MEMOIRS
of the
American Mathematical Society

Number 521

Prime Ideals in Skew and q-Skew Polynomial Rings

K. R. Goodearl
E. S. Letzter

May 1994 • Volume 109 • Number 521 (first of 5 numbers) • ISSN 0065-9266

American Mathematical Society

Recent Titles in This Series

521 **K. R. Goodearl and E. S. Letzter,** Prime ideals in skew and q-skew polynomial rings, 1994
520 **Tom Ilmanen,** Elliptic regularization and partial regularity for motion by mean curvature, 1994
519 **William M. McGovern,** Completely prime maximal ideals and quantization, 1994
518 **René A. Carmona and S. A. Molchanov,** Parabolic Anderson problem and intermittency, 1994
517 **Takashi Shioya,** Behavior of distant maximal geodesics in finitely connected complete 2-dimensional Riemannian manifolds, 1994
516 **Kevin W. J. Kadell,** A proof of the q-Macdonald-Morris conjecture for BC_n, 1994
515 **Krzysztof Ciesielski, Lee Larson, and Krzysztof Ostaszewski,** $\mathcal{I}$-density continuous functions, 1994
514 **Anthony A. Iarrobino,** Associated graded algebra of a Gorenstein Artin algebra, 1994
513 **Jaume Llibre and Ana Nunes,** Separatrix surfaces and invariant manifolds of a class of integrable Hamiltonian systems and their perturbations, 1994
512 **Maria R. Gonzalez-Dorrego,** $(16, 6)$ configurations and geometry of Kummer surfaces in $\mathbb{P}^3$, 1994
511 **Monique Sablé-Tougeron,** Ondes de gradients multidimensionnelles, 1993
510 **Gennady Bachman,** On the coefficients of cyclotomic polynomials, 1993
509 **Ralph Howard,** The kinematic formula in Riemannian homogeneous spaces, 1993
508 **Kunio Murasugi and Jozef H. Przytycki,** An index of a graph with applications to knot theory, 1993
507 **Cristiano Husu,** Extensions of the Jacobi identity for vertex operators, and standard $A_1^{(1)}$-modules, 1993
506 **Marc A. Rieffel,** Deformation quantization for actions of R^d, 1993
505 **Stephen S.-T. Yau and Yung Yu,** Gorenstein quotient singularities in dimension three, 1993
504 **Anthony V. Phillips and David A. Stone,** A topological Chern-Weil theory, 1993
503 **Michael Makkai,** Duality and definability in first order logic, 1993
502 **Eriko Hironaka,** Abelian coverings of the complex projective plane branched along configurations of real lines, 1993
501 **E. N. Dancer,** Weakly nonlinear Dirichlet problems on long or thin domains, 1993
500 **David Soudry,** Rankin-Selberg convolutions for $\mathrm{SO}_{2\ell+1} \times \mathrm{GL}_n$: Local theory, 1993
499 **Karl-Hermann Neeb,** Invariant subsemigroups of Lie groups, 1993
498 **J. Nikiel, H. M. Tuncali, and E. D. Tymchatyn,** Continuous images of arcs and inverse limit methods, 1993
497 **John Roe,** Coarse cohomology and index theory on complete Riemannian manifolds, 1993
496 **Stanley O. Kochman,** Symplectic cobordism and the computation of stable stems, 1993
495 **Min Ji and Guang Yin Wang,** Minimal surfaces in Riemannian manifolds, 1993
494 **Igor B. Frenkel, Yi-Zhi Huang, and James Lepowsky,** On axiomatic approaches to vertex operator algebras and modules, 1993
493 **Nigel J. Kalton,** Lattice structures on Banach spaces, 1993
492 **Theodore G. Faticoni,** Categories of modules over endomorphism rings, 1993
491 **Tom Farrell and Lowell Jones,** Markov cell structures near a hyperbolic set, 1993
490 **Melvin Hochster and Craig Huneke,** Phantom homology, 1993
489 **Jean-Pierre Gabardo,** Extension of positive-definite distributions and maximum entropy, 1993

(Continued in the back of this publication)

This page intentionally left blank

MEMOIRS
of the
American Mathematical Society

Number 521

Prime Ideals in Skew and q-Skew Polynomial Rings

K. R. Goodearl
E. S. Letzter

May 1994 • Volume 109 • Number 521 (first of 5 numbers) • ISSN 0065-9266

American Mathematical Society
Providence, Rhode Island

1991 *Mathematics Subject Classification.*
Primary 16D30, 16P40, 26S36.

Library of Congress Cataloging-in-Publication Data

Goodearl, K. R.

Prime ideals in skew and q-skew polynomial rings / K. R. Goodearl, E. S. Letzter.

p. cm. – (Memoirs of the American Mathematical Society, ISSN 0065-9266; no. 521)

"Volume 109, number 521 (first of 5 numbers)."

Includes bibliographical references.

ISBN 0-8218-2583-6

1. Ideals (Algebra) 2. Polynomial rings. 3. Noetherian rings. I. Letzter, E. S. (Edward S.), 1958– . II. Title. III. Series.

QA3.A57 no. 521
[QA247]
510 s–dc20 94-4145
[512′.4] CIP

Memoirs of the American Mathematical Society

This journal is devoted entirely to research in pure and applied mathematics.

Subscription information. The 1994 subscription begins with Number 512 and consists of six mailings, each containing one or more numbers. Subscription prices for 1994 are $353 list, $282 institutional member. A late charge of 10% of the subscription price will be imposed on orders received from nonmembers after January 1 of the subscription year. Subscribers outside the United States and India must pay a postage surcharge of $25; subscribers in India must pay a postage surcharge of $43. Expedited delivery to destinations in North America $30; elsewhere $92. Each number may be ordered separately; *please specify number* when ordering an individual number. For prices and titles of recently released numbers, see the New Publications sections of the *Notices of the American Mathematical Society*.

Back number information. For back issues see the *AMS Catalog of Publications*.

Subscriptions and orders should be addressed to the American Mathematical Society, P. O. Box 5904, Boston, MA 02206-5904. *All orders must be accompanied by payment.* Other correspondence should be addressed to Box 6248, Providence, RI 02940-6248.

Memoirs of the American Mathematical Society is published bimonthly (each volume consisting usually of more than one number) by the American Mathematical Society at 201 Charles Street, Providence, RI 02904-2213. Second-class postage paid at Providence, Rhode Island. Postmaster: Send address changes to Memoirs, American Mathematical Society, P. O. Box 6248, Providence, RI 02940-6248.

Printed in the United States of America.

This volume was printed directly from author-prepared copy.

♾ The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability.

♻ Printed on recycled paper.

10 9 8 7 6 5 4 3 2 1 99 98 97 96 95 94

Contents

Abstract.

New methods are developed to describe prime ideals in skew polynomial rings $S = R[y; \tau, \delta]$, for automorphims τ and τ-derivations δ of a noetherian coefficient ring R. A complete description is given under the additional assumptions that R is an algebra over a field k on which τ and δ act trivially and that $\tau^{-1}\delta\tau = q\delta$ for some nonzero element $q \in k$. These hypotheses abstract behavior found in many quantum algebras, such as q-Weyl algebras and coordinate rings of quantum matrices, and specific examples along these lines are considered in detail. Finally, we prove, for a general class of n-fold iterated skew polynomial extensions (including the 1-fold extensions of the first sentence), that there cannot exist chains of more than $n + 1$ prime ideals all lying over a common prime ideal of the coefficient ring.

Key Words and Phrases: Noetherian ring, prime ideal, ring extension, skew polynomial ring, quantum algebra.

1. Introduction.

The study of ideals and representations of an algebra S is often based on the dual techniques of induction and restriction between S and a suitable subalgebra R. In some of the most extensively studied cases, S is equal to (or is a factor ring of) a skew polynomial ring $R[y;\tau,\delta]$. In fact, S is often obtained through an iterated skew polynomial construction $k[y_1][y_2;\tau_2,\delta_2]\cdots[y_n;\tau_n,\delta_n]$ over a base field k. Classical instances include Weyl algebras, enveloping algebras of solvable Lie algebras, group algebras of polycyclic groups, and the enveloping algebra $U(\mathrm{sl}_2(k))$. Examples currently under scrutiny are q-Weyl algebras, enveloping algebras of solvable Lie superalgebras, the q-enveloping algebra $U_q(\mathrm{sl}_2(k))$, and certain other quantum groups (e.g., [**13**], [**32**], [**38**]). In the "unmixed" cases – those of skew polynomial rings $R[y;\tau,\delta]$ where either τ or δ is trivial – a vast existing literature provides a thorough understanding of most of the classical iterated skew polynomial rings. Many of these methods, however, cannot be extended to the case where both τ and δ are nontrivial, as easy examples show. Thus different tools are required for investigations of the more recent instances of iterated skew polynomial rings.

Our primary aim is to introduce new methods for analyzing the structure of noetherian skew polynomial rings and, in particular, to develop means to find and classify the prime ideals in such rings. We obtain the most precise results for the *q-skew extensions* introduced in [**13**], namely skew polynomial rings $R[y;\tau,\delta]$ in which $\tau^{-1}\delta\tau = q\delta$ for some nonzero scalar q. In this setting – which includes for example the skew polynomial extensions appearing in q-Weyl algebras, enveloping algebras of solvable Lie superalgebras, and coordinate rings of quantum matrices – we present a complete description of the prime ideals. Next, we test our analysis of the q-skew case by considering in more detail some

Received by the editor October 22, 1991, and in revised form December 28, 1992.

The research of the first author was partially supported by a grant from the National Science Foundation.

Most of the second author's research for this paper was supported by a National Science Foundation postdoctoral research fellowship. Part of the research was done during October and November of 1990 while visting the Weizmann Institute of Science; the hospitality of this institute is gratefully acknowledged.

of the examples that motivated our study. In particular, we provide concise classifications of the prime ideals in q-Weyl algebras, and we study the prime ideals and factors of the coordinate ring of quantum 2×2 matrices. Also, our techniques yield a classification of the prime ideals in certain rings abstracted from enveloping algebras of solvable Lie superalgebras, leading to new proofs and generalizations of results in [**32**]. (In a sequel to this paper [**15**], we employ our analysis of the prime ideals in q-skew extensions to prove that every prime factor of the coordinate ring of quantum $n \times n$ matrices is an integral domain when q is not a root of unity; see also [**14**].)

In the final section of the paper, we return to general noetherian iterated skew polynomial extensions $S = R[y_1; \tau_1, \delta_1] \cdots [y_n; \tau_n, \delta_n]$ and extend a classical incomparability result to this setting. Under the assumption that R is invariant under τ_i, τ_i^{-1}, and δ_i for $1 \leq i \leq n$, we prove that S cannot contain a chain of more than $n+1$ prime ideals which all lie over a single prime ideal of R.

The following more detailed description of our strategy and results is to serve as a guideline for the paper.

1.1. We concentrate on skew polynomial rings (Ore extensions) of the form $S = R[y; \tau, \delta]$ where R is a noetherian ring, τ is an automorphism and δ is a τ-derivation of R. (Detailed background information is presented in section 2.) The ring S is generated by R and y subject only to the relations $yr = \tau(r)y + \delta(r)$ for $r \in R$. A standard version of the Hilbert Basis Theorem shows S to be noetherian; we shall use this fact without further mention. Complete descriptions of the prime ideals of S are already known in the cases when either τ or δ is trivial; we sketch them in (1.2). Moreover, full descriptions have been obtained when R is simple [**29**], and when R is commutative [**19**], [**13**].

1.2. For the cases $S = R[y; \delta]$ and $S = R[y; \tau]$, the standard techniques of contraction, reduction modulo an ideal, and localization give an immediate grip on the prime ideals. In the first case, a prime ideal P of S contracts to a δ-prime ideal I of R, the factor ring R/I has an artinian classical quotient ring A, and P can be written as the inverse image of a prime ideal of $A[y; \delta]$. Moreover, the prime ideals of $A[y; \delta]$ can be completely described on the basis of old methods due to Jacobson [**20**]. In the second case, there are two possibilities. Any prime ideal P of S which contains y must contract to a prime ideal Q of R, and then $P = Q + yS$. Otherwise, P contracts to a τ-prime ideal I of R, and P

can be written as the inverse image of a prime ideal of $A[y;\tau]$, where A is the Goldie quotient ring of R/I. Here too, classical methods provide a complete description of the prime ideals of $A[y;\tau]$.

Several immediate problems arise when one attempts to extend these techniques to the general case. First, if P is a prime ideal of S, the contraction $P \cap R$ need not be stable under either τ or δ (even when $y \notin P$). Second, no criteria are known for deciding which ideals of R are contractions of prime ideals of S. Third, even those prime ideals that can be obtained from skew polynomial rings $A[y;\tau,\delta]$ over an artinian ring A cannot be described by classical methods, which break down when τ is nontrivial and the radical of A is nonzero.

1.3. Our first tactic is to modify the classical modes of relating prime ideals of S to prime ideals of R. We start by proving that for any prime ideal P of S, the set of prime ideals of R minimal over $P \cap R$ is contained within a single τ-orbit of spec$R$. Consequently, the standard noncommutative notion of "lying over" (a prime ideal P of S *lies over* a prime ideal Q of R when Q is minimal over $P \cap R$) establishes a well-defined function from spec$S$ to the collection of τ-orbits of spec$R$.

The following specialization of the concept of "lying over" leads to a sharper picture. We shall say that a prime ideal P of S lies *directly* over a prime ideal Q of R provided P lies over Q and Q is an annihilator prime of S/P viewed as a right R-module. (In particular, P lies directly over Q when P contracts to Q.) Any prime ideal P of S lies directly over at least one prime ideal Q of R, and we further prove that the set of prime ideals of R minimal over $P \cap R$ has the form $\{Q, \tau^{-1}(Q), \ldots, \tau^{-m}(Q)\}$. Moreover, if the τ-orbit of Q is infinite, then P is the only prime ideal of S lying directly over Q, and P does not lie directly over any prime ideal of R distinct from Q. Thus "lying directly over" establishes a well-defined function from spec$S$ to the quotient space of spec$R$ modulo finite τ-orbits (cf. (1.4ii)).

To analyze the prime ideals of S lying directly over a given prime ideal Q of R, we study the induced bimodule $A \otimes_R S$ where A is the Goldie quotient ring of R/Q. We prove that a prime ideal P of S lies directly over Q if and only if some A-S-bimodule factor of $A \otimes_R S$ is a torsionfree faithful right (S/P)-module. In case the τ-orbit of Q is infinite, $A \otimes_R S$ has a unique maximal proper sub-bimodule B, and the right annihilator of $(A \otimes_R S)/B$ in S is the unique prime ideal of S lying directly over Q. This approach was influenced by analogous methods used in [**31**] and [**32**].

A considerably more detailed picture is obtained when the sub-bimodules of $A \otimes_R S$ can be computed. We carry out a full analysis in the case of q-skew extensions, as follows.

1.4. To simplify the discussion of the q-skew case, suppose that R is an algebra over a field k on which τ and δ act trivially. Moreover, assume that $\tau^{-1}\delta\tau = q\delta$ for some nonzero $q \in k$. We then say that (τ, δ) is a *q-skew derivation* and that S is a *q-skew polynomial ring* over R. (In general, we only require q to be a central unit of R such that $\tau(q) = q$ and $\delta(q) = 0$.) The analysis falls into several cases, depending on the characteristic of k and on whether q is a root of unity.

(i) We first sketch the outcome in the most accessible situations, namely those occurring when q is not a root of unity or when $q = 1$ and char$k = 0$. In these cases, the contraction to R of any prime ideal of S is either a δ-stable τ-prime ideal or a prime ideal which is not τ-stable; hence, we can catalog the prime ideals of S according to their contractions. If Q is a prime ideal of R which is not τ-stable, then at most one prime ideal of S contracts to Q, and there is an effective criterion to determine when this event occurs: there exists a prime ideal of S contracting to Q if and only if δ induces an inner τ-derivation from R to the Goldie quotient ring A of R/Q. In the positive case, the unique prime ideal of S contracting to Q can be exhibited as the right annihilator of an explicit simple A-S-bimodule factor of $A \otimes_R S$. Now suppose that J is a δ-stable τ-prime ideal of R. Then there is at least one prime ideal of S contracting to J, and the actions of τ and δ on the Goldie quotient ring A of R/J determine how many prime ideals of S contract to J. If δ is inner on A but no positive power of τ is inner on A, then the set of prime ideals of S contracting to J consists of either a single prime ideal or a pair of comparable prime ideals. On the other hand, if δ and some positive power of τ are inner on A, then the set of prime ideals of S contracting to J is homeomorphic to either spec$K[z]$ or spec$K[z, z^{-1}]$, where K is the field of central (τ, δ)-constants in A. (When A is simple, these results are based on the work of Leroy and Matczuk [**29**].)

(ii) To view the results given in (i) from a different perspective, we impose an equivalence relation $\sim$ on spec$R$, where $Q \sim Q'$ provided Q and Q' either coincide or belong to a common finite τ-orbit of spec$R$. As mentioned in (1.3), "lying directly over" establishes a single-valued map $\Phi\colon \operatorname{spec} S \to \operatorname{spec} R/\sim$. In the situation discussed in (i), our results lead to sharp descriptions of the image and fibers of Φ. In particular, each fiber of Φ is either a finite set or homeomorphic to $\operatorname{spec} K[z]$, plus or minus finitely many points.

(iii) If q is a root of unity, say a primitive t^{th} root, we no longer have a complete characterization of those ideals of R which occur as contractions of prime ideals of S. However, we can give a detailed description of the image and fibers of the map Φ introduced in (ii). For instance, each fiber of Φ is either finite or homeomorphic to $\operatorname{spec} K^{\ell}[z]$, plus or minus finitely many points, for a suitable positive integer ℓ and field K of constants. In addition, we obtain effective criteria to distinguish among the different possibilities for the fibers of Φ, as well as criteria to determine which equivalence classes appear in the image of Φ. These criteria are similar to those in (i), the main difference being that certain powers of δ must be taken into account. Specifically, the criteria involve δ^t as well as δ, and if $\operatorname{char} k = p > 0$ they involve δ, δ^t, δ^{tp}, $\delta^{tp^2}, \ldots$ For example, suppose that $\operatorname{char} k = 0$. Then a prime ideal Q of R with infinite τ-orbit occurs in the image of Φ if and only if δ^n induces an inner τ^n-derivation from R to the Goldie quotient ring of R/Q, for either $n = 1$ or $n = t$. On the other hand, a finite τ-orbit $\mathcal{X}$ in $\operatorname{spec} R$ occurs in the image of Φ if and only if the intersection of the prime ideals in $\mathcal{X}$ is stable under δ^t.

1.5. For a more detailed summary of our main results, we refer the reader to section 11. The main examples are found in sections 6, 12, 13, 14, which respectively contain analyses of prime ideals in the quadratic (-1)-skew extensions occurring in enveloping algebras of solvable Lie superalgebras, of irreducible representations of arbitrary q-skew extensions, of prime ideals in q-Weyl algebras, and of prime factors of the coordinate ring of quantum 2×2 matrices.

2. Preliminaries for $S = R[y; \tau, \delta]$.

The aim of this section is to sketch some basic background results on skew derivations and skew polynomial rings which will be used throughout the paper. In particular, we discuss the q-skew derivations and q-skew polynomial rings studied in [**13**], and we outline one class of examples.

Throughout this section R will denote a noetherian ring.

2.0. (i) We will rely on basic results from the theory of noetherian rings. Included are properties of right Krull dimension and classical Krull dimension, Goldie's Theorem, and elementary facts concerning localization at Ore sets. The reader is referred to [**16**], [**21**], or [**34**] for more information.

(ii) We denote the right Krull dimension of R by rKdim$R$, the classical Krull dimension of R by clKdim$R$, the set of prime ideals of R by spec$R$, and the set of maximal ideals of R by max$R$. Endow spec$R$ with the Jacobson (Zariski) topology, and each subset with the subspace topology. If Q is a prime ideal of R which is the annihilator of a submodule of an R-module M, then Q is said to be an *annihilator prime* of M. Now suppose that a ring S contains R as a subring. If P is a prime ideal of S and Q is a prime ideal of R then P is said to *lie over* Q provided Q is minimal over $P \cap R$. If in addition Q is an annihilator prime of the right R-module $(S/P)_R$, we shall say that P lies *directly* over Q (see (5.1), (5.2)). Finally, "$\subset$" will denote proper containment.

We next briefly summarize some basic information about skew polynomial rings, much of which is found in greater detail in [**13**, Sections 1 and 2].

2.1. (i) Let τ denote a ring endomorphism of R. A *(left) τ-derivation* δ of R is an additive map from R to itself such that $\delta(ab) = \tau(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. The pair (τ, δ) is referred to as a *(left) skew derivation.*

(ii) Let X denote a subset of R, and let Σ denote a set of functions from R to itself. If $\sigma(X) \subseteq X$ for all $\sigma \in \Sigma$, we say that X is Σ-*stable*. A Σ-*ideal* of R is an ideal which is Σ-stable. A proper Σ-ideal K of R is Σ-*prime* if for each pair of Σ-ideals I, J with $IJ \subseteq K$ it follows that $I \subseteq K$ or $J \subseteq K$. We refer to the collection of Σ-prime ideals of R as spec$^\Sigma R$, and we endow it with the obvious generalization of the Jacobson topology. The ring R is said to be Σ-*prime* provided (0) is a Σ-prime ideal, and when R is nonzero it is said to be Σ-*simple* provided (0) and R are the only Σ-ideals of R. Let τ be an endomorphism of R, and let δ be a left τ-derivation. If Σ is equal to $\{\tau\}$, $\{\delta\}$, or $\{\tau,\delta\}$, we use τ, δ, or (τ,δ) respectively.

(iii) Let S be a ring containing R as a subring. We say that S is a *skew polynomial ring* over R or that S is an *Ore extension* of R if there exists $y \in S$ for which S is a free left R-module with basis $1, y, y^2, \ldots$ such that $yR \subseteq Ry+R$. In this case, there exist an endomorphism τ of R and a τ-derivation δ on R such that $yr = \tau(r)y + \delta(r)$ for all $r \in R$. To summarize this data we write $S = R[y;\tau,\delta]$; if τ is the identity we write $S = R[y;\delta]$, while if δ is zero we write $S = R[y;\tau]$. We will denote the polynomial ring over R in one commuting indeterminate by $R[y]$.

(iv) Let τ be an endomorphism of R, and let $a \in R$. The assignment $r \mapsto ar - \tau(r)a$ is a left τ-derivation, denoted by δ_a, and is said to be an *inner* τ-derivation. Note that every τ-ideal of R is stable under δ_a. It follows from [**13**, 1.5c] that $R[y;\tau,\delta_a] = R[y-a;\tau]$.

(v) Let τ be an endomorphism of R and let δ be a τ-derivation. Suppose that I is an ideal of $R[y;\tau,\delta]$ and that $I \cap R$ is τ-stable. For $a \in I \cap R$ we then have both $ya \in I$ and $\tau(a)y \in I$, whence $\delta(a) \in I \cap R$. Thus $I \cap R$ is a (τ,δ)-ideal of R in this case.

(vi) If τ is an automorphism of R, δ is a τ-derivation, and I is a (τ,δ)-ideal of R then $IS = SI$ is an ideal of S. Letting τ and δ denote their induced actions on R/I, we have $S/IS \cong (R/I)[y;\tau,\delta]$.

(vii) Let τ be an inner automorphism of R, and let δ be a τ-derivation. Choosing an invertible element $u \in R$ such that $\tau(r) = u^{-1}ru$ for each $r \in R$, we see from an easy calculation that $u\delta$ is a derivation of R. Suppose further that $u\delta$ is an inner derivation, so that there exists $v \in R$ such that $u\delta(r) = vr - rv$ for all $r \in R$. We see then that

$$\delta(r) \;=\; u^{-1}(vr - rv) \;=\; (u^{-1}v)r - \tau(r)(u^{-1}v),$$

for all $r \in R$, and hence δ is an inner τ-derivation.

2.2. For the remainder of this section, we assume that $S = R[y;\tau,\delta]$ where τ is an automorphism of R.

(i) If I is an ideal of R such that $\tau(I) \subseteq I$, then $\tau(I) = I$. (See for example [**12**, p. 337].)

(ii) [**12**, Remarks 4*, 5*, p. 338] A proper τ-ideal I of R is τ-prime if and only if there exist a prime ideal Q minimal over I and a positive integer n such that $\tau^n(Q) = Q$ and $I = Q \cap \tau(Q) \cap \cdots \cap \tau^{n-1}(Q)$. In particular, a τ-prime ideal is semiprime.

(iii) [**5**, 2.1] If I is a δ-stable τ-prime ideal of R then IS is a prime ideal of S.

2.3. Suppose that R is (τ, δ)-prime, and let $\mathcal{C}$ denote the set of regular elements of R.

(i) It follows from [**13**, 2.3] that the prime radical N of R is a τ-prime ideal, that $\mathcal{C}$ is a denominator set of R, and that $R\mathcal{C}^{-1}$ is an artinian (τ, δ)-simple ring. By [**13**, 1.3], (τ, δ) extends uniquely to $R\mathcal{C}^{-1}$. Moreover, from [**13**, 1.4] it follows that $\mathcal{C}$ is a denominator set of S and that $S\mathcal{C}^{-1} = (R\mathcal{C}^{-1})[y; \tau, \delta]$. Finally, note that $R\mathcal{C}^{-1}/N\mathcal{C}^{-1}$ is a τ-simple ring, by (2.2ii).

(ii) In view of (i), let $A = R\mathcal{C}^{-1}$, let $B = A[y; \tau, \delta] = S\mathcal{C}^{-1}$, and let $\varphi: S \to B$ denote the natural embedding. Let $\mathcal{Y}$ denote the set of those prime ideals of S which lie over minimal prime ideals of R. Note that $\mathcal{Y}$ is precisely the set of prime ideals of S which are disjoint from $\mathcal{C}$ (since the minimal prime ideals of R are exactly the prime ideals of R disjoint from $\mathcal{C}$). Hence, $\mathcal{Y} = \{\varphi^{-1}(P) \mid P \in \operatorname{spec} B\}$. (See for example [**16**, 9.22].) Therefore, in order to describe the set of prime ideals of S which lie over minimal prime ideals of R, it suffices to describe $\operatorname{spec} B$. We first discuss the case that δ is inner on A.

(iii) Let A and B be as in (ii), and suppose that δ is inner on A. By (2.1iv), $B = A[x; \tau]$ for a suitable element $x \in B$; moreover, A is τ-simple.

A complete description of $\operatorname{spec} B$, which we sketch for the reader's convenience and to establish notation, may now be deduced from well-known results and techniques. (Most of this description is in fact implicit in [**20**, p. 38] and depends only on the τ-simplicity of A. This description has also recently appeared, in a more general setting, in [**10**].) To begin, it is immediate that the set of central elements of A which are fixed by τ is a field, say k.

