Solutions to Problem Set I

(September 15)

1. In this exercise we need to show that if preferences $\succsim$ on $\mathbb{R}^L_+$ satisfy continuity and $x \succsim z \succsim y$, then there is an m in the line segment connecting x and y, s.t. $m \sim z$.
   It is best to present the problem graphically, assuming $L = 2$. By definition of continuity of preferences

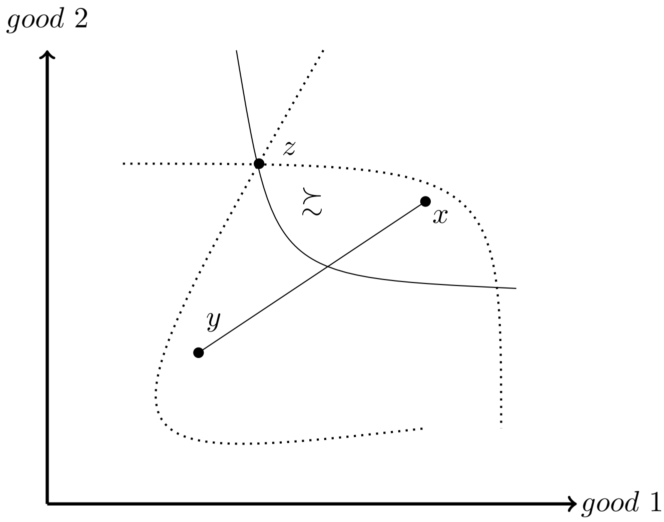

Figure 1: Continuity of Preferences

   the set $\succsim$ must be closed on $\mathbb{R}^L_+$. So, the bundle x must be inside of the set $\succsim z$ or on its border - indifference curve. If it is on indifference curve then $x \sim z$, so $x = m$. If x is inside of the set "at least as good as z", the indifference curve should be crossing the line connecting x and y (again by continuity). More detailed explanation will be given in class.

2. In this problem we need to show that if preferences $\succsim$ on $\mathbb{R}^L_+$ satisfy continuity and monotonicity, then the function t, s.t. $x \sim (t(x), ..., t(x))$ is continuous.
   Function $t(\cdot)$ is continuous if for all open intervals (a, b):

$$\left\{x \in \mathbb{R}^L_+ : a > t(x) > b\right\}$$

   is an open set. By the construction of t, $(t(x), ..., t(x)) \sim x$. Thus the above condition says that:

$$\left\{x \in \mathbb{R}^L_+ : (a, ..., a) \succ (t(x), ..., t(x)) \succ (b, ..., b)\right\} = \left\{x \in \mathbb{R}^L_+ : (a, ..., a) \succ x \succ (b, ..., b)\right\}$$

   is an open set. By the continuity of preferences, $\left\{x \in \mathbb{R}^L_+ : (b, ..., b) \succsim x\right\}$ and $\left\{x \in \mathbb{R}^L_+ : x \succsim (a, ..., a)\right\}$ are closed sets. Hence their complements $\left\{x \in \mathbb{R}^L_+ : (a, ..., a) \succ x\right\}$ and $\left\{x \in \mathbb{R}^L_+ : x \succ (b, ..., b)\right\}$ are open sets. Then their union is also an open set, i.e.

$$\left\{x \in \mathbb{R}^L_+ : (a, ..., a) \succ x \succ (b, ..., b)\right\}$$

is an open set.

3. The preferences of Leontief form: $(x_1, x_2) \succsim (y_1, y_2)$ on $\mathbb{R}^2_+$ if $\min\{x_1, x_2\} \geq \min\{y_1, y_2\}$ are represented by the utility function $u(x_1, x_2) = \min\{x_1, x_2\}$. Such function is obviously non-differentiable due to the kink in the indifference curves, as it can be seen in figure below. It is easy to see that the budget constraint always intersects the indifference curve at the kink.
Why? Because when you are at the kink, increasing the amount of just one good does not increase the utility. At the same time, increasing the consumption of a good always leads to a cost increase. Thus, the Marshallian demands are found at the kink, and the following equation must hold.

