Signals and Systems

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Solution

{**Exercise from Chapter 1**}:

Consider a discrete-time system with input $x[n]$ and output $y[n]$ related by

$$y[n] = x[3n-1]$$

is this system

(a) Memoryless?

(b) Time-invariant?

(c) Linear?

(d) Causal?

(e) Stable?

Try to prove.

Solution

⇒ (1) The output $y[n]$ is not only dependent on the input at the same time.

So. the system is Memory system.

(2) $x_1[n] \longrightarrow y_1[n] = x_1[3n-1]$

$x_2[n] = x_1[n-n_0] \longrightarrow y_2[n] = x_1[3n-n_0-1]$

而 $y_1[n-n_0] = x_1[3n-3n_0-1]$ $\qquad y_2[n] \neq y_1[n-n_0]$

so, the system is Time varying.

(3) $x_1[n] \longrightarrow y_1[n] = x_1[3n-1]$

$x_2[n] \longrightarrow y_2[n] = x_2[3n-1]$

令 $x_3[n] = a x_1[n] + b x_2[n]$.

则 $y_3[n] = x_3[3n-1] = a x_1[3n-1] + b x_2[3n-1] = a y_1[n] + b y_2[n]$

So the system is linear.

(4). if $n=3$. then $y[3] = x[8]$.

the output at time of $n=3$ depends on a future value of input $x[8]$.

So, the system is not causal.

(5). If $x[n]$ is bounded, $x[3n-1]$ is just a time Scaling version of $x[n]$, so $y[n] = x[3n-1]$ is also bounded.

then, the system is stable.

Solution

(a) The output $y[n]$ is not only dependent on the input at the same time, so the system is a memory system.

(b) $$x_1[n] \;\rightarrow\; y_1[n] = x_1[3n-1]$$

$$let\;\; x_2[n] = x_1[n-n_0] \;\rightarrow\; y_2[n] = x_1[3n-n_0-1] \neq y[n-n_0]$$

So the system is Time varying system.

(c) $$x_1[n] \;\rightarrow\; y_1[n] = x_1[3n-1]$$

$$x_2[n] \;\rightarrow\; y_2[n] = x_2[3n-1]$$

$$let\;\; x_3[n] = ax_1[n] + bx_2[n] \;\rightarrow\; y_3[n] = x_3[3n-1] = ax_1[3n-1] + bx_2[3n-1]$$

So the system is linear.

Solution

(d) If n=3, then y[3] = x[8]

The output at time of n=3 depends on a future value of input x[8].

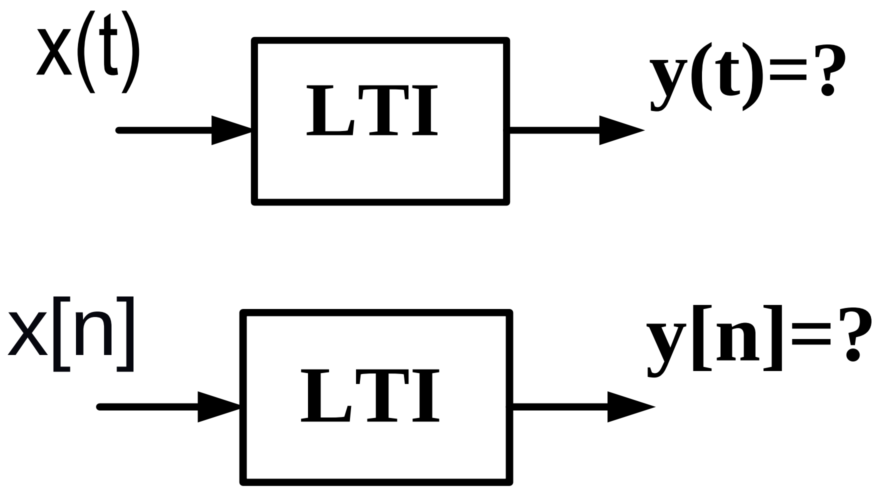
So the system is not causal.

(e) If x[n] is bounded, $x[3n-1]$ is just a time scaling and shifting version of $x[n]$, so $y[n]=x[3n-1]$is also bounded.

So the system is stable.

2. Linear Time-Invariant Systems

Problem:



Key points of analysis:

Signals decompostion: basic signal (impulse)

Response synthesis: basic response (impuse response)

2.1 Discrete-time LTI system: The convolution sum

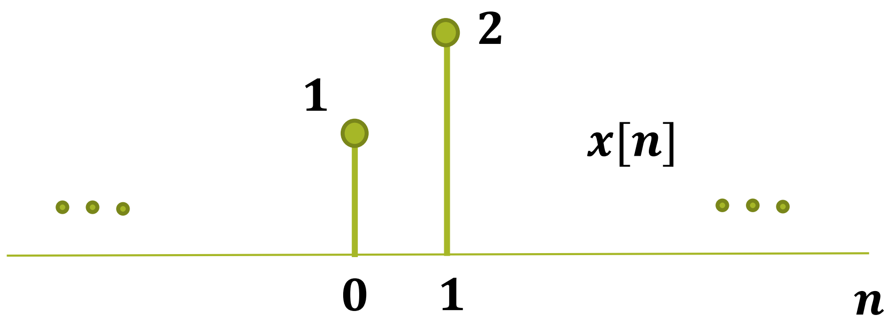
2.1.1 The Representation of Discrete-time Signals in Terms of Impulses

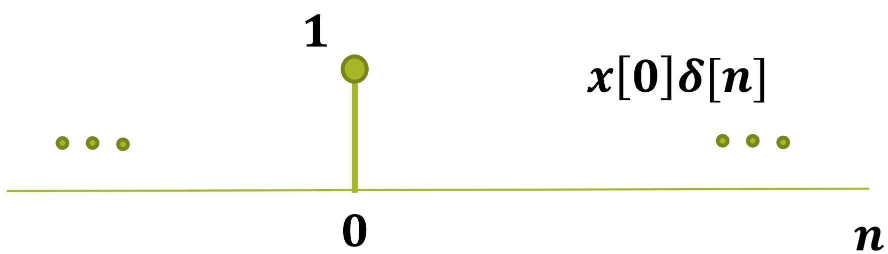
$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] +$$

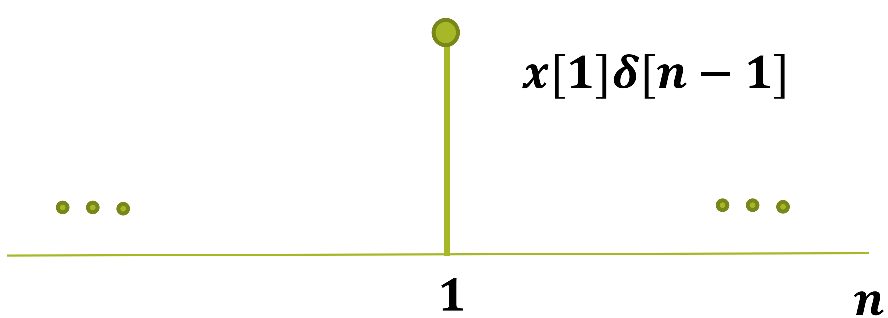
$$x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

If $\boldsymbol{x[n] = u[n]}$ **, then** $\boldsymbol{u[n] = \sum_{k=0}^{+\infty} \delta[n-k]}$

2 Linear Time-Invariant Systems







2.1.2 The Discrete-time Unit Impulse Response and the Convolution Sum Representation of LTI Systems

(1) Unit Impulse(Sample) Response

$$x[n] = \delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = h[n]$$

Unit Impulse Response: $h[n]$

(2) Convolution Sum of LTI System

Question:

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]=?$$

Solution:

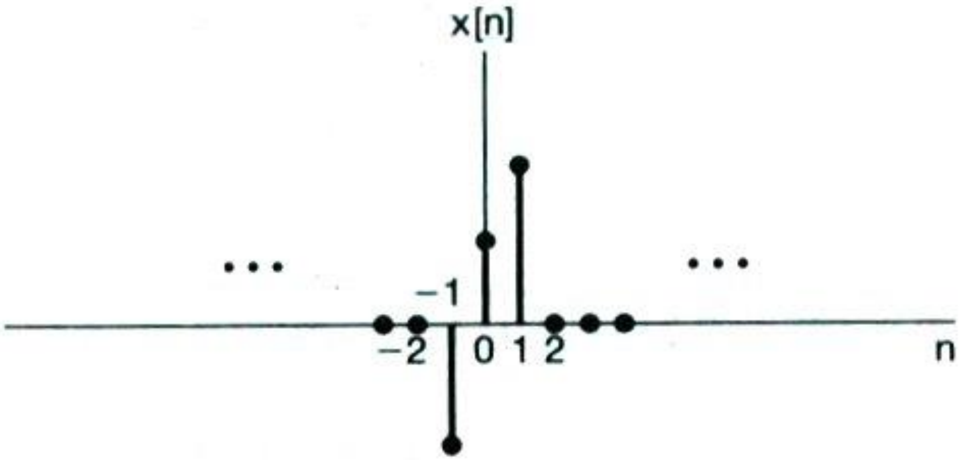
$$\delta[n] \longrightarrow h[n]$$

$$\delta[n\text{-}k] \longrightarrow h[n\text{-}k]$$

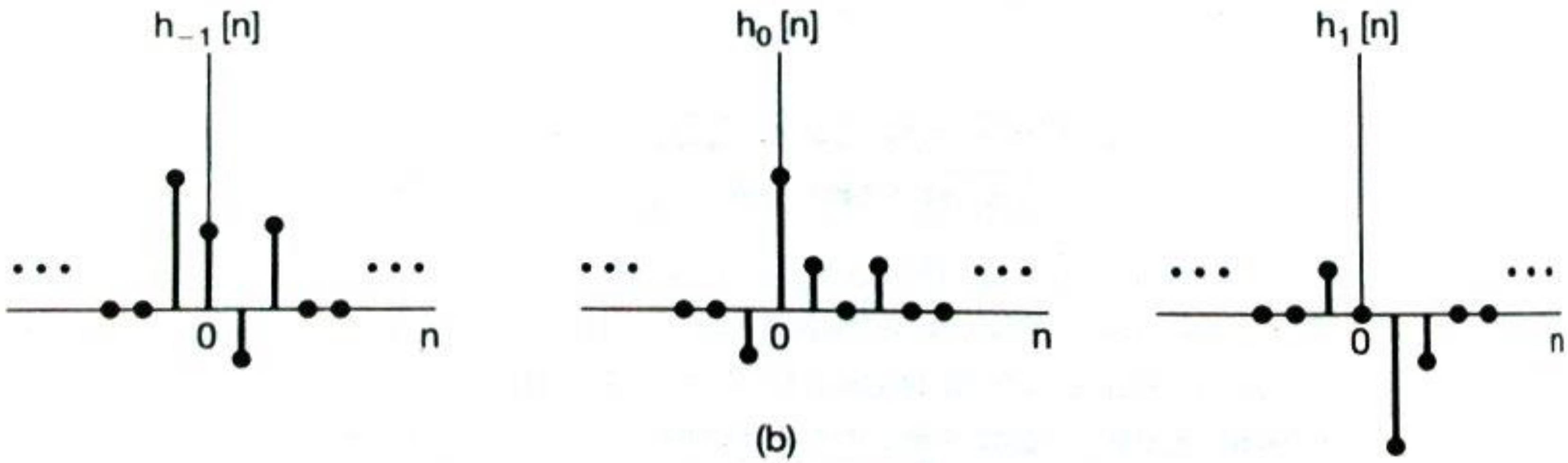
$$x[k]\delta[n\text{-}k] \longrightarrow x[k]\, h[n\text{-}k]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \longrightarrow y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

2 Linear Time-Invariant Systems

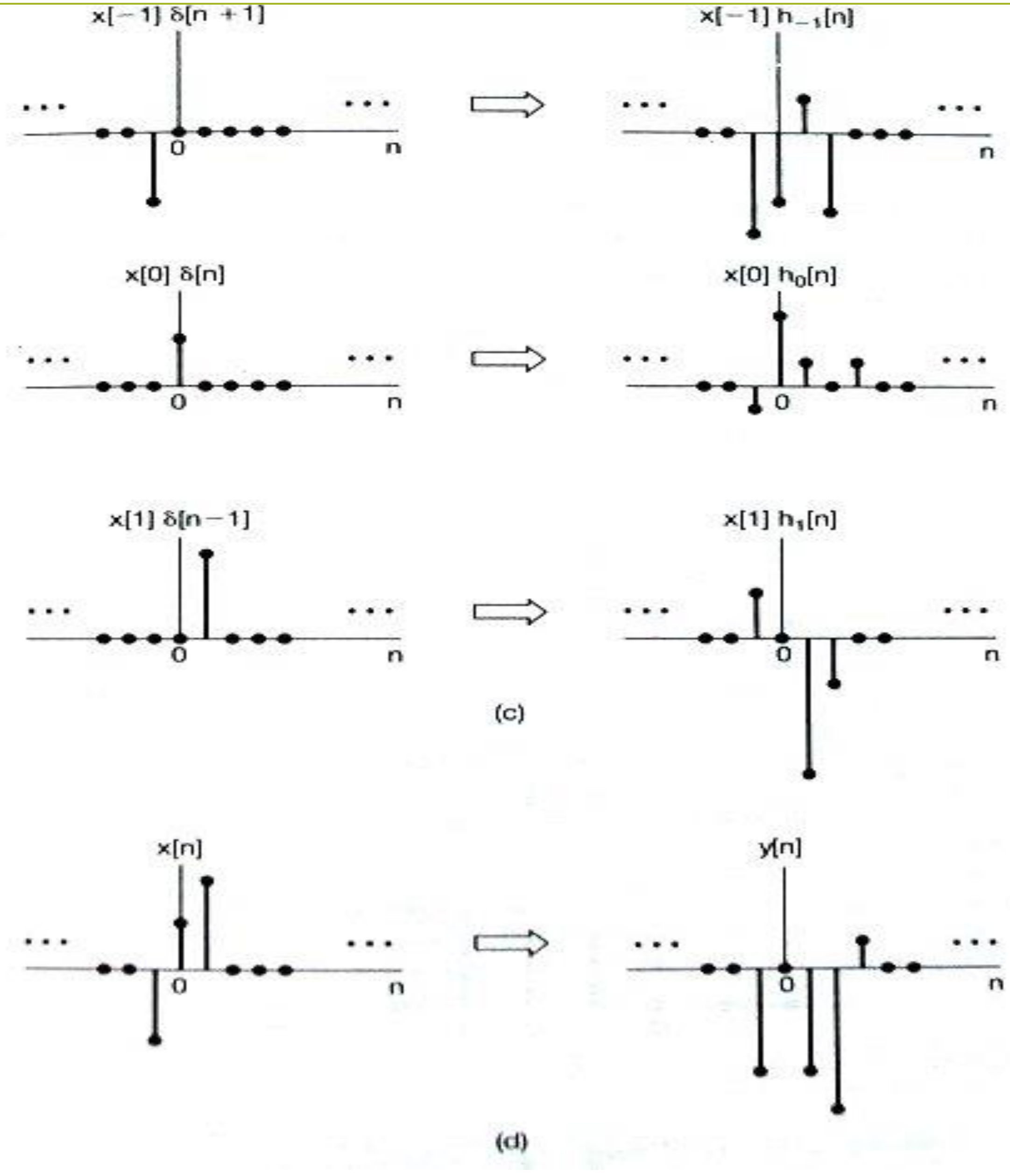


(a)



