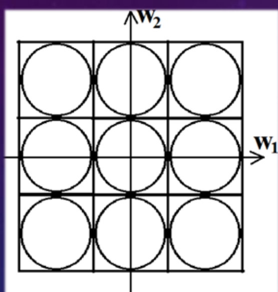
SAMPLING EFFICIENCY

- Rectangular samping



$$U = \begin{bmatrix} 2f_m & 0 \\ 0 & 2f_m \end{bmatrix}$$

$$U^T V = 2\pi I$$

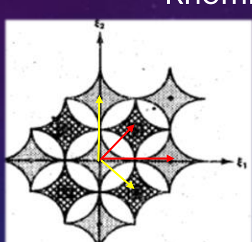
$$\frac{2\pi}{T} = f_m$$

$$V = \begin{bmatrix} \frac{\pi}{f_m} & 0 \\ 0 & \frac{\pi}{f_m} \end{bmatrix} = \begin{bmatrix} \frac{T}{2} & 0 \\ 0 & \frac{T}{2} \end{bmatrix}$$

SAMPLING EFFICIENCY

$$\begin{bmatrix} \frac{f_m}{\pi} & 0 \\ \frac{f_m}{2\pi} & \frac{f_m}{2\pi} \end{bmatrix} V = I$$

- A different sampling strategy more efficient for other spectral shapes
  - Rhombus



$$U = \begin{bmatrix} 2f_m & f_m \\ 0 & f_m \end{bmatrix}$$

$$U^T V = 2\pi I$$

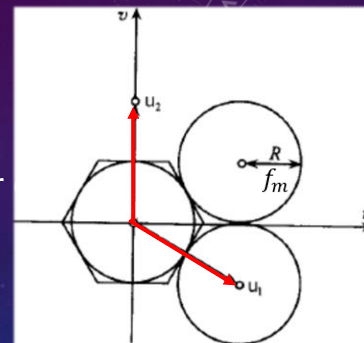
$$\frac{2\pi}{T} = f_m$$

$$V = \begin{bmatrix} \frac{\pi}{f_m} & 0 \\ -\frac{\pi}{f_m} & \frac{2\pi}{f_m} \end{bmatrix} = \begin{bmatrix} \frac{T}{2} & 0 \\ -\frac{T}{2} & T \end{bmatrix}$$

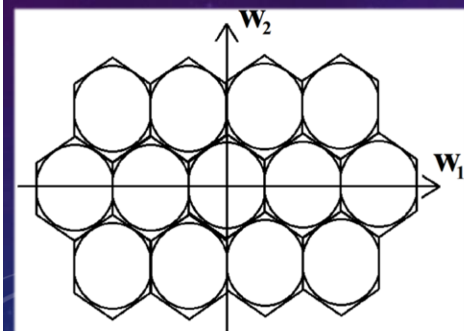
$$\begin{bmatrix} \frac{\sqrt{3} f_m}{2\pi} & -\frac{f_m}{2\pi} \\ 0 & \frac{f_m}{\pi} \end{bmatrix} V = I$$

SAMPLING EFFECIENCY

- A different sampling strategy more efficient for shapes
  - Disk



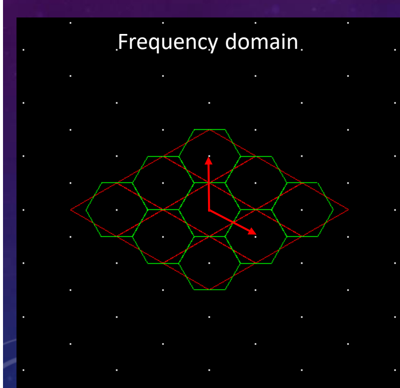
$$U^T V = 2\pi I$$

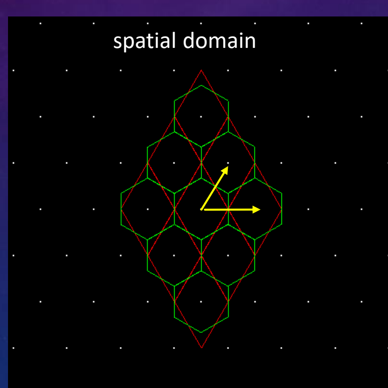


$$U = \begin{bmatrix} \sqrt{3} f_m & 0 \\ -f_m & 2 f_m \end{bmatrix} \quad V = \begin{bmatrix} \frac{2\pi}{\sqrt{3} f_m} & \frac{\pi}{\sqrt{3} f_m} \\ 0 & \frac{\pi}{f_m} \end{bmatrix} = \begin{bmatrix} \frac{T}{\sqrt{3}} & \frac{T}{2\sqrt{3}} \\ 0 & \frac{T}{2} \end{bmatrix}$$