$$x_1 = x_2$$

We will solve for x_2 in the budget constraint $p_1x_1 + p_2x_2 = w$ and substitute into the above equation to get:

$$x_1 = x_2 = \frac{w}{p_1 + p_2}$$

As for the indirect utility function, plug the Marshallian demand functions into the direct utility function to get:

$$v(p, w) = \frac{w}{p_1 + p_2}$$

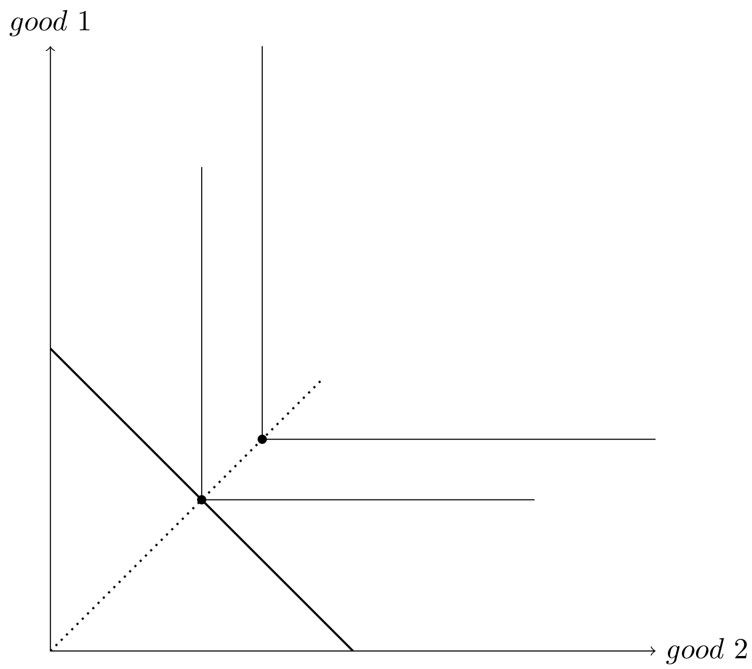

Figure 2: Leontief Preferences.

4. Let consumer's preferences be represented by a utility function u reflecting constant elasticity of substitution (CES):

$$u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}, \quad \text{where} \quad \rho \in (0, 1)$$

Let product price be p_1 and p_2 and income w.

(a) Construct a Lagrangian that reflects consumer's optimization problem: Consumer solves:

$$\max_{x_1, x_2} u(x_1, x_2) = \max_{x_1, x_2} \ (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$$

$$\text{s.t.} \quad p_1 x_1 + p_2 x_2 \leq w$$

So, Lagrangian for this problem will be:

$$\mathcal{L}(\mathbf{x}, \lambda) \ = \ (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} + \lambda(w - \ p_1 x_1 - \ p_2 x_2)$$

(b) Find the first order conditions (FOCs) for optimality:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= (x_1^\rho + x_2^\rho)^{\frac{1-\rho}{\rho}} x_1^{\rho-1} - \lambda p_1 = 0, \\ \frac{\partial \mathcal{L}}{\partial x_2} &= (x_1^\rho + x_2^\rho)^{\frac{1-\rho}{\rho}} x_2^{\rho-1} - \lambda p_2 = 0, \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= w - \ p_1 x_1 - \ p_2 x_2 = 0 \end{aligned}$$

(c) Identify the Marshallian demand functions $x_i(\mathbf{p}, w)$:
From the first two FOCs:

$$\begin{aligned} \frac{(x_1^\rho + x_2^\rho)^{\frac{1-\rho}{\rho}} x_1^{\rho-1}}{p_1} &= \frac{(x_1^\rho + x_2^\rho)^{\frac{1-\rho}{\rho}} x_2^{\rho-1}}{p_2} \quad \Longrightarrow \\ x_1 &= x_2 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}} \end{aligned}$$

Plugging this into the third first order condition we can solve for x_2:

$$\begin{aligned} w = p_1 x_2 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}} + p_2 x_2 \qquad &\Longrightarrow \quad = x_2 \left(p_1 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}} + p_2\right) \\ x_2 &= \frac{p_2^{1/(\rho-1)} w}{p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}} \\ x_1 &= \frac{p_1^{1/(\rho-1)} w}{p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}} \end{aligned}$$

We can define a parameter $r = \frac{\rho}{(\rho-1)}$ and rewrite Marshallian demands as:

$$x_1(\mathbf{p}, w) = \frac{p_1^{r-1} w}{p_1^r + p_2^r} \quad \text{and} \quad x_2(\mathbf{p}, w) = \frac{p_2^{r-1} w}{p_1^r + p_2^r}$$

