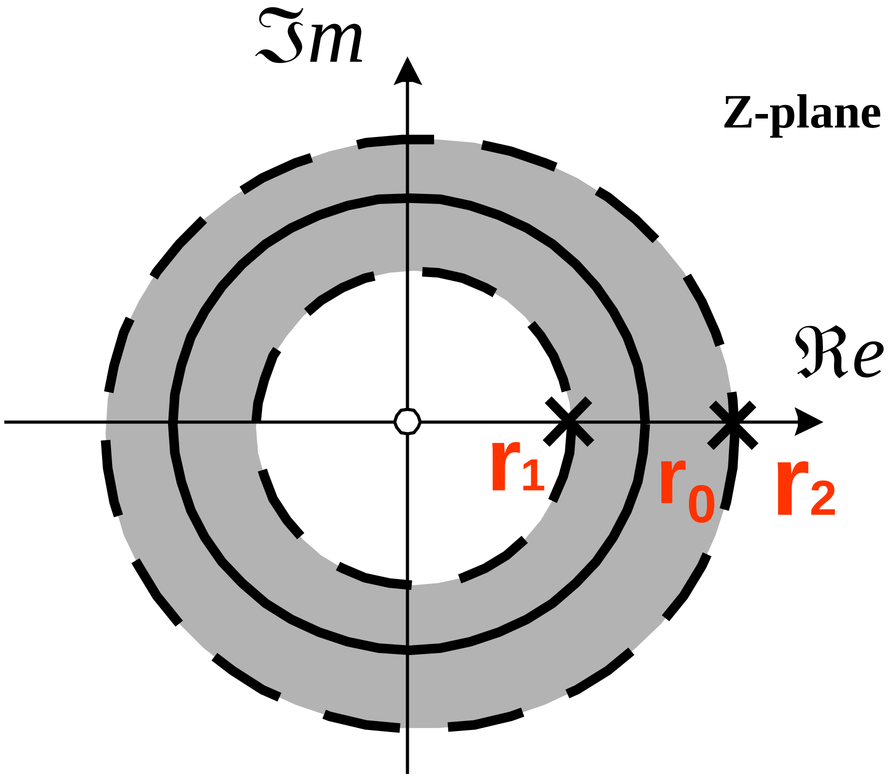
The ROC of a right-sided signal

Property5: If x[n] is left sided, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also in the ROC.

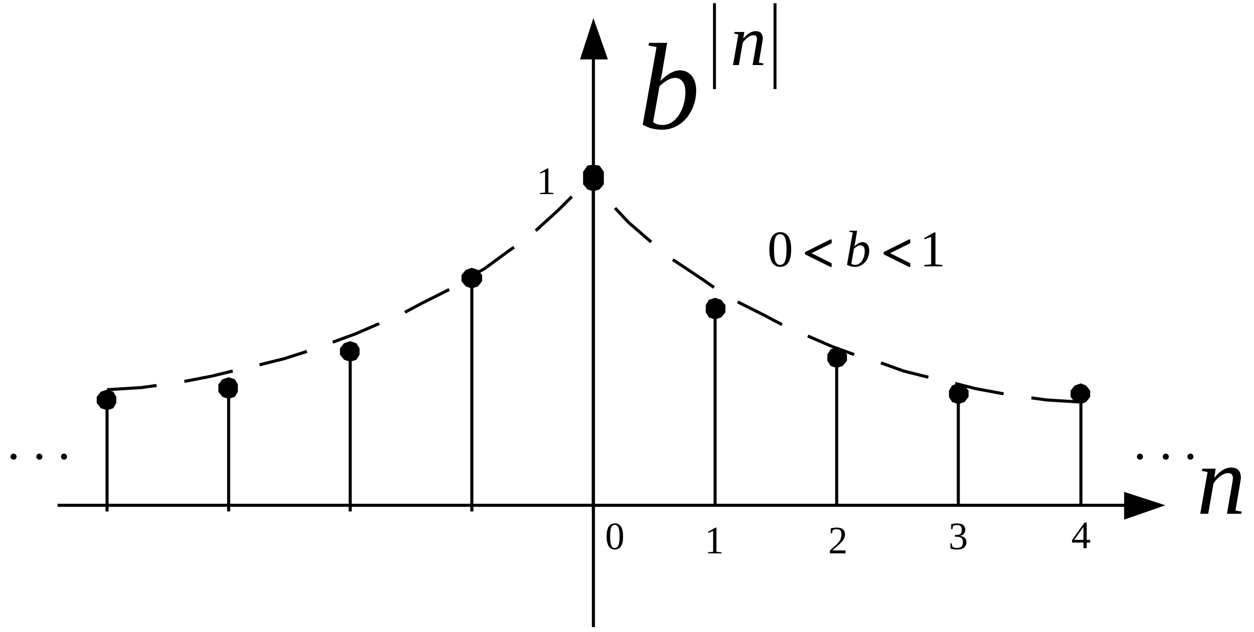


The ROC of a left-sided signal

Property6: If x[n] is two sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$. Normally, $r_1 < |z| < r_2$. ($r_1 < r_2$)



Example 10.7 $x[n] = b^{|n|},\ for\ 0 < b < 1$



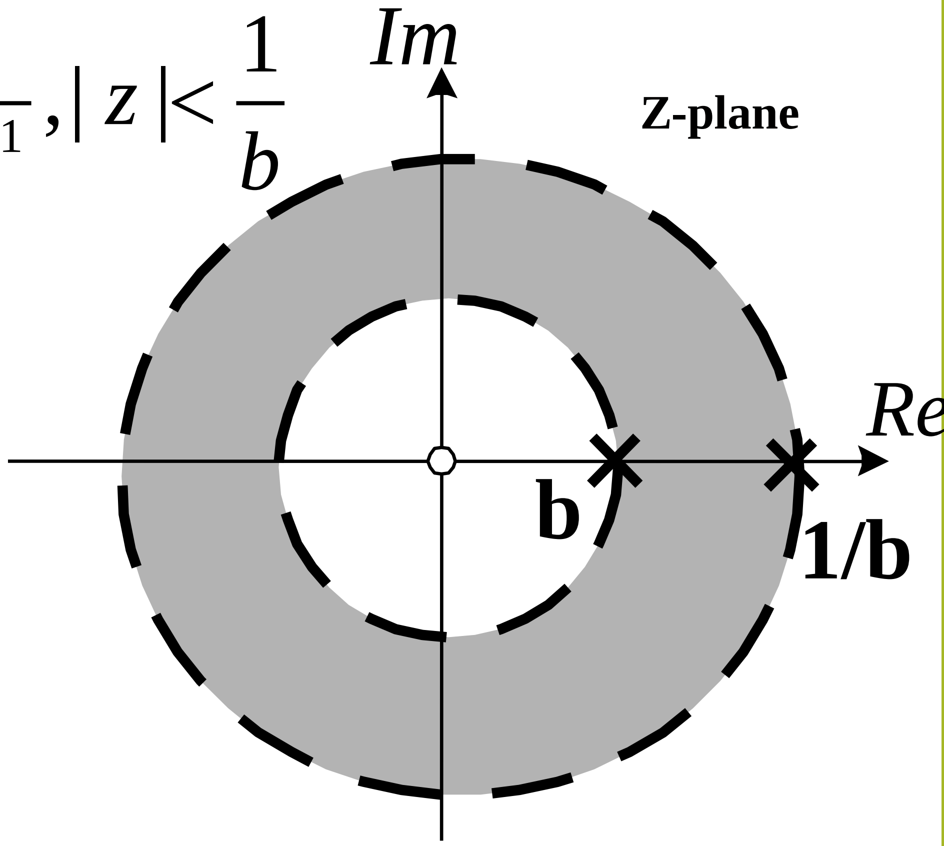
$$x[n] = b^{n}u[n] + b^{-n}u[-n-1]$$

$$b^n u[n] \xleftrightarrow{Z} \frac{1}{1-bz^{-1}}, |z| > b$$

$$b^{-n} u[-n-1] \xleftrightarrow{Z} \frac{-1}{1-b^{-1}z^{-1}}, |z| < \frac{1}{b}$$

$$X(z) = \frac{1}{1-bz^{-1}} - \frac{1}{1-b^{-1}z^{-1}},$$

$$b < |z| < \frac{1}{b}$$



When $b > 1, X(z)$ **is not existed.**

Property7: If the Z transform X(z) of x[n] is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of X(z) are contained in the ROC.

Property8: If the Z transform X(z) of x[n] is rational, and if x[n] is right sided, then the ROC is the region in the z-plane outside the outmost pole---I.e., outside the circle of radius equal to the largest magnitude of the poles of X(z). Furthermore, if it is causal, then the ROC also includes $z=\infty$.

Property9: If the z- transform X(z) of x[n] is rational, and if x[n] is left sided, then the ROC is

the region in the z-plane inside the innermost nonzero pole---I.e., inside the circle of radius equal to the smallest magnitude of the poles of X(z) other than any at z=0 and extending inward to and possibly including z=0.

In particular. if x[n] is anticausal, then the ROC also includes z=0.

Example 10.8

$$X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})},$$

There are 3 possible ROCs

	ROC1	ROC2	ROC3
	$\lvert z\rvert > 2$	$\lvert z\rvert < \frac{1}{3}$	$\frac{1}{3} < \lvert z\rvert < 2$
x[n]	**right sided**	**left sided**	**two sided**

$$X(z)=\frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}=\frac{-2}{1-\frac{1}{3}z^{-1}}+\frac{3}{1-2z^{-1}},$$

ROC1: $|z|>2,\quad x[n]=-2\left(\frac{1}{3}\right)^{n}u[n]+3(2)^{n}u[n]$

ROC2: $|z|<\frac{1}{3},\quad x[n]=2\left(\frac{1}{3}\right)^{n}u[-n-1]-3(2)^{n}u[-n-1]$

ROC3: $\frac{1}{3}<|z|<2,\quad x[n]=-2\left(\frac{1}{3}\right)^{n}u[n]-3(2)^{n}u[-n-1]$

10.3 The Inverse Z-Transform

From $X(z) = X(re^{j\omega}) = DTFT\{r^{-n}x[n]\}$

$$\therefore r^{-n}x[n] = F^{-1}\{X(re^{j\omega})\}$$

$$= \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{jn\omega}d\omega$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})(re^{j\omega})^n d\omega$$

Let $z = re^{j\omega}$ $\quad d\omega = \frac{1}{jz} dz$

So

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

Integration around a counterclockwise closed circular contour centered at the origin and with radius *r*.

The calculation for inverse Z-Transform:

(1) Integration of complex function by equation.

(2) Compute by Partial Fraction Expansion .

