UESTC3001 Dynamics & Control
Lecture 6

Characteristics and Performance of Feedback Control Systems – II

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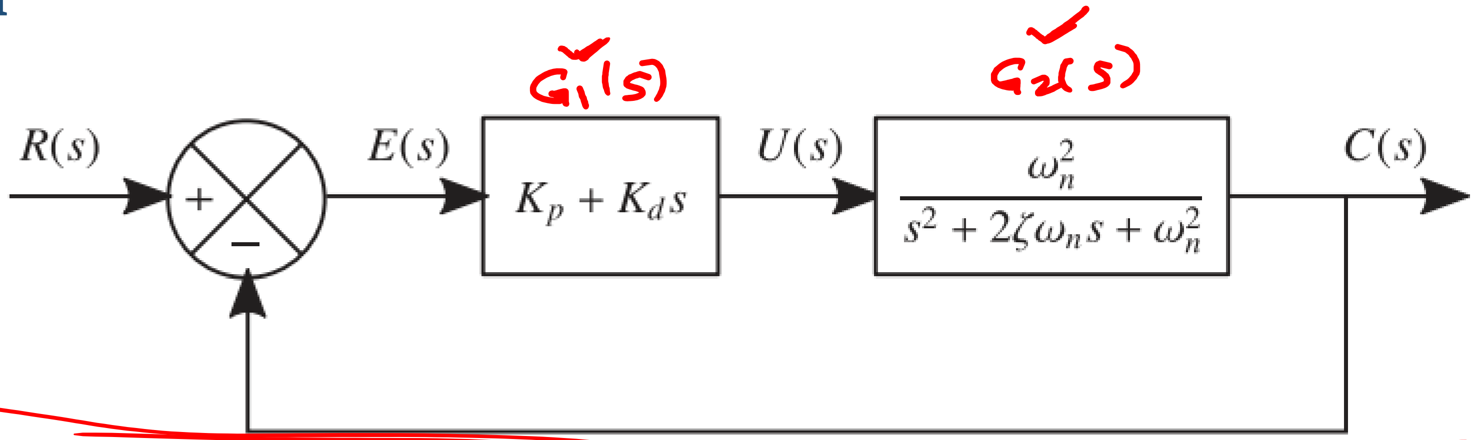
Outline

- Proportional + Derivative Control of a Second-Order System and Effect on a Second-Order System
- Integral Control of a First-Order/Second-Order System and Effect on a First-Order/Second-Order System
- Proportional + Integral Control of a First-Order System and Effect on a First-Order System
- Proportional + Derivative + Integral Control of a First-Order /Second-Order System and Effect on a First-Order System

Proportional Plus Derivative (PD) Control of a Second-Order System



$$\frac{C(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)}$$

$$R(s) = \frac{1}{s}$$

$$\frac{C(s)}{R(s)} = \frac{(K_p + K_d s)\omega_n^2}{s^2 + (2\zeta\omega_n + K_d\omega_n^2)s + (1+K_p)\omega_n^2}$$

$$c_{ss} = \lim_{s\to 0} sC(s) = \lim_{s\to 0} \cancel{s} \times \frac{1}{\cancel{s}} \frac{(K_p + \underbrace{K_d s}_{\to 0})\omega_n^2}{\underbrace{s^2 + (2\zeta\omega_n + K_d\omega_n^2)s}_{\to 0} + (1+K_p)\omega_n^2}$$

$$= \frac{K_p \cancel{\omega_n^2}}{(1+K_p)\cancel{\omega_n^2}} = \frac{K_p}{1+K_p}$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+G_1(s)G_2(s)}$$

$$R(s) = \frac{1}{s}$$

$$\frac{E(s)}{R(s)} = \left(\frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{s^2 + (2\zeta\omega_n + K_d\omega_n^2)s + (1+K_p)\omega_n^2} \right)$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \cancel{s} \times \frac{1}{\cancel{s}} \times \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{s^2 + (2\zeta\omega_n + K_d\omega_n^2)s + (1+K_p)\omega_n^2}$$