(b)

2 Linear Time-Invariant Systems



(c)

(d)

So $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ **(Convolution Sum)**

Or $y[n] = x[n] * h[n]$

(3) Calculation of Convolution Sum

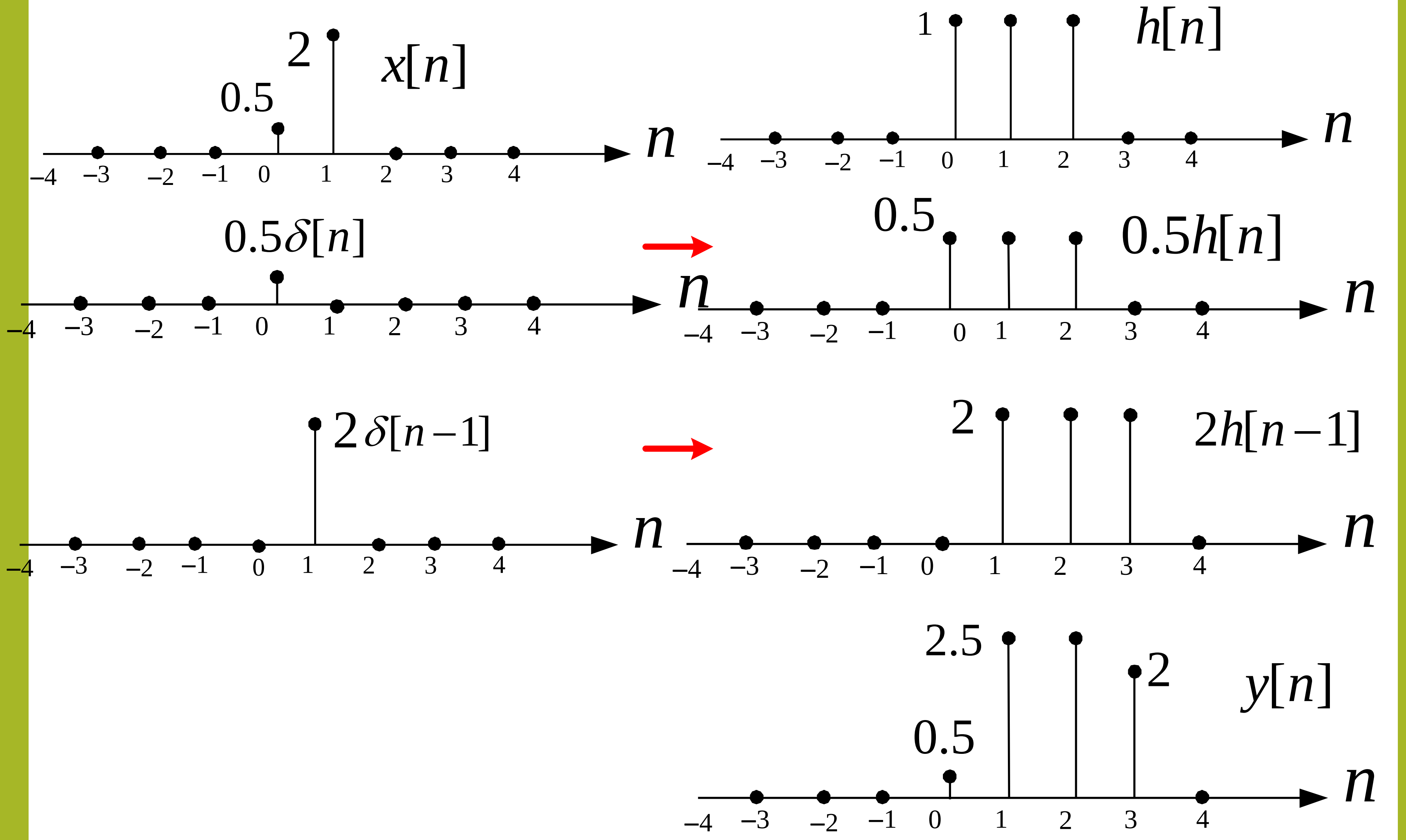
Time Inverse: $h[k] \longrightarrow h[-k]$

Time Shift: $h[-k] \longrightarrow h[n-k]$

Multiplication: $x[k]h[n-k]$

Summing: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$

Example 2.2 **In graphic form,**



In another words,

$$x[n] = 0.5\delta[n] + 2\delta[n-1]$$

$$y[n] = x[n] * h[n] = 0.5h[n] + 2h[n-1]$$

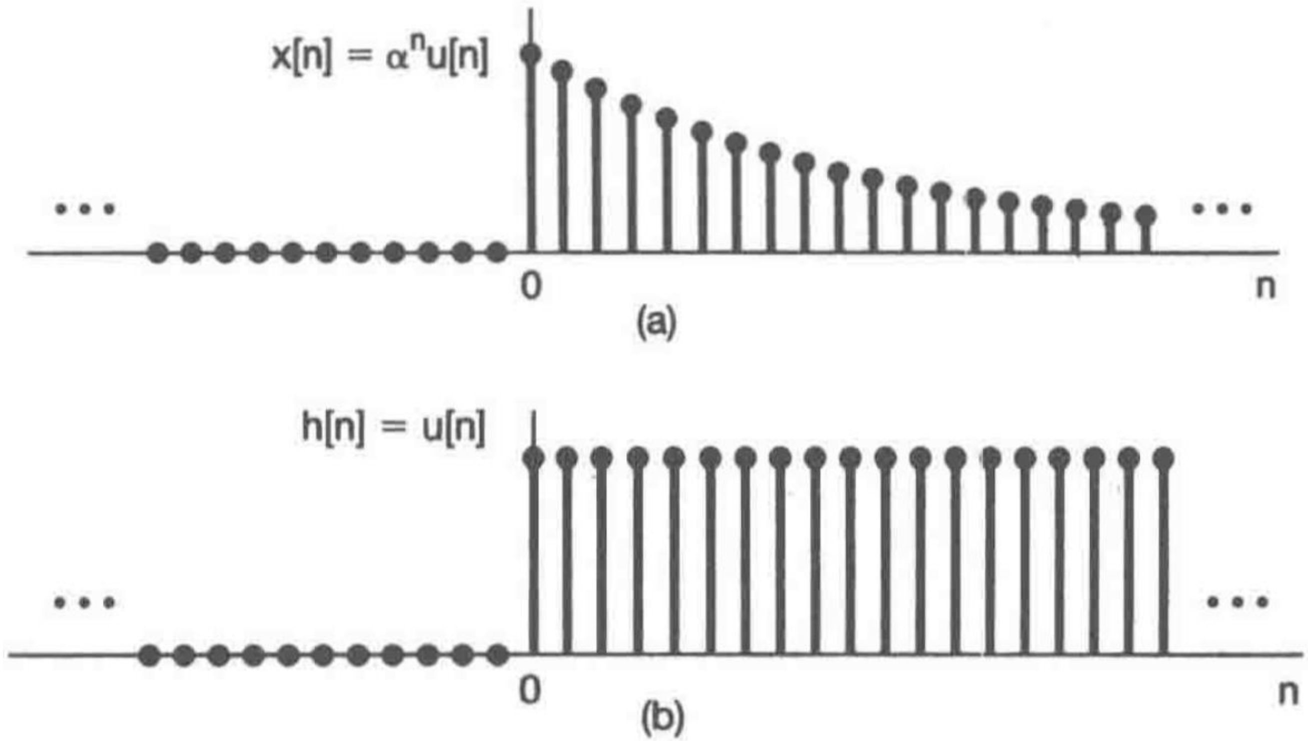
That means ,

$$\delta[n] * h[n] = h[n]$$

$$A\delta[n-k] * h[n] = Ah[n-k]$$

Example 2.3 $x[n] = \alpha^n u[n], h[n] = u[n]$

$0 < \alpha < 1$, $y[n] = ?$

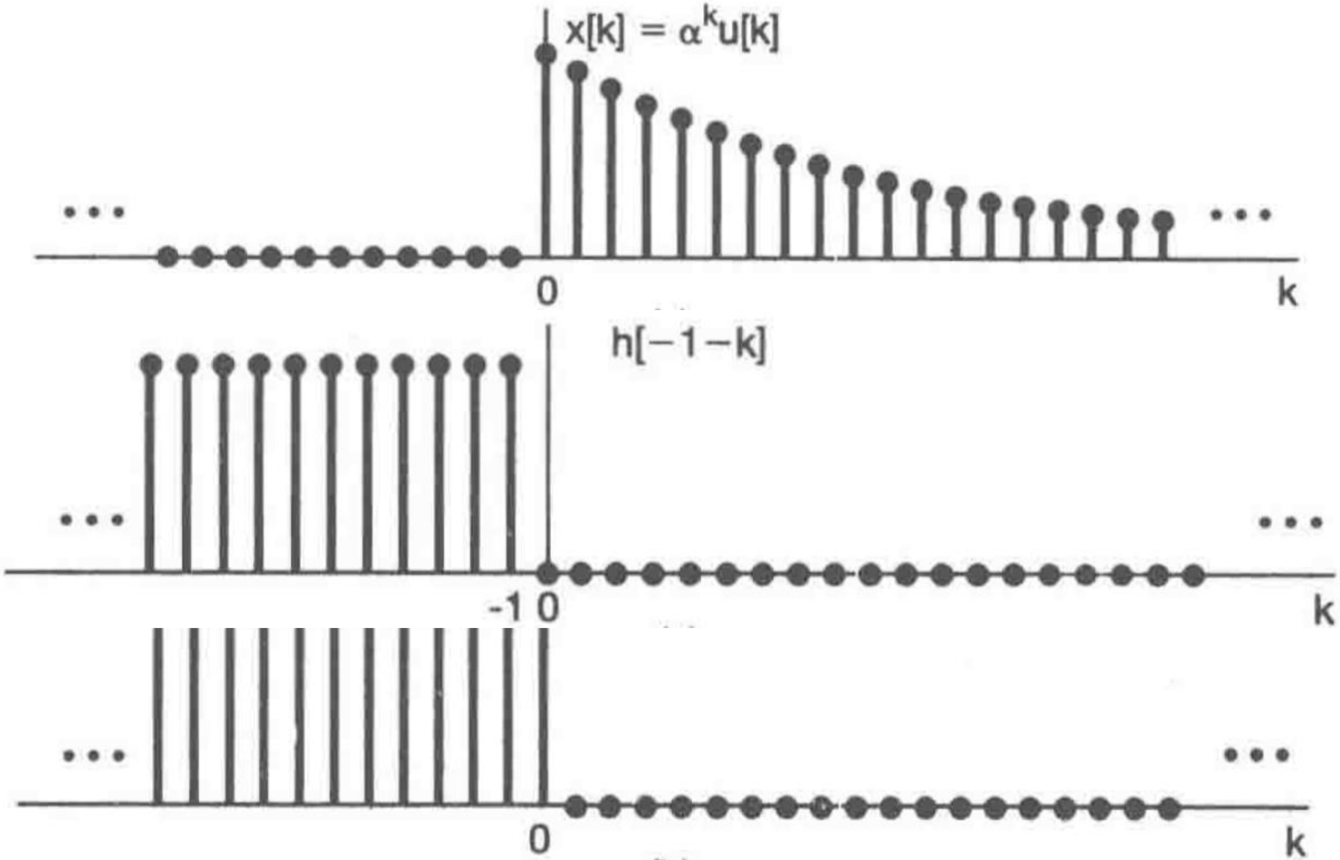


(a)

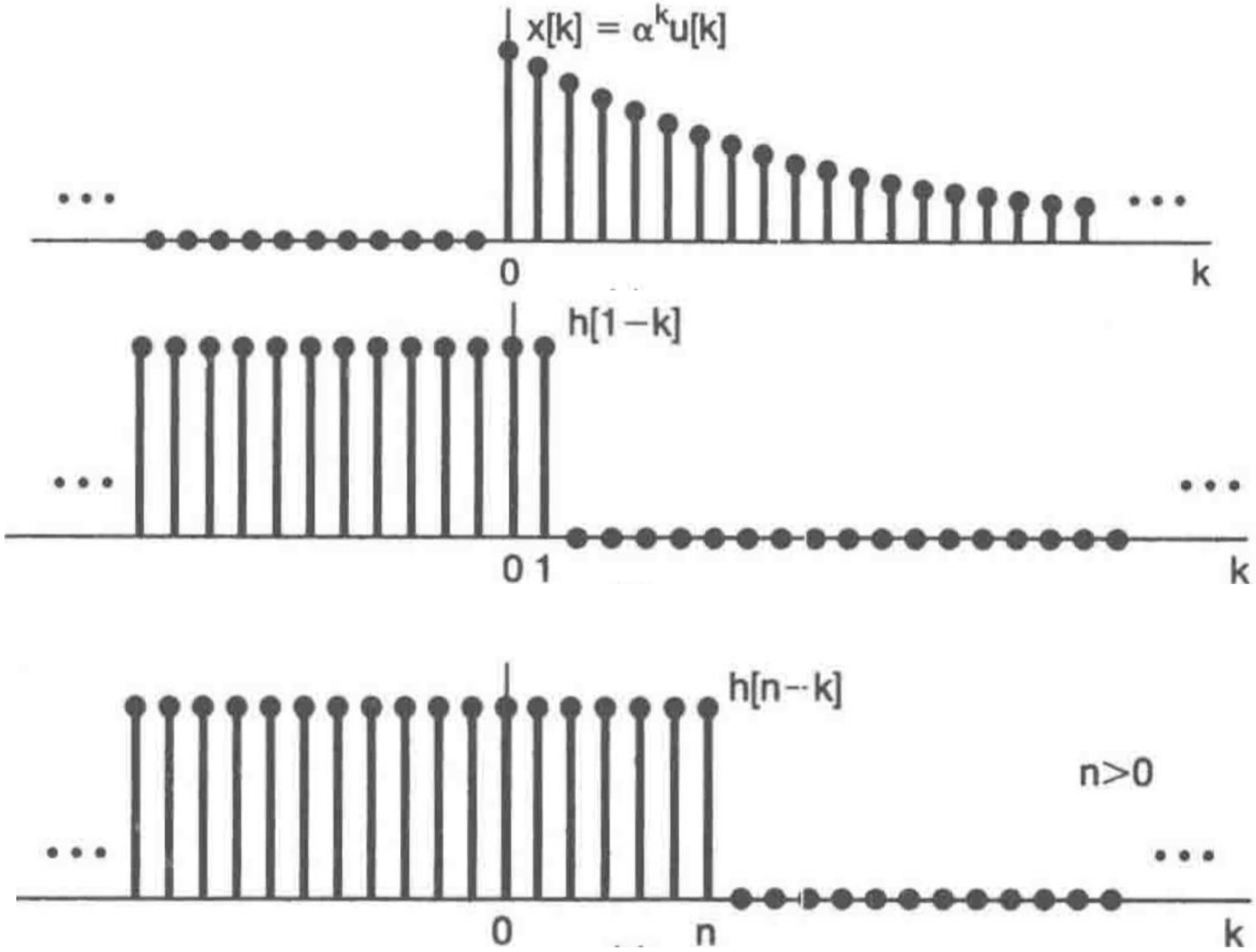
(b)

