Solutions to Problem Set II

(September 29)

1. *Elasticity* of a factor k at x is:
$$\varepsilon_k = \frac{\partial f(\mathbf{x})}{\partial x_k}\frac{x_k}{f(\mathbf{x})}$$

   (a) *Elasticity of a factor* k gives the percentage change in output $f(\mathbf{x})$ per 1 percentage change in the amount of input of factor k (assuming the amount of all other inputs are constant).

   (b) In case of Cobb-Douglas production function: $f(x_1, x_2) = \beta x_1^{\alpha} x_2^{1-\alpha}$, where $\beta > 0, 0 < \alpha < 1$, the output elasticities are:
$$\begin{aligned}\varepsilon_1 &= \frac{\partial f(\mathbf{x})}{\partial x_1}\frac{x_1}{f(\mathbf{x})} = \alpha\beta x_1^{\alpha-1}x_2^{1-\alpha}\frac{x_1}{\beta x_1^{\alpha}x_2^{1-\alpha}} = \alpha \\ \varepsilon_2 &= \frac{\partial f(\mathbf{x})}{\partial x_2}\frac{x_2}{f(\mathbf{x})} = (1-\alpha)\beta x_1^{\alpha}x_2^{-\alpha}\frac{x_2}{\beta x_1^{\alpha}x_2^{1-\alpha}} = 1-\alpha\end{aligned}$$

   (c) In case of CES production function: $f(x_1, x_2) = (x_1^{\rho} + x_2^{\rho})^{\frac{1}{\rho}}$, where $\rho > 0$, the output elasticities are:
$$\begin{aligned}\varepsilon_1 &= \frac{\partial f(\mathbf{x})}{\partial x_1}\frac{x_1}{f(\mathbf{x})} = (x_1^{\rho}+x_2^{\rho})^{\frac{1-\rho}{\rho}}x_1^{\rho-1}\frac{x_1}{(x_1^{\rho}+x_2^{\rho})^{\frac{1}{\rho}}} = \frac{x_1^{\rho}}{(x_1^{\rho}+x_2^{\rho})} \\ \varepsilon_2 &= \frac{\partial f(\mathbf{x})}{\partial x_2}\frac{x_2}{f(\mathbf{x})}(x_1^{\rho}+x_2^{\rho})^{\frac{1-\rho}{\rho}}x_2^{\rho-1}\frac{x_2}{(x_1^{\rho}+x_2^{\rho})^{\frac{1}{\rho}}} = \frac{x_2^{\rho}}{(x_1^{\rho}+x_2^{\rho})}\end{aligned}$$

   NB:
$$(x_1^{\rho}+x_2^{\rho})^{\frac{1-\rho}{\rho}}x_1^{\rho-1}\frac{x_1}{(x_1^{\rho}+x_2^{\rho})^{\frac{1}{\rho}}} = \frac{\left((x_1^{\rho}+x_2^{\rho})^{\frac{1}{\rho}}\right)^{(1-\rho)}}{(x_1^{\rho}+x_2^{\rho})^{\frac{1}{\rho}}}x_1^{\rho} = (x_1^{\rho}+x_2^{\rho})^{-\frac{\rho}{\rho}}x_1^{\rho} = \frac{x_1^{\rho}}{(x_1^{\rho}+x_2^{\rho})}$$

2. First we have to show that marginal rate of technical substitution (MRTS) of a homothetic f at x is the same as at tx for any $t > 0$. By definition of MRTS:
$$MRTS_{ij}(\mathbf{x}) = \frac{\partial f(\mathbf{x})/\partial x_i}{\partial f(\mathbf{x})/\partial x_j}$$
Again, we need to prove that slopes of isoquants along rays from the origin are constant or:
$$\frac{\partial f(t\mathbf{x})/\partial x_i}{\partial f(t\mathbf{x})/\partial x_j} = \frac{\partial f(\mathbf{x})/\partial x_i}{\partial f(\mathbf{x})/\partial x_j}$$
By definition of homotheticity and using chain and constant factor rule:
$$\frac{\partial f(t\mathbf{x})/\partial x_i}{\partial f(t\mathbf{x})/\partial x_j} = \frac{\partial g(h(t\mathbf{x}))/\partial x_i}{\partial g(h(t\mathbf{x}))/\partial x_j} = \frac{g'(h(t\mathbf{x}))\frac{\partial h(t\mathbf{x})}{\partial x_i}}{g'(h(t\mathbf{x}))\frac{\partial h(t\mathbf{x})}{\partial x_j}} = \frac{\frac{\partial h(t\mathbf{x})}{\partial x_i}}{\frac{\partial h(t\mathbf{x})}{\partial x_j}} = \frac{t\frac{\partial h(\mathbf{x})}{\partial x_i}}{t\frac{\partial h(\mathbf{x})}{\partial x_j}} = \frac{\frac{\partial h(\mathbf{x})}{\partial x_i}}{\frac{\partial h(\mathbf{x})}{\partial x_j}}$$