$\to 0$

$$= \frac{\cancel{\omega_n^2}}{(1+K_p)\cancel{\omega_n^2}}$$

$$= \frac{1}{1+K_p}$$

Effect of Proportional Plus Derivative (PD) Control on a Second-Order System

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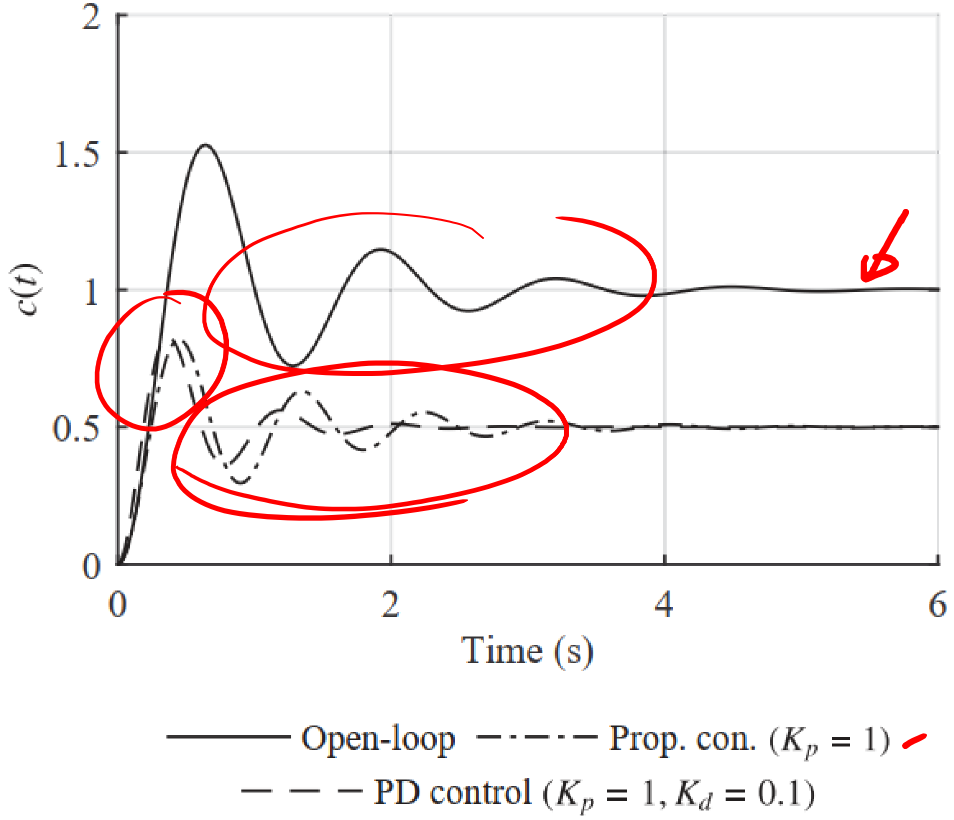
$$s^2 + 2\zeta_{cl}\omega_{n,cl}s + \omega_{n,cl}^2 = 0$$

$$c(t) = \frac{K_p}{K_p+1}\left[1 - e^{-\zeta_{cl}\omega_{n,cl}t}\left(\cos\omega_{d,cl}t + \frac{K_p\zeta_{cl} - K_d\omega_{n,cl}}{K_p\sqrt{1-\zeta_{cl}^2}}\sin\omega_{d,cl}\right)\right]$$



$$\omega_{n,cl} = \omega_n\sqrt{K_p+1}$$

$$\zeta_{cl} = \frac{\zeta + \frac{1}{2}K_d\omega_n}{\sqrt{K_p+1}}$$

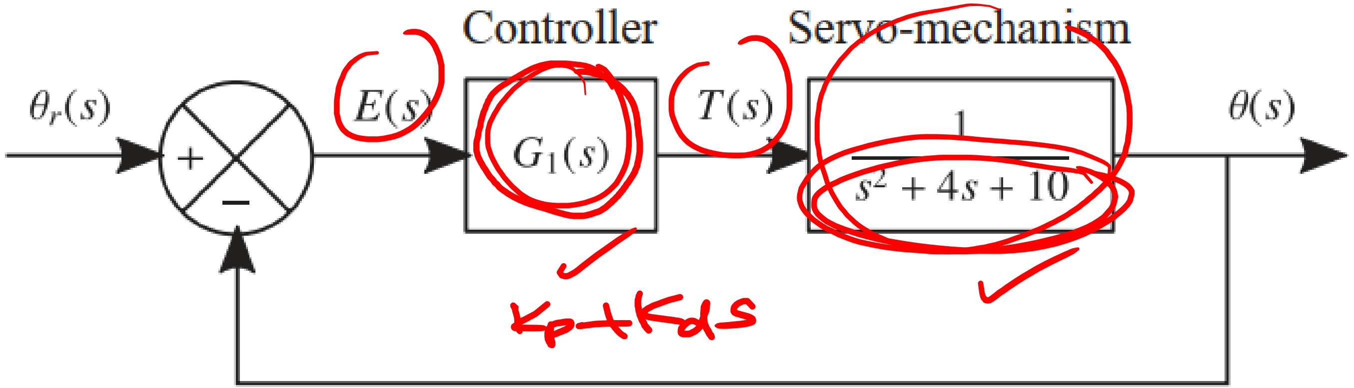
Exercise 1

The angular output of a servo-mechanism θ is controlled as indicated in the figure below where θ_r is the required (reference) displacement. The controller transfer function is given by $G_1(s)$, the torque it applies to the servo-mechanism is denoted by $T(s)$ and the error signal $E(s)$.

(a) Calculate the damping and natural frequency of the uncontrolled open-loop system. What is the maximum overshoot of the uncontrolled system in response to a unit step change in torque $T(s)$?

(b) Proportional control with gain $K_p = 10$ is applied. Calculate the closed-loop natural frequency and damping. What is the steady state error in response to a unit step input in $\theta_r(s)$? Calculate the resulting maximum overshoot.

(c) Derivative action with gain $K_d = 4$ is now added to the proportional action. Calculate the closed-loop natural frequency and damping. What is the steady state error in response to a unit step input in $\theta_r(s)$? Calculate the resulting maximum overshoot.



