Solutions to Problem Set I

(October 3, 2014)

1. (a) **Not necessarily.** If the cost of search is the same for all buyers, firms anticipate that in equilibrium no one will choose to see more prices and no price dispersion will occur (Diamond's paradox). If the cost differ, consumers will choose to observe more prices and there will be dispersion.

   (b) **Not necessarily.** Think about the stag hunt game.

   (c) **Not necessarily.** For example, in case of the second degree price discrimination (when the seller offers different price/quality bundles to different types of consumers, making sure that each particular type self selects into the bundle meant for her) the high type buyers purchase the socially optimal quality ($u'(q) = mc(q)$), maximizing her surplus.

   (d) **False.** Suppose β is very close to one. Then the optimal s_i will be very close to v. Hence, as a buyer I don't have any incentive to acquire additional price information, because I already know that the price will be high, so there is little point in sampling twice.

2. There is a market consisting of two segments. Fraction $0 < \alpha < 1$ has individual demand curve given by $d(p) = 10 - 2p$ and fraction $1 - \alpha$ have unit demands if $p \leq 3$ and zero otherwise.

   (a) The demand curves will be drawn in class. Note that the demand curves will have a horizontal gap at the point where $p = 3$.

   (b) First, lets prove that the optimal uniform price cannot be greater than 3, i.e. optimal price always satisfies the condition $p \in [0, 3]$.
   Suppose that the optimal uniform price is greater than 3. Then the firms profit (assuming zero marginal cost) is $\pi = \alpha p(10 - 2p)$. This function is decreasing for all values of p ($\pi' = \alpha(10 - 4p) < 0$ if $p \in (3, 5]$), if $p = 3 + \varepsilon$, the profit will be at its maximum possible value for the price range, that is $\pi = 10\alpha(3 + \varepsilon) - 2\alpha(3 + \varepsilon)^2 = 12\alpha - 2\alpha\varepsilon - 2\alpha\varepsilon^2$. The producer can set the price exactly at 3 to capture all segment $(1 - \alpha)$ getting the profit $11\alpha + 1$. Obviously, for any $\alpha \in [0, 1]$, the latter expression is greater. So the optimal price cannot be above 3. Having this in mind, we set up the maximization problem for the monopolist:

   $$\max_{0 \leq p \leq 3} \pi(p) = \alpha p(10 - 2p) + (1 - \alpha)p = 9\alpha p - 2\alpha p^2 + p$$

   Yielding the first order condition:

   $$9\alpha - 4\alpha p^* + 1 = 0 \quad \Longrightarrow \quad p^* = \frac{1 + 9\alpha}{4\alpha}$$

   Second order condition yields $-4\alpha < 0$, so indeed the p^* yields the maximum profit.
   Note that for certain low values of α it still might be optimal to set the uniform price at 3. To

check for which values α the profit maximizing price will be 3 we plug $p^* = 3$ into the derivative of the profit with respect to price. We arrive at $-3\alpha + 1$, the derivative is positive, so the profit is increasing given $p^* = 3$ until $-3\alpha + 1 \geq 0$. So if $\alpha \leq \frac{1}{3}$ the optimal price is 3 and if $\alpha > \frac{1}{3}$ the optimal price is $p^* = \frac{1+9\alpha}{4\alpha}$.

(c) In order to compute the optimal price for the segment α, we set up the firm maximization problem, ignoring the fact segment $1 - \alpha$ exists. Lets call the price charged from the consumers with linear demand p_{lin} and from consumers with unit demand p_{unit}.
Solving: $\max_{p_{lin} \geq 0} \pi(p_{lin}) = p_{lin}(10 - 2p_{lin})$, yields $p^*_{lin} = \frac{10}{4} = 2.5$. For the segment with unit demand for prices below 3 the monopolist sets the price in order to capture all consumer surplus, so $p^*_{unit} = 3$.

(d) Remember that the third degree price discrimination is the situation when the producer observes some signal related to consumer preferences and can use this signal to charge different price from different segment. In this case, suppose monopolist can perfectly discriminate between two segments. So the firm charges $p_{unit} = 3$ from $1 - \alpha$ segment. And the price for α consumers is $p = 2.5$. The producer surplus in this case will be equal to:

$$PS = \alpha p_{lin}(10 - 2p_{lin}) + p_{unit}(1 - \alpha) = \alpha 2.5(10 - 2 \cdot 2.5) + 3(1 - \alpha) = 9.5\alpha + 3$$

The consumer surplus to the segment with the unit demand and $p_{unit} = 3$ obviously equals to 0. The consumer surplus to the segment with linear demand equals to the area of the shape above the price line and below the demand curve. Since the demand is linear it will be a triangle[1]. Note that the maximum price is 5. The consumer surplus for the segment α is then:

$$CS_{lin} = \frac{1}{2}(p_{max} - p_{lin})(10 - 2p_{lin}) = \frac{1}{2}(5 - 2.5)(10 - 2 \cdot 2.5) = 6.25$$

So the total surplus in case of third degree price discrimination will be $PS + CS_{lin} = 9.5\alpha + 9.25$.