If t=1, then:

$$\frac{\partial f(\mathbf{x})/\partial x_i}{\partial f(\mathbf{x})/\partial x_j} = \frac{\partial f(\mathbf{x})/\partial x_i}{\partial f(\mathbf{x})/\partial x_j} = \frac{\partial h(\mathbf{x})/\partial x_i}{\partial h(\mathbf{x})/\partial x_j}$$

Next, we have to show that the Cobb-Douglas and CES production functions are homothetic. By definition of homotheticity it must be that the production function $f(\cdot) = g(h(\cdot))$ has the following properties: $h(\cdot)$ is homogeneous and $g(\cdot)$ is monotonically increasing.
Let $h(x) = \beta x_1^{\alpha} x_2^{1-\alpha}$. Then $h(tx) = \beta(tx_1)^{\alpha}(tx_2)^{1-\alpha} = \beta t^{\alpha+1-\alpha} x_1^{\alpha} x_2^{1-\alpha} = t\beta x_1^{\alpha} x_2^{1-\alpha} = th(x)$. And let $g(y) = y$.

CES: Let $h(x) = (x_1^{\rho} + x_2^{\rho})^{1/\rho}$. Then $h(tx) = ((tx_1)^{\rho} + (tx_2)^{\rho})^{1/\rho} = [t^{\rho}(x_1^{\rho} + x_2^{\rho})]^{1/\rho} = t(x_1^{\rho} + x_2^{\rho})^{1/\rho} = th(x)$. Also let $g(y) = y$.

3. In this exercise we need to prove Hotelling lemma. Profit function $\pi(p, w)$ of the firm is:

$$\pi(p, w) = pf(y(p, w)) - w \cdot y(p, w)$$

where $y(p, w)$ is the optimal production given the output price p and the input price w (f is the profduction function). Hotelling's Lemma:

$$\begin{aligned} \frac{\partial \pi(p, w)}{\partial p} &= f(y(p, w)) \\ \frac{\partial \pi(p, w)}{\partial w_\ell} &= -y_\ell(p, w) \end{aligned}$$

Proof: By FOC, $y(p, w)$ satisfies

$$p\frac{\partial f(y(p, w))}{\partial y_\ell} = w_\ell \text{ for all inputs } \ell$$

Thus (envelopte argument)

$$\begin{aligned} \frac{\partial \pi(p, w)}{\partial p} &= \frac{\partial [pf(y(p, w)) - w \cdot y(p, w)]}{\partial p} \\ &= f(y(p, w)) + \sum_\ell \frac{\partial y_\ell(p, w)}{\partial p}[p\frac{\partial f(y(p, w))}{\partial y_\ell} - w_\ell] \\ &= f(y(p, w)) \end{aligned}$$

and similarly for ∂w_ℓ.

4. (a) The cost minimization problem is:

$$\min_{x_1 \geq 0, x_2 \geq 0} w_1 x_1 + w_2 x_2 \quad \text{s.t.} \quad \beta x_1^{\alpha} x_2^{1-\alpha} \geq y$$

Lagrangean to the cost minimization problem is:

$$\mathcal{L}(\mathbf{x}, \lambda) = w_1 x_1 + w_2 x_2 + \lambda(y - \beta x_1^{\alpha} x_2^{1-\alpha})$$

(b) First, assuming interior solution (i.e. $x_1, x_2 > 0$ at optimum) lets solve for the conditional factor demands:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= w_1 - \lambda\alpha\beta x_1^{\alpha-1}x_2^{1-\alpha} = 0 \quad \Longrightarrow \quad w_1 = \lambda\alpha\beta x_1^{\alpha-1}x_2^{1-\alpha} \\
\frac{\partial \mathcal{L}}{\partial x_2} &= w_2 - \lambda(1-\alpha)\beta x_1^{\alpha}x_2^{-\alpha} = 0 \quad \Longrightarrow w_2 \quad = \lambda(1-\alpha)\beta x_1^{\alpha}x_2^{-\alpha} \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= y - \beta x_1^{\alpha}x_2^{1-\alpha}) = 0 \quad \Longrightarrow \quad y = \beta x_1^{\alpha}x_2^{1-\alpha}
\end{aligned}$$