$$O/L\ T/f = G_1(s)G_2(s) = G_1(s)\cdot\frac{1}{s^2+4s+10}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$Ch^{c}\,E \Rightarrow s^2 + 4s + 10 = 0$$

$$\omega_n^2 = 10 \Rightarrow \omega_n = 3.162\ rad/s$$

$$2\zeta\omega_n = 4 \Rightarrow \zeta = \frac{4}{2\times 3.162} = 0.6325$$

$$M_p = 100\, e^{-\zeta\omega_n t_p}\,\%$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$M_p = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

$$t_p = \frac{\pi}{\omega_d}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\omega_d = 2.4492$$

$$t_s = \frac{3}{\zeta\omega_n} = \frac{3}{0.6325\times 3.162} = 1.5\ s$$

$$M_p = e^{-\left(\frac{\pi\times 0.6325}{\sqrt{1-0.6325^2}}\right)} = 7.6892\%$$

$$t_p = \frac{\pi}{\omega_d} = 1.2827\ s$$

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b)

$$C/L\ T/f = \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)} = \frac{10\times\left(\frac{1}{s^2+4s+10}\right)}{1+10\times\left(\frac{1}{s^2+4s+10}\right)} = \frac{10}{s^2+4s+20}.$$

$$s^2+2\zeta\omega_n s+\omega_n^2=0$$

$$s^2+4s+20=0$$

$$\omega_n^2=20 \Rightarrow \omega_n = 4.472\ rad/s$$

$$2\zeta\omega_n=4 \Rightarrow \zeta=\frac{4}{2\times 4.472}=0.4472$$

$$\frac{E(s)}{R(s)}=\frac{1}{1+G_1(s)G_2(s)}=\left(\frac{s^2+4s+10}{s^2+4s+20}\right)$$

$$E(s)=\frac{1}{s}\times\left(\frac{s^2+4s+10}{s^2+4s+20}\right)$$

$$e_{ss}=\lim_{s\to 0} sE(s)=\lim_{s\to 0}\cancel{s}\,\frac{1}{\cancel{s}}\left(\frac{s^2+4s+10}{s^2+4s+20}\right)=\frac{10}{20}$$

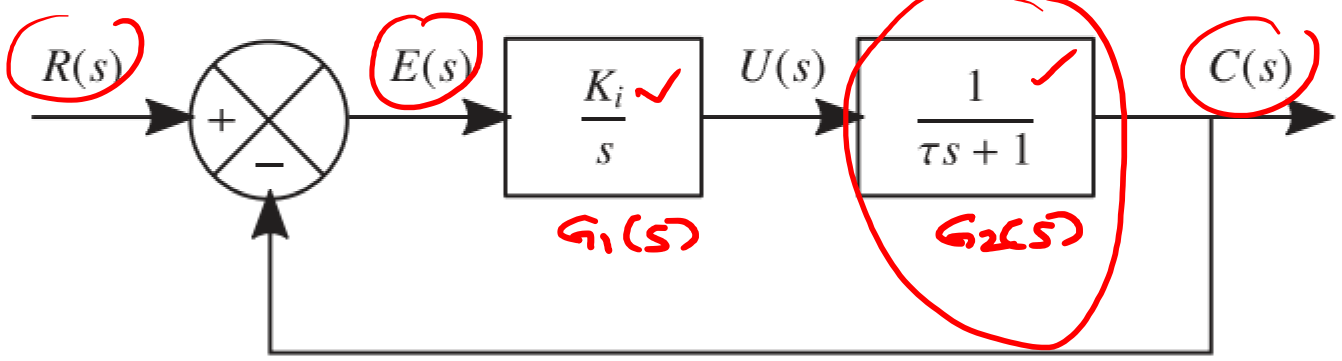
$$\downarrow 0$$

$$=0.5$$

$M_p = 20.789\ \%$

c) $G(s) \rightarrow K_p + K_d s$

Integral Control of a First-Order System



$$\frac{C(s)}{R(s)} = ?$$

$$\frac{E(s)}{R(s)} = ?$$

$$R(s) = 1/s$$

C_{ss} ? E_{ss} ?

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2}{1 + G_1 G_2} = \left(\frac{K_i}{\tau s^2 + s + K_i}\right)$$

$R(s) = 1/s$

$$C(s) = \frac{1}{s}\left(\frac{K_i}{\tau s^2 + s + K_i}\right)$$

$$C_{ss} = \lim_{s \to 0} s\,C(s) = \lim_{s \to 0} \cancel{s} \times \frac{1}{\cancel{s}} \times \left(\frac{K_i}{\underbrace{\tau s^2 + s}_{0} + K_i}\right) = \frac{K_i}{K_i}$$

$$= 1$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+G_1G_2} = \frac{s(\tau s+1)}{\tau s^2+s+K_i}$$

$R(s) \to 1/s$

$$E(s) = \frac{1}{s}\,\frac{s(\tau s+1)}{\tau s^2+s+K_i}$$

$$e_{ss} = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0} \cancel{s} \times \frac{1}{\cancel{s}} \times \frac{\overbrace{s(\tau s+1)}^{\to 0}}{\underbrace{\tau s^2+s}_{\to 0}+K_i} = \frac{0}{K_i}$$