(3) power-series expansion

Example 10.9 The inverse ZT is wanted, if

$$X(z)=\frac{3-\frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})},|z|>\frac{1}{3}$$

Partial Fraction Expansion in z^{-1} form

$$X(z)=\frac{1}{(1-\frac{1}{4}z^{-1})}+\frac{2}{(1-\frac{1}{3}z^{-1})},|z|>\frac{1}{3}$$

we get

$$x_1[n]=(\frac{1}{4})^n u[n]\xleftrightarrow{Z}\frac{1}{(1-\frac{1}{4}z^{-1})},|z|>\frac{1}{4}$$

$$x_2[n]=2(\frac{1}{3})^n u[n]\xleftrightarrow{Z}\frac{2}{(1-\frac{1}{3}z^{-1})},|z|>\frac{1}{3}$$

$$x[n] = x_1[n] + x_2[n] = (\frac{1}{4})^n u[n] + 2(\frac{1}{3})^n u[n]$$

Example 10.10

Same form of X(z), but with different ROC .

$$X(z) = \frac{1}{(1 - \frac{1}{4}z^{-1})} + \frac{2}{(1 - \frac{1}{3}z^{-1})}, \frac{1}{4} < |z| > \frac{1}{3}$$

$$x_1[n] = (\frac{1}{4})^n u[n] \xleftrightarrow{\;Z\;} \frac{1}{(1-\frac{1}{4}z^{-1})}, |z| > \frac{1}{4}$$

$$x_2[n] = -2(\frac{1}{3})^n u[-n-1]$$

$$x_2[n] \xleftrightarrow{\;Z\;} \frac{2}{(1-\frac{1}{3}z^{-1})}, |z| < \frac{1}{3}$$

So, we get $$x[n] = x_1[n] + x_2[n]$$

$$= (\frac{1}{4})^n u[n] - 2(\frac{1}{3})^n u[-n-1]$$

Normally, Partial Fraction Expansion of rational X(z)

$$X(z) = \sum_{i=1}^{m} \frac{A_i}{1 - a_i z^{-1}}$$

Example 10.12

$$X(z) = 4z^{2} + 2 + 3z^{-1}, 0 < |z| < \infty$$

From the power-series definition of ZT, we get:

$$x[n] = \begin{cases} 4, & n = -2 \\ 2, & n = 0 \\ 3, & n = 1 \\ 0, & otherwise \end{cases}$$

So, $x[n] = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1]$

$$\delta[n+n_0] \xleftrightarrow{Z} z^{n_0}, \quad 0 \le |z| < \infty$$

Example 10.13 **Consider** $X(z)=\dfrac{1}{1-az^{-1}}$

ROC1: $|z|>|a| \quad \therefore |az^{-1}|<1$

$$x[n]=\{1,a,a^2,\ldots\}=a^n u[n]$$

$$\because \frac{1}{1-az^{-1}}=1+az^{-1}+a^2z^{-2}+\ldots$$

$$\therefore x[n]=\delta[n]+a\delta[n-1]+a^2\delta[n-2]+\ldots$$

$$=a^n u[n]$$

ROC₂: $|z| < |a| \qquad \therefore |az^{-1}| > 1$

$$x[n] = \{\ldots, -a^{-2}, -a^{-1}, 0\} = -a^{-n}u[-n-1]$$

$$\because \frac{1}{1-az^{-1}} = -a^{-1}z - a^{-2}z^{2} - \ldots$$

$$\therefore x[n] = -a^{-1}\delta[n+1] - a^{-2}\delta[n+2] + \ldots$$

$$= -a^{-n}u[-n-1]$$

Example 10.14 $X(z) = \log(1 + az^{-1}), |z| > |a|$

$$x[n] = ?$$

$$\log(1+v) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} v^n}{n}, |v| < 1$$

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

$$x[n] = \begin{cases} \dfrac{(-1)^{n+1} a^n}{n}, n \geq 1 \\ 0, n \leq 0 \end{cases}$$

10.4 The Properties of Z-Transform (10.5)

10.4.1 Linearity

If $$x_1[n] \xleftrightarrow{Z} X_1(z),$$ **R_1**

$$x_2[n] \xleftrightarrow{Z} X_2(z),$$ **R_2**

Then

$$ax_1[n] + bx_2[n] \xleftrightarrow{Z} aX_1(z) + bX_2(z),$$

ROC containing $R_1 \cap R_2$

Note: (1) normally, common ROC (overlap).

(2) $R_1 \cap R_2$ may be larger than R_1 or R_2

For example:

$$x_1[n] = a^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, |z| > |a|$$

$$x_2[n] = a^n u[n-1] \overset{Z}{\longleftrightarrow} \frac{az^{-1}}{1-az^{-1}}, |z| > |a|$$

$$x_1[n] - x_2[n] = \delta[n] \overset{Z}{\longleftrightarrow} 1, -\infty < |z| < \infty$$

10.4.2 Time Shifting

If $$x[n] \xleftrightarrow{Z} X(z),$$ **R**

Then $$x[n-n_0] \xleftrightarrow{Z} z^{-n_0} X(z),$$ **R**

Example From:

$$a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}} = \frac{z}{z-a}, |z| > |a|$$

We can get

$$a^{n-1} u[n-1] \xleftrightarrow{Z} \frac{z^{-1}}{1-az^{-1}} = \frac{1}{z-a}, |z| > |a|$$

10.4.3 Scaling in z-Domain

If $x[n] \xleftrightarrow{Z} X(z),$ **R**

Then $z_0^{\,n} x[n] \xleftrightarrow{Z} X(z/z_0),$ **|z0| R**

Why?

Specially, $e^{j\omega_0 n} x[n] \xleftrightarrow{Z} X(e^{-j\omega_0} z),$ **R**

Example $\omega_0 = \pi$

$$(-1)^n x[n] \xleftrightarrow{Z} X(-z),$$ **R**

10.4.4 Time Reversal

If $$x[n] \xleftrightarrow{Z} X(z),$$ **R**

Then $$x[-n] \xleftrightarrow{Z} X(\frac{1}{z}),$$ **1/R**

Example

From:

$$a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, |z| > |a|$$

We can get

$$a^{-n}u[-n] \xleftrightarrow{Z} \frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}}, |z| < |a|^{-1}$$

Furthermore, $a^{-1} \to a$

$$a^{n}u[-n] \xleftrightarrow{Z} \frac{-(a/z)}{1-az^{-1}}, |z| < |a|$$

Delay the sequence,we can get

$$-a^{n}u[-n-1] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, |z| < |a|$$

10.4.5 Time Expansion

If $x[n] \xleftrightarrow{Z} X(z),$ **R**

Then, $x_{(k)}[n] \xleftrightarrow{Z} X(z^k),$ $\mathbf{R}^{1/k}$ **Why?**

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{, if } n \text{ is a multiple of } k \\ 0 & \text{, if } n \text{ is not a multiple of } k \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} \Longrightarrow X(z^k) = \sum_{n=-\infty}^{+\infty} x[n](z^k)^{-n}$$

10.4.6 Conjugation

If $x[n] \xleftrightarrow{Z} X(z),$ **R**

Then, $x^*[n] \xleftrightarrow{Z} X^*(z^*),$ **R**

If $x[n]$ **is real,** $X(z) = X^*(z^*)$

If X(z) has a pole (or zero) at $z = z_0$**,**

It must also have a pole (or zero) at $Z = Z_0^*$

10.4.7 The Convolution Property

If $$x_1[n] \xleftrightarrow{Z} X_1(z),$$ **R1**

Then $$x_2[n] \xleftrightarrow{Z} X_2(z),$$ **R2**

$$x_1[n] * x_2[n] \xleftrightarrow{Z} X_1(z)X_2(z),$$

Containing R1 ∩R2

Note: if R1 ∩R2=Ø, $X_1(z)X_2(z)$ **does not exist.**

Example 10.15

If $$y[n]=x[n]*h[n]$$

where $$h[n]=\delta[n]-\delta[n-1]$$

$$h[n]=\delta[n]-\delta[n-1]\xleftrightarrow{Z}1-z^{-1}$$

ROC: entire z-plane except the origin

$$x[n]\xleftrightarrow{Z}X(z),R$$

$$y[n]\xleftrightarrow{Z}(1-z^{-1})X(z),$$ **R**

Example

If $x[n] \xleftrightarrow{Z} X(z),$ **R**

Then

$$g[n] = \sum_{k=-\infty}^{n} x[k] = x[n] * u[n] \xleftrightarrow{Z} G(z)$$

$$G(z) = X(z)\frac{1}{1-z^{-1}},$$ **R ∩ |z|>1**

10.4.8 Differentiation in the z-Domain

If $x[n] \xleftrightarrow{Z} X(z),$ **R**

Then, $nx[n] \xleftrightarrow{Z} -z\dfrac{dX(z)}{dz},$ **R**

Example From:

$$a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, |z| > |a|$$

We can get

$$na^n u[n] \xleftrightarrow{Z} -z\frac{d}{dz}\left[\frac{1}{1-az^{-1}}\right] = \frac{az^{-1}}{(1-az^{-1})^2},$$

Example: The inverse ZT is wanted.

$$x[n] \overset{Z}{\longleftrightarrow} X(z) = \ln(1 - \frac{1}{2} z^{-1}) \quad |z| > \frac{1}{2}$$

$$nx[n] \overset{Z}{\longleftrightarrow} -z\frac{dX(z)}{dz} = -\frac{1}{2}\frac{z^{-1}}{1-\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2}$$

$$(\frac{1}{2})^{n-1}u[n-1] \overset{Z}{\longleftrightarrow} \frac{z^{-1}}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$nx[n] = -\frac{1}{2}(\frac{1}{2})^{n-1}u[n-1] \quad x[n] = -\frac{1}{n}(\frac{1}{2})^{n}u[n-1]$$

10.4.9 The Initial- /Final-Value Theorem

If $x[n]=0,\ for \quad n<0.$ **Its ZT** $X(z),$

ROC:|z|> r$_1$.

Then $x[0]=\lim_{z\to\infty} X(z),$

Furthermore, If ROC of (z-1)X(z) include the unit circle of z-plane, then

$$\lim_{n\to\infty} x[n]=x[\infty]=\lim_{z\to 1}(z-1)X(z),$$

Example 10.19

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

ROC:|z|> r$_1$.

Then $x[0] = ?$

Also, by using the initial-value theorem

$$x[0] = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{1 - \frac{3}{2}z^{-1}}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)} = 1$$

10.5 Table 10.1-Properties of z-transform

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	z_0R
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R

10.5.9 Initial Value Theorem

If $x[n] = 0$ for $n < 0$, then

$$x[0] = \lim_{z \to \infty} X(z)$$

10.5 Table 10.2 -Some ZT Pairs

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m>0$) or ∞ (if $m<0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

10.6 Analysis and Characterization of LTI systems Using ZT (including 10.4 10.7)

Consider an LTI system:

$$\xrightarrow[X(z)]{\mathrm{x}[n]} \boxed{\begin{matrix} h[n] \\ H(z) \end{matrix}} \xrightarrow[Y(z) = X(z)\,H(z)]{y[n] = x[n] * h[n]}$$

$$(z^n) \qquad\qquad (H(z)z^n)$$

10.6.1 Causality

(1)A causal system $\Longrightarrow H(z)$**, ROC:** **(|z|> r_1)**

$$(h[n]=0, n<0)$$

exterior of a circle

(including infinity)

(2) For rational $H(z)=\dfrac{N(z)}{D(z)}$,

A causal system $\Longleftrightarrow$ **(a) ROC: (|z|> r_1)**

exterior of a circle outside

the outmost pole(r_1)

(b) The order of the numerator $N(z)$ cannot be greater than the order of the denominator $D(z)$.

Example 10.20

$$H(z)=\frac{z^3-2z^2+z}{z^2+\frac{1}{4}z+\frac{1}{8}}$$

ROC: $|z|>\frac{1}{2}$

not causal system

Example 10.21

$$H(z)=\frac{2-2.5z^{-1}}{(1-0.5z^{-1})(1-2z^{-1})}$$

causal system

$$=\frac{2z^2+2.5z}{z^2-2.5z+1}\ ,|z|>2$$

10.6.2 Stability

(1)A stable system $\Leftrightarrow H(z)$, **ROC: Includes |z|=1 (the unit circle)**