(d) Construct the indirect utility function:

$$v(\mathbf{p}, w) = (x_1(\mathbf{p}, w)^\rho + x_2(\mathbf{p}, w)^\rho)^{\frac{1}{\rho}} = \left[\left(\frac{p_1^{r-1}w}{p_1^r + p_2^r}\right)^\rho + \left(\frac{p_2^{r-1}w}{p_1^r + p_2^r}\right)^\rho\right]^{\frac{1}{\rho}} = w\left[\frac{p_1^r + p_2^r}{(p_1^r + p_2^r)^\rho}\right]^{\frac{1}{\rho}}$$
$$v(\mathbf{p}, w) = w(p_1^r + p_2^r)^{-\frac{1}{r}}$$

(e) Identify the Hicksian demand function:
To find Hicksian demands $x_i^h(\mathbf{p}, u)$ we will set up the expenditure minimization problem:

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2$$

$$\text{s.t. } u - (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} = 0 \text{ and } x_1, x_2 \geq 0$$

So, Lagrangian for this problem will be:

$$\mathcal{L}(\mathbf{x}, \lambda) = p_1 x_1 + p_2 x_2 + \lambda[u - (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}]$$

(f) Find the first order conditions (FOCs) for optimality:

$$\frac{\partial \mathcal{L}}{\partial x_1} = p_1 - \lambda(x_1^\rho + x_2^\rho)^{\frac{1-\rho}{\rho}} x_1^{\rho-1} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = p_2 - \lambda(x_1^\rho + x_2^\rho)^{\frac{1-\rho}{\rho}} x_2^{\rho-1} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = u - (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} = 0$$

From the first two FOCs we get.

$$x_1 = x_2 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}$$

Plugging this into the third first order condition and some algebra we arrive at:

$$x_1^h(\mathbf{p}, u) = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_1^{r-1} \quad \text{and} \quad x_2^h(\mathbf{p}, u) = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_2^{r-1}$$

(g) Construct the expenditure function:
We can form the expenditure function using two approaches. *First* we will rearrange the expression for the indirect utility plugging in u in place of $v(\mathbf{p}, w)$ and e($\mathbf{p}$, u) in place of w:

$$u = e(\mathbf{p}, u)(p_1^r + p_2^r)^{-\frac{1}{r}} \quad \Longrightarrow \quad e(\mathbf{p}, u) = u(\mathbf{p}, u)(p_1^r + p_2^r)^{\frac{1}{r}}$$

We can also apply Sheppard's lemma to identify Hicksian demand from here.
Second we can plug the Hicksian demands into the budget constraint and find e($\mathbf{p}$, u) from there.
For more details see **Examples 1.1, 1.2, 1.3** in Jehle&Reny (2nd edition).

5. (a) Preferences are homothetic if $x \succsim y$ implies that $tx \succsim ty$ for all consumption bundles x, y. Suppose $x = (x_1, x_2)$ and $y = (y_1, y_2)$. So, if there are two goods and if the consumer prefers bundle x to bundle y, i.e. he prefers $(x_1, x_2) to (y_1, y_2)$, he automatically prefers bundle $(2x_1, 2x_2)$ to $(2y_1, 2y_2)$ This implies that the consumer's preferences only depend on the ratio of amount of good 1 to good 2. Homothetic preferences can be represented by the diagram below.

   If the consumer has homothetic preferences, then income offer curves (lines that depict the