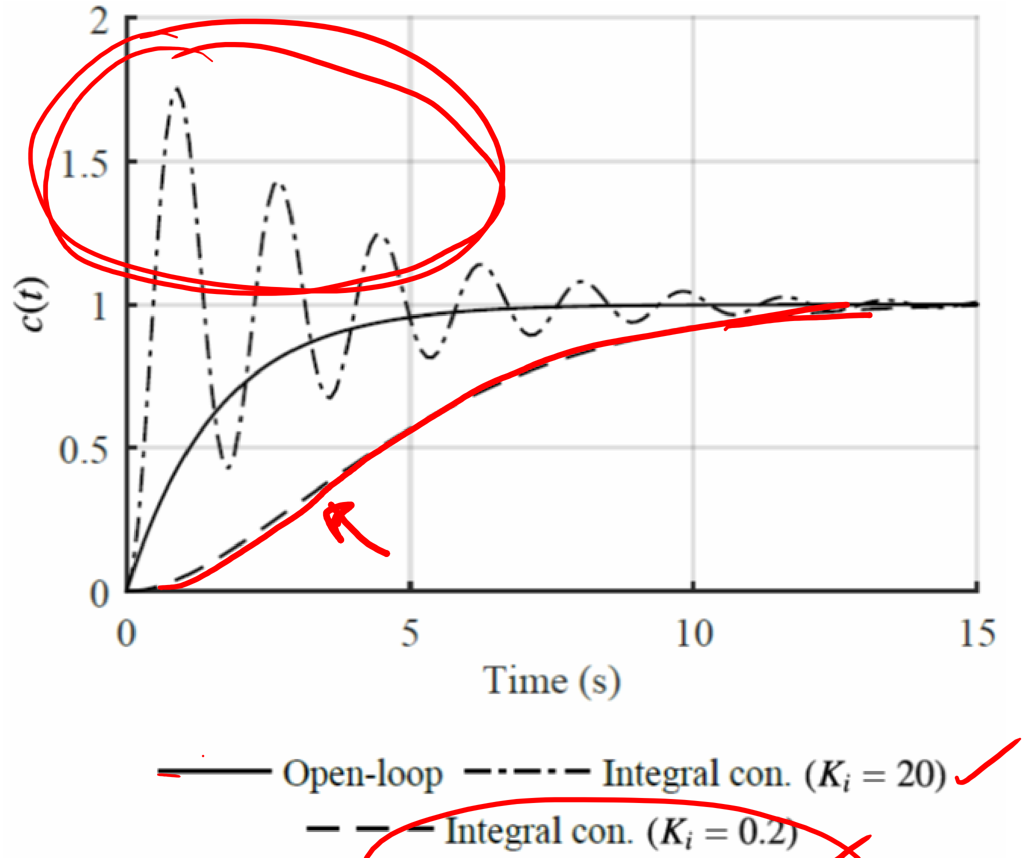
$$= 0$$

Effect of Integral Control on a First-Order System

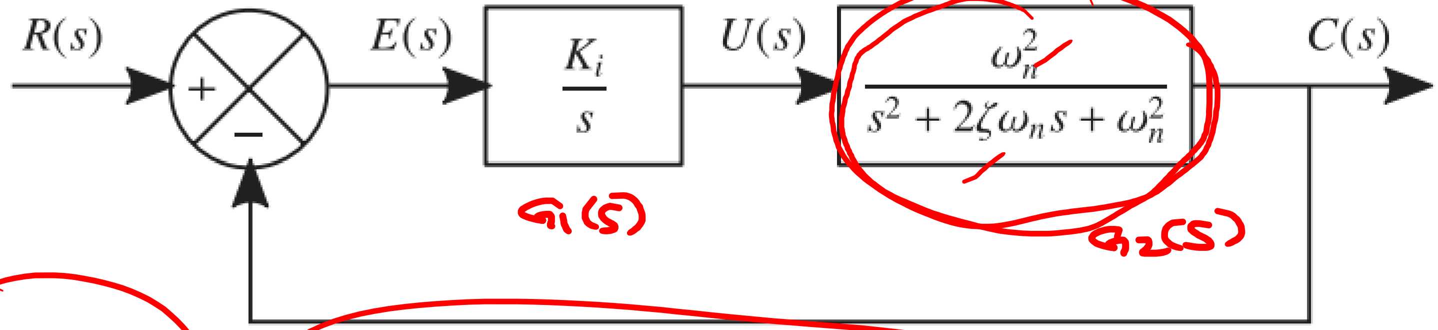
$$s^2 + 2\zeta cl\omega_{n,cl}s + \omega_{n,cl}^2 = 0$$

$$\omega_{n,cl} = \sqrt{\frac{K_i}{\tau}}$$

$$\zeta_{cl} = \frac{1}{2\sqrt{\tau K_i}}$$



Integral Control of a Second-Order System



$G_1(s)$ $\quad$ $G_2(s)$

$c(t)$

$$\frac{C(s)}{R(s)} = \frac{G_1G_2}{1+G_1G_2} = \left(\frac{K_i\omega_n^2}{s^3+2\zeta\omega_n s^2+\omega_n^2 s+K_i\omega_n^2}\right)$$

$$R(s)=\frac{1}{s} \Longrightarrow \quad C(s)=\frac{1}{s}\times\left(\quad\right)$$

$$c_{ss} = \lim_{s\to 0} s\,C(s) = \lim_{s\to 0} \cancel{s}\times\frac{1}{\cancel{s}}\times\frac{K_i\omega_n^2}{\underbrace{s^3+2\zeta\omega_n s^2+\omega_n^2 s}_{0}+K_i\omega_n^2}$$

$$=\frac{K_i\omega_n^2}{K_i\omega_n^2} = 1$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+G_1G_2} = \left(\frac{s(s^2+2\zeta\omega_n s+\omega_n^2)}{s^3+2\zeta\omega_n s^2+\omega_n^2 s+K_i\omega_n^2}\right)$$