Next, it is easy to see that $Q + Bx$ is a prime ideal of B for each prime ideal Q of A. The ideals of this form account for all of the prime ideals of B that contain x. Also, if A is not prime then the prime ideals in B of this form are exactly the prime ideals which contract to prime ideals of A, since a prime ideal of B which does not contain x will necessarily contract to the zero ideal of A by, for example, [**34**, 10.6.4ii,iii]. One consequence of these assertions is the following: If τ is extended to some automorphism of B, also denoted τ, and if

A is not prime, then the set of prime ideals of B which contract to prime ideals of A consists of a single finite τ-orbit of spec$B$.

We now see that to describe spec$B$ it suffices to specify the prime ideals of $E = A[x, x^{-1}; \tau]$, and it follows from [**24**, Theorem 1] that E is simple if and only if no positive power of τ is inner on A. Moreover, E is simple if and only if its center is k.

Now assume that E as above is not simple. From [**24**, Theorem 0] it follows that there exist an element $u \in A$ and a positive integer ℓ such that the following condition (ℓ) holds: u is invertible, $\tau(u) = u$, and $\tau^{\ell}(a) = u^{-1}au$ for all $a \in A$. (Observe that a nonzero $u \in A$ satisfies (ℓ) exactly when ux^{ℓ} is a central element of E.) Now assume that ℓ is minimal, that is, no elements of A satisfy (ℓ') for $0 < \ell' < \ell$. If vx^m is a central element of E for some integer m and some nonzero $v \in A$, then it is routine to check that $\ell \mid m$. We may now deduce that the center of B is $k[z]$, where $z = ux^{\ell}$, and that the center of E is $k[z, z^{-1}]$.

Let J be an arbitrary nonzero ideal of E, let $I = J \cap B$, and let n be the minimal degree of nonzero polynomials in I. It follows, for example, from the proof of [**24**, Theorem 0] that I contains a monic polynomial $f(x) \in B$ of degree n such that $xf(x) = f(x)x$ and $f(x)a = \tau^n(a)f(x)$ for all $a \in A$. From the minimality of n it follows that $I = Bf(x)$ and $J = Ef(x)$. Also from the minimality of n it follows that $f(x)$ has a nonzero constant term c. Now $ca = \tau^n(a)c$ for all $a \in A$, and $xc = cx$. Therefore, c is invertible and $c^{-1}f(x)$ is a central element of E. In particular, J is generated by a single element in the center of E.

We thus conclude that there are mutually inverse homeomorphisms between spec$E$ and spec$k[z, z^{-1}]$ obtained by contraction and extension. Similarly, there is a continuous closed finite-to-one surjection of spec$B$ onto spec$k[z]$ such that each fiber is a singleton except for the preimage of (z), and if R is prime this map is a homeomorphism. Furthermore, if τ extends to some automorphism of B then the fiber over (z) consists of a single finite τ-orbit of spec$B$.

(iv) Again let $A = RC^{-1}$ and $B = A[y; \tau, \delta] = SC^{-1}$, and suppose that R is prime. By (ii), the set of prime ideals of S which contract to zero can be identified with spec$B$. Since A is simple artinian, a description of spec$B$ may be deduced from [**29**, 2.2, 2.9]. This description may be briefly summarized as follows. Let k denote the subfield of the center of A consisting of central (τ, δ)-constants, that is, central elements c such that $\delta(c) = 0$ and $\tau(c) = c$. Then one of the following three possibilities occurs: (1) B is simple; (2) spec$B$ consists of exactly two prime ideals, one of which is zero; (3) there exists a central element $z \in B$, transcendental over k, such that each nonzero prime ideal of B is equal to $f(z)B$ for a unique monic irreducible polynomial $f(z) \in k[z]$. In the third case, contraction to $k[z]$ produces a homeomorphism between spec$B$ and spec$k[z]$. Also, the third case occurs if and only if B is not simple and some positive power of τ is inner on A. Alternately, this case occurs if and only if there exist a positive integer n, an element $\alpha \in A$, and a monic polynomial $f(y) \in B$ such that τ^n is inner on A and $f(y)y = (y+\alpha)f(y)$ while also $f(y)a = \tau^n(a)f(y)$ for all $a \in A$.

(v) It is easy to choose R and τ such that R is τ-prime but not prime, and the following such choice will be useful later. Let $k = \mathbf{Q}(z)$, and let τ be the $\mathbf{Q}$-algebra automorphism of $k[x]$ such that $\tau(x) = -x$ and $\tau(z) = z + \lambda$, for some fixed $\lambda \in \mathbf{Q}$. Finally, let $R = k[x]/(x^2 - 1)$, and let τ also denote the induced automorphism of R. (Note that there is a nonzero power of τ which is inner on R if and only if $\lambda = 0$.)

(vi) To completely analyze the prime ideals of the ring B defined in (ii) would require a description of spec$B$ in the case when R is not prime and δ is not inner on A. (It will follow from (3.6) that this case occurs if and only if the radical of A is nonzero.) While it is tempting to conjecture that some analog of the descriptions given in (iii) and (iv) should hold in the general setting of (ii), no such result is known. However, in the q-skew case, to be defined in (2.4), a description of spec$B$ of the expected form can be obtained using the results of the following section (see (10.1.6)). The availability of such a description is one of the main reasons why we are able to build a much better picture of the prime spectra of q-skew polynomial rings than of arbitrary skew polynomial rings.

2.4. We next describe some basic properties of q-skew derivations.

(i) Let (τ, δ) be a skew derivation on R. An element q of R is termed a *(τ, δ)-constant* if $\tau(q) = q$ and $\delta(q) = 0$. If q is a fixed central (τ, δ)-constant and $\delta\tau = q\tau\delta$, then we say that (τ, δ, q) (or (τ, δ)) is a *(left) q-skew derivation* on R. We refer in this situation to S as a *q-skew extension* of R, or as a *q-skew polynomial ring* over R.

(ii) Let (τ, δ, q) be as in (i). Observe that if q is invertible then

$$(q^{-1}y)\tau(r) = q^{-1}[(\tau^2(r))y + \delta\tau(r)] = (\tau^2(r))(q^{-1}y) + \tau\delta(r)$$

for all $r \in R$. Hence, τ extends to an automorphism of S mapping y to $q^{-1}y$.

(iii) Let (τ, δ, q) be as in (i), and let K be an ideal of R such that $q^i + q^{i-1} + \cdots + 1 \notin K$ for all $i = 0, 1, \ldots$. Then K is (τ, δ)-prime if and only if it is a δ-stable τ-prime ideal [**13**, 6.5]. Furthermore, if K is a minimal τ-prime ideal of R, then K must be δ-stable [**13**, 6.4].

2.5. For the remainder of this section, we assume that (τ, δ, q) is a q-skew derivation, for some fixed central invertible (τ, δ)-constant $q \in R$.

(i) For carrying out computations in S, we use the following q-Leibniz Rules noted in [**13**, 6.2]:

$$y^n a = \sum_{i=0}^{n} \tbinom{n}{i}_q \tau^{n-i}\delta^i(a) y^{n-i}$$

$$\delta^n(ab) = \sum_{i=0}^{n} \tbinom{n}{i}_q \tau^{n-i}\delta^i(a)\delta^{n-i}(b)$$

for all $a, b \in R$ and all nonnegative integers n, where $\binom{n}{i}_q$ is the evaluation at $t = q$ of the polynomial function

$$\binom{n}{i}_t = \frac{(t^n - 1)(t^{n-1} - 1)\cdots(t-1)}{(t^i - 1)(t^{i-1} - 1)\cdots(t-1)(t^{n-i} - 1)(t^{n-i-1} - 1)\cdots(t-1)}.$$

(See [**13**, Section 6] for a discussion of the q-binomial coefficients $\binom{n}{i}_q$.)

(ii) Several identities are useful for dealing with the q-binomial coefficients. For instance,

$$\tbinom{n}{i}_q = \tbinom{n-1}{i}_q + q^{n-i}\tbinom{n-1}{i-1}_q = \tbinom{n-1}{i-1}_q + q^i\tbinom{n-1}{i}_q$$

for all integers $n > i > 0$ [**2**, p. 35], [**39**, p. 26].

(iii) A convenient way to obtain further identities is to work within a skew polynomial ring $R[X][Y; \sigma]$ where σ is the automorphism of the polynomial ring $R[X]$ such that $\sigma(X) = qX$ and $\sigma(r) = r$ for $r \in R$. As noted by several authors (see for example [**9**, Satz 1] for one case),

$$(X + Y)^n = \sum_{i=0}^{n} \tbinom{n}{i}_q X^i Y^{n-i}$$

for all nonnegative integers n. (This formula can be obtained by an easy induction on n, using (ii).)

2.6. (i) We must distinguish between the cases in which q is or is not a root of unity, since this determines which of the q-binomial coefficients might vanish. For example, if $q^i + q^{i-1} + \cdots + 1$ is invertible in R for all positive integers i, then $\binom{n}{i}_q$ is invertible for $n \geq i \geq 0$. This includes the case that $q = 1$ and R is a $\mathbf{Q}$-algebra.

(ii) If t is an integer greater than 1 such that $q^{t-1} + q^{t-2} + \cdots + 1 = 0$ while $q^{i-1} + q^{i-2} + \cdots + 1$ is invertible for $i = 1, \ldots, t-1$, then $\binom{t}{i}_q = 0$ for $i = 1, \ldots, t-1$ (see e.g. [**13**, 7.3]). Note, on the other hand, that $\binom{n}{i}_q$ is invertible for $0 \leq i \leq n < t$.

(iii) Let t be as in (ii), and construct $R[X][Y;\sigma]$ as in (2.5iii). Then we have $(X+Y)^t = X^t + Y^t$, and so for any positive integer m we obtain

$$(X+Y)^{tm} = \sum_{j=0}^{m} \binom{m}{j} X^{tj} Y^{tm-tj}.$$

Consequently, $\binom{tm}{tj}_q = \binom{m}{j}$ for $j = 0, \ldots, m$, while $\binom{tm}{i}_q = 0$ for any i in $\{0, 1, \ldots, tm\}$ which is not divisible by t.

(iv) Continuing with the notation of (iii), observe for $n = 0, \ldots, t-1$ that

$$(X+Y)^{tm+n} = \sum_{i=0}^{n} \binom{n}{i}_q X^i Y^{tm+n-i} + [\text{terms of } Y\text{-degree} < tm],$$

whence $\binom{tm+n}{i}_q = \binom{n}{i}_q$ for $i = 0, \ldots, n$. Thus $\binom{tm+n}{i}_q$ is invertible for $0 \leq i \leq n < t$. In particular, $\binom{h}{1}_q$ is invertible for any positive integer h which is not divisible by t.

2.7. We label three possible cases into which R and q might fall. In case q is a nonzero element of a field contained within R as a subring (with the same identity), these cases cover all possibilities. Recall that we are already assuming that q is invertible in R.

Case $(*, 0)$: $q^i + q^{i-1} + \cdots + 1$ is invertible in R for all positive integers i. This includes the case that R is a $\mathbf{Q}$-algebra and $q = 1$.

Case $(0, t)$: R is a $\mathbf{Q}$-algebra and t is an integer such that $t > 1$ and $q^{t-1} + q^{t-2} + \cdots + 1 = 0$ while $q^{i-1} + q^{i-2} + \cdots + 1$ is invertible in R for $i = 1, \ldots, t-1$.

Case (p, t): p is a prime integer, R is a $\mathbf{Z}/p\mathbf{Z}$-algebra, and t is an integer such that $t > 1$ and $q^{t-1} + q^{t-2} + \cdots + 1 = 0$ while $q^{i-1} + q^{i-2} + \cdots + 1$ is invertible in R for $i = 1, \dots, t-1$. This includes the case that $\mathbf{Z}/p\mathbf{Z} \hookrightarrow R$ while $q = 1$ and $t = p$.

2.8. One class of examples of q-skew polynomial rings consists of the quantized Weyl algebras $A_1(T, q)$ studied for example in [**37**] and [**13**, Section 8]. (The quantum Weyl algebras studied in [**17**] are finite normalizing extensions of localizations of $A_n(\mathbf{C}, q)$, for q not a root of unity; see also [**27**].) Here $A_1(T, q) = T[x][y; \tau, \delta]$ where q is a central unit of the ring T, while τ is the automorphism of the polynomial ring $T[x]$ such that τ is the identity on T and $\tau(x) = qx$, and δ is the (unique) τ-derivation on $T[x]$ such that $\delta(T) = 0$ and $\delta(x) = 1$. We will also require the following modification of this construction obtained by allowing τ to act nontrivially on T. (The modified construction is similar to that studied by Jordan in [**25**]; however, the skew polynomial rings produced by his methods are generally not q-skew.)

Start with a ring T, an automorphism τ of T, and a central unit $q \in T$ such that $\tau(q) = q$. First construct the skew polynomial ring $T_1 = T[x; \tau^{-1}]$, and then extend τ to an automorphism of T_1 for which $\tau(x) = qx$. Let d be an arbitrary central element of T. We claim that there is a (unique) τ-derivation δ on T_1 such that $\delta = 0$ on T while $\delta(x) = d$. To see this, let $\phi \colon T \to M_2(T_1)$ be the map given by the rule $\phi(a) = \begin{pmatrix} \tau(a) & 0 \\ 0 & a \end{pmatrix}$, and set $c = \begin{pmatrix} qx & d \\ 0 & x \end{pmatrix}$. From the equations $xa = \tau^{-1}(a)x$ for $a \in T$, it follows that $c\phi(a) = \phi\tau^{-1}(a)c$ for all $a \in T$. Consequently, ϕ extends uniquely to a ring homomorphism $T_1 \to M_2(T_1)$ such that $\phi(x) = c$. Hence, there is a map $\delta \colon T_1 \to T_1$ such that $\phi(u) = \begin{pmatrix} \tau(u) & \delta(u) \\ 0 & u \end{pmatrix}$ for all $u \in T_1$, and since ϕ is a ring homomorphism, δ is the desired τ-derivation on T_1.

Now set $d = 1$. (The existence of τ-derivations δ on T_1 with more general values for $\delta(x)$ will only be used in section 14.) We thus obtain a ring $A_1(T, \tau, q) = T[x; \tau^{-1}][y; \tau, \delta]$ generated by $T \cup \{x, y\}$ and determined by the relations $yx = qxy + 1$ together with $xa = \tau^{-1}(a)x$ and $ya = \tau(a)y$ for all $a \in T$. Since $\tau^{-1}\delta\tau$ and $q\delta$ are τ-derivations on T_1 which agree on $T \cup \{x\}$, we conclude that $\tau^{-1}\delta\tau = q\delta$. Therefore (τ, δ) is a q-skew derivation on $T[x; \tau^{-1}]$.

2.9. We close this section by describing $\operatorname{spec} A_1(T, q)$, for a noetherian ring T, when it is further assumed that $q^i - 1$ is invertible for each positive integer i (cf. [**13**, 8.4] for the case that T is a field). The case when $q^t = 1$ for some integer $t > 1$ requires the techniques of subsequent sections, and is discussed in section 13.

Let $S = A_1(T, q)$, let $R = T[x]$, and let $u = [y, x] = (q-1)xy + 1$. We claim that

$$\operatorname{spec} S = \{uS + QS \mid Q \in \operatorname{spec} T[x] \text{ and } x \notin Q\} \cup \{IS \mid I \in \operatorname{spec} T\}.$$

To prove the above equality, first note from [**13**, 8.2] that u is a normal element of S and that the natural homomomorphism $T[x] \to S/uS$ extends to an isomorphism of $T[x, x^{-1}]$ onto S/uS. Thus the set of prime ideals of S which contain u is equal to $\{uS + QS \mid Q \in \operatorname{spec} T[x], x \notin Q\}$.

Next, let P be a prime ideal of S which does not contain u. We must prove that $P = IS$ for some prime ideal I of T, and we do so by adapting the proof of [**3**, 2.2(2)]. First, since $(P \cap T)R$ is a (τ, δ)-ideal of R, it follows that $(P \cap T)S$ is an ideal of S contained in P, and we may therefore assume without loss of generality that $P \cap T = 0$. It now suffices to show that $P = 0$; so assume that $P \neq 0$. Choose a nonzero polynomial in P, say $f(x, y)$, such that its total degree (i.e., the sum of the x-degree and y-degree) is minimal among nonzero polynomials in P. Note that the total degree of $f(x, y)$ is greater than zero. Straightforward calculations using (2.5i) show that either $[x, f(x, y)]$ or $[y, f(x, y)]$ may be represented as $ug(x, y)$, where $g(x, y)$ is a nonzero polynomial having total degree less than $f(x, y)$. However, because u is normal and P is prime, either $u \in P$ or $g(x, y) \in P$. Since both possibilties lead to a contradiction, we conclude that $P = 0$.

3. Tau-delta-prime coefficient rings.

Throughout let $S = R[y;\tau,\delta]$ where R is a noetherian ring and τ is an automorphism of R. In studying the prime ideals of S that lie over a given prime ideal Q of R, we will see that it is often useful to first factor out the largest (τ,δ)-ideal of R that is contained in Q. Assuming this procedure has been done, R becomes a (τ,δ)-prime ring. Further, as discussed in (2.3), the prime ideals of S that lie over minimal prime ideals of R can then be more efficiently studied after localizing R to its artinian classical quotient ring, which is (τ,δ)-simple. In order to take full advantage of these reductions, we now examine the structure of S in the case where R is (τ,δ)-prime, with particular emphasis on the case where R is artinian and (τ,δ)-simple. In the latter situation, provided that certain q-skew hypotheses hold, we prove that S is isomorphic to a matrix ring over a skew polynomial ring of the form $(R/J(R))[y';\tau^n,\delta']$.

If R is (τ,δ)-prime and commutative, it follows from [**13**, 3.3] that S is prime. We begin by showing that in general, if R is (τ,δ)-prime but noncommutative, then S is not necessarily even semiprime. However, if R is (τ,δ)-prime and τ extends to an automorphism of S, then S must be semiprime (3.3), although not necessarily prime (10.5).

Example 3.1. *An artinian (τ,δ)-simple ring R such that $S = R[y;\tau,\delta]$ is not semiprime.*

We use a modification of [**13**, 2.8]. Let k be a field, and let α denote the automorphism of k^4 given by the rule $\alpha(a,b,c,d) = (b,c,d,a)$. Let $U = k^4[x;\alpha]$, and let $T = k^4[x,x^{-1};\alpha]$. Extend α to an automorphism of T by setting $\alpha(x) = x$. Let $\tau = \alpha^{-1}$, and let δ be the inner τ-derivation of T given by

$$\delta(t) = (0,0,0,1)x^{-1}t - \tau(t)(0,0,0,1)x^{-1} = [(0,0,0,1),\tau(t)]x^{-1},$$

for $t \in T$. Note for $(a,b,c,d) \in k^4$ and any integer n that

$$\delta((a,b,c,d)x^n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4, \\ (0,0,-b,c)x^{n-1} & \text{if } n \equiv 1 \pmod 4, \\ (0,-a,0,c)x^{n-1} & \text{if } n \equiv 2 \pmod 4, \\ (-d,0,0,c)x^{n-1} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

In particular, $\delta(U) \subseteq U$. Since $\tau(x^4) = x^4$ and $\delta(x^4) = 0$, we see that Ux^4 is a (τ,δ)-ideal of U.

To see that Ux^4 is a maximal (τ, δ)-ideal, consider a (τ, δ)-ideal I of U properly containing Ux^4. There exists an element $u \in I \setminus Ux^4$ such that $u = (a, b, c, d)x^3$ for some nonzero $(a, b, c, d) \in k^4$. By replacing u if necessary by $\tau(u)$, $\tau^2(u)$, or $\tau^3(u)$, we may assume that $d \neq 0$. Hence, replacing u with $(0, 0, 0, d^{-1})u$ allows us to assume that $u = (0, 0, 0, 1)x^3$. Then $\delta^3(u) = (0, 0, -1, 0)$, and we conclude that $I = U$ and that Ux^4 is a maximal (τ, δ)-ideal of U.

Let $R = U/Ux^4$; then R is a 16-dimensional k-algebra. Allowing τ and δ to denote the induced actions on R, we see that R is (τ, δ)-simple. Let $\overline{x}$ denote the image of x in R, and let $v = (1, 0, 0, 0)\overline{x}$. Note that

$$\begin{aligned} yv &= (0, 1, 0, 0)\overline{x}y \ , \\ y^2v &= (0, 0, 1, 0)\overline{x}y^2 + (0, 0, -1, 0)y \ , \\ y^3v &= (0, 0, 0, 1)\overline{x}y^3 \ , \\ y^4v &= vy^4 \ . \end{aligned}$$

It follows that $v\overline{x}^2y^tv = \overline{x}^3(0, 0, 0, 1)y^tv = 0$ for all nonnegative integers t, since the equality holds for $t = 0, 1, 2, 3$. Since $v\overline{x}^3 = 0$, we conclude that $v\overline{x}^2Sv\overline{x}^2 = 0$. In particular, S is not semiprime. □

The situation improves if τ extends to an automorphism of S, which we will also denote by τ. Note however that the extension need not be unique.

Lemma 3.2. *If I and J are nonzero ideals of S such that $I = \mathrm{ann}_S J$ and $J = \mathrm{ann} I_S$, then $I \cap R$ and $J \cap R$ are nonzero.*

Proof: This follows immediately from the proof of [**13**, 3.2]. □

Part (i) of the following proposition was proved under the assumption that τ and δ commute by Voskoglou [**40**, 2.6].

Proposition 3.3. *Suppose that R is (τ, δ)-prime and that τ extends to an automorphism of S.*

(i) *S is τ-prime.*

(ii) *If R is (τ, δ)-simple then S is a finite direct product of pairwise isomorphic prime rings.*

Proof: (i) If S is not τ-prime then there exist nonzero τ-ideals I and J of S such that $IJ = 0$. Since $\mathrm{ann}_S J$ and $\mathrm{ann} I_S$ are τ-ideals we may assume that

$J = \mathrm{ann}I_S$ and $I = \mathrm{ann}_S J$. From (3.2) it follows that $I \cap R$ and $J \cap R$ are nonzero, and from (2.1v) that $I \cap R$ and $J \cap R$ are (τ, δ)-ideals of R. Since $(I \cap R)(J \cap R) = 0$, this contradicts the assumption that R is (τ, δ)-prime.

(ii) From (i) and (2.2ii) it follows that S is semiprime with distinct minimal prime ideals $P, \tau(P), \ldots, \tau^m(P)$ for some nonnegative integer m. We are done if $m = 0$. Now suppose that $m > 0$, and let $A = P \cap \tau(P) \cap \cdots \cap \tau^{m-1}(P)$. Observe that the ideals $A, \tau(A), \ldots, \tau^m(A)$ are independent, and set $J = A \oplus \tau(A) \oplus \cdots \oplus \tau^m(A)$. Then J is a nonzero τ-ideal of S. Since $A = \mathrm{ann}(\tau^m(P))_S$, it follows from (3.2) that $A \cap R \neq 0$, and so $J \cap R \neq 0$. By (2.1v), $J \cap R$ is a (τ, δ)-ideal. Hence, $J \cap R = R$ by the (τ, δ)-simplicity of R, and consequently $J = S$.

We now see that S is equal to the direct sum of ideals $A \oplus \tau(A) \oplus \cdots \oplus \tau^m(A)$. But $\mathrm{ann}A_S = \tau^m(P)$. Therefore, as a ring, $A \cong S/\tau^m(P) \cong S/P$. Hence, as rings, $\tau^i(A) \cong S/P$, for $0 \leq i \leq m$. Part (ii) follows. □

Remark 3.4. (i) We will give a more detailed description of the minimal prime ideals of S, under the hypotheses of (3.3), in (9.7).

(ii) Suppose that (τ, δ, q) is a q-skew derivation with q invertible in R, and that R is (τ, δ)-prime. It follows from (2.4ii) and (3.3i) that S is τ-prime. If $q^i + q^{i-1} + \cdots + 1 \neq 0$ for all positive integers i then it follows from (2.4iii) and (2.2iii) that S is prime. In (10.5) we will see that if $q^n + q^{n-1} + \cdots + 1 = 0$ for some positive integer n, then S need not be prime.

We next establish some conditions under which δ must be inner.

Lemma 3.5. [**8**, 1.4; **42**, Proposition 5, p. 249] *Suppose there exist orthogonal central idempotents $e_1, \ldots, e_n$ in R, for some $n > 1$, such that $e_1 + \cdots + e_n = 1$. Suppose further that $\tau(e_i) = e_{i+1}$, for $1 \leq i \leq n-1$, and that $\tau(e_n) = e_1$. Then δ is inner.* □

Proposition 3.6. *If R is (τ, δ)-prime and δ is not inner, then R is indecomposable as a ring.*

Proof: Since R is noetherian it contains only finitely many central idempotents, and the minimal central idempotents are permuted by τ. Now let e denote a central idempotent fixed by τ. Then for each $r \in R$, it follows that $\delta(er) = e\delta(r) + \delta(e)r$. In particular, $\delta(e) = \delta(e^2) = e\delta(e) + \delta(e)e$, and so $\delta(eR) \subseteq eR$. Hence, eR and $(1 - e)R$ are (τ, δ)-ideals. Since R is (τ, δ)-prime, we see that either $e = 0$ or $e = 1$. Consequently, the sum over a τ-orbit of minimal central idempotents must equal 1, so there exists precisely one such orbit.

We may now conclude that there exist orthogonal minimal central idempotents $e_1, \ldots, e_n \in R$ such that $e_1 + \cdots + e_n = 1$ and such that $\tau(e_i) = \tau(e_{i+1})$ for $1 \leq i \leq n$, where subscript arithmetic is performed modulo n. If $n > 1$, it follows from (3.5) that δ is inner. Hence $n = 1$ and R is indecomposable. $\square$

The rest of this section is mostly devoted to developing procedures for reducing, in the q-skew case, to settings where the coefficient ring is semisimple. In [**13**, Section 7] it is shown that in certain situations when (τ, δ) is a q-skew derivation, the skew polynomial ring S is isomorphic to a matrix ring over an ordinary differential operator ring. In particular, if R is commutative, artinian, local, δ-simple with maximal ideal M, and if $(1 - \tau)(R) \subseteq M$, then $S \cong M_t((R/M)[y'; \delta'])$ for some t and some derivation δ' on R/M [**13**, 7.8]. Using similar methods, we prove here that if R is artinian and (τ, δ)-simple and (τ, δ) is a q-skew derivation, then S is isomorphic to a matrix ring over a skew polynomial ring over $R/J(R)$. (Cf. also [**13**, 4.6].) In the case $(*, 0)$, there is nothing to prove, since then R is semisimple by (2.4iii).

Observe that when R is (τ, δ)-simple, the set of central (τ, δ)-constants in R forms a subfield of the center of R. Hence, if (τ, δ) is a q-skew derivation, one of the cases described in (2.7) must hold.

