Solutions to Problem Set II

(October 16-17, 2014)

1. (a) **False.** Cost of capacity should be allocated only when the capacity constraint is binding.

   (b) **True.** Suppose there are two periods. The monopolist will always set the monopoly price in the last period. Using the backward induction, the price in the first period cannot be lower than that in the second period, otherwise everyone would buy in the first period already. So, the price in the first period must be the same or higher. If some consumer is impatient he will buy in the first period at the higher price, everyone else buys in the last period.

   (c) **True.** If the buyer is time inconsistent and naive. Or if the consumers are primitive robots that make decisions /act without taking into consideration the strategy of the seller.

   (d) **False.** Think about revenue equivalence theorem.

   (e) **False.** As the number of bidders increases, the auction becomes competitive, so very little surplus is left to the buyers. The revenue to the seller is increasing a the number of bidders go up.

2. (a) The plotting will be made in class. Season w is the peak season, since the demand curve is higher then.

   (b) In this problem the monopolist has to set the same price for the both seasons. If capacity binds in the high season w, we will have $q^w = k$, the common price for both seasons is $p^{w,s} = 10 - q^w = 10 - k$. Then $q^s = \frac{10-p}{3} = \frac{10-10+k}{3} = \frac{k}{3}$. Then the profit maximization will be:

$$\max_k \pi(k) = k(10-k) + \tfrac{k}{3}(10-k) - fk$$

   From the first order condition: $10 - 2k + \frac{10}{3} - \frac{2k}{3} - f = 0 \implies k^* = \frac{40-3f}{8}$. And the optimal price will be $p^{ws} = 10 - k^* = \frac{40+3f}{8}$.

   (c) So, if the capacity constraint is binding only in season w the profit in this season is $p^w \cdot q^w = \frac{40+3f}{8} \cdot \frac{40-3f}{8} = \frac{40^2-9f^2}{64}$. The profit in season s is $\frac{50}{6}$ (the producer gets the normal monopoly profit, so $q^s = \frac{10}{6}$ and $p^s = 5$). So, the total profit is: $\pi_{wbind} = \frac{40^2-9f^2}{64} + \frac{50}{6}$. Next let's write down the profit of the monopolist in case the capacity constraint is binding in both seasons and different prices are charged in different seasons:

$$\pi(k) = k(10-k) + k(10-3k) - fk$$

   Maximizing this expression with respect to k, we get that $k^{**} = \frac{20-f}{8}$. If we plug this expression back into the profit function we get:

$$\pi_{bothbind}(k) = 20k^{**} - 4k^{**2} - fk^{**} = 20 \cdot \frac{20-f}{8} - 4\left(\frac{20-f}{8}\right)^2 - f \cdot \frac{20-f}{8} = \frac{(20-f)^2}{16}$$

If we equate π_{wbind} and $\pi_{bothbind}$ we can find the threshold value of $\hat{f}$, below which the capacity constraint binds only in high season w.

(d) So, we know that in this case $f < \hat{f}$, the capacity in winter will be $\frac{40-3f}{8}$. The supply at season s, since the capacity does not bind will be $\frac{10}{6}$.
In case $f > \hat{f}$, the capacity constraint is binding in both seasons. Capacity in summer and winter is $k^{**} = \frac{20-f}{8}$, using this we can calculate the supplies.

3. In this exercise we have two types of buyers $i \in \{H, L\}$, that have different valuations of the instant delivery of the books from Amazon.com. s is the actual delivery time and the preferences of consumers are given by:

$$v(\theta^i, s) = \begin{cases} \theta^i(1-s), & \text{if} \quad s_i \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) An example that explains such preferences would be a story of two students taking Advanced Micro course. 3 weeks before the final exam they decide to acquire "Advanced Micro" book from Amazon to get ready for the exam. Student H values the urgent delivery of the book more since he is currently enrolled in the Master's program in Economics and takes this Micro course for credit. H would like to get a nice grade for the course. Student L has in fact already got his master degree in marketing. L believes that learning Micro would help him in his work and he does not care about the grade. Both of the students would actually like to read the book before the exam, but type H would definitely like to start reading ASAP.