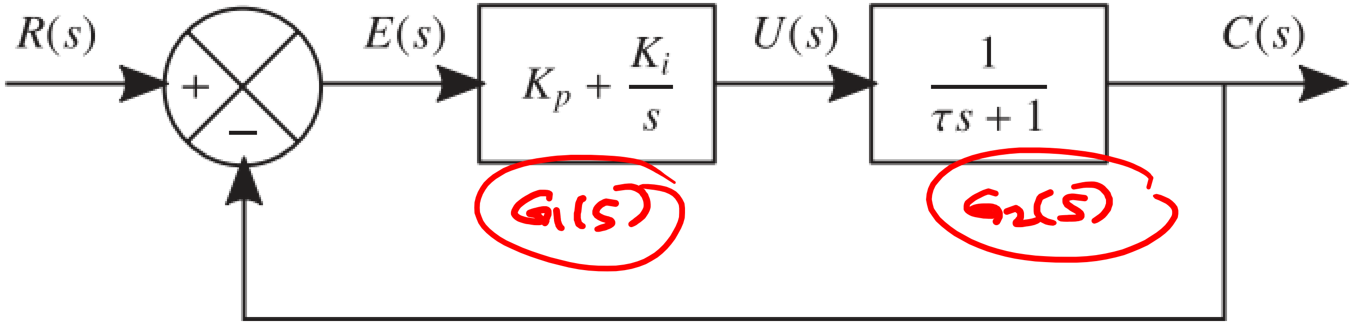
$$R(s) = \frac{1}{s} \Rightarrow E(s) = \underline{\frac{1}{s}\times\left(\quad\right)}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \cancel{s}\times\cancel{\frac{1}{s}}\times \overbrace{\frac{s(s^2+2\zeta\omega_n s+\omega_n^2)}{\underbrace{s^3+2\zeta\omega_n s^2+\omega_n^2 s}_{0}+K_i\omega_n^2}}^{0}$$

$$= \frac{0}{K_i\omega_n^2}$$

$$= 0$$

PI Control of a First-Order System



$$R(s) = \frac{1}{s}$$

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2}{1 + G_1 G_2} = \frac{K_p s + K_i}{\tau s^2 + (K_p + 1)s + K_i}$$

$$C_{ss} = 1$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+G_1G_2} = \frac{s(\tau s+1)}{\tau s^2+(K_p+1)s+K_i}$$