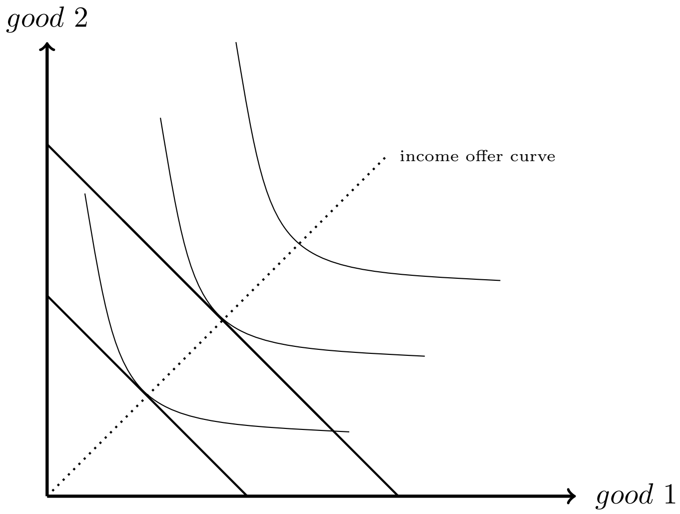

Figure 3: Homothetic Preferences

bundles of goods demanded at different income levels) are just straight lines through the origin . This means that if income is scaled up or scaled down by the amount $t > 0$, the demand bundle scales up or down by the same amount. It might be established analytically, but it is clear from the diagram: if the indifference curve is tangent to the budget line at some (x_1, x_2) given income w, then the indifference curve through (tx_1, tx_2) is tangent to the budget line where the income level is tw and the prices are the same. So, homothetic preferences imply that if you say double income you double the demand for each good.

   (b) The same logic works for scaling all the prices up by some positive constant. This is equivalent to scaling the income w down by the same constant, so the effect would be similar to that in (a) but in the opposite direction: for example, if say you double all prices you decrease the demand for each good by two times.

   (c) Using similar logic and noticing that increasing all prices and income by the same constant will not shift the budget line, we can conclude that demand would not change in this case: $x(tp, tw) = x(p, w) \quad \forall \quad t > 0$. In fact this result does not require the preferences to be homothetic (the mere existence of utility representation and strict convexity of preferences suffice). For more formal proof of this result see **Theorem 1.10** in Jehle&Reny (2nd edition).

6. If there is no tax the optimal consumption bundle corresponds to the point a on the diagram. If there is the income tax t_w the optimal bundle will move from point a to point b. Income tax simply shifts the budget line towards the origin, but the slope of the budget line does not change. The utility level drops from u_0 to u_1.

On the other hand if there is a tax on consumption of good x: t_x, it implies that the price of good x is increasing relative to the price of good y. This causes the budget line to become steeper. At the point c (where two "after tax" budget lines cross), the revenues from the income tax are equal to the revenues from the consumption tax: $t_w w = t_x x = g$. This is because the value of consumption at the point c is the same as at the point b, so the same amount was taken from the consumer by the government in both cases. The consumption tax caused the utility level decrease to u_2, which is lower than u_1. The difference between these utility levels is the dead weight loss in utility terms.
Point d is on the same indifference curve as point c, so the government can actually keep the consumer's utility level constant by imposing income tax even greater than the consumption tax.
The effect of consumption tax on choosing the optimal bundle can be broken into two parts: *income effect*, denoted by move from the original point a to d. And *substitution effect* due to the change in relative prices. It is represented by the move along the indifference curve from point d to point c.
For more details see *Intermediate Public Economics* book by Jean Hindriks and Gareth Myles.

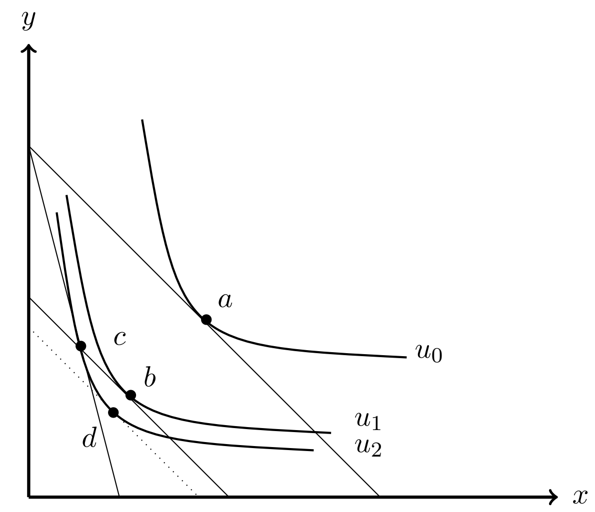

Figure 4: Income vs. Good Consumption Tax

7. In this exercise we know that the preferences $\succsim$ have the utility representation u. We need to show that $\succsim$ are convex iff the utility function u representing $\succsim$ is *quasiconcave*.
First, we will prove that if there is the utility representation, the convexity of preferences implies (is sufficient) for the utility function to be *quasiconcave*. Let's do this for bundles x, y such that $x \sim y$ and $x \neq y$. It is given that there exists the utility representation u for $\succsim$, so:

$$x \succsim y \quad \Longleftrightarrow \quad u(x) \geq u(y) \quad \forall \;\; x, y \in X$$