(b) The cost of delivery at time s is $c(s) = 2(1-s)^2$ for $0 \leq s \leq 1$, and $c(s) = 0$ for $s > 1$. Let's take the derivative of $c(\cdot)$: $c'(s) = 4s - 4$. So the cost of shipping is decreasing as the time of delivery increases. This makes sense, think about air vs. sea shipping.

(c) If the monopolist can observe the type of the buyer, he can perfectly price discriminate, i.e. set the shipping tariff to capture all the consumer surplus: $\theta^i(1-\hat{s}^i) = \hat{t}^i$. So, for each type $i \in \{H, L\}$ the monopolist will be solving:

$$\max_{\hat{s}^i} \pi(\hat{s}^i) = \hat{t}^i - 2(1-\hat{s}^i)^2 = \theta^i(1-\hat{s}^i) - 2(1-\hat{s}^i)^2$$

From the first order condition we get $\hat{s}^{i*} = 1 - \frac{\theta^i}{4}$ and $\hat{t}^{i*} = \frac{(\theta^i)^2}{4}$

(d) Given the first degree price discrimination solution, it is easy to see from the incentive compatibility constraint of the hight type, that there it does not bind for the tariff-delivery time combination $(\hat{s}^H, \hat{t}^H)$ that is meant for him:

$$\begin{aligned} \theta^H(1-\hat{s}^H) - \hat{t}^H \quad &\overbrace{\geq}^{?} \quad \theta^H(1-\hat{s}^L) - \hat{t}^L && \Longrightarrow \\ \theta^H(1-1-\frac{\theta^H}{4}) - \frac{(\theta^H)^2}{4} \quad &\overbrace{\geq}^{?} \quad \theta^H(1-1-\frac{\theta^L}{4}) - \frac{(\theta^L)^2}{4} && \Longrightarrow \\ -\frac{(\theta^H)^2}{2} \quad &\not\geq \quad -\frac{\theta^H\theta^L}{4} - \frac{(\theta^L)^2}{4} \end{aligned}$$

Now, for concreteness it is assumed that $\theta^H = 3$ and $\theta^L = 1$ and there are λ buyers of the high type. If we plot the idifference curves of the consumers in s, t coordinate space it will be evident that they cross only once, so it is possible to offer the type specific menu of contracts. Let's solve the problem making sure that individual rationality constraint will bind for the low type: $\theta^L(1 - s^L) = t^L \implies (1 - s^L) = t^L$. The incentive compatibility constraint of the hight type must bind, so: $\theta^H(1-s^H)-t^H = \theta^H(1-s^L)-t^L \implies 3(1-s^H)-t^H = 3(1-s^L)-(1-s^L) \implies t^H = 3(1 - s^H) - 2(1 - s^L)$.
So, now the monopolist Amazon solves:

$$\max_{s^H, s^L} \quad \lambda(t^H - c(s^H)) + (1 - \lambda)(t^L - c(s^H)) \quad \implies$$
$$\max_{s^H, s^L} \quad \lambda(3(1 - s^H) - 2(1 - s^L) - 2(1 - s^H)^2) + (1 - \lambda)((1 - s^L) - 2(1 - s^L)^2)$$

From the first order conditions we can solve for s^H and s^L and optimal tariffs. So, $s^H = \frac{3}{4}$ and $s^L = 4 \cdot \frac{\lambda-3}{\lambda-1}$. Solving for the optimal tariffs we get: $t^L = 1 - s^L = \frac{11-3\lambda}{\lambda-1}$ and $t^H = 3(1 - s^H) - 2(1 - s^L) = \frac{3}{4} - \frac{11-3\lambda}{\lambda-1}$.