$1/s$

$E(s)$ ✓

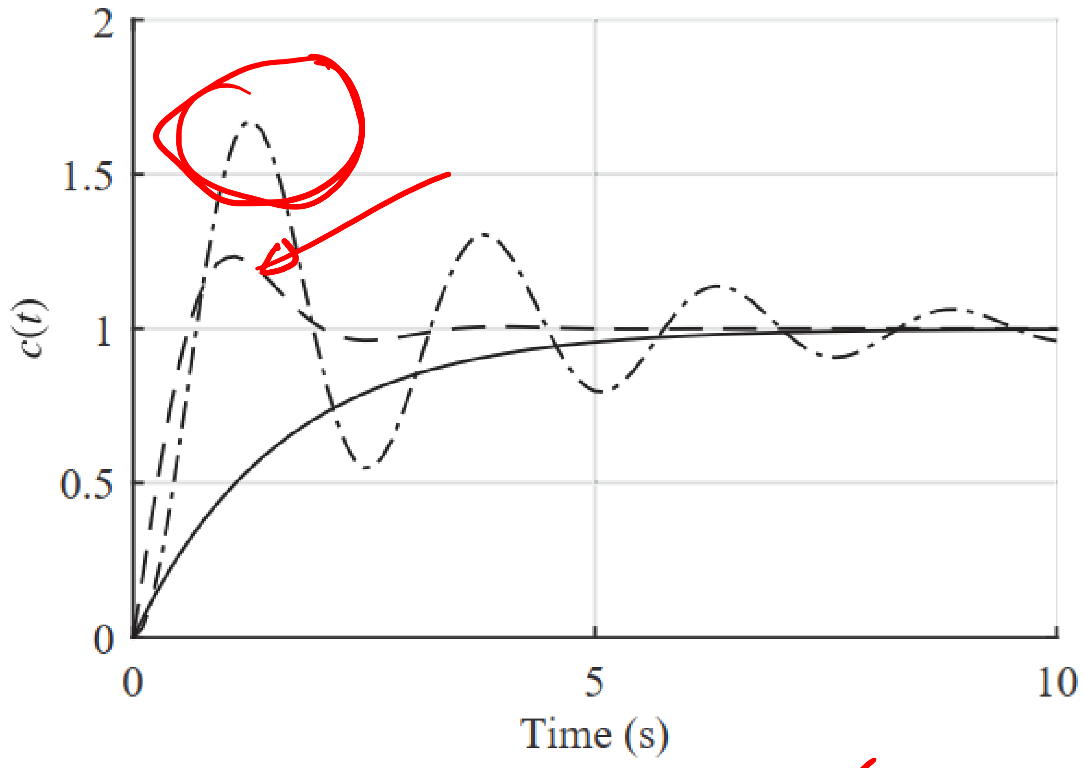
$$e_{ss} = 0$$

Effect of PI Control on a First-Order System

$$s^2 + 2\zeta cl\omega_{n,cl}s + \omega_{n,cl}^2 = 0$$

$$\omega_{n,cl} = \sqrt{\frac{K_i}{\tau}}$$

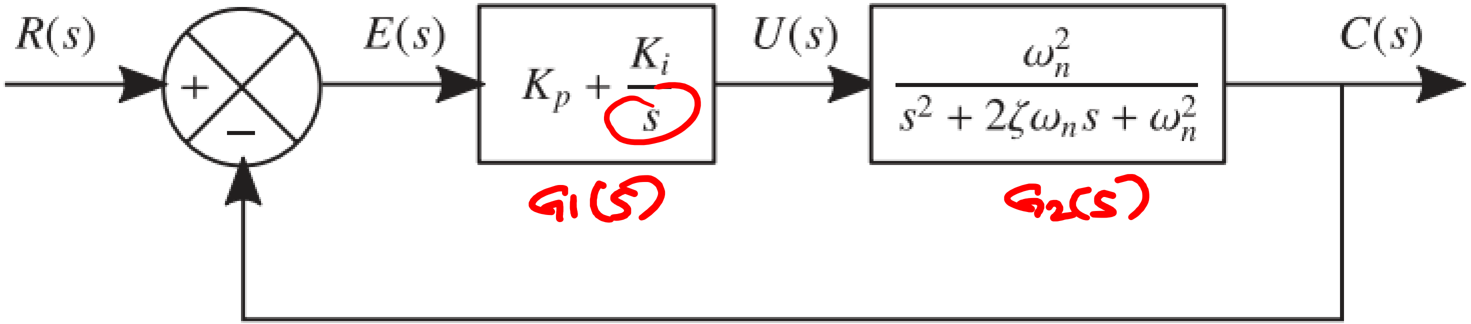
$$\zeta_{cl} = \frac{K_p + 1}{2\sqrt{\tau K_i}}$$



Open-loop — · — · — Integral con. (K_i = 10)

— — — PI control (K_p = 3, K_i = 10)

PI Control of a Second-Order System



$$\frac{C(s)}{R(s)} = \frac{(K_i + K_p s)\,\omega_n^2}{s^3 + 2\zeta\omega_n s^2 + \omega_n^2 (K_p + 1)s + K_i \omega_n^2}$$

$R(s) = 1/s$

$$C_{ss} = 1$$

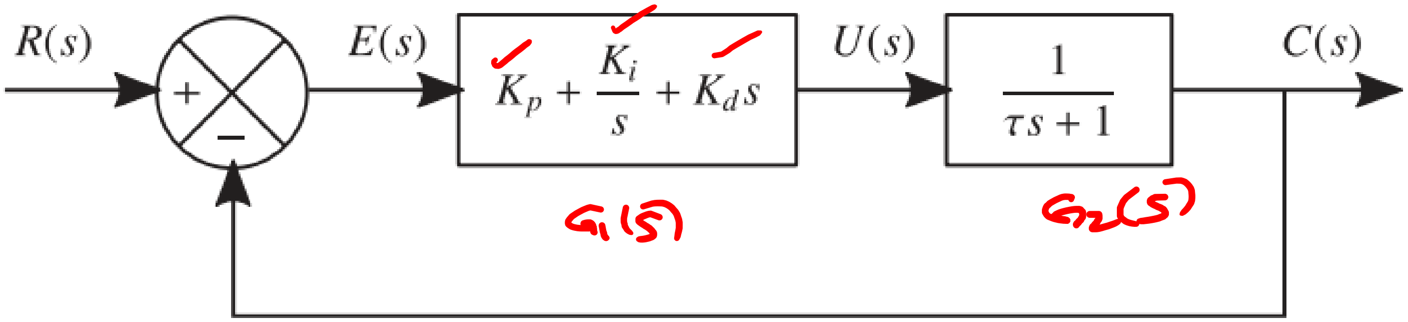
$$\frac{E(s)}{R(s)} = \frac{s(s^2+2\zeta\omega_n s+\omega_n^2)}{s^3+2\zeta\omega_n s^2+\omega_n^2(K_p+1)s+K_i\omega_n^2}$$

$1/s$

$E(s)$ ✓

$$e_{ss} = 0$$

PID Control of a First-Order System



$$\frac{C(s)}{R(s)} = \frac{K_d s^2 + K_p s + K_i}{(\tau + K_d)s^2 + (K_p + 1)s + K_i}$$

$\frac{1}{s}$

$$C_{ss} = 1$$

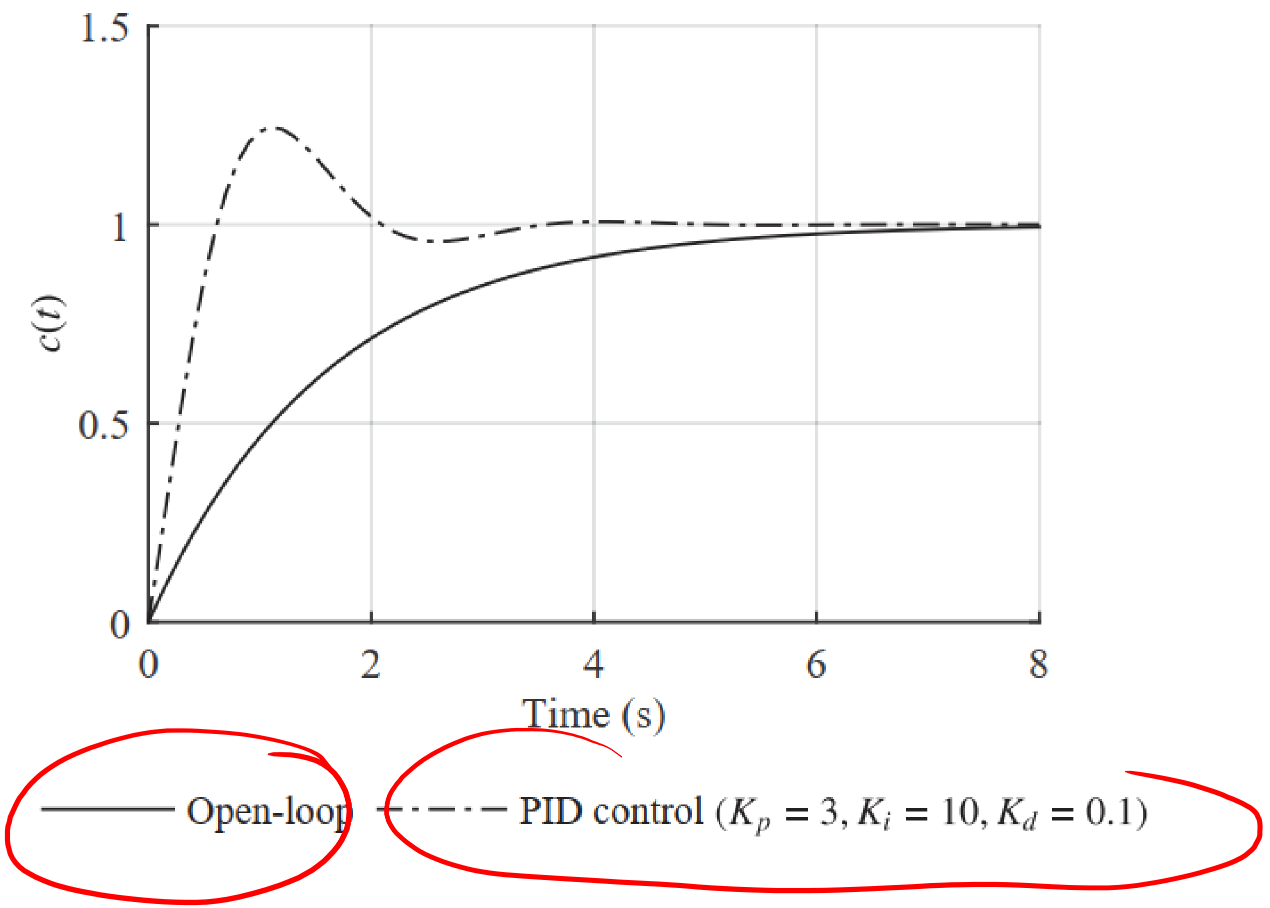
$$\frac{E(s)}{R(s)} = \frac{s(\tau s+1)}{(\tau + K_d)s^2 + (K_p+1)s + K_i}$$