From the first two FOC's:

$$\begin{aligned}
\frac{w_1}{\alpha x_1^{\alpha-1}x_2^{1-\alpha}} = \frac{w_2}{(1-\alpha)x_1^{\alpha}x_2^{-\alpha}} \quad &\Longrightarrow \quad \alpha x_1^{\alpha-1}x_2^{1-\alpha}w_2 = (1-\alpha)x_1^{\alpha}x_2^{-\alpha}w_1 \quad \Longrightarrow \\
\alpha\frac{x_2}{x_1}w_2 = (1-\alpha)w_1 \quad &\Longrightarrow \quad x_2 = \frac{(1-\alpha)x_1w_1}{\alpha w_2}
\end{aligned}$$

Plugging the expression for x_2 into the third FOC, solve for x_1 and then for x_2:

$$\begin{aligned}
y = \beta x_1^{\alpha}x_2^{1-\alpha} \quad \Longrightarrow \quad y = \beta x_1^{\alpha}\left(\frac{w_1}{w_2}\frac{(1-\alpha)}{\alpha}x_1\right)^{1-\alpha} \quad &\Longrightarrow \quad x_1 = \frac{y}{\beta}\left(\frac{w_1}{w_2}\frac{(1-\alpha)}{\alpha}\right)^{\alpha-1} \\
\therefore x_2 = \frac{w_1}{w_2}\frac{(1-\alpha)}{\alpha}\frac{y}{\beta}\left(\frac{w_1}{w_2}\frac{(1-\alpha)}{\alpha}\right)^{\alpha-1} &= \frac{y}{\beta}\left(\frac{w_1}{w_2}\frac{(1-\alpha)}{\alpha}\right)^{\alpha}
\end{aligned}$$

Now, lets solve for the cost function:

$$\begin{aligned}
c(w_1, w_2, y) = w_1x_1 + w_2x_2 &= w_1\frac{y}{\beta}\left(\frac{w_1}{w_2}\frac{(1-\alpha)}{\alpha}\right)^{\alpha-1} + w_2\frac{y}{\beta}\left(\frac{w_1}{w_2}\frac{(1-\alpha)}{\alpha}\right)^{\alpha} \\
&= \frac{y}{\beta}\left(\frac{w_1}{w_2}\frac{(1-\alpha)}{\alpha}\right)^{\alpha}\left[w_1\left(\frac{w_1}{w_2}\frac{(1-\alpha)}{\alpha}\right)^{-1} + w_2\right] = \frac{y}{\beta}\left(\frac{w_1}{w_2}\frac{(1-\alpha)}{\alpha}\right)^{\alpha}w_2\left(\frac{1}{1-\alpha}\right)
\end{aligned}$$

At the end we arrive at:

$$c(w_1, w_2, y) = \frac{y}{\beta}w_1^{\alpha}w_2^{1-\alpha}(1-\alpha)^{\alpha-1}\alpha^{-\alpha} = \frac{y}{\beta}w_1^{\alpha}w_2^{1-\alpha}K \ \text{ where } \ K = (1-\alpha)^{\alpha-1}\alpha^{-\alpha}$$

(c) It is easy to spot that given production function $f(x_1, x_2) = \beta x_1^{\alpha}x_2^{1-\alpha}$ demonstrates constant returns to scale (CRS). In case the production function is CRS a single optimal production schedule does not exists. If we try to set up the profit maximization problem, we will see that since the cost is linear in output there is not unique solution to the problem :

$$\max_{y\geq 0}\,(p - \beta w_1^{\alpha}w_2^{1-\alpha}K)y$$

In this case any quantity of output is profit maximizing as long as $p = c(w_1, w_2, 1)$. CRS implies that scaling up the input scales up the output and costs by the same amount. So the only possible situation for a firm is to receive zero profits and supply the output as long as the costs are covered.