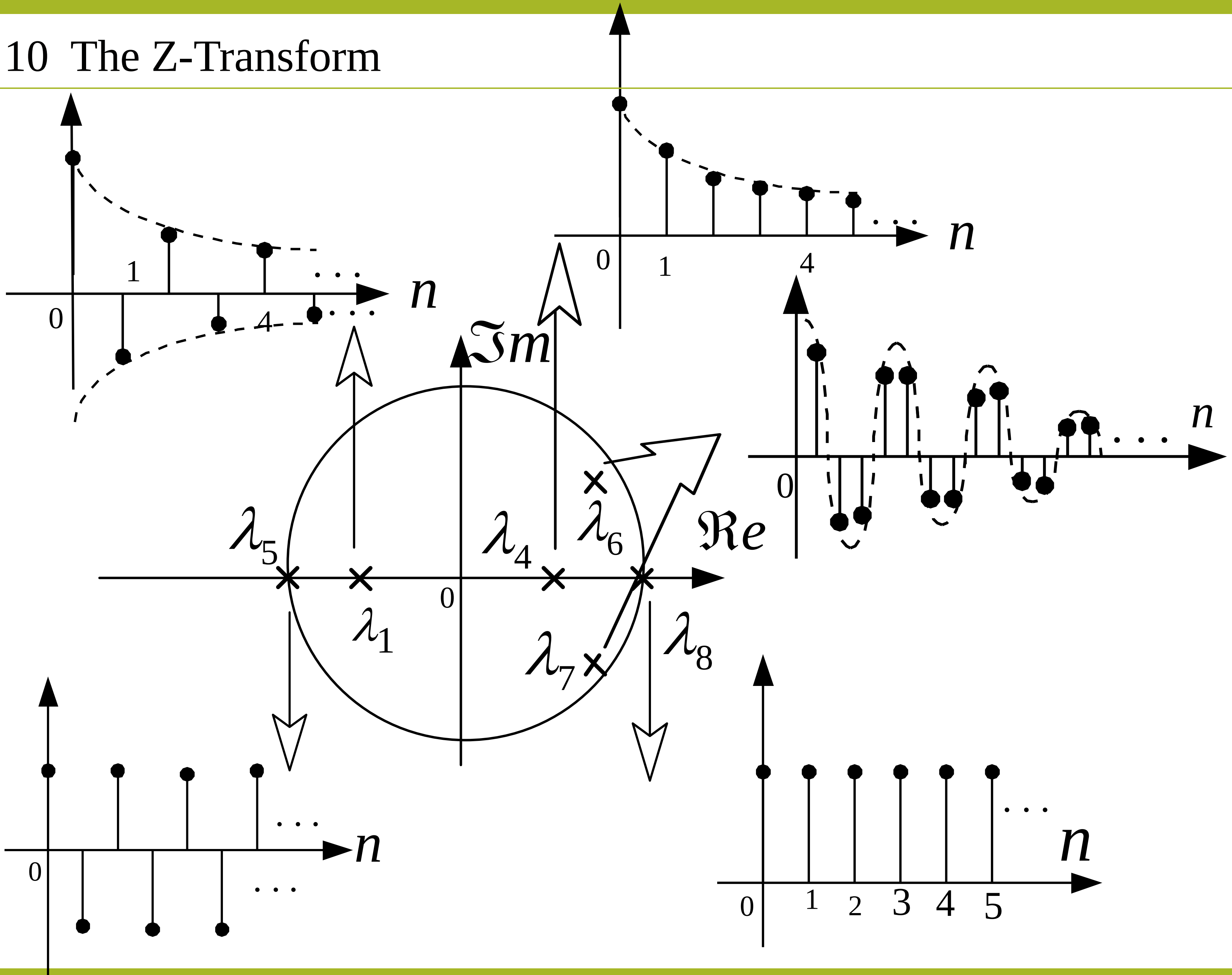
(2) A causal stable system with rational

$$H(z)=\frac{N(z)}{D(z)},\quad \Leftrightarrow$$

All poles lies inside the unit circle of z-plane

Why?

Example 10.22~24 Read by yourself!



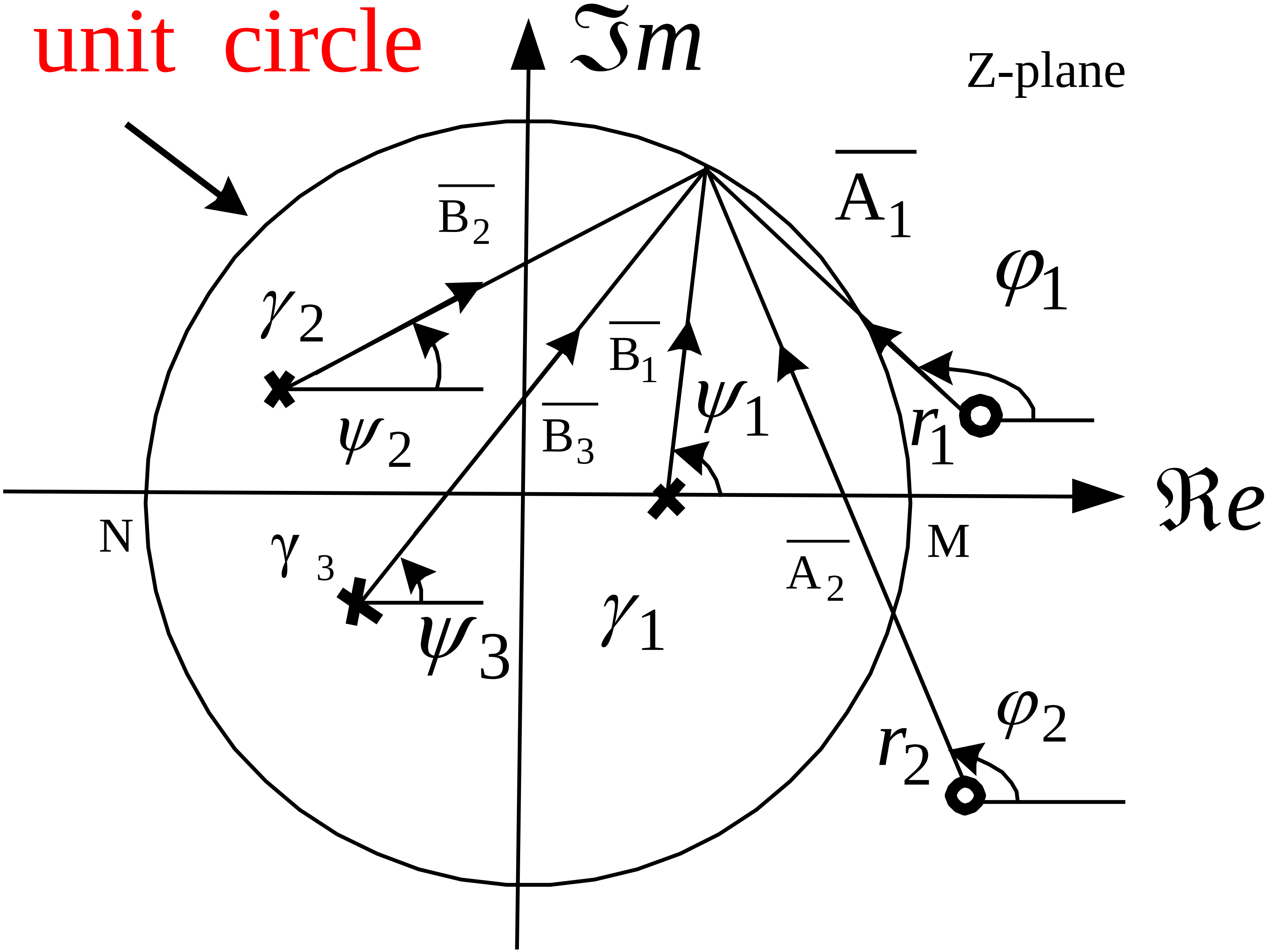
n
0
1
4
ℑm
ℜe
0
λ5
λ1
λ4
λ6
λ7
λ8
n
0
1
2
3
4
5

10.6.3 Pole-Zero Plot of $H(z)$ and Evaluation of Frequency Response $H(e^{j\omega})$ (10.4)

$$H(z)=\frac{N(z)}{D(z)}$$

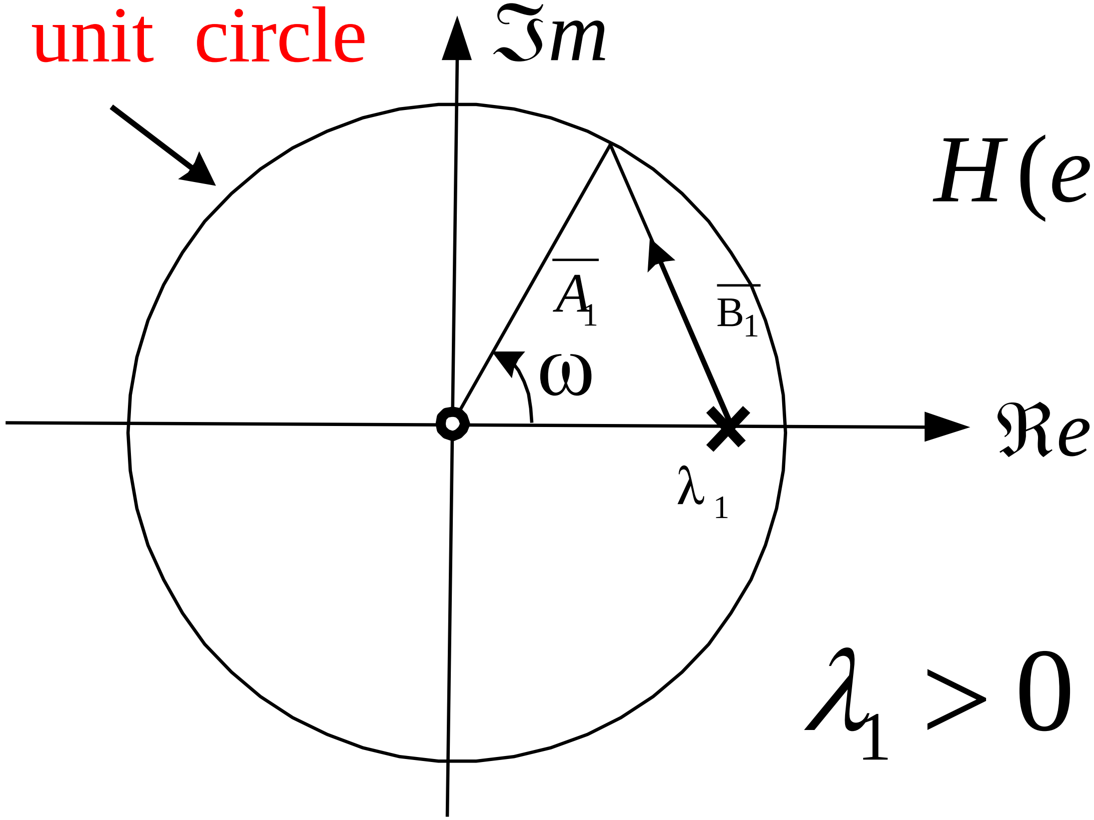
$$H(z)=\frac{b_0\prod_{i=1}^{M}(z-\gamma_i)}{\prod_{i=1}^{N}(z-\lambda_i)}$$

$$H(e^{j\omega})=\frac{b_0\prod_{i=1}^{M}(e^{j\omega}-\gamma_i)}{\prod_{i=1}^{N}(e^{j\omega}-\lambda_i)}$$


unit circle
ℑm
Z-plane
$\overline{A_1}$
$\overline{B_2}$
φ_1
γ_2
$\overline{B_1}$
ψ_1
ψ_2
$\overline{B_3}$
r_1
ℜe
N
γ_3
M
$\overline{A_2}$
γ_1
ψ_3
φ_2
r_2

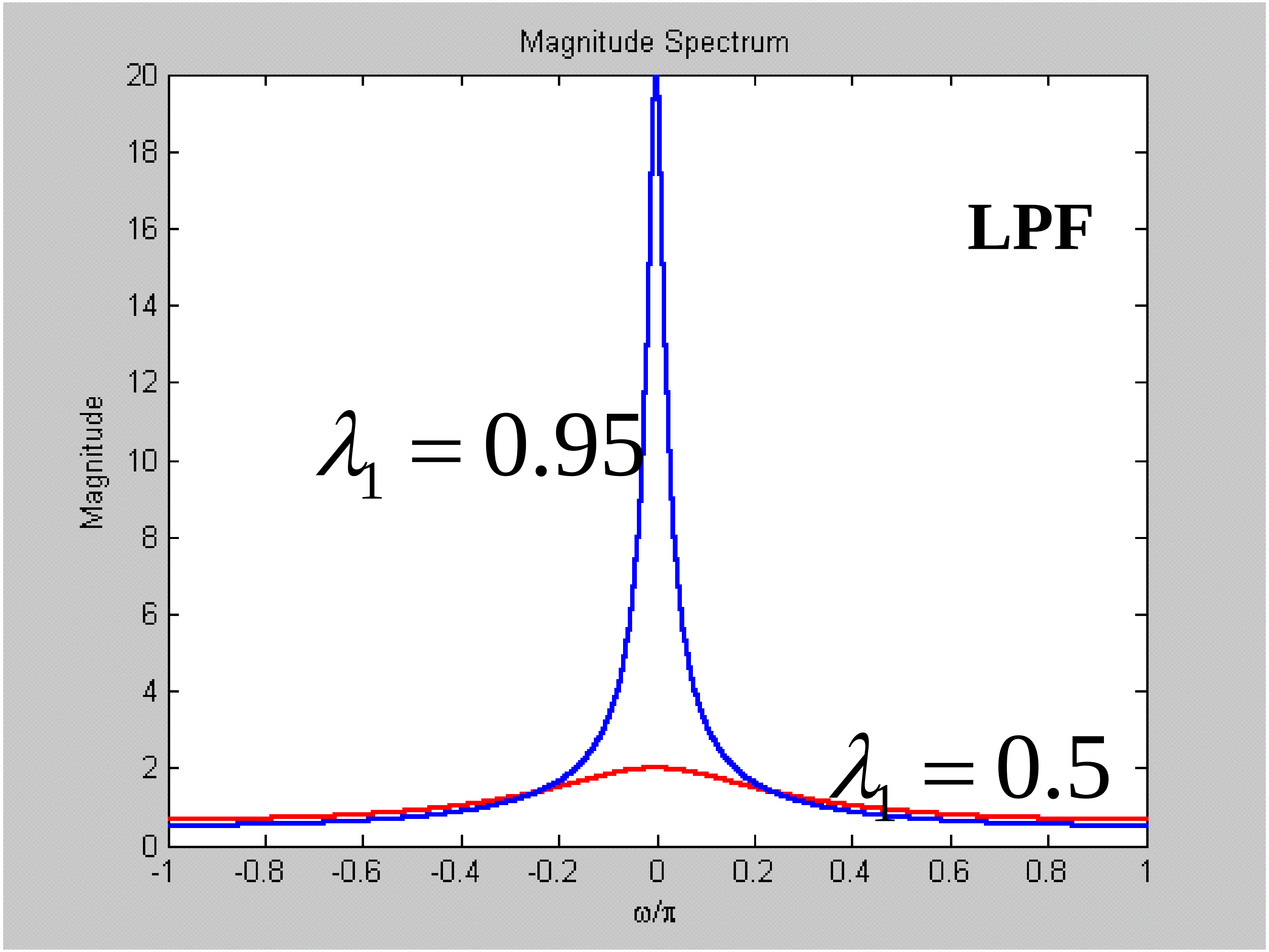
Example (10.4.1 first order systems)

