UESTC3001 Dynamics & Control
Lecture 4

Control System Stability

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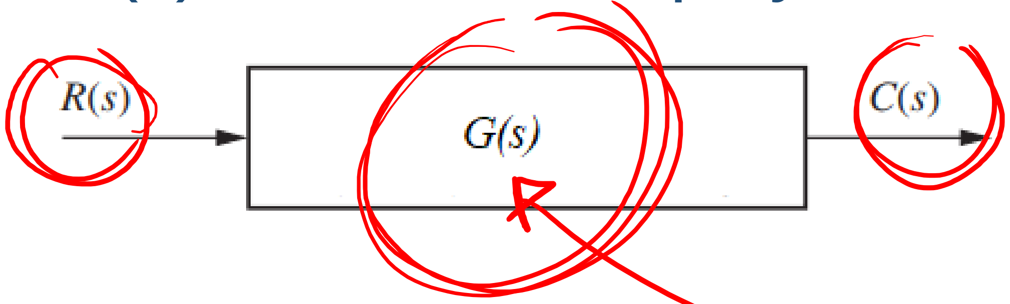
Outline

- Properties of the Transfer Function
- System Stability

Properties of the Transfer Function

- Separate the input, system, and output
- Algebraically combine mathematical representations of subsystems
- G(s) is the ratio of polynomials in s domain



$$G(s) = \frac{C(s)}{R(s)} = \frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)}$$

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$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

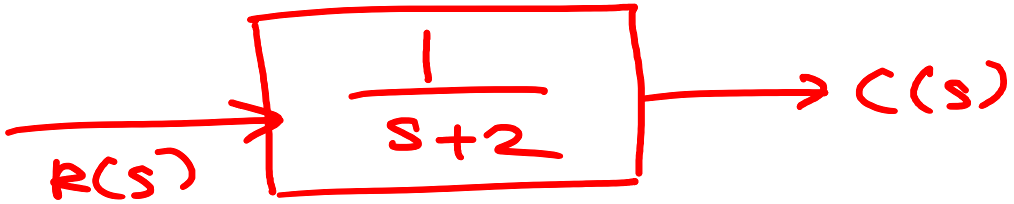
$$sC(s) + 2C(s) = R(s)$$

$$C(s)\{s+2\} = R(s)$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

order 1

1st order T/f



Poles, Zeros, Characteristic Equation

- Fundamental to the analysis and design of control systems
- Simplifies the evaluation of a system’s response
- Poles - roots of the denominator of the transfer function
- Zeros - roots of the numerator of the transfer function
- Characteristic Equation – denominator polynomial set to zero

$$\frac{C(s)}{R(s)} = \frac{(b_n s^n + b_{n-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)}$$

$$= \frac{(s - b_m)(s - b_{m-1}) \cdots (s - b_0)}{(s - a_n)(s - a_{n-1}) \cdots (s - a_0)}$$

$$G(s) = \frac{(s-3)(s-5)\cancel{(s-2)}}{\cancel{(s-2)}(s-6)(s-7)(s)}$$

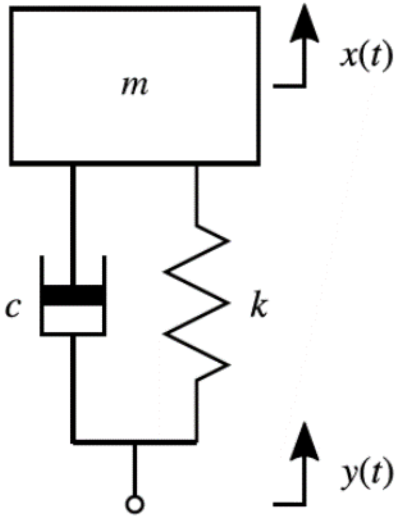
← zero

← poles

Zeros ⇒ $s = 3$, $s = 5$

Poles ⇒ $s = 6$, $s = 7$, $s = 0$

Exercise: Poles, Zeros, Characteristic Equation



$$G(s) = \frac{X(s)}{Y(s)} = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ζ ✓
ω_n ✓
→ 0

Find poles and zeros:

i) Case a: overdamped ($\zeta = 1.25$)

ii) Case b: underdamped ($\zeta = 0.4$)

