DIGITAL IMAGE PROCESSING

IMAGE TRANSFORMS

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IMAGE TRANSFORMS

- Definition
- Properties
- Realization
- Application

IMAGE TRANSFORMS

- Why?

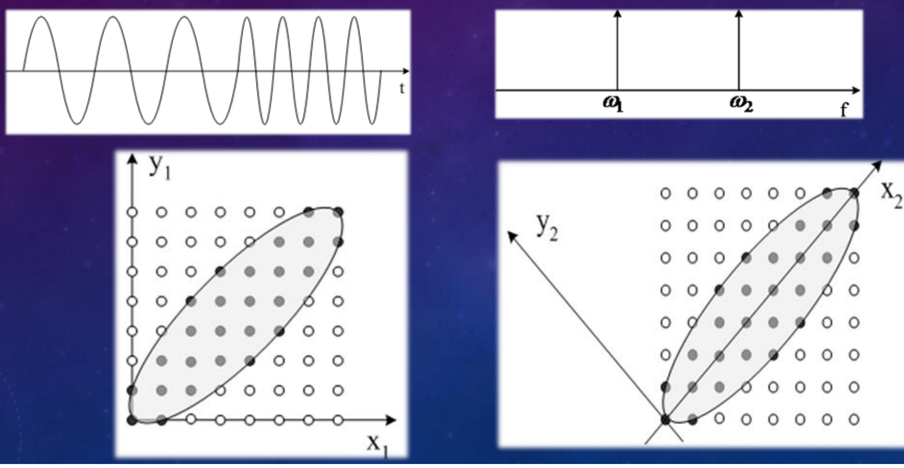
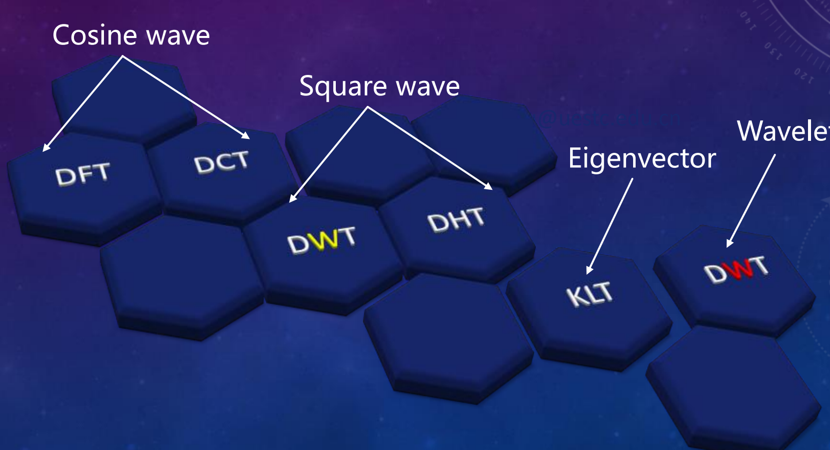


IMAGE TRANSFORMS



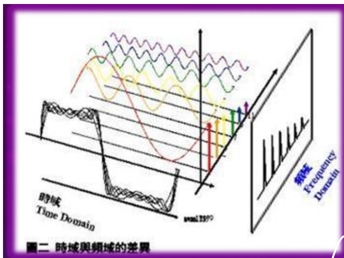
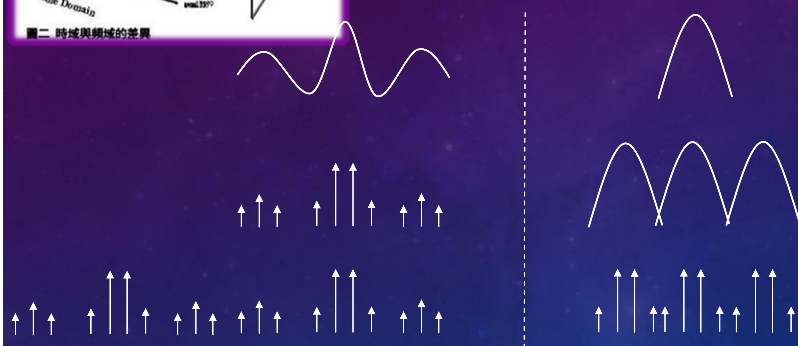


IMAGE TRANSFORMS - DFT



$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-j2\pi\omega t}dt$$

$$F(u)=\sum_{k=0}^{N-1} f(k)e^{-j2\pi uk/N}$$

- The relationship between discrete time/space domain and discrete frequency domain