Proposition 3.7. *Assume that R is artinian and (τ, δ)-simple, and that (τ, δ) is a q-skew derivation. If J is the prime radical of R, then $S_S \cong (S/JS)^\ell$ for some positive integer ℓ, and thus $S \cong M_\ell(\operatorname{End}(S/JS)_S)$. In fact, $\ell = \operatorname{length} R_R / \operatorname{length}(R/J)_R$.*

Proof: Set $J_0 = R$ and $J_j = \{r \in R \mid \delta^i(r) \in J \text{ for all } i = 0, \ldots, j-1\}$ for $j > 0$. By [**13**, 6.3], each J_j is a τ-ideal of R, and $\bigcap_j J_j$ is a (τ, δ)-ideal. Then $\bigcap_j J_j = 0$ by (τ, δ)-simplicity. Since R is artinian, there must now be a positive integer ℓ such that $J_\ell = 0$ while $J_{\ell-1} \neq 0$.

For $j > 0$, observe that δ induces an additive embedding

$$f_j \colon J_j/J_{j+1} \to J_{j-1}/J_j.$$

Since δ is a q-skew τ-derivation, we see that f_j is a right R-module homomorphism satisfying $f_j(rb) = \tau(r) f_j(b)$ and $f_j \tau(b) = q \tau f_j(b)$ for all $r \in R$ and $b \in J_j/J_{j+1}$. Hence, f_j maps τ-stable R-R-sub-bimodules of J_j/J_{j+1}

onto τ-stable R-R-sub-bimodules of J_{j-1}/J_j. Since R/J is τ-simple by (2.3i), it now follows by induction that $f_1, \dots, f_{\ell-1}$ are isomorphisms of right R-modules. Now $\operatorname{length}(J_j/J_{j+1})_R = \operatorname{length}(R/J)_R$ for $j = 0, \dots, \ell-1$, and thus $\operatorname{length} R_R = \ell.\operatorname{length}(R/J)_R$.

In particular, $\delta^{\ell-1}$ induces a right R-module isomorphism $f_1 f_2 \cdots f_{\ell-1}$ from $J_{\ell-1}$ onto R/J. Hence, there exists $z \in J_{\ell-1}$ such that $\delta^{\ell-1}(z) \equiv 1 (\operatorname{mod} J)$ while also $J_{\ell-1} = zR$ and $zJ = 0$. Since $\delta^i(z) \in J$ for $i < \ell - 1$, we have

$$y^{\ell-1}z = \sum_{i=0}^{\ell-1} \binom{\ell-1}{i}_q \tau^{\ell-1-i}\delta^i(z)y^{\ell-1-i} \equiv 1 \pmod{JS},$$

by (2.5i), and consequently $zy^{\ell-1}z = z$. Thus the element $e = zy^{\ell-1}$ is an idempotent in S such that $eS = zS = J_{\ell-1}S$. For $j = 0, \dots, \ell-1$ we have

$$J_jS/J_{j+1}S \cong (J_j/J_{j+1}) \bigotimes\nolimits_R S \cong (R/J) \bigotimes\nolimits_R S \cong S/JS$$

as right S-modules. In particular, $eS = J_{\ell-1}S \cong S/JS$, and so all these right S-modules are projective. Therefore $S_S \cong (S/JS)^\ell$. $\square$

Lemma 3.8. *Assume that (τ, δ) is a q-skew derivation satisfying either $(0, t)$ or (p, t) of (2.7). Let U denote the subring of S generated by $R \cup \{y^t\}$*

(i) *δ^t is a τ^t-derivation of R, and δ^t commutes with τ. In particular, (τ^t, δ^t) is a 1-skew derivation of R.*

(ii) *If case $(0, t)$ holds for (τ, δ), then case $(*, 0)$ holds for (τ^t, δ^t). In this case, all (τ^t, δ^t)-prime ideals of R are τ^t-prime, and all minimal τ^t-prime ideals of R are δ^t-stable.*

(iii) *The subring U is equal to $R[y^t; \tau^t, \delta^t]$. Moreover, S is a free right or left U-module, with basis $\{1, y, \dots, y^{t-1}\}$, and the automorphism τ of S defined in (2.4ii) restricts to an automorphism of U fixing y^t.*

(iv) *Let δ also denote the inner τ-derivation $s \mapsto ys - \tau(s)y$ on S, since this τ-derivation agrees with δ on R. Then δ restricts to a q-skew τ-derivation on U. Finally, the identity map on U extends to a surjective ring homomorphism $\phi: U[\theta; \tau, \delta] \to S$ sending θ to y, and $\ker \phi$ is the ideal of $U[\theta; \tau, \delta]$ generated by the normal element $\theta^t - y^t$.*

Proof: That δ^t commutes with τ is immediate from the fact that $q^t = 1$. By (2.6ii), $\binom{t}{i}_q = 0$ for $i = 1, \dots, t-1$. Hence, it follows from (2.5i) that δ^t is a τ^t-derivation, and that $y^t r = \tau^t(r)y^t + \delta^t(r)$ for all $r \in R$. Statements (i) and (iii) now follow. If $(0,t)$ holds for (τ,δ), then $\mathbf{Q} \hookrightarrow R$, and it is clear that (τ^t,δ^t) falls under case $(*,0)$. Therefore (ii) follows from (2.4iii).

It remains to prove (iv). Since $\delta(R) \subseteq R \subseteq U$ and $\delta(y^t) = 0$, it follows that $\delta(U) \subseteq U$, and thus δ does restrict to a τ-derivation on U. Moreover, since the τ-derivations $\tau^{-1}\delta\tau$ and $q\delta$ agree on $R \cup \{y\}$, we see that $\tau^{-1}\delta\tau = q\delta$ on S. Thus (τ,δ) is a q-skew derivation on either S or U.

It is clear that there exists a surjective ring homomorphism $\phi: U[\theta;\tau,\delta] \to S$ as described, and that $\theta^t - y^t \in \ker\phi$. Further, it follows from (iii) that $\theta^t - y^t$ generates $\ker\phi$ as a right or left ideal of $U[\theta;\tau,\delta]$. We conclude in particular that $\theta^t - y^t$ is a normal element of $U[\theta;\tau,\delta]$, and (iv) is proved. $\square$

We include the trivial case $J = 0$ in the following theorem in order to make the notation more convenient for later use.

Theorem 3.9. *Assume that R is artinian and (τ,δ)-simple, and that (τ,δ) is a q-skew derivation satisfying $(0,t)$ of (2.7). Let J denote the prime radical of R, and set $B = S/JS$. If $J = 0$, set $n = 1$, while if $J \neq 0$, set $n = t$. Then set $U = R[y^n;\tau^n,\delta^n]$.*

(i) *J is a (τ^n,δ^n)-ideal of R, and thus $JU = UJ$ is a nilpotent ideal of U.*

(ii) *$U/JU \cong \mathrm{End}B_S$ via left multiplication.*

(iii) *$S \cong M_\ell(U/JU) \cong M_\ell((R/J)[y^n;\tau^n,\delta^n])$ for some positive integer ℓ.*

(iv) *There are inverse homeomorphisms $\varphi: \mathrm{spec}S \to \mathrm{spec}U$ and $\psi: \mathrm{spec}U \to \mathrm{spec}S$ given by the rules $\varphi(P) = \mathrm{l.ann}_U(B/BP)$ and $\psi(P') = \mathrm{r.ann}(B/P'B)_S$.*

(v) *Extend τ to automorphisms of U and S as in (2.4ii) and (3.8iii). Then $\varphi(\tau(P)) = \tau(\varphi(P))$ and $\psi(\tau(P')) = \tau(\psi(P'))$ for all $P \in \mathrm{spec}S$ and $P' \in \mathrm{spec}U$.*

Proof: These results are all trivial in case $J = 0$. Assume now that $J \neq 0$, and so $n = t$.

(i) Since J is semiprime and τ^t-stable, it must be an intersection of minimal τ^t-prime ideals, all of which are δ^t-stable by (3.8ii). Thus J is a (τ^t,δ^t)-ideal.

(ii) Let $\mathbf{I} = \mathbf{I}_S(JS)$ denote the idealizer of JS in S, and recall that $\mathbf{I}/JS \cong \mathrm{End}B_S$ via left multiplication. Since $U \cap JS = JU$, it thus suffices to show that $\mathbf{I} = U + JS$.

Since J is a (τ^t, δ^t)-ideal, $y^t J \subseteq JU$, from which we obtain $U + JS \subseteq \mathbf{I}$. If this inclusion is proper, choose $s \in \mathbf{I} \setminus (U + JS)$ of minimal degree, say degree d. Write

$$s = ay^d + by^{d-1} + [\text{terms of lower degree}]$$

for some $a, b \in R$. The leading term ay^d cannot be in $U + JS$, for then we would have $s - ay^d \in \mathbf{I}$ and consequently $s - ay^d \in U + JS$ by the minimality of d. Thus $a \notin J$ and d is not divisible by t. In particular, $\binom{d}{1}_q$ is invertible by (2.6iv).

In the notation of the proof of (3.7), δ induces a right R-module isomorphism from J/J_2 onto R/J because J is nonzero. Hence, there exists $c \in J$ such that $\delta(c) \equiv 1(\text{mod} J)$. Since $s \in \mathbf{I}$, we have $sc \in JS$. On the other hand,

$$sc = a\tau^d(c)y^d + \left[\binom{d}{1}_q a\tau^{d-1}\delta(c) + b\tau^{d-1}(c)\right]y^{d-1} + [\text{terms of lower degree}],$$

and so $\binom{d}{1}_q a\tau^{d-1}\delta(c) \in J$, because $b\tau^{d-1}(c) \in J$. Since $\binom{d}{1}_q$ is invertible and $\delta(c) \equiv 1$ $(\text{mod} J)$, we conclude that $a \in J$, a contradiction. Therefore $\mathbf{I} = U + JS$, as desired.

(iii) This follows immediately from (ii) and (3.7).

(iv) Set $E = \text{End} B_S$. As a right S-module, B is a finitely generated projective generator, because of (3.7). Hence, the functor $B \bigotimes_S (-)$ provides a category equivalence from S-Mod to E-Mod. Standard Morita theory (e.g. [**1**, 21.11]) now yields inverse lattice isomorphisms between the ideal lattices of S and E, given by $P \mapsto P' = \text{l.ann}_E(B/BP)$ and $P' \mapsto P = \text{r.ann}(B/P'B)_S$, such that S/P is Morita equivalent to E/P' for each P. These maps then restrict to inverse homeomorphisms between $\text{spec} S$ and $\text{spec} E$. Using (ii), we obtain corresponding homeomorphisms between $\text{spec} S$ and $\text{spec}(U/JU)$. Since JU is a nilpotent ideal of U, part (iv) follows.

(v) Because J is τ-stable, the action of τ on S induces an automorphism of B. Thus $\tau(\text{ann}_U(B/BP)) = \text{ann}_U(B/B\tau(P))$ and $\tau(\text{ann}(B/P'B)_S) = \text{ann}(B/\tau(P')B)_S$ for $P \in \text{spec} S$ and $P' \in \text{spec} U$, and (v) follows. $\square$

In (3.10–13) we consider the case (p, t).

Remark 3.10. The analog of (3.9) in the case (p, t) is more involved, even if $q = 1$. For example, let $R = k(u)[v]/(v^2)$ where k is a field of characteristic 2, let τ be the identity automorphism of R, and let δ be the derivation $u(\frac{\partial}{\partial u} + \frac{\partial}{\partial v})$ on R. Then $\delta^i(v) = u$ for all $i > 0$, and so the prime radical J of R is not invariant under any positive power of δ. Note, however, that J is invariant under $\delta^2 + \delta$. As we shall see in (3.12), it follows that our skew polynomial ring S is isomorphic to a matrix ring over $(R/J)[y^2 + y; \delta^2 + \delta]$.

Lemma 3.11. *Assume that R is artinian and (τ, δ)-simple, and that (τ, δ) is a q-skew derivation satisfying (p, t) of (2.7). Set $N = \{1, t, tp, tp^2, \ldots\}$, and let J denote the prime radical of R. Then there exist integers $n_1 < \cdots < n_k = n$ in N and elements $c_1, \ldots, c_{k-1}, c_k = 1$ in R such that:*

(i) *J is invariant under the map $\zeta = \sum_{i=1}^{k} c_i \delta^{n_i}$.*

(ii) *If $m_1 < \cdots < m_h$ in N and $r_1, \ldots, r_h \in R$ such that J is invariant under the map $\rho = \sum_{i=1}^{h} r_i \delta^{m_i}$, then either $r_h \in J$ or $m_h \geq n$.*

(iii) *$\tau(c_i) \equiv c_i \pmod{J}$ for all $i = 1, \ldots, k-1$ such that $n_i > 1$. If $n_1 = 1 < n$, then $\tau(c_1) \equiv qc_1 \pmod{J}$.*

(iv) *$c_i r \equiv \tau^{n-n_i}(r) c_i \pmod{J}$ for all $i = 1, \ldots, k-1$ and all $r \in R$.*

(v) *ζ induces a q^n-skew τ^n-derivation $\overline{\zeta}$ on R/J.*

(vi) *Let $s \in S$ such that $sJ \subseteq JS$. If the leading coefficient of s is not in J, then $n \mid \deg(s)$.*

Proof: In case $J = 0$, it is clear that the lemma holds with $n = n_1 = 1$. Assume now that $J \neq 0$.

Recall from (3.8i) that (τ^t, δ^t) is a 1-skew derivation on R. Since $\mathbf{Z}/p\mathbf{Z} \hookrightarrow R$, it follows using (2.5i) that δ^d is a τ^d-derivation for each $d \in N$. Hence, each δ^d, for $d \in N$, induces a right R-module homomorphism $f_d \colon J/J^2 \to R/J$. Since $\operatorname{Hom}((J/J^2)_R, (R/J)_R)$ is a finitely generated left R-module, the left R-submodule generated by the f_d must also be finitely generated.

Let n be the least integer in N such that f_n is either zero or a left R-linear combination of the f_d with $d < n$, say $f_n = -c_1 f_{n_1} - \cdots - c_{k-1} f_{n_{k-1}}$ for some integers $n_1 < \cdots < n_k = n$ in N and some elements $c_1, \ldots, c_{k-1}$ in R. (We allow the possibility that $k = 1$, in case $f_n = 0$.) Setting $c_k = 1$, we obtain $\sum_{i=1}^{k} c_i f_{n_i} = 0$, and (i) follows.

Note that since J is nonzero, it cannot be invariant under δ. Hence, $n > 1$, and so $t \mid n$ and $q^n = 1$.

(ii) Assume that $r_h \notin J$. Since R/J is τ-simple (2.3i), there exist elements u_0, v_0, ..., u_g, v_g in R such that $\sum_{j=0}^{g} u_j\tau^j(r_h)v_j \equiv 1 \pmod{J}$. For $j = 0,\ldots,g$, observe that J is invariant under $\tau^j\rho\tau^{-j}$, and that $\tau^j\rho\tau^{-j} = \sum_{i=1}^{h} q^{-m_ij}\tau^j(r_i)\delta^{m_i}$. Then J is invariant under the composition of $\tau^j\rho\tau^{-j}$ with left multiplication by $\tau^{-m_h}(v_j)$, from which we infer that J is invariant under the map

$$\rho_j = \sum_{i=1}^{h} q^{-m_ij}\tau^j(r_i)\tau^{m_i-m_h}(v_j)\delta^{m_i}.$$

Finally, J is invariant under $\rho' = \sum_{j=0}^{g} q^{m_ij}u_j\rho_j$. Since $\rho' = \sum_{i=1}^{h} r_i'\delta^{m_i}$ for some $r_i' \in R$ with $r_h' \equiv 1 \pmod{J}$, we conclude from the minimality of n that $m_h \geq n$.

(iii) Note first that J is invariant under $\tau\zeta\tau^{-1} = \sum_{i=1}^{k} q^{-n_i}\tau(c_i)\delta^{n_i}$, and hence also under the map

$$\zeta - q^n\tau\zeta\tau^{-1} = \sum_{i=1}^{k-1}(c_i - q^{n-n_i}\tau(c_i))\delta^{n_i}.$$

from (ii) it follows that $c_i - q^{n-n_i}\tau(c_i) \in J$ for $1 \leq i \leq k-1$. Now $t \mid n$, and $t \mid n_i$ as long as $n_i > 1$. In this case, $q^{n-n_i} = 1$ and so $\tau(c_i) \equiv c_i \pmod{J}$. If $n_1 = 1$, then $q^{n-n_1} = q^{-1}$, and so $\tau(c_1) \equiv qc_1 \pmod{J}$ in this case.

(iv) Given $r \in R$, observe that J is invariant under the composition of ζ with left multiplication by r, from which it follows that J is invariant under the map $\zeta_r = \sum_{i=1}^{k} c_i\tau^{n_i}(r)\delta^{n_i}$. Hence, J is invariant under the map

$$\tau^n(r)\zeta - \zeta_r = \sum_{i=1}^{k-1}\big(\tau^n(r)c_i - c_i\tau^{n_i}(r)\big)\delta^{n_i}.$$

In view of (ii), we must have $\tau^n(r)c_i - c_i\tau^{n_i}(r) \in J$ for $i = 1,\ldots,k-1$, and (iv) follows.

(v) That the induced map $\overline{\zeta}$ is a τ^n-derivation on R/J follows from (iv) together with the fact that each δ^{n_i} is a τ^{n_i}-derivation on R. For $i = 1,\ldots,k$, the composition of τ^n with $c_i\delta^{n_i}$ equals $\tau^n(c_i)\tau^n\delta^{n_i}$. Since $q^n = 1$, we have $\tau^n \circ (c_i\delta^{n_i}) = \tau^n(c_i)\delta^{n_i}\tau^n$. In view of (iii), $\tau^n(c_i) \equiv c_i \pmod{J}$, and hence $\tau^n \circ (c_i\delta^{n_i}) - c_i\delta^{n_i}\tau^n$ maps R into J. Thus τ^n commutes with $\overline{\zeta}$ on R/J.

(vi) We may assume that $s = a_1y + \ldots + a_my^m$ for some $a_i \in R$ with $m > 0$ and $a_m \notin J$. For those $j \in \{1,\ldots,m\}$ which are not divisible by t, set $j^* = 1$, and note from (2.6iv) that $\binom{j}{j^*}_q$ is invertible in R. For $j \in \{1,\dots,m\}$ such that $t \mid j$, write $j = tp^{\mu_j}\lambda_j$ for some nonnegative integer μ_j and some positive integer λ_j not divisible by p, and set $j^* = tp^{\mu_j}$. Using (2.6iii) together with the analogous result for ordinary binomial coefficients in characteristic p, we find that

$$\binom{j}{j^*}_q = \binom{p^{\mu_j}\lambda_j}{p^{\mu_j}} = \binom{\lambda_j}{1} = \lambda_j$$

while $\binom{j}{i}_q = 0$ for $i = 1,\dots,j^*-1$. Again, $\binom{j}{j^*}_q$ is invertible in R.

Let $K = \{j \in \{1,\dots,m\} \mid a_j \notin J\}$ and $\kappa = \max\{j - j^* \mid j \in K\}$, and note that $\kappa \ge m - m^*$. Then set $L = \{j \in K \mid j - j^* = \kappa\}$. For $j \in K$ and $c \in J$, we have

$$y^j c = \tau^j(c)y^j + \binom{j}{j^*}_q \tau^{j-j^*}\delta^{j^*}(c)y^{j-j^*} + [\text{terms of degree less than } \kappa].$$

Since $sc \in JS$, we now infer that

$$\sum_{j\in L} \binom{j}{j^*}_q a_j\tau^\kappa\delta^{j^*}(c) \in J.$$

For $j \in L$, set $b_j = \binom{j}{j^*}_q q^{-\kappa j^*} a_j$, and note that $b_j \notin J$. Moreover,

$$\sum_{j\in L} b_j\delta^{j^*}\tau^\kappa(c) \in J$$

for all $c \in J$, and so J is invariant under the map $\sum_{j\in L} b_j\delta^{j^*}$.

Finally, let $h = \max L$. Since $b_j \notin J$ and $j^* = j - \kappa$ for all $j \in L$, we conclude from (ii) that $h^* \ge n$. We also have $h - h^* = \kappa \ge m - m^*$, and so $m^* \ge h^* \ge n$. Since $n, m^* \in N$, it follows that $n \mid m^*$, and therefore $n \mid m$. □

Theorem 3.12. *Assume that R is artinian and (τ,δ)-simple, and that (τ,δ) is a q-skew derivation satisfying (p,t) of (2.7). Let J denote the prime radical of R, and set $B = S/JS$. Let $n_1,\dots,n_k, n$, $c_1,\dots,c_k$, $\zeta,\overline{\zeta}$ be as in (3.11), and set $z = \sum_{i=1}^k c_iy^{n_i}$.*

(i) *The set $V = JS + \sum_{m=0}^\infty Rz^m$ is a subring of S, and JS is an ideal in V.*

(ii) *$V/JS \cong (R/J)[z;\tau^n,\overline{\zeta}]$, and $V/JS \cong \mathrm{End}B_S$ via left multiplication.*

(iii) *$S \cong M_\ell(V/JS)$ for some positive integer ℓ.*

(iv) *There are inverse homeomorphisms* $\operatorname{spec}S \to \operatorname{spec}(V/JS) \to \operatorname{spec}S$ *given by the rules* $P \mapsto \text{l.ann}_{V/JS}(B/BP)$ *and* $P' \mapsto \text{r.ann}(B/P'B)_S$.

Proof: (i) It follows from (2.6iii) that $\binom{n_i}{j}_q = 0$ for $1 \le i \le k$ and $1 \le j \le n_i - 1$, and then from (2.5i) that $y^{n_i}r = \tau^{n_i}(r)y^{n_i} + \delta^{n_i}(r)$ for $1 \le i \le k$ and $r \in R$. In view of (3.11iv), we obtain $zr \equiv \tau^n(r)z + \zeta(r) \pmod{JS}$ for all $r \in R$, and (i) follows.

(ii) From the relations $zr \equiv \tau^n(r)z + \zeta(r) \pmod{JS}$ just noted, we see that $V/JS = \overline{R}[z; \tau^n, \overline{\zeta}]$ where $\overline{R}$ denotes the image of R in V/JS. Since R/J is naturally isomorphic to $\overline{R}$, the first part of (ii) follows.

To prove the second part, it suffices to show that the idealizer $\mathbf{I} = \mathbf{I}_S(JS)$ is equal to V. Since JS is an ideal in V, we certainly have $V \subseteq \mathbf{I}$. If this inclusion is proper, choose $s \in \mathbf{I} \setminus V$ of minimal degree, say degree m. By the minimality of m, the leading coefficient c of s is not in J. Hence, $n \mid m$ by (3.11vi). But now $s' = s - cz^{m/n}$ is an element of $\mathbf{I}$ with degree less than m. By minimality, $s' \in V$, and then $s \in V$, a contradiction. Therefore $V = \mathbf{I}$, as desired.

(iii) This is immediate from (ii) and (3.7).

(iv) This follows from (ii) as in the proof of (3.9iv). □

The conclusions of (3.12) improve considerably if we impose the hypothesis that no positive power of τ induces an inner automorphism of R/J, as follows.

Corollary 3.13. *Assume that R is artinian and (τ, δ)-simple, and that (τ, δ) is a q-skew derivation satisfying (p, t) of (2.7). Let J denote the prime radical of R, and set $B = S/JS$. Assume that no positive power of τ induces an inner automorphism of R/J. Then there exists a positive integer n in the set $N = \{1, t, tp, tp^2, \dots\}$ such that*

(i) *J is a (τ^n, δ^n)-ideal of R, but J is not δ^d-stable for any $d < n$ in N.*

(ii) *Set $U = R[y^n; \tau^n, \delta^n]$. Then $JU = UJ$ is a nilpotent ideal of U.*

(iii) *$U/JU \cong \operatorname{End}B_S$ via left multiplication, and $S \cong M_\ell(U/JU)$ for some positive integer ℓ.*

(iv) *There are inverse homeomorphisms $\varphi\colon \operatorname{spec}S \to \operatorname{spec}U$ and $\psi\colon \operatorname{spec}U \to \operatorname{spec}S$ given by the rules $\varphi(P) = \text{l.ann}_U(B/BP)$ and $\psi(P') = \text{r.ann}(B/P'B)_S$.*

(v) *Extend τ to automorphisms of U and S as in (2.4ii) and (3.8iii). Then $\varphi(\tau(P)) = \tau(\varphi(P))$ and $\psi(\tau(P')) = \tau(\psi(P'))$ for all $P \in \operatorname{spec}S$ and $P' \in \operatorname{spec}U$.*

Proof: Let $n_1, \ldots, n_k, n$, $c_1, \ldots, c_k$, $\zeta, \overline{\zeta}$, z, V be as in (3.11), (3.12). Note from (2.6iii) and (2.5i) that δ^n is a τ^n-derivation on R, and that $y^n r = \tau^n(r)y^n + \delta^n(r)$ for all $r \in R$.

We claim that $c_1, \ldots, c_{k-1} \in J$. Once this is established, we may replace $c_1, \ldots, c_{k-1}$ by 0, yielding $\zeta = \delta^n$ and $z = y^n$, and then $V = JS + U$. Hence, (i) will follow from (3.11i,ii), and (iii), (iv) from (3.12ii-iv). Statement (ii) is clear.

Thus it remains to show that $c_i \in J$ for any i in $\{1, \ldots, k-1\}$. Set $c = c_i + J$ and $m = n - n_i$. From (3.11iii,iv), we have either $\tau(c) = c$ or $\tau(c) = qc$, and $cr = \tau^m(r)c$ for all $r \in R$. Hence, cR is a τ-ideal of R/J. By assumption, τ^m does not induce an inner automorphism of R/J, and so c cannot be invertible in R/J. Thus $cR \neq R/J$, and so by τ-simplicity (2.3i) we conclude that $c = 0$. Therefore $c_i \in J$, as desired.

Part (v) follows as in the proof of (3.9v). □

In the case that R is an algebra over a subfield k consisting of central (τ, δ)-constants, we shall be interested in the finite dimensional representations of S. To this end, we need to understand finite dimensional (τ, δ)-simple factor algebras of R. Provided k is algebraically closed, the Noether-Skolem Theorem (e.g., [**18**, 4.3.1]) simplifies the discussion in the following manner.

Lemma 3.14. *Assume that R is an algebra over an algebraically closed subfield k consisting of central (τ, δ)-constants. If R is finite dimensional and τ-simple, then δ is an inner τ-derivation of R and some positive power of τ is an inner automorphism of R.*

Proof: As in the proof of (3.6), there exist orthogonal minimal central idempotents $e_1, \ldots, e_n \in R$ such that $e_1 + \ldots + e_n = 1$ and such that $\tau(e_i) = \tau(e_{i+1})$ for $i = 1, \ldots, n$, where subscripts are considered modulo n. From (2.2ii) it follows that each Re_i is a simple k-algebra. Since R is finite dimensional and k is algebraically closed, the center of Re_i must be ke_i. The restriction of τ^n to Re_i is a k-algebra automorphism of Re_i, and by the Noether-Skolem Theorem it must be inner. Therefore τ^n is an inner automorphism of R.