$$h[n] = \lambda_1^n u[n] \quad H(z) = \frac{1}{1-\lambda_1 z^{-1}} = \frac{z}{z-\lambda_1}$$

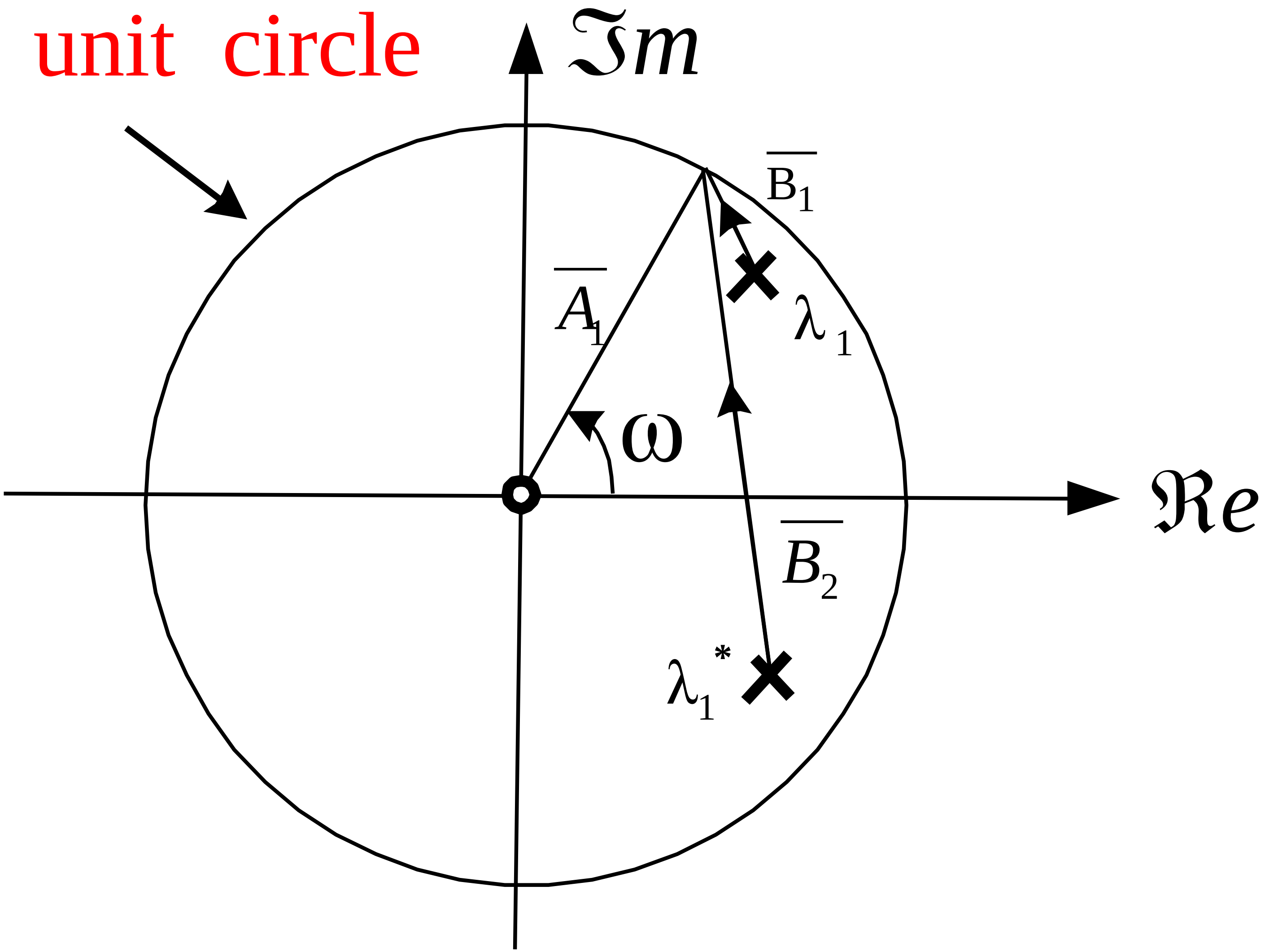


$$H(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega}-\lambda_1}$$

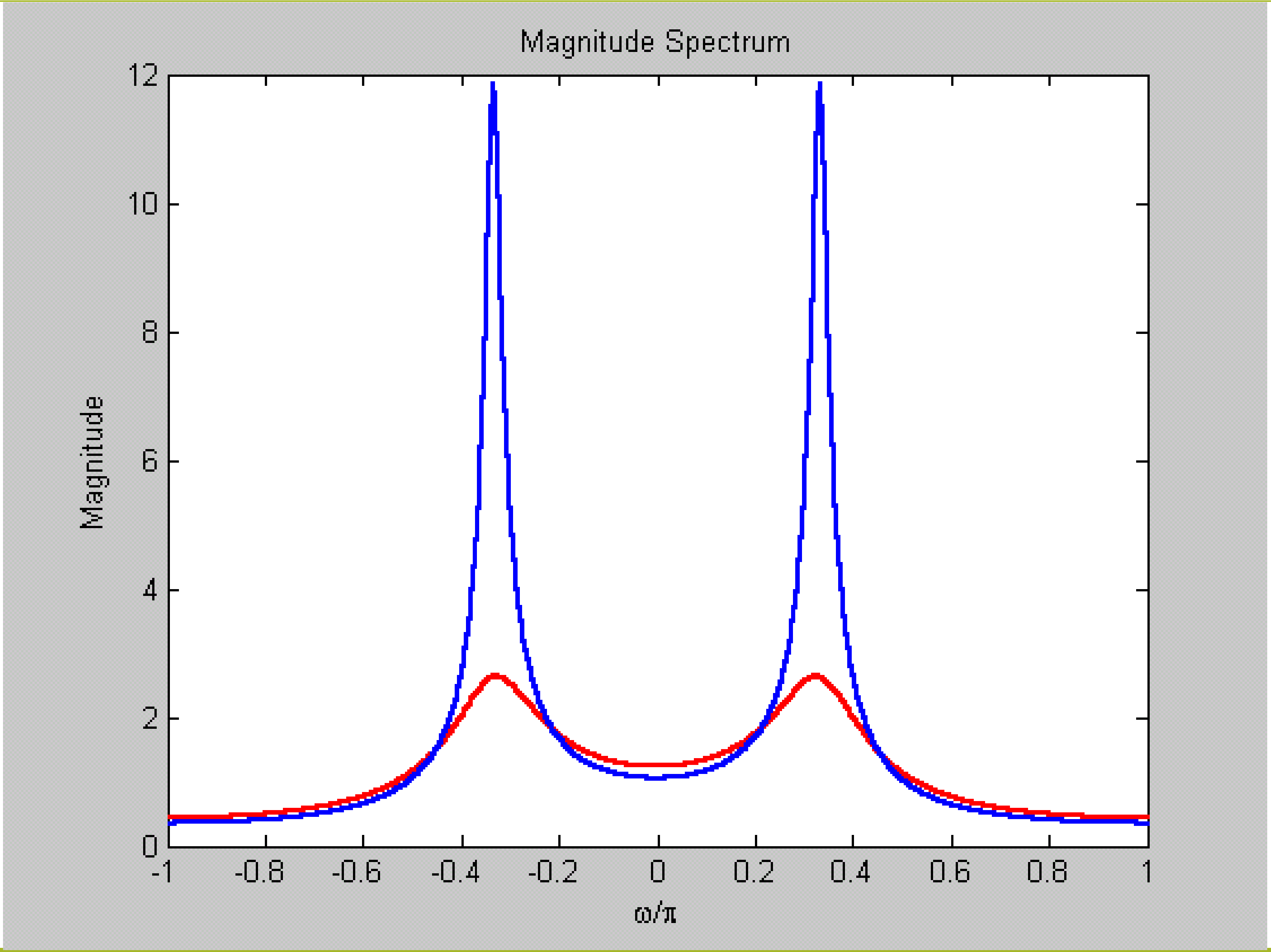
$$\lambda_1 > 0$$


Magnitude Spectrum
LPF
$\lambda_1 = 0.95$
$\lambda_1 = 0.5$
Magnitude
ω/π
20
18
16
14
12
10
8
6
4
2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Example (10.4.2 second order systems)

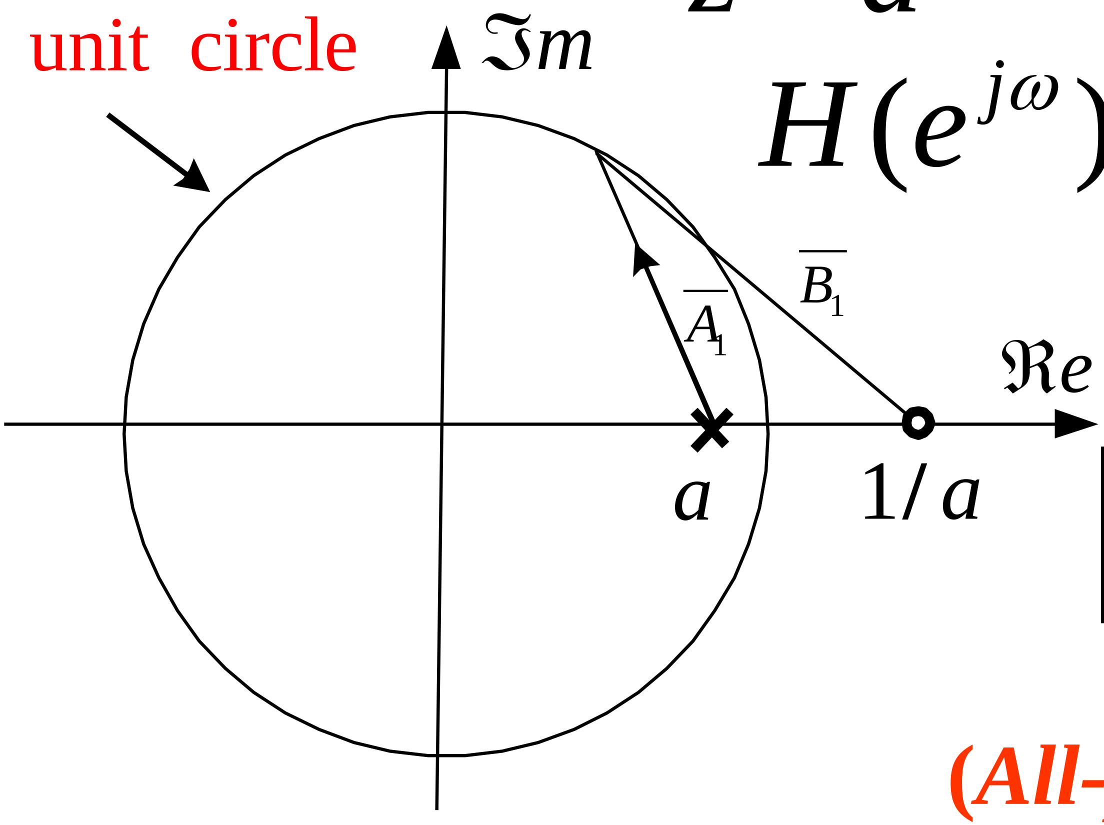


10 The Z-Transform



Example: A causal and stable system

$$H(z) = \frac{z-(1/a)}{z-a} \qquad 0<a<1$$



$$H(e^{j\omega}) = \frac{e^{j\omega}-(1/a)}{e^{j\omega}-a}$$

$$\left|H(e^{j\omega})\right| = 1/a$$

(*All-pass system*)

10.6.4 LTI Systems Characterized by Linear Constant-Coefficient Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Z-transform:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{Y(z)}{X(z)}$$

(rational)

Usually, a practical system is causal and stable.