4. (a) In *all-pay auction* all bidders must pay their own bid b_i and the bidder with highest bid wins the object. Obviously, bidding your own valuation is not optimal because in case you bid your own valuation and win the auction your pay off is zero and if you lose the pay-off will be negative. So, in the case of *all pay auction* underbidding seems like the right thing to do. But how much should you underbid? Let's derive the optimal bidding strategy formally. Ignoring ties, the pay-offs of the bidder i is:

$$u_i(v_i, .., v_k, b_i.., b_k) = \begin{cases} v_i - b_i, & \text{if} \quad b_i > \max_{j \neq i} b_j \\ -b_i, & \text{otherwise} \end{cases}$$

So, the expected pay-off from participation in the *all-pay auction* and bidding b_i for player i is: $\Pr(b_i > \max_{j \neq i} b_j)v_i - b_i$. Let's assume that the bidding strategy of the player i is some increasing function of his valuation - $\gamma(v_i)$ and that other players use similar strategy (i.e. we are after a symmetric equilibrium here). Note that $v_i = \gamma^{-1}(b_i)$.
Keeping in mind that the valuations are private and uniform[1], the probability of posting the winning bid for player i is: $\Pr(b_i > \max_{j \neq i} b_j) = \Pr(\max_{j \neq i} \gamma(v_j) < b_i) = F(\gamma^{-1}(b_i))^{N-1} =$

[1]also if X is random variable and $Y = g(X)$, where g is some increasing function, then $\Pr(Y < a) = F_y(Y) = F_x(g^{-1}(y))$

$(\gamma^{-1}(b_i))^{N-1}$. Writing down and solving the bidder's maximization problem we get:

$$\begin{aligned}
\max_{b_i} \quad & (\gamma^{-1}(b_i))^{N-1}v_i - b_i \\
\text{FOC:} \quad & (N-1)v_i(\gamma^{-1}(b_i))^{N-2}\frac{\mathrm{d}\gamma^{-1}(b_i)}{\mathrm{d}b_i} - 1 = 0 \\
(N-1) \quad & \gamma^{-1}(b_i)(\gamma^{-1}(b_i))^{N-2}\frac{\mathrm{d}\gamma^{-1}(b_i)}{\mathrm{d}b_i} - 1 = 0 \\
(N-1) \quad & (\gamma^{-1}(b_i))^{N-1}\frac{\mathrm{d}\gamma^{-1}(b_i)}{\mathrm{d}b_i} = 1 \qquad | \times \frac{N}{N} \\
\tfrac{(N-1)}{N} \quad & N(\gamma^{-1}(b_i))^{N-1}\frac{\mathrm{d}\gamma^{-1}(b_i)}{\mathrm{d}b_i} = 1
\end{aligned}$$

Noting that $N(\gamma^{-1}(b_i))^{N-1}\frac{\mathrm{d}\gamma^{-1}(b_i)}{\mathrm{d}b_i} = (\gamma^{-1}(b_i)^N)'$ and integrating both sides we get:

$$\frac{(N-1)}{N}(\gamma^{-1}(b_i))^N = b_i + C$$

We know that the bidder with valuation 0 will optimally bid zero, so at the end we arrive at:

$$\frac{(N-1)}{N}(\gamma^{-1}(b_i))^N = b_i + C \qquad \Longleftrightarrow \qquad b_i = \frac{N-1}{N}v_i^N.$$

So, in case of all pay auction the optimal bid is very low compared to the true valuation of the bidder[2].

(b) Here we are asked if bidding the true valuation is still a best response to all possible bids by other bidders if there is a reservation price in such auction.The seller submits (possibly at random) a bid b_0. The auctioned item cannot be sold at a price lower than b_0, so bidders with valuation $v_i < b_0$ will not participate in the auction, since they cannot make any positive profit. For the remaining bidders whose valuations are above the reservation price b_0 the optimal bidding strategy will be the same as in the standard second price auction, i.e. bidding own valuation.

(c) In this question we are trying to find out if bidding ones true valuation is a best response to all possible bids by other bidders in case of the *third price auction*. Suppose that $b(v_k) = v_k$ for all $k \neq i$. That is, all player i's opponents bid their true valuation. Then player i with valuation $v_k < v_i < v_l$ should bid above v_l to win the auction and get a payoff of $v_i - v_k > 0$ instead of 0 which he would get by bidding truthfully. Hence, bidding true valuation is not optimal strategy in this game.