2 Linear Time-Invariant Systems

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$



2 Linear Time-Invariant Systems



2 Linear Time-Invariant Systems

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} \alpha^k u[k]u[n-k] = \{\sum_{k=0}^{n} \alpha^k\}u[n]$$

$$= \frac{1-\alpha^{(n+1)}}{1-\alpha} u[n]$$

Example **(Similar to Ex 2.4)**

$$x[n] = u[n - N_1] - u[n - N_2 - 1]$$

Length $N_2 - N_1 + 1$

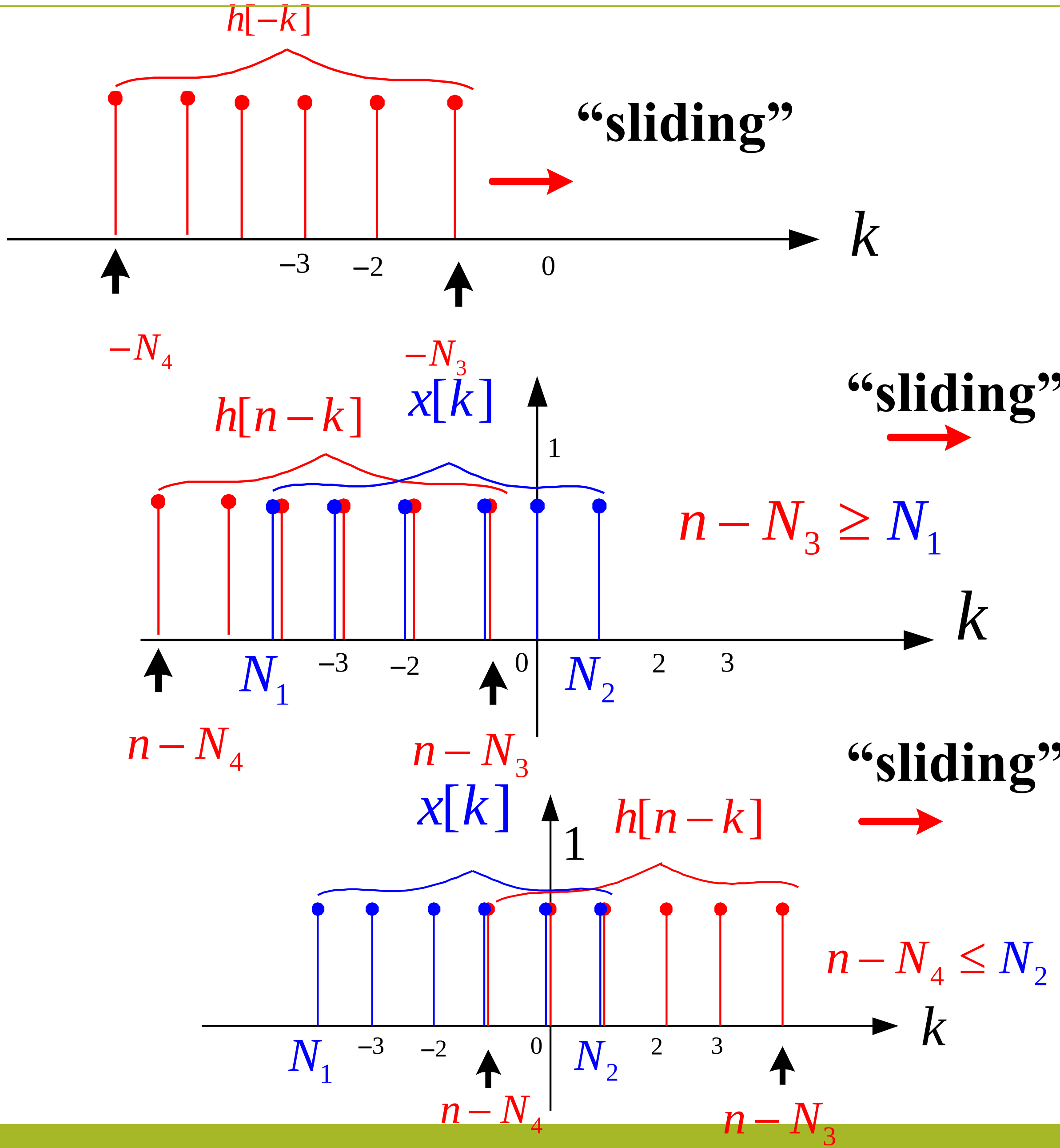
$$n\text{:} \quad [N_1, N_2] \qquad N_1 < N_2$$

$$h[n] = u[n - N_3] - u[n - N_4 - 1]$$

Length $N_4 - N_3 + 1$

$$n : [N_3, N_4] \qquad N_3 < N_4$$

2 Linear Time-Invariant Systems



2 Linear Time-Invariant Systems

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[n] = \begin{cases} 0, & n > N_2 + N_4, or, n < N_1 + N_3 \\ \neq 0, & elsewhere \end{cases}$$

Convergence of convolution

$$u[n]*u[-n]=\ ?$$

Not existed

$$u[n]*1=u[n]*\{u[n-1]+u[-n]\}=\ ?$$

Not existed

2 Linear Time-Invariant Systems

Example

$$x[n] = z_0^n, \quad \longrightarrow \boxed{h[n]} \longrightarrow \quad y[n]$$

$$-\infty < n < \infty$$

$$h[n] = \gamma^n u[n]$$

$$= \frac{z_0}{z_0 - \gamma} z_0^n, |z_0| > |\gamma|$$

$$y[n] = x[n] * h[n]$$

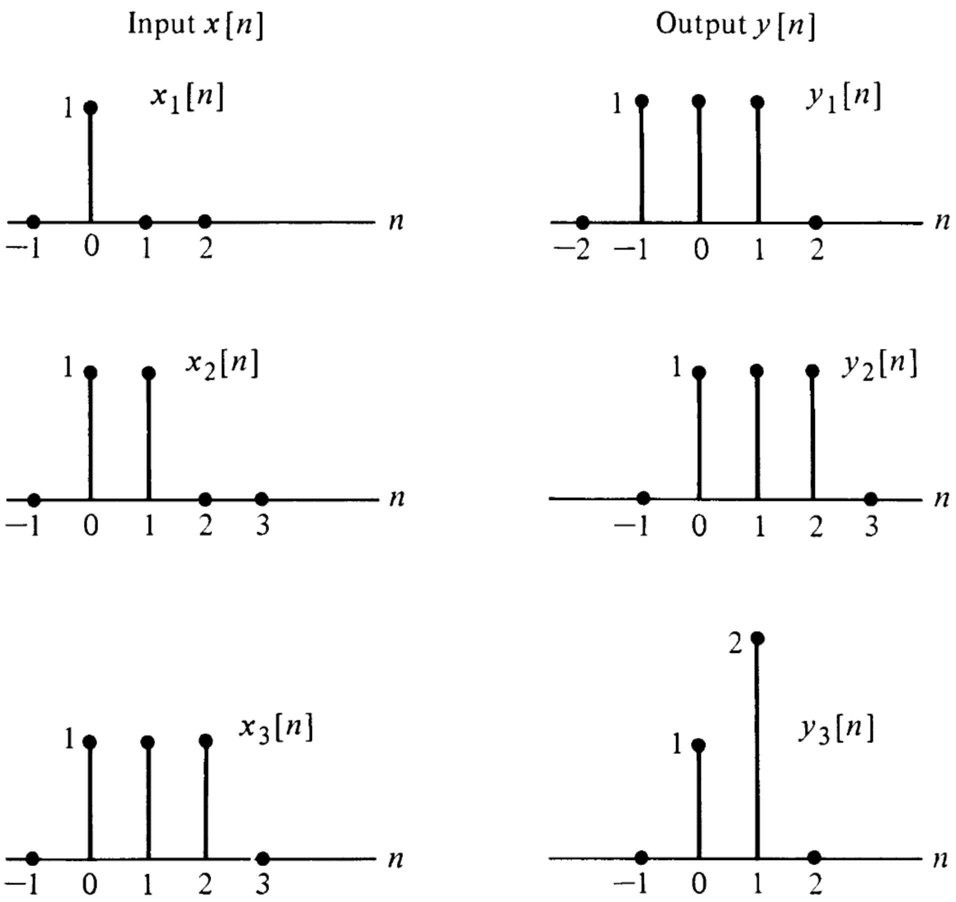
$$= \sum_{k=0}^{\infty} \gamma^k {z_0}^{n-k} = z_0^n \sum_{k=0}^{\infty} (\gamma / z_0)^k$$

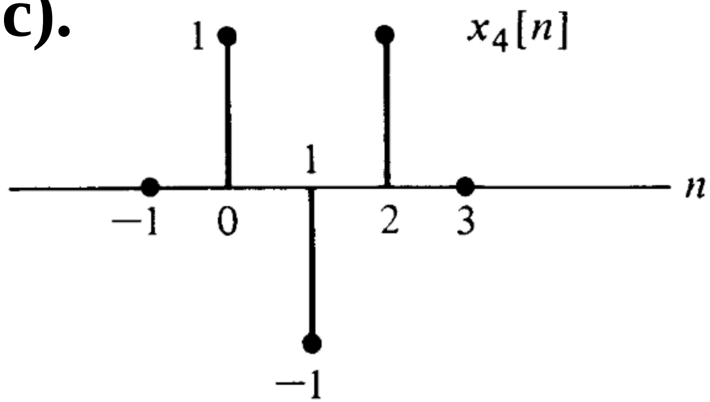
$$= \frac{z_0}{z_0 - \gamma} z_0^n, |z_0| > |\gamma|$$

Resume of convolution sum

$x[n]$	$h[n]$	$y[n] = x[n] * h[n]$
Causal(right-side)	Causal(right-side)	Causal(right-side)
Time-limited $[N_1, N_2]$	Time-limited $[N_3, N_4]$	Time-limited $[N_1 + N_3, N_2 + N_4]$
Causal(right-side)	Anti-Causal(left-side)	two-side or not existed
$z_0^n, -\infty < n < \infty$	$\gamma^n u[n]$	$\dfrac{z_0}{z_0 - \gamma} z_0^n, \lvert z_0 \rvert > \lvert \gamma \rvert$

Example: **Suppose that a discrete-time linear system has outputs $y[n]$ for the given inputs $x[n]$ as shown in the figure. Then Determine(a)(b) and (c).**





(a)Express $x_4[n]$ as a linear combination of $x_1[n]$, $x_2[n]$, and $x_3[n]$.

(b) Using the fact that the system us linear, determine $y_4[n]$.

(c)From the input-output pairs, determine whether the system is time-invariant.

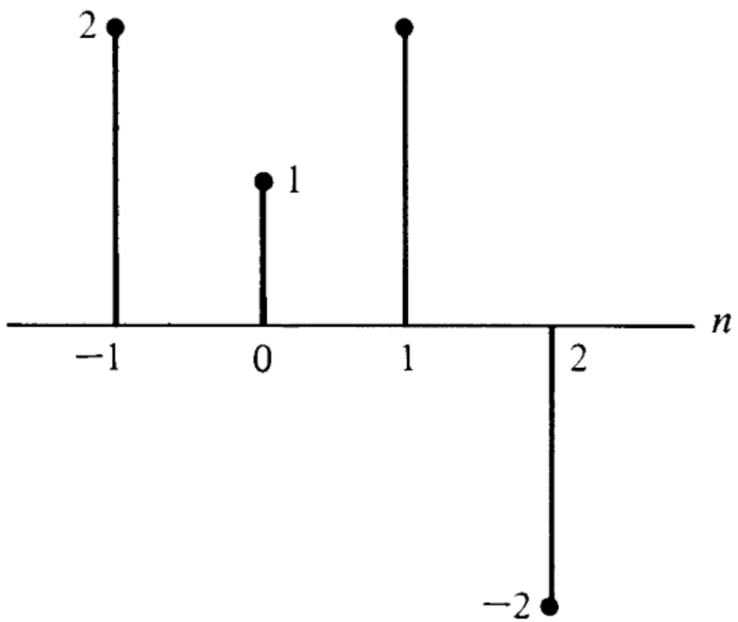
Solution:

(a) $x_4[n] = 2x_1[n] - 2x_{2[n]} + x_3[n]$

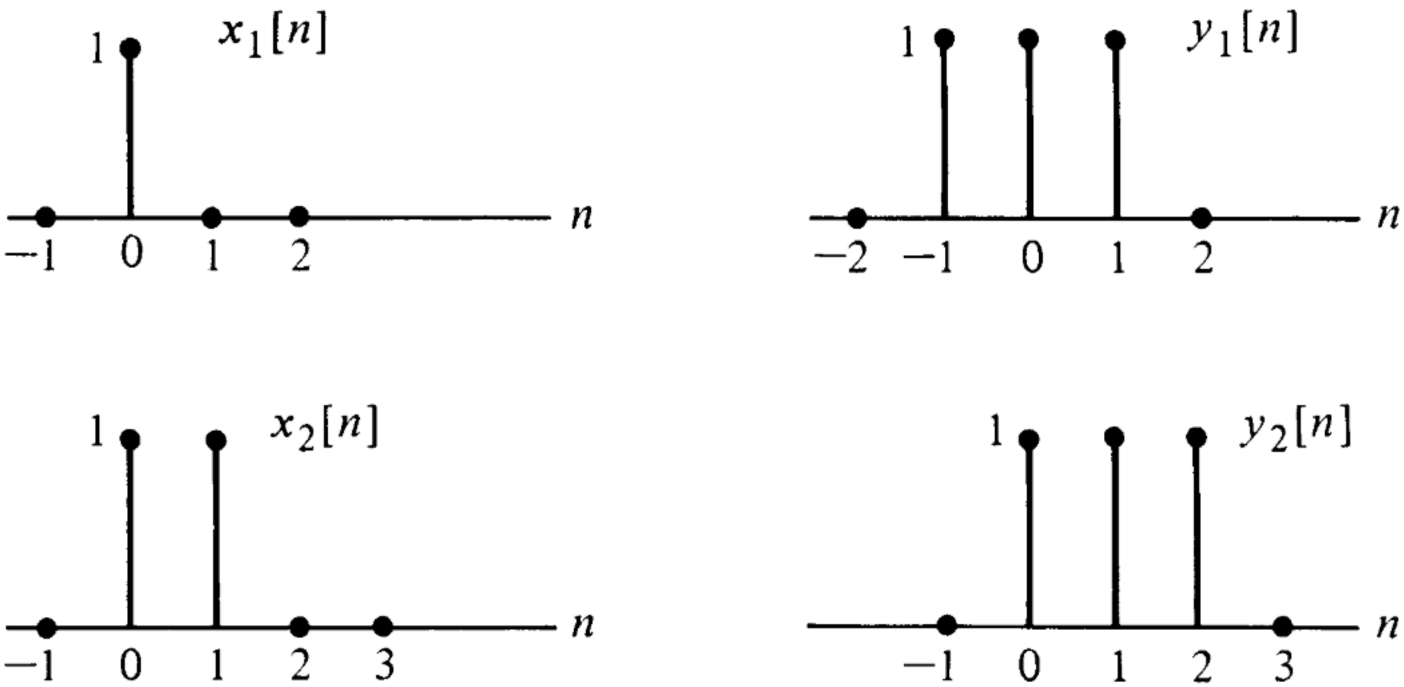
(b) Using superposition,

$y_4[n] = 2y_1[n] - 2y_{2[n]} + y_3[n]$

So the figure is depicted as following.



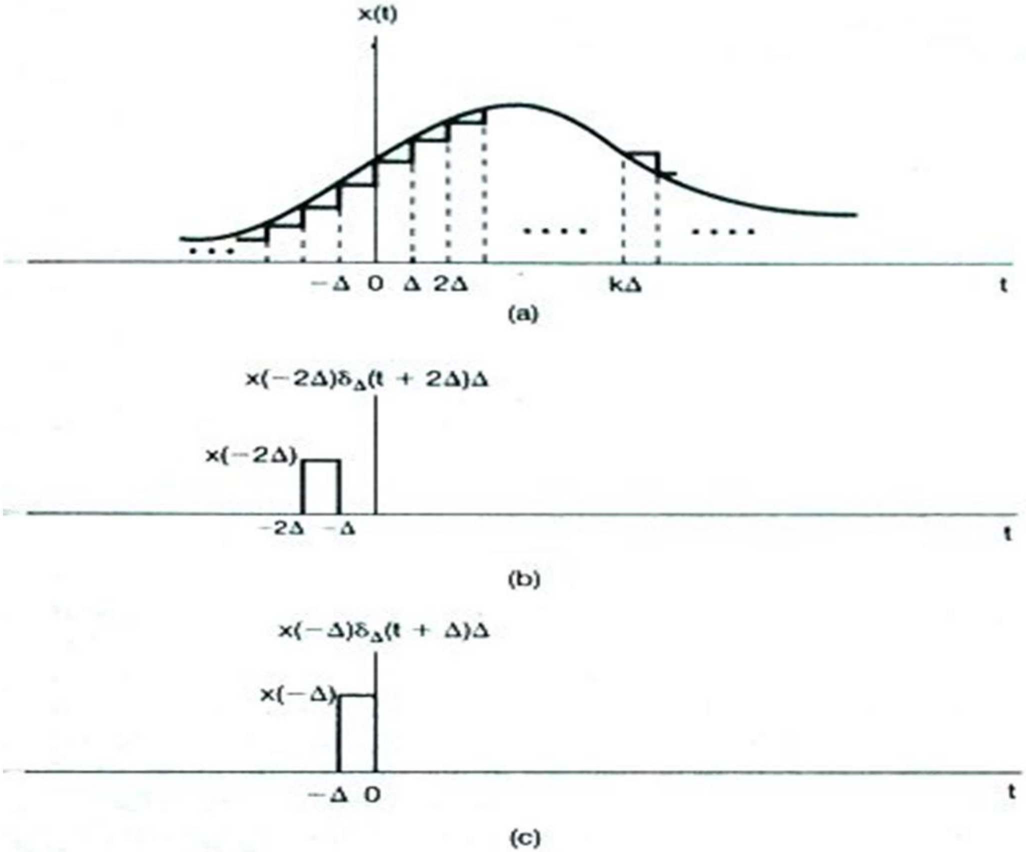
(c) The system is not time-invariant because an input $x_2[n] = x_1[n] + x_1[n-1]$ does not produce an output $y_2[n] \neq y_1[n] + y_1[n-1]$.



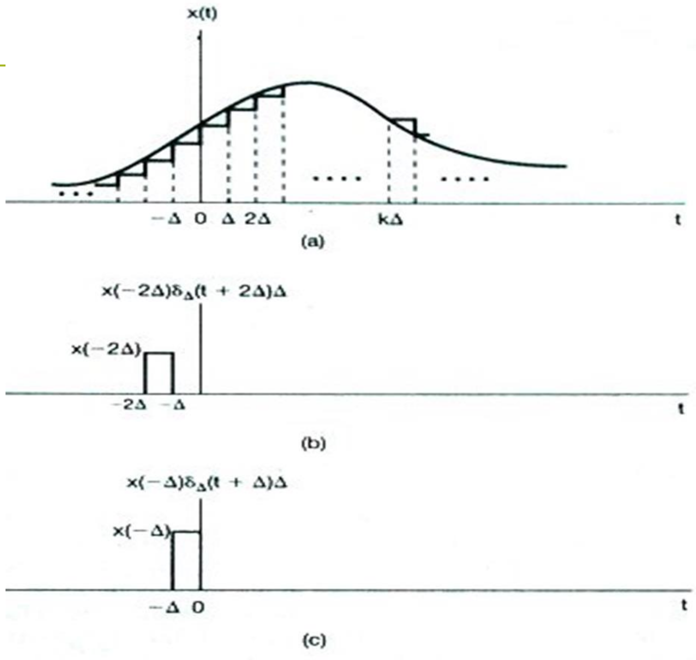
2.2 Continuous-time LTI system:

The convolution integral

2.2.1 The Representation of Continuous-time Signals in Terms of Impulses



2 Linear Time-Invariant Systems



Define $$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t \le \Delta \\ 0, & otherwise \end{cases}$$

We have the expression:

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\Delta\delta_\Delta(t - k\Delta)$$

Therefore: $$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\Delta\delta_\Delta(t - k\Delta)$$

or

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

Sifting property of continuous-time impulse

2.2.2 The Continuous-time Unit impulse Response and the convolution Integral Representation of LTI Systems

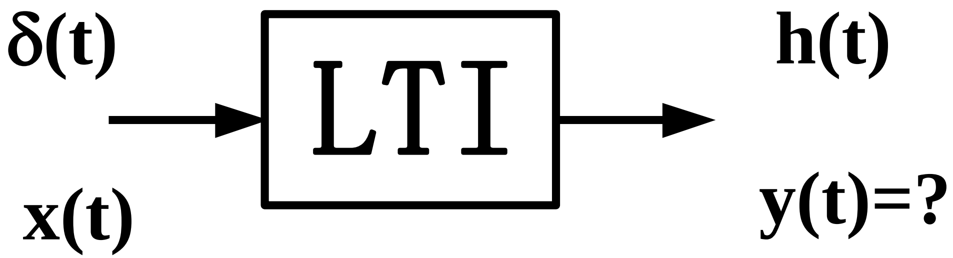
(1) Unit Impulse Response

$$x(t) = \delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = h(t)$$

(2) The Convolution of LTI System

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = ?$$

2 Linear Time-Invariant Systems



Because of $$\boldsymbol{x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau}$$

So $$\boldsymbol{y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau}$$

(Convolution Integral)

Or $$\boldsymbol{y(t) = x(t) * h(t)}$$

(3) Computation of Convolution Integral

Time Reversal: $\boldsymbol{h(\tau) \longrightarrow h(-\tau)}$

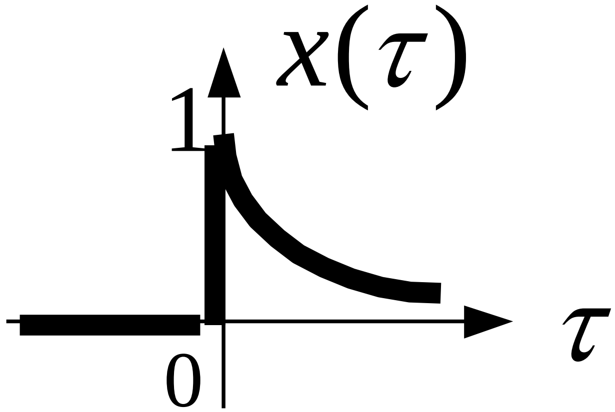
Time Shift: $\boldsymbol{h(-\tau) \longrightarrow h(t-\tau)}$

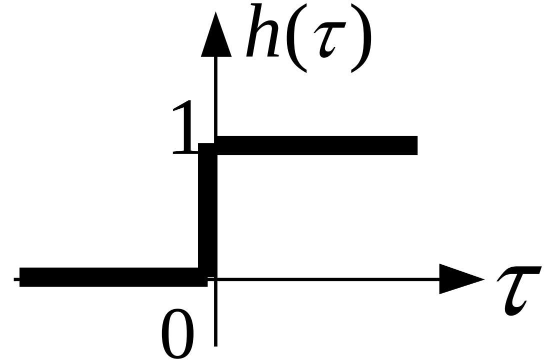
Multiplication: $\boldsymbol{x(\tau)h(t-\tau)}$

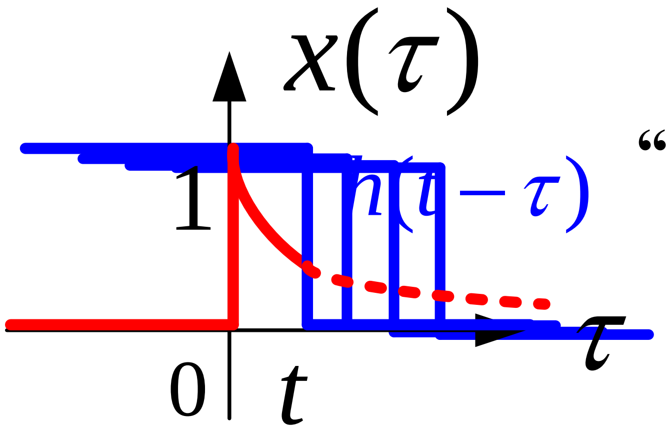
Integrating: $\boldsymbol{y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau}$

Example 2.6

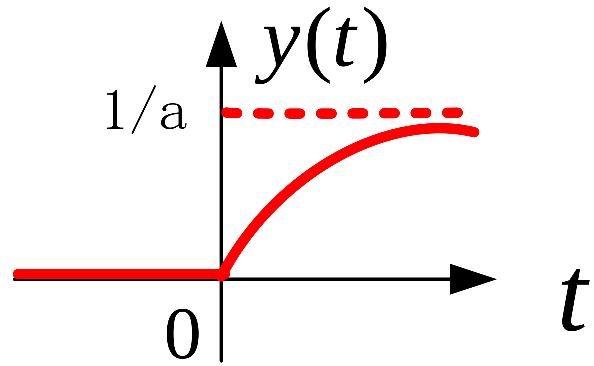
$$y(t) = x(t) * h(t) = e^{-at}u(t) * u(t), a > 0$$







“sliding”



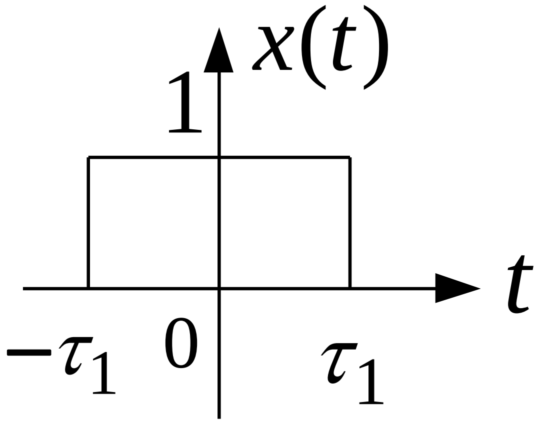
$$= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau = [\int_{0}^{t} e^{-a\tau}d\tau]u(t)$$

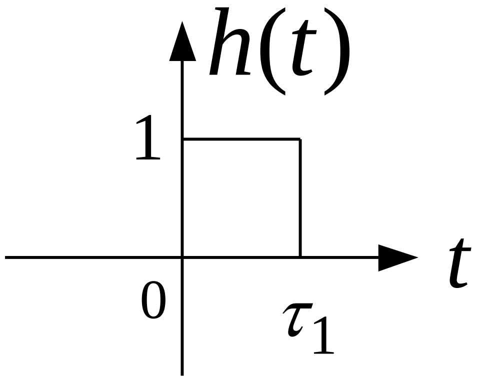
$$For\ all\ values\ of\ t,$$

$$y(t) = \frac{1}{a}[1 - e^{-at}]u(t)$$

Example:

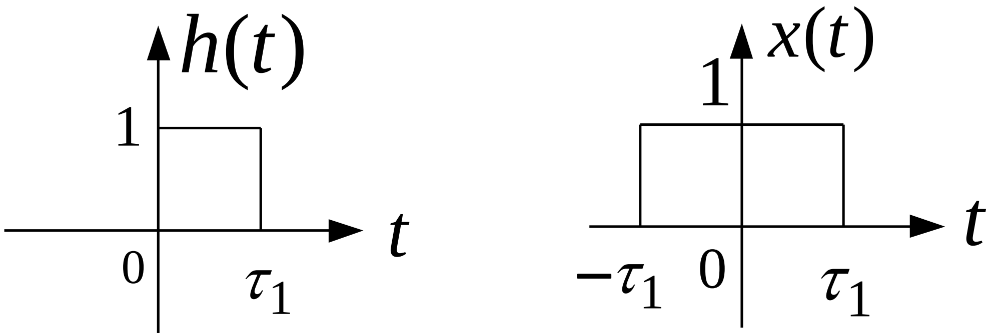
Consider an LTI system with input $x(t)$ and unit impulse response $h(t)$, compute and sketch the output signal $y(t)$.