Example 10.25

$$y[n]-\frac{1}{2}y[n-1]=x[n]+\frac{1}{3}x[n-1]$$

$$H(z)=\frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

ROC1: $|z|>\frac{1}{2}$

$$h[n]=(\frac{1}{2})^{n}u[n]+\frac{1}{3}(\frac{1}{2})^{n-1}u[n-1]$$

ROC₂: $|z| < \frac{1}{2}$

$$H(z) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$h[n] = -(\frac{1}{2})^n u[-n-1] - \frac{1}{3}(\frac{1}{2})^{n-1} u[-n]$$

10.6.5 Examples Relating System Behavior to the System Function

Example 10.26

information 1: $$x_1[n] = (1/6)^n u[n]$$

$$\Rightarrow \quad y_1[n] = \left[a\left(\frac{1}{2}\right)^n + 10\left(\frac{1}{3}\right)^n \right] u[n]$$

information 2: if $$x_2[n] = (-1)^n$$

$$\Rightarrow \quad y_2[n] = \frac{7}{4}(-1)^n$$

From information 1, we get:

$$X_1(z)=\frac{1}{1-\frac{1}{6}z^{-1}},|z|>\frac{1}{6}$$

$$Y_1(z)=\frac{a}{1-\frac{1}{2}z^{-1}}+\frac{10}{1-\frac{1}{3}z^{-1}},|z|>\frac{1}{2}$$

$$H(z)=\frac{Y_1(z)}{X_1(z)}=\frac{[(a+10)-(5+\frac{a}{3})z^{-1}][1-\frac{1}{6}z^{-1}]}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$$

From **information 2**, **we know :** $H(-1)=\frac{7}{4}$

$\therefore a=-9$

$$H(z)=\frac{(1-2z^{-1})(1-\frac{1}{6}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$$

$$H(z)=\frac{1-\frac{13}{6}z^{-1}+\frac{1}{3}z^{-2}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

ROC: $|z|>1/2$

$$y[n]-\frac{5}{6}y[n-1]+\frac{1}{6}y[n-2]=x[n]-\frac{13}{6}x[n-1]+\frac{1}{3}x[n-2]$$

Example 10.27 a stable,causal system

$H(z)$ **(rational) contains a pole,** $z = 1/2$

a zero somewhere on the unit circle

Other zeros and poles are unknown.

Whether can we definitely say that it is true or false each of following statements?

(a) $F\{(\frac{1}{2})^n h[n]\}$ **converges.** **T**

(b) $H(e^{j\omega}) = 0,$ **for some** ω**.** **T**

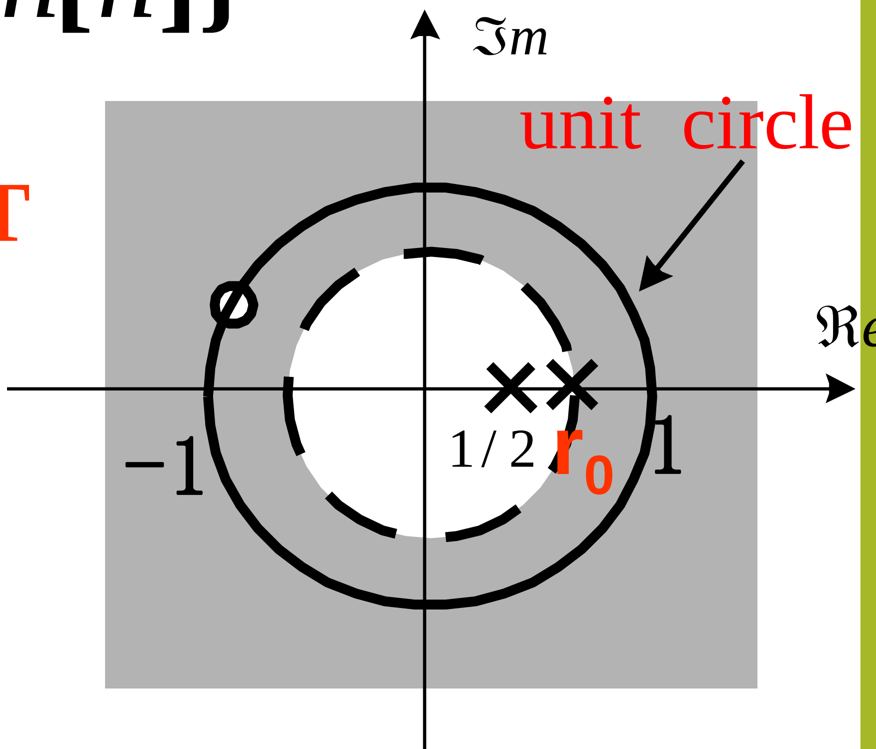
(c) $h[n]$ **has finite duration.** F

(d) $h[n]$ **is real.** Insufficient information

(e) $g[n] = n\{h[n] * h[n]\}$
is the impulse response of a stable system. T

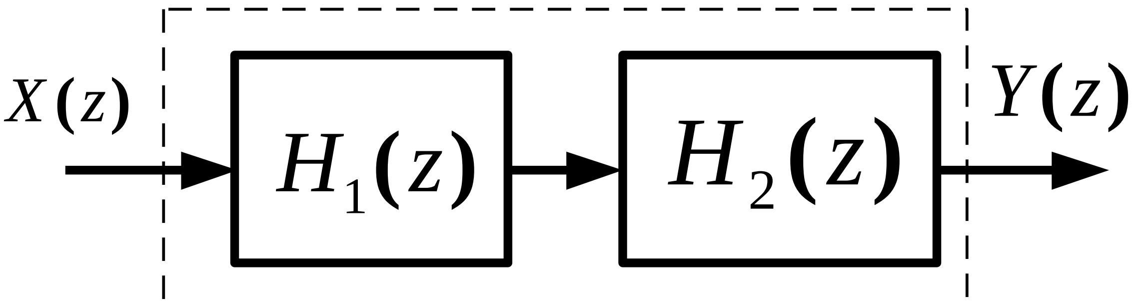
$$G(z) = -z\frac{d}{dz}H(z)^2$$

$$= -2zH(z)[\frac{d}{dz}H(z)]$$



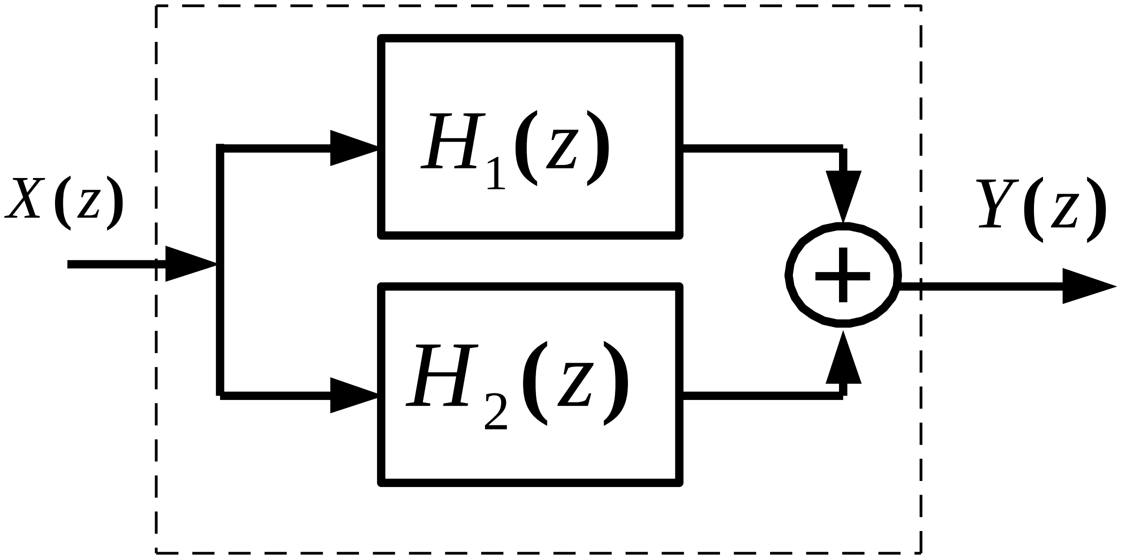
10.7 System Function Algebra and Block Diagram Representations (10.8)

10.7.1 System Functions for Interconnections of LTI Systems



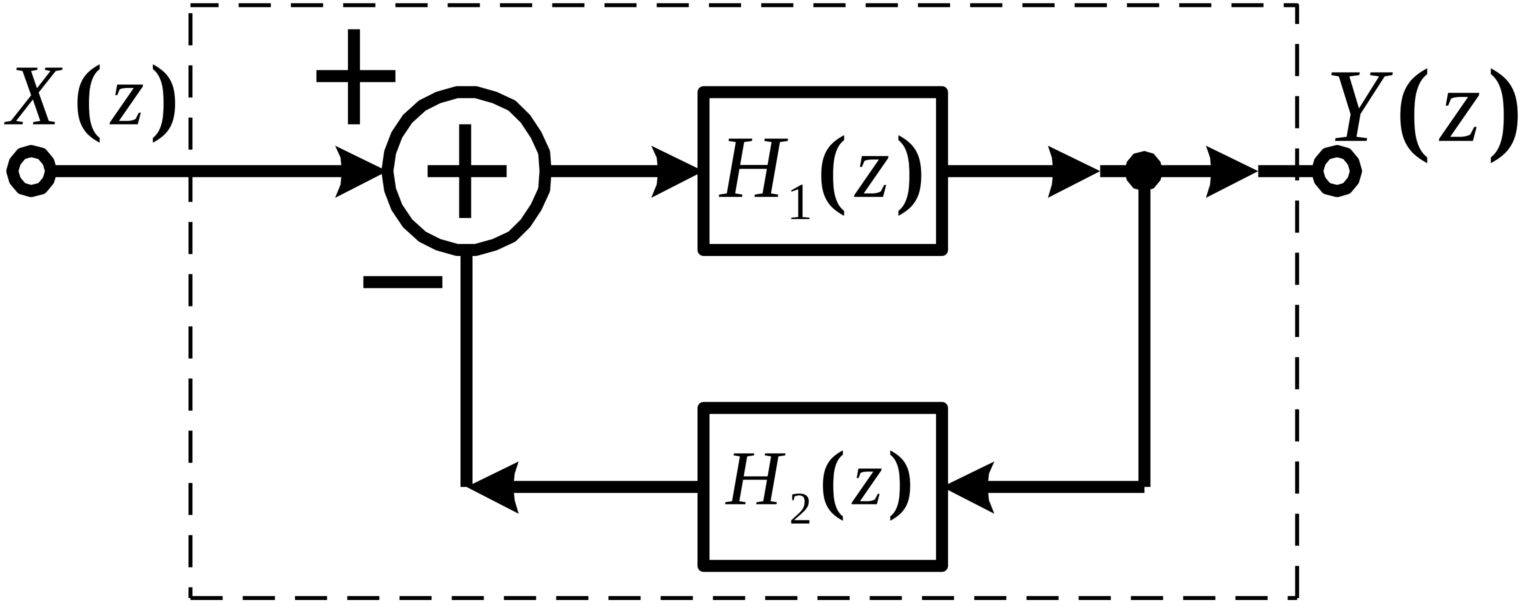
$$H(z) = H_1(z)H_2(z)$$

Series(cascade)



$$H(z) = H_1(z) + H_2(z)$$

Parallel

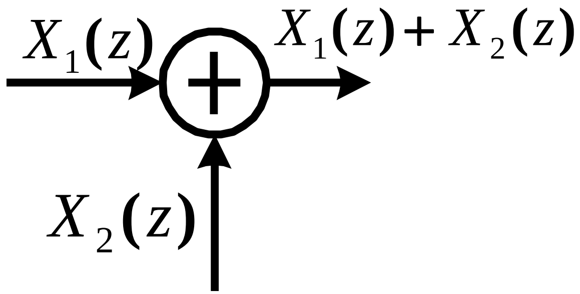


$$H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

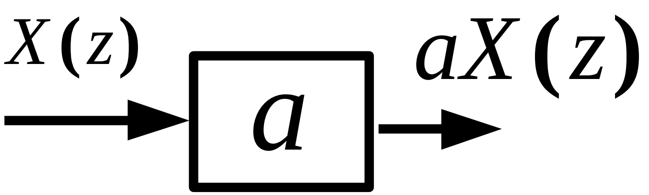
Feed-back

10.7.2 Block Diagram Representation for causal LTI Systems Described by Difference Equations and Rational System Functions