In case there is a uniform pricing there are two possible cases. For $\alpha > \frac{1}{3}$: $p^* = \frac{1+9\alpha}{4\alpha}$ and the consumers surplus will consist of two parts:
First the unit demand consumers will get the surplus $3 - \frac{1+9\alpha}{4\alpha}$. Second, the linear demand segment will get $\frac{1}{2}(p_{max} - p^*)(10 - 2p^*) = \frac{1}{2}(5 - \frac{1+9\alpha}{4\alpha})(10 - 2\frac{1+9\alpha}{4\alpha}) = \left(5 - \frac{1+9\alpha}{4\alpha}\right)^2$. So:

$$CS = 3 - \frac{1+9\alpha}{4\alpha} + \left(5 - \frac{1+9\alpha}{4\alpha}\right)^2$$

Producer surplus in case $\alpha > \frac{1}{3}$ is: $PS = \alpha p^*(10 - 2p^*) + (1 - \alpha)p^* = 9\alpha p^* - 2\alpha p^{*2} + p^*$

$$PS = 9\alpha \frac{1+9\alpha}{4\alpha} - 2\alpha \left(\frac{1+9\alpha}{4\alpha}\right)^2 + \frac{1+9\alpha}{4\alpha}$$

Comparing the total surplus for this case and third degree price discrimination we can find α. For $\alpha \leq \frac{1}{3}$ the optimal uniform price is 3. The surplus of the unit demand consumers is zero.

[1](easier to find than integrating in case the demand is non linear)

The consumer surplus of linear demand segment is $\frac{1}{2}(p_{max} - p^*)(10 - 2p^*) = (5 - 3)^2 = 4$. The profit of the producer is $\alpha p^*(10 - 2p^*) + (1 - \alpha)p^* = 9\alpha + 3$.
Now we can clearly see that in case $\alpha \leq \frac{1}{3}$ the third degree price discrimination yields a higher level of total surplus $(9.25\alpha + 9.5 > 9\alpha + 3)$.

(e) We've already proved in part (b), that it is not optimal for the monopolist to charge the price higher than 3. And charging such price is the only way to screen the unit demand consumers. So, now the producer will charge the uniform price from all consumers. Depending of α the monopolist will either charge 3 or the optimal uniform price $p^* = \frac{1+9\alpha}{4\alpha}$.

3. Let's call channel 1 - "HBO" and channel 2 - "ESPN" for convenience.

(a) There are four possible cases for valuations and associated probabilities , that is:

$$\begin{aligned}
Pr(v(HBO) &= v^h \text{ and } v(ESPN) = v^h) = \alpha^2 \\
Pr(v(HBO) &= v^h \text{ and } v(ESPN) = v^l) = \alpha(1 - \alpha) \\
Pr(v(HBO) &= v^l \text{ and } v(ESPN) = v^h) = (1 - \alpha)\alpha \\
Pr(v(HBO) &= v^l \text{ and } v(ESPN) = v^l) = (1 - \alpha)^2
\end{aligned}$$

(b) The cable firm can choose different modes of pricing: selling each channel separately, pure bundling (sell only a bundle of two channels) and mixed bundling (selling bundles of a single channel and a bundle of two channels).
I will focus on the case of mixed bundling, that is there are 3 packages available for consumers: {HBO}, {ESPN}, {HBO, ESPN}. The pricing strategy is the list of prices for each of these bundles. In this case the firm has 12 possible pricing strategies for such bundles (we don't differentiate between pricing the bundle at $v^h + v^l$ or $v^l + v^h$).
Let's look at some of the pricing strategies and calculate the profits from each. At the end it will be possible to compare the profits from each pricing strategy and choose the optimal.
Denote $P(\cdot)$ - price of a bundle. So, suppose one pricing strategy is the following:
($P(\{\text{HBO}\}) = v^h$, $P(\{\text{ESPN}\}) = v^h$, $P(\{\text{HBO,ESPN}\}) = v^h + v^h$), then the profit from such pricing strategy will be equal to: $\alpha^2(v^h + v^h) + 2\alpha(1 - \alpha)v^h$. In this case consumers that value one of the channels high and another low will choose one channel bundles, and the consumer that values both channels high will choose a two channel bundle, consumer that value both of the channels low are not served.
Another example of the pricing strategy would be: ($P(\{\text{HBO}\}) = v^l$, $P(\{\text{ESPN}\}) = v^h$, $P(\{\text{HBO,ESPN}\}) = v^l + v^h$), then the profit from such pricing strategy will be equal to: $(1-\alpha)^2 v^l + 0 + (\alpha^2 + 2\alpha(1-\alpha))(v^h + v^l)$. In this case consumers that value both of the channels low, will go for the bundle with only one low priced channel. Consumers that value one of the channels high and another low will together with consumers with both channels valued high go for the two channel bundle.

