UESTC4004
Digital Communications

Demodulation and Detection

Student Feedback Meetings

- Lead GTA, Bang Huang will schedule meetings with volunteer students to gather feedback about course delivery
- Your feedback is important for the improvement
- The meetings will take place after each 5 lectures block
- Students volunteering to provide feedback are requested to contact Bang Huang 15320295034@163.com

Lecture Preview

- Decoding the received signal
  - Baseband Demodulation
    - Matched Filter
  - Baseband Detection
    - Maximum likelihood detector

Detection of Binary Signal in Gaussian Noise

- For any binary channel, the transmitted signal over a symbol interval (0,T) is:

$$s_i(t) = \begin{cases} s_0(t) \;\; 0 \le t \le T & \textit{for a binary } 0 \\ s_1(t) \;\; 0 \le t \le T & \textit{for a binary } 1 \end{cases}$$

- The received signal r(t) degraded by noise n(t) and possibly degraded by the impulse response of the channel $h_c(t)$, is

$$r(t) = s_i(t) * h_c(t) \; + \; n(t) \qquad i = 0,1$$

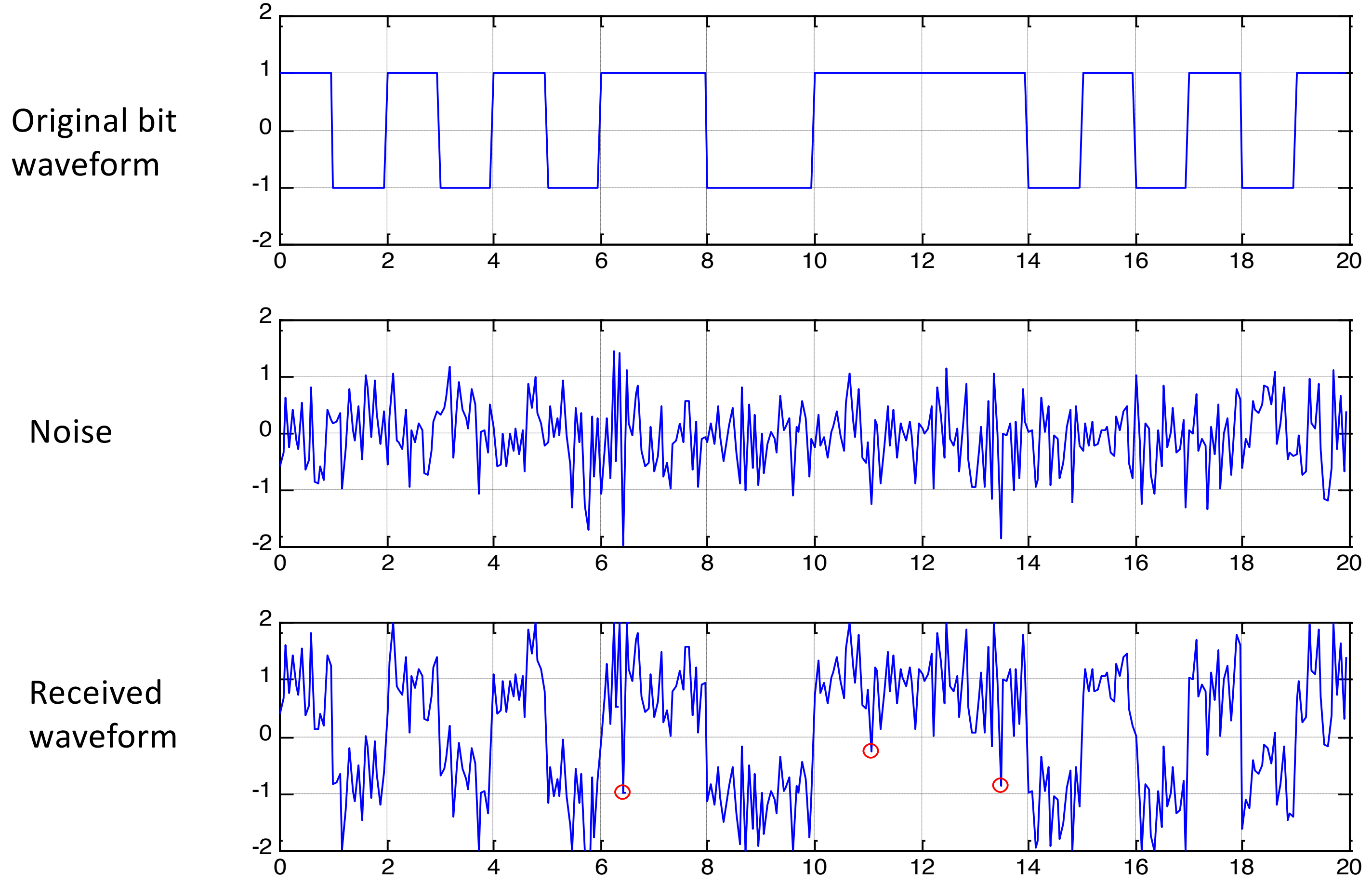
where *n(t)* is assumed to be zero mean AWGN process with probability density function

$$p(n_0) = \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{n_0}{\sigma_0}\right)^2 \right]$$

