Z-Transform

The Z-transform is a mathematical tool which is used to convert the difference equations in discrete time domain into the algebraic equations in z-domain. Mathematically, if $x(n)$ is a discrete time function, then its Z-transform is defined as,

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Final Value Theorem of Z-Transform

The final value theorem of Z-transform enables us to calculate the steady state value of a sequence $x(n)$, i.e., $x(\infty)$ directly from its Z-transform, without the need for finding its inverse Z-transform.

Statement - If $x(n)$ is a causal sequence, then the final value theorem of Z-transform states that if,

$$x(n) \overset{ZT}{\leftrightarrow} X(z)$$

And if the Z-transform X(z) has no poles outside the unit circle, and it has no higher poles on the unit circle centred at the origin of the z-plane, then,

$$x(\infty) = \lim_{n\to\infty} x(n) = \lim_{z\to 1} (\mathrm{z}-1)X(z)$$

Proof

From the definition of Z-transform of a causal sequence, we have,

$$Z[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

And

$$Z[x(n+1)] = \sum_{n=0}^{\infty} x(n)z^{-n} = zX(z) - zx(0)$$

$$\therefore Z[x(n+1)] - Z[x(n)] = \sum_{n=0}^{\infty} x(n)z^{-n} - \sum_{n=0}^{\infty} x(n)z^{-n}$$

$$\Rightarrow Z[x(n+1)] - Z[x(n)] = zX(z) - zx(0) - X(z)$$

$$\therefore (\mathrm{z}-1)X(z) - zx(0) = \sum_{n=0}^{\infty} [x(n+1) - x(n)]z^{-n}$$

$$\Rightarrow (\mathrm{z}-1)X(z) - zx(0) = [x(1) - x(0)]z^{0} + [x(2) - x(1)]z^{-1} + [x(3) - x(2)]z^{-2} + \ldots$$

Now taking limit $z \to 1$ on both the sides, we get,

$$\lim_{\mathrm{z}\to 1} [(\mathrm{z}-1)X(z) - zx(0)] = \lim_{\mathrm{z}\to 1} \left\{ [x(1) - x(0)]z^{0} + [x(2) - x(1)]z^{-1} + [x(3) - x(2)]z^{-2} + \ldots \right\}$$

$$\Rightarrow \lim_{\mathrm{z}\to 1} [(\mathrm{z}-1)X(z)] - x(0) = x(1) - x(0) + x(2) - x(1) + x(3) - x(2) + \ldots + x(\infty) - x(\infty - 1)$$

$$\Rightarrow \lim_{z\to 1}[(z-1)X(z)] - x(0) = x(\infty) - x(0)$$

$$\therefore x(\infty) = \lim_{z\to 1}[(z-1)X(z)]$$

Numerical Example (1)

Find $x(\infty)$ if X(z) is given by,

$$X(z) = \frac{z^2}{(z-1)(z-0.3)}$$

Solution

The given Z-transform of the sequence is,

$$X(z) = \frac{z^2}{(z-1)(z-0.3)}$$

Now, using the final value theorem for Z-transform $\left[\text{i. e}, x(\infty) = \lim_{z\to 1}[(z-1)X(z)]\right]$,we get,

$$x(\infty) = \lim_{z\to 1}(z-1)\left[\frac{z^2}{(z-1)(z-0.3)}\right] = \lim_{z\to 1}\left[\frac{z^2}{(z-0.3)}\right]$$

$$\therefore x(\infty) = \left[\frac{1}{(1-0.3)}\right] = 1.43$$

Numerical Example (2)

Using the final value theorem, calculate $x(\infty)$ if X(z) is given by,

$$X(z) = \frac{z+1}{3\,(z-1)(z+0.4)}$$

Solution

The given Z-transform of the sequence is,

$$X(z) = \frac{z+1}{3\,(z-1)(z+0.4)}$$

$$\therefore (z-1)X(z) = \frac{z+1}{3(z+0.4)}$$

As we can see, $(z-1)X(z)$ has no poles on or outside the unit circle. Therefore, using the final value theorem for Z-transform, we have,

$$x(\infty) = \lim_{z\to 1}\left[\frac{z+1}{3(z+0.4)}\right] = \left[\frac{1+1}{3(1+0.4)}\right]$$

$$\therefore x(\infty) = \left[\frac{2}{3\times 1.4}\right] = 0.48$$