Applying similar reasoning to all possible combinations of prices/channels and depending on parameters one can identify the profit maximizing pricing strategy.

4. (a) In this problem the consumer has logarithmic utility: $\ln(q+1)-t$ and the producer has linear cost of production $c(q)=cq$, where $0<c<1$. We know that in the case of first best $u'(q)=mc(q)$ or the sum of consumer and producer surplus is at maximum for each type, so:

$$\max_q \ \theta\ln(q+1)-cq \quad \Longrightarrow \quad \frac{\theta}{q^*+1}-c=0 \quad \Longrightarrow \quad q^*=\frac{\theta-c}{c}$$

Second order condition yields: $-\frac{\theta}{(q^*+1)^2}<0$.

(b) Now there are two types of buyers $\theta^H=3$ and $\theta^L=1$. λ is the fraction of buyers of type θ^H and $(1-\lambda)$ is the fraction of consumers of type θ^L. The incentive compatibility (IC) constraints for both types will look like:

$$3\ln(q^h+1)-t^h \geq 3\ln(q^l+1)-t^l$$
$$\ln(q^l+1)-t^l \geq \ln(q^h+1)-t^h$$

The individual rationality (IR) or participation constraints are:

$$3\ln(q^h+1)-t^h \geq 0$$
$$\ln(q^l+1)-t^l \geq 0$$

At the optimum the IC of the high type and IR of the low type bind. If we plug the expression for t^l type into IC of the high type and solve for t^h, we get $t^h=3\ln(q^h+1)-2\ln(q^l+1)$. Rearranging we see that high type's surplus or information rent is: $3\ln(q^h+1)+t^h=2\ln(q^l+1)$

(c) Monopolist solves:

$$\max_{q^h,q^l,t^h,t^l} \lambda(t^h-cq^h)+(1-\lambda)(t^l-cq^l)$$

Subject to binding IC of high type and binding IR of the low type. Plugging the expressions for t^h and t^l into the monopolist's problem we get:

$$\max_{q^h,q^l} \lambda(3\ln(q^h+1)-2\ln(q^l+1)-cq^h)+(1-\lambda)(\ln(q^l+1)-cq^l)$$

First order conditions yield:

$$\frac{3\lambda}{q^h+1}-c=0 \quad \Longrightarrow \quad q^h=\frac{3\lambda-c}{c}$$
$$-\frac{2\lambda}{q^l+1}+\frac{1-\lambda}{q^l+1}-(1-\lambda)c=0 \quad \Longrightarrow \quad q^l=\frac{(1-3\lambda)-(1-\lambda)c}{(1-\lambda)c}$$

The transfers will be:

$$\begin{aligned} t^h &= 3\ln(q^h+1) - 2\ln(q^l+1) = 3\ln\left(\frac{3\lambda - c}{c} + 1\right) - 2\ln\left(\frac{(1-3\lambda)-(1-\lambda)c}{(1-\lambda)c} + 1\right) \\ &= 3\ln\left(\frac{3\lambda}{c}\right) - 2\ln\left(\frac{1-3\lambda}{(1-\lambda)c}\right) \\ t^l = \ln(q^l+1) &= \ln\left(\frac{(1-3\lambda)-(1-\lambda)c}{(1-\lambda)c} + 1\right) = \ln\left(\frac{1-3\lambda}{(1-\lambda)c}\right) \end{aligned}$$

So the optimal menu is:

$$\left\{\left(\frac{3\lambda - c}{c},\ 3\ln\left(\frac{3\lambda}{c}\right) - 2\ln\left(\frac{1-3\lambda}{(1-\lambda)c}\right)\right), \left(\frac{(1-3\lambda)-(1-\lambda)c}{(1-\lambda)c},\ \ln\left(\frac{1-3\lambda}{(1-\lambda)c}\right)\right)\right\}$$

q^l is positive when $\frac{(1-3\lambda)-(1-\lambda)c}{(1-\lambda)c} > 0$ iff $\lambda < \frac{1-c}{3-c}$.