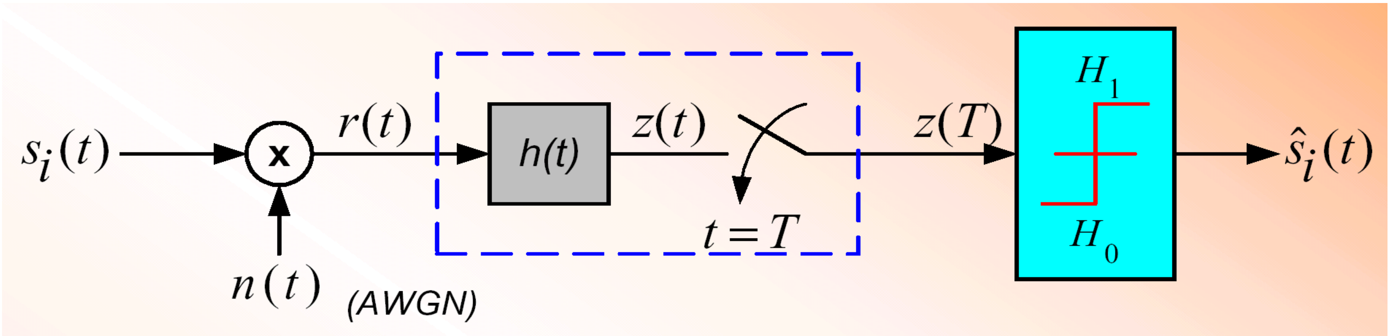
- For ideal distortionless channel where $h_c(t)$ is an impulse function and convolution with $h_c(t)$ produces no degradation, r(t) can be represented as:

$$r(t) = s_i(t) \; + \; n(t) \qquad i = 0,1 \qquad 0 \le t \le T$$

Detection of Binary Signal in Gaussian Noise



Demodulation and Detection



Matched Filter

Detector

- The digital receiver performs two basic functions:
  - Demodulation by using **matched filter** *h(t)*, to recover a waveform to be sampled at $t = nT$.
  - Detection through **detector**, decision-making process of selecting possible digital symbol

$$z(T) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_0$$

where H_1 and H_0 are the two possible binary hypothesis

Matched Filter

最大
为了消除noise

- The input to the matched filter *h(t)* is the received signal *r(t)*

$$r(t) = s_i(t) \;+\; n(t) \qquad i = 0,1 \qquad 0 \le t \le T$$

- Let $s_1(t)=a_1$ and $s_0(t)=a_o$ and $\sigma_0{}^2$ is the noise variance

- The ratio of instantaneous signal power to average noise power , (S/N)T, at a time t=T, out of the sampler is:

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2}$$

$$\frac{S}{N} = \frac{E_b}{N_0}\cdot\frac{e}{W}$$

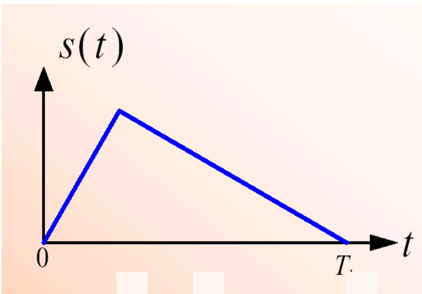
- We need to achieve maximum $(S/N)_T$
- It is the matching filter h(t) which does this job.

Matched filter

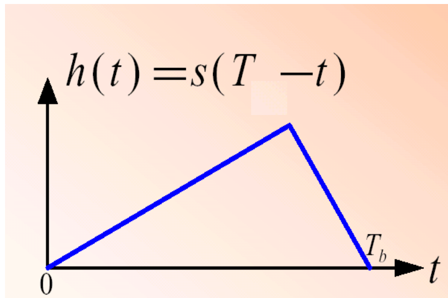
- Matched filter is a filter that is matched to the waveform *s(t)*, for producing maximum output signal-to-noise ratio. Matched filter has an impulse response

$$h(t) = \begin{cases} ks(T-t) & 0 \le t \le T \\ 0 & else\ where \end{cases}$$

  - *h(t)* is a shifted and inverted version of the original signal waveform
  - k is some arbitrary constant