If $n > 1$, then δ is inner by (3.6). Now suppose that $n = 1$; then R is a central simple k-algebra, and τ is an inner automorphism of R. Moreover, another application of the Noether-Skolem Theorem shows that every k-linear derivation of R is inner (e.g., [**18**, Proposition, p. 100]). Therefore, by (2.1vii), δ is an inner τ-derivation on R. $\square$

4. Each prime ideal of S is associated to a unique τ-orbit in spec$R$.

In this section we apply to skew polynomial rings the methods of [**7**] and [**41**] used to study additivity principles. The main results of the section concern skew polynomial rings $S = R[y;\tau,\delta]$ where R is noetherian and τ is an automorphism. In this setting, we prove that for each prime ideal P of S, the prime ideals of R minimal over $P \cap R$ are contained within a single τ-orbit of spec$R$. (A refined version of this assertion will be presented in (5.12).) We also prove that the ring $R/(P\cap R)$ has an artinian classical quotient ring embedded naturally into the Goldie quotient ring of S/P.

4.1. We begin with a specialization of a result of Warfield [**41**, Lemma 2]. Let T be a noetherian ring, and let R be a noetherian subring of T. Let P be a prime ideal of T, let C be the Goldie quotient ring of T/P, and let $\varphi\colon T \to C$ be the canonical ring homomorphism. Fix an R-C-bimodule composition series $0 = C_0 \subset \cdots \subset C_m = C$ for C, and let $Q_i = \operatorname{ann}_R(C_i/C_{i-1})$ for $1 \le i \le m$. Recall that the right or left annihilator of a simple bimodule is a prime ideal.

It is immediate that $P \cap R \subseteq Q_i$ for $1 \le i \le m$. Also, if Q is a prime ideal of R minimal over $P\cap R$, then $Q \in \{Q_1, \ldots, Q_m\}$ since $Q_1 \cdots Q_m \subseteq P \cap R \subseteq Q$.

Lemma. *For $1 \le i \le m$ there exist R-R-sub-bimodules L_i and M_i of ${}_RC_R$ satisfying the following properties:*

(i) $C_{m-1} \subseteq M_i \subset L_i \subseteq C$.

(ii) $Q_m = \operatorname{ann}_R(L_i/M_i)$ *and* $Q_i = \operatorname{ann}(L_i/M_i)_R$.

(iii) L_i/M_i *is a torsionfree* (R/Q_m)-(R/Q_i)*-bimodule.*

(iv) $Q_mT \subseteq \varphi^{-1}(C_{m-1}) \subseteq \varphi^{-1}(M_i) \subset \varphi^{-1}(L_i) \subseteq \varphi^{-1}(C) = T$.

In particular, there exists an R-R-bimodule subfactor of T/Q_mT which is a nonzero torsionfree (R/Q_m)-(R/Q_i)-bimodule.

Proof: This follows directly from [**41**, Lemma 2] and its proof. □

4.2. Let $S = R[y;\tau,\delta]$ where R is noetherian and τ is an automorphism of R. Let B denote an arbitrary R-R-bimodule subfactor of S, and let $b \in B$. It is routine to verify that $R.b.R$ is noetherian as both a right and left R-module.

4.3. Let R and S be rings. If ψ is a ring endomorphism of R and M is a right R-module, then let M^ψ denote the new R-module obtained by the action

$m * r = m.\psi(r)$ for $r \in R$ and $m \in M$. If M is an S-R-bimodule, then M^ψ denotes the S-R-bimodule obtained in similar fashion.

Lemma. *Let R be a noetherian ring, let ψ be an automorphism of R, let Q be a prime ideal of R, and let A denote the Goldie quotient ring of R/Q. Then ${}_A(A^\psi)_R$ is a simple A-R-bimodule.*

Proof: The case where ψ is the identity is proved in the first paragraph of the proof of [**16**, 7.27]. The general case follows since the collection of sub-bimodules of ${}_A(A^\psi)_R$ is identical to the collection of sub-bimodules of ${}_AA_R$. $\square$

4.4. Let $S = R[y;\tau,\delta]$ where R is noetherian and τ is an automorphism. Let P be a prime ideal of S, and let C be the Goldie quotient ring of S/P. Setting $T = S$, choose $C_0, \ldots, C_m$ and $Q_1, \ldots, Q_m$ as in (4.1). In particular, $\{Q_1, \ldots, Q_m\}$ contains the set of prime ideals of R minimal over $P \cap R$.

Proposition. *Fix $i \in \{1, \ldots, m\}$. Then there exists a non-negative integer ℓ such that $\tau^\ell(Q_i) = Q_m$.*

Proof: Let A be the Goldie quotient ring of R/Q_m. From (4.1) we see that there exists an R-R-bimodule subfactor B of S/Q_mS such that the left annihilator of B is Q_m, the right annihilator of B is Q_i, and such that B is torsionfree on each side as an (R/Q_m)-(R/Q_i)-bimodule. Using (4.2), we may further assume that B is finitely generated on each side. Next, choose a simple A-R-sub-bimodule M in $A \otimes_R B$. Note that Q_i is the annihilator of M_R, and also note that ${}_AM_R$ is an A-R-bimodule subfactor of ${}_A[A \otimes_R (S/Q_mS)]_R$.

Let V denote the A-R-bimodule $A \otimes_R S = A \otimes_R (S/Q_mS)$. Note that the set $\{1 \otimes y^j \mid j \geq 0\}$ forms a free basis for ${}_AV$. Also note, for $r \in R$ and $j \geq 0$, that

$$(1 \otimes y^j).r \ \in \ \tau^j(r).(1 \otimes y^j) + \sum_{k<j} A.(1 \otimes y^k).$$

Hence, there is a filtration $0 = V_0 \subset V_1 \subset \cdots$ of A-R-sub-bimodules of V such that $V = \bigcup_j V_j$, and such that ${}_A(V_{j+1}/V_j)_R \cong {}_A(A^{\tau^j})_R$ for $j \geq 0$.

Recall from (4.3) that each ${}_A(A^{\tau^j})_R$ is a simple bimodule. Hence, there exists a non-negative integer ℓ such that ${}_AM_R \cong {}_A(A^{\tau^\ell})_R$. Consequently, $Q_i = \tau^{-\ell}(Q_m)$, and the lemma follows. (A more thorough examination of the structure of V will occur in section 5.) $\square$

4.5. We next provide a lemma adapted directly from [**7**, 5.1] and [**41**, Corollary 1]. Let T be a prime noetherian ring, let S be a noetherian subring of T, and let R be a noetherian subring of S. Let C be the Goldie quotient ring of T, let $0 = C_0 \subset \cdots \subset C_m = C$ be an R-C-bimodule composition series for C, and let $Q_i = \operatorname{ann}_R(C_i/C_{i-1})$, for $1 \leq i \leq m$.

Lemma. *Suppose that $Q_1, \ldots, Q_m$ are all minimal prime ideals of R and that $R.t.R$ is noetherian as a right and left R-module for each $t \in T$.*

(i) *R has an artinian classical quotient ring.*

(ii) *If $\mathcal{C}$ is the set of regular elements of R then $\mathcal{C}$ is an Ore set of regular elements of S.*

Proof: Let N denote the prime radical of R and let $\mathcal{C}(N)$ denote the set of elements of R regular modulo N. First observe as follows that each element of $\mathcal{C}(N)$ is a regular element of T. Suppose that $x \in \mathcal{C}(N)$ and that $xt = 0$ for some nonzero element $t \in T$. Choose i such that $\bar{t} = t + C_i/C_{i-1}$ is nonzero. From [**16**, 6.4] it follows that x is regular modulo Q_i. However, it follows from [**21**, 5.1.1] that the simple R-C-bimodule C_i/C_{i-1} must be torsionfree as a left (R/Q_i)-module. But this last statement contradicts the fact that $x\bar{t} = 0$. Hence, $\mathcal{C}(N)$ is a subset of the set of regular elements of T, since T is left and right noetherian. Consequently, $\mathcal{C}(N)$ is a subset of the sets of regular elements of both R and S.

Now let $\mathcal{C}$ be the set of regular elements of R. We see from the above paragraph and [**16**, 10.8] that $\mathcal{C} = \mathcal{C}(N)$. Hence, (i) follows from Small's Theorem; see for example [**16**, 10.10].

The proof of (ii) follows from [**7**, 5.1] since $\mathcal{C}$ is a set of regular elements of S. $\square$

We close with our primary localization-theoretic result for skew polynomial rings.

Theorem 4.6. *Let $S = R[y; \tau, \delta]$ where R is a noetherian ring and τ is an automorphism of R. Let P be a prime ideal of S, let $F = S/P$, and let $E = R/(P \cap R)$. Let $\mathcal{C}$ denote the set of regular elements of E.*

(i) *$\mathcal{C}$ is an Ore set of E, and $E\mathcal{C}^{-1} = \mathcal{C}^{-1}E$ is the artinian quotient ring of E.*

(ii) *$\mathcal{C}$ is an Ore set of regular elements of F. In particular, F naturally embeds into its quotient ring $F\mathcal{C}^{-1} = \mathcal{C}^{-1}F$, and the artinian quotient ring of E naturally embeds into the Goldie quotient ring of F.*

Proof: Let $C_0, \ldots, C_m = C$ and $Q_1, \ldots, Q_m$ be as in (4.1), with $T = S$. As noted in (4.1), $\{Q_1, \ldots, Q_m\}$ contains the set of prime ideals of R minimal over $P \cap R$. Let d denote the classical Krull dimension of R/Q_m. It follows from (4.4) that the rings R/Q_i for $1 \leq i \leq m$ all have classical Krull dimension equal to d. It is therefore straightforward to deduce that $\{Q_1, \ldots, Q_m\}$ is precisely the set of prime ideals of R minimal over $P \cap R$. The theorem now follows from (4.2) and (4.5). $\square$

Remark 4.7. In the situation of (4.6), the Additivity Principle can be applied to show that if Q is any prime ideal of R minimal over $P \cap R$, then $\operatorname{rank}(R/Q)$ divides $\operatorname{rank}(S/P)$. We delay this result until (5.14), in order to present a more precise version.

5. Annihilator primes and induced bimodules.

We assume throughout that S is a skew polynomial ring $R[y;\tau,\delta]$ where R is a noetherian ring and τ is an automorphism of R. Building on the previous section, we give a more detailed and more constructive analysis of the prime ideals of R minimal over a contraction $P \cap R$, where P is an arbitrary prime ideal of S. Our investigation is based on an analysis of the annihilator primes in R of $(S/P)_R$ (note since R is noetherian that the set of such prime ideals is never empty), and on an analysis of the annihilator primes of factor bimodules of the bimodules $A \otimes_R S$, where A is the Goldie quotient ring of a prime factor ring of R. The latter annihilator primes will also play a fundamental role in the sequel, when we study the prime ideals of S that lie over a given prime ideal of R.

The approach in this section is in part adapted from [**31**] and [**32**].

Definition 5.1. Let L be a ring and let K be a subring of L. If Q is a prime ideal of K and P is a prime ideal of L which lies over Q then we say that P *lies directly over* Q provided Q is an annihilator prime of the right K-module $(L/P)_K$.

Remark 5.2. (i) Let L be a ring and let K be a noetherian subring. Let P be a prime ideal of L lying over a prime ideal Q of K. If $P \cap K$ is a semiprime ideal, Q is an annihilator prime of $K/(P \cap K)$ (e.g., [**16**, 6.1d]), and therefore P lies directly over Q. Consequently, if L is a noetherian ring and is a normalizing extension of K, then a prime ideal P of L lies over a prime ideal Q of K if and only if P lies directly over Q. (See for example[**16**, 7.27] for the fact that all prime ideals of L contract to semiprime ideals of K in this case.) In particular, "lying directly over" is equivalent to "lying over" for a ring extension of the form $R \hookrightarrow R[y;\tau]$.

(ii) Suppose that $S = R[y;\delta]$. It was proved in [**23**, 1.3iii, 2.2iii] that if P is a prime ideal of S then the prime radical in R of $P \cap R$ is a prime ideal. (Observe that this conclusion also follows from (4.4).) Consequently, "lying over" and "lying directly over" are equivalent in this case also.

(iii) We will see in (7.7) that the realization of $U(\mathrm{sl}_2(k))$ as an iterated skew polynomial ring provides a well-known example where "lying directly over" is strictly stronger than "lying over". Furthermore, the following example shows that these concepts need not coincide even for a (-1)-skew extension. Set $S =$

$A_1(k[z], \tau, -1)$ as in (2.8), where k is a field and τ is the k-algebra automorphism of the polynomial ring $k[z]$ sending z to $z+1$. Then $S = R[y; \tau, \delta]$, where $R = k[z][x; \tau^{-1}]$. Note that S is the k-algebra given by generators x, y, z and relations

$$xz = (z-1)x, \qquad yz = (z+1)y, \qquad yx = -xy+1.$$

Since these same relations are satisfied by the matrix units e_{12}, e_{21}, e_{11} in $M_2(k)$, there is a k-algebra homomorphism $\phi\colon S \to M_2(k)$ such that

$$\phi(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \phi(y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \phi(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly ϕ is surjective, and so $P = \ker\phi$ is a prime ideal of S. It is easily checked that

$$P \cap R = x^2R + xzR + z(z-1)R = Q_1Q_0$$

where $Q_1 = xR + (z-1)R$ and $Q_0 = xR + zR$. Thus P lies over Q_1 and Q_0. Now observe that $\phi^{-1}(\left(\begin{smallmatrix} 0 & k \\ 0 & k \end{smallmatrix}\right))/P$ is an essential right R-submodule of S/P with right annihilator Q_0. Hence, Q_0 is the unique annihilator prime of $(S/P)_R$. Therefore P lies directly over Q_0, but P does not lie directly over Q_1.

Note that in the preceding example the prime ideals of R minimal over $P \cap R$ form a "connected segment" of the τ-orbit of Q_0, namely the set $\{Q_0, \tau^{-1}(Q_0)\}$. This illustrates a general pattern which will be presented in (5.12).

The next lemma shows that for $R \hookrightarrow S$ there is some redundancy in the above definition and that each prime ideal of S lies directly over some prime ideal of R.

Lemma 5.3. *Let P be a prime ideal of S. Choose an annihilator prime Q of $(S/P)_R$, and let C denote the Goldie quotient ring of S/P.*

(i) *P lies over Q.*

(ii) *There exists a simple R-C-bimodule factor of C with left annihilator equal to Q.*

Proof: (i) First, it is immediate that $P \cap R \subseteq Q$. Now let E be the artinian quotient ring of $R/(P \cap R)$; the existence of E is guaranteed by (4.6). Also following (4.6), identify E with its image in C. Let I be the extension in E of the prime ideal $Q/(P \cap R)$, and note that $cI = 0$ for some nonzero $c \in S/P$. Therefore, $I \neq E$, and it follows that Q is minimal over $P \cap R$.

(ii) Retaining the notation of the preceding paragraph, first observe that C/IC is a nonzero E-C-bimodule. Since E is artinian and I is a maximal ideal of E, it follows that C/IC is faithful and torsionfree as a left (E/I)-module, and hence also as a left (R/Q)-module. That there exists a simple (R/Q)-C-bimodule factor of C/IC with left annihilator Q follows from [**16**, 7.20]. Statement (ii) follows. □

A refinement of the following lemma will appear in (5.12).

Lemma 5.4. *Let P be a prime ideal of S, and let Q_α and Q_β be distinct prime ideals of R.*

(i) *If P lies over Q_α and lies directly over Q_β, then there exists a positive integer s such that $\tau^s(Q_\alpha) = Q_\beta$.*

(ii) *If P lies directly over both Q_α and Q_β, then Q_α and Q_β are contained within a single finite τ-orbit.*

Proof: (i) Let C denote the Goldie quotient ring of S/P. In view of (5.3ii), there exists an R-C-bimodule composition series $0 = C_0 \subset \cdots \subset C_m = C$ for C such that $Q_\beta = \operatorname{ann}_R(C/C_{m-1})$. Part (i) now follows from (4.4).

(ii) This follows from (i). □

Compare the following lemma with [**31**, 4.1, 4.2].

5.5. For a given prime ideal Q of R, let $\mathcal{A}(Q)$ denote the Goldie quotient ring of R/Q, and let $\mathcal{V}(Q) = \mathcal{A}(Q) \otimes_R S$. We identify $\mathcal{V}(Q)$ with $\mathcal{A}(Q) \otimes_R (S/QS)$ when convenient. Note that since S/QS is a free left (R/Q)-module, $(S/QS)_S$ embeds in $\mathcal{V}(Q)_S$, and consequently $\operatorname{ann}[\mathcal{V}(Q)]_S = \operatorname{ann}(S/QS)_S$.

Lemma. *Let P be a prime ideal of S, let Q be a prime ideal of R, and let $A = \mathcal{A}(Q)$. Then P lies directly over Q if and only if there exists an A-S-bimodule factor M of $\mathcal{V}(Q)$ such that $P = \operatorname{ann} M_S$ and such that M is torsionfree as a right (S/P)-module.*

Proof: Suppose first that P lies directly over Q. It follows from (5.3ii) that there exists a simple bimodule factor U of ${}_RC_C$ with left annihilator equal to Q, where C is the Goldie quotient ring of S/P. It follows from [**21**, 5.1.1] that U is torsionfree as a left (R/Q)-module, and it is immediate that U is torsionfree as a right (S/P)-module. Let N be the image of the canonical R-S-bimodule map from S into U. Then N is a nonzero (R/Q)-(S/P)-bimodule which is torsionfree on each side. Hence, letting $M = A \otimes_R N$ completes one direction of the proof.

Now suppose that there exists an A-S-bimodule factor M of $\mathcal{V}(Q)$ such that $P = \text{ann}M_S$ and such that M is torsionfree as a right (S/P)-module. Denote images in M with an overbar. Note since $1 \otimes 1$ generates M as an A-S-bimodule that $\overline{1 \otimes 1} \neq 0$. Therefore, from (4.3) it follows that $A.(\overline{1 \otimes 1}).R$ is isomorphic as an A-R-bimodule to ${}_AA_R$. In particular, $\text{ann}[A.(\overline{1 \otimes 1}).R]_R = Q$. Hence, Q is an annihilator prime of M_R. Let B denote the canonical R-S-bimodule image of S in M. Thus B is an R-S-bimodule factor of S/QS, with right annihilator in S equal to P, and B is torsionfree on each side as an (R/Q)-(S/P)-bimodule. Also, Q is an annihilator prime of B_R, since $\overline{1 \otimes 1} \in B$. From [**16**, 6.19] it follows that $B_S \hookrightarrow [\bigoplus_{i=1}^t (S/P)]_S$ for some t. Therefore Q is an annihilator prime of $(S/P)_R$. That P lies directly over Q now follows from (5.3i). $\square$

5.6. Let Q be a prime ideal of R, let $A = \mathcal{A}(Q)$, and let

$$V \;=\; \mathcal{V}(Q) \;=\; (A \otimes 1) \oplus (A \otimes y) \oplus (A \otimes y^2) \oplus \cdots.$$

We now establish some elementary properties of V which will be required later. By a *polynomial* f in V we mean an element $a_n \otimes y^n + \cdots + a_0 \otimes 1 \in V$, with $a_0, \ldots, a_n \in A$. The elements $a_0, \ldots, a_n$ are termed *(left) coefficients* of f, and a_i is the left coefficient of y^i for $0 \leq i \leq n$. If $a_n \neq 0$ then the *degree* of f is n, and if $a_n = 1$ then f is said to be *monic.*

Lemma. *Let B denote an A-S-sub-bimodule of V.*

(i) *If B contains a polynomial of degree n then it contains a monic polynomial of degree n.*

(ii) *B contains a unique monic polynomial f of minimal degree, and $B = A.f.S$.*

(iii) *Every proper A-S-bimodule factor of V has finite length as a left A-module.*

(iv) *V is uniform as an A-S-bimodule.*

Proof: (i) Suppose that $f = a_n \otimes y^n + \cdots + a_0 \otimes 1 \in B$ for $a_0, \ldots, a_n \in A$ and $a_n \neq 0$. Since $A.f.R \subseteq B$, we see that

$$aa_nr \otimes y^n + (\text{certain terms of degree} < n) \in B,$$

for all $a \in A$ and $r \in R$. Part (i) now follows from the simplicity of ${}_AA_R$, noted in (4.3).

(ii) This follows from a procedure mimicking the Euclidean algorithm.

(iii) This follows from (ii).

(iv) Suppose that V is not uniform as an A-S-bimodule. Then there exist nonzero A-S-sub-bimodules M and N such that $M \cap N = 0$. But then V embeds as an A-S-bimodule into $(V/M) \oplus (V/N)$, which by (iii) has finite length as a left A-module. This last statement is impossible, and (iv) follows. □

Corollary 5.7. *Let Q be a prime ideal of R, let $A = \mathcal{A}(Q)$, and let $V = \mathcal{V}(Q)$.*

(i) Let P be a prime ideal of S lying directly over Q, and suppose that $P \neq \operatorname{ann} V_S$. Then there exists a simple A-S-bimodule factor M of V such that $P = \operatorname{ann} M_S$, such that M is torsionfree as a right (S/P)-module, and such that M is of finite length as a left A-module.

(ii) Suppose there exists a proper simple A-S-bimodule factor M of V. Then $P = \operatorname{ann} M_S$ is a prime ideal of S lying directly over Q.

Proof: (i) From (5.5) it follows that there exists a proper A-S-bimodule factor M' of V such that $P = \operatorname{ann} M'_S$ and such that $M'_{S/P}$ is torsionfree. From (5.6iii) it follows that ${}_AM'$ has finite length, and so M' has finite length as an A-S-bimodule. Part (i) now follows from [**16**, 7.20].

(ii) Since M is a simple A-S-bimodule, it is clear that P is a prime ideal of S. It follows from (5.6iii) that M has finite length as a left A-module, and from [**21**, 5.1.1] it follows that M is torsionfree as a right (S/P)-module. Part (ii) now follows from (5.5). □

5.8. Let P be a prime ideal of S. From (4.6) it follows that the set $\mathcal{C}$ of regular elements of $R/(P \cap R)$ is an Ore set of regular elements in $R/(P \cap R)$ and in S/P, and that the ring $E = [R/(P \cap R)]\mathcal{C}^{-1}$ is artinian. Let C be the Goldie quotient ring of S/P, and let $F = [S/P]\mathcal{C}^{-1}$.

Now suppose that P lies directly over a prime ideal Q of R. Let $A = \mathcal{A}(Q)$ and $V = \mathcal{V}(Q)$. From (5.5) it follows that there exists an A-S-bimodule factor M of V such that $\operatorname{ann} M_S = P$ and such that M is torsionfree as a right (S/P)-module. If M has finite length as a left A-module, then recall that the right action of S/P on M extends uniquely to a right C-module action which provides an A-C-bimodule structure for M (see, e.g., [**16**, 7.8]). In this case we define an A-E-bimodule structure on M via the right action of E inherited from the above action of C.

Lemma. *If M has finite length as a left A-module then*

(i) *There exists an A-R-bimodule composition series $0 = M_0 \subset \cdots \subset M_n = M$ such that M_i/M_{i-1} is isomorphic as an A-R-bimodule to ${}_A(A^{\tau^{i-1}})_R$, for $1 \leq i \leq n$.*

(ii) *M and C have finite length as right E-modules.*

Proof: Let n equal the length of M as an A-R-bimodule. Let $M_0 = 0$. Denote images in M with an overbar, and for each integer $i \geq 0$ let M_i denote the A-R-bimodule $A.(\overline{1 \otimes y^{i-1}}) + \cdots + A.(\overline{1 \otimes 1})$. Observe that $M = \bigcup_i M_i$.

(i) For $t \in \{1, \ldots, n\}$, observe from (4.3) that M_t/M_{t-1} is either zero or isomorphic as an A-R-bimodule to the simple bimodule ${}_A(A^{\tau^{t-1}})_R$. Noting that right multiplication by y induces a left A-module epimorphism from M_i/M_{i-1} onto M_{i+1}/M_i, for all $i \geq 1$, we see that M_t/M_{t-1} cannot be zero since n is the length of M as an A-R-bimodule. It follows that $0 = M_0 \subset \cdots \subset M_n = M$ is an A-R-bimodule composition series for M with the desired properties.

(ii) Set $M' = (\overline{1 \otimes 1}).E + \cdots + (\overline{1 \otimes y^{n-1}}).E$ and observe that M' is an R-E-sub-bimodule of M, having finite length as a right E-module. By [**16**, 7.8], M' is divisible as a left (R/Q)-module, whence M' is a left A-submodule of M. Consequently, $M' = M$, and therefore M_E has finite length. It follows that the unique (up to isomorphism) simple right C-module is finitely generated over E, and therefore C_E has finite length. $\square$

Lemma 5.9. *Retain the notation of* (5.8).

(i) *If C is not finitely generated as a right E-module, then $M = V$. In particular, $P = \operatorname{ann} V_S = \operatorname{ann}(S/QS)_S$.*

(ii) *If $P = \operatorname{ann}(S/QS)_S$, there is a positive integer m such that $\tau^m(Q) = Q$, and such that $\{Q, \ldots, \tau^{m-1}(Q)\}$ is the set of prime ideals of R minimal over $P \cap R$.*

Proof: (i) This follows from (5.6iii) and (5.8iii).

(ii) Observe first that $P \cap R \subseteq \bigcap_{i \leq 0} \tau^i(Q)$. Next, note from (5.4) that the prime ideals of R minimal over $P \cap R$ all lie in the τ-orbit of Q. Given $i \leq 0$, the prime ideal $\tau^i(Q)$ must contain a prime ideal minimal over $P \cap R$, say $\tau^j(Q)$. Since $R/\tau^i(Q)$ and $R/\tau^j(Q)$ have the same classical Krull dimension, $\tau^i(Q) = \tau^j(Q)$. Thus $\tau^i(Q)$ is minimal over $P \cap R$ for all $i \leq 0$. As there are only finitely many prime ideals minimal over $P \cap R$, we conclude that the τ-orbit of Q is finite. Part (ii) follows. □

Corollary 5.10. *Let P be a prime ideal of S, let C be the Goldie quotient ring of S/P, and let E be the artinian quotient ring of $R/(P \cap R)$. Suppose that C is not finitely generated as a right E-module. If Q is a prime ideal of R, then P lies directly over Q if and only if $P = \operatorname{ann}(S/QS)_S$ and S/QS is torsionfree as a right (S/P)-module.*

Proof: This follows from (5.5) and (5.9i). □

Corollary 5.11. *Let Q be a prime ideal of R, and let P_0 and P_1 be prime ideals of S such that $P_0 \subset P_1$. Suppose that P_0 and P_1 both lie over Q.*

(i) *There exists a nonnegative integer m such that P_0 lies directly over $\tau^m(Q)$ and such that $P_0 = \operatorname{ann}[S/\tau^m(Q)S]_S$.*

(ii) *The τ-orbit of Q is finite.*

(iii) *Let C_1 be the Goldie quotient ring of S/P_1, and let E_1 be the artinian quotient ring of $R/(P_1 \cap R)$. If τ extends to an automorphism of S, then C_1 is of finite length as a right E_1-module.*

Proof: Let Q_0 be an annihilator prime of $(S/P_0)_R$, and let Q_1 be an annihilator prime of $(S/P_1)_R$. By (5.3i), P_0 lies directly over Q_0, and P_1 lies directly over Q_1. Then by (5.4i), there exist nonnegative integers m, n such that $\tau^m(Q) = Q_0$ and $\tau^n(Q) = Q_1$.