5. In this exercise we have to consider the partial equilibrium in a single market with identical consumers

and firms. The price on the market is p. Here we will assume that consumer's budget constraint does not bind, so the Marshallian demand can be found by maximizing the utility with respect to q, so:

$$\max_q u(q) = \ln q - pq \quad \Longrightarrow \quad q_d^* = \frac{1}{p} \text{ (note that here } q_d^* \text{ is decreasing in prices)}$$

we can also write down the same expression in the form of inverse demand or: $p(q) = \dfrac{1}{q}$

Note that I also added the subscripts to q to differentiate between quantity demanded and supplied. Solving the firm's problem we get the optimal supply:

$$\max_q \pi(q) = pq - q^2 \quad \Longrightarrow \quad q_s^* = \frac{p}{2} \text{ (note that here } q_s^* \text{ is increasing in prices)}$$

We also can rewrite the expression above and state that the marginal cost in this case is: $mc(q) = 2q$

To compute consumer's indirect utility and maximal profit of a firm we simply plug the expression for q_d^* in the utility function and q_s^* into the profit function:

$$\begin{aligned} \text{Indirect utility:} \qquad v(p) &= \ln q_d^* - pq_d^* = \ln\frac{1}{p} - 1 \\ \text{Maximum profit:} \qquad \pi^*(p) &= pq_s^* - q_s^{*2} = p\,\frac{p}{2} - \frac{p^2}{4} = \frac{p^2}{4} \end{aligned}$$

If we sum indirect utility and maximum profit and we obtain the sum of consumer and producer surplus. Which can also be written down in the form of an integral: $\int_0^q [p(\xi) - mc(\xi)]\mathrm{d}\xi$ (for details see Jehle&Reny book Chapter 4).
Taking the derivative of this sum with respect to price yields:

$$\left(\ln\frac{1}{p} - 1 + \frac{p^2}{4}\right)' = -\frac{1}{p} + \frac{p}{2} = \frac{-2+p^2}{2p}$$

This expression is negative for $p < \sqrt{2}$, meaning that the sum of indirect utility and profit is a decreasing function till p reaches $\sqrt{2}$. Which is the local minimum of this function (the second order condition yields $\frac{1}{p^2} + 1 > 0$). Also note that in equilibrium, when we equate demand and supply: $q_d(p^*) = q_s(p^*)$ or alternatively $mc(q) = p(q)$, so we arrive at $p^* = \sqrt{2}$ - an equilibrium price. The intuition behind this observation is that whenever the marginal cost and price differ (or $q_s \neq q_s$), there is the room for improvement in total surplus.

6. In this exercise we need to consider a VAT in the setting identical to that in previous problem. So, the price consumer pays in store is p, and the price that the firm charges before tax is $p(1-t)$. The amount pt is paid to the government fer each unit of good sold. The consumers problem and its solution remains exactly the same as in the previous problem, so $q_d^* = \frac{1}{p}$. The firms problem, however is slightly different:

$$\max_q \pi(q) = p(1-t)q - q^2 \quad \Longrightarrow \quad q_s^* = \frac{p(1-t)}{2}$$

We already know that in equilibrium $q_d(p^*) = q_s(p^*)$, so solving for the p^* again, we get:

$$\frac{p^*(1-t)}{2} = \frac{1}{p^*} \quad \Longleftrightarrow \quad p^{*2} = \frac{2}{1-t} \quad \Longleftrightarrow \quad p^* = \sqrt{\frac{2}{1-t}}$$

Obviously, new after tax p^* is higher than p^* from the previous exercise: $\sqrt{2} < \sqrt{\frac{2}{1-t}}$, because $t \in (0,1)$. The demand requested in this case is $q_d^* = \frac{1}{p^*} = \frac{1}{\sqrt{2/1-t}} = \frac{1-t}{\sqrt{2}}$, which is lower than the equilibrium demand in previous exercise ($q_s^* = \frac{1}{\sqrt{2}}$). The price is approaching infinity and the quantity sold to zero, if $t \to 1$ From the graph it will be easy to see that dead-weight loss (DWL) from the commodity tax is equal to the area of the region ACD. The greater the tax increase the greater is the area of DWL. Note that the consumer surplus decreases after tax, whereas the producer surplus remains the same.