5. The fundamental difference of this problem from the standard non-linear pricing setting is that in this case each consumer has two decision variables: he will choose tariffs from the tariff menu offered and also the optimal quantity to consume given each tariff.

   (a) Let $q_i(p)$ be a chosen quantity of type i consumer, given per unit price p and if i buys positive quantity[2]. $q_i(p)$ should solve $p = \frac{\partial \theta^i v(q)}{\partial q_i}$. At the end the usage fee p^i must be equal to marginal utility of type i.
   In the case of linear preferences and the optimal quantity chosen: $p^1 = \theta^1$ and $p^2 = \theta^2$, the marginal utility that the consumer gets from each unit of good is constant. Obviously, if the monopolist wants to reserve zero surplus for the low type, he should set the fixed fee for him f^1 to zero. The tariff offered for the low type will be always more attractive to the high type and the monopolist cannot ensure that the high type will choose the tariff/quantity meant for him. So, the low type will not be served.
   If the monopolist decides to serve only the high type the problem with two part tariff implementation is that (because of constant marginal utility) the high type (as well as the low type) will always ask for as much as he can. On the other hand the costs of production are convex, so the marginal cost for monopolist is increasing. So, the two part tariff in this case cannot be implemented.

   (b) In the situation where the utility is concave it is actually possible to implement two-part tariff pricing. $p_i = \frac{\theta^i v(q)}{\partial q}$, i.e. $q_i = \left(\frac{\theta^i}{4p^i}\right)^2$. Remember that the consumer chooses both tariffs and quantities. We will offer the low type such tariff structure so that he will choose his optimal quantity using the standard IR. In the IC for the high type we must ensure that facing the tariffs of the low type, he will still be better off by choosing his own optimal tariff and corresponding

[2] I assume that $\theta^1 < \theta^2$. I also assume that $\lambda^1 + \lambda^2 = 1$, so $\lambda^1 = 1 - \lambda^2 = \lambda$

quantity. For this case the firm's problem can be formulated as:

$$\begin{aligned}
\max_{p^1,p^2,f^1,f^2} \lambda(p^1 q_1(p^1) - cq_1(p^1) + f^1) \quad &+ \quad (1-\lambda)(p^2 q_2(p^2) - cq_2(p^2) + f^2) \quad \text{s.t.} \\
\text{IR for low type binds: } \theta^1 v(q_1(p^1)) - p^1 q_1(p^1) - f^1 \quad &= \quad 0 \\
\text{IC for high type binds: } \theta^2 v(q_2(p^2)) - p^2 q_2(p^2) - f^2 \quad &= \quad \theta^2 v(q_2(p^1)) - p^1 q_2(p^1) - f^1
\end{aligned}$$

We can express the fixed parts of the tariff f_1 and f_2 from the constraints and plug them into the objective function.

$$\begin{aligned}
f^1 &= \theta^1 v(q_1(p^1)) - p^1 q_1(p^1) \\
f^2 &= \theta^2 v(q_2(p^2)) - p^2 q_2(p^2) - \theta^2 v(q_2(p^1)) + p^1 q_2(p^1) - f^1
\end{aligned}$$

So now firm solves:

$$\begin{aligned}
\max_{p^1,p^2} \quad \lambda \quad & [p^1 q_1(p^1) - cq_1(p^1) + f^1] + (1-\lambda)[p^2 q_2(p^2) - cq_2(p^2) + \theta^2 v(q_2(p^2)) - p^2 q_2(p^2) - \theta^2 v(q_2(p^1)) \\
+ \quad & p^1 q_2(p^1) - f^1]
\end{aligned}$$

After going through some tedious algebra we will find that the profit maximizing usage fee for the high type p^2 will be equal to the firms marginal cost (remember there is always no distortion on the top). Low type's price will be distorted to make sure the hight type chooses his tariff and quantity (firm will charge low type the usage fee above its marginal cost). It is also likely that the optimal quantity chosen by the low type will be smaller than in the case where firm directly chooses the quantity/tariff menus (not two part tariff menus).

Another potential approach to solving this exercise would be solving it as if the firm chooses quantity/tariff. Since we know that q is optimal quantity for the consumer given the per unit fee, we can calculate the fixed fee at the end if the know the quantity and tariff. However, this approach is flawed, as high type's IC will not take into consideration that he will choose his optimal quantity facing the tariff of the low type. To understand the point better graphs will be drawn in class.