By convexity:

$$x \succsim y \quad \Longrightarrow \quad tx + (1-t)y \succsim y \quad \forall \;\; t \in [0,1] \quad \text{so:}$$
$$tx + (1-t)y \succsim y \quad \Longleftrightarrow \quad u(tx + (1-t)y) \geq u(y) = \min\{u(x), u(y)\} \quad \forall \;\; t \in [0,1]$$

Second, we will prove that if there is the utility representation, the *quasiconcavity* of the utility function implies the convexity of preferences. Since u is *quasiconcave*, by definition:

$$u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$$

In other words:

$$\begin{aligned} u(x) \geq u(y) \quad &\Longrightarrow \quad u(tx + (1-t)y) \geq u(y) \quad \forall \;\; t \in [0,1] \\ u(x) \geq u(y) \quad &\Longrightarrow \quad x \succsim y \\ u(tx + (1-t)y) \geq u(y) \;&\Longrightarrow\; tx + (1-t)y \succsim y \quad \forall \;\; t \in [0,1] \end{aligned}$$

So, the preferences are convex.

8. (a) Normal good is a good that has increasing demand when consumer's income's increasing. Or, more formally, a good is normal if $\frac{\partial x_i(\mathbf{p},w)}{\partial w} \geq 0$ Having this in mind lets start with the utility maximization problem

$$\max_{\mathbf{x} \in \mathbb{R}^{\mathbf{L}}_{+}} u(\mathbf{x}) = \sum_{i=1}^{L} u_i(x_i)$$
$$\text{s.t. } \mathbf{p} \cdot \mathbf{x} \leq w$$

Lagrangian will be:

$$\mathcal{L}(\mathbf{x}, \lambda) = \sum_{i=1}^{L} u_i(x_i) + \lambda(w - \sum_{i=1}^{L} p_i x_i)$$

The FOC's are:

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial u_i(x_i)}{\partial x_i} - \lambda p_i = 0 \quad \Longrightarrow \quad u_i' = \lambda p_i \;\; \forall i \in [1, L]$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\sum_{i=1}^{L} p_i x_i + w \leq 0 \text{ and } \lambda \left[\sum_{i=1}^{L} p_i x_i - w \right] = 0$$

The preferences are strictly monotonous, there is no money left on the table, so that the budget constraint holds with equality. We will also consider only interior solutions. So, we can write down the first order condition with respect to λ as:

$$\sum_{i=1}^{L} p_i x_i = w$$

At the end solving for λ's we can simply write down that:

$$u_i'(x_i) = \frac{p_i}{p_j} u_j'(x_j) = \frac{p_i}{p_k} u_k'(x_k) = ... = \frac{p_i}{p_L} u_L'(x_L)$$

In this exercise $u_i''(x_i) < 0 \; \forall \; x_i \geq 0$. Also, remember that the second derivative is simply a derivative of the first derivative for any function. Suppose the income w increased and prices remained the same. Then the demand for at least one good should increase, because consumer wants to spend her entire income according to Walras law. Suppose it is the good i. We got $u_i'(x_i) = \frac{p_i}{p_j} u_j'(x_j)$ from our FOC's. As $x_i(\mathbf{p}, w)$ increased, negative second derivative implies that right hand side of the expression decreased (price ratio is constant). This also implies that left hand side of the expression decreased as well, thus $x_j(\mathbf{p}, y)$ also increased. Applying similar reasoning to the next good k etc., we can conclude that all goods must be normal. None of the goods can be inferior with the additively separable utility function.

