Solutions to Problem Set III

(October 13, 2014)

1. By definition of *strictly convex preferences*, the preference relation $\succeq$ on X is strictly convex if for every x, we have that $y \succeq x$, $z \succeq x$, and $y \neq z$ implies that $\alpha y - (1-\alpha)z \succ x \;\; \forall \alpha \in (0,1)$.
By the definition of *convex preferences*, the preference relation $\succeq$ on X is convex if for every $x \in X$, the upper contour set $\{y \in X : y \succeq x\}$ is convex; that is if $y \succeq x$, $z \succeq x$, and $y \neq z$ then $\alpha y - (1-\alpha)z \succeq x \;\; \forall \alpha \in [0,1]$. Comparing the two definitions it is easy to spot that strictly convex preferences imply that there cannot be linear segments in the consumer's indifference curves.
It is given in the problem set up that x - initial allocation in the economy is already Pareto effi-

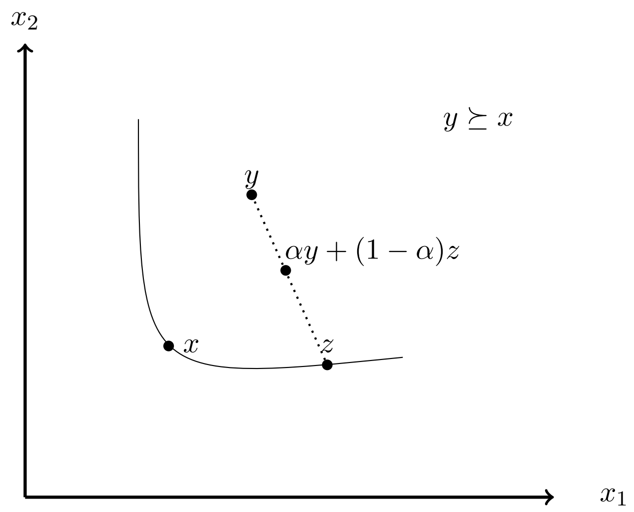


Figure 1: Strictly Convex Preferences

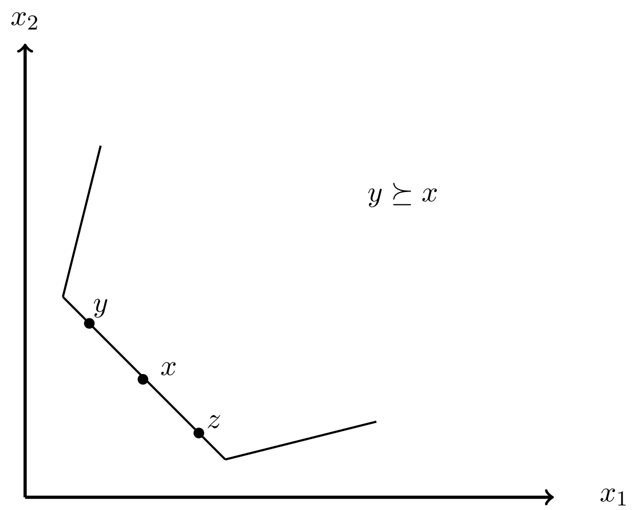


Figure 2: Convex Preferences

cient. Then by the Second Fundamental Theorem of Welfare Economics, it constitutes the Walrasian

equilibrium. Since the preferences of all consumers are strictly convex, and we argued that their indifference curves cannot have linear segments, i.e. they are *"strictly curves"*, it is obvious that they are tangents only at one point, i.e. the Walrasian equilibrium is unique.
Another approach to this problem will be to assume that there is another Walrasian equilibrium $\tilde{x}$. Since x and $\tilde{x}$ are Pareto efficient it must be that $x \sim \tilde{x}$. If x and $\tilde{x}$ were distinct, a convex combination of the two bundles would be feasible and strictly preferred by every agent. This contradicts the assumption that x is Pareto efficient.