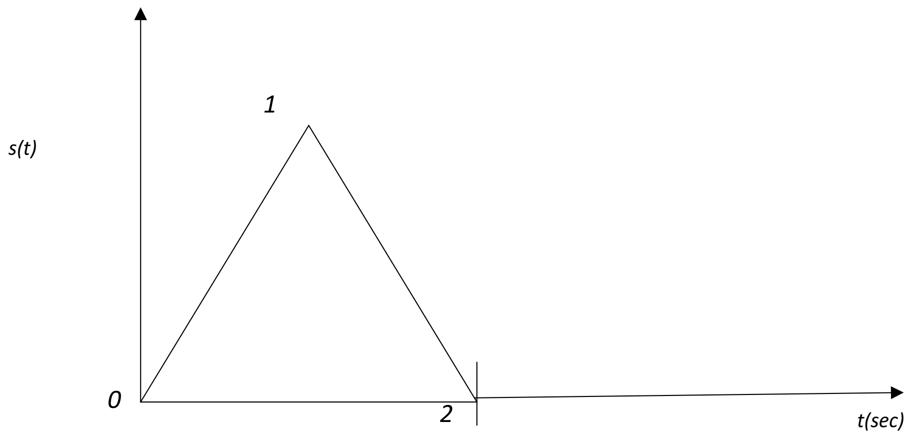
Signal Waveform



Impulse response of matched filter

Example Matched Filter

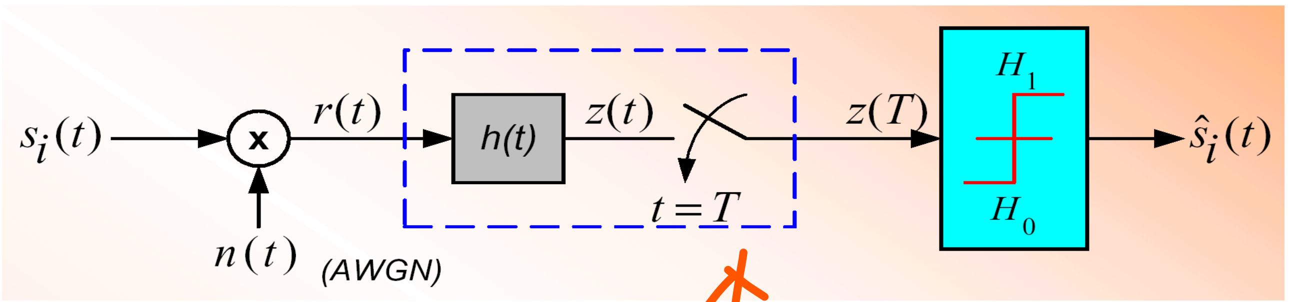
- Assuming the symbol waveform *s(t)* of duration 2 sec in figure below, write only the equation of the impulse response *h(t)* of the corresponding matching filter. Also plot *h(t)*.



最终结论是 $P_B = Q\left(\frac{a_1 - a_0}{2\sigma_0}\right)$

Detection

- Matched filter reduces the received signal to a single variable *z(T), after which the detection of symbol is carried out*
- We will us a ***maximum likelihood detector*** that allows us to
  - formulate the decision rule that operates on the data
  - optimize the detection criterion



抽样检测

$$z(T) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_0$$

How to Choose the threshold?

- Maximum Likelihood Ratio test and Maximum a posteriori (MAP) criterion:
- If

$$p(s_0|z) > p(s_1|z) -> H_0$$

- else

$$p(s_1|z) > p(s_0|z) -> H_1$$

- Problem is that a posteriori probability are not known.
- Solution: Use Bay's theorem:

$$p(s_i|z) = \frac{p(z|s_i)p(s_i)}{p(z)}$$

$$\Rightarrow \frac{p(z|s_1)P(s_1)}{P(z)} \underset{H0}{\overset{H1}{\gtrless}} \frac{p(z|s_0)P(s_0)}{P(z)} \Rightarrow p(z|s_1)P(s_1) \underset{H0}{\overset{H1}{\gtrless}} p(z|s_0)P(s_0)$$

How to Choose the threshold?

定判据

- MAP criterion:

$$\boxed{L(z) \triangleq \frac{p(z|s_1)}{p(z|s_0)} \underset{H0}{\overset{H1}{\gtrless}} \frac{P(s_0)}{P(s_1)} \Leftarrow likelihood\ ratio\ test\ (LRT)}$$

- When the two signals, $s_0(t)$ and $s_1(t)$, are equally likely, i.e., $P(s_0) = P(s_1) = 0.5$, then the decision rule becomes

$$L(z) = \frac{p(z|s_1)}{p(z|s_0)} \underset{H0}{\overset{H1}{\gtrless}} 1 \Leftarrow \max likelihood\ ratio\ test$$

- This is known as maximum likelihood ratio test because we are selecting the hypothesis that corresponds to the signal with the maximum likelihood.