2 Linear Time-Invariant Systems

Solution:



$$y(t) = x(t) * h(t) = [u(t) - u(t-\tau_1)] * [u(t+\tau_1) - u(t-\tau_1)]$$

$$= \int_{-\infty}^{\infty} [u(\tau) - u(\tau-\tau_1)][u(t-\tau+\tau_1) - u(t-\tau-\tau_1)]d\tau$$

$$= \int_{-\infty}^{\infty} u(\tau)u(t-\tau+\tau_1)d\tau + \int_{-\infty}^{\infty} u(\tau-\tau_1)u(t-\tau-\tau_1)d\tau$$

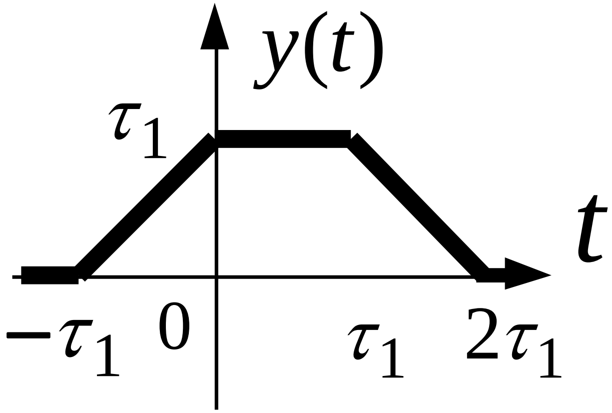
$$- \int_{-\infty}^{\infty} u(\tau)u(t-\tau-\tau_1)d\tau - \int_{-\infty}^{\infty} u(\tau-\tau_1)u(t-\tau+\tau_1)d\tau$$

$$= [\int_0^{t+\tau_1} 1 d\tau] u(t+\tau_1) + [\int_{\tau_1}^{t-\tau_1} 1 d\tau] u(t-2\tau_1)$$

$$-[\int_0^{t-\tau_1} 1 d\tau] u(t-\tau_1) - [\int_{\tau_1}^{t+\tau_1} 1 d\tau] u(t)$$

$$= (t+\tau_1)u(t+\tau_1) + (t-2\tau_1)u(t-2\tau_1)$$

$$-(t-\tau_1)u(t-\tau_1) - tu(t)$$



Example (Similar to Ex 2.8)

$$y(t) = x(t) * h(t) = e^{at}u(-t) * u(t-3), a > 0$$

$$= \int_{-\infty}^{\infty} e^{a\tau}u(-\tau)u(t-\tau-3)d\tau = \begin{cases} \int_{-\infty}^{t-3} e^{a\tau}d\tau = \frac{1}{a}e^{a(t-3)}, t-3<0 \\ \int_{-\infty}^{0} e^{a\tau}d\tau = \frac{1}{a}, \quad t-3 \geq 0 \end{cases}$$

$$\textbf{\textit{or}}, y(t) = \frac{1}{a}u(t-3) + \frac{1}{a}e^{a(t-3)}u(3-t)$$

Note:when $a \leq 0$, $y(t) = \infty$

Convergence of convolution

$$u(t) * u(-t) = \quad ?$$

Not existed

$$u(t) * 1 = u(t) * \{u(t) + u(-t)\} = \quad ?$$

Not existed

Resume of convolution Integral

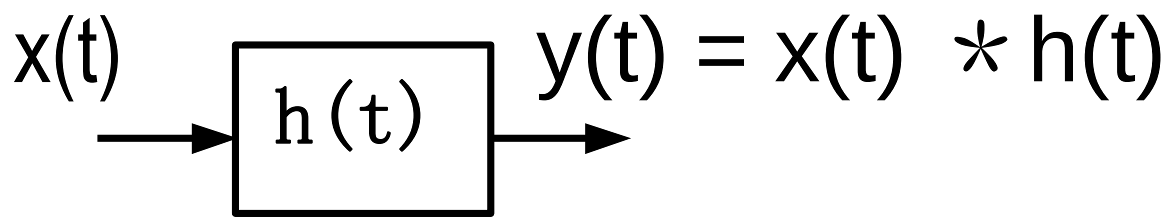
$x(t)$	$h(t)$	$y(t) = x(t) * h(t)$
Causal(right-side)	**Causal(right-side)**	**Causal(right-side)**
Time-limited (t_1, t_2)	**Time-limited** (t_3, t_4)	**Time-limited** $(t_1 + t_3, t_2 + t_4)$
Causal(right-side)	**Anti-Causal(left-side)**	**two-side or not existed**
$e^{s_1 t}, -\infty < t < \infty$	$e^{s_2 t} u(t)$	$\frac{1}{s_1 - s_2} e^{s_1 t}, \mathrm{Re}[s_1] > \mathrm{Re}[s_2]$

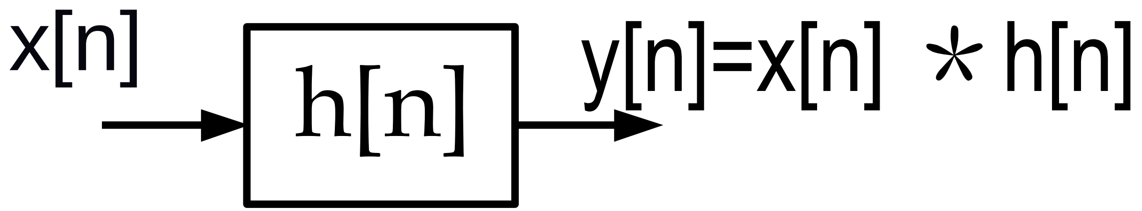
2.3 Properties of Linear Time Invariant System

Convolution formula:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

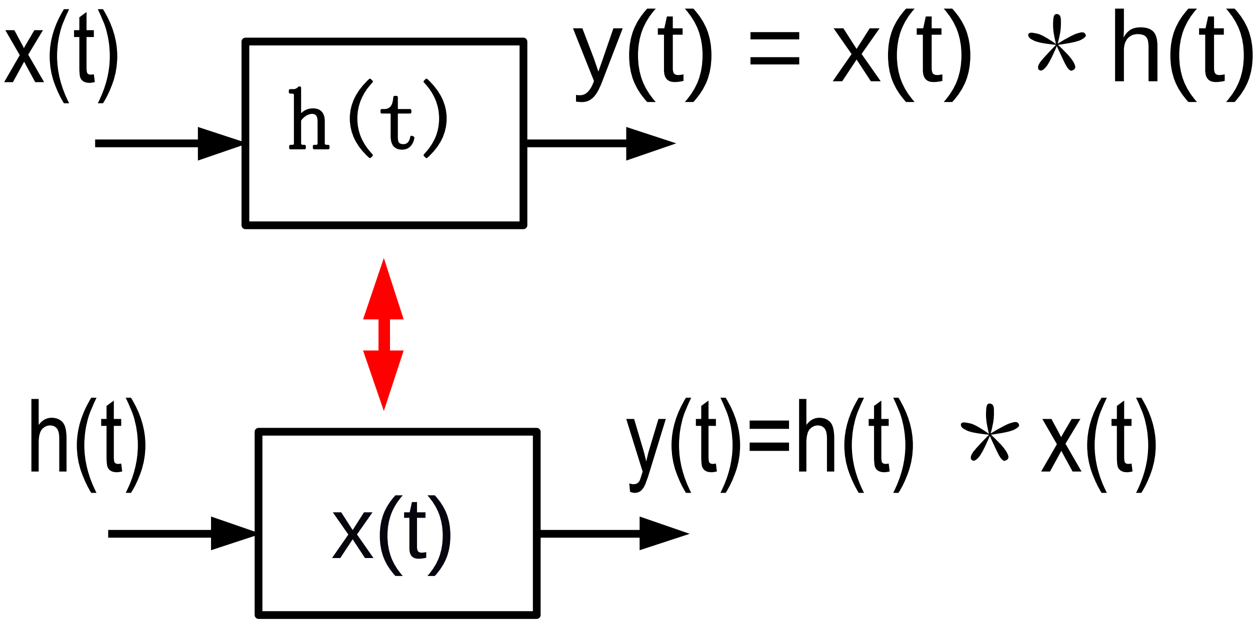




2.3.1 The Commutative Property

Discrete time: $x[n] * h[n] = h[n] * x[n]$

Continuous time: $x(t) * h(t) = h(t) * x(t)$



How to prove?

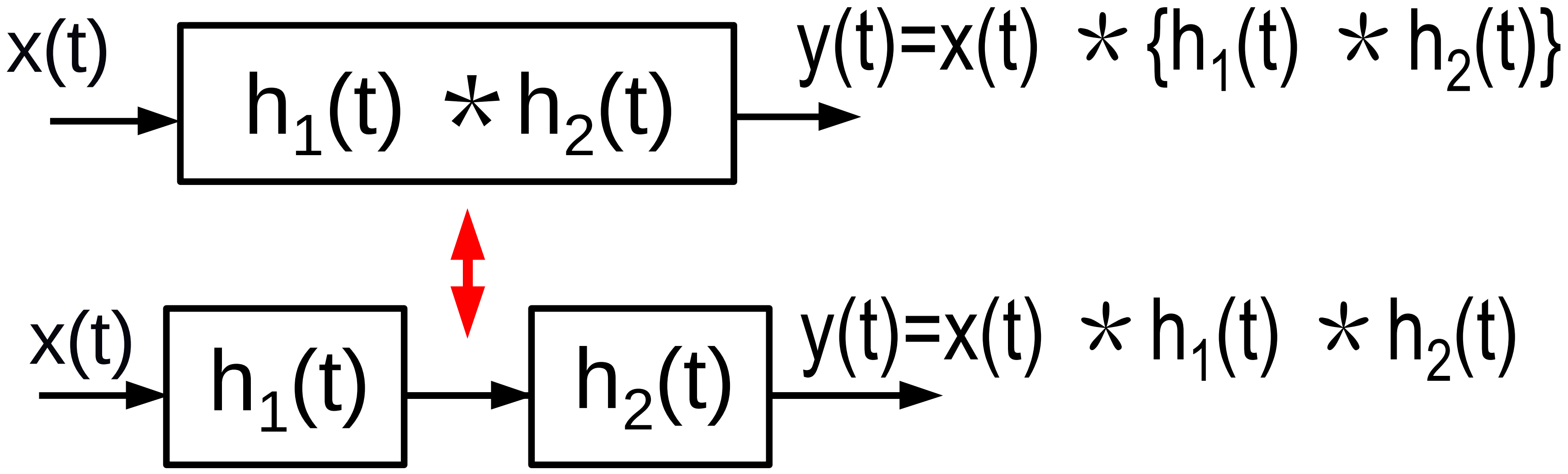
2.3.2 The Associative Property

Discrete time:

$$x[n] * \{h_1[n] * h_{2[n]}\} = \{x[n] * h_1[n]\} * h_2[n]$$

Continuous time:

$$x(t) * \{h_1(t) * h_{2(t)}\} = \{x(t) * h_1(t)\} * h_2(t)$$



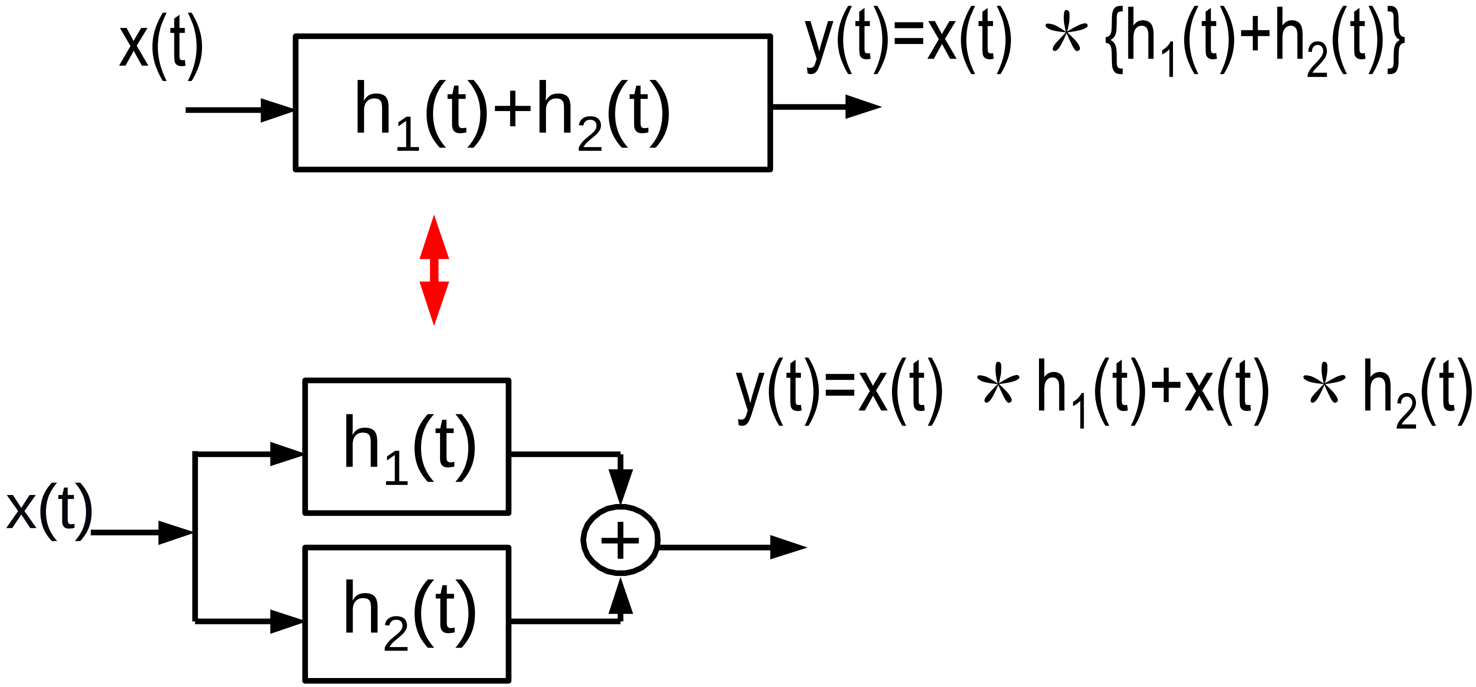
2.3.3 The Distributive Property

Discrete time:

$$x[n] * \{h_1[n] + h_{2[n]}\} = x[n] * h_1[n] + x[n] * h_2[n],$$

Continuous time:

$$x(t) * \{h_1(t) + h_{2(t)}\} = x(t) * h_1(t) + x(t) * h_2(t)$$



Example 2.10

$$x[n] = (\frac{1}{2})^n u[n] + 2^n u[-n], \qquad h[n] = u[n]$$

Let $x_1[n] = (\frac{1}{2})^n u[n] \quad , x_2[n] = 2^n u[-n]$

Then

$$y[n] = (x_1[n] + x_2[n]) * h[n]$$

$$y_1[n] = x_1[n] * h[n], \quad y_2[n] = x_2[n] * h[n]$$

$$y[n] = y_1[n] + y_2[n]$$

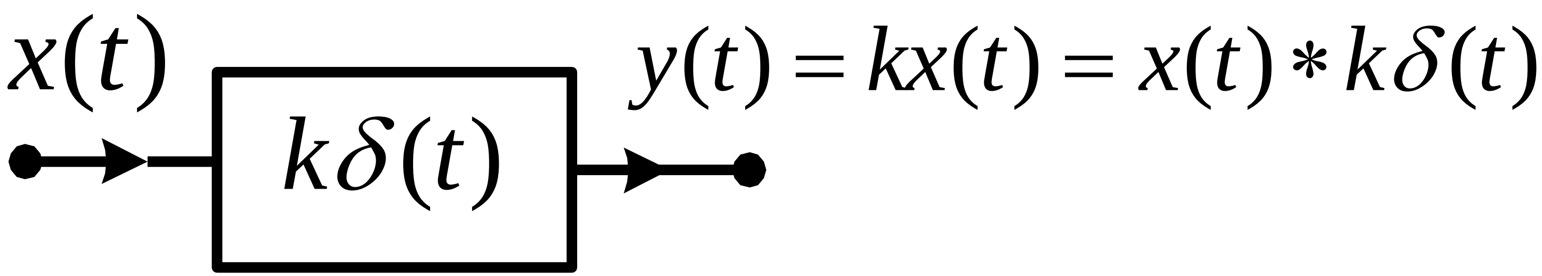
So, $y[n]$ **can be obtained.**

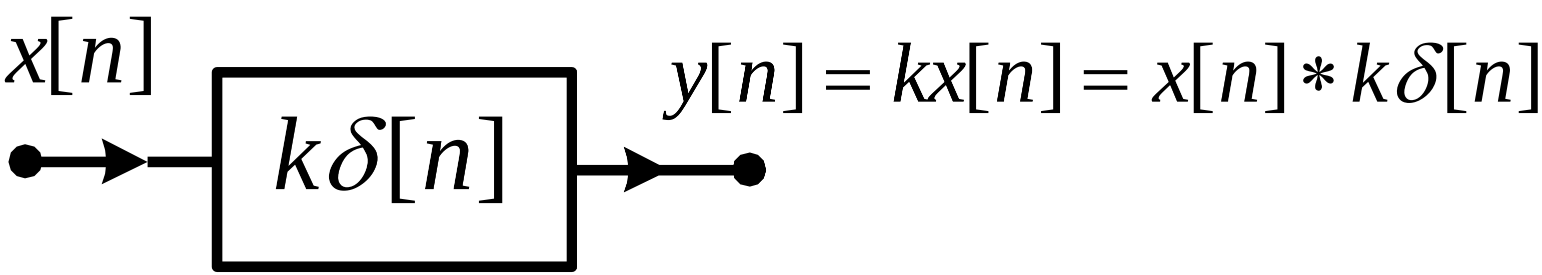
2.3.4 LTI system with and without Memory

Memoryless system:

D-T: $h[n] = k\delta[n]$, $y[n] = kx[n]$

C-T: $h(t) = k\delta(t)$, $y(t) = kx(t)$



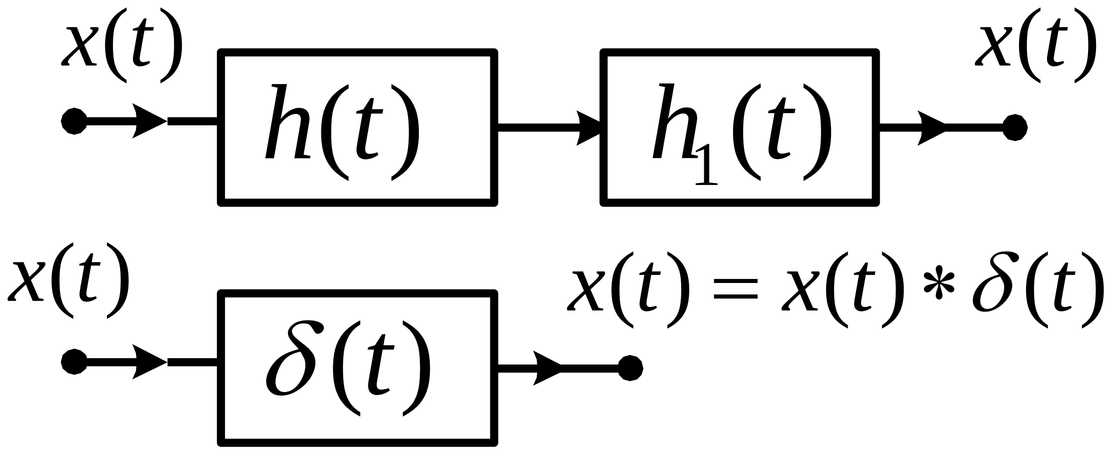


Imply that: $x(t) * \delta(t) = x(t)$ **and** $x[n] * \delta[n] = x[n]$

2.3.5 Invertibility of LTI system

Original system: $h(t)$

Reverse system: $h_1(t)$



So, for the invertible system:

$h(t) * h_1(t) = \delta(t)$ **or** $h[n] * h_1[n] = \delta[n]$

2.3.5 Invertibility of LTI system

Ex. 1: $LTI\ system: y(t) = x(t - t_0)$

$$\Rightarrow h(t) = \delta(t - t_0)$$

$$inverse\ system:\ h_1(t) = \delta(t + t_0)$$

Ex. 2: $LTI\ system: h[n] = u[n]$

$$\Rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]u[n-k]$$

$$inverse\ system:\ h_1[n] = \delta[n] - \delta[n-1]$$

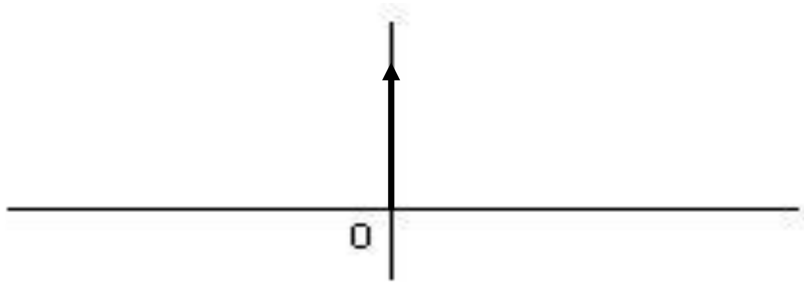
2.3.6 Causality for LTI system

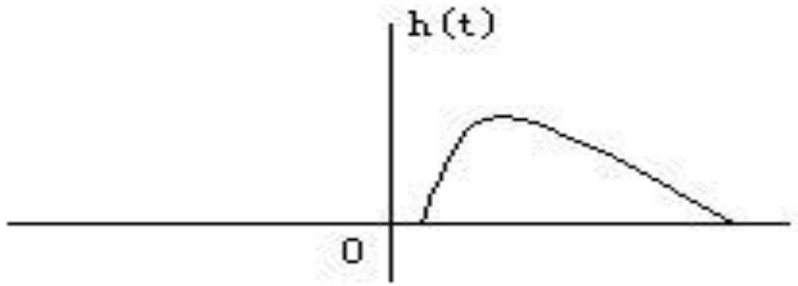
Discrete time system satisfy the condition:

$$h[n] = 0 \; for \; n < 0$$

Continuous time system satisfy the condition:

$$h(t) = 0 \; for \; t < 0$$





2.3.7 Stability for LTI system

Definition of stability: Every bounded input produces a bounded output.

Discrete time system:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]\,,\boldsymbol{or},\sum_{k=-\infty}^{+\infty} x[n-k]h[k]$$

If $|x[n]|$<B, the condition for $|y[n]|$<A is

$$\sum_{k=-\infty}^{+\infty} |h[k]| < +\infty$$

Absolutely summable

Because:

$$|y[n]| \le \sum_{k=-\infty}^{+\infty} |x[n-k]||h[k]| < B \sum_{k=-\infty}^{+\infty} |h[k]|$$

$$if \quad \sum_{k=-\infty}^{+\infty} |h[k]| < +\infty, \quad then \quad |y[n]| < A$$

Continuous time system:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \ , or \ , \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau$$

If |x(t)|<B, the condition for |y(t)|<A is

$$\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < +\infty$$

Absolutely integrable

Because:

$$|y(t)| \le \int_{-\infty}^{+\infty} |x(t-\tau)|\,|h(\tau)|\, d\tau < B\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau$$

$$if \quad \int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < +\infty, \quad then \quad |y(t)| < A$$

Example 2.13

Pure time shift system **stable**

$$y[n] = x[n - n_0] \qquad h[n] = \delta[n - n_0]$$

$$y(t) = x(t - t_0) \qquad h(t) = \delta(t - t_0)$$

accumulator $$y[n] = \sum_{k=-\infty}^{n} x[k] \qquad h[n] = u[n]$$

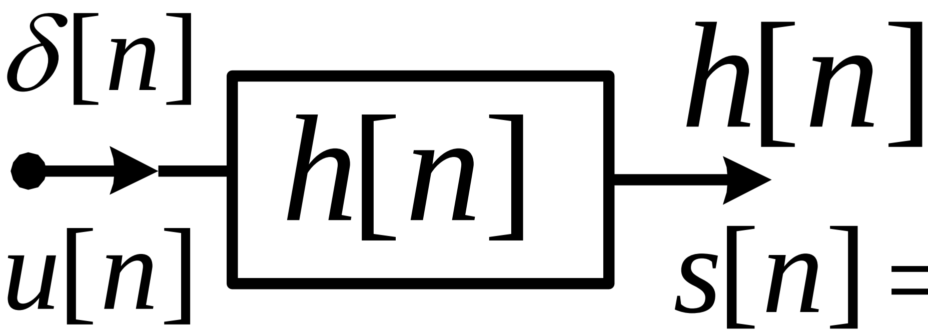
unstable

integrator $$y(t) = \int_{-\infty}^{t} x(\tau) d\tau \qquad h(t) = u(t)$$

unstable

2.3.8 The Unit Step Response of LTI system

Discrete time system:



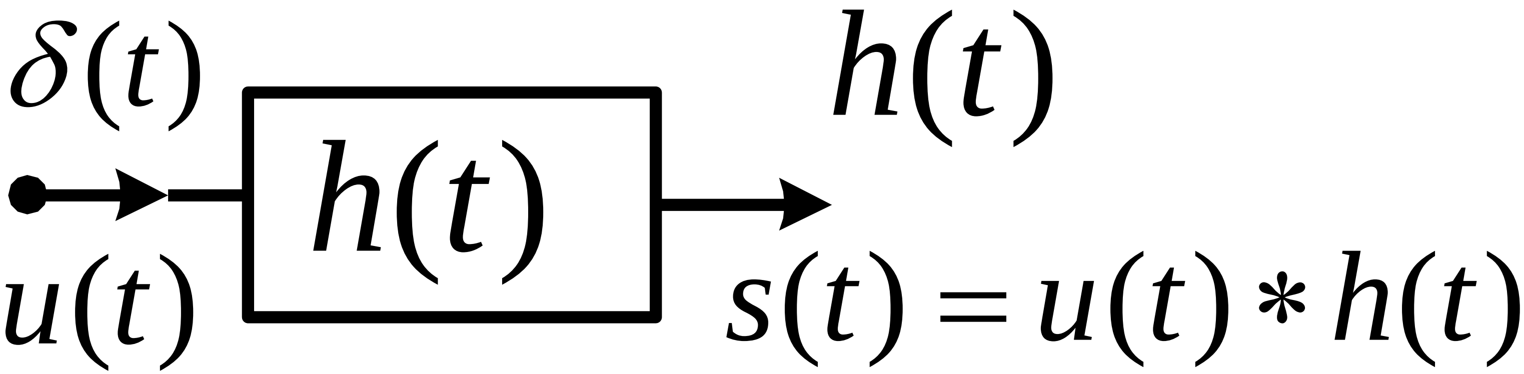
$$s[n] = u[n] * h[n]$$

$$s[n] = \sum_{k=-\infty}^{n} h[k], \; or$$

$$h[n] = s[n] - s[n-1]$$

2 Linear Time-Invariant Systems

Continuous time system:



$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau = h^{(-1)}(t) \ , or \quad h(t) = s'(t)$$

2.3.9(2.5.4) Convolution integral with a Singularity Functions

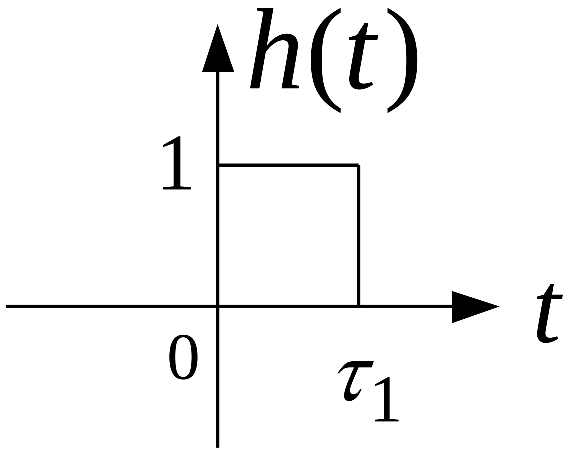
$$\left.\begin{array}{l}(1)\ \ \boldsymbol{x(t) * \delta(t) = x(t)} \\ (2)\ \ \boldsymbol{x(t) * \delta(t-t_0) = x(t-t_0)} \\ (3)\ \ \boldsymbol{x(t-t_1) * \delta(t-t_2) = x(t-t_1-t_2)}\end{array}\right\}\ \text{Key points}$$