2. Equal division allocation is Pareto efficient by definition in this case. The consumers have identical preferences, so there is no way to make someone strictly better off without making someone else strictly worse off. This can be shown more formally.
Define: $\hat{x}_k = \frac{1}{n}\sum_{i=1}^{n} \omega_k^i$, where $\hat{x}_k$ is an element of a k-dimensional vector $\hat{\mathbf{x}}$. The equal division is an allocation where $\mathbf{x^i} = \hat{\mathbf{x}}$. Since all consumers have identical preferences the equal division allocation gives: $u^i(\mathbf{x}^i){=}u^j(\mathbf{x}^j){=}u(\hat{\mathbf{x}})$, where $u(\cdot)$ is a common utility function. Suppose $(\tilde{\mathbf{x}}^1, ..., \tilde{\mathbf{x}}^n)$ is some feasible allocation such that $u(\tilde{\mathbf{x}}^i){\geq}u(\hat{\mathbf{x}})\ \ \forall i$, and $u(\tilde{\mathbf{x}}^i){>}u(\hat{\mathbf{x}})$ for at least one i. This implies that:

$$\frac{1}{n}\sum_{i=1}^{n} u(\tilde{\mathbf{x}}^i) > u(\hat{\mathbf{x}})$$

As $u(\cdot)$ is strictly concave:

$$u\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{\mathbf{x}}^i\right) > \frac{1}{n}\sum_{i=1}^{n} u(\tilde{\mathbf{x}}^i) \quad \Longrightarrow \quad u\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{\mathbf{x}}^i\right) > u(\hat{\mathbf{x}}) \tag{1}$$

Since the allocation $(\tilde{\mathbf{x}}^1, ..., \tilde{\mathbf{x}}^n)$ is feasible: $\frac{1}{n}\sum_{i=1}^{n}\tilde{\mathbf{x}}^i = \frac{1}{n}\sum_{i=1}^{n}\tilde{\omega}^i$, which implies that:

$$\frac{1}{n}\sum_{i=1}^{n}\tilde{\mathbf{x}}^i = \hat{\mathbf{x}}$$

Hence (1) implies that $u(\hat{\mathbf{x}}) > u(\hat{\mathbf{x}})$, which leads us to a contradiction. So, there cannot be a feasible allocation $(\tilde{\mathbf{x}}^1, ..., \tilde{\mathbf{x}}^n)$, such that $u(\tilde{\mathbf{x}}^i){\geq}u(\hat{\mathbf{x}})\ \ \forall i$, and $u(\tilde{\mathbf{x}}^i){>}u(\hat{\mathbf{x}})$ for at least one i. So, the equal division allocation is indeed Pareto optimal if all consumers have identical, strictly concave utility functions.

3. In this problem we are looking at the Robinson Crusoe economy, so one firm and one consumer, lets call him Robinson, is carrying out all production and all consumption in this setting. His utility is Cobb-Douglas: $u(x, \ell) = x\ell$, where x is the amount of consumption good and ℓ is the amount of leisure. Robinson's production function is $x = y^\beta$, where y is the amount of labor input and $\beta \in (0, 1)$. Finally the resource constraint on the amount of leisure and labor is $\ell + y = 1$. The output price is p and the input (labor) price w. I also assume that Robinson does not have any other endowment (e.g. does not have any x).

(a) Robinson's consumer maximization problem is:

$$\max_{x,\ell} u(x,\ell) = \max_{x,\ell} \; x\ell$$

$$\text{s.t.} \quad px + w\ell = w + \pi(w,p)$$

Firm's problem is:

$$\max_{y} \pi(y,p,w) = \max_{y} \; py^{\beta} - wy$$

(b) To solve the consumer's problem let's solve for x from the constraint and plug x into the objective function. We get:

$$\max_{\ell} u(\ell) = \max_{x,\ell} \; \left(\frac{w + \pi(w,p) - w\ell}{p}\right)\ell$$

The first order condition for *the consumer's problem* is:

$$-\frac{w}{p}\ell + \frac{w + \pi(w,p) - w\ell}{p} = 0 \quad \Longrightarrow \quad w\ell + w + \pi(w,p) - w\ell = 0 \quad \Longrightarrow$$
$$\ell = \frac{w + \pi(w,p)}{2w}$$

Now we will plug in the expression for ℓ into the constraint and solve for x:

$$x = \frac{w + \pi(w,p) - w(\frac{w+\pi(w,p)}{2w})}{p} \quad \Longrightarrow \quad x = \frac{w + \pi(w,p)}{2p}$$