SAMPLING EFFICIENCY

- Hexagonal sampling





$$U = \begin{bmatrix} \sqrt{3} f_m & 0 \\ -f_m & 2 f_m \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{2\pi}{\sqrt{3} f_m} & \frac{\pi}{\sqrt{3} f_m} \\ 0 & \frac{\pi}{f_m} \end{bmatrix} = \begin{bmatrix} \frac{T}{\sqrt{3}} & \frac{T}{2\sqrt{3}} \\ 0 & \frac{T}{2} \end{bmatrix}$$

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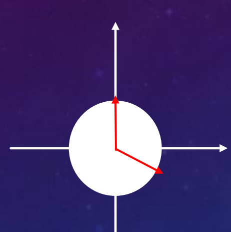
SAMPLING EFFICIENCY

- Sampling density
  - Lossless condition
  - Number of samples per unit area
- Sampling density is inversely proportional to $\det(V)$
- For rectangular sampling

$$S.D. = \frac{1}{\det(V)} = \frac{\det(U)}{4\pi^2} \qquad V = \begin{bmatrix} \frac{\pi}{f_m} & 0 \\ 0 & \frac{\pi}{f_m} \end{bmatrix} \qquad \det(V) = |V| = \frac{\pi^2}{f_m^2}$$

SAMPLING EFFECIENCY

- Hexagonal sampling



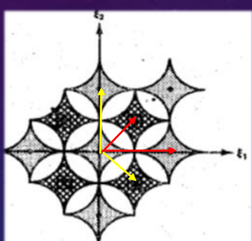
$$V = \begin{bmatrix} \frac{2\pi}{\sqrt{3} f_m} & \frac{\pi}{\sqrt{3} f_m} \\ 0 & \frac{\pi}{f_m} \end{bmatrix} \qquad \det(V) = |V| = \frac{2\pi^2}{\sqrt{3} f_m^2}$$

$$\rho_{H/R} = \frac{|V_R|}{|V_H|} = \frac{\sqrt{3}}{2} \approx 0.866 \qquad \frac{2}{\sqrt{3}} \approx 1.155$$

SAMPLING EFFICIENCY

- Interlaced sampling

$$U = \begin{bmatrix} 2f_m & f_m \\ 0 & f_m \end{bmatrix}$$



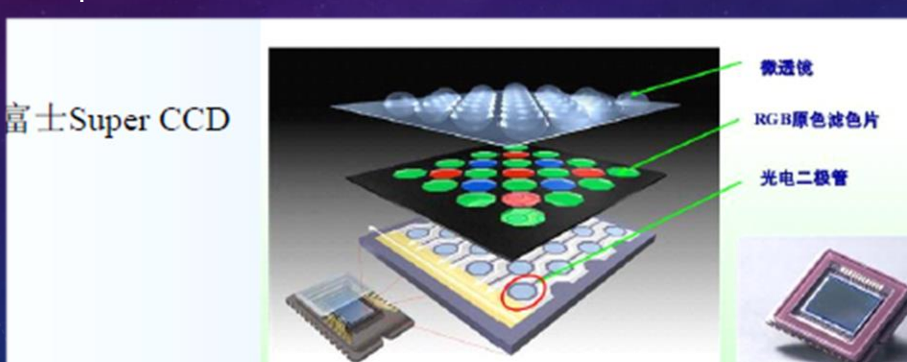
$$V = \begin{bmatrix} \dfrac{\pi}{f_m} & 0 \\ -\dfrac{\pi}{f_m} & \dfrac{2\pi}{f_m} \end{bmatrix}$$

$$\det(V) = |V| = \frac{2\pi^2}{f_m^2}$$

$$\rho_{I/R} = \frac{|V_R|}{|V_I|} = \frac{1}{2}$$

SAMPLING ON NON-RECTANGULAR LATTICES

- Non-rectangular lattices hexagonal
- Complex reconstruction function



SAMPLING & RECONSTRUCTION - LIMITS

- overlapping
- LPF before sampling

Ideal bandlimited signal

- Reconstruction function is impractical，since sinc function has small attenuation(衰减)
- Simple approximation techniques are employed