(b) In this exercise we have to show that for additive utility the cross-price derivatives of demands are proportional to income derivatives:

$$\frac{\partial x_i(\mathbf{p}, w)/\partial p_k}{\partial x_j(\mathbf{p}, w)/\partial p_k} = \frac{\partial x_i(\mathbf{p}, w)/\partial w}{\partial x_j(\mathbf{p}, w)/\partial w}.$$

We start by differentiating the first order conditions with respect to w:

$$\begin{aligned} u_i''(x_i)\frac{\partial x_i}{\partial w} &= p_i \frac{\partial \lambda}{\partial w} \quad \forall i \in [1, L] \\ \sum_{i=1}^{L} p_i \frac{\partial x_i}{\partial w} &= 1 \end{aligned}$$

We can rewrite and plug the expressions for $\frac{\partial x_i}{\partial w}$ into the last equation:

$$\sum_{i=1}^{L} p_i \left(\frac{p_i}{u_i''(x_i)} \right) = \frac{1}{\partial \lambda / \partial w} \tag{8.1}$$

Now we will differentiate the original first order conditions with respect to p_j:

$$\begin{aligned} u_i''(x_i)\frac{\partial x_i}{\partial p_j} &= p_i \frac{\partial \lambda}{\partial p_j} \\ u_j''(x_j)\frac{\partial x_j}{\partial p_j} &= \lambda + p_j \frac{\partial \lambda}{\partial p_j} \\ x_k + p_j \frac{\partial x_j}{\partial p_j} + p_i \frac{\partial x_i}{\partial p_j} &= 0 \end{aligned}$$

After some algebraic manipulations we arrive at:

$$p_i \left(\frac{p_i}{u_i''(x_i)} \right) + p_j \left(\frac{p_j}{u_j''(x_j)} \right) = \frac{-x_j - p_j(\lambda / u_j''(x_j))}{\partial \lambda / \partial p_j} \tag{8.2}$$

Then will differentiate the original first order conditions with respect to p_i and get:

$$p_i \left(\frac{p_i}{u_i''(x_i)} \right) + p_j \left(\frac{p_j}{u_j''(x_j)} \right) = \frac{-x_i - p_i(\lambda / u_i''(x_i))}{\partial \lambda / \partial p_i} \tag{8.3}$$

Comparing equations (8.1), (8.2) and (8.3) and keeping in mind that $\frac{p_k}{u_k''(x_k)} = \frac{\partial x_k}{\partial w}\frac{\partial w}{\partial \lambda}$ we get:

$$\begin{aligned}\frac{1}{\partial\lambda/\partial w} &= \frac{-x_i - p_i(\lambda/u_i''(x_i))}{\partial\lambda/\partial p_i} = \frac{-x_j - p_j(\lambda/u_j''(x_j))}{\partial\lambda/\partial p_j} = \frac{-x_k - p_k(\lambda/u_k''(x_k))}{\partial\lambda/\partial p_k} \implies \\ \frac{\partial\lambda}{\partial p_k} &= \frac{\partial\lambda}{\partial w}\left(-x_k - p_k\frac{\lambda}{u_k''(x_k)}\right) = \frac{\partial\lambda}{\partial w}\left(-x_k - \frac{\lambda}{\partial\lambda/\partial w}\frac{\partial x_k}{\partial w}\right)\end{aligned}$$

Remember that $\frac{\partial x_i}{\partial p_k} = \frac{p_i}{u_i''(x_i)}\frac{\partial\lambda}{\partial p_k}$, hence:

$$\frac{\partial x_i}{\partial p_k} = -\frac{p_i}{u_i''(x_i)}\frac{\partial\lambda}{\partial w}\left(x_k + p_k\frac{\lambda}{\partial\lambda/\partial w}\frac{\partial x_k}{\partial w}\right) = -\frac{\partial x_i}{\partial w}\left(x_k + p_k\frac{\lambda}{\partial\lambda/\partial w}\frac{\partial x_k}{\partial w}\right) \quad (i \neq k)$$

This also holds for any other $j \neq i$:

$$\frac{\partial x_j}{\partial p_k} = \frac{\partial x_j}{\partial w}\left(x_k + p_k\frac{\lambda}{\partial\lambda/\partial w}\frac{\partial x_k}{\partial w}\right) \quad (j \neq k)$$

so that:

$$\frac{\partial x_i/\partial p_k}{\partial x_j/\partial p_k} = \frac{\partial x_i/\partial w}{\partial x_i/\partial w}.$$

If we rearrange the equation above such restriction means that the change in demand for good i induced by the change in price for good k is proportional to the change in demand for good i induced by the change in income.