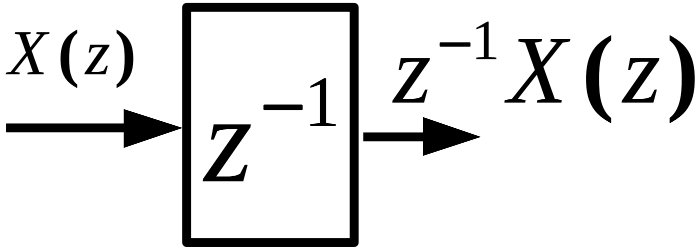
Basic elements:



adder



multiplication



unit delay

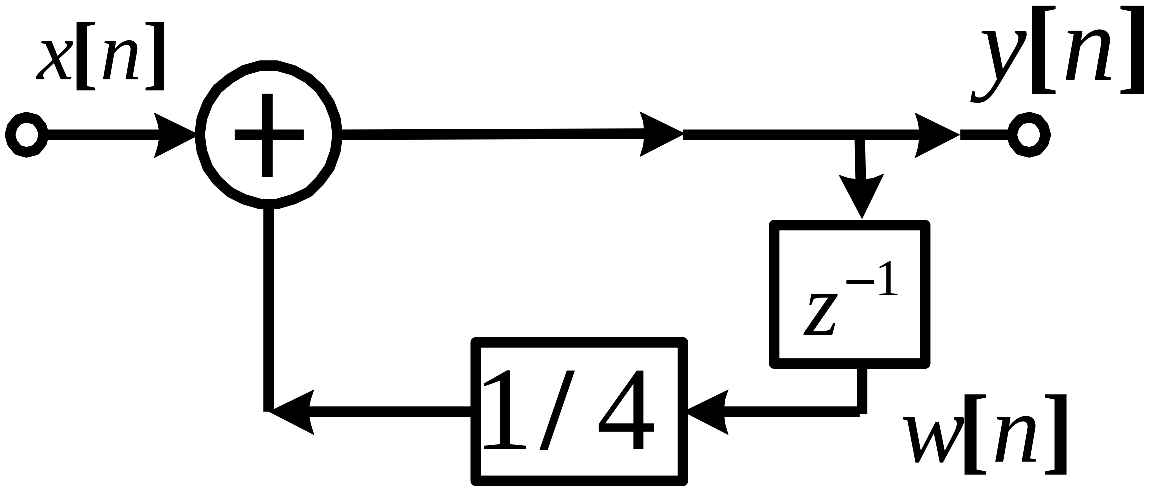
Example 10.28

$$H(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} = \frac{Y(z)}{X(z)}$$

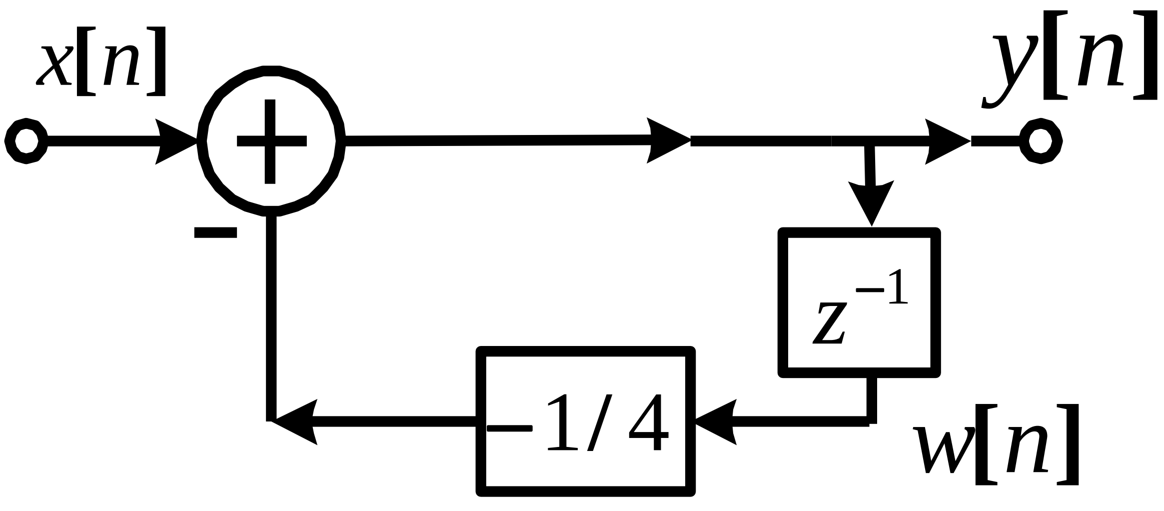
$$y[n] - \frac{1}{4}y[n-1] = x[n]$$

Let $W(z) = z^{-1}Y(z), \quad w[n] = y[n-1]$

Then, $Y(z) = X(z) + \frac{1}{4}W(z)$



Equivalent representation

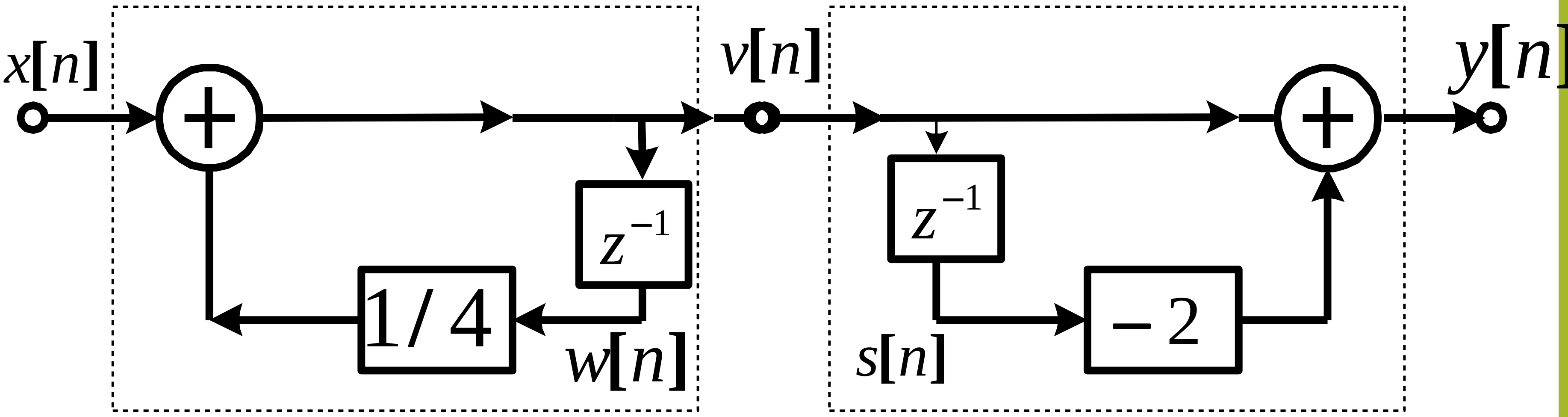


Example 10.29 $$H(z)=\frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}}=\frac{Y(z)}{X(z)}$$

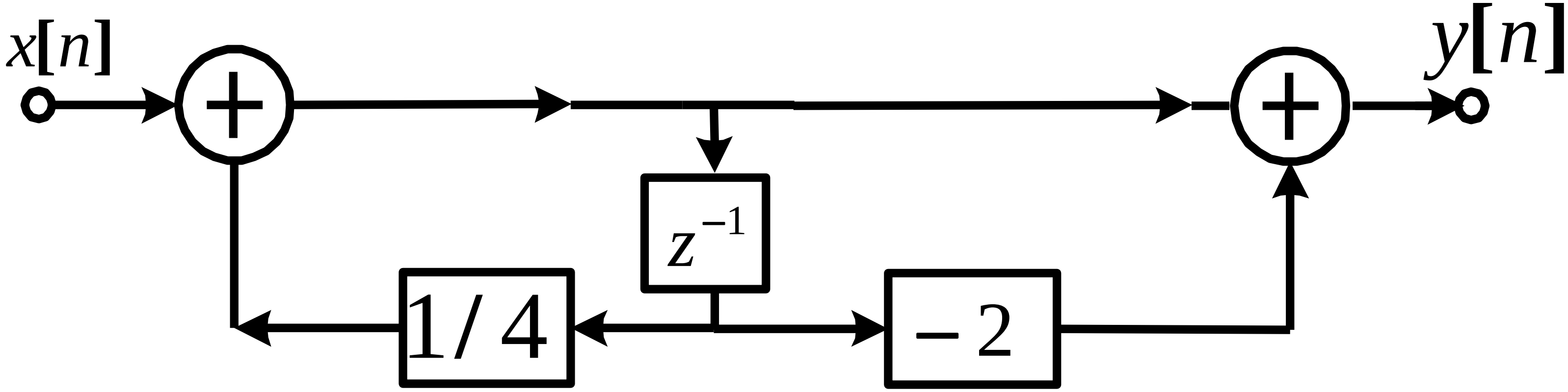
$$H(z)=\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)(1-2z^{-1})=\frac{Y(z)}{X(z)}$$