(d) In this case we have a common value option. The pay-off to the buyer i, ignoring ties is:

$$u_i(v_i,..,v_k,b_i,..,b_k) = \begin{cases} \sum_i v_i - b_i, & \text{if} \quad b_i > \max_{j\neq i} b_j \\ 0, & \text{otherwise} \end{cases}$$

First, for simplicity let's assume that there $N = 2$. We will find a symmetric Bayesian Nash equilibrium in such linear strategies i.e., we assume that the strategies are of the form $b_i = 2v_i \ \forall \ i \in \{1,2\}$. So, then the expected pay-off of the bidder i is:

[2] the average bid in this case is $\int_0^1 \gamma(v)\mathrm{d}v = \int_0^1 \frac{N-1}{N}v^N\mathrm{d}v = \frac{N-1}{N}\frac{1}{N+1}v^{n+1}|_0^1 = \frac{N-1}{N(N+1)}$. If N is large it's around $\frac{1}{N}$

$$\begin{aligned} Eu_i(b_i, b_j, v_i, v_j) &= \Pr(b_i > 2v_j)\, E(v_i + v_j - E(2v_j | b_i > 2v_j)) \\ &= \Pr\left(\frac{b_i}{2} > v_j\right)\left[v_i - E\left(v_j \mid \frac{b_i}{2} > v_j\right)\right] \\ &= \frac{b_i}{2}\left[v_i - \frac{b_i}{4}\right]. \end{aligned}$$

Maximizing w.r.t. b_i we have the FOC

$$\begin{aligned} \tfrac{\partial Eu_i(\cdot)}{\partial b_i} &= \tfrac{v_i}{2} - \tfrac{b_i}{4} =_{set} 0 \\ \iff \quad b_i &= 2v_i \end{aligned}$$

In a more general case where the number of bidders is N if we maximize the following expression with respect to b_i, we can verify the optimal bidding strategy for i[3]:

$$\begin{aligned} E[u_i(\cdot)] &= \Pr(b_i > \max_{j\neq i} b_j)(v_i + \sum_{j\neq i} E(v_j | b_i > b_j(v_j)) - \max_{j\neq i} b_j) \\ &= \Pr\left(\max_{j\neq i} v_j < \frac{b_i}{2}\right)\left(v_i + \sum_{j\neq i} E\left(v_j | v_j < \frac{b_i}{2}\right) - E\left(\max_{j\neq i} v_j | \max_{j\neq i} v_j < \frac{b_i}{2}\right)\right) \end{aligned}$$

5. (a) If $p = 0$ we are back to the "vanilla" second price sealed bid auction, where the pay-off to the bidder of type i (ignoring the ties) is:

$$u_i(v_i, v_j, b_i, b_j) = \begin{cases} v_i - b_j, & \text{if} \quad b_i > b_j \\ 0, & \text{otherwise} \end{cases}$$

Keeping in mind that the dominant strategy in the second price auction is to bid your own valuation, the expected pay-off will be[4]:

$$\begin{aligned} E[u(v_i, v_j, b_i, b_j)] &= \Pr(b_i > b_j)(v_i - b_j) = F(v_i)(v_i - E(v_j | v_j < v_i)) = v_i\left(v_i - \frac{1}{v_i}\int_0^{v_i} v_j f(v_j) \mathrm{d}v_j\right) \\ &= v_i\left(v_i - \frac{1}{v_i}\frac{1}{2}\, v_j^2\Big|_0^{v_i}\right) = v_i\left(v_i - \frac{v_i}{2}\right) = \frac{v_i^2}{2} \end{aligned}$$

(b) If $p > 0$ pay-offs to bidder i are:

$$u_i(b_i, b_j, v_i, v_j) = \begin{cases} 0, & \text{if } v_i \leq p \\ v_i - p, & \text{if } v_i > p \text{ and } v_j \leq p \\ v_i - b_j - p, & \text{if } b_i > b_j > p \\ -p, & \text{if } b_j > b_i > p \end{cases}$$