(i) Let $\mathcal{C}$ be the set of regular elements of $R/(P_0 \cap R)$. By (4.6), $\mathcal{C}$ is an Ore set in both $R/(P_0 \cap R)$ and S/P_0, and $[R/(P_0 \cap R)]\mathcal{C}^{-1}$ is the artinian quotient ring of $R/(P_0 \cap R)$. In particular, it follows that $Q/(P_0 \cap R)$ is disjoint from $\mathcal{C}$, whence P_1/P_0 is disjoint from $\mathcal{C}$, and so P_1/P_0 extends to a non-minimal prime ideal of $[S/P_0]\mathcal{C}^{-1}$. Consequently, $[S/P_0]\mathcal{C}^{-1}$ is not artinian, and so it cannot be a finitely generated right module over the artinian ring $[R/(P_0 \cap R)]\mathcal{C}^{-1}$. Therefore the Goldie quotient ring of S/P_0 is not a finitely generated right module over the artinian quotient ring of $R/(P_0 \cap R)$. Hence, $P_0 = \operatorname{ann}(S/Q_0S)_S$ by (5.9i).

(ii) This now follows from (5.9ii).

(iii) Suppose that C_1 does not have finite length as a right E_1-module. Then $P_1 = \operatorname{ann}\,(S/Q_1S)_S$, by (5.9i). Since $\tau^{m-n}(Q_1) = Q_0$, an extended automorphism τ^{m-n} maps Q_1S onto Q_0S, whence $\tau^{m-n}(P_1) = \operatorname{ann}(S/Q_0S)_S = P_0$.

This last conclusion implies that S/P_1 and S/P_0 have the same classical Krull dimension, which is impossible. Therefore C_1 does have finite length as a right E_1-module. □

We now present a description of the minimal prime ideals of R modulo the contraction of a prime ideal of S. This result is a refinement of [**13**, 3.9] and [**19**, 4.3, 4.4]. Recall from (5.3) that each prime ideal of S lies directly over some prime ideal of R.

Theorem 5.12. *Let P be a prime ideal of S lying directly over a prime ideal Q of R. Denote the Goldie quotient ring of R/Q by A, and set $V = A \otimes_R S$.*

(i) *There exists a positive integer n such that*

$$[\tau^{-(n-1)}(Q)][\tau^{-(n-2)}(Q)]\cdots[\tau^{-1}(Q)]Q \subseteq P,$$

and such that $\{Q, \tau^{-1}(Q), \ldots, \tau^{-(n-1)}(Q)\}$ is the set of prime ideals of R minimal over $P \cap R$. (This list of prime ideals may have repetitions if the τ-orbit of Q is finite.)

(ii) *Choose an A-S-bimodule factor M of V as in (5.5) such that $P = \operatorname{ann} M_S$ and such that M is torsionfree as a right (S/P)-module. If M has finite length as a left A-module then n can be chosen equal to the length of ${}_AM$ divided by the length of ${}_AA$.*

(iii) *Let C be the Goldie quotient ring of S/P, and let E be the artinian classical quotient ring of $R/(P \cap R)$. If C is not finitely generated as a right E-module then n may be chosen such that $\tau^n(Q) = Q$.*

(iv) *If the τ-orbit of Q is infinite, then Q is the unique annihilator prime of $(S/P)_R$, and $\tau^{-(n-1)}(Q)$ is the unique annihilator prime of ${}_R(S/P)$. In particular, P lies directly over exactly one prime ideal of R.*

Proof: Choose an A-S-bimodule factor M of V as in (5.5) such that $P = \operatorname{ann} M_S$ and such that M is torsionfree as a right (S/P)-module.

(i) (ii) (iii) Assume first that M has finite length as a left A-module. Let $0 = M_0 \subset \cdots \subset M_n = M$ be an A-R-bimodule composition series for M as in (5.8), and observe that $n = \operatorname{length}({}_AM)/\operatorname{length}({}_AA)$. Now $\operatorname{ann} M_R = P \cap R$. Further, $\operatorname{ann}(M_i/M_{i-1})_R = \tau^{-(i-1)}(Q)$ for $1 \leq i \leq n$, whence

$$[\tau^{-(n-1)}(Q)][\tau^{-(n-2)}(Q)]\cdots[\tau^{-1}(Q)]Q \subseteq P \cap R$$

and $P \cap R \subseteq \tau^{-(i-1)}(Q)$ for $1 \leq i \leq n$. By (5.4i), the prime ideals of R minimal over $P \cap R$ all lie in the τ-orbit of Q. In particular, the classical Krull dimension of R modulo a prime ideal minimal over $P \cap R$ is equal to the classical Krull dimension of $R/\tau^j(Q)$ for any integer j. Hence the prime ideals of R minimal over $P \cap R$ are precisely $Q, \tau^{-1}(Q), \ldots, \tau^{-(n-1)}(Q)$. Moreover, in the present case (5.8ii) says that C is finitely generated as a right E-module.

Now consider the case that M does not have finite length as a left A-module. In this case, $M = V$ by (5.6iii), and so $P = \operatorname{ann}(S/QS)_S$. Now by (5.9ii) there is a positive integer m such that $\tau^m(Q) = Q$ and such that $\{Q, \ldots, \tau^{m-1}(Q)\}$ is the set of prime ideals of R minimal over $P \cap R$. In particular, it follows that

$$(Q \cap \tau(Q) \cap \cdots \cap \tau^{m-1}(Q))^r \subseteq P \cap R$$

for some positive integer r. Set $n = mr$. Then $\tau^n(Q) = Q$, the set of prime ideals minimal over $P \cap R$ can be written as $\{Q, \tau^{-1}(Q), \ldots, \tau^{-(n-1)}(Q)\}$, and

$$[\tau^{-(n-1)}(Q)][\tau^{-(n-2)}(Q)] \cdots [\tau^{-1}(Q)]Q \subseteq (Q \cap \tau(Q) \cap \cdots \cap \tau^{m-1}(Q))^r \subseteq P.$$

(iv) By (5.4ii), Q is the only annihilator prime of $(S/P)_R$. Since S^{op} is isomorphic to $R^{\text{op}}[y; \tau^{-1}, -\delta\tau^{-1}]$ (see [**13**, 1.5]), we may apply (i) in S^{op}. Thus if Q' is an annihilator prime of ${}_R(S/P)$, there exists a positive integer n' such that $\{Q', \tau(Q'), \ldots, \tau^{n'-1}(Q')\}$ is the set of prime ideals minimal over $P \cap R$. Now Q is in the τ-orbit of Q', and so the τ-orbit of Q' is infinite; thus Q' is the unique annihilator prime of ${}_R(S/P)$. Finally, from

$$\{Q', \tau(Q'), \ldots, \tau^{n'-1}(Q')\} = \{Q, \tau^{-1}(Q), \ldots, \tau^{-(n-1)}(Q)\}$$

we conclude that $Q' = \tau^{-(n-1)}(Q)$. $\square$

When τ extends to an automorphism of S we can turn (5.12) around to obtain the following information about the prime ideals of S lying over a given prime ideal of R. A more precise formulation of part (ii) of the following corollary will be obtained in (7.9).

Corollary 5.13. *Assume that τ extends to an automorphism of S. Let Q be a prime ideal of R, and suppose that there exists a prime ideal P of S lying over Q.*

(i) *There exist nonnegative integers r and s such that $\tau^{-r}(P)$ lies directly over Q and such that $\tau^{-r}(P), \tau^{-r+1}(P), \ldots, \tau^{s}(P)$ all lie over Q.*

(ii) *If the τ-orbit of Q is infinite then r and s can be chosen so that $\tau^{i}(P)$ does not lie over Q for $i < -r$ or for $i > s$. Further, in this case $\tau^{i}(P)$ does not lie directly over Q for any $i \neq -r$, and $r+s+1$ is equal to the number of distinct prime ideals of R minimal over $P \cap R$.*

Proof: By (5.3), P lies directly over some prime ideal Q_0 of R. Then by (5.12i), there exists a positive integer n such that the set of prime ideals of R minimal over $P \cap R$ is $\{Q_0, \tau^{-1}(Q_0), \ldots, \tau^{-(n-1)}(Q_0)\}$. Then $Q = \tau^{-r}(Q_0)$ for some nonnegative integer $r \leq n-1$. Thus if $s = n-1-r$, the prime ideals of R minimal over $P \cap R$ are precisely $\tau^{r}(Q), \tau^{r-1}(Q), \ldots, \tau^{-s}(Q)$. Since P lies over each of these prime ideals, we find that $\tau^{-r}(P), \tau^{-r+1}(P), \ldots, \tau^{s}(P)$ all lie over Q. Moreover, since P lies directly over $\tau^{r}(Q)$ we see that $\tau^{-r}(P)$ lies directly over Q, and (i) is proved. Now suppose the τ-orbit of Q is infinite. Then P does not lie over $\tau^{j}(Q)$ for $j > r$ or for $j < -s$, and there are exactly $r+s+1$ distinct prime ideals of R minimal over $P \cap R$. Further, by (5.12iv), P does not lie directly over $\tau^{j}(Q)$ for any $j \neq r$, and (ii) follows. □

We conclude this section by combining our present results with the Additivity Principle to derive the following relationships between the ranks of prime factors of S and R. These results should be compared with [**13**, 3.1(II)], which says that if R is commutative and P is a prime ideal of S lying over a prime ideal Q of R which is not τ-stable, then S/P is a commutative domain. Part (ii) of the following appears to be new even when R is commutative.

Theorem 5.14. *Let P be a prime ideal of S lying over a prime ideal Q of R, and let n be the number of distinct prime ideals of R minimal over $P \cap R$.*

(i) $\operatorname{rank}(S/P) = m.\operatorname{rank}(R/Q)$ *for some integer $m \geq n$. If the τ-orbit of Q is infinite, then $m = n$. If $m = 1$, then $P \cap R = Q$.*

(ii) *If $P \cap R = Q$ and the τ-orbit of Q is infinite, then the natural embedding $R/Q \hookrightarrow S/P$ extends to an isomorphism from the Goldie quotient ring of R/Q onto the Goldie quotient ring of S/P.*

Proof: Let C denote the Goldie quotient ring of S/P. It was verified in the proof of (4.6) that the prime ideals of R that occur as left annihilators of R-C-bimodule composition factors of C are precisely the prime ideals minimal over

$P \cap R$. By the Additivity Principle (e.g., [**16**, 7.25]),

$$\operatorname{rank}(S/P) \;=\; \sum_{i=1}^{n} z_i.\operatorname{rank}(R/Q_i),$$

where $z_1, \dots, z_n$ are positive integers and $Q_1, \dots, Q_n$ are the prime ideals of R minimal over $P \cap R$. Since these Q_i all lie in the τ-orbit of Q (by (5.4)), the factor rings R/Q_i all have the same rank as R/Q, and the first part of (i) follows. The last statement of (i) follows from [**16**, 7.26].

Now assume that the τ-orbit of Q is infinite. Since P lies directly over some Q_i (by (5.3)), there is no loss of generality in assuming that P lies directly over Q. Let $A = \mathcal{A}(Q)$ and $V = \mathcal{V}(Q)$. By (5.9ii), $P \neq \operatorname{ann}(S/QS)_S$, and so by (5.7i), there exists a simple A-S-bimodule factor M of V such that $P = \operatorname{ann} M_S$, such that M is torsionfree as a right (S/P)-module, and such that M has finite length as a left A-module. Now from (5.12ii) it follows that $\operatorname{length}_A M = n.\operatorname{length}_A A = n.\operatorname{rank}(R/Q)$. (Since the τ-orbit of Q is infinite, the integer n occurring in (5.12i) must equal the number of distinct prime ideals of R minimal over $P \cap R$.) On the other hand, $\operatorname{rank}(S/P)$ divides $\operatorname{length}_A M$ by [**16**, 7.22], and hence $\operatorname{rank}(S/P) \leq n.\operatorname{rank}(R/Q)$. Thus $m = n$ in this case, and the second part of (i) is proved.

Finally, assume that $P \cap R = Q$ and that the τ-orbit of Q is infinite. It follows from (5.12ii) that $\operatorname{length}_A M = \operatorname{length}_A A$. Hence, M has length 1 as an A-R-bimodule, and so ${}_A M_R \cong {}_A A_R$ by (5.8i). Thus the natural map of R/Q into the ring $E = \operatorname{End}_A M$, given by right multiplication, extends to an isomorphism of A onto E. On the other hand, since M is an A-C-bimodule as in (5.8), the natural map of S/P into E extends to an embedding $\phi\colon C \to E$. Observe that the image in E of any regular element of R/Q is invertible in E, hence regular in $\phi(C)$, and consequently invertible in $\phi(C)$. Therefore $E = \phi(C)$, and part (ii) follows. $\square$

6. Prime ideals in Quadratic (-1)-skew extensions.

As a digression and illustration of our methods thus far, we turn to the ring extensions used in [**32**] to classify the prime and primitive ideals in enveloping algebras of (complex finite dimensional) solvable Lie superalgebras. A detailed characterization of the prime ideals in these extension rings, which are factors of (-1)-skew polynomial rings, is central to the approach in [**32**]. However, the results obtained therein are only proved for algebras of finite Gelfand-Kirillov dimension over a field of characteristic not equal to two. In this section we obtain some of the main results without relying on the additional hypotheses.

6.1. (i) Let R be a noetherian ring, τ an automorphism of R, and δ a τ-derivation of R. Further assume that (τ, δ) is a (-1)-skew derivation of R, and that there exists a (τ, δ)-constant element $x \in R$ such that $\delta^2(r) = xr - \tau^2(r)x$ for all $r \in R$. Set $S = R[y; \tau, \delta]$, observe that $y^2 - x$ is a normal element of S, and set $T = S/\langle y^2 - x\rangle$. We shall also use y to denote the coset $y + \langle y^2 - x\rangle$ in T.

(ii) Since $\langle y^2 - x\rangle = S(y^2 - x) = (y^2 - x)S$, the ring T is free on each side as an R-module, with basis $\{1, y\}$.

(iii) Extend τ to S as in (2.4ii), so that $\tau(y) = -y$. Since $\langle y^2 - x\rangle$ is a τ-stable ideal of S we see that τ also induces an automorphism of T mapping y to $-y$.

(iv) Let Q be a prime ideal of R, let A denote the Goldie quotient ring of R/Q, and let $U = A \otimes_R T$. Note from (ii) that there is a series of A-R-bimodules

$$0 \subset A.(1 \otimes 1).R \subset A.(1 \otimes 1).R + A.(1 \otimes y).R = U.$$

In particular, it follows from (4.3) that U has length two as an A-R-bimodule, that one composition factor is isomorphic to ${}_AA_R$, and that the other composition factor is isomorphic to ${}_AA^{\tau}_R$.

The following generalization of [**32**, 2.10ii] says in essence that direct lying over produces a bijection, modulo finite τ-orbits, between spec$T$ and spec$R$.

Theorem 6.2. *Let R, τ, T be as in* (6.1).

(i) *Let Q be a prime ideal of R. Then there exists at least one prime ideal of T lying directly over Q. If two distinct prime ideals P_α and P_β of T lie directly over Q, then $\tau(Q) = Q$ and $P_\alpha = \tau(P_\beta) = \tau^2(P_\alpha)$.*

(ii) *Let P be a prime ideal of T. Then P lies directly over at least one prime ideal of R. If P lies directly over two distinct prime ideals Q_α and Q_β of R, then $\tau(P) = P$ and $Q_\alpha = \tau(Q_\beta) = \tau^2(Q_\alpha)$.*

Proof: (i) Let A denote the Goldie quotient ring of R/Q, and let $U = A \otimes_R T$. We may identify U with V/VI where $V = \mathcal{V}(Q) = A \otimes_R S$ and $I = S(y^2 - x)$. By (6.1iv), U is a nonzero A-T-bimodule of finite length as a left A-module. It therefore follows from (5.7ii) that at least one prime ideal of T lies directly over Q.

Now suppose that there exist two distinct prime ideals P_α and P_β in T lying directly over Q. In view of (6.1iv), U has length no greater than two as an A-T-bimodule. It therefore follows from (5.5) that P_α and P_β are the respective right annihilators of non-isomorphic simple A-T-bimodule factors W_α and W_β of U. Let $W \in \{W_\alpha, W_\beta\}$, and observe that W has length one as an A-R-bimodule. Furthermore, the image of $1 \otimes 1$ in W must be nonzero, and so we must have ${}_AW_R \cong {}_AA_R$, and consequently $\operatorname{ann} W_R = Q$. In particular, P_α and P_β both contract to Q. Since any A-T-bimodule composition series for U of length two will also be an A-R-bimodule composition series, it further follows from (6.1iv) that at least one of W_α or W_β must be isomorphic as an A-R-bimodule to ${}_AA^\tau_R$. Combining the above isomorphisms, we see that A is isomorphic to A^τ as an A-R-bimodule. Thus $\tau(Q) = Q$.

Let $\widetilde{P}_\alpha$ and $\widetilde{P}_\beta$ denote the respective preimages in S of P_α and P_β. It suffices to show that $\tau(\widetilde{P}_\alpha) = \widetilde{P}_\beta$ and that $\tau(\widetilde{P}_\beta) = \widetilde{P}_\alpha$. Since $Q = \widetilde{P}_\alpha \cap R$, it follows from (2.1v) that Q is δ-stable as well as τ-stable. We may therefore assume without loss of generality, in view of (2.3i,ii), that $Q = 0$ and $R = A$. Now T has length two as an R-R-bimodule, from which we see that $P_\alpha \cap P_\beta = 0$ and $\widetilde{P}_\alpha \cap \widetilde{P}_\beta = I$. Also, observe that T is artinian, and hence $\widetilde{P}_\alpha$ and $\widetilde{P}_\beta$ are incomparable.

Choose a nonzero polynomial $f(y) \in \widetilde{P}_\alpha$ of minimal degree n. Since R is simple, we may assume without loss of generality that $f(y)$ is monic. Moreover, since n is minimal, it follows from the Division Algorithm that $\widetilde{P}_\alpha = Sf(y)$. Since $\widetilde{P}_\alpha$ properly contains I, we must have $f(y) = y + a$ for some $a \in R$, and so $\widetilde{P}_\alpha = S(y + a)$. Likewise, $\widetilde{P}_\beta = S(y + b)$ for some $b \in R$. Now since

$$y^2 + (a + \tau(b))y + (ab + \delta(b)) = (y + a)(y + b) \in \widetilde{P}_\alpha \cap \widetilde{P}_\beta = I = S(y^2 - x),$$

we must have $a+\tau(b)=0$. Thus $\tau(y+b)=-y-a$, and likewise $\tau(y+a)=-y-b$. Therefore $\tau(\widetilde{P}_\alpha)=\widetilde{P}_\beta$ and $\tau(\widetilde{P}_\beta)=\widetilde{P}_\alpha$, as desired.

(ii) That P lies directly over at least one prime ideal of R follows from (5.3). Now suppose that P lies directly over distinct prime ideals Q_α and Q_β of R, with respective Goldie quotient rings A_α and A_β. For $j=\alpha,\beta$, it follows from (5.5) that there exists an A_j-S-bimodule factor M_j of $A_j\otimes_R T$ such that $P=\operatorname{ann}(M_j)_S$ and such that M_j is torsionfree as a right (S/P)-module. Observe that as a left A_j-module, M_j has finite length, and in fact length at most $2.\text{length}A_j$. Since there are (at least) two distinct prime ideals of R minimal over $P\cap R$, it follows from (5.12i,ii) that the set of prime ideals of R minimal over $P\cap R$ is precisely $\{Q_j,\tau^{-1}(Q_j)\}$. Therefore $Q_\alpha=\tau(Q_\beta)=\tau^2(Q_\alpha)$.

Finally, since P lies directly over Q_β we see that $\tau(P)$ lies directly over Q_α. By (i), there cannot exist two distinct prime ideals of T lying directly over Q_α, and consequently $\tau(P)=P$. □

The following generalizes [**32**, 2.10i].

Corollary 6.3. *Let R, τ, T be as in* (6.1).

(i) *If Q is a prime ideal of R, then the set of prime ideals of T lying over Q is contained within a single τ-orbit of* spec$T$.

(ii) *If P is a prime ideal of T, then the set of prime ideals of R minimal over $P\cap R$ is contained within a single τ-orbit of* spec$R$.

Proof: (i) Suppose that P is a prime ideal of T lying over Q. It follows from (5.13) that $\tau^{-n}(P)$ lies directly over Q, for some positive integer n. Part (i) now follows from (6.2i).

(ii) This follows from (5.12i). □

Remark 6.4. Let R, τ, T be as in (6.1). If P is a prime ideal of T lying over a prime ideal Q of R, then since T is finitely generated on each side as an R-module, it follows that P is right primitive if and only if Q is right primitive. (See [**30**, 1.4] or [**16**, 7.17].) Consequently, (6.2) provides a classification of the right primitive ideals of T.

7. Prime ideals in S associated to infinite orbits. The general case.

Assume throughout that $S = R[y; \tau, \delta]$ where R is noetherian and τ is an automorphism of R. In this section we initiate a general investigation of the prime ideals of S that lie over a given prime ideal Q of R, concentrating first on the case that the τ-orbit of Q is infinite. In (7.5), we show that if P is a prime ideal of S that lies directly over Q and if the τ-orbit of Q is infinite, then P and Q uniquely determine each other. (Recall that if P is a prime ideal of S lying over Q, then P lies *directly* over Q provided Q is an annihilator prime of $(S/P)_R$.) Under the assumption that τ extends to an automorphism of S, we further provide a constructive description of the set of prime ideals of S lying over Q, in (7.9).

7.1. Recall from (5.12i) that for each prime ideal P of S, the prime ideals of R minimal over $P \cap R$ lie in a single τ-orbit of spec$R$. For brevity we refer to this orbit as the *τ-orbit associated to P*.

Lemma 7.2. *Let T be a prime noetherian ring, let A be the Goldie quotient ring of T, and let α be an automorphism of T. Suppose that ${}_A A_T$ is isomorphic as an A-T-bimodule to ${}_A(A^\alpha)_T$. Then α induces an inner automorphism of A.*

Proof: Let v be the image of 1 under a bimodule isomorphism ${}_A A_T \to {}_A(A^\alpha)_T$. Then $Av = A$ and $tv = v\alpha(t)$ for all $t \in T$. In particular, v is a unit of A, and so $\alpha(t) = v^{-1}tv$ for all $t \in T$. □

We continue to use the notations $\mathcal{A}(Q)$ and $\mathcal{V}(Q)$ introduced in (5.5).

Proposition 7.3. *Let Q be a prime ideal of R, let $A = \mathcal{A}(Q)$, and let $V = \mathcal{V}(Q)$.*

(i) *Suppose that B is a nonzero proper A-S-sub-bimodule of V, and let n be the minimal degree for nonzero polynomials in B. If there does not exist an integer i in $\{1, \ldots, n-1\}$ such that τ^i maps Q to itself and induces an inner automorphism of A, then there is a unique maximal proper A-S-sub-bimodule of V containing B.*

(ii) *Suppose that there exists no nonzero power of τ which maps Q to itself and which induces an inner automorphism of A. Then there is a unique maximal proper (possibly zero) A-S-sub-bimodule of V.*

Proof: (i) Let $V_i = A.(1 \otimes y^i) + \cdots + A.(1 \otimes 1) \subseteq V$, for all $i \geq 0$, and set $V_{-1} = 0$. For $i \geq 0$, we observe that $V_i/V_{i-1} \cong A^{\tau^i}$ as A-R-bimodules, and so V_i/V_{i-1} is a simple A-R-bimodule by (4.3). It now follows from (7.2) that the bimodules V_i/V_{i-1} for $0 \leq i \leq n-1$ are pairwise non-isomorphic. By (5.6), B contains a monic polynomial of degree n, and hence $B + V_{n-1} = V$. Thus in any A-R-bimodule composition series for V/B, at most one composition factor can be isomorphic to ${}_AA_R$.

Suppose that K and L are distinct maximal proper A-S-sub-bimodules of V containing B. Then V/B has an A-S-bimodule factor isomorphic to $(V/K) \oplus (V/L)$. However, V/K contains an A-R-sub-bimodule isomorphic to ${}_AA_R$ generated by the image of $1 \otimes 1$, and the same holds true for V/L, whence V/B has an A-R-bimodule subfactor isomorphic to ${}_A(A \oplus A)_R$. We have obtained a contradiction, and part (i) follows.

(ii) Since any two nonzero A-S-sub-bimodules of V have nonzero intersection by (5.6iv), part (ii) follows from part (i). $\square$

Definition 7.4. Let Q be a prime ideal of R, and suppose there exists no nonzero power of τ which maps Q to itself and which induces an inner automorphism of $\mathcal{A}(Q)$. In light of (7.3), there is a unique simple $\mathcal{A}(Q)$-S-bimodule factor of $\mathcal{V}(Q)$, and we denote this simple bimodule factor by $\mathcal{W}(Q)$. In particular, $\mathcal{W}(Q)$ is defined whenever the τ-orbit of Q is infinite.

Theorem 7.5. (I) *Let P be a prime ideal of S whose associated τ-orbit in* spec$R$ *is infinite.*

(1) *There exists exactly one prime ideal Q of R such that P lies directly over Q.*

(2) *P is the unique prime ideal of S lying directly over Q.*

(3) $P = \operatorname{ann}[\mathcal{W}(Q)]_S$.

(4) *$\mathcal{W}(Q)$ is a proper factor of $\mathcal{V}(Q)$ and has finite length as a left $\mathcal{A}(Q)$-module.*

(5) *The number of prime ideals of R minimal over $P \cap R$ is equal to the length of ${}_{\mathcal{A}(Q)}[\mathcal{W}(Q)]$ divided by the length of ${}_{\mathcal{A}(Q)}\mathcal{A}(Q)$.*

(II) *Let Q be a prime ideal of R whose τ-orbit is infinite.*

(1) *If $\mathcal{W}(Q)$ is a proper factor of $\mathcal{V}(Q)$, then $P = \operatorname{ann}[\mathcal{W}(Q)]_S$ is a prime ideal of S lying directly over Q. Moreover, P is the only prime ideal*

of S lying directly over Q, and P does not lie directly over any prime ideal of R distinct from Q.