From the first order condition for *the firms problem* we can solve for conditional factor demand for y:

$$\beta p y^{\beta-1} - w = 0 \quad \Longrightarrow \quad y = \left(\frac{w}{\beta p}\right)^{\frac{1}{\beta-1}} = \left(\frac{\beta p}{w}\right)^{\frac{1}{1-\beta}}$$

Consequently the firms profit is[1]:

$$\begin{aligned}\pi(p,w) &= p\left(\frac{\beta p}{w}\right)^{\frac{\beta}{1-\beta}} - w\left(\frac{\beta p}{w}\right)^{\frac{1}{1-\beta}} = \left(\frac{\beta p}{w}\right)^{\frac{1}{1-\beta}}\left[p\left(\frac{\beta p}{w}\right)^{-1} - w\right] \\ &= \left(\frac{\beta p}{w}\right)^{\frac{1}{1-\beta}}\left[\left(\frac{w}{\beta}\right) - w\right] = \frac{1-\beta}{\beta}w\left(\frac{\beta p}{w}\right)^{\frac{1}{1-\beta}}\end{aligned}$$

The production function will be:

$$x = y^{\beta} = \left(\frac{\beta p}{w}\right)^{\frac{\beta}{1-\beta}} \tag{2}$$

(c) In this part of the exercise we need to solve the Walrasian equilibrium prices p^* and w^*, that is find such p^* and w^* that will clear the markets. As the aggregate excess demand is homogeneous

[1] note that $p\left(\frac{\beta p}{w}\right)^{\frac{\beta}{1-\beta}} = \left((\beta p^{\frac{1-\beta}{\beta}} p)/w\right)^{\frac{\beta}{1-\beta}} = \left(\frac{\beta p^{\frac{1}{\beta}}}{w}\right)^{\frac{\beta}{1-\beta}}$

of degree zero, and the equilibrium prices must be positive, without loss of generality we can set the price of the consumption good x - p^* equal to one: $p^* = 1$. Now plugging in the expression for the profit $\pi(w,p)$ into expression for ℓ and keeping in mind that market for leisure should clear, i.e. $\ell + y = 1$, we can solve for w^*:

$$\begin{aligned}
\frac{w+\pi(w,p)}{2w} + \left(\frac{\beta p}{w}\right)^{\frac{1}{1-\beta}} &= 1 \quad \Longrightarrow \quad \frac{w + \frac{1-\beta}{\beta} w \left(\frac{\beta p}{w}\right)^{\frac{1}{1-\beta}}}{2w} + \left(\frac{\beta p}{w}\right)^{\frac{1}{1-\beta}} = 1 \\
\frac{1 + \frac{1-\beta}{\beta}\left(\frac{\beta p}{w}\right)^{\frac{1}{1-\beta}}}{2} + \left(\frac{\beta p}{w}\right)^{\frac{1}{1-\beta}} &= 1 \quad \Longrightarrow \quad \left(\frac{\beta p}{w}\right)^{\frac{1}{1-\beta}} \left(\frac{1-\beta}{\beta} + 2\right) = 1 \\
w^* &= \left(\frac{\beta}{1+\beta}\right)^{\beta-1} \beta = \frac{\beta^\beta}{(1+\beta)^{\beta-1}} > 0
\end{aligned}$$

(d) To calculate the equilibrium utility, we first need to calculate x and ℓ given the equilibrium prices. So:

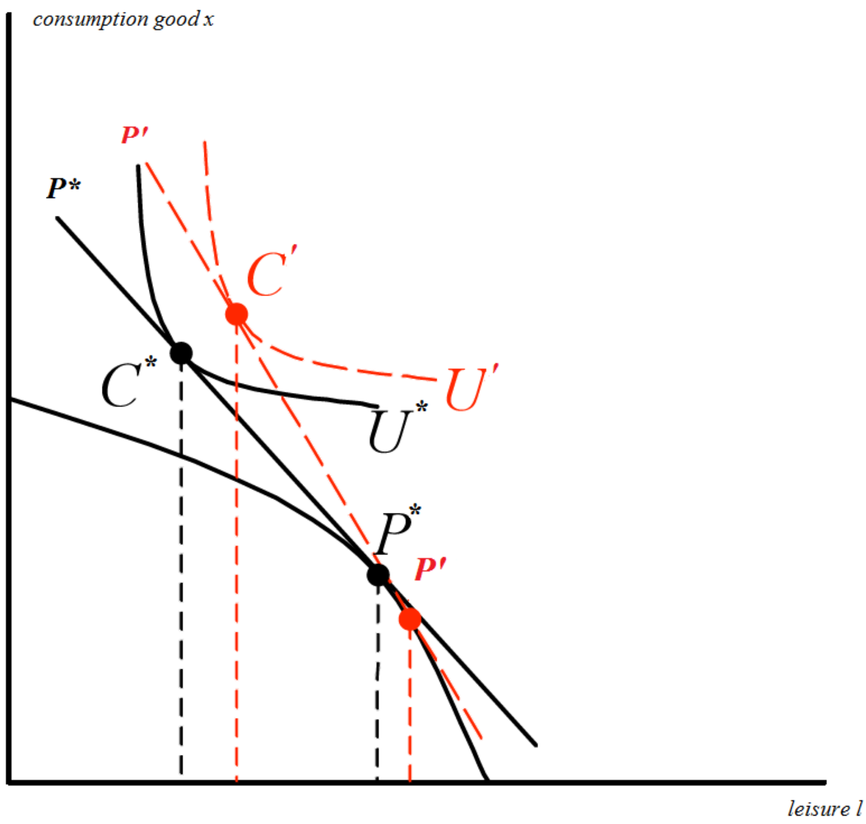
$$\begin{aligned}
x &= \left(\frac{\beta p}{w}\right)^{\frac{\beta}{1-\beta}} = \left(\frac{\beta \cdot 1}{\beta^\beta/(1+\beta)^{\beta-1}}\right)^{\frac{\beta}{1-\beta}} = \left(\frac{\beta(1+\beta)^{\beta-1}}{\beta^\beta}\right)^{\frac{\beta}{1-\beta}} = \left(\frac{\beta}{1+\beta}\right)^{\beta} \\
\ell &= \frac{w+\pi(w,p)}{2w} = \frac{1 + \frac{1-\beta}{\beta}\left(\frac{\beta p}{w}\right)^{\frac{1}{1-\beta}}}{2} = \frac{1 + \frac{1-\beta}{\beta}\left[\left(\frac{1+\beta}{\beta}\right)^{\beta}\right]^{\frac{1}{1-\beta}}}{2} = \frac{1}{2}\left(1 + \frac{1-\beta}{\beta}\left(\frac{1+\beta}{\beta}\right)^{\frac{\beta}{1-\beta}}\right)
\end{aligned}$$

So, the utility is:

$$u(x,\ell) = x\ell = \frac{1}{2}\frac{\beta^\beta}{(1+\beta)^{\beta-1}}\left(1 + \frac{1-\beta}{\beta}\left(\frac{1+\beta}{\beta}\right)^{\frac{\beta}{1-\beta}}\right)$$

If you take derivative of this expression with respect to β, you will see that $u'(\cdot)$ is increasing with β. It makes sense intuitively since higher β implies that the production possibility frontier is higher.

4. If Robinson participates in the larger market, the production possibility frontier of his firm remains the same, but the relative price is given to Robinson exogenously. So, the equilibrium prices in the economy are defined by the slope of the new iso-profit line, where it is tangent to both his production possibility frontier and Robinson's indifference curve. Suppose the new price ratio is $\frac{p'}{w'}$. The new price ratio can coincide with the previous price ratio i.e. $\frac{p'}{w'} = \frac{p^*}{w^*}$, then Robinson gets as much utility as in exercise 3. The new price ratio can be different, i.e. $\frac{p'}{w'} \lessgtr \frac{p^*}{w^*}$. In this case there is an opportunity for arbitrage, since Robinson can re-optimize his choices (for example, given the new exogenous prices if consumption good is more expensive relative to leisure Robinson can choose to work more and sell the extra consumption good he produces on the market) and the new indifference curve tangent to the iso-profit line will be higher.
In the figure, the relative price of leisure ℓ rises from $P^* = \frac{p^*}{w^*}$ to $P' = \frac{p'}{w'}$, causing production to move to the right, from P^* to P'. With the indifference curves shown, consumption also moves to the right, from C^* to C'.