$1/s$

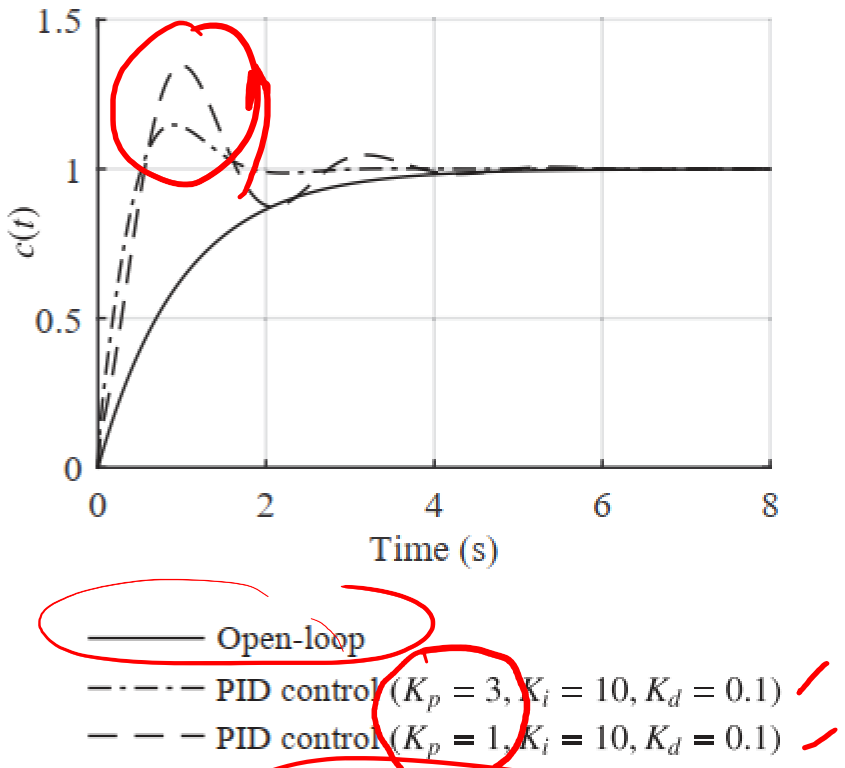
$$e_{ss} = 0$$

Effect of PID Control on a First-Order System

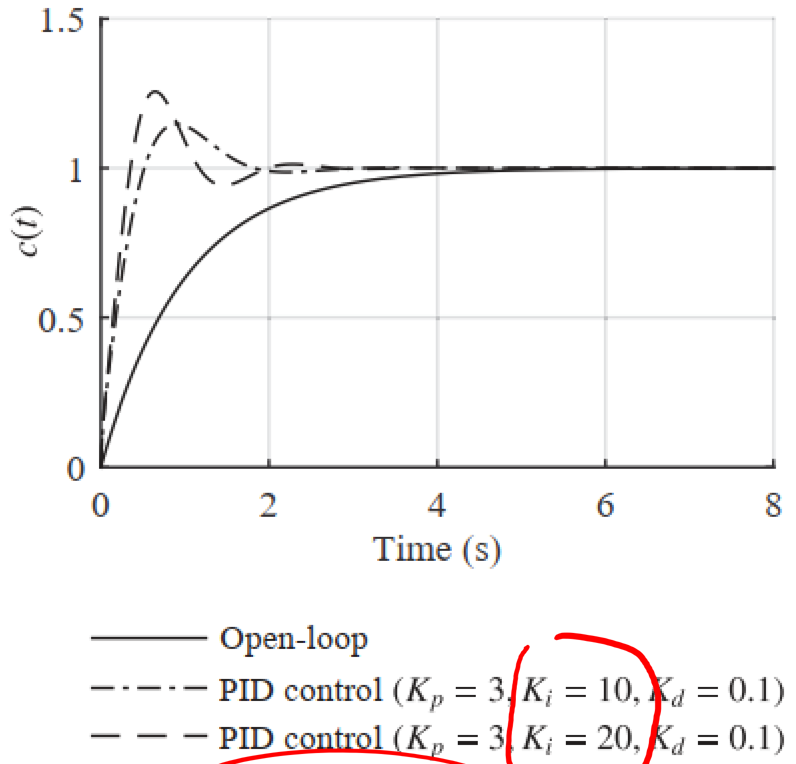
$$s^2 + 2\zeta cl\omega_{n,cl}s + \omega_{n,cl}^2 = 0 \qquad \omega_{n,cl} = \sqrt{\frac{K_i}{\tau + K_d}} \qquad \zeta_{cl} = \frac{K_p + 1}{2\sqrt{K_i(\tau + K_d)}}$$