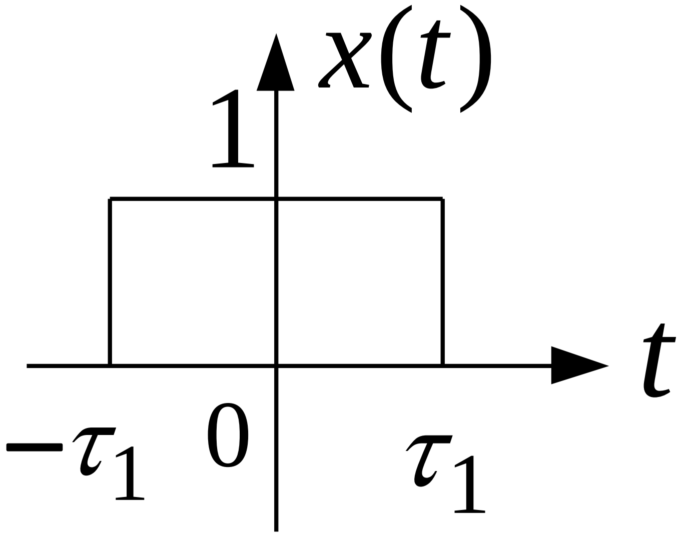
$$(4)\ \ x(t) * \delta'(t) = x'(t)$$

$$(5)\ \ x(t) * u(t) = x^{(-1)}(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

$$(6)\ \ x(t) * h(t) = x'(t) * h^{(-1)}(t)$$

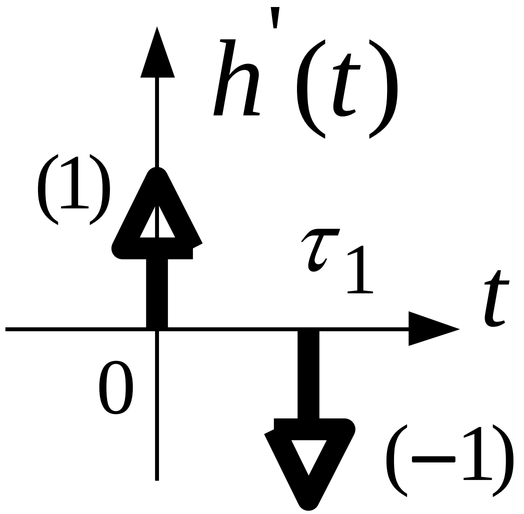
Example

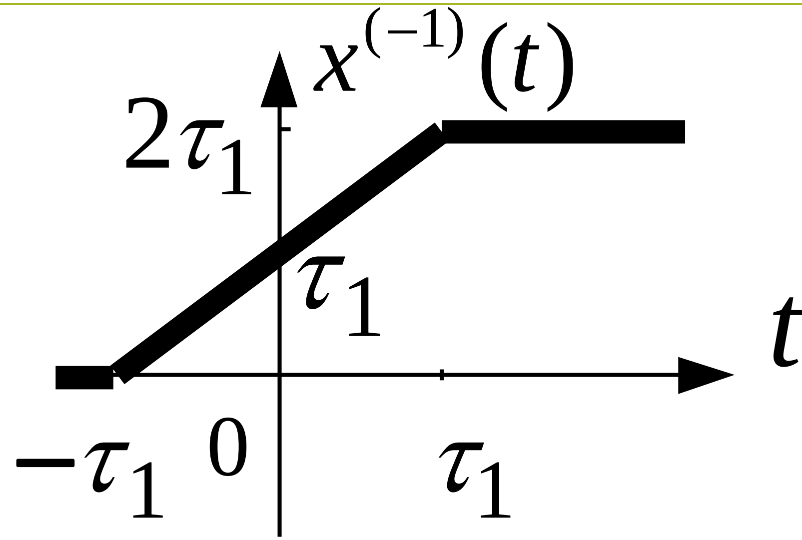




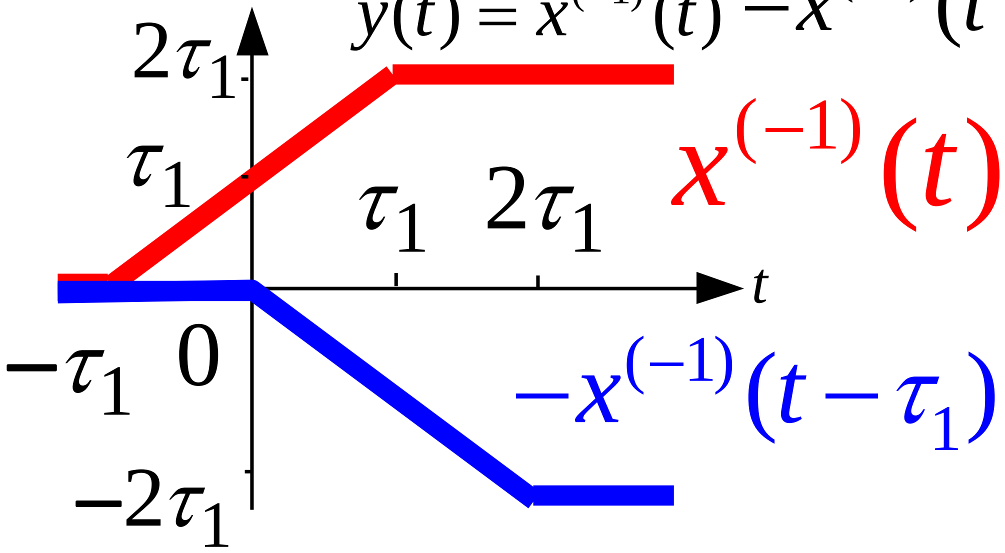
Sketch $$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

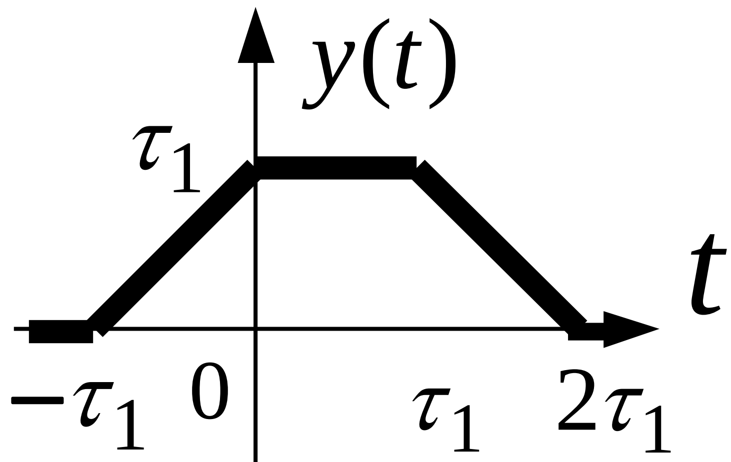
2 Linear Time-Invariant Systems





$$y(t) = x^{(-1)}(t) - x^{(-1)}(t-\tau_1)$$





2.4 Causal LTI Systems Described by Differential and Difference Equation

Discrete time system: Difference Equation
Continuous time system: Differential Equation

2.4.1 Linear Constant-Coefficient Differential Equation

A general Nth-order linear constant-coefficient differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{dx^k(t)}{dt^k}$$

2.4.2 Linear Constant-Coefficient Difference Equation

A general Nth-order linear constant-coefficient difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

or

$$a_N y[n-N] + a_{N-1} y[n-(N-1)] + \cdots + a_1 y[n-1] + a_0 y[n]$$
$$= b_M x[n-M] + b_{M-1} x[n-(M-1)] + \cdots + b_1 x[n-1] + b_0 x[n]$$

2.4.3 Block Diagram Representations of First-order Systems Described by Differential and Difference Equation

(1) Discrete time system

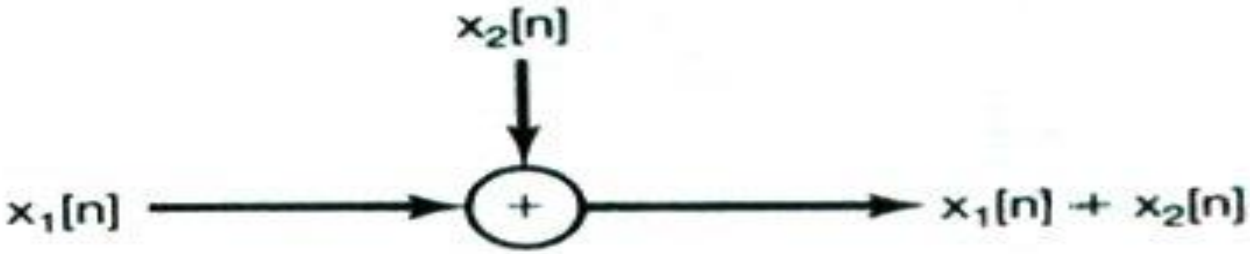
Basic elements:

A. An adder

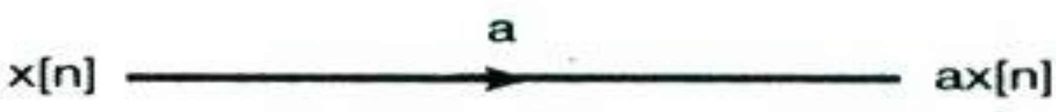
B. Multiplication by a coefficient

C. A unit delay

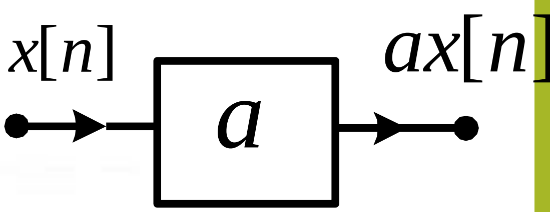
Basic elements:

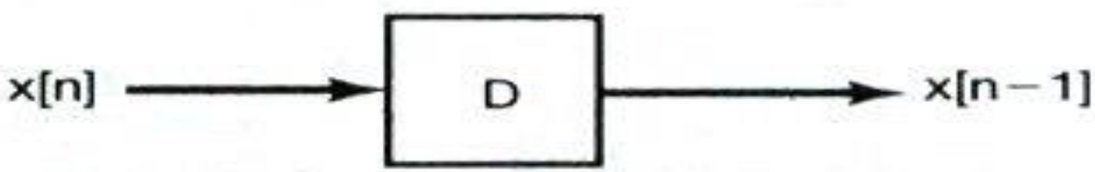


(a)



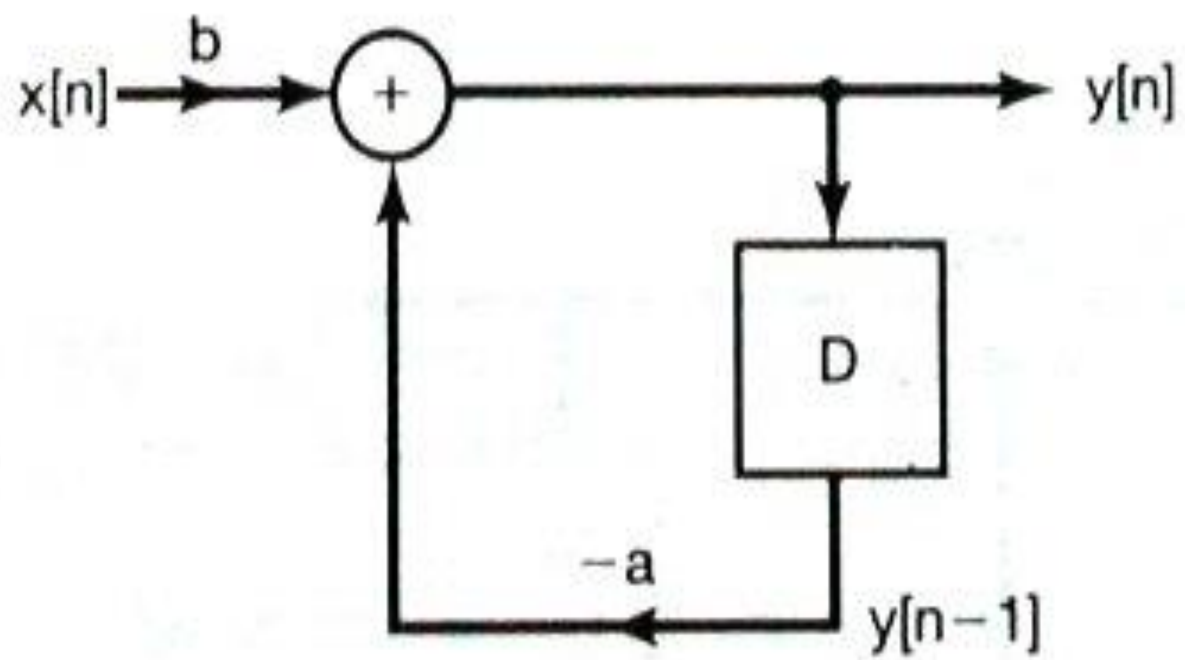
(b)





(c)

Example: $y[n] + ay[n-1] = bx[n]$



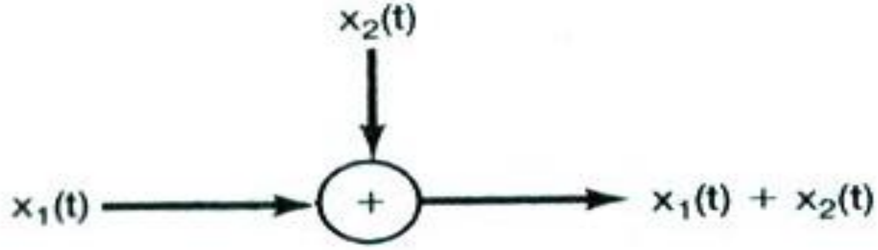
(2) Continuous time system

Basic elements:

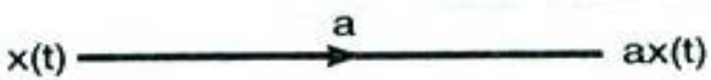
A. An adder
B. Multiplication by a coefficient
C. An (differentiator) integrator