Ideal low pass reconstruction Filter

- Sampling aperture --- loss of high frequency component

Ideal 2-D function

- Practical sampling and reconstruction of an image is different

QUANTIZATION

- Uniform
- Symmetric
- With Memory
- Scalar

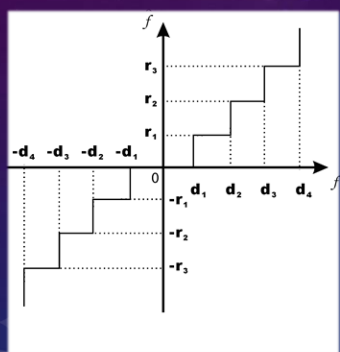
- Non-uniform
- Asymmetric
- Memoryless
- Vector

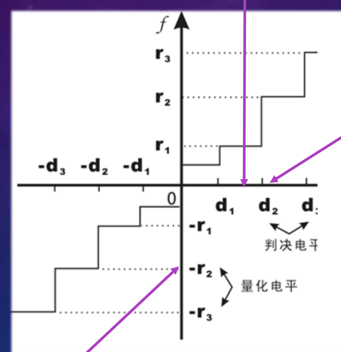
The conversion process between analog samples and discrete-valued samples is called quantization

QUANTIZATION

quantization interval $\Delta_i = d_{i+1} - d_i$

$d \in [d_i, d_{i+1})$ $\quad d \to r_i$





decision level d_i

quantization error $e = d - Q(d)$

quantization noise $\varepsilon^2 = E[e^2]$

reconstruction level r_i

QUANTIZATION

- Scalar quantization ---- For a given number of reconstruction levels k, design a quantizer and minimize its loss
  - Optimal mean square error quantizer(最优均方误差量化器) ---- minimum MSE

  Random variable z， with continuous PDF $p(z)$， quantized as q_i

$$\varepsilon^2 = \sum_{i=0}^{k-1} \int_{z_i}^{z_{i+1}} (z - q_i)^2 p(z)dz \to \begin{cases} \dfrac{\partial \varepsilon^2}{\partial q_i} = 0 \\ \dfrac{\partial \varepsilon^2}{\partial z_i} = 0 \end{cases}$$

$$q_i = \frac{\int_{z_i}^{z_{i+1}} zp(z)dz}{\int_{z_i}^{z_{i+1}} p(z)dz}$$

$$z_i = \frac{q_{i-1} + q_i}{2}$$

QUANTIZATION

- Scalar quantization
  - uniform (linear) quantization

Separate $[z_0, z_k)$ into k sub-intervals, each of which is of the same length, uniform quantization is the optimal quantization if z is uniformly distributed.

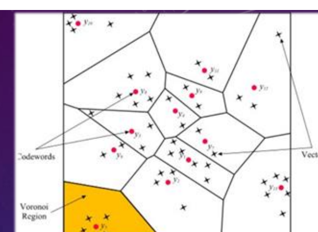
$$L = (z_k - z_0)/k \qquad q_i = (z_i + z_{i+1})/2 \qquad z_i = (q_{i-1} + q_i)/2$$

$$p(z) = \begin{cases} \dfrac{1}{(z_k - z_0)}, & z_0 \le z < z_k \\ 0 & otherwise \end{cases}$$

$$\varepsilon^2 = \sum_{i=0}^{k-1} \int_{z_i}^{z_{i+1}} (z - q_i)^2 p(z)dz$$

$$= \sum_{i=0}^{k-1} \int_{z_i}^{z_{i+1}} (z - q_i)^2 \frac{1}{kL} dz = \frac{L^2}{12}$$

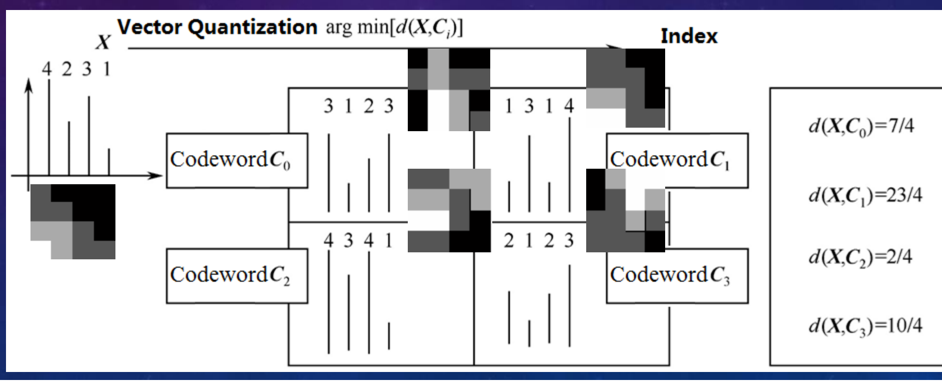
QUANTIZATION



- Vector Quantization(VQ) ---- Also called *block quantization* or *pattern matching quantization*, works by encoding values from a MD vector space into a finite set of values from a discrete subspace of lower dimension
- A vector in subspace requires less storage space, so the data is compressed. Due to the density matching property of vector quantization, the compressed data has errors that are inversely proportional to density
- The transformation is usually done by projection or by using a codebook

QUANTIZATION

- Vector Quantization(VQ) ---- works by encoding values from a MD vector space into a finite set of values from a discrete subspace of lower dimension.