Let $$V(z)=\frac{1}{1-\frac{1}{4}z^{-1}}X(z)$$

$$(1-2z^{-1})V(z)=Y(z) \qquad y[n]=v[n]-2v[n-1]$$



$$w[n] = s[n] = v[n-1]$$



Canonic form:

the number of delayer = the order of the difference equation

Example 10.30

$$H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

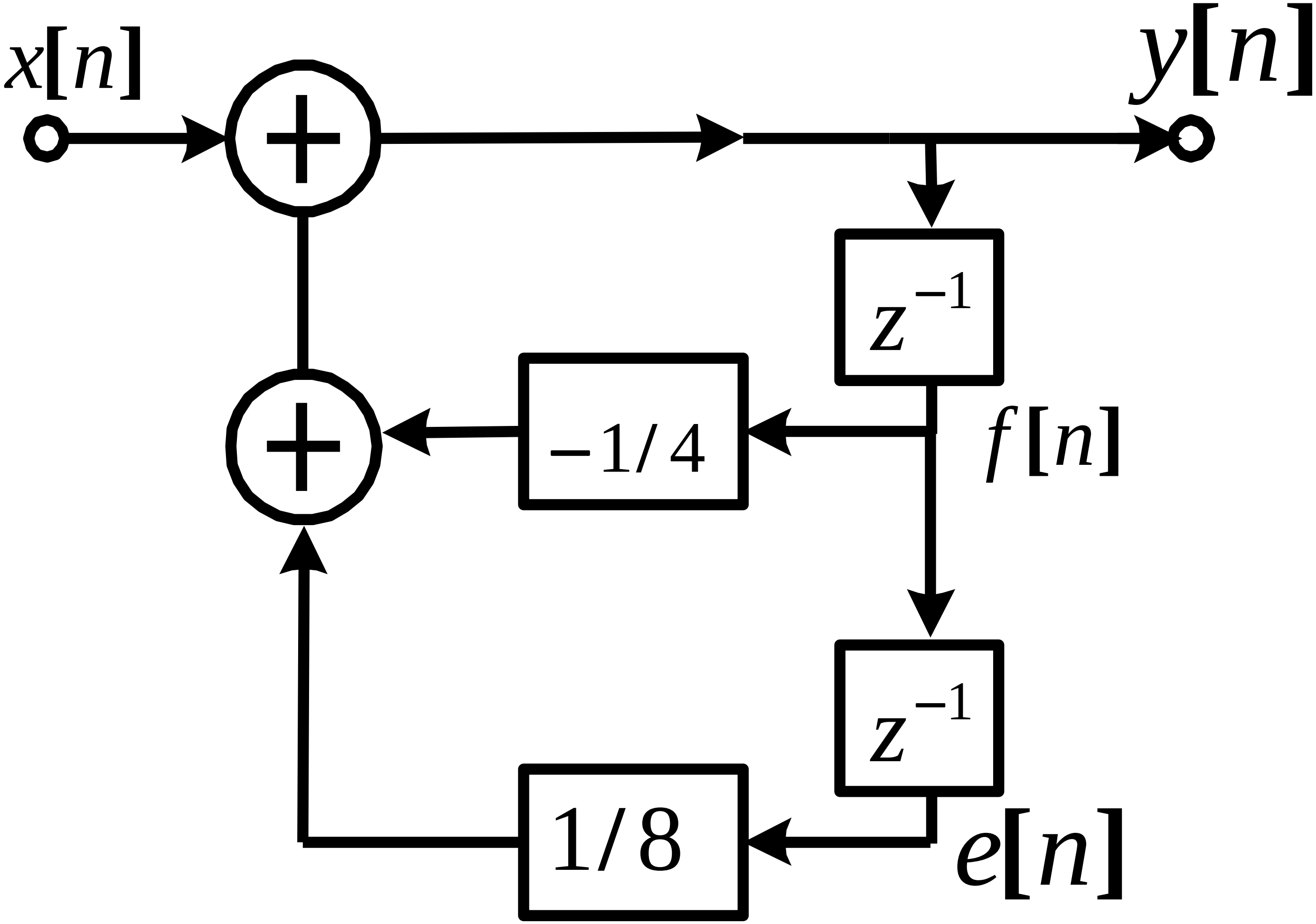
$$y[n]+\frac{1}{4}y[n-1]-\frac{1}{8}y[n-2] = x[n]$$

(1) *direct-form*

Let $F(z) = z^{-1}Y(z)$

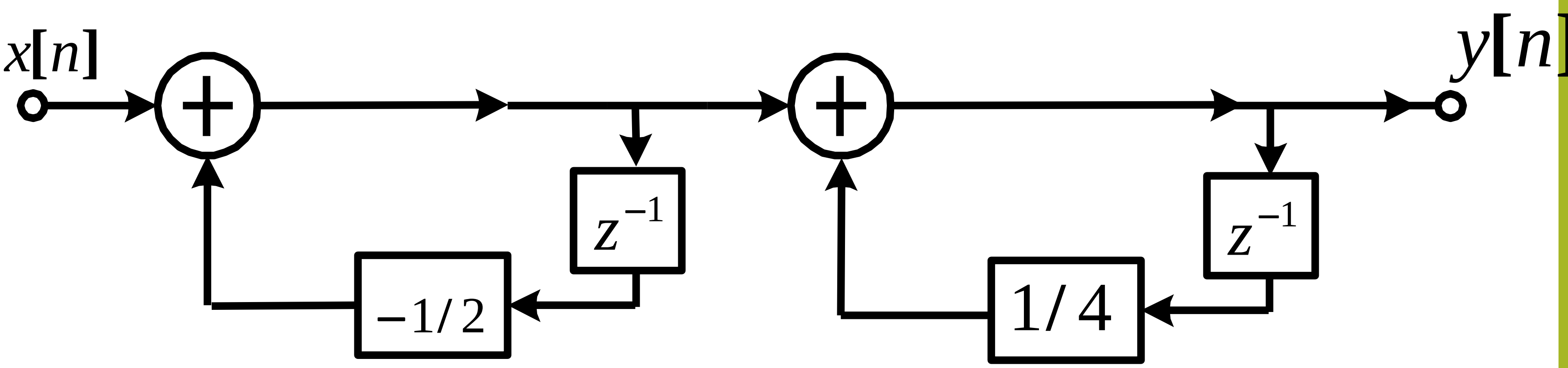
$$E(z) = z^{-1}F(z) = z^{-2}Y(z)$$

Then, $$Y(z) = X(z) - \frac{1}{4}F(z) + \frac{1}{8}E(z)$$



(2) ***cascade-form***

$$H(z)=\frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}=\frac{1}{(1+\frac{1}{2}z^{-1})}\times\frac{1}{(1-\frac{1}{4}z^{-1})}$$

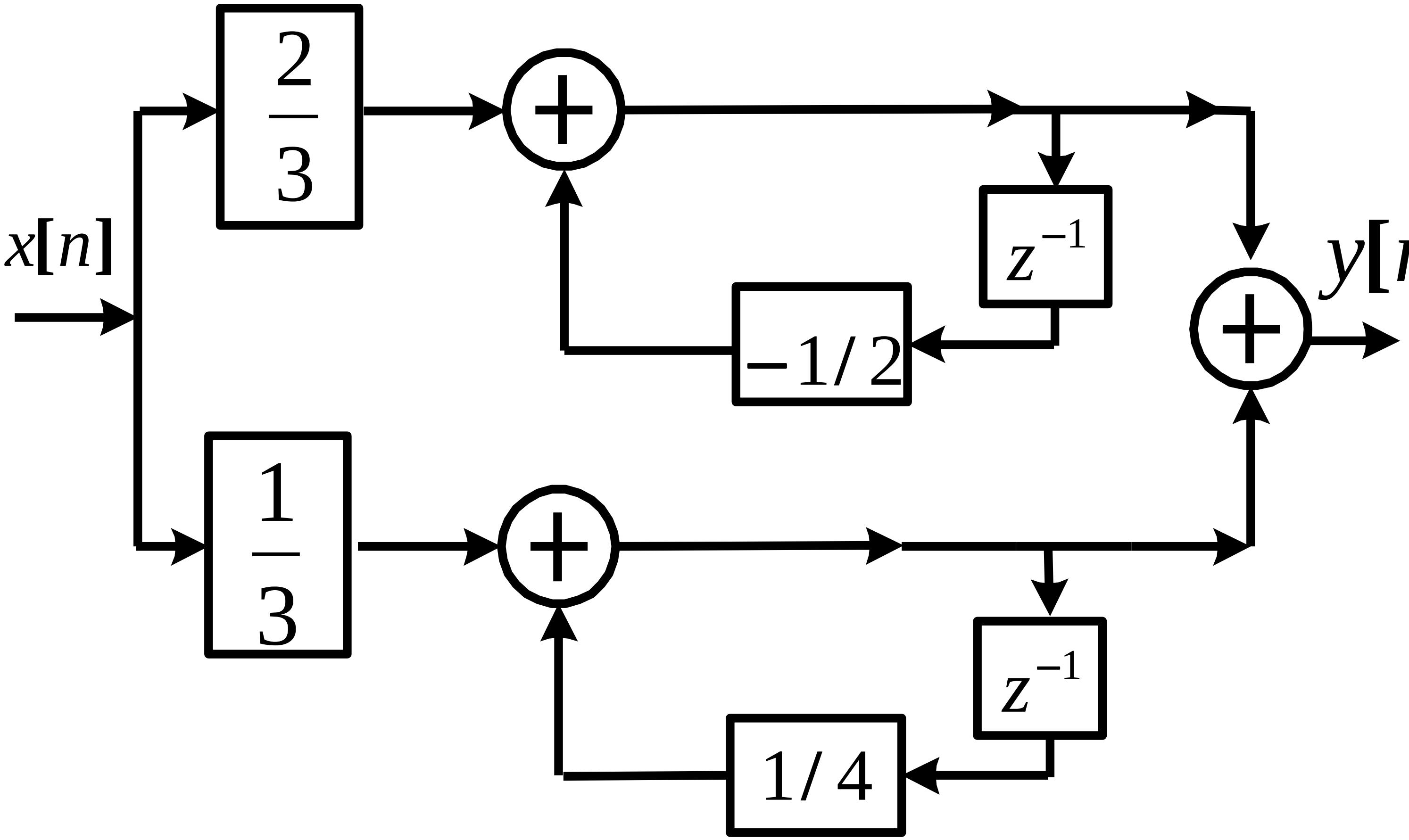


(3) ***parallel-form***

Partial Fraction Expansion in z^{-1} form

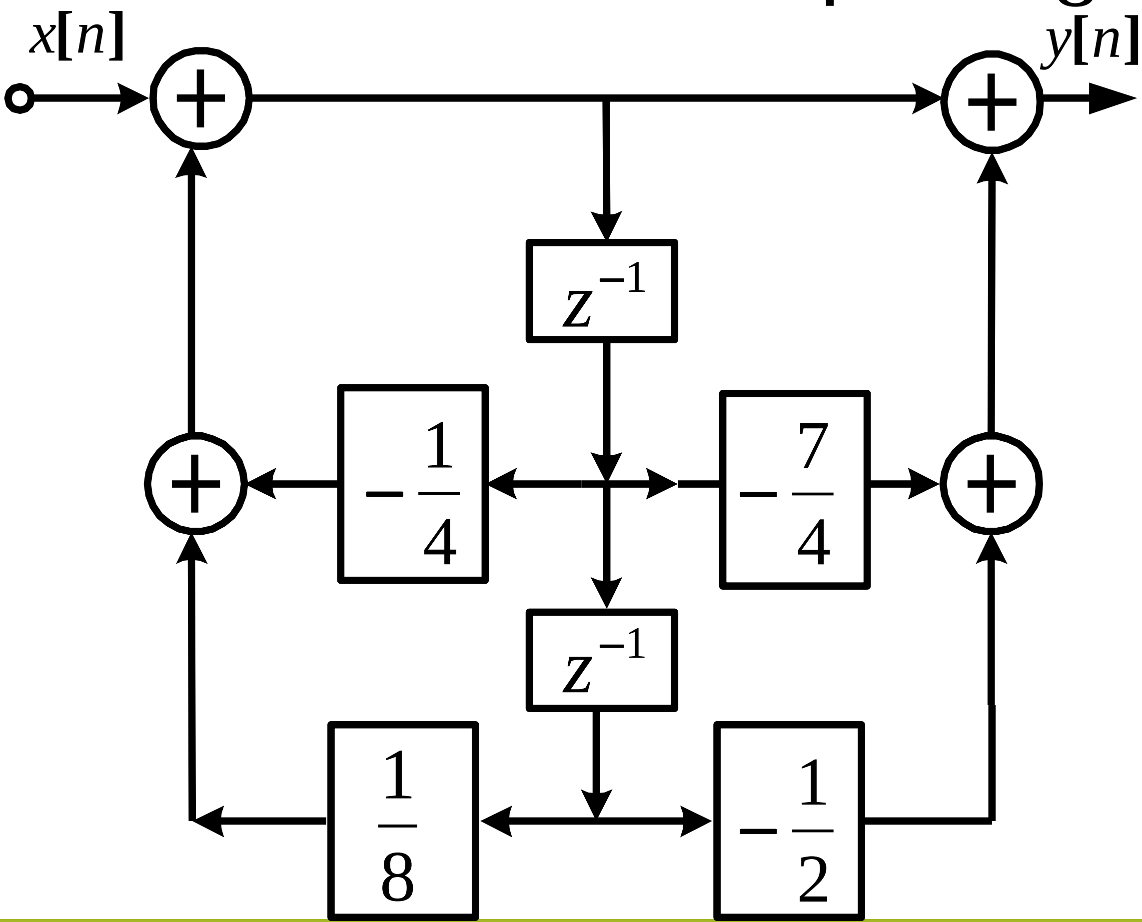
$$H(z)=\frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$