(2) *If $\mathcal{W}(Q)$ is not a proper factor of $\mathcal{V}(Q)$, then no prime ideal of S lies directly over Q.*

Proof: First, (I.1) follows from (5.3) and (5.4). Next, it follows from (5.9ii) that $P \neq \mathrm{ann}[\mathcal{V}(Q)]_S$. It then follows from (5.7i) and (7.3) that $P = \mathrm{ann}[\mathcal{W}(Q)]_S$ and that $\mathcal{W}(Q)$ is a proper factor of $\mathcal{V}(Q)$ with finite length as a left $\mathcal{A}(Q)$-module. This proves (I.3) and (I.4). Part (I.5) is now clear from (5.12ii). For (I.2), consider a prime ideal P' of S lying directly over Q. Since the τ-orbit of Q is infinite, $P' \neq \mathrm{ann}[\mathcal{V}(Q)]_S$, by (5.9ii). Hence, (5.7i) shows that P' is the right annihilator of some simple $\mathcal{A}(Q)$-S-bimodule factor of $\mathcal{V}(Q)$. By the uniqueness of $\mathcal{W}(Q)$, we conclude that $P' = \mathrm{ann}[\mathcal{W}(Q)]_S = P$.

To prove (II.1), assume that $\mathcal{W}(Q)$ is a proper factor of $\mathcal{V}(Q)$. It follows from (5.7ii) that $P = [\mathrm{ann}\mathcal{W}(Q)]_S$ is a prime ideal lying directly over Q. The remainder of (II.1) follows from (I.2) and (5.4ii). Finally, (II.2) follows from (I.4). $\square$

Corollary 7.6. *If P_α and P_β are distinct prime ideals of S whose associated τ-orbits are infinite, then the ideals $P_\alpha \cap R$ and $P_\beta \cap R$ have distinct prime radicals in R.*

Proof: This follows from (5.12) and (7.5.I.2). $\square$

Example 7.7. (i) It is not unusual for infinitely many prime ideals of S to lie over a given prime ideal of R whose τ-orbit is infinite. Let k be an algebraically closed field of characteristic zero, and let $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ be a basis for $\mathbf{g} = \mathrm{sl}_2(k)$. (The reader is referred, for example, to [**11**] for definitions and basic results.) Let $S = k\langle e, h, f\rangle$ denote the enveloping algebra of $\mathbf{g}$. Let R denote the enveloping algebra of the Borel subalgebra generated by e and h; then R is the associative k-subalgebra of S generated by e and h. The set of co-finite-dimensional prime ideals of R is $\{\langle e, h-\alpha\rangle \mid \alpha \in k\}$. It is well known that $S = R[f; \tau, \delta]$, where τ is the k-algebra automorphism of R such that $\tau(h) = h + 2$ and $\tau(e) = e$, while δ is the k-linear τ-derivation of R such that $\delta(h) = 0$ and $\delta(e) = -h$. Note that each co-finite-dimensional prime ideal of R has an infinite τ-orbit. For any nonnegative integer r, there is exactly one irreducible $(r+1)$-dimensional (left) representation of S; denote its kernel by P_r.

The prime ideals of R minimal over $P_r \cap R$ are

$$\langle e, h-r\rangle, \ \langle e, h-(r-2)\rangle, \ \ldots, \ \langle e, h+(r-2)\rangle, \ \langle e, h+r\rangle.$$

In particular, when r is even P_r lies over $\langle e, h\rangle$, and when r is odd P_r lies over $\langle e, h-1\rangle$. Hence, each one of the infinitely many co-finite-dimensional prime ideals of S lies over either $\langle e, h\rangle$ or $\langle e, h-1\rangle$. On the other hand, by (7.5.II.1) there is only one prime ideal of S lying directly over each of $\langle e, h\rangle$ and $\langle e, h-1\rangle$. Therefore, "lying over" and "lying directly over" are not equivalent in this situation.

(ii) Retaining the notation of (i), let $P = P_r$. If Q is the unique annihilator prime of ${}_R(S/P)$, then Q is the annihilator of a highest weight vector of the unique $(r+1)$-dimensional simple left S-module. Note that R/Q is isomorphic to k. Let $\mathcal{V} = {}_S(S/SQ)_R$. It follows from the left handed versions of (7.3) and (7.5) that $\mathcal{V}$ has a unique simple S-R-bimodule factor $\mathcal{W}$, and that $P = \operatorname{ann}_S\mathcal{V}$. Now note that $\mathcal{V}$ is isomorphic as a left S-module to a Verma module of highest weight r. Further note, since the canonical map $k \to R/Q$ is an isomorphism, that the lattice of left S-submodules of $\mathcal{V}$ is identical to the lattice of S-R-sub-bimodules of $\mathcal{V}$. Consequently, $\mathcal{W}$ is the unique simple left S-factor of $\mathcal{V}$. Thus for the co-finite-dimensional prime ideals of this example, (7.5) is a reformulation of part of the basic representation theory of $\mathrm{sl}_2(k)$.

(iii) In the notation of (i), if $\alpha \in k$ is not an integer, then no prime ideal of S lies over $\langle e, h-\alpha\rangle$. □

7.8 Now suppose that τ extends to an automorphism of S. Observe for any prime ideal Q of R that the automorphism τ induces a ring isomorphism of $\mathcal{A}(Q)$ onto $\mathcal{A}(\tau(Q))$, and hence the extended τ induces additive isomorphisms of $\mathcal{V}(Q)$ onto $\mathcal{V}(\tau(Q))$ and of $\mathcal{W}(Q)$ onto $\mathcal{W}(\tau(Q))$. In particular, $\tau(\operatorname{ann}[\mathcal{W}(Q)]_S) = \operatorname{ann}[\mathcal{W}(\tau(Q))]_S$. Similar statements hold for any power of τ.

We now show that the pattern of (7.7i) does not occur when τ extends to an automorphism of S.

Theorem 7.9. *Assume that τ extends to an automorphism of S, and let Q be a prime ideal of R whose τ-orbit is infinite.*

(i) *There exists a prime ideal of* S *lying over* Q *if and only if* $\mathcal{W}(Q)$ *is a proper factor of* $\mathcal{V}(Q)$.

(ii) *If* $\mathcal{W}(Q)$ *is a proper factor of* $\mathcal{V}(Q)$, *then the set of prime ideals of* S *lying over* Q *is* $\{P, \tau(P), \ldots, \tau^{n-1}(P)\}$ *where* $P = \operatorname{ann}[\mathcal{W}(Q)]_S$ *and* n *equals the length of* ${}_{\mathcal{A}(Q)}[\mathcal{W}(Q)]$ *divided by the length of* ${}_{\mathcal{A}(Q)}\mathcal{A}(Q)$. *The prime ideals* $P, \tau(P), \ldots, \tau^{n-1}(P)$ *are all distinct, and* $n = \operatorname{rank}(S/P)/\operatorname{rank}(R/Q)$.

Proof: (i) If $\mathcal{W}(Q)$ is a proper factor of $\mathcal{V}(Q)$, then there exists a prime ideal of S lying over Q by (7.5.II). Conversely, suppose that there exists a prime ideal P of S lying over Q. In view of (7.5.I), P lies directly over exactly one prime ideal Q' of R, and $\mathcal{W}(Q')$ is a proper factor of $\mathcal{V}(Q')$. But Q' and Q are in the same τ-orbit by (5.4i). Since τ extends to S it follows that $\mathcal{W}(Q)$ is a proper factor of $\mathcal{V}(Q)$.

(ii) By (7.5.II), P is the unique prime ideal of S lying directly over Q. It follows from (5.12i,ii) that the prime ideals of R minimal over $P \cap R$ are precisely Q, $\tau^{-1}(Q)$, ..., $\tau^{-(n-1)}(Q)$. Consequently, $P, \tau(P), \ldots, \tau^{n-1}(P)$ all lie over Q. Since $\tau^i(P)$ lies directly over $\tau^i(Q)$ for each i, and since $\tau^i(Q) \neq \tau^j(Q)$ for $i \neq j$, it follows from (7.5.I.1) that $\tau^i(P) \neq \tau^j(P)$ for $i \neq j$.

Now suppose that P' is a prime ideal of S lying over Q. from (7.5.I) it follows that P' lies directly over exactly one prime ideal Q' of R and that $P' = \operatorname{ann}[\mathcal{W}(Q')]_S$. By (5.4i), $Q' = \tau^i(Q)$ for some nonnegative integer i. As observed in (7.8), $\operatorname{ann}[\mathcal{W}(Q')]_S = \tau^i(\operatorname{ann}[\mathcal{W}(Q)]_S)$, that is, $P' = \tau^i(P)$. Since Q is minimal over $P' \cap R$, we conclude that $\tau^{-i}(Q)$ is minimal over $P \cap R$, and therefore $i \in \{0, 1, \ldots, n-1\}$. This proves that $P, \tau(P), \ldots, \tau^{n-1}(P)$ are precisely the prime ideals of S lying over Q.

The final statement of (ii) follows from (5.14i). □

7.10. If R is commutative, it follows from [**13**, 3.1] that for each prime ideal P of S whose associated τ-orbit in spec$R$ is infinite, $P \cap R$ is equal to a prime ideal Q of R, and P is the only prime ideal of S lying over Q. That neither of these statements hold when R is noncommutative follows from (7.7). The following example shows that these statements need not hold when (τ, δ) is a (-1)-skew derivation.

Example. Let k be an algebraically closed field of characteristic zero. Let $V = k\langle s, t, u, v\rangle$ be the enveloping algebra of the completely solvable Lie superalgebra pl$(1, 1)$ with basis $s = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), t = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), u = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), v = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$. (See [**4**] and [**32**] for details.) Basic properties of V may be deduced from the embedding of V into $M_2(k[X,Y])$ via the map

$$s \mapsto \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}, \quad t \mapsto \begin{pmatrix} Y & 0 \\ 0 & Y-1 \end{pmatrix}, \quad u \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad v \mapsto \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}.$$

Let R be the subalgebra of V generated by s, t, and u, and let U denote the subalgebra generated by s and t. Note that U is a commutative polynomial ring in two variables, and note that u is a nilpotent normal element of R. Hence, the maximal ideals of R are the ideals $Q_{(\alpha,\beta)} = Ru + R(s-\alpha) + R(t-\beta)$ for $\alpha, \beta \in k$.

Let τ be the k-algebra automorphism of R determined by $\tau(s) = s$, $\tau(t) = t+1$, $\tau(u) = -u$. It follows from [**32**, 2.2i,iii] that there is a k-linear τ-derivation $\delta\colon R \to R$ determined by $\delta(s) = \delta(t) = 0$ and $\delta(u) = s$, and that $\delta\tau = (-1)\tau\delta$. Moreover, $R[y;\tau,\delta]/\langle y^2\rangle$ is isomorphic to V via the assignment $y \mapsto v$, by [**32**, 3.2]. Let $S = R[y;\tau,\delta]$. Note that the τ-orbit of each maximal ideal of R is infinite. Consider the irreducible representation of S given by

$$s \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad t \mapsto \begin{pmatrix} \beta & 0 \\ 0 & \beta-1 \end{pmatrix}, \quad u \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad v \mapsto \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix},$$

for $\alpha, \beta \in k$, and $\alpha \neq 0$. Let P be the kernel of this representation. Note that the prime ideals of R minimal over $P \cap R$ are precisely $Q_{(\alpha,\beta)}$ and $Q_{(\alpha,\beta-1)}$. In particular, the τ-orbit of spec$R$ associated to P is infinite, and $P \cap R$ is not prime. Moreover, $u(t-\beta) \notin P \cap R$ and so $Q_{(\alpha,\beta)} \cap Q_{(\alpha,\beta-1)} \neq P \cap R$. Therefore $P \cap R$ is not even semiprime. Finally, we observe that two prime ideals of S lie over $Q_{(\alpha,\beta)}$, namely P and $\tau^{-1}(P)$, where we have extended τ to S as in (2.4ii). $\square$

Recall that there is a *link* from a prime ideal Q_α of R to a prime ideal Q_β of R, denoted by $Q_\alpha \rightsquigarrow Q_\beta$, if there exists a nonzero (R/Q_α)-(R/Q_β)-bimodule factor of $(Q_\alpha \cap Q_\beta)/Q_\alpha Q_\beta$ which is torsionfree on each side. (See [**16**], [**21**], or [**34**] for more information.) We next observe that a prime ideal of S whose associated τ-orbit in spec$R$ is infinite lies over a link-connected subset of spec$R$.

Proposition 7.11. *If P is a prime ideal of S lying over a prime ideal Q of R, and the τ-orbit of Q is infinite, then either $P \cap R = Q$ or $\tau^i(Q) \rightsquigarrow \tau^{i+1}(Q)$ for all integers i.*

Proof: Suppose that $P \cap R \neq Q$. From (5.12) it follows that there are nonnegative integers m and n such that

$$[\tau^{-m}(Q)][\tau^{-(m-1)}(Q)] \cdots Q \cdots [\tau^{n-1}(Q)][\tau^{n}(Q)] \subseteq P \cap R$$

and that $\{\tau^{-m}(Q), \ldots, \tau^{n}(Q)\}$ is the set of prime ideals of R minimal over $P \cap R$, listed without repetition. Moreover, since $P \cap R \neq Q$, at least one of m or n is positive. To prove the proposition it suffices to prove that $\tau^{n-1}(Q) \rightsquigarrow \tau^{n}(Q)$. By (4.6), $R/(P \cap R)$ has an artinian classical quotient ring A. Let J and K denote the respective extensions of $\tau^{n-1}(Q)/(P \cap R)$ and $\tau^{n}(Q)/(P \cap R)$ to A. It now suffices to show that $J \rightsquigarrow K$. Note that there exists a nonzero ideal I of A, induced from $[\tau^{-m}(Q)][\tau^{-(m-1)}(Q)] \cdots [\tau^{n-2}(Q)]$, such that $IJK = 0$, such that $IJ \neq 0$, and such that $IJ \neq I$. Since A is artinian, $0 \subset IJ \subset I$ is an affiliated series for the right A-module I, with affiliated primes K and J. However, K is the only annihilator prime of I_A by (5.4). It now follows from an application of the Main Lemma of Jategaonkar (see [**21**, 9.1.2] or [**16**, 11.1]) that $J \rightsquigarrow K$, as desired. □

Corollary 7.12. *If R is a finitely generated module over its center and P is a prime ideal of S whose associated τ-orbit in* spec$R$ *is infinite, then $P \cap R$ is a prime ideal of R.*

Proof: It is well known that if R is a finite module over its center Z then at most finitely many prime ideals of R can have a common intersection with Z. (E.g., from incomparability [**34**, 10.4.15], one deduces that the prime ideals of R contracting to a prime ideal I of Z are minimal over IR.) Therefore, since linked prime ideals in R have the same contraction in Z (e.g., [**16**, 11.7]), at most finitely many prime ideals of R can belong to a link-connected component of spec$R$. The corollary now follows from (7.11). □

8. Prime ideals in S associated to infinite orbits. The q-skew case.

We carry over the assumptions and notation of section 7, specializing to the case that (τ, δ) is a q-skew derivation. In this case the description given in (7.9) for prime ideals of S lying over a given prime ideal of R with infinite τ-orbit improves considerably, and in (8.5) we obtain a complete analysis assuming that one of the conditions $(*, 0)$, $(0, t)$, (p, t) of (2.7) holds.

Throughout this section, we shall assume that (τ, δ, q) is a q-skew derivation, for some fixed central invertible (τ, δ)-constant $q \in R$. Hence, as noted in (2.4ii), τ extends to an automorphism of S, such that $\tau(y) = q^{-1}y$. We rely heavily on the various q-computations summarized in (2.5) and (2.6).

Recall the notation of (5.5), and assume temporarily that Q is a prime ideal of R whose τ-orbit is infinite. In (7.9) we learned that the prime ideals of S lying over Q are precisely determined by the unique proper simple $\mathcal{A}(Q)$-S-bimodule factor $\mathcal{W}(Q)$ of $\mathcal{V}(Q)$. We therefore seek an exact criterion for the existence of proper nonzero sub-bimodules of $\mathcal{V}(Q)$. Such a criterion is given in (8.4).

When the τ-orbit of Q is finite, there is also a connection – although not as close – between the prime ideals lying over Q and the simple bimodule factors of $\mathcal{V}(Q)$ (cf. (5.5)). Thus we will later require information as in the preceding paragraph for prime ideals of R whose τ-orbit is finite. For this reason the first three results of this section are more general than immediately necessary.

Part of the proof of the following lemma is modelled on proofs of the simplicity criterion for a differential operator ring $R[\theta; \delta]$ in characteristic p as in [**22**, Theorem 4.1.6] and [**33**, Theorem 4]. (Parallel arguments were used in proving (3.11vi).)

Lemma 8.1. *Let Q be a prime ideal of R, let $A = \mathcal{A}(Q)$ and $V = \mathcal{V}(Q)$, and denote images in A with an overbar. Assume that one of the cases $(*, 0)$, $(0, t)$, (p, t) holds, and in these respective cases set $N = \{1\}$, $N = \{1, t\}$, $N = \{1, t, tp, tp^2, \ldots\}$. Suppose that there exists a monic polynomial $f \in V$, with positive degree m, such that $fs = \tau^m(s)f$ for all $s \in R$. Then there exist integers $n_1 < n_2 < \cdots < n_k = n$ in N, with $n \leq m$, and elements $c, d_1, d_2, \ldots, d_{k-1}$ in A such that*

$$\overline{\delta^n(r)} + \sum_{i=1}^{k-1} d_i \delta^{n_i}(r) \;=\; \tau^n(r)c - cr$$

and $d_i r = \tau^{n-n_i}(r)d_i$ *for all* $r \in R$ *and all* $i = 1, \dots, k-1$.

Proof: Write $f = 1 \otimes y^m + a_{m-1} \otimes y^{m-1} + \cdots + a_0 \otimes 1$ for some $a_0, \dots, a_{m-1} \in A$. Given $s \in R$, the left coefficients of y^{m-1} in fs and $\tau^m(s)f$ are equal, and so it follows from (2.5i) that

$$\binom{m}{1}_q \overline{\tau^{m-1}\delta(s)} + a_{m-1}\tau^{m-1}(s) \;=\; \tau^m(s)a_{m-1} \;.$$

Since $\delta\tau = q\tau\delta$, it must also hold that $u\overline{\delta\tau^{m-1}(s)} + a_{m-1}\tau^{m-1}(s) = \tau^m(s)a_{m-1}$ for the central (τ,δ)-constant $u = \binom{m}{1}_q q^{1-m}$. Now let $r = \tau^{m-1}(s)$, and observe from the above that $u\overline{\delta(r)} + a_{m-1}r = \tau(r)a_{m-1}$. Thus if u is invertible in R, we can take $n = n_1 = 1$ and $c = a_{m-1}u^{-1}$.

We now assume that u is not invertible in R. Then either $(0,t)$ or (p,t) must hold, and $t \mid m$ by (2.6iv). Set $a_m = 1$, and set

$$\alpha \;=\; \max\bigl(\{0\} \cup \{i \in \{0, \dots, m-1\} \mid a_i \neq 0 \text{ and } t \text{ does not divide } i\}\bigr).$$

Then set $J = \{\alpha\} \cup \{j \in \{\alpha+1, \dots, m\} \mid a_j \neq 0\}$.

Consider $j \in J$ such that $j > \alpha$; then $t \mid j$. In case $(0,t)$, write $j = t\lambda_j$ for an appropriate positive integer λ_j, and set $j^* = t$. Then λ_j is invertible in R. Using (2.6iii), we find that $\binom{j}{j^*}_q = \lambda_j$ while $\binom{j}{i}_q = 0$ for $i = 1, \dots, j^*-1$. In case (p,t), write $j = tp^{\mu_j}\lambda_j$ for some nonnegative integer μ_j and some positive integer λ_j not divisible by p, and set $j^* = tp^{\mu_j}$. Again, λ_j is invertible in R. Using (2.6iii) together with the analogous result for ordinary binomial coefficients in characteristic p, we find that

$$\binom{j}{j^*}_q \;=\; \binom{p^{\mu_j}\lambda_j}{p^{\mu_j}} \;=\; \binom{\lambda_j}{1} \;=\; \lambda_j$$

while $\binom{j}{i}_q = 0$ for $i = 1, \dots, j^*-1$. Finally, set $\alpha^* = 1$ and $\lambda_\alpha = \binom{\alpha}{1}_q$.

Let $\kappa = \max\{j - j^* \mid j \in J\}$ and $K = \{j \in J \mid j - j^* = \kappa\}$, and note that $\kappa \geq m - m^* \geq 0$ and $\kappa \geq \alpha - \alpha^* = \alpha - 1$. We shall compare coefficients of $y^m, y^{m-1}, \dots, y^\kappa$ in the identity $fs = \tau^m(s)f$. Note that $a_j = 0$ for those $j = m, m-1, \dots, \kappa+1$ not in J. For $j \in J$ such that $j > \alpha$, we compute that

$$y^j s \;=\; \tau^j(s)y^j + \lambda_j\tau^{j-j^*}\delta^{j^*}(s)y^{j-j^*} + [\text{terms of degree less than } \kappa].$$

By our choice of α^* and λ_α, the equation above also holds for $j = \alpha$. Thus

$$fs = \sum_{j=\kappa}^{m} (a_j \tau^j(s) \otimes y^j) + \Big(\sum_{j \in K} \lambda_j a_j \tau^\kappa \delta^{j^*}(s) \Big) \otimes y^\kappa$$
$$+ \text{[terms of degree less than } \kappa\text{]}.$$

Consequently, $a_j \tau^j(s) = \tau^m(s) a_j$ for $j = \kappa + 1, \ldots, m$ and

$$a_\kappa \tau^\kappa(s) + \sum_{j \in K} \lambda_j a_j \tau^\kappa \delta^{j^*}(s) \;=\; \tau^m(s) a_\kappa.$$

For all $r \in R$, it follows that $a_j r = \tau^{m-j}(r) a_j$ for $j = \kappa + 1, \ldots, m$ and

$$\sum_{j \in K} q^{-j^* \kappa} \lambda_j a_j \delta^{j^*}(r) \;=\; \tau^{m-\kappa}(r) a_\kappa - a_\kappa r.$$

Label the elements of the set $\{j^* \mid j \in K\}$ in ascending order as $n_1 < n_2 < \cdots < n_k = n$, and note that these n_i all lie in N. Then $n \le m$ and $K = \{n_1 + \kappa, \ldots, n_k + \kappa\}$. Set $b_i = q^{-n_i \kappa} \lambda_{n_i + \kappa} a_{n_i + \kappa}$ for $i = 1, \ldots, k$. For all $r \in R$, we now have

$$\sum_{i=1}^{k} b_i \delta^{n_i}(r) \;=\; \tau^{m-\kappa}(r) a_\kappa - a_\kappa r$$

and $b_i r = \tau^{m - n_i - \kappa}(r) b_i$ for $i = 1, \ldots, k$.

The only way in which a_j could vanish for some $j \in J$ is in case $j = \alpha = 0$ and $a_0 = 0$. However, if $\alpha = 0$ then $\alpha - \alpha^* < 0$ and so $\alpha \notin K$. Thus $a_j \neq 0$ for all $j \in K$. In particular, it follows that $b_k \neq 0$. From the identity $b_k r = \tau^{m-n-\kappa}(r) b_k$ we see that Ab_k is an A-R-sub-bimodule of A. Hence, $Ab_k = A$ by the bimodule simplicity noted in (4.3), and so b_k is invertible in A. Setting $c = b_k^{-1} a_\kappa$ and $d_i = b_k^{-1} b_i$ for all i, we therefore conclude that

$$\overline{\delta^n(r)} + \sum_{i=1}^{k-1} d_i \delta^{n_i}(r) \;=\; \tau^n(r) c - cr$$

and $d_i r = \tau^{n - n_i}(r) d_i$ for all $r \in R$ and all $i = 1, \ldots, k-1$, as desired. $\square$

Remark 8.2. If some $d_i \neq 0$ in (8.1), we see from the identity $d_i r = \tau^{n-n_i}(r) d_i$ that Ad_i is a nonzero A-R-sub-bimodule of A. It follows from (4.3) that $Ad_i =$

A, so d_i is invertible in A. Hence, it follows from the identity $d_i r = \tau^{n-n_i}(r) d_i$ that $\tau^{n-n_i}(Q) = Q$ and that $\tau^{n-n_i}(a) = d_i a d_i^{-1}$ for all $a \in A$.

Therefore if we impose the hypothesis that no positive power of τ which maps Q to itself induces an inner automorphism of A, the situation of the preceding paragraph cannot occur. The conclusion of (8.1) thus simplifies considerably under this hypothesis, namely that there exist $n \in N$ and $c \in A$ such that $\overline{\delta^n(r)} = \tau^n(r)c - cr$ for all $r \in R$. (Of course the proof of (8.1) also simplifies under this hypothesis.)

Proposition 8.3. *Let Q be a prime ideal of R, let $A = \mathcal{A}(Q)$ and $V = \mathcal{V}(Q)$, and denote images in A with an overbar. Assume that one of the cases $(*, 0)$, $(0, t)$, (p, t) holds, and in these respective cases set $N = \{1\}$, $N = \{1, t\}$, $N = \{1, t, tp, tp^2, \dots\}$.*

(i) *There exists a proper nonzero A-S-sub-bimodule in V if and only if the following condition $[n]$ holds for some $n \in N$.*

$[n]$: *There exist integers $n_1 < n_2 < \cdots < n_k = n$ in N and elements $c, d_1, \dots, d_{k-1}$ in A such that*

$$\overline{\delta^n(r)} + \sum_{i=1}^{k-1} d_i \delta^{n_i}(r) \;=\; \tau^n(r)c - cr$$

and $d_i r = \tau^{n-n_i}(r) d_i$ for all $r \in R$ and all $i = 1, \dots, k-1$.

(ii) *If n is the least integer in N for which $[n]$ holds, then any proper A-S-sub-bimodule of V which contains a monic polynomial of degree n is maximal. In particular,*

$$A.(1 \otimes y^n + \sum_{i=1}^{k-1} d_i \otimes y^{n_i} + c \otimes 1).S$$

is a maximal proper A-S-sub-bimodule of V.