QUANTIZATION

- Optimal matching： minimize the expected absolute/squared quantization error
  - For a random variable X with PDF $p(X)$

Expected Quantization Error:

$$\varepsilon = \sum_i e(X, X_i)\, p(X)$$

$$e(\boldsymbol{X}, \boldsymbol{X}_i) = \|\boldsymbol{X} - \boldsymbol{X}_i\|^j \quad j = 1, 2, \ldots p, \ldots$$

Taxicab-norm
Euclidean-norm
p-norm
Infinite-norm

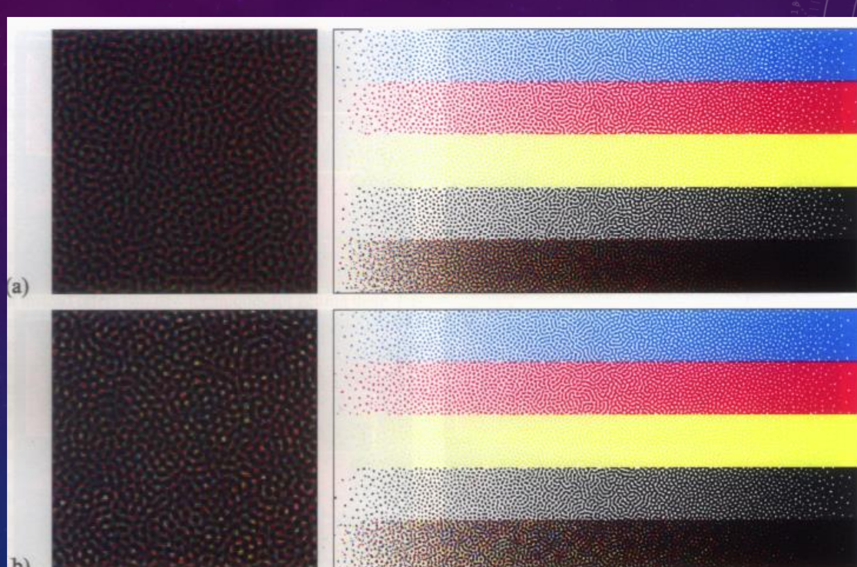
$$MAE:\ e(X, X_i) = \frac{1}{k}\sum_{m=1}^{k}\left|x(m) - \hat{x}_i(m)\right|$$

$$MSE:\ e(X, X_i) = \frac{1}{k}\sum_{m=1}^{k}\left[x(m) - \hat{x}_i(m)\right]^2$$

HALFTONING

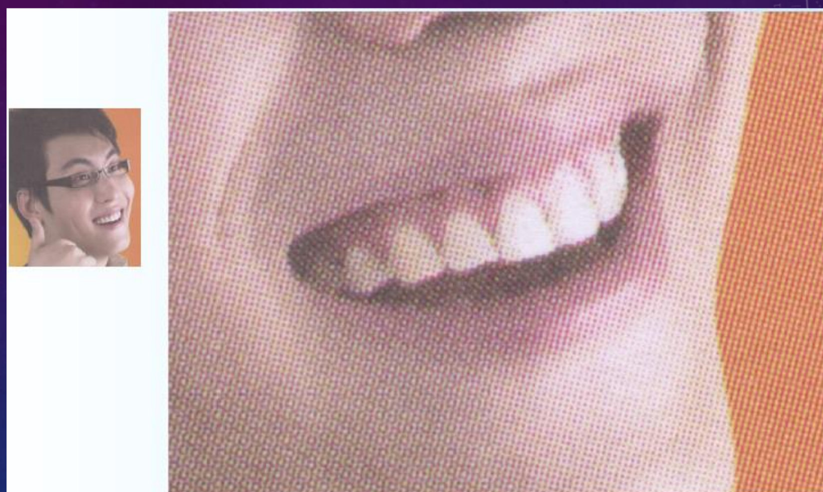
- Halftoning
  - Mainly used in printing/reprography(复印)
  - The halftone process reduces visual reproductions to an image that is printed with only one color of ink
  - Question: try to convert a grayscale image into a binary one(black and white).
  - This reproduction relies on a basic optical illusion: the tiny halftone dots are blended into smooth tones by the human eye.