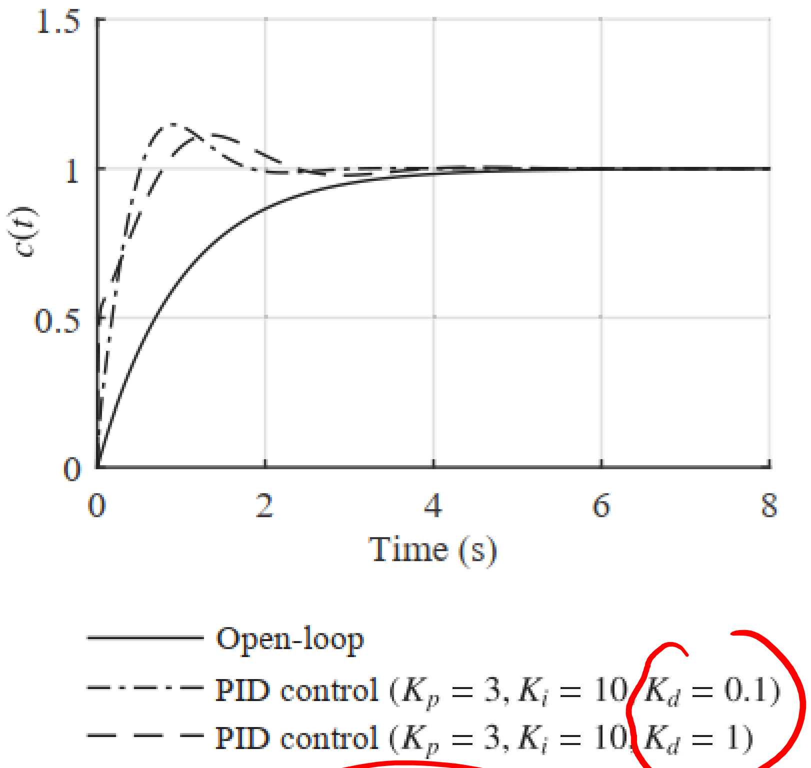
Effect of PID Control on a First-Order System cont.



(a) Proportional action.

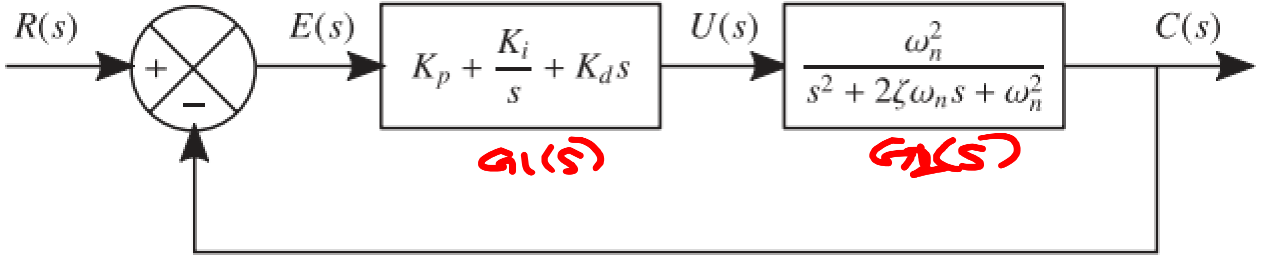


(b) Integral action.



(c) Derivative action.

PID Control of a Second-Order System



$$\frac{C(s)}{R(s)} = \frac{(K_i + K_p s + K_d s^2)\omega_n^2}{s^3 + (2\zeta\omega_n + K_d\omega_n^2)s^2 + \omega_n^2(K_p+1)s + K_i\omega_n^2}$$

Ys

$C(s) = R$

$C_{ss} = 1$

PID Control of a Second-Order System cont.

$$\frac{E(s)}{R(s)} = \frac{s(s^2 + 2\xi\omega_n s + \omega_n^2)}{s^3 + (2\xi\omega_n + K_d\omega_n^2)s^2 + \omega_n^2(K_p+1)s + K_i\omega_n^2}$$

Y_s

$E(s)$ ✓

$$e_{ss} = 0$$

Summary

- PD Control of a Second-Order System and Effect on a Second-Order System
- Integral Control of a First-Order/Second-Order System and Effect on a First-Order/Second-Order System
- PI Control of a First-Order System and Effect on a First-Order System
- PID Control of a First-Order /Second-Order System and Effect on a First-Order System

Reference:

-Control Systems Engineering, 7th Edition, N.S. Nise

-UESTC3001 2019/20 Notes, J. Le Kernec