[3] while solving this optimization, note that $\sum_{j\neq i} E(v_j | b_i > b_j(v_j)) = (N-1)\left(\frac{b_i}{2}\right)$ and $E\left(\max_{j\neq i} v_j | \max_{j\neq i} v_j < \frac{b_i}{2}\right)$ is the second highest order statistic for N independent draws from our uniform distribution F

[4] remember that the expected value of the distribution truncated from the top is $E[x|X < y] = \frac{\int_{-\infty}^{y} x g(x) \mathrm{d}x}{F(y)}$

We know already that despite the fact that there is an entrance fee, conditional on entry, bidding one's own valuation ($b_i = v_i$) is the dominant bidding strategy (because p is a sunk cost).
If player i believes that j will only enter iff $v_j \geq v_i$, then i's expected value of participating in such auction will be:

$$\begin{aligned} E[u(v_i, v_j)] &= \Pr(v_j \geq v_j)(-p) + \Pr(v_i \geq v_j)(v_i - p) = (1 - F(v_i))(-p) + F(v_i)(v_i - p) \\ &= (1 - v_i)(-p) + v_i^2 - v_i p = v_i^2 - p \end{aligned}$$

(c) Here we know that player i participates in the auction (and pays p) iff $v_i \geq v(p)$. In other words player i follows a cut-off strategy $v(p)$. Then expected pay-off of player j conditional on bidding is:

$$\begin{aligned} E\left[u_j\left(v_j, v_i\right)\right] &= -p \Pr\left(v_i \geq v_j\right) + \Pr\left(v_i \leq v(p)\right)\left(v_j - p\right) \\ &\quad + \Pr\left(v(p) \leq v_i \leq v_j\right)\left(v_j - E\left[v_i | v(p) \leq v_i \leq v_j\right] - p\right) \end{aligned} \tag{1}$$

If we tackle probabilities and expectation one by one we get:

$$\Pr\left(v_i \geq v_j\right) = 1 - F\left(v_j\right) = 1 - v_j$$

$$\Pr\left(v_i \leq v(p)\right) = F\left(v(p)\right) = v(p)$$

$$\Pr\left(v(p) \leq v_i \leq v_j\right) = F\left(v_j\right) - F\left(v(p)\right) = v_j - v(p)$$

$$E\left[v_i | v(p) \leq v_i \leq v_j\right] = \frac{1}{\Pr(v(p) \leq v_i \leq v_j)} \int_{v(p)}^{v_j} v_i f\left(v_i\right) \mathrm{d}v_i = \frac{(v_j)^2 - (v)^2}{2(v_j - v(p))}$$

We can rewrite 1 as:

$$\begin{aligned} E\left[u_j\left(v_j, v_i\right)\right] &= -p\left(1 - v_j\right) + v(p)\left(v_j - p\right) + \left(v_j - v(p)\right)\left[v_i - \frac{(v_j)^2 - (v(p))^2}{2(v_j - v(p))} - p\right] \\ &= -p\left(1 - v_j\right) - pv(p) + \tfrac{1}{2}v_j^2 + \tfrac{1}{2}(v(p))^2 - (v_j - v(p))p \\ &= -p + \tfrac{1}{2}v_j^2 + \tfrac{1}{2}(v(p))^2 \end{aligned}$$

In a symmetric equilibrium if player j also follows a cut-off strategy, then whenever $v_j = v(p)$, she must be indifferent between bidding and getting $E\left[u_j\left(v_j, v_i\right)\right]$ and not bidding and getting 0, so :

$$\begin{aligned} -p + \tfrac{1}{2}v_j^2 + \tfrac{1}{2}(v(p))^2 &= -p + \tfrac{1}{2}(v(p))^2 + \tfrac{1}{2}(v(p))^2 = -p + (v(p))^2 =_{set} 0 \\ \Leftrightarrow \qquad v(p)^* &= \sqrt{p}. \end{aligned}$$