2 Linear Time-Invariant Systems

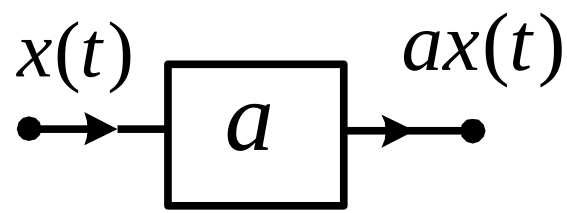
Basic elements:

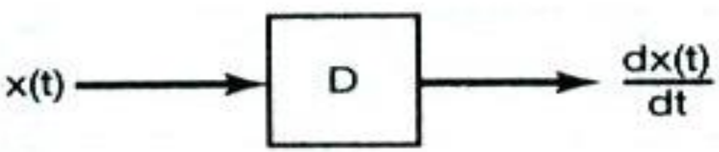


(a)

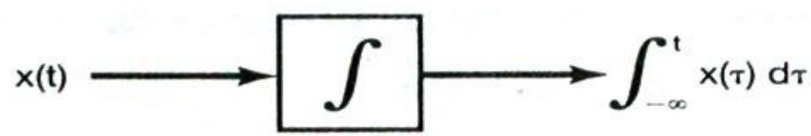


(b)





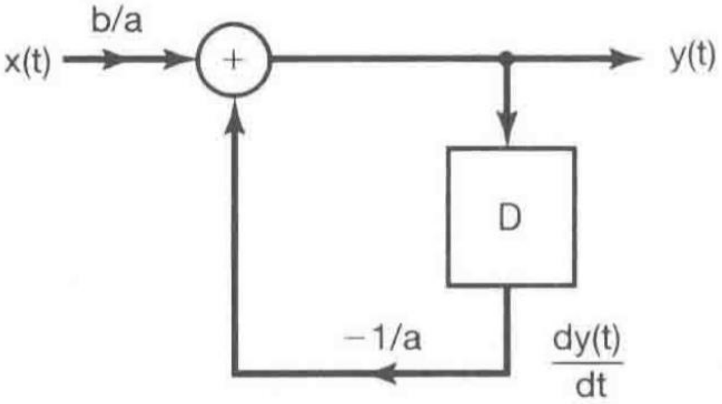
(c)



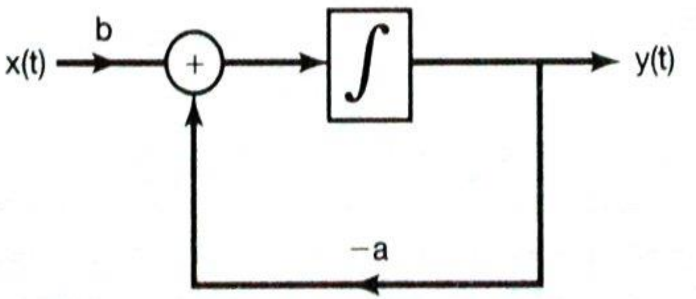
2 Linear Time-Invariant Systems

Example: $$\frac{d}{dt}y(t) + ay(t) = bx(t)$$

$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$$



$$y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)]\, d\tau$$



2.4.3 How to analysis Differential and Difference Equation

1. Deriving $h(t)$ **and** $h[n]$

Example $$\frac{d}{dt}y(t)+ay(t)=bx(t)$$

$$\frac{d}{dt}h(t)+ah(t)=b\delta(t)$$

$$[\frac{d}{dt}h(t)]e^{at}+[ae^{at}]h(t)=b\delta(t)e^{at}=b\delta(t)$$

$$\frac{d}{dt}[h(t)e^{at}]=b\delta(t) \qquad h(t)e^{at}=bu(t)$$

$$\boldsymbol{h(t)=be^{-at}u(t)}$$ **Eigen root:** $-a$

Example 1.10: Balance in a bank account

y[n]-1.01y[n-1]=x[n] **y[n]=0,n<0**

$$h[n]-1.01h[n-1]=\delta[n]$$

$$n=0, h[0]-1.01h[-1]=\delta[0]=1 \qquad h[0]=1$$

$$n=1, h[1]-1.01h[0]=\delta[1]=0 \qquad h[1]=1.01$$

$$n=2, h[2]-1.01h[1]=\delta[2]=0 \qquad h[1]=(1.01)^2$$

....

$$n=k, h[k]-1.01h[k-1]=\delta[k]=0 \qquad h[k]=(1.01)^k$$

$$h[n]=(1.01)^n u[n]$$

Eigen root: 1.01

2. Deriving response

Example: $\frac{d}{dt}y(t) + 2y(t) = x(t)$

Input: $x(t) = e^{3t}u(t)$ **Initial condition:** $y(0^{-}) = 1$

$$y(t) = ?, t \geq 0$$

Solution: ***homogeneneous equation***

$$\frac{d}{dt}y(t) + 2y(t) = 0$$

homogeneneous solution

$$y_h(t) = Ce^{-2t}$$

Eigen root: -2

Zero-Input response: $y_{zi}(t) = Ce^{-2t}, t \geq 0$

From $y(0^-) = y_{zi}(0^-) = y_{zi}(0^+) = 1$ $\quad C = 1$

$$y_{zi}(t) = e^{-2t}, t \geq 0$$

Forced response: $\quad y_{zc}(t) = x(t) * h(t)$

(Zero-condition response or Zero-state response) $\quad h(t) = e^{-2t}u(t)$

$$y_{zc}(t) = x(t) * h(t) = \int_{-\infty}^{\infty} e^{3\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau$$

$$= e^{-2t}\{\int_{0}^{t} e^{5\tau}d\tau\}u(t) = \frac{1}{5}(e^{3t} - e^{-2t})u(t)$$

General response: $y(t) = y_{zi}(t) + y_{zc}(t), t \ge 0$

$$y(t) = e^{-2t} + \frac{1}{5}(e^{3t} - e^{-2t})u(t), t \ge 0$$

$$= \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}, t \ge 0$$

forced response **natural response**

If $x(t) = e^{3t}, -\infty < t < \infty$

$$y_{zc}(t) = x(t) * h(t) = \int_{-\infty}^{\infty} e^{3(t-\tau)} e^{-2\tau} u(\tau) d\tau$$

$$= e^{3t}\{\int_{0}^{\infty} e^{-5\tau} d\tau\} = \frac{1}{5}e^{3t}, -\infty < t < \infty$$

Classification of LTI system response

General response

= Zero-condition response + Zero-Input response

= Forced response + Natural response

Example $$\frac{d}{dt}y(t)+2y(t)=x(t)$$

Input: $\boldsymbol{x(t)=u(t)}$ **Initial condition:** $y(0^-)=1$

$$y(t)=?,t\geq 0$$

Solution: **Eigen root: -2**

Zero-Input response:

$$y_{zi}(t)=e^{-2t},t\geq 0$$

Zero-condition response:

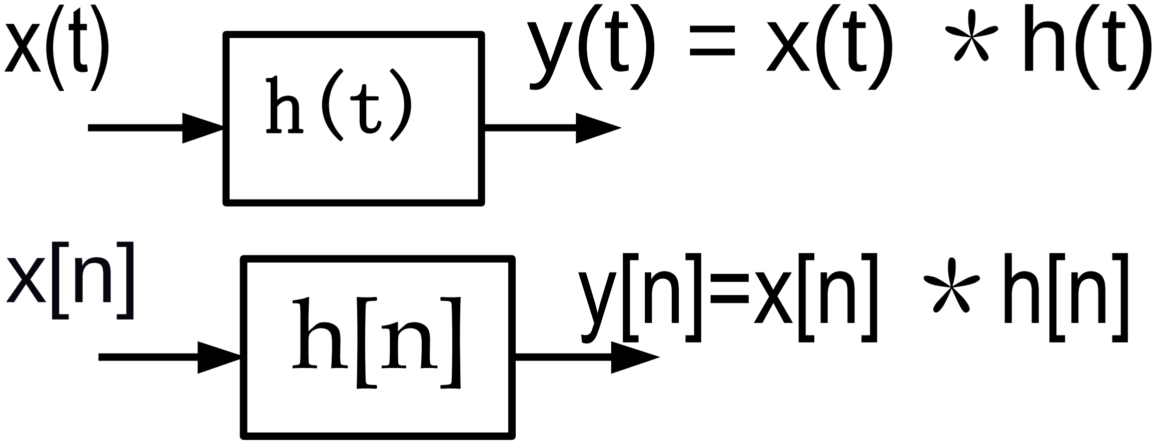
$$y_{zc}(t) = x(t) * h(t)$$

$$= u(t) * e^{-2t}u(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-2t}u(t)$$

General response: $y(t) = y_{zi}(t) + y_{zc}(t), t \geq 0$

$$y(t) = \frac{1}{2}u(t) + \frac{1}{2}e^{-2t}, t \geq 0$$

Resume of Chapter 2



Key points of analysis:

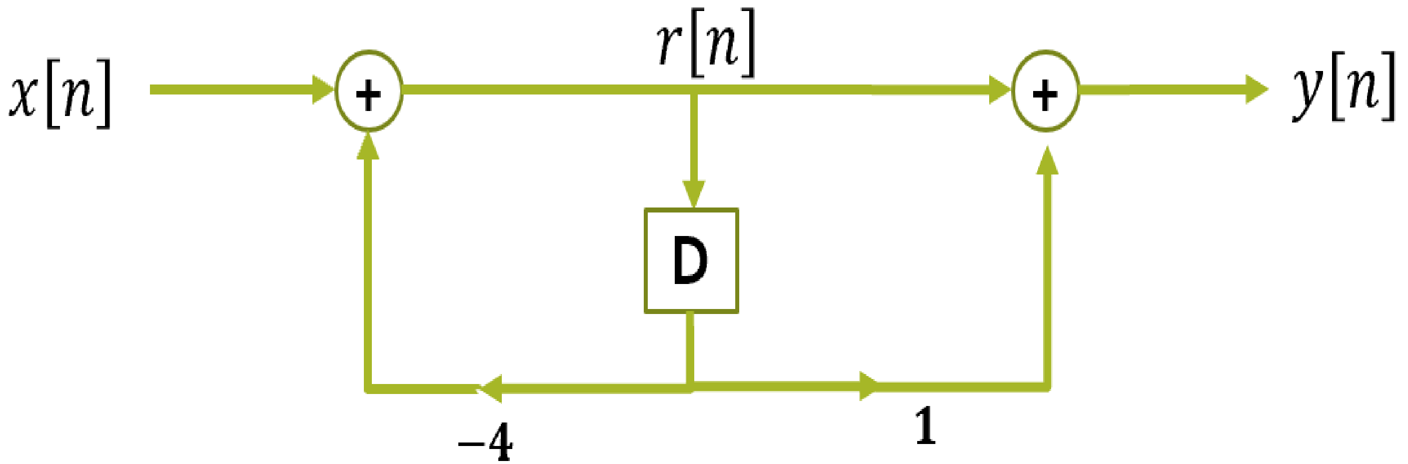
Signals decompostion

$$x(t)=\int_{-\infty}^{+\infty}x(\tau)\delta(t-\tau)d\tau \qquad x[n]=\sum_{k=-\infty}^{+\infty}x[k]\delta[n-k]$$

Response synthesis

$$y(t)=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau \qquad y[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]$$

Example: Consider the block diagram in the figure. The system is causal and is initially at rest.



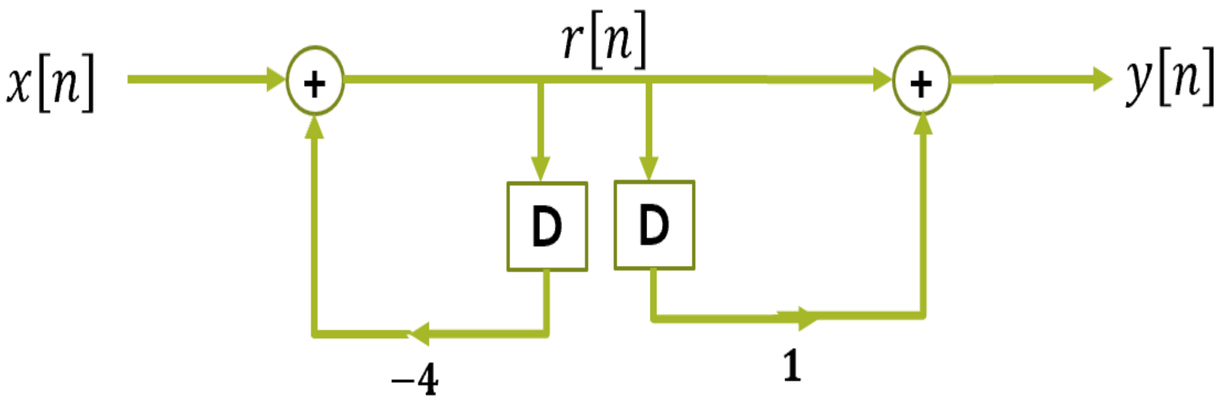
(a) Find the difference equation relating $x[n]$ and $y[n]$.

(b) For $x[n] = \delta[n]$, find $r[n]$ for all n.

(c) Find the system impulse response.

Solution:

(a) In the following figure we convert the block diagram from the original figure to direct form I.



$r[n]$ is given by $r[n] = x[n] - 4r[n-1]$

while $y[n] = r[n] + r[n-1]$

Substituting for $r[n]$ yields

$$y[n] + 4y[n-1] = x[n] + x[n-1]$$

(b) The relation between $x[n]$ and $r[n]$ is $r[n] = -4r[n-1] + x[n]$. For such a simple equation, we solve it recursively when $\delta[n] = x[n]$.

n	$\delta[n]$	$r[n-1]$	$r[n]$
<0	0	0	0
0	1	0	1
1	0	1	-4
2	0	-4	16
3	0	16	-64

We see that $r[n] = (-4)^n u[n]$.

(c) Since $r[n] = (-4)^n u[n]$, and $y[n] = r[n] + r[n-1]$

So $y[n] = (-4)^n u[n] + (-4)^{n-1} u[n-1]$

Now $y[n] = h[n]$,when $x[n] = \delta[n]$,

so $h[n] = (-4)^n u[n] + (-4)^{n-1} u[n-1]$

This expression for $h[n]$ can be further simplified:

$$h[n] = (-4)^n u[n] + (-4)^{n-1} u[n-1]$$

Or $h[n] = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ -3(-4)^{n-1}, & n > 0 \end{cases}$

Thus, $h[n] = \delta[n] - 3(-4)^{n-1} u[n-1]$

Homework list for Chapter 2:

5, 7, 10, 11, 12, **19**, 20, **23**, **40**, 46, 47

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