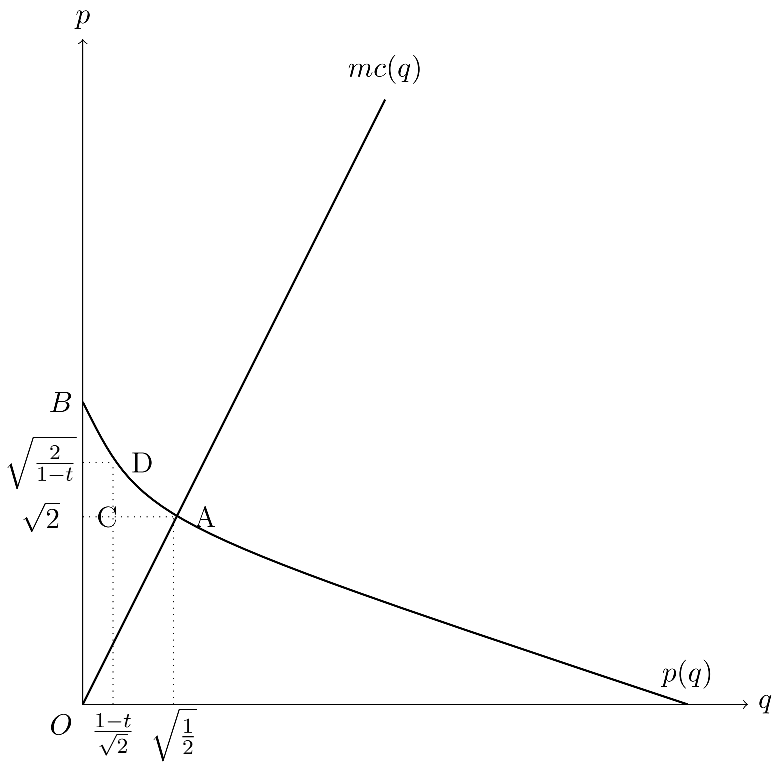


Figure 1: Welfare loss from tax

7. In this problem we need to consider a two person two good exchange economy. The initial endowments are $\omega^1 = (1,0)$ and $\omega^2 = (0,1)$.

   (a) In this exercise the preferences of consumers 1 and 2 are:

$$\begin{aligned} u^1(x_1^1, x_2^1) &= x_1^1 + x_2^1 \\ u^2(x_1^2, x_2^2) &= (x_1^2)^{\alpha}(x_2^2)^{1-\alpha} \end{aligned}$$

   The Pareto optimal allocations require that the marginal rates of substitution for consumer 1 and consumer 2 between goods 1 and 2 are the same, or:

$$\frac{\partial u^1/\partial x_1^1}{\partial u^1/\partial x_2^1} = \frac{\partial u^2/\partial x_1^2}{\partial u^2/\partial x_2^2}$$

It is clear from the first utility function that consumers MRS^1 is always equal to 1. Since the marginal rates of substitution must be the same we can already see that relative price should be $\frac{p_1}{p_2} = 1$. But lets actually solve for the equilibrium allocation at this point. In case of the second consumer:

$$\mathrm{MRS}^2 = \frac{\partial u^2/\partial x_1^2}{\partial u^2/\partial x_2^2} = \left(\frac{\alpha}{1-\alpha}\right)\frac{x_2^2}{x_1^2} = \mathrm{MRS}^1 = 1$$