$$=\frac{2/3}{(1+\frac{1}{2}z^{-1})}+\frac{1/3}{(1-\frac{1}{4}z^{-1})}$$


$x[n]$
$\frac{2}{3}$
$\frac{1}{3}$
z^{-1}
$-1/2$
z^{-1}
$1/4$
$y[n]$

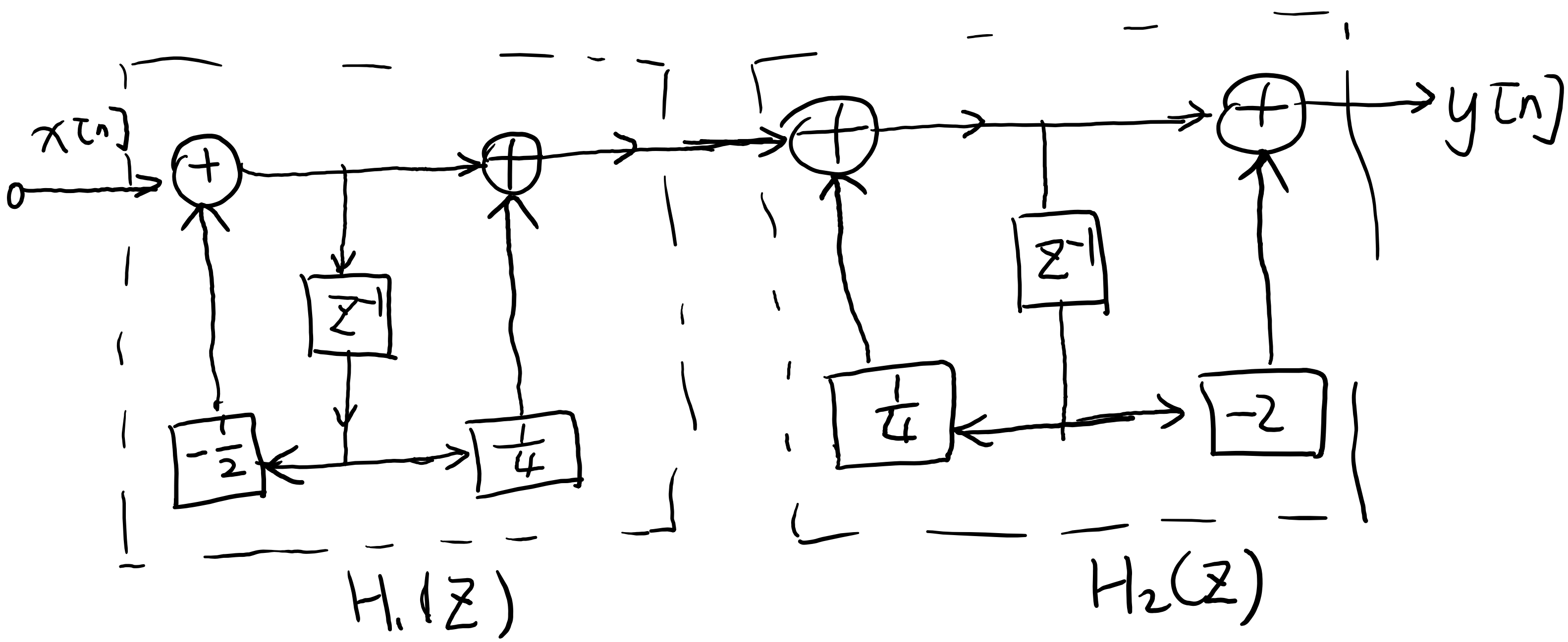
Example:

$$H(z)=\frac{1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$



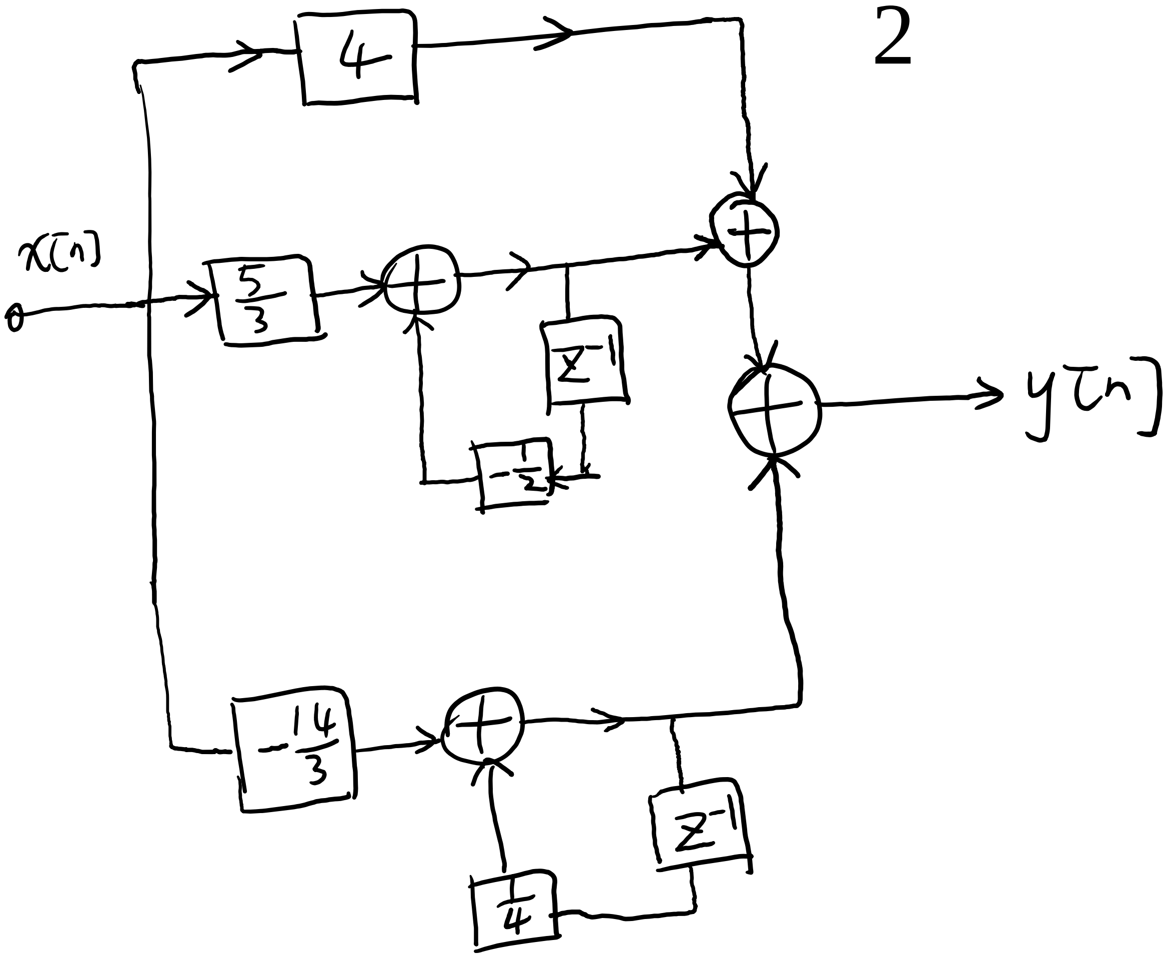
Cascade-form:

$$H(z)=\left(\frac{1+\frac{1}{4}z^{-1}}{1+\frac{1}{2}z^{-1}}\right)\left(\frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}}\right)$$



parallel-form:

$$H(z) = 4 + \frac{5/3}{1+\frac{1}{2}z^{-1}} - \frac{14/3}{1-\frac{1}{4}z^{-1}}$$



10.8 The Unilateral ZT(10.9)

$$X(z)=\sum_{n=0}^{\infty}x[n]z^{-n}$$

or

$$x[n]u[n]\xleftrightarrow{Z}X(z),\qquad |z|>r_1$$

$$x[n]\xleftrightarrow{UZ}X(z)=UZ\{x[n]\}$$

The most properties of Unilateral ZT are same as ZT, except delay and advance in the time Domain.

TABLE 10.3 PROPERTIES OF THE UNILATERAL z-TRANSFORM

Property	Signal	Unilateral z-Transform
—	$x[n]$	$\mathcal{X}(z)$
—	$x_1[n]$	$\mathcal{X}_1(z)$
—	$x_2[n]$	$\mathcal{X}_2(z)$
Linearity	$ax_1[n] + bx_2[n]$	$a\mathcal{X}_1(z) + b\mathcal{X}_2(z)$
Time delay	$x[n-1]$	$z^{-1}\mathcal{X}(z) + x[-1]$
Time advance	$x[n+1]$	$z\mathcal{X}(z) - zx[0]$
Scaling in the z-domain	$e^{j\omega_0 n}x[n]$ $z_0^n x[n]$ $a^n x[n]$	$\mathcal{X}(e^{-j\omega_0}z)$ $\mathcal{X}(z/z_0)$ $\mathcal{X}(a^{-1}z)$
Time expansion	$x_k[n] = \begin{cases} x[m], & n = mk \\ 0, & n \neq mk \quad \text{for any } m \end{cases}$	$\mathcal{X}(z^k)$
Conjugation	$x^*[n]$	$\mathcal{X}^*(z^*)$
Convolution (assuming that $x_1[n]$ and $x_2[n]$ are identically zero for $n < 0$)	$x_1[n] * x_2[n]$	$\mathcal{X}_1(z)\mathcal{X}_2(z)$
First difference	$x[n] - x[n-1]$	$(1 - z^{-1})\mathcal{X}(z) - x[-1]$
Accumulation	$\sum_{k=0}^{n} x[k]$	$\frac{1}{1 - z^{-1}}\mathcal{X}(z)$
Differentiation in the z-domain	$nx[n]$	$-z\frac{d\mathcal{X}(z)}{dz}$

Initial Value Theorem

$$x[0] = \lim_{z \to \infty} \mathcal{X}(z)$$

If $$x[n]u[n] \xleftrightarrow{Z} X(z),$$ **|z|> r_1**

Then

$$x[n-1]u[n] \xleftrightarrow{Z} z^{-1}X(z) + x[-1],$$

|z|> r_1

$$x[n+1]u[n] \xleftrightarrow{Z} zX(z) - zx[0],$$

|z|> r_1

Solving difference equations using the Unilateral ZT

Example 10.33 $$x[n] = a^{n+1}u[n+1]$$

In this case, the unilateral and bilateral transforms are not equal, since $x[-1] = 1 \neq 0$.

By time shifting property: $$X(z) = \frac{z}{1-az^{-1}}, |z| > |a|$$

In contrast, the unilateral transform:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} a^{n+1}z^{-n}$$

$$= \frac{a}{1-az^{-1}}, |z| > |a|$$

Example: Balance in a bank account from month to month:

balance --- $y[n]$

interest --- 0.5%

net deposit --- $x[n]=100u[n-1]$

so $y[n+1]=y[n]+0.5\%\cdot y[n]+x[n]$

or $y[n+1]-1.005y[n]= x[n]$ -----(A)

$y[0]=10000$, **initial condition(borrowed at $n=0$).**

Solution: Using the Unilateral ZT to Eq.(A)

$$zY(z) - zy[0] - 1.005Y(z) = X(z)$$

$$Y(z) = \frac{100}{(z-1.005)(z-1)} + \frac{zy[0]}{z-1.005}$$

$$Y_{zc}(z) \qquad\qquad Y_{zi}(z)$$

zero − state response *zero − input response*

$$Y(z)=\frac{2\times 10000}{z-1.005}-\frac{2\times 10000}{z-1}+\frac{10000z}{z-1.005}$$

$$y[n]=[2\times 10^4\times 1.005^{n-1}-2\times 10^4]u[n-1]$$

$$+10^4\times 1.005^n,\ n\geq 1$$

Resume of Chapter 10

Key points of analysis:

$$x[n] \xrightarrow{X(z)} \boxed{\begin{matrix} h[n] \\ H(z) \end{matrix}} \xrightarrow{Y(z)=X(z)H(z)} y[n]=x[n]*h[n]$$

$$(z^n) \qquad\qquad (H(z)z^n)$$

Key points of caculation:

1. Properties and Basic ZT Pairs

2. Partial Fraction Expansion

3. Block Diagram Representation

Homework list for Chapter 10

2, 3, 6, 10(a), **24**, **31**, **47**