IMAGE TRANSFORMS – DFT2

- DFT for a discrete image with the size of $M\times N$:

$$F(u,v)=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

$$F(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\frac{(ux+vy)}{N}}$$

- IDFT :

$$f(x,y)=\frac{1}{\sqrt{MN}}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

$$f(x,y)=\frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi\frac{(ux+vy)}{N}}$$

IMAGE TRANSFORMS – DFT2

- $F(u, v)$ is usually a complex number : $F(u,v) = R(u,v) + jI(u,v) = |F(u,v)|\, e^{j\phi(u,v)}$

Magnitude/Spectrum : $|F(u,v)| = [R^2(u,v) + I^2(u,v)]^{1/2}$

Phase angle : $\phi(u,v) = \arctan\left[\dfrac{I(u,v)}{R(u,v)}\right]$

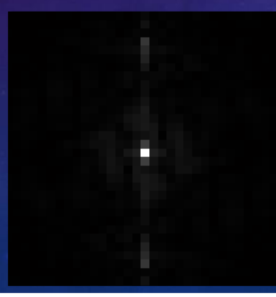
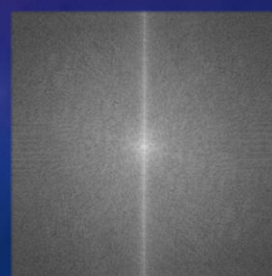
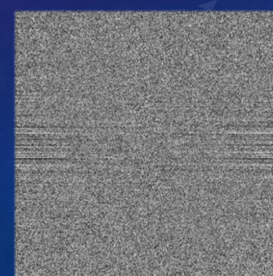





