Linear Algebra Review

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Some Things to Know First

- Welcome to Linear Algebra!
- *Dimensionality*: This just tells us what size our vector or matrix is. Dimensionality is always denoted like *(row x column)*. This is the same order Python and R use when slicing by index so it's important that you know this.
- *Scalar*: This is just what we call numbers. A scalar has (1×1) dimensionality.
- Here are some very common scalars:

$$7 \quad 69 \quad 420 \quad 1923$$

Some Things to Know First

- *Vector*: This is an ordered list of numbers where the order *does* matter. Vectors can be written vertically (column vectors) or horizontally (row vector). Vertical vectors will have a dimensionality of $(n \times 1)$, and horizontal $(1 \times n)$.
- Here are some vectors:

$$\underset{(4 \times 1)}{\begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}} \quad \underset{(n \times 1)}{\begin{bmatrix} k_0 \\ k_1 \\ \vdots \\ k_n \end{bmatrix}} \quad \underset{(1 \times 7)}{\begin{bmatrix} 1 & 1 & 2 & 3 & 5 & 8 & 13 \end{bmatrix}}$$

Some Things to Know First

- *Matrix*: We should all know this, but this is an ordered collection of numbers with dimensionality $(n \times n)$. We can think of this like a full dataset.

- Here are two examples:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,k} \\ x_{2,1} & x_{2,2} & & \\ \vdots & & & \\ x_{n,1} & & & x_{n,k} \end{bmatrix}$$

$$(3 \times 3) \qquad\qquad (n \times k)$$

Simple Addition

- Maybe self-explanatory, but with addition and subtraction, both (or all) vectors/ matrices need to be the same dimensions and you just add the elements in the same position. This is also called the entry-wise sum

$$\begin{array}{ccccc} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} & + & \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} & = & \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \\ (3 \times 1) & + & (3 \times 1) & = & (3 \times 1) \end{array}$$

Simple Addition

- Same thing applies with matrices

$$\begin{array}{ccccc} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & + & \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} & = & \begin{bmatrix} 5 & 7 \\ 9 & 11 \end{bmatrix} \\ (2 \times 2) & + & (2 \times 2) & = & (2 \times 2) \end{array}$$

- In Python you can add a single number to a vector or matrix, but Python will expand the single number to the proper dimensions to add it. This is called vectorization and is incredibly useful. (Usually we will do this when we want to apply a function to every unit in a matrix or vector)

Multiplication

- Multiplication is a little more complicated, but intuitive.
- First, we can multiply a vector or matrix by a scalar, and we get a vector or matrix of the same dimensionality.

$$
\begin{array}{ccccc}
4 & \times & \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} & = & \begin{bmatrix} 12 \\ 20 \\ 28 \end{bmatrix} \\
(1 \times 1) & \times & (3 \times 1) & = & (3 \times 1) \\
 & & & & \\
2 & \times & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} & = & \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix} \\
(1 \times 1) & \times & (2 \times 3) & = & (2 \times 3)
\end{array}
$$

Multiplication

- For vector and matrix multiplication, dimensionality becomes very important.
- First, let's just look at vectors.
- You can multiply equal dimension vectors, but for our examples, we are going to multiply row by column vectors to help once we get to matrices. The result is called the *inner product* or the *dot product*. For example:

$$\begin{array}{ccccc} \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} & \times & \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} & = & 56 \\ (1 \times 3) & \times & (3 \times 1) & = & (1 \times 1) \end{array}$$

Multiplication

- To multiply vectors you multiply the corresponding entries of both vectors and sum them. Abstractly this looks like this:

$$z = \sum_{i=1}^{n} v_i \times x_i$$

  Where v and x are the two vectors of the same length
- Another way:

$$z = v_1 \times x_1 \; + \; v_2 \times x_2 \; + \ldots + v_n \times x_n$$

Multiplication

- So returning to our example:

$$\begin{bmatrix} 2 & 4 & 6 \end{bmatrix} \times \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = 56$$

$$(1 \times 3) \times (3 \times 1) = (1 \times 1)$$

$$2 \times 2 + 4 \times 4 + 6 \times 6 = 56$$

Multiplication

- You can also multiply vectors by matrices (don't worry, this is the last step before we get to matrix-matrix multiplication).
- This is where the dimensionality becomes a concern. Remember, the column number of the first vector/ matrix and row number of the second must match because we are multiplying the i^{th} row of the first vector/matrix by each column in the second matrix.
- The result of this multiplication will be a vector with dimensions (1^{st} matrix rows $\times$ 2^{nd} matrix columns)
- This is very easy to remember if you write it out, so let's practice quickly:

Multiplication - Dimensionality Practice

- What is the dimensionality of the product of these two vectors/ matrices?
- $({\color{red}1} \times 4) \;\times\; (4 \times {\color{red}3})$
    - Solution: (1×3)
- $(1 \times 15) \;\times\; (15 \times 12)$
    - Solution: (1×12)
- $(1 \times 3) \;\times\; (4 \times 6)$
    - Solution: Undefined
    - Remember the inner dimensions must match, or you cannot multiply them. This rule holds for both vector-matrix and matrix-matrix multiplication.
- $(3 \times 3) \;\times\; (3 \times 6)$
    - Solution: (3×6)
- $(4 \times 6) \;\times\; (6 \times 2) \;\times\; (2 \times 8)$
    - Solution: (4×8)