Note, $\omega_n = 4$ rad/s

a) $G(s) = \dfrac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

a) $\zeta = 1.25$; $\omega_n = 4$

$$G(s) = \frac{2 \times 1.25 \times 4\, s + 4^2}{s^2 + 2 \times 1.25 \times 4 \times s + 4^2} = \frac{10s + 16}{s^2 + 10s + 16}$$

zeros

characteristic eq^n

poles

$10s + 16 = 0 \implies$ zeros $\implies s = -16/10 = -1.6$

$s^2 + 10s + 16 = 0 \implies$ poles $\implies s = -8$ & $s = -2$

$(s+8)(s+2) = 0$

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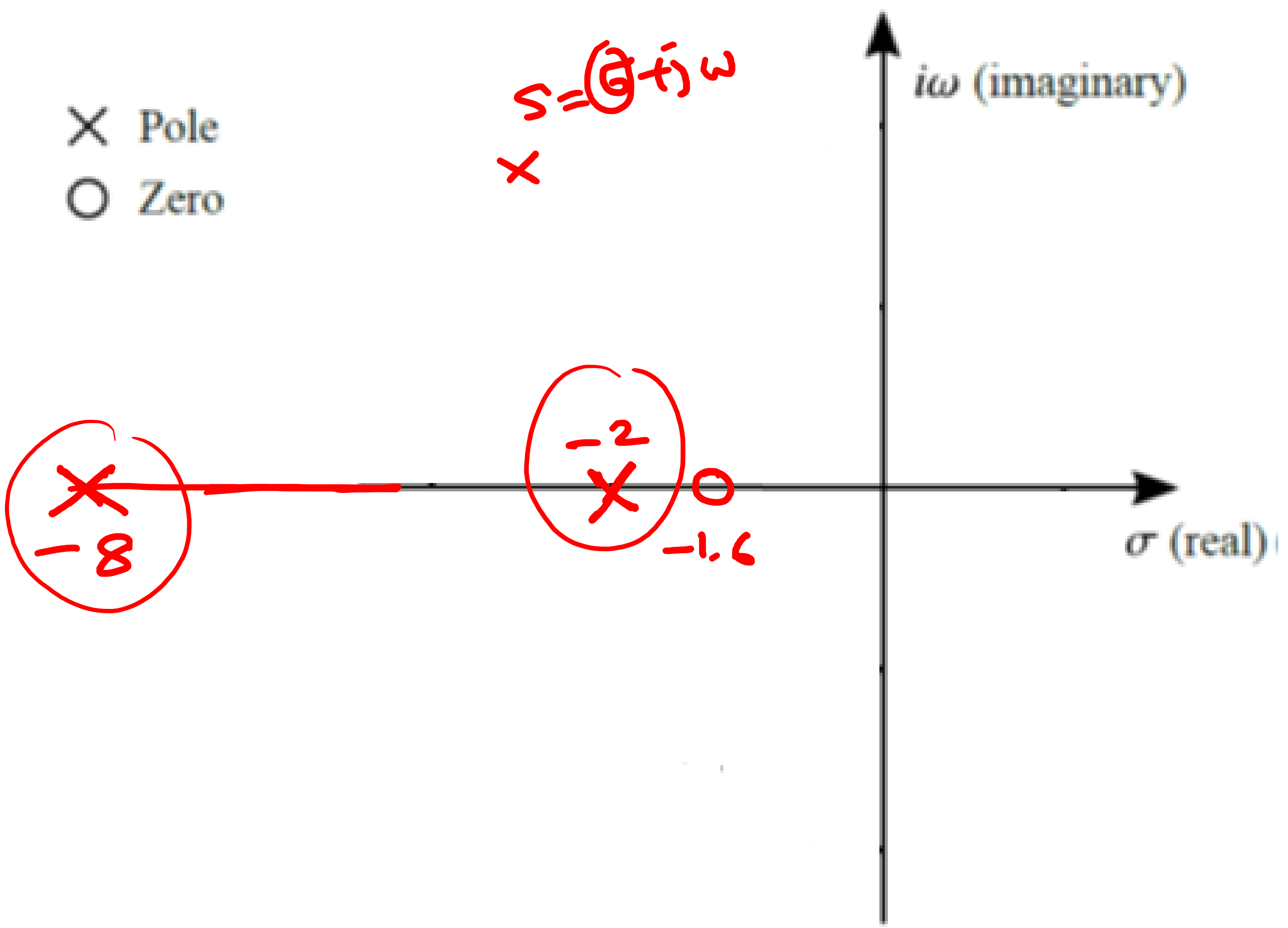
b) $\zeta = 0.4$; $\omega_n = 4$ rad/s

$$G(s) = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{2 \times 0.4 \times 4 + 4^2}{s^2 + 2 \times 0.4 \times 4 + 4^2} =$$

Complex Plane (s-plane)