Proof: Suppose first that there exists a proper nonzero A-S-sub-bimodule B of V. By (5.6ii), B contains a unique monic polynomial f of minimal degree, say m, and since $B \neq V$ it must follow that $m > 0$. For any $s \in R$, observe that $fs - \tau^m(s)f$ is contained in B and has degree less than m. Hence, $fs - \tau^m(s)f = 0$, by (5.6i). Thus $fs = \tau^m(s)f$ for all $s \in R$. It follows from (8.1) that $[n]$ holds for some $n \in N$.

Conversely, suppose that $[n]$ holds for some $n \in N$, and set

$$b = 1 \otimes y^n + \sum_{i=1}^{k-1} d_i \otimes y^{n_i} + c \otimes 1 \in V.$$

Using (2.6ii,iii), we infer that $\binom{n_i}{j}_q = 0$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n_i - 1$. Consequently, we conclude from $[n]$ that $br = \tau^n(r)b$ for all $r \in R$. Then $A.b.S = \sum_{i=0}^{\infty} A.b.y^i$, and therefore $A.b.S$ is a proper nonzero A-S-sub-bimodule of V.

Finally, suppose that n is the least integer in N for which $[n]$ holds, and that B is a proper A-S-sub-bimodule of V containing a monic polynomial of degree n. If V contains an A-S-sub-bimodule B' such that $B \subset B' \subset V$, then the unique monic polynomial f of minimal degree in B' has degree $m < n$. But then (8.1) implies that $[n']$ holds for some $n' < n$, contradicting our assumption. Therefore B is a maximal proper A-S-sub-bimodule of V. □

Corollary 8.4. *Retain the notation and assumptions of* (8.3), *and assume further that no positive power of τ which maps Q to itself induces an inner automorphism of A. Then there exists a proper nonzero A-S-sub-bimodule in V if and only if there exist $n \in N$ and $c \in A$ such that $\overline{\delta^n(r)} = \tau^n(r)c - cr$ for all $r \in R$. Moreover, if n is the least integer in N for which the preceding identity holds, then $A.(1 \otimes y^n + c \otimes 1).S$ is the unique maximal proper A-S-sub-bimodule of V.*

Proof: That V has a unique maximal proper A-S-sub-bimodule is proved in (7.3). Therefore, in view of (8.2), the corollary follows directly from (8.3). □

We can now offer the following full description of the prime ideals of S associated to infinite τ-orbits in spec$R$.

Theorem 8.5. *Let Q be a prime ideal of R whose τ-orbit is infinite, let A be the Goldie quotient ring of R/Q, and denote images in A with an overbar.*

(I) *Assume that Case* $(*, 0)$ *holds.*

(1) *There exists a prime ideal of S lying over Q if and only if there exists an element $c \in A$ such that $\overline{\delta(r)} = \tau(r)c - cr$ for all $r \in R$.*

(2) *If there exists an element $c \in A$ as in* (1), *then there is exactly one prime ideal of S lying over Q, namely $P = \operatorname{ann}[\mathcal{W}(Q)]_S$ where $\mathcal{W}(Q) =$*

$\mathcal{V}(Q)/A.(1 \otimes y + c \otimes 1).S$. Moreover, $P \cap R = Q$, so P lies over no prime ideal of R distinct from Q, and $\operatorname{rank}(S/P) = \operatorname{rank}(R/Q)$.

(3) *If there exists $c_0 \in R$ such that $\delta(r) \equiv \tau(r)c_0 - c_0 r \pmod{Q}$ for all $r \in R$, then the unique prime ideal of S lying over Q is equal to $Q + (y + c_0)S = Q + S(y + c_0)$.*

(II) *Assume that either Case $(0,t)$ or Case (p,t) holds. In the first case set $N = \{1, t\}$, while in the second set $N = \{1, t, tp, tp^2, \ldots\}$.*

(1) *There exists a prime ideal of S lying over Q if and only if there exist $n \in N$ and $c \in A$ such that $\overline{\delta^n(r)} = \tau^n(r)c - cr$ for all $r \in R$.*

(2) *If there exist n, c as in (1), with n minimal, then the set of prime ideals of S lying over Q is $\{P, \tau(P), \ldots, \tau^{n-1}(P)\}$ where $P = \operatorname{ann}[\mathcal{W}(Q)]_S$ and $\mathcal{W}(Q) = \mathcal{V}(Q)/A.(1 \otimes y^n + c \otimes 1).S$. Moreover, P is the unique prime ideal of S lying directly over Q, and does not lie directly over any prime ideal of R distinct from Q. Also,* $\operatorname{rank}(S/P) = n.\operatorname{rank}(R/Q)$.

Proof: In (7.9) we saw that there exists a prime ideal P of S lying over Q if and only if $\mathcal{W}(Q)$ is a proper factor of $\mathcal{V}(Q)$. Also, if $\mathcal{W}(Q)$ is proper then $P = \operatorname{ann}[\mathcal{W}(Q)]_S$, $\tau(P)$, $\ldots$, $\tau^{n-1}(P)$ are the distinct prime ideals lying over Q, where n is equal to the length of ${}_{\mathcal{A}(Q)}\mathcal{W}(Q)$ divided by the length of ${}_{\mathcal{A}(Q)}\mathcal{A}(Q)$. Moreover, $\operatorname{rank}(S/P) = n.\operatorname{rank}(R/Q)$ for n and P as above. Also, by (5.14), if $n = 1$ then $P \cap R = Q$. Parts (I.1), (I.2), (II.1), (II.2) of the theorem now follow from (8.4) and (7.5.I).

To prove (I.3), observe that $(y+c_0)r - \tau(r)(y+c_0) = \delta(r) + c_0 r - \tau(r)c_0 \in Q$ for all $r \in R$. It follows that the set $P' = Q + (y + c_0)S = Q + S(y + c_0)$ is an ideal of S. Moreover, $P' \cap R = Q$ and $R + P' = S$, whence $S/P' \cong R/Q$. Therefore $P' \in \operatorname{spec} S$. $\square$

Remark 8.6. (i) In case (I.2) of (8.5), the bimodule $\mathcal{W}(Q)$ can be constructed directly from A. Namely, the A-R-bimodule structure of A can be extended to an A-S-bimodule structure such that $a.y = -ac$ for all $a \in A$, and then $A \cong \mathcal{W}(Q)$ as A-S-bimodules.

(ii) Example (7.10) gives an instance of (8.5.II), Case $(0,t)$ in which $n = t = 2$.

Example 8.7. *A q-skew derivation (τ, δ, q) of a noetherian ring R, satisfying the hypotheses of (8.5.I), where there exists a prime ideal Q of R with infinite τ-orbit such that no prime ideal of $S = R[y; \tau, \delta]$ lies over Q.*

The example is adapted from [**6**] and [**36**, 2.2]. Let k be a field of characteristic zero, and let $R = k\{x, z\}/\langle xz - qzx\rangle$ for some nonzero $q \in k$ which is not a root of unity. Then $R = k[z][x; \tau^{-1}]$, where τ is the k-algebra automorphism of $k[z]$ such that $\tau(z) = q^{-1}z$, and we extend τ to an automorphism of R such that $\tau(x) = qx$. By (2.8), there exists a q-skew τ-derivation δ on R such that $\delta = 0$ on $k[z]$ and $\delta(x) = 1$.

Let Q denote the prime ideal $\langle x, z - \lambda\rangle$ of R, for some nonzero $\lambda \in k$, and let $A = R/Q \cong k$. Note that the τ-orbit of Q is infinite. Since $\delta(x) = 1$ while $\tau(x), x \in Q$, there does not exist $c \in A$ such that $\overline{\delta(x)} = \tau(x)c - cx$. Therefore by (8.5.I.1), no prime ideal of S lies over Q. We remark that, in contrast, if $Q' = \langle z, x - \mu\rangle$ for some $\mu \in k$, then the prime ideal $\langle z, x - \mu, (q-1)xy + 1\rangle$ in S lies over Q'. (See (2.9).) $\square$

9. Prime ideals in S associated to finite orbits. The general case.

Once again, assume throughout that $S = R[y;\tau,\delta]$ where R is a noetherian ring and τ is an automorphism of R. In this section we provide preliminary results toward a general description of the prime ideals of S lying over a prime ideal Q of R whose τ-orbit is finite. (A detailed analysis for the q-skew case is obtained in the next section.) Under the assumption that τ extends to an automorphism of R, we give in (9.6) a precise description of the minimal elements in the set of prime ideals of S lying over Q.

9.1. Let $\mathcal{Y}$ denote the collection of prime ideals of S whose associated τ-orbits in spec$R$, as defined in (7.1), are finite. Let $\mathcal{X}$ denote an arbitrary finite τ-orbit of spec$R$. Assuming a knowledge of the τ-prime ideals of R, the task of describing $\mathcal{Y}$ reduces to describing the collection $\mathcal{Y}_{\mathcal{X}}$ of prime ideals in S which lie over some member of $\mathcal{X}$. Moreover, it follows from (5.12) that a prime ideal P is contained in $\mathcal{Y}_{\mathcal{X}}$ if and only if P lies directly over some prime ideal in $\mathcal{X}$.

We first deal with a situation analogous to (7.5.II). The notation of (7.4) will again be used.

Proposition 9.2. *Let Q be a prime ideal of R whose τ-orbit is finite, and let A be the Goldie quotient ring of R/Q. Suppose that no positive power of τ which maps Q to itself induces an inner automorphism of A.*

(i) *If P is a prime ideal of S lying directly over Q then either $P = \operatorname{ann}(S/QS)_S$ or $P = \operatorname{ann}[\mathcal{W}(Q)]_S$. Consequently, at most two distinct prime ideals of S lie directly over Q, and if two distinct prime ideals of S lie directly over Q they must be comparable.*

(ii) *If $\mathcal{W}(Q)$ is a proper factor of $\mathcal{V}(Q)$, then $P_1 = \operatorname{ann}[\mathcal{W}(Q)]_S$ is a prime ideal of S lying directly over Q.*

(iii) *Suppose that $P_0 = \operatorname{ann}(S/QS)_S$ is a prime ideal of S. If $\mathcal{V}(Q)$ is torsionfree as a right (S/P_0)-module, then P_0 lies directly over Q.*

Proof: (i) This follows from (5.7i) and (7.3).

(ii) This follows from (5.7ii).

(iii) This is immediate from (5.5). $\square$

Remark 9.3. Let Q satisfy the hypotheses of (9.2), and let P be a prime ideal of S which lies directly over Q. In contrast to (7.5.II), it is not unusual for P

to lie directly over prime ideals of R distinct from Q. For instance, suppose that R is τ-prime but not prime, and that Q and Q' are distinct minimal prime ideals of R. (E.g., let R and τ be as in (2.3v), and let $\delta = 0$.) From (2.2iii) it follows that S is prime, and from (2.2ii) it follows that R is semiprime. Letting P denote the zero ideal of S, it follows from [**16**, 6.1] that P lies directly over both Q and Q'.

Corollary 9.4. *Let Q be an arbitrary prime ideal of R. If more than two prime ideals of S lie directly over Q then the τ-orbit of Q is finite and some positive power of τ induces an inner automorphism of the Goldie quotient ring of R/Q.*

Proof: This follows from (7.5.I.2) and (9.2i). □

Compare the following with (7.7i) and (7.9ii).

Corollary 9.5. *Let Q be a prime ideal of R such that $\tau^n(Q) = Q$ for some positive integer n but such that no positive power of τ which maps Q to itself induces an inner automorphism of the Goldie quotient ring of R/Q. Then at most $2n$ prime ideals of S lie over Q.*

Proof: By (5.12i), any prime ideal lying over Q lies directly over some prime ideal in the τ-orbit of Q. The conclusion now follows from (9.4). □

We next consider the case when τ extends to an automorphism of S. If I is a (τ, δ)-prime ideal of R, and τ extends to an automorphism of S, then recall from (3.3) that IS is a τ-prime ideal of S. Also, note that the largest (τ, δ)-ideal of R contained within a given prime ideal of R is (τ, δ)-prime.

Proposition 9.6. *Suppose that τ extends to an automorphism of S. Let Q be a prime ideal of R such that $\tau^n(Q) = Q$ for some positive integer n. Let I be the maximal (τ, δ)-ideal contained within Q, and let $Q_i = \tau^i(Q)$ for $0 \le i \le n-1$.*

(i) *If there exists a prime ideal P of S lying over Q, then P contains IS and Q is minimal over I.*

(ii) *Suppose that Q is minimal over I. Then each of the ideals $P_i = \mathrm{ann}(S/Q_iS)_S$ for $0 \le i \le n-1$ is a prime ideal lying over Q, and every prime ideal of S lying over Q contains at least one of these P_i.*

Proof: (i) Let $N = \bigcap_i Q_i$. It follows from (5.12) that P lies directly over Q_j for some $j \in \{0, \ldots, n-1\}$, and that $N^m \subseteq P \cap R$ for some positive integer m. Set

$J = \bigcap_{i \in \mathbf{Z}} \tau^i(P)$, and observe that $J \cap R \subseteq P \cap R \subseteq Q$. Since N^m is τ-stable, it follows from the inclusion $N^m \subseteq P \cap R$ that $N^m \subseteq J \cap R$. Consequently, Q is minimal over $J \cap R$. Since J is a τ-ideal, $J \cap R$ is a (τ, δ)-ideal, as in (2.1v). Because Q is minimal over $J \cap R$, we now conclude that Q is minimal over I.

Since P lies directly over Q_j, (5.5) implies that P is the right annihilator of an $\mathcal{A}(Q_j)$-S-bimodule factor of $\mathcal{V}(Q_j) = \mathcal{A}(Q_j) \otimes_R (S/Q_jS)$. Now note that $I \subseteq Q_j$, and recall that $SI = IS$. Therefore, $I \subseteq \operatorname{ann}[\mathcal{V}(Q_j)]_R$, and we see that $P \supseteq IS$.

(ii) We may assume without loss of generality that $I = IS = 0$, and hence that R is (τ, δ)-prime. It then follows from (2.3i) and (2.2ii) that the prime radical of R is equal to $Q_0 \cap \cdots \cap Q_{n-1}$. By (3.3), S is τ-prime, and from (2.2ii) it follows that S is semiprime. Moreover, by (2.3i, ii), we may assume without loss of generality that R is artinian. (Recall from (5.5) that $\operatorname{ann}(S/Q_iS)_S = \operatorname{ann}[\mathcal{V}(Q_i)]_S$ for each i.)

Since S is semiprime, it follows from Goldie's theorem and [**16**, 10.13] that there exists a right affiliated series $0 = S_0 \subset \cdots \subset S_m = S$ for S such that the right annihilator in S of S_j/S_{j-1} is a minimal prime ideal for $1 \leq j \leq m$. Since S is not finitely generated as a right R-module, there exists at least one minimal prime ideal P' of S such that S/P' is not finitely generated as a right R-module. Set C' equal to the Goldie quotient ring of S/P'.

Now P' lies directly over some prime ideal Q' of R, and it follows from (5.9i) that $P' = \operatorname{ann}(S/Q'S)_S$. Since R is artinian, Q' is a minimal prime ideal of R, and hence $Q' \in \{Q_i\}$. From the equalities $\tau^i(\operatorname{ann}(S/QS)_S) = \operatorname{ann}(S/\tau^i(Q)S)_S$, it follows that the τ-orbit of P' in $\operatorname{spec} S$ is $\{P_0, P_1, \ldots, P_{n-1}\}$. Since S is τ-prime, we now conclude from (2.2ii) that the minimal prime ideals of S are precisely $P_0, \ldots, P_{n-1}$.

Finally, by (5.9ii) we have $P' \cap R \subseteq \tau^i(Q)$ for all i, and consequently $P_i \cap R \subseteq Q$ for all i. Therefore the prime ideals P_i all lie over Q. $\square$

Corollary 9.7. *Suppose that R is (τ, δ)-prime and that τ extends to an automorphism of S. Let Q be a minimal prime ideal of R, and suppose that $\tau^n(Q) = Q$ for some positive integer n. Then the minimal prime ideals of S are precisely the ideals $\operatorname{ann}[S/\tau^i(Q)S]_S$ for $i = 0, \ldots, n-1$.* $\square$

Corollary 9.8. *Suppose that τ extends to an automorphism of S. Let Q be a prime ideal of R such that $\tau^n(Q) = Q$ for some positive integer n, let A denote* *the Goldie quotient ring of R/Q, and let Y denote the set of prime ideals of S which lie over at least one prime ideal in the τ-orbit of Q. Assume that no positive power of τ which fixes Q induces an inner automorphism of A. Then Y equals the union of at most two τ-orbits in* spec$S$, *each containing no more than n prime ideals.*

Proof: Suppose that Y is nonempty. Then by (9.6ii) the subset

$$X = \{\mathrm{ann}[S/\tau^i(Q)S]_S \mid 0 \le i \le n-1\}$$

of Y is a τ-orbit of spec$S$. Next suppose that there exists a prime ideal $P \in Y \setminus X$, and recall the notation of (7.4). From (5.12) it follows that P lies directly over some Q' in the τ-orbit of Q. It therefore follows from (9.2i) that $P = \mathrm{ann}[\mathcal{W}(Q')]_S$ and that $\mathcal{W}(Q')$ is a proper factor of $\mathcal{V}(Q')$. In view of (7.8), the subset $Y \setminus X$ is equal to the τ-orbit $\{\mathrm{ann}[\mathcal{W}(\tau^i(Q))]_S \mid 0 \le i \le n-1\}$. The corollary follows. $\square$

10. Prime ideals in S associated to finite orbits. The q-skew case.

We now specialize the discussion of the previous section. Throughout, let $S = R[y;\tau,\delta]$, where R is a noetherian ring and τ is an automorphism of R. Further assume that (τ,δ) is a q-skew derivation of R for a fixed central invertible (τ,δ)-constant q. Extend τ to an automorphism of S as in (2.4ii), so that $\tau(y) = q^{-1}y$. In this section we give a detailed description, assuming that one of the cases $(*,0)$, $(0,t)$, (p,t) from (2.7) holds, of those prime ideals in S whose associated τ-orbits in spec$R$ are finite.

Following (9.1), to obtain the above description we provide a full analysis of those prime ideals in S lying over at least one prime ideal in a given arbitrary finite τ-orbit of spec$R$, and our approach, in essence, is to reduce to the cases outlined in (2.3iii,iv). In (10.1) we outline a complete algorithm for this analysis, after which some supporting results and consequences are proved. In particular, explicit criteria for the existence of prime ideals of S lying over a given prime ideal of R with finite τ-orbit are established.

Algorithm 10.1. *Assume that Q is a prime ideal of R whose τ-orbit is finite, and that one of the cases $(*,0)$, $(0,t)$, (p,t) holds. The prime ideals of S lying over prime ideals in the τ-orbit of Q are determined by the following procedure.*

(1) *There exists a prime ideal of S lying over Q if and only if Q is minimal over a (τ,δ)-ideal.* This follows from (9.6). For more explicit criteria in the three current cases, see (10.3i), (10.8i), (10.9i).

(2) *If there are prime ideals of S lying over Q, reduce to the case that R is artinian and (τ,δ)-simple.* First factor out the largest (τ,δ)-ideal of R contained within Q. Then R is a (τ,δ)-prime ring, and under the assumption that there exist prime ideals of S lying over Q, it follows from (1) that Q is a minimal prime ideal of R. By (2.3i,ii), R has an artinian classical quotient ring $R\mathcal{C}^{-1}$, and the prime ideals of S lying over minimal prime ideals of R correspond exactly to the prime ideals of $(R\mathcal{C}^{-1})[y;\tau,\delta]$. Thus after localizing with respect to the regular elements of R, we are in the situation that R is artinian and (τ,δ)-simple. In particular, since by (2.3i) the radical J of R is τ-prime, we must have $J = \bigcap_i \tau^i(Q)$.

(3) *Suppose that R is semisimple but not simple.* In this case, R is decomposable as a ring, and hence δ is an inner τ-derivation, by (3.6). A complete determination of spec$S$ for this case is available from (2.3iii) as follows. First,

let n denote the number of (minimal) prime ideals of R, and let k denote the subfield of central τ-constants in R. (Since δ is inner, the elements of k are actually (τ, δ)-constants.) If no positive power of τ is an inner automorphism of R, then spec$S$ consists of the zero ideal and a single τ-orbit containing exactly n prime ideals. Second, if some positive power of τ is inner then a continuous closed surjection of spec$S$ onto spec$k[z]$ may be constructed, as in (2.3iii), such that each fiber is a singleton except for the preimage of (z), which consists of a single τ-orbit in spec$S$ of cardinality n.

(4) *If R is simple, determine whether S is simple.* See (10.2i) for a criterion. Of course, when R is simple, all prime ideals of S lie over Q. If R and S are both simple, then (0) is the unique prime ideal of S lying over Q.

(5) *Suppose that R is simple, while S is not simple.* In this case, if no positive power of τ is inner then spec$S$ consists of exactly two points; see (10.2ii). On the other hand, if some positive power of τ is inner, a homeomorphism can be constructed from spec$S$ onto spec$k[z]$, as in [**29**, 2.2, 2.9] or (2.3iv), where k is the subfield of central (τ, δ)-constants in R.

(6) *Suppose that R is not semisimple.* This is only possible in cases $(0, t)$ and (p, t), by (2.4iii). By (3.8), δ^t is a 1-skew τ^t-derivation on R, and the subring U of R generated by $R \cup \{y^t\}$ has the form $U = R[y^t; \tau^t, \delta^t]$. Suppose first that $(0, t)$ holds. Then by (3.9), J is a (τ^t, δ^t)-ideal of R, and a homeomorphism of spec$S$ onto spec(U/JU) which commutes with the actions of τ on spec$S$ and spec(U/JU) may be constructed as in (3.9iv). Since U/JU is isomorphic to $(R/J)[y^t; \tau^t, \delta^t]$, it is a finite direct product of rings of the form $U_j = R_j[y^t; \tau^t, \delta^t]$ where R_j is semisimple artinian and τ^t-simple. (Observe as in the proof of (3.6) that all τ^t-ideals of R/J are δ^t-stable.) Hence, spec(U_j) can be determined using steps (3), (4), (5). Finally, assume that case (p, t) holds. Then by (3.11) and (3.12), there is a 1-skew τ^n-derivation $\overline{\zeta}$ on R/J, for some $n \in \{1, t, tp, tp^2, \ldots\}$, such that the ring $E = \operatorname{End}(S/JS)_S$ has the form $(R/J)[z; \tau^n, \overline{\zeta}]$ and a homeomorphism between spec$S$ and spec$E$ may be constructed as in (3.12iv). (The description of E improves considerably if no positive power of τ induces an inner automorphism of R/J – see (3.13).) As in the previous case, E is a finite direct product of skew polynomial rings whose prime spectra can be determined using steps (3), (4), (5). □

Part (ii) of the following proposition is a refinement (in the q-skew case) of the last part of [**29**, 2.3]. Note that the definition of $[n]$ in (i) is consistent with (8.3).

Proposition 10.2. *Suppose that R is simple artinian. Assume that one of the cases $(*, 0)$, $(0, t)$, (p, t) of (2.7) holds, and in these respective cases set $N = \{1\}$, $N = \{1, t\}$, $N = \{1, t, tp, tp^2, \ldots\}$.*

(i) *The ring S is not simple if and only if the following condition $[n]$ holds for some $n \in N$.*

$[n]$: *There exist integers $n_1 < n_2 < \cdots < n_k = n$ in N and elements $c, d_1, \ldots, d_{k-1}$ in R such that*

$$\delta^n(r) + \sum_{i=1}^{k-1} d_i \delta^{n_i}(r) \;=\; \tau^n(r)c - cr$$

and $d_i r = \tau^{n-n_i}(r) d_i$ for all $r \in R$ and all $i = 1, \ldots, k-1$.

(ii) *Suppose that S is not simple, and that no positive power of τ is an inner automorphism of R. If n is the least integer in N for which $[n]$ holds, then $S.(y^n + c) = (y^n + c).S$ is the unique nonzero prime ideal of S.*

Proof: Observe that $R = \mathcal{A}(0)$ and $S = \mathcal{V}(0)$, in the notation of (5.5).

(i) It follows immediately from (8.3i) that if S is not simple then $[n]$ holds for some $n \in N$. Conversely, assume that $[n]$ holds for some $n \in N$. Then by (8.3i) there exists a proper nonzero R-S-sub-bimodule of S. Let M be a simple R-S-bimodule factor of S, and let $P = \text{ann} M_S$. From (5.6iii) and [**21**, 5.1.1] it follows that M has finite length as a left R-module and that M is torsionfree as a right (S/P)-module. Then (5.8iii) implies that S/P has finite length as a right R-module. In particular, $P \neq 0$, and thus S is not simple.

(ii) First observe that $d_1, \ldots, d_{k-1} = 0$ (cf. (8.2)). Now let $z = y^n + c$. As in the second paragraph of the proof of (8.3), we see that $Rz = zR$, and then from (8.4) it follows that $zS = RzS$ is the unique maximal proper R-S-sub-bimodule of S. Hence, the set $J = \{w \in S \mid Sw \subseteq zS\}$ is the unique maximal ideal of S. However, S has right Krull dimension at most one by [**34**, 6.5.4i]. Therefore J is the unique nonzero prime ideal of S. Finally, by [**28**, 2.6] and the discussion in [**28**, p. 259], it follows that $Sz = zS$. Thus $J = Sz$. □

Theorem 10.3. *Assume that condition $(*,0)$ holds. Let Q be a prime ideal of R whose τ-orbit is finite, let $J = \bigcap_i \tau^i(Q)$, and let A denote the Goldie quotient ring of R/J.*

(i) *There exists a prime ideal of S lying over Q if and only if J is δ-stable.*

(ii) *If J is δ-stable, then $JS = SJ$ is a prime ideal of S lying over Q, and every prime ideal of S lying over Q contains JS. Moreover, every prime ideal of S lying over Q contracts either to Q or to J.*

(iii) *If J is δ-stable, and if δ does not induce an inner τ-derivation on A, then JS is the unique prime ideal of S lying over Q.*

(iv) *Assume that J is δ-stable, and that δ induces an inner τ-derivation on A. If no positive power of τ induces an inner automorphism of A, there are precisely two prime ideals of S lying over Q, while if some positive power of τ does induce an inner automorphism of A, there are infinitely many prime ideals of S lying over Q.*

(v) *If J is δ-stable but not prime, then δ induces an inner τ-derivation on A.*

Proof: If I is the maximal (τ,δ)-ideal contained within Q, then by (9.6i) any prime ideal of S lying over Q must contain the ideal IS. Thus there is no loss of generality in assuming that $I = 0$. Now R is a (τ,δ)-prime ring, and it follows from $(*,0)$ and (2.4iii) that R is actually τ-prime. Hence, Q is a minimal prime ideal of R if and only if $J = 0$, that is, if and only if J is δ-stable. By (9.6), Q is minimal if and only if there exists a prime ideal of S lying over Q, and (i) is proved.