5. (This exercise and the solution are possibly subject to change). In this exercise $X = \{x_1, x_2, x_3\}$ is the set of pure outcomes where vNM preferences $\succeq$ satisfy $x_1 \succ x_2 \succ x_3$ and a lottery 1_k offers x_k with certainty. We have to show that if $1_2 \sim \alpha \cdot 1_1 + (1-\alpha)1_3$ for some α, then $\alpha \in (0,1)$.

   Following the proof in Kreps' *Notes on the Theory of Choice*:

   Lets define: $\alpha^* = \sup\{\alpha \in (0,1) : 1_2 \succeq \alpha \cdot 1_1 + (1-\alpha)\cdot 1_3\}$. By the definition of α^*, if $1 \geq \alpha > \alpha^*$, then $\alpha \cdot 1_1 + (1-\alpha)\cdot 1_3 \succ 1_2$. If $0 \leq \alpha < \alpha^*$, then $1_2 \succ \alpha \cdot 1_1 + (1-\alpha)\cdot 1_3$. To see this note that if $0 \leq \alpha < \alpha^*$ there exists some α', s.t. $0 \leq \alpha < \alpha' \leq \alpha^*$ and $1_2 \succeq \alpha' \cdot 1_1 + (1-\alpha')\cdot 1_3$ by definition of α^*. Since $\alpha < \alpha'$, this implies that $1_2 \succeq \alpha' \cdot 1_1 + (1-\alpha')\cdot 1_3 \succ \alpha \cdot 1_1 + (1-\alpha)\cdot 1_3$. Let's look at three possible cases:

   First, suppose that $\alpha^* \cdot 1_1 + (1-\alpha^*)\cdot 1_3 \succ 1_2 \succ 1_3$, then by NM1 (Continuity), there exists some $\beta \in (0,1)$, s.t. $\beta(\alpha^* \cdot 1_1 + (1-\alpha^*)\cdot 1_3) + (1-\beta)\cdot 1_3 = \beta\alpha^* \cdot 1_1 + (1-\beta\alpha^*)\cdot 1_3 \succ 1_2$. But $\beta\alpha* < \alpha*$, so by previous argument $1_2 \succ \beta\alpha^* \cdot 1_1 + (1-\beta\alpha^*)\cdot 1_3$. We've arrived at contradiction. Next, suppose that $1_1 \succ 1_2 \succ \alpha^* \cdot 1_1 + (1-\alpha^*)\cdot 1_3$, then again by NM1, there exists some $\beta \in (0,1)$, s.t. $1_2 \succ \beta(\alpha^* \cdot 1_1 + (1-\alpha^*)\cdot 1_3) + (1-\beta)\cdot 1_1 = (1-\beta(1-\alpha^*))\cdot 1_1 + (\beta(1-\alpha^*))\cdot 1_3$. As $1-\beta-\beta\alpha^* > \alpha* \iff 1 > \alpha^*$, we have by previous argument $(1-\beta(1-\alpha^*))\cdot 1_1 + (\beta(1-\alpha^*))\cdot 1_3 \succ 1_2$. We've arrived at contradiction.

   The above arguments leave the only possibility: $1_2 \sim \alpha^* \cdot 1_1 + (1-\alpha^*)\cdot 1_3$

   Alternative proof:

   We are given that: $1_2 \sim \alpha \cdot 1_1 + (1-\alpha)\cdot 1_3$ for some α.

   By continuity: $\forall\, \beta \in (0,1) : \beta \cdot x_1 + (1-\beta)\cdot x_3 \succ x_2$ or $\beta 1_1 + (1-\beta)\cdot 1_3 \succ 1_2$. We can rewrite this as $\beta \cdot 1_1 + (1-\beta)\cdot 1_3 \succ \alpha \cdot 1_1 + (1-\alpha)\cdot 1_3$.

Suppose $\alpha \geq 1$:
By independence, if we add to both sides another lottery $x = -\beta \cdot 1_1 - (1-\alpha) \cdot 1_3$ we get:

$$(1 - \beta - 1 + \alpha) \cdot 1_3 \succ (\alpha - \beta) \cdot 1_1 \quad \Longrightarrow 1_3 \succ 1_1$$