IMAGE TRANSFORMS – DFT2

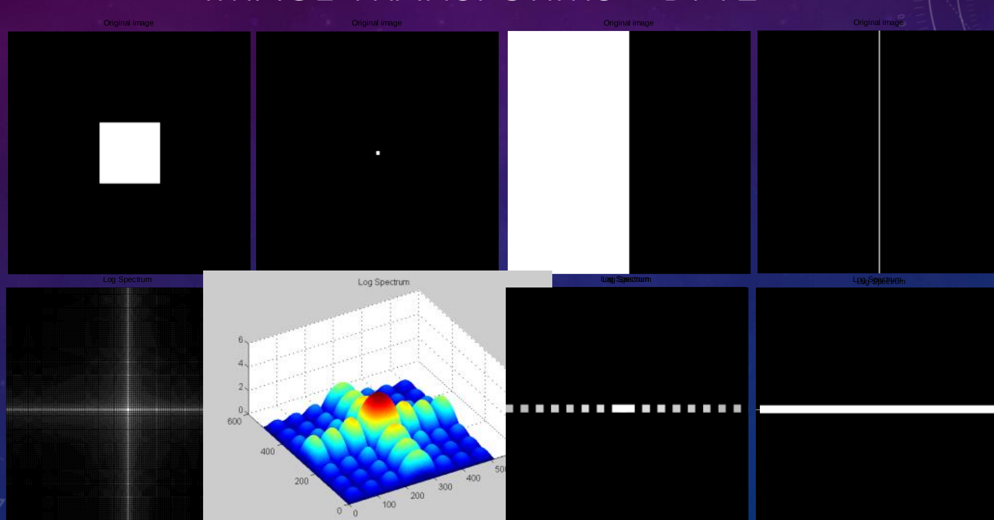


IMAGE TRANSFORMS – DFT2

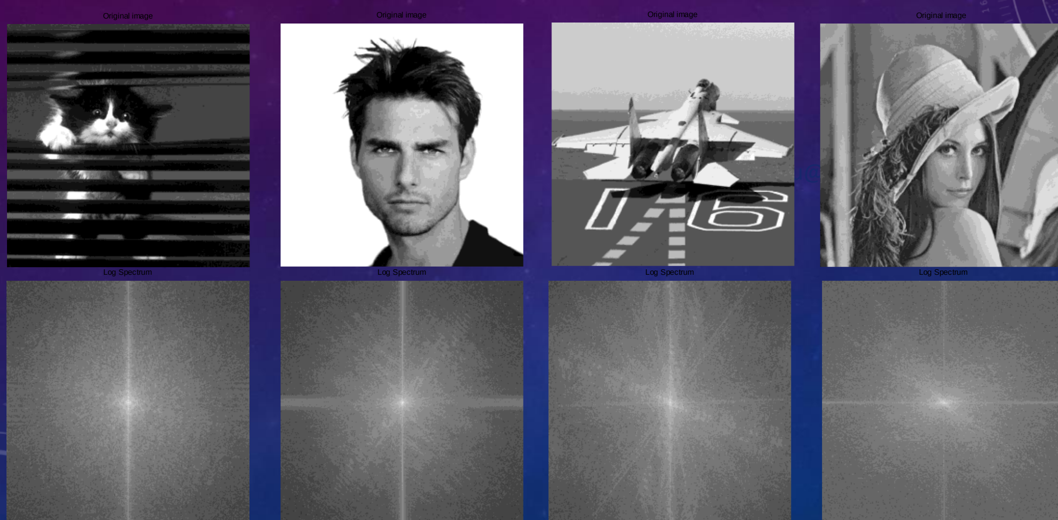


IMAGE TRANSFORMS – DFT2

Euler's formula

$$e^{-jx} = \cos x + j\sin x$$

- $|F(u, v)|$ has even symmetry about the origin

$$|F(u,v)| = |F(-u,-v)|$$

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\frac{(ux+vy)}{N}}$$

$$F^*(u,v) = F(-u,-v)$$

$$F(u,v) = F^*(-u,-v)$$

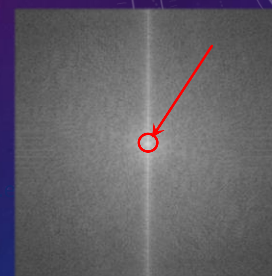
$$F^*(-u,-v) = \left[\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{j2\pi\frac{(ux+vy)}{N}}\right]^*$$

$$= \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f^*(x,y)e^{-j2\pi\frac{(ux+vy)}{N}}$$

the FT of a real function is conjugate symmetric

$$F(u,v) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M+vy/N)}$$

IMAGE TRANSFORMS – DFT2



$$mean2(f) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

$$f_o(x,y) = \frac{1}{\sqrt{MN}} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M+vy/N)}$$

$$= \frac{1}{\sqrt{MN}} \sum_{u=0}^{0} \sum_{v=0}^{0} F(0,0) e^{j2\pi(0x/M+0y/N)}$$

$$= \frac{1}{\sqrt{MN}} \sum_{u=0}^{0} \sum_{v=0}^{0} \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

- What is the DC component of $F(\mathrm{u}, \mathrm{v})$?

IMAGE TRANSFORMS – DFT2

- $|F(u, v)|$ has even symmetry about the origin
- The DC component $F(0, 0)$ corresponds to the average value of an image

IMAGE TRANSFORMS – DFT2

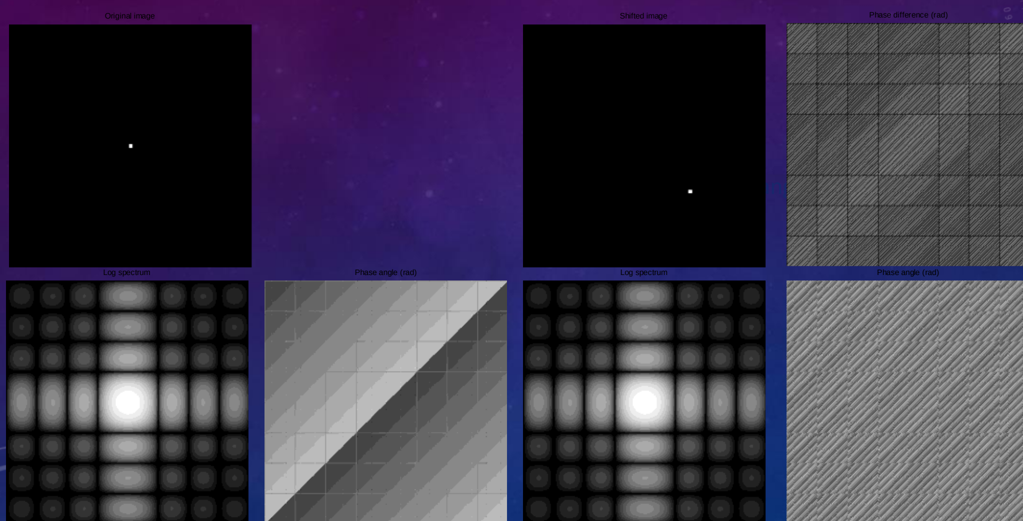
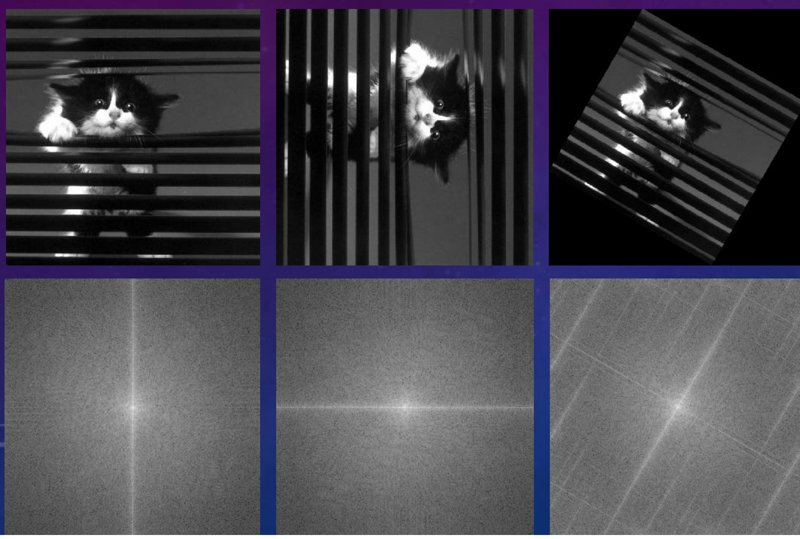


IMAGE TRANSFORMS – DFT2

- |F(u, v)| has even symmetry about the origin
- The DC component F(0, 0) corresponds to the average value of an image
- Translation has no effect on the spectrum of F(u, v)

$$\frac{1}{N}\sum_{x=0}^{N-1} f(x-x_0)e^{-j\frac{2\pi}{N}xu} = \frac{1}{N}\sum_{z=0}^{N-1} f(z)e^{-j\frac{2\pi}{N}zu}e^{-j\frac{2\pi}{N}x_0u} = F(u)e^{-j\frac{2\pi}{N}x_0u}$$

IMAGE TRANSFORMS – DFT2



If $f(x, y)$ rotates, $F(u, v)$ will rotate by the same angle

IMAGE TRANSFORMS – DFT2

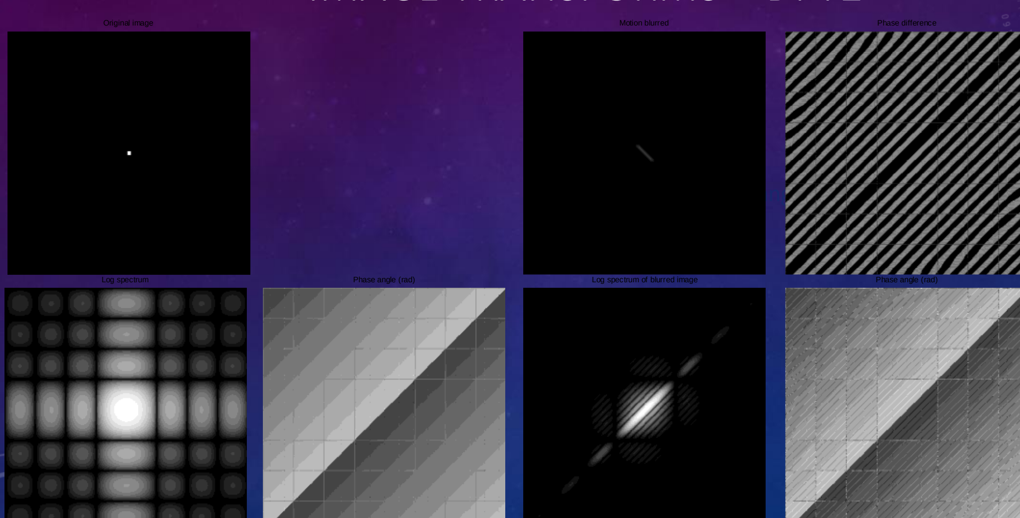


IMAGE TRANSFORMS – DFT2

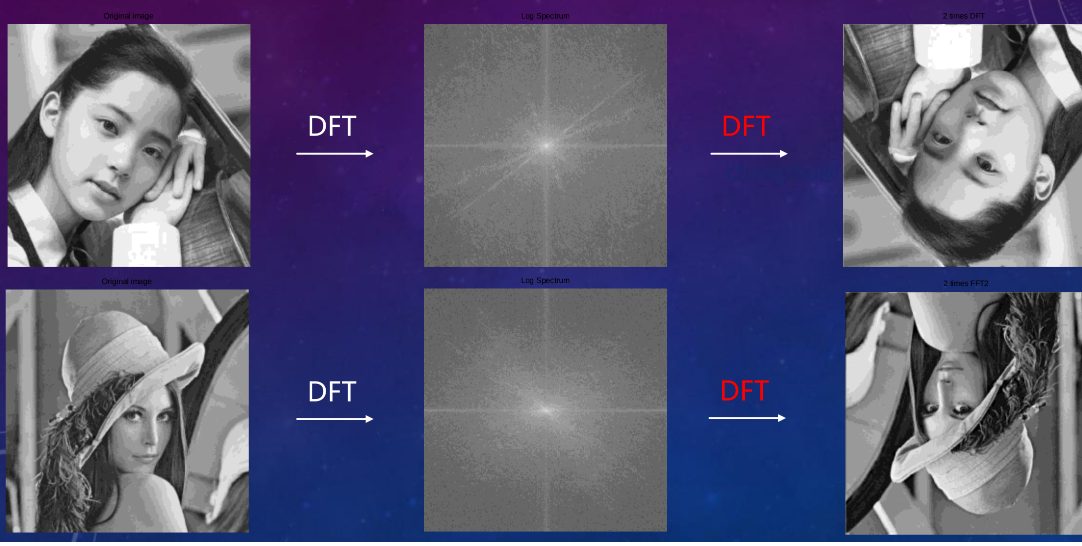


IMAGE TRANSFORMS – DFT2

$$f'(x,y) = f(-x,-y)$$

- |F(u, v)| has even symmetry about the origin
- The DC component F(0, 0) corresponds to the average value of an image
- Translation has no effect on the spectrum of F(u, v)
- Applying DFT twice = rotate image by 180 degrees

$$g(y) = f(-y)$$

$$g(y) \rightarrow F'(v)$$

$$f'(x) = \frac{1}{N}\sum_{u=0}^{N-1} F(u) e^{-j\frac{2\pi}{N}xu} = \frac{1}{N}\sum_{u=0}^{N-1}\left[\frac{1}{N}\sum_{x=0}^{N-1} f(x) e^{-j\frac{2\pi}{N}xu}\right] e^{-j\frac{2\pi}{N}xu}$$

$v = -u$

$y = -x$

$$= \frac{1}{N}\sum_{v=0}^{N-1}\left[\frac{1}{N}\sum_{y=0}^{N-1} f(-y) e^{-j\frac{2\pi}{N}yv}\right] e^{j\frac{2\pi}{N}xv} = \frac{1}{N}\sum_{v=0}^{N-1} F'(v) e^{j\frac{2\pi}{N}xv} = g(x) = f(-x)$$

IMAGE TRANSFORMS – DFT2

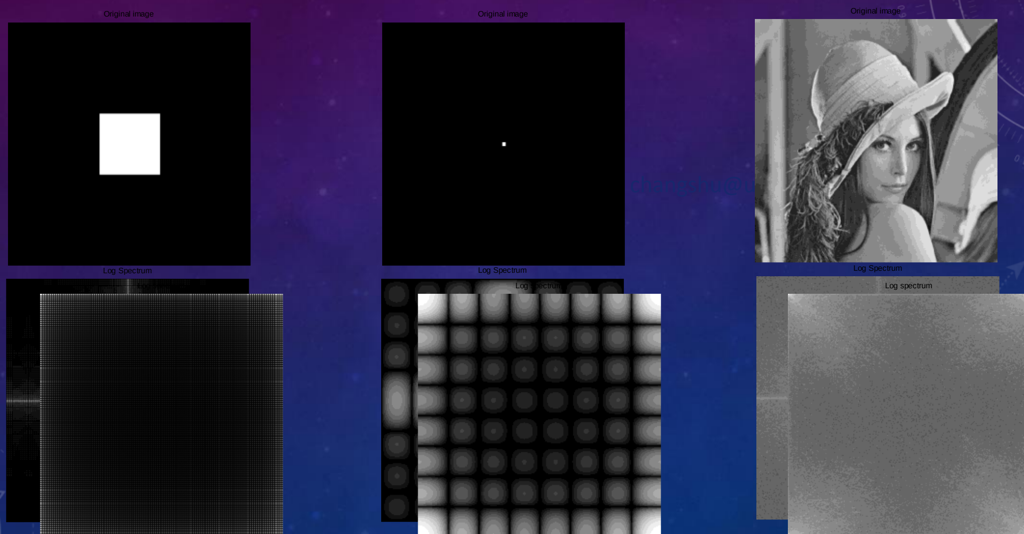


IMAGE TRANSFORMS – DFT2

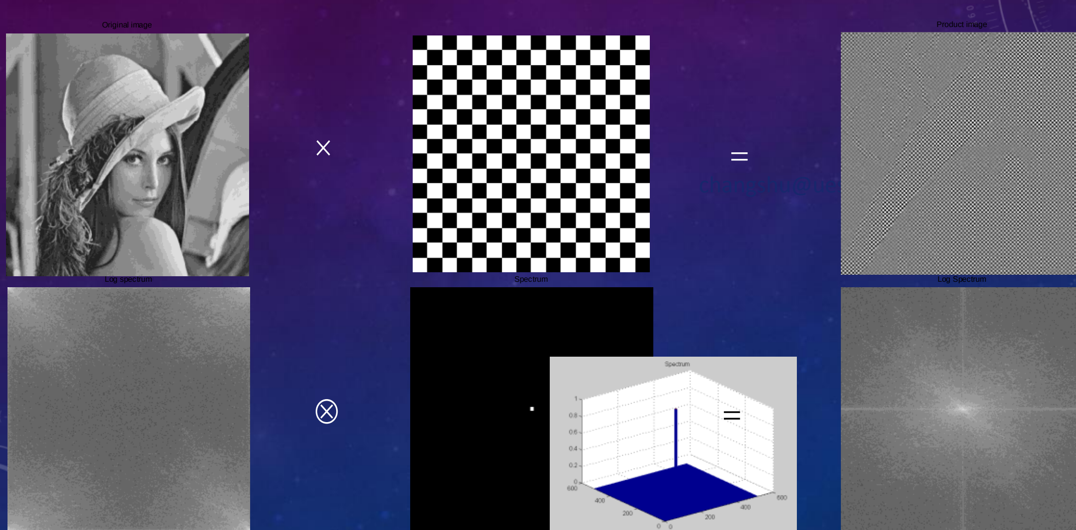


IMAGE TRANSFORMS – DFT2

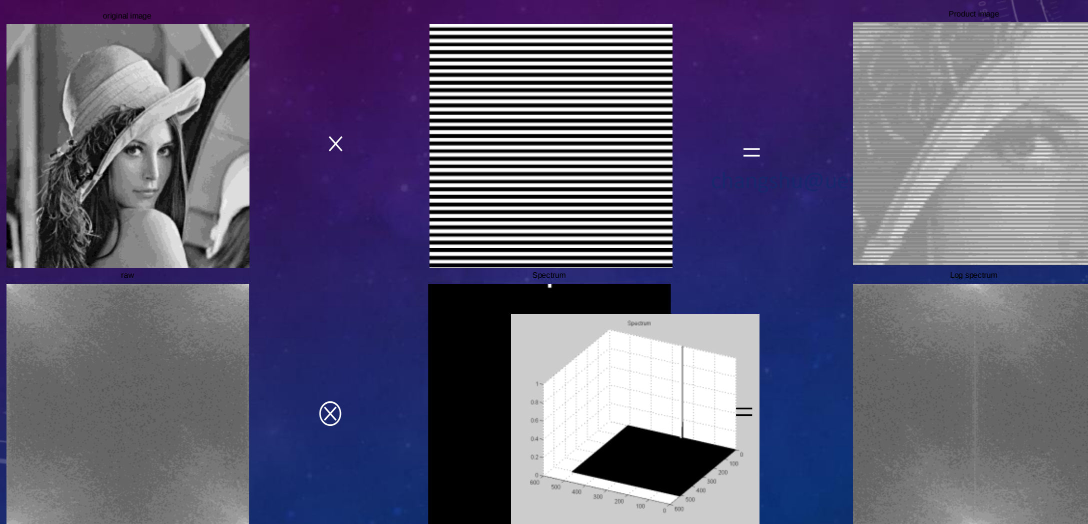


IMAGE TRANSFORMS – DFT2

- |F(u, v)| has even symmetry about the origin
- The DC component F(0, 0) corresponds to the average value of an image
- Translation has no effect on the spectrum of F(u, v)
- Applying DFT twice = image rotated 180 degrees
- Centering the Fourier transform:

$$DFT[f(x,y)(-1)^{x+y}] = F(u - M/2, v - N/2)$$

$$F(u - N/2) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}x(u-N/2)} = \frac{1}{N}\sum_{x=0}^{N-1}(-1)^x f(x)\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}xu}$$

IMAGE TRANSFORMS – DFT2

$$F(u,v)=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

$$F(u,v)=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi ux/M}e^{-j2\pi vy/N}$$

$$F(u,v)=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}[\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi vy/N}]e^{-j2\pi ux/M}$$

$$F(u,v)=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1} F(x,v)e^{-j2\pi ux/M}$$

$$f(x,y)$$
$$\downarrow \sum_{y=0}^{N-1}(\bullet)e^{-j2\pi vy/N}$$
$$F(x,v)$$
$$\downarrow \sum_{x=0}^{M-1}(\bullet)e^{-j2\pi ux/M}$$
$$F(u,v)$$

IMAGE TRANSFORMS – DFT2

- Separability:

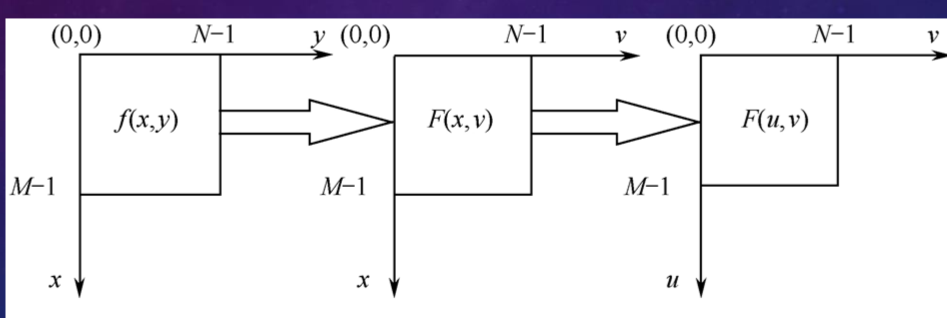


IMAGE TRANSFORMS – DFT2

- Separability:

Computing 2D DFT by computing 1D DFT twice!

$$f(x,y) \rightarrow F_{\mathrm{col}}[f(x,y)] = F(x,v) \rightarrow F(x,v)^T$$
$$\rightarrow F_{col}[F(x,v)^T] = F(u,v)^T \rightarrow F(u,v)$$

IMAGE TRANSFORMS – DFT2

- Separability :

$$f(x,y) \rightarrow F_{Row}[f(x,y)] = F(u,y) \rightarrow F(u,y)^T \rightarrow F_{Row}[F(u,y)^T] = F(u,v)^T \rightarrow F(u,v)$$

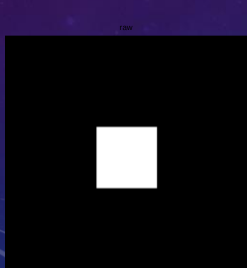
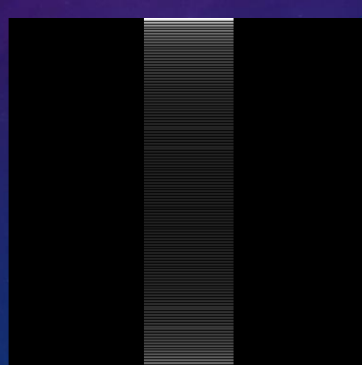
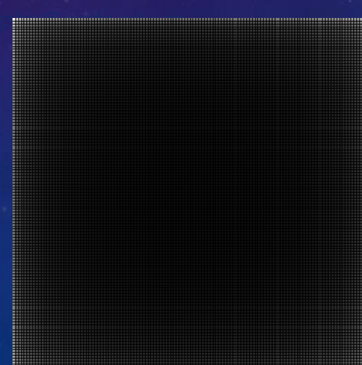




IMAGE TRANSFORMS – DFT2

- Separability :

$$f(x,y) \rightarrow F_{Col}[f(x,y)] = F(x,v) \rightarrow F(x,v)^T \rightarrow F_{Col}[F(x,v)^T] = F(u,v)^T \rightarrow F(u,v)$$

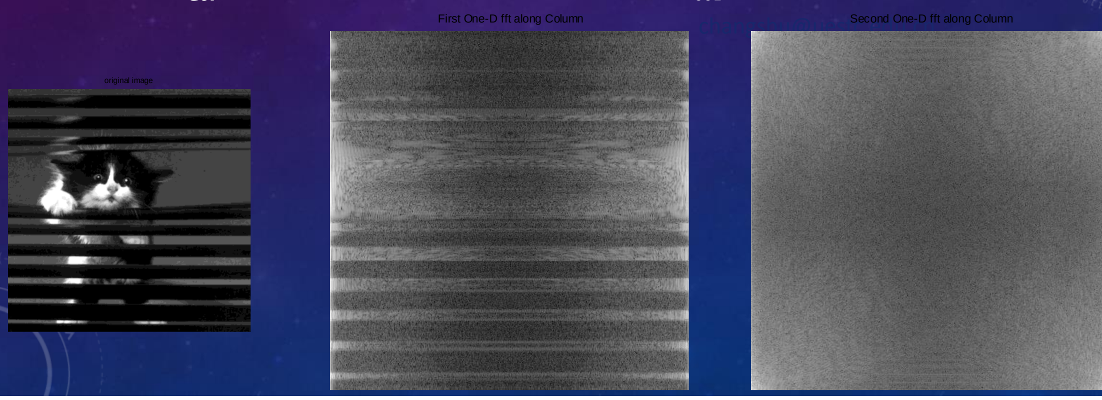


IMAGE TRANSFORMS – DFT2

- Separability :

$$f(x,y) \rightarrow F_{Col}[f(x,y)] = F(u,y) \rightarrow F(u,y)^T \rightarrow F_{Col}[F(u,y)^T] = F(u,v)^T \rightarrow F(u,v)$$

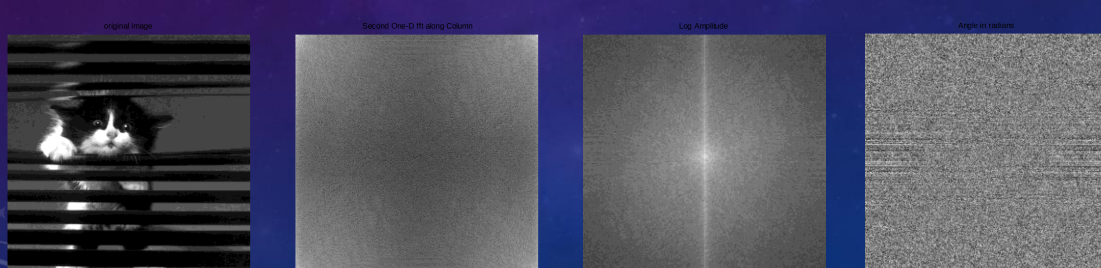


IMAGE TRANSFORMS – DFT2 GENERALIZATION

- In the general form, image transform can be expressed as :
- 1-D linear transform :

Transform variable

Forward transform : $$T(u) = \sum_{x=0}^{N-1} f(x)g(x,u) \qquad u = 0,1,\dots,N-1$$

Forward transformation kernel

Inverse transform : $$f(x) = \sum_{u=0}^{N-1} T(u)h(x,u) \qquad x = 0,1,\dots,N-1$$

Inverse transformation kernel

- 2-D linear transform :

Forward transform : $$T(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)g(x,y,u,v) \qquad \begin{array}{l} u = 0,1,\dots,M-1 \\ v = 0,1,\dots,N-1 \end{array}$$

Inverse transform : $$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} T(u,v)h(x,y,u,v) \qquad \begin{array}{l} x = 0,1,\dots,M-1 \\ y = 0,1,\dots,N-1 \end{array}$$

IMAGE TRANSFORMS – DFT2 GENERALIZATION

- Separable transform in the general form
- Transform kernel : 2D transform → Two 1D transform

$$g(x,y,u,v) = g_1(x,u)g_2(y,v)$$

- 1D transform across y dimension of *f(x,y)* :

$$T(x,v) = \sum_{y=0}^{N-1} f(x,y)g_2(y,v) \qquad x,v = 0,1,\dots M-1$$

- 1D transform across x dimension *of T(x,v)* :

$$T(u,v) = \sum_{x=0}^{M-1} T(x,v)g_1(x,u) \qquad u,v = 0,1,\dots N-1$$

IMAGE TRANSFORMS – DFT2 GENERALIZATION

- DFT2 kernel： $g(x,y,u,v)=\exp\left[-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right]$

$$g(x,y,u,v)=g_1(x,u)g_1(y,v)=\exp\left[-j2\pi\frac{ux}{M}\right]\cdot\exp\left[-j2\pi\frac{vy}{N}\right]$$

- DFT2 kernel：

$$F(u,v)=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\exp\left[-j\frac{2\pi\cdot ux}{M}\right]\cdot\sum_{y=0}^{N-1}f(x,y)\exp\left[-j\frac{2\pi\cdot vy}{N}\right]$$

IMAGE TRANSFORMS – DFT2 GENERALIZATION

$$\begin{aligned}g(x,y,u,v)&=\exp\left[-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right]\\&=\cos\left[2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right]-j\sin\left[2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right]\end{aligned}$$

- Working in complex domain sometimes would be impractical if we had to implement those equations directly even by FFT
- Can we take the real part of the transformation kernel only ? A new transform in real domain —— Discrete Cosine Transform