Multiplication

- So let's actually multiply some vectors and matrices

$$\begin{bmatrix} 2 & 4 & 6 \end{bmatrix} \quad \times \quad \begin{bmatrix} 2 & 1 \\ 4 & 1 \\ 6 & 1 \end{bmatrix}$$

$$(1 \times 3) \qquad \times \qquad (3 \times 2)$$

- First, what will be the dimensionality of the product?
- Remember the rule for multiplication is every row by every column (following the same procedure as vector multiplication)

$$\begin{aligned} 2 \times 2 \ + \ 4 \times 4 \ + \ 6 \times 6 \ &= \ 56 \\ 2 \times 1 \ + \ 4 \times 1 \ + \ 6 \times 1 \ &= \ 12 \end{aligned}$$

- Solution: $\begin{bmatrix} 56 & 12 \end{bmatrix}$

Multiplication Practice

- Now you try:

$$\begin{bmatrix} 3 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$$

- Solution: $\begin{bmatrix} 13 & 37 \end{bmatrix}$

$$\begin{bmatrix} 1 & 10 \\ 2 & 9 \\ 3 & 8 \end{bmatrix} \times \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

- Solution: $\begin{bmatrix} 54 \\ 53 \\ 52 \end{bmatrix}$

Multiplication Practice

$$\begin{bmatrix} 4 & 7 & 9 \\ 3 & 1 & 5 \\ 2 & 6 & 4 \\ 5 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 5 \\ 8 \\ 2 \\ 4 \end{bmatrix}$$

- Solution: Undefined. You cannot multiply a (4×3) matrix by a (4×1) matrix

Matrix Multiplication

- If you understand vector-matrix multiplication, matrix-matrix multiplication is the exact same. The results are going to be matrices derived the exact same way
- Two important properties of matrices:
- Matrices are *non-commutative*, meaning $\boldsymbol{AB} \neq \boldsymbol{BA}$.
- Matrices are *associative*, meaning $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$

Matrix Multiplication Practice

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

- Step-by-step:

$$\begin{aligned} W_{1,1} &= 1 \times 1 + 2 \times 2 &= 5 \\ W_{1,2} &= 1 \times 3 + 2 \times 4 &= 11 \\ W_{2,1} &= 3 \times 1 + 4 \times 2 &= 11 \\ W_{2,2} &= 3 \times 3 + 4 \times 4 &= 25 \end{aligned}$$

- Solution:

$$\begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$$

Matrix Multiplication Practice

- Solve these equations

$$\begin{bmatrix} -2 & 3 \\ 5 & -1 \end{bmatrix} \times \begin{bmatrix} 4 & 10 \\ 6 & 7 \end{bmatrix}$$

- Solution

$$\begin{bmatrix} 10 & 1 \\ 14 & 43 \end{bmatrix}$$

Matrix Multiplication Practice

$$\begin{bmatrix} 5 & 3 \\ -4 & 8 \\ 0 & 9 \end{bmatrix} \times \begin{bmatrix} 0 & -5 & 5 \\ 5 & 9 & -7 \end{bmatrix}$$

- Solution:

$$\begin{bmatrix} 15 & 2 & 4 \\ 40 & 92 & -76 \\ 45 & 81 & -63 \end{bmatrix}$$

Transpose

- *Transpose*: This flips a matrix over its diagonal, switching the row and column indices. This is usually written $\boldsymbol{A}^\intercal$ or $\boldsymbol{A}'$.
  - For example:

$$\begin{aligned} \boldsymbol{A} &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \\ \boldsymbol{A}^\intercal &= \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \end{aligned}$$

- Notice: The diagonal elements are the same. The dimensions of the matrices are reversed.
- Now let's compare the coordinates of the number 4. In $\boldsymbol{A}$, 4 is in the (2,1) position. But in $\boldsymbol{A}^\intercal$, it is in the (1,2) position.

Transpose and the Identity Matrix

- Here are few more useful things to know about transposes:
- $(\boldsymbol{A}^\intercal)^\intercal = \boldsymbol{A}$
- When $\boldsymbol{A} = \boldsymbol{A}^\intercal$, this is called a symmetric matrix.
- Multiplying a matrix by its transpose (written $\boldsymbol{A}'\boldsymbol{A}$) will give you a symmetric matrix.

$$\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} \times \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 40 & 42 \\ 42 & 40 \end{bmatrix}$$

- For more on the properties of transposes, see `https://en.wikipedia.org/wiki/Transpose`

Identity Matrix

- One of the most common symmetric matrices is called the *identity* matrix.
- An identity matrix is the equivalent of the number 1 for matrices. Any matrix $\boldsymbol{A}$ times the identity matrix is the same ($\boldsymbol{A}$). It looks like this:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Thus:

$$\begin{bmatrix} 4 & 9 \\ 1 & 4 \\ 6 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 1 & 4 \\ 6 & 2 \end{bmatrix}$$

Singularity

- *Singularity* is a circumstance where one or more columns in a matrix can perfectly predict another column in the same matrix.
- This can sometimes be a problem in linear regression and makes the system of equations unsolvable.
- In practical terms, if you have singularities in your dataset (matrix), Python will return an error and you will have to drop the column that is perfectly correlated to the others.
- Mathematically, this is a problem because you cannot invert a singular matrix. But what does that mean?

Inverse

- The *inverse* of a matrix is the same as the idea of an inverse for regular numbers. Any number times its inverse is 1. A matrix times its inverse is the identity matrix: $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$
- Non-square matrices do not have an inverse.
- There are multiple ways to find the inverse of a matrix including Cholesky Decomposition, Eigendecomposition, and the Cayley-Hamilton method.
- For more information on invertible matrices, see `https://en.wikipedia.org/wiki/Invertible_matrix#Methods_of_matrix_inversion`

Inverse: A Simple Example

- We can solve an easy example with simple polynomial algebra:

$$\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} \times \begin{bmatrix} x & p \\ y & q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} 2x + 3y &= 1 \\ 2p + 3q &= 0 \\ 5x + 8y &= 0 \\ 5p + 8q &= 1 \end{aligned}$$

$$\begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$$