Discussion

- What will change in MAP criterion if the probabilities of s_0 and s_1 are not equal?

Detection of symbol

Assume that input noise is a Gaussian random process and receiving filter is linear

$$p(n_0) = \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{n_0}{\sigma_0}\right)^2\right]$$

- Then output is another Gaussian random process

$$p(z \mid s_0) = \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a_0}{\sigma_0}\right)^2\right]$$

$$p(z \mid s_1) = \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a_1}{\sigma_0}\right)^2\right]$$

Where $\sigma_0{}^2$ is the noise variance

为是noise的方差.

$$L(z) = \frac{p(z|s_1)}{p(z|s_0)} \underset{H0}{\overset{H1}{\gtrless}} 1 \Leftarrow \max \textit{likelihood ratio test}$$

- Substituting the pdfs

$$L(z) = \frac{p(z \mid s_1)}{p(z \mid s_0)} \underset{H0}{\overset{H1}{\gtrless}} 1 \Rightarrow \frac{\frac{1}{\sigma_0\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma_0}(z-a_1)^2\right]}{\frac{1}{\sigma_0\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma_0}(z-a_0)^2\right]} \underset{H0}{\overset{H1}{\gtrless}} 1$$

- Hence:

$$\exp\left[\frac{z(a_1-a_0)}{\sigma_0^2}-\frac{(a_1^2-a_0^2)}{2\sigma_0^2}\right]\begin{matrix}>\\<\end{matrix}1$$

- Taking the log of both sides will give

$$\Lambda=\ln\{L(z)\}=\frac{z(a_1-a_0)}{\sigma_0^2}-\frac{(a_1^2-a_0^2)}{2\sigma_0^2}\begin{matrix}H1\\>\\<\\H0\end{matrix}0$$

$$\Rightarrow\frac{z(a_1-a_0)}{\sigma_0^2}\begin{matrix}H1\\>\\<\\H0\end{matrix}\frac{(a_1^2-a_0^2)}{2\sigma_0^2}=\frac{(a_1+a_0)(a_1-a_0)}{2\sigma_0^2}$$

$$\text{threshold } \boxed{\gamma_0 = \frac{a_1 + a_0}{2}}$$

- Hence

$$z \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma_0^2 (a_1 + a_0)(a_1 - a_0)}{2\sigma_0^2 (a_1 - a_0)} \qquad z \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(a_1 + a_0)}{2} \triangleq \gamma_0$$

where z is the *minimum error criterion* and γ_0 is *optimum threshold*

- For antipodal signal, $s_1(t) = - s_0(t) \Rightarrow a_1 = - a_0$

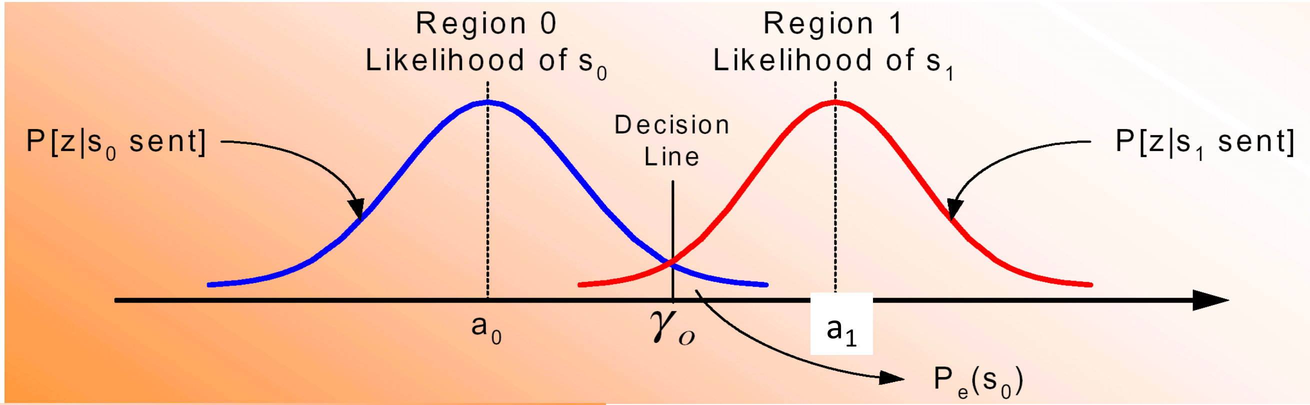
$$z \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