(d) To compute the expected revenue to a seller and optimal participation fee p first lets assume that the seller values object at 0. First note that the probability $v_i \in [0, \sqrt{p}]$ equals $\sqrt{p}$, thus probability that $v_i \in [\sqrt{p}, 1]$ is $(1 - \sqrt{p})$. So, taking into consideration all possibilities (i.e. both participate or only one participates) the seller gets expected revenue of $2p(1 - \sqrt{p})^2 + 2p(1 - \sqrt{p})\sqrt{p} = 2p(1 - \sqrt{p})$ from participation fees only.
The only case where the seller gets positive revenue from the bids is when both players participate. Now, suppose player i's bid is higher than that of player j, but as it is the second price

auction, what the seller expects to get from the winner is the second biggest bid:

$$\Pr\left(\sqrt{p} \leq v_i \leq 1\right) \Pr\left(\sqrt{p} \leq v_j \leq v_i\right) E\left[v_j \mid \sqrt{p} \leq v_j \leq v_i \leq 1\right]$$

$$= \left(1-\sqrt{p}\right)\left(v_i-\sqrt{p}\right) \int_{\sqrt{p}}^{1} \int_{\sqrt{p}}^{v_i} x f\left(x \mid \sqrt{p} \leq x \leq v_i\right) f\left(v_i \mid \sqrt{p} \leq v_i \leq 1\right) dv_i dx$$

$$= \left(1-\sqrt{p}\right)\left(v_i-\sqrt{p}\right) \int_{\sqrt{p}}^{1} \int_{\sqrt{p}}^{v_i} x \left(\frac{f(x)}{\Pr\left(\sqrt{p} \leq v_j \leq v_i\right)}\right)\left(\frac{f(v_i)}{\Pr\left(\sqrt{p} \leq v_i \leq 1\right)}\right) dv_i dx$$

$$= \int_{\sqrt{p}}^{1} \int_{\sqrt{p}}^{v_i} x f\left(x\right) f\left(v_i\right) dv_i dx$$

$$= \int_{\sqrt{p}}^{1} \frac{(v_i)^2-\left(\sqrt{p}\right)^2}{2} f\left(v_i\right) dv_i$$

$$= \frac{1}{2}\left[\frac{1}{3}\left(v_i\right)^3-\left(\sqrt{p}\right)^2 v_i\right]_{\sqrt{p}}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{3}-\left(\sqrt{p}\right)^2-\frac{1}{3}\left(\sqrt{p}\right)^3+\left(\sqrt{p}\right)^3\right]$$

$$= \frac{1}{6}-\frac{1}{2}\left(\sqrt{p}\right)^2+\frac{1}{3}\left(\sqrt{p}\right)^3$$

$$= \frac{1}{6}-\frac{1}{2}p+\frac{1}{3}p^{\frac{3}{2}}$$

By symmetry we will have a similar term if jth bidder wins. So, seller's expected profit is:

$$\begin{aligned} E[\pi(f)] &= 2f(1-\sqrt{f})+2\left[\frac{1}{6}-\frac{1}{2}f+\frac{1}{3}f^{\frac{3}{2}}\right] \\ &= 2f(1-\sqrt{f})+\frac{1}{3}-f+\frac{2}{3}f^{\frac{3}{2}} \\ &= f-\frac{4}{3}f^{\frac{3}{2}}+\frac{1}{3}. \end{aligned} \tag{2}$$

Taking the first order condition from 2 with respect to f yields:

$$1 = 2(p^*)^{\frac{1}{2}} \quad \Leftrightarrow \quad p^* = \frac{1}{4} \tag{3}$$

It is optimal for the seller to set participation fee $p^* = \frac{1}{4}$. However, there may be an issue of inefficiency if p^* hinders the allocation of the object to the bidder who values it the most (if $v_i, v_j \in [0, p^*)$). Nevertheless, introducing a participation fee is profitable to the seller even though it involves the risk that no sale takes place.