CMYK color plane



The Density of tiny dots are different in these color ramps

Color printing



HALFTONING

- One color of ink -> Grayscale image
- Optic illusion
- Perceptual grayscale ----- ratio of the inked pixels within a cell
- Resolution is sacrificed

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DITHER

cyclical

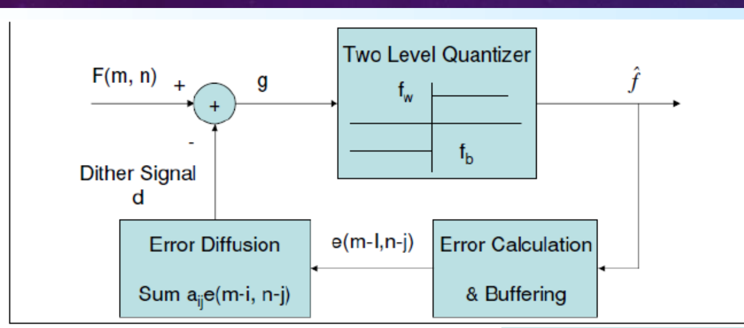
- Dither is an intentionally applied form of noise used to randomize quantization error
- Dither siganl d(m,n) could be obtained by applying dither matrix Dk(m,n) repeatedly.

$$D^4 = \begin{bmatrix} 8 & 136 & 40 & 168 \\ 200 & 72 & 232 & 104 \\ 56 & 184 & 24 & 152 \\ 248 & 120 & 216 & 88 \end{bmatrix}$$

$$H_2 = \left[\begin{array}{cccc|cccc} 52 & 44 & 36 & 124 & 132 & 140 & 148 & 156 \\ 60 & 4 & 28 & 116 & 200 & 228 & 236 & 164 \\ 68 & 12 & 20 & 108 & 212 & 252 & 244 & 172 \\ 76 & 64 & 92 & 100 & 204 & 196 & 188 & 180 \\ \hline 132 & 140 & 148 & 156 & 52 & 44 & 36 & 124 \\ 200 & 228 & 236 & 164 & 60 & 4 & 28 & 116 \\ 212 & 252 & 244 & 172 & 68 & 12 & 20 & 108 \\ 204 & 196 & 188 & 180 & 76 & 84 & 92 & 100 \end{array}\right]$$

Figure 4.25 Two halftone patterns. Repeat periodically to obtain the full size array. H_2 is called 45° halftone screen because it repeats two 4×4 basic patterns at $\pm 45°$ angles.

HALFTONING USING ERROR DIFFUSION



$$d(m,n) = \sum_{i,j} \alpha_{ij} e(m-i, n-j) / \sum_{i,j} \alpha_{ij}$$

$$\{\alpha_{ij}\} = \begin{bmatrix} 1 & 3 & 5 & 3 & 1 \\ 3 & 5 & 7 & 5 & 3 \\ 5 & 7 & * & & \end{bmatrix}$$

The dither signal is generated from a weighted sum of the quantization errors of previously scanned signal.

* Is the current pixel location

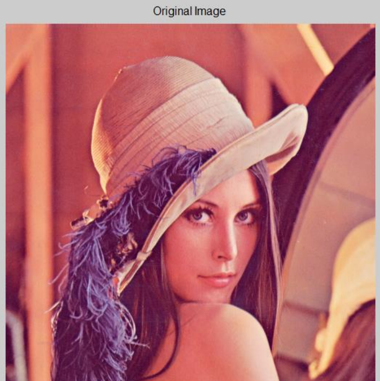
Note: can be used for more than 2-level quantizer

DITHER

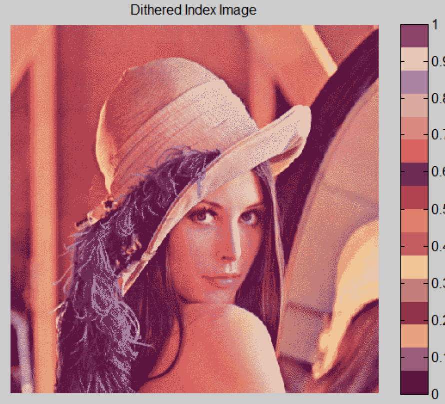
Floyd-Steinberg error diffusion

DITHER

Original Image



Dithered Index Image



SUMMARY

- Digitalization
  - Sampling（rectangular/hexagonal, sampling density/efficiency)
  - Quantization（vector quantization, halftone, dither)
- To balance sampling and quantization, we have to take the image content into consideration

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