Which is a contradiction, as we know that $1_1 \succ 1_3$.
By continuity: $\forall\, \gamma \in (0,1) : x_2 \succ \gamma \cdot x_1 + (1-\gamma) \cdot x_3$ or $1_2 \succ \gamma 1_1 + (1-\gamma) \cdot 1_3$. We can rewrite this as $\alpha \cdot 1_1 + (1-\alpha) \cdot 1_3 \succ \gamma \cdot 1_1 + (1-\gamma) \cdot 1_3$.
Suppose $\alpha \leq 0$:
By independence, if we add to both sides another lottery $y = -\gamma \cdot 1_1 - (1-\alpha) \cdot 1_3$ we get:

$$(1 - \alpha - 1 + \gamma) \cdot 1_3 \succ (\gamma - \alpha) \cdot 1_1 \quad \Longrightarrow 1_3 \succ 1_1$$

Which is a contradiction, as we know that $1_1 \succ 1_3$. So it must be that $\alpha \in (0,1)$

6. Let's assume that the initial wealth of the consumer is w. Also, the insurance where the marginal cost of insurance is equal to the probability of accident is called *actuarially fair*. Then the expected utility maximization problem will look like:

$$\max_x (1-\beta)u(w - \beta x) + \beta u(w - L + x - \beta x) \quad \text{s.t.} \quad w - \beta x \geq 0$$

The first order condition for the optimum are:

$$\begin{aligned} -\beta(1-\beta)u'(w - \beta x^*)) \quad &+ \quad (1-\beta)\beta u'(w - L + x - \beta x^*) = 0, \quad \text{if} \quad x^* > 0 \quad \Longrightarrow \\ u'(w - \beta x^*)) \quad &= \quad u'(w - L + (1-\beta)x^*) \end{aligned}$$

Since the agent is risk averse, i.e. his Bernoulli utility function $u(\cdot)$ is strictly concave, it must be the case that $u'(\cdot)$ is strictly decreasing, it is thus injection, that is $u'(a) = u'(b) \implies a = b$, so it must be:

$$w - \beta x^* = w - L + (1-\beta)x^*$$

So, at the optimum the coverage is $x^* = L$. This implies that if the insurance is actuarially fair, the decision maker insures completely. The individual's final wealth is then $w - \beta L$ regardless if the loss has occurred.

7. By definition of risk aversion (see Jehle&Reny book page 105), the agent is risk averse at g if:

$$u(E[g]) > E[u(g)] \quad \Longleftrightarrow \quad u\left(\sum_{i=1}^{n} p_i x_i\right) > \sum_{i=1}^{n} p_i u(x_i)$$

where $g = (p_1 \circ x_1, ..., p_n \circ x_n)$ - a simple gamble. Following the inequality called **Jensen's inequality** (see Jehle&Reny book page 109), for any strictly concave function $u(\cdot)$:

$$E[u(x)] < u(E[x]) \quad \Longleftrightarrow \quad \sum_{i=1}^{n} p_i u(x_i) < u\left(\sum_{i=1}^{n} p_i x_i\right)$$

So, by Jensen's inequality the consumers Bernoulli utility function $u(\cdot)$ must be strictly concave.

8. Assume that the agent always rejects a gamble where he wins W with probability 1/2 and loses L with probability 1/2 where $W > L$ (i.e. rejects under all initial wealth levels). Then, for all wealth levels x,

$$\frac{1}{2}u(x+W) + \frac{1}{2}u(x-L) \leq u(x).$$

Equivalently, for all wealth levels x,

$$u(x+W) - u(x) \leq u(x) - u(x-L)$$

or

$$\frac{u(x+W) - u(x)}{W} \leq \frac{L}{W} \cdot \frac{u(x) - u(x-L)}{L}.$$

Thus the marginal utility u' drops by at least W/L in every additional $W+L$. The first $W+L$ generates utility $u(W+L)$. By summing over maximal utility additions from a sequence of wealth additions of size $W+L$, the utility after a wealth addition of size $T(W+L)$ is bounded by

$$u(T(W+L)) \leq \sum_{t=0}^{T} u(W+L)\left(\frac{L}{W}\right)^t$$

Since $L < W$, this number converges to

$$\begin{aligned} \lim_{T\to\infty} u(T(W+L)) &\leq u(W+L)\sum_{t=0}^{\infty}\left(\frac{L}{W}\right)^t \\ &= \frac{u(W+L)}{1-\frac{L}{W}} \\ &= \frac{u(W+L)W}{W-L} \end{aligned}$$

Hence the utility is bounded.