(c) Remember that $u_i''(x_i) < 0$, so the consumer has decreasing marginal utility from all goods but good 1 (i.e. the money), which equals to 1. If the income increases drastically and the prices remain constant, because of decreasing marginal utilities for all other goods the consumer will consume all the goods until the marginal utility of consumption of the most expensive good will be equal to 1 over the price of that good. Beyond that point the consumer will put all his wealth into good 1 or money, so basically he will save.

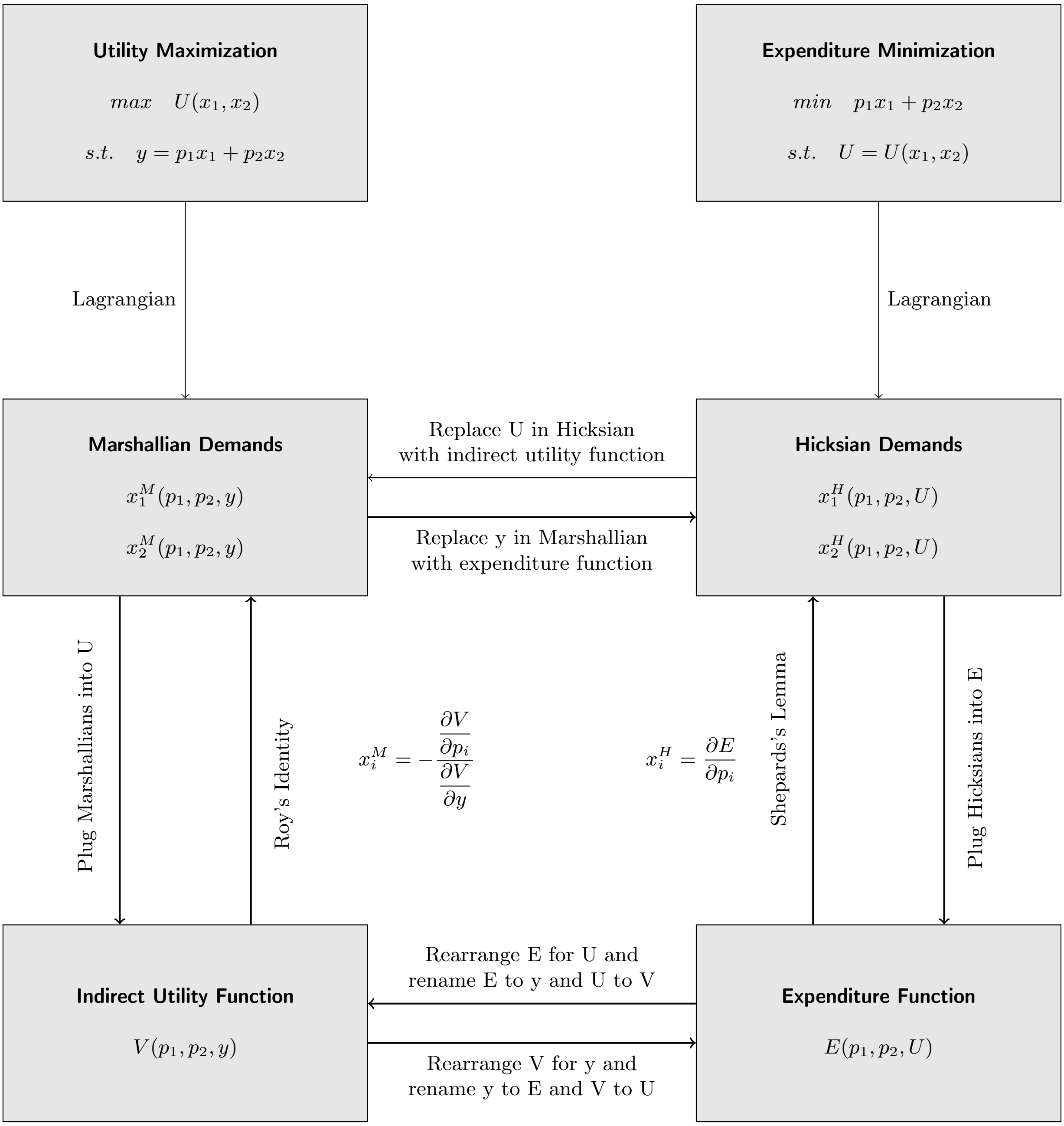

Figure 5: Duality Overview