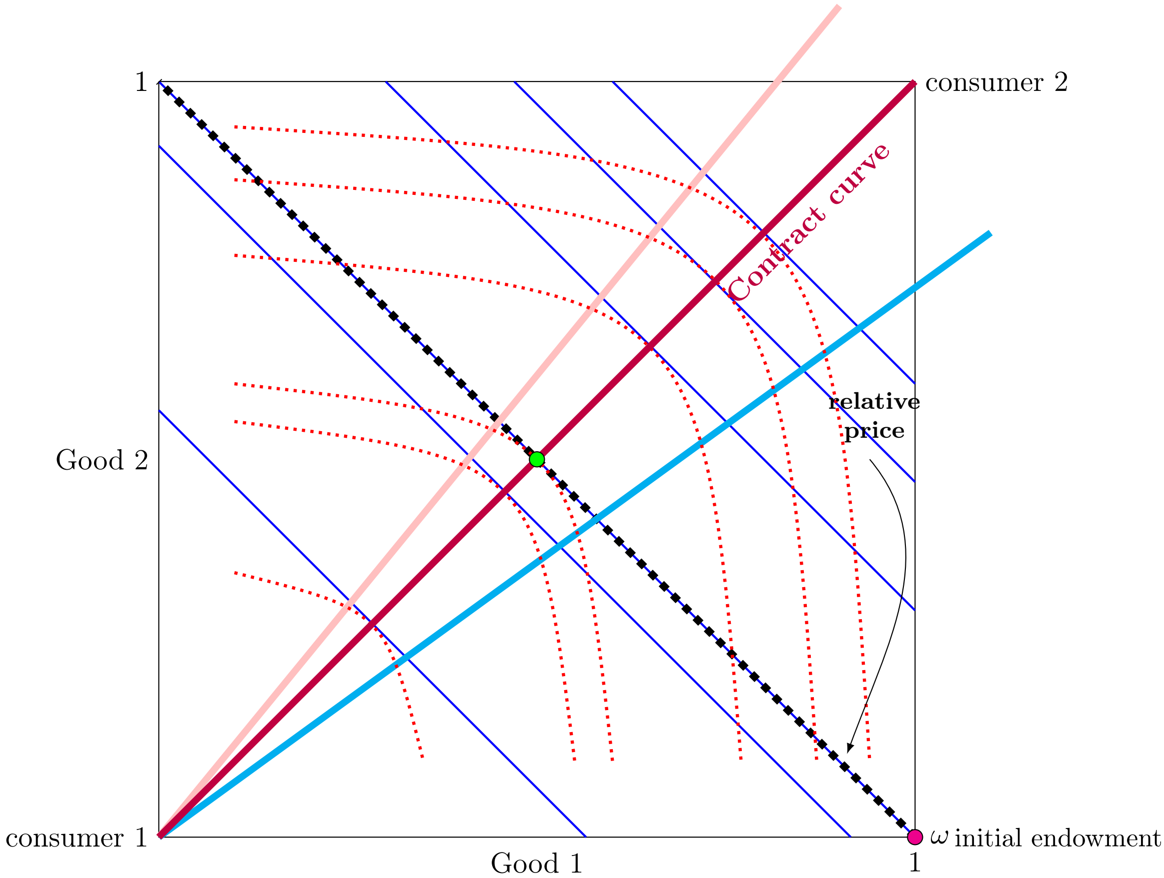
Rewriting x_1^2 from the previous equation, noting that $p_1 = p_2$ and plugging these expressions into the second consumer's budget constraint $p_1 x_1^2 + p_2 x_2^2 = p_2$ we get:

$$\left(\frac{\alpha}{1-\alpha}\right) x_2^2 p_2 + p_2 x_2^2 = p_2 \quad \Longrightarrow \quad \left(\frac{\alpha}{1-\alpha}\right) x_2^2 + x_2^2 = 1 \quad \Longrightarrow$$
$$x_2^2 = 1-\alpha \quad \text{and} \quad x_1^2 = \alpha$$

Market clearing for goods one and two imply that: $x_1^1 + x_1^2 = 1$ and $x_2^1 + x_2^2 = 1$, so:

$$x_1^1 = 1-\alpha \quad \text{and} \quad x_2^1 = \alpha$$

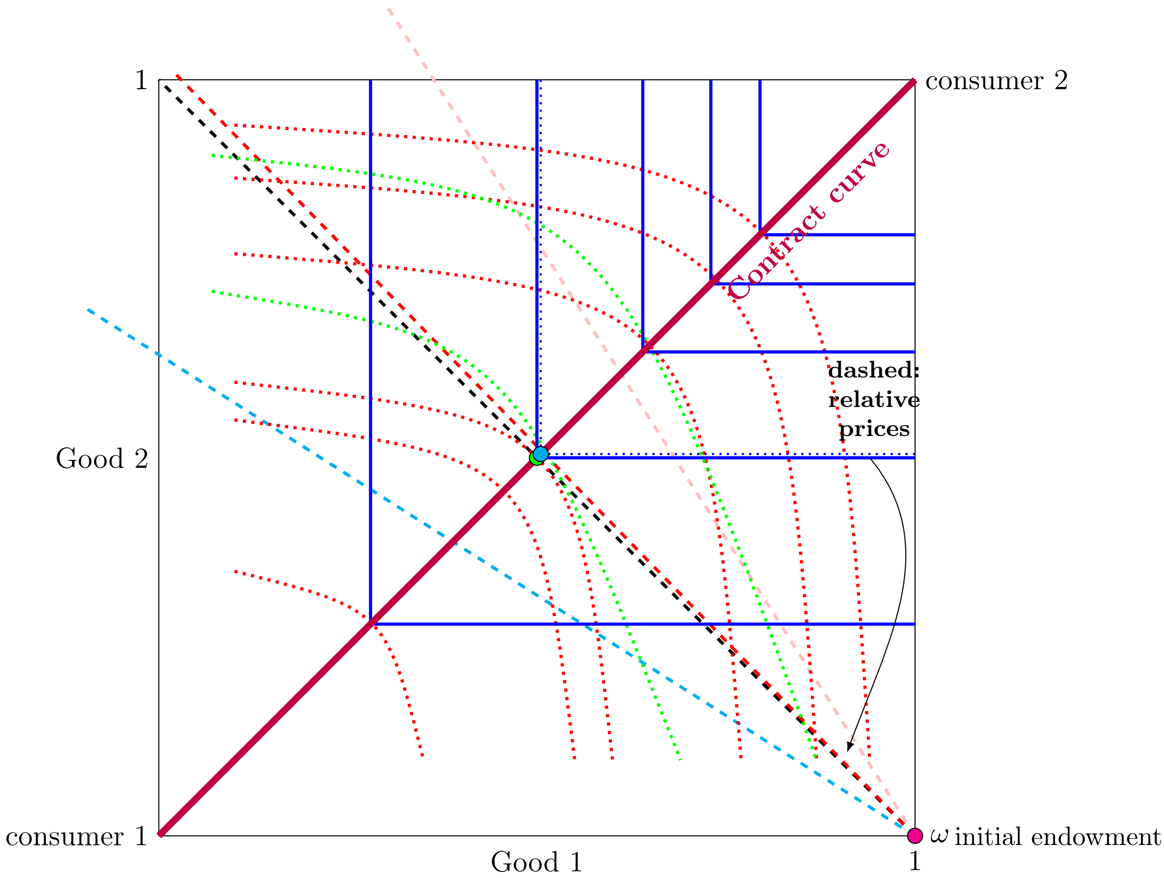
To summarize the equilibrium allocations in this economy are:
$(x_1^1, x_2^1) = (1-\alpha, \alpha)$ and $(x_1^2, x_2^2) = (\alpha, 1-\alpha)$.
Note that depending on the values of parameter α the slope of the contract curve may be greater than, less than or equal to 1. So, depending on α the contract curve in this case can be any ray coming from the consumer 1 origin and crossing the box (e.g. pink and cyan lines) The magenta thick line on the diagram corresponds to the case when the slope of the contract curve is exactly equal to 1. Note that Walrasian equilibrium point will always lay on the relative price line with $p_1 = p_2$ (on the diagram - thick dotted diagonal line). The green point on the line depicts the Walrasian equilibrium point in case $\alpha = \frac{1}{2}$.



(b) In this exercise the preferences of consumers 1 and 2 are:

$$
\begin{aligned}
u^1(x_1^1, x_2^1) &= \min\{x_1^1, x_2^1\} \\
u^2(x_1^2, x_2^2) &= (x_1^2)^{\alpha}(x_2^2)^{1-\alpha}
\end{aligned}
$$

In case of consumer 1, the MRS depends on how much of each good she is consuming. At optimum the consumption of consumer 1 is $x_1^1 = x_2^1$. The Pareto efficient allocations correspond to the points where the corners of the first consumer's indifference curves cross the second consumer's indifferent curves. The location of these points will depend on the parameter α. Red dotted indifference curves correspond to the situation where $\alpha = \frac{1}{2}$. If α equals $\frac{1}{2}$ then the price line as well as the equilibrium point will be identical to that in the previous exercise. However, if for example, $\alpha > \frac{1}{2}$, then the indifference curves of the consumer 2 will look more like green dotted curves (there is more weight given to good 1, so the indifference curves are "steeper" (closer) relative to Good 1 axis). In this case the optimal price line will correspond to the dashed red line, because the point where the indifference curves of consumer 1 and consumer 2 cross is higher (equilibrium will correspond to the cyan point where dotted green and dotted blue indifference curves intersect). Depending on the value of parameter α the price line can be any ray coming from the initial endowment point and crossing the Edgeworth box (as, for example pink or cyan dashed lines).


1
consumer 2
Contract curve
dashed: relative prices
Good 2
consumer 1
Good 1
1
ω initial endowment