$\zeta = 1.25$

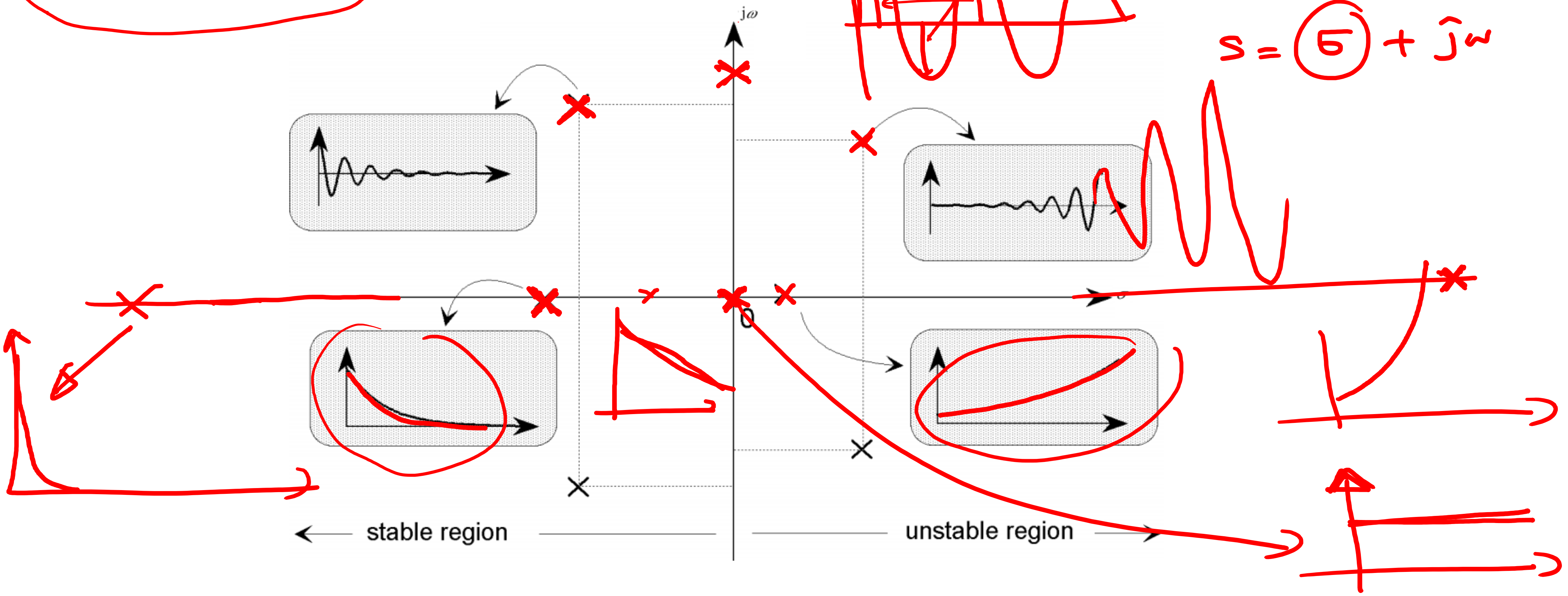
$\Downarrow$

stable.

Stability (LTIs)

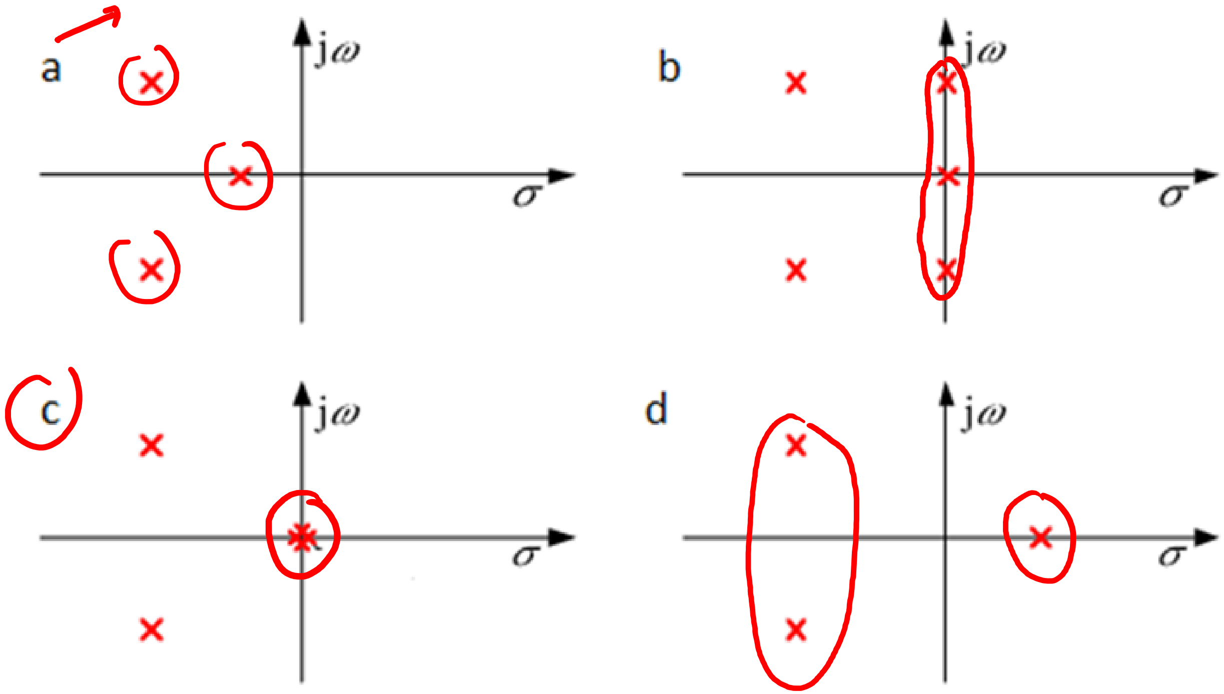
- System is stable if its transient response decays.
- (σ<0) → stable ; (σ>0) → unstable; (σ=0) → marginally stable