To prove (ii)–(v), we assume that $J = 0$. Localization then allows us to replace R with A, by (2.3i, ii), and thus there is no loss of generality in assuming that R is semisimple and τ-simple. Since R is τ-simple, S is prime by (2.2iii), and so (0) is a prime ideal of S lying over Q. This proves the first part of (ii); the last part will be shown in proving (iii) and (iv). If R is not prime, then it is decomposable as a ring, and it follows from (3.6) that δ is inner, establishing (v).

(iii) Under these hypotheses, R is simple because of (v), and then S is a simple ring by (10.2i). In this case, the prime ideal (0) in S contracts to $Q = 0$ in R.

(iv) In this case $S = R[z;\tau]$ for a suitable element z, by (2.1iv), and $Q + zS$ is a nonzero prime ideal of S lying over Q and contracting to Q. Note, moreover, that $Q + zS$ is the only prime ideal of S that contains z and also lies over Q. If no positive power of τ is inner, then $R[z, z^{-1}; \tau]$ is simple by [**24**, Theorem 1], and hence $Q + zS$ is the only nonzero prime ideal of S lying over Q. On the other hand, if some positive power of τ is inner, then by (2.3iii), $R[z, z^{-1}; \tau]$ has infinitely many prime ideals, all of which contract to (0) in R. Thus in this case S contains infinitely many prime ideals lying over Q, and all except $Q + zS$ contract to J. □

The following corollary was proved for commutative noetherian coefficient rings in [**13**, 6.7].

Corollary 10.4. *Suppose that $(*, 0)$ holds and that P is an arbitrary prime ideal of S. Then $P \cap R$ is a semiprime ideal of R. More precisely, $P \cap R$ is either a prime ideal which is not τ-stable or a τ-prime ideal which is δ-stable.*

Proof: First, if $P \cap R$ is a τ-stable prime ideal of R, then $P \cap R$ is clearly τ-prime, and it is δ-stable by (2.1v). Next, if the τ-orbit in spec$R$ associated to P is infinite, then $P \cap R$ is prime by (8.5.I.2). Now suppose that P lies over a prime ideal Q of R whose τ-orbit is finite. By (10.3i,ii), Q is minimal over a δ-stable τ-prime ideal J, and P contracts to either Q or J. The corollary follows. □

We now develop analogs for (10.3) under assumptions $(0, t)$ and (p, t). In these cases certain complications arise which are not present in case $(*, 0)$. First, as seen for example in [**13**, 8.6.Ic], there may exist (τ, δ)-prime ideals of R which are not τ-prime. Second, in contrast to the case where either R is commutative (cf. [**13**, 3.3]) or R satisfies $(*, 0)$, a (τ, δ)-prime ideal of R need not induce to a prime ideal of S, as the next example demonstrates.

Example 10.5. *Let $t > 1$ be an integer. There exists a δ-simple artinian choice of R, satisfying either $(0, t)$ or (p, t), such that S is not prime.*

Let k be a field which contains a primitive $\mathrm{t^{th}}$-root of unity, say q. Then construct $A_1(k^t, \tau, q) = T_1[y; \tau, \delta]$ as in (2.8), where $T_1 = k^t[x; \tau^{-1}]$ and τ is the automorphism of k^t which sends $(a_1, \ldots, a_t)$ to $(a_t, a_1, \ldots, a_{t-1})$. By [**13**, 1.1], $\delta(x^j) = (\sum_{i=0}^{j-1} q^i)x^{j-1}$ for all positive integers j. In particular, $\delta(x^t) = 0$ while for $1 \leq j \leq t-1$ we have $\delta(x^j) = u_j x^{j-1}$ for some nonzero $u_j \in k$. Thus $x^t T_1$ is a (τ, δ)-ideal of T_1, and we claim that it is a maximal proper δ-ideal. Any

δ-ideal I of T_1 that properly contains x^tT_1 must contain an element of the form cx^{t-1} for some nonzero element $c \in k^t$. It follows that I contains the nonzero element $b = \delta^{t-1}(cx^{t-1}) = \tau^{t-1}(c)u_{t-1}u_{t-2}\cdots u_1$. Since $\delta(ax) = \tau(a)$ for all $a \in k^t$, and since k^t is τ-simple, we obtain $1 \in I$. Therefore x^tT_1 is a maximal proper δ-ideal, as claimed.

Now set $R = T_1/x^tT_1$, let (τ, δ) denote the induced q-skew derivation on R, and let $\overline{x} = x + x^tT_1$. Then R is a δ-simple artinian ring, and $\overline{x}R$ is the radical of R.

By (3.8), δ^t is a τ^t-derivation on R and S has a subring U of the form $R[y^t; \tau^t, \delta^t]$. Observe that $\delta^t(\overline{x}) = 0$, from which it follows that $\overline{x}R$ is a δ^t-ideal of R. Since $\delta(k^t)$ is zero, δ^t induces the zero map on $R/\overline{x}R$, and thus $U/\overline{x}U \cong k^t[y^t]$. From either (3.9) or (3.12) (depending on the characteristic of k), we conclude that $S \cong M_\ell(k^t[y^t])$ for some positive integer ℓ. Therefore S is a direct product of t prime rings. □

Remark 10.6. In the case where $t = 2$, the construction in (10.5) produces R, τ, δ isomorphic to the ring with skew derivation constructed in [**13**, 2.7]. As seen in (10.5), S is in this case a direct product of two prime rings, and thus S must contain a nontrivial central idempotent. Indeed, such a central idempotent is easy to write down, namely $(-1, 1)xy + (1, 0)$. The existence of such nontrivial central idempotents in skew polynomial rings was also discovered by Bergman and Isaacs in response to a question of Kamal [**26**, 6.4].

Lemma 10.7. *Assume that R is artinian and (τ, δ)-simple, and that one of conditions $(0, t)$, (p, t) holds. Set $N = \{1, t\}$ in the first case and $N = \{1, t, tp, tp^2, \dots\}$ in the second. Let J denote the prime radical of R, let $n \in N$, and assume that J is δ^n-stable. Let $J_1, \dots, J_m$ be the distinct τ^n-prime ideals of R, and set $A_j = \bigcap_{i \neq j} J_i$ for $1 \leq j \leq m$.*

(i) *$J_1, \dots, J_m$ are all δ^n-stable.*

(ii) *$R/J = (A_1/J) \oplus \dots \oplus (A_m/J)$, and each A_j/J is a (τ^n, δ^n)-ideal of R/J.*

(iii) *As rings, each A_j/J is τ^n-simple.*

Proof: Statement (iii) and the first part of (ii) are clear. There are orthogonal central idempotents $e_1, \dots, e_m$ in R/J such that $e_j(R/J) = A_j/J$ for each j. Since A_j/J is τ^n-stable, $\tau^n(e_j) = e_j$.

It follows from (2.6ii,iii) and (2.5i) that δ^n is a τ^n-derivation on R. Hence, $\delta^n(e_j) = \delta^n(e_j^2) = e_j\delta^n(e_j) + \delta^n(e_j)e_j$, and consequently A_j/J is δ^n-stable. This establishes (ii). Similarly, $J_j/J = (1-e_j)(R/J)$ is δ^n-stable, and therefore J_j is δ^n-stable. This proves (i). $\square$

Theorem 10.8. *Assume that condition $(0,t)$ holds. Let Q be a prime ideal of R such that $\tau^r(Q) = Q$ for some positive integer r, set $J = Q \cap \tau(Q) \cap \cdots \cap \tau^{r-1}(Q)$, and let A denote the Goldie quotient ring of R/J.*

(i) *There exists a prime ideal of S lying over Q if and only if J is δ^t-stable.*

(ii) *If J is δ^t-stable, then the ideals $P_i = \operatorname{ann}[S/\tau^i(Q)S]_S$ for $0 \le i \le r-1$ are prime ideals of S lying over Q, and every prime ideal of S lying over Q contains one of these P_i. In addition, if J is δ-stable then the P_i all coincide, while if J is not δ-stable, then the number of distinct P_i equals the number of distinct τ^t-orbits contained in the τ-orbit of Q.*

(iii) *If J is δ^t-stable, and if δ^t does not induce an inner τ^t-derivation on A, then $P_0, \ldots, P_{r-1}$ are the only prime ideals of S lying over Q.*

(iv) *Assume that J is δ^t-stable, and that δ^t induces an inner τ^t-derivation on A. Then there is at least one prime ideal of S different from $P_0, \ldots, P_{r-1}$ that lies over Q. If no positive power of τ induces an inner automorphism of A, there are at most $2r$ prime ideals of S lying over Q. If some positive power of τ does induce an inner automorphism of A, there are infinitely many prime ideals of S lying over Q.*

(v) *Assume that J is δ^t-stable, but that Q is not τ^t-stable. Then δ^t induces an inner τ^t-derivation on A.*

Proof: In view of (3.8i,ii), we see that (τ^t, δ^t) is a 1-skew derivation on R satisfying $(*, 0)$. Because of (9.6i), there is no loss of generality in assuming that (0) is the largest (τ,δ)-ideal contained in Q. Now R is a (τ,δ)-prime ring, and by (9.6) there exists a prime ideal of S lying over Q if and only if Q is a minimal prime ideal of R.

(i) If Q is a minimal prime ideal of R, then J is an intersection of minimal τ^t-prime ideals, and thus J is δ^t-stable by (3.8ii).

Conversely, assume that J is δ^t-stable. Since δ^t is a τ^t-derivation, it follows that J^t is δ^t-stable. Moreover, $\delta^j(J^t) \subseteq J$ for $0 \le j < t$, and hence $\delta^j(J^t) \subseteq J$ for all $j \ge 0$. Since the set $\{r \in R \mid \delta^j(r) \in J \text{ for all } j \ge 0\}$ forms a (τ,δ)-ideal contained within Q, we obtain $J^t = 0$. Therefore Q is a minimal prime ideal of R.

To prove (ii)–(v), we assume that J is δ^t-stable, and hence that Q is a minimal prime ideal. In view of (2.3i,ii), we may replace R by its artinian quotient ring. Thus there is no loss of generality in assuming that R is artinian and (τ,δ)-simple. Further, the radical of R is τ-prime by (2.3i), whence J equals the radical of R and R/J is τ-simple. If $J=0$, set $n=1$, while if $J\neq 0$, set $n=t$. In either case, J is δ^n-stable, and we may write $R/J=(A_1/J)\oplus\ldots\oplus(A_m/J)$ as in (10.7). Set $U=R[y^n;\tau^n,\delta^n]$, and observe that JU is a nilpotent ideal of U.

(ii) The first statement is immediate from (i) and (9.6). Moreover, because of our current reductions, $P_0,\ldots,P_{r-1}$ are precisely the minimal prime ideals of S, by (9.7).

By (3.9iii), $S\cong M_\ell(U/JU)$ for some positive integer ℓ. Further, U/JU is isomorphic to the direct sum of the rings $(A_j/J)[y^n;\tau^n,\delta^n]$. Since A_j/J is τ^n-simple, $(A_j/J)[y^n;\tau^n,\delta^n]$ is prime by (2.2iii). Therefore U/JU has precisely m minimal prime ideals, and hence the same holds for S. Since m is the number of τ^n-orbits contained in the τ-orbit of Q, part (ii) follows.

(v) Under this assumption, Q is not τ^n-stable, from which it follows that the rings A_j/J are semisimple artinian but not simple. Then by (3.6), δ^n induces an inner τ^n-derivation on each A_j/J, and hence δ^n induces an inner τ^n-derivation on R/J. In case $n=1$, it of course follows that δ^t induces an inner τ^t-derivation on R/J.

(iii) If $n=1$, note that δ does not induce an inner τ-derivation on A. In either case, from the argument just given for (v) it follows that Q must be τ^n-stable. Consequently, the rings A_j/J are simple artinian. Since τ transitively permutes the A_j, and since δ^n does not induce an inner τ^n-derivation on R/J, we see that δ^n cannot induce an inner τ^n-derivation on any A_j/J. Hence, by (10.2i) the rings $(A_j/J)[y^n;\tau^n,\delta^n]$ are all simple. Thus U/JU is a finite direct product of simple rings. In particular, clKdim$U=0$. Since S is finitely generated as a left (or right) U-module (3.8iii), it follows from [**16**, 10.6] that clKdim$S=0$. Thus the prime ideals of S lying over Q are all minimal, and so by (ii) the only possibilities are $P_0,\ldots,P_{r-1}$.

(iv) Suppose first that some positive power of τ is inner on R/J, and that $J=0$. If δ is an inner τ-derivation of R, then by (2.3iii), there are infinitely many prime ideals in S that contract to (0) in R, and these all lie over Q. If δ is not inner, then it follows from (3.6) that R is simple artinian. Further, by (10.2i), S is not simple, and so we conclude from [**29**, 2.2, 2.9] that spec$S$ is infinite. Since R is simple, all prime ideals of S lie over Q in this case.

Next suppose that some positive power of τ is inner on R/J, and that $J \neq 0$. Then also some positive power of τ^n is inner on R/J, and it follows from (2.3iii) that there are infinitely many prime ideals in U/JU contracting to each τ^n-prime ideal J_j/J in R/J. By (3.9iv) there is a homeomorphism $P' \mapsto P = \text{r.ann}(S/P'S)_S$ from spec$U$ onto spec$S$. Since $P \cap R \subseteq P'S \cap R \subseteq P' \cap R$ for all $P' \in \text{spec}U$, we again obtain infinitely many prime ideals of S lying over Q.

Finally, suppose that no positive power of τ is inner on R/J. It follows that no positive power of τ which maps Q to itself is inner on R/Q, and thus by (9.5) at most $2r$ prime ideals of S can lie over Q. If δ^n is an inner τ^n-derivation on R/J, then U/JU may be identified with $(R/J)[z;\tau^n]$ for some element z, and then $(Q/J) + z(U/JU)$ is a non-minimal prime ideal of U/JU lying over Q/J. On the other hand, if δ^n is not inner on R/J, then R/J is simple artinian by (3.6), and it follows from (10.2ii) that there is a unique non-minimal prime ideal in U/JU. In either case, we obtain a non-minimal prime ideal P' in U lying over Q. By (3.9iv), the ideal $P = \text{r.ann}(S/P'S)_S$ is a non-minimal prime ideal of S, and since $P \cap R \subseteq P' \cap R \subseteq Q$ we conclude that P lies over Q. □

The analog of (10.8) for the case (p,t) is unfortunately more involved to state in general. However, the statements improve considerably under the hypothesis that no positive power of τ induces an inner automorphism of the Goldie quotient ring of R/J, and so we first give a precise result in that situation.

Theorem 10.9. *Assume that condition (p,t) holds; set $N = \{1, t, tp, tp^2, \ldots\}$. Let Q be a prime ideal of R such that $\tau^r(Q) = Q$ for some positive integer r, and set $J = Q \cap \tau(Q) \cap \cdots \cap \tau^{r-1}(Q)$. Let A denote the Goldie quotient ring of R/J, and assume that no positive power of τ induces an inner automorphism of A.*

(i) *There exists a prime ideal of S lying over Q if and only if J is δ^n-stable for some $n \in N$. Altogether there are at most $2r$ prime ideals of S lying over Q.*

Assume for the remainder that there does exist a prime ideal of S lying over Q, and let n be the least integer in N such that J is δ^n-stable.

(ii) *The ideals $P_i = \text{ann}[S/\tau^i(Q)S]_S$ for $0 \leq i \leq r-1$ are prime ideals of S lying over Q, and every prime ideal of S lying over Q contains one of these P_i.*

In addition, the number of distinct P_i equals the number of distinct τ^n-orbits contained in the τ-orbit of Q.

(iii) *If δ^d does not induce an inner τ^d-derivation on A for any $d \geq n$ in N, then $P_0, \ldots, P_{r-1}$ are the only prime ideals of S lying over Q.*

(iv) *If δ^d induces an inner τ^d-derivation on A for some $d \geq n$ in N, there is at least one prime ideal of S different from $P_0, \ldots, P_{r-1}$ that lies over Q.*

(v) *If Q is not τ^n-stable, then δ^n induces an inner τ^n-derivation on A.*

Proof: Since the proof is almost identical to the proof of the corresponding parts of (10.8), we omit most of the details. Of course, (3.13) is to be used in place of (3.9). In proving (i), if there exists a prime ideal of S lying over Q, then Q is minimal over a (τ, δ)-prime ideal I of R, and $(J/I)\mathcal{C}^{-1}$ is the radical of $(R/I)\mathcal{C}^{-1}$, where $\mathcal{C} = \mathcal{C}_R(I)$. By (3.13i), $(J/I)\mathcal{C}^{-1}$ is δ^n-stable for some $n \in N$, and since R/J is $\mathcal{C}$-torsionfree it follows that J is δ^n-stable. In proving (iii), there are two cases to consider when applying (10.2i). If $n = 1$, (10.2i) can be applied to the rings $(A_j/J)[y; \tau, \delta]$ because δ^d does not induce an inner τ^d-derivation on A_j/J for any $d \in N$. If $n > 1$, then δ^n is a 1-skew τ^n-derivation on R satisfying condition (p,p). In this case, (10.2i) can be applied to the rings $(A_j/J)[y^n; \tau^n, \delta^n]$ because $(\delta^n)^{p^i}$ does not induce an inner $(\tau^n)^{p^i}$-derivation on A_j/J for any positive integer i. □

We conclude the section with a brief discussion of some aspects of the general analog of (10.8) in the case (p,t).

10.10. Assume that condition (p,t) holds, and set $N = \{1, t, tp, tp^2, \ldots\}$. Let Q be a prime ideal of R such that $\tau^r(Q) = Q$ for some positive integer r, and set $J = Q \cap \tau(Q) \cap \cdots \cap \tau^{r-1}(Q)$. Let A denote the Goldie quotient ring of R/J, and suppose that some positive power of τ induces an inner automorphism of A.

(i) If J is δ^n-stable for some $n \in N$, it follows as in (10.8i) that there exists a prime ideal of S lying over Q. However, as shown by the example mentioned in (3.10), it is possible for there to exist prime ideals of S lying over Q even if J is not stable under any positive power of δ.

(ii) Now suppose that there does exist a prime ideal of S lying over Q. With the (now) usual reductions, we may assume that R is artinian and (τ, δ)-simple, that J is the radical of R, and that R/J is τ-simple. Let $n_1, \ldots, n_k, n$,

$c_1, \ldots, c_k$, $\zeta, \overline{\zeta}$ be as in (3.11). Denoting images in A with an overbar, we obtain

$$\overline{\delta^n(j)} + \sum_{i=1}^{k-1} \overline{c_i}\delta^{n_i}(j) \;=\; 0$$

for all $j \in J$. This condition seems to be the best possibility for a criterion for the existence of prime ideals of S lying over Q.

(iii) As in the proof of (10.8), it follows from (9.6) and (9.7) that the ideals $P_i = \operatorname{ann}[S/\tau^i(Q)S]_S$ for $0 \le i \le r-1$ are prime ideals of S lying over Q, that every prime ideal of S lying over Q contains one of these P_i, and that the P_i are precisely the minimal prime ideals of S. As in (10.7), the ring R/J is a direct sum $(A_1/J) \oplus \ldots \oplus (A_m/J)$ of $(\tau^n, \overline{\zeta})$-ideals where m is the number of distinct τ^n-prime ideals of R and each A_j/J is τ^n-simple as a ring. It follows from (2.2iii) and (3.12ii,iii) that S is a direct sum of m prime rings, and therefore as before, the number of distinct P_i is precisely m.

(iv) If Q is not τ^n-stable, it follows from (3.6) that $\overline{\zeta}$ is an inner τ^n-derivation on R/J.

(v) A criterion for the existence of prime ideals of S other than $P_0, \ldots, P_{r-1}$ lying over Q can be derived from (10.2i). First set $N_n = N$ in case $n = 1$ while $N_n = \{1, p, p^2, \ldots\}$ in case $n > 1$. If there exist integers $u_1 < \cdots < u_h = u$ in N_n and elements $c, d_1, \ldots, d_{h-1}$ in A such that

$$\overline{\zeta}^u(a) + \sum_{i=1}^{h-1} d_i \overline{\zeta}^{u_i}(a) \;=\; \tau^{nu}(a)c - ca$$

and $d_i a = \tau^{n(u-u_i)}(a)d_i$ for all $a \in A$ and all $i = 1, \ldots, h-1$, then there exist infinitely many prime ideals of S lying over Q. On the other hand, if this condition does not hold, then $P_0, \ldots, P_{r-1}$ are the only prime ideals of S lying over Q.

11. Classification of prime ideals in q-skew extensions.

The combined results of sections 8 and 10 provide detailed descriptions of the prime ideals in q-skew extension algebras. Our aim here is to summarize, as succinctly as possible, the key aspects of these characterizations. In order to achieve this goal we omit the explicit constructions of prime ideals, and of maps between spectra, presented in the earlier sections.

Throughout, let $S = R[y; \tau, \delta]$, where R is a noetherian ring equipped with an automorphism τ. Further assume that (τ, δ) is a q-skew derivation of R for a fixed central invertible (τ, δ)-constant q, and that one of the hypotheses $(*, 0)$, $(0, t)$, or (p, t) of (2.7) holds. As in (2.4ii), extend τ to an automorphism of S sending y to $q^{-1}y$.

We first consider the situation when $(*, 0)$ holds; in this case it is possible to catalogue the prime ideals of S according to their contractions in R.

Theorem 11.1. *Assume that case* $(*, 0)$ *of* (2.7) *holds.*

(I) *If P is a prime ideal of S, then the contraction of P to R is either a prime ideal which is not τ-stable or a τ-prime ideal which is δ-stable.*

(II.1) *Let Q be a prime ideal of R which is not τ-stable, let A denote the Goldie quotient ring of R/Q, and denote images in A with an overbar.*

 (i) *There exists a prime ideal of S contracting to Q if and only if there exists an element $c \in A$ such that $\overline{\delta(r)} = \tau(r)c - cr$ for all $r \in R$.*

 (ii) *There exists at most one prime ideal of S contracting to Q.*

(II.2) *Let J be a δ-stable τ-prime ideal of R, let A denote the Goldie quotient ring of R/J, let k denote the field of central (τ, δ)-constants of A, and let z be an indeterminate.*

 (i) *Suppose that J is prime.*

 (a) *If the τ-derivation of A induced by δ is not inner, then there exists exactly one prime ideal of S contracting to J.*

 (b) *If the τ-derivation of A induced by δ is inner, but no positive power of τ induces an inner automorphism of A, then there exist exactly two distinct prime ideals of S contracting to J, and these prime ideals are comparable.*

 (c) *If the τ-derivation of A induced by δ is inner, and some positive power of τ does induce an inner automorphism of A, then the set of prime ideals of S contracting to J is homeomorphic to* spec $k[z]$.

(ii) *Suppose that J is not prime.*

(a) *If no positive power of τ induces an inner automorphism of A, then there exists exactly one prime ideal of S contracting to J.*

(b) *If some positive power of τ does induce an inner automorphism of A, then the set of prime ideals of S contracting to J is homeomorphic to* $\operatorname{spec} k[z, z^{-1}]$.

Proof: (I) This is (10.4).

(II.1i) Let $V = \mathcal{V}(Q) = A \otimes_R S$. First suppose that there exists $c \in A$ such that $\overline{\delta(r)} = \tau(r)c - cr$ for all $r \in R$. Let $W = V/[A.(1 \otimes y + c \otimes 1).S]$. From (8.3ii) we conclude that W is a simple A-S-bimodule factor of V. Since $W_R \cong A_R$, we see that $\operatorname{ann} W_S$ is a prime ideal which contracts to Q.

Now suppose that there exists a prime ideal P of S contracting to Q. In particular, P lies directly over Q. Since Q is not τ-stable, it follows from (5.9ii) that $P \neq \operatorname{ann} V_S = \operatorname{ann}(S/QS)_S$, and therefore, by (5.7i), there exists a proper nonzero A-S-bimodule factor of V. Hence from (8.3i) it follows that there exists $c \in A$ such that $\overline{\delta(r)} = \tau(r)c - cr$ for all $r \in R$.

(II.1ii) If the τ-orbit of Q is infinite, the conclusion follows from (8.5.I.2). Now suppose that the τ-orbit of Q is finite, and that there exists at least one prime ideal of S contracting to Q. By (10.1.2), we may assume without loss of generality that R is (τ, δ)-simple artinian and that Q is a minimal (nonzero) prime ideal of R. Furthermore, from (2.4iii) it follows that R is semisimple artinian. Since R is not simple, it follows from (3.6) that δ is inner. It now follows from the description in (2.3iii) that there exists exactly one prime ideal of S contracting to Q.

(II.2) We may assume that $J = 0$, and by (2.3i,ii) we may assume that $R = A$.

(II.2i) Parts (a) and (b) follow from (10.2), and part (c) follows from (2.3iii).

(II.2ii) First, it follows from (3.6) that δ is inner on A. Parts (a) and (b) now follow from (2.3iii). $\square$

Although in cases $(0, t)$ and (p, t) we have not calculated the exact nature of $P \cap R$ for an arbitrary prime ideal P of S, we are able to characterize $\operatorname{spec} S$ in these cases by other means. Recall if P is a prime ideal of S lying over a prime ideal Q of R, that P lies *directly* over Q provided Q is an annihilator prime of $(S/P)_R$.

Theorem 11.2. *Assume that case* $(0, t)$ *of* (2.7) *holds.*

(I) *Each prime ideal P of S lies directly over at least one prime ideal of R, and if P lies directly over two distinct prime ideals then they are contained within a single finite τ-orbit of* spec$R$.

(II.1) *Let Q be a prime ideal of R whose τ-orbit is infinite, let A denote the Goldie quotient ring of R/Q, and denote images in A with an overbar. There exists a prime ideal of S lying directly over Q if and only if there exist $c \in A$ and $m \in \{1, t\}$ such that $\overline{\delta^m(r)} = \tau^m(r)c - cr$ for all $r \in R$. Moreover, at most one prime ideal of S lies directly over Q.*

(II.2) *Let n be a positive integer, let $\mathcal{X}$ be a τ-orbit in* spec$R$ *of cardinality n, and let $\mathcal{Y}$ denote the set of prime ideals in S lying directly over at least one ideal in $\mathcal{X}$. Set $J = \bigcap \mathcal{X}$, and let A denote the Goldie quotient ring of R/J.*

(i) *If J is not δ^t-stable, then $\mathcal{Y}$ is empty.*

(ii) *If J is δ^t-stable but δ^t does not induce an inner τ^t-derivation of A, then $\mathcal{Y}$ is equal to a τ-orbit in* spec$S$ *of cardinality no greater than n.*

(iii) *Suppose that J is δ^t-stable and that δ^t induces an inner τ^t-derivation of A. If no positive power of τ induces an inner automorphism of A, then $\mathcal{Y}$ equals the union of two τ-orbits in* spec$S$, *each of cardinality no greater than n.*

(iv) *Suppose that J is δ^t-stable, that δ^t induces an inner τ^t-derivation of A, and that some positive power of τ induces an inner automorphism