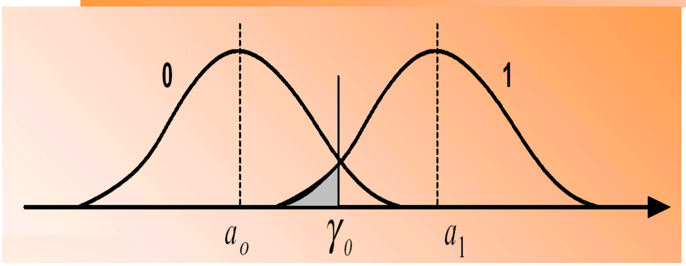
Discussion

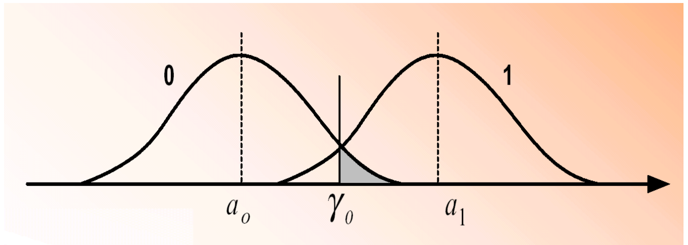
- If the transmitted symbols are not equally likely, would it make any impact on the value of optimal threshold?

Probability of error

- Error will occur if
  s_1 is sent $\rightarrow$ s_0 is received or s_0 is sent $\rightarrow$ s_1 is received







$$P(e|s_1) = \int_{-\infty}^{\gamma_0} p(z|s_1)\, dz$$

$$P(e|s_0) = \int_{\gamma_0}^{\infty} p(z|s_0)\, dz$$

Probability of Error

- The total probability of error is the sum of the errors

$$P_B = \sum_{i=1}^{2} P(e, s_i) = P(e \mid s_1)P(s_1) + P(e \mid s_0)P(s_0)$$

- If we consider equi-probable transmission i.e., $P(s_1)=P(s_0)=0.5$ then

$$P_B = \frac{1}{2}(P(e|s_1) + P(e|s_0))$$

- By symmetry $P(e|s_1) = P(e|s_0)$

- Therefore

$$P_B = P(e|s_1) = P(e|s_0))$$

- Numerically, P_B is the area under the tail of either of the conditional distributions $p(z|s_1)$ or $p(z|s_2)$

$$P_B = \int_{\gamma_0}^{\infty} \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a_0}{\sigma_0}\right)^2\right] dz$$

$$\Rightarrow u = \frac{(z-a_0)}{\sigma_0}$$

误码率公式

$$= \int_{\frac{(a_1-a_0)}{2\sigma_0}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{u^2}{2}\right] du$$

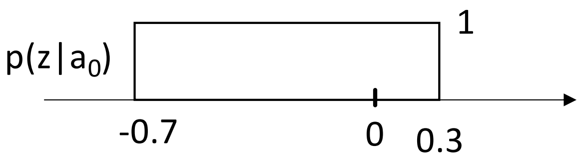
$$\boxed{P_B = Q\left(\frac{a_1-a_0}{2\sigma_0}\right)} \Leftarrow equation\ B.18$$

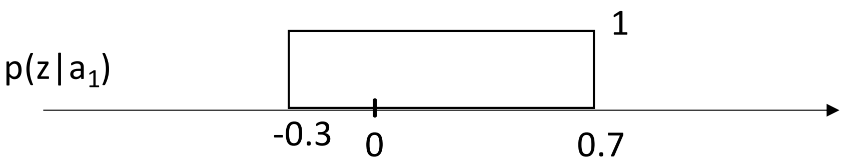
- The above equation cannot be evaluated in closed form (Q-function)
- Hence a good approximation for x>3 only,

$$\boxed{Q(x) \cong \frac{1}{x\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right]}$$

Example

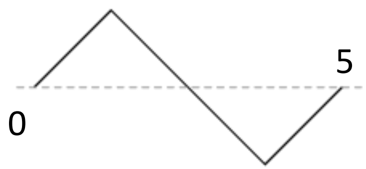
- Find the probability of bit error for a binary signaling with equally probable signals a_0 = -1 and a_1 =1 and noise distribution of the received signal is given as below:





Review Questions

- What is the purpose of a matched filter?
- For a given waveform s(t), plot the matched filter impulse response h(t).



- Will the probability of error be more when the transmitted bits amplitudes $a_i(t)$ are ±5 instead of ±1? Justify please.