$s = \sigma + j\omega$



Exercise

Consider the below s plane pole plots and comment on the expected form of stability for each system.



a) all poles are on LH s-plane ⇒ stable.

b) marginally stable

c) " "

d) unstable.

Routh's Stability Criterion

- Allow to find stability without solving the characteristic equation

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

- A necessary (but not sufficient) condition for stability is that all the coefficients of the characteristic polynomial be positive.
- A system is stable if and only if all the elements in the first column of the Routh array are positive.

Routh Array

$> 0 \Rightarrow$ poles $\rightarrow$ LHS

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

s^n:	1	a_2	a_4	a_6
s^{n-1}:	a_1	a_3	a_5	a_7
s^{n-2}:	b_1	b_2	b_3	b_4
$\vdots$	c_1	c_2	c_3	
s^2:				
s^1:				
s^0:				

$$b_1 = \frac{-\begin{vmatrix} 1 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1} \qquad b_2 = \frac{-\begin{vmatrix} 1 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1}$$

$$b_3 = \frac{-\begin{vmatrix} 1 & a_6 \\ a_1 & a_7 \end{vmatrix}}{a_1}$$

$$c_1 = \frac{-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}}{b_1}$$

$$c_2 = \frac{-\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}}{b_1}$$

$$c_3 = \frac{-\begin{vmatrix} a_1 & a_7 \\ b_1 & b_4 \end{vmatrix}}{b_1}$$

*

1
+ 5
− 3
+ 2

1
1

Exercise: Routh's Stability Criterion

Consider the below polynomial:

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

Determine the stability of the system.

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

$\Rightarrow$ we have all the coefficient   all the coefficient (+) ve ✓

s^6	:	1	3	1	4
s^5	:	4	2	4	0
s^4	:	$\dfrac{-\begin{vmatrix}1 & 3\\ 4 & 2\end{vmatrix}}{4} = 2.5$	0	4	
s^3	:	2	$-12/5$		
s^2	:	3	4		
s^1	:	$-76/15$	0		
s^0	:	4			

$-\dfrac{\begin{vmatrix}1 & 1\\ 4 & 4\end{vmatrix}}{4} = \dfrac{-0}{4}$

$-\dfrac{\begin{vmatrix}1 & 4\\ 4 & 0\end{vmatrix}}{4} = \dfrac{16}{4}$

$-\dfrac{\begin{vmatrix}1 & 0\\ 4 & 0\end{vmatrix}}{4} = 0$

$-\dfrac{\begin{vmatrix}2.5 & 0\\ 2 & -12/5\end{vmatrix}}{2}$

$-\dfrac{\begin{vmatrix}4 & 2\\ 2.5 & 0\end{vmatrix}}{2.5} = \dfrac{5}{2.5}$

$-\dfrac{\begin{vmatrix}2.5 & 4\\ 2 & 0\end{vmatrix}}{2}$

$-\dfrac{\begin{vmatrix}4 & 4\\ 2.5 & 4\end{vmatrix}}{2.5} = \dfrac{-(16-10)}{2.5}$

$+3 \rightarrow -76/15 \rightarrow +4$

$\Rightarrow$ unstable.

$\Rightarrow$ 2 poles on RHS

Summary

- Poles, Zeros, Characteristic Equation
- Characteristics of System Stability
- Routh's Stability Criterion

Reference:
-Control Systems Engineering, 7th Edition, N.S. Nise
-UESTC3001 2019/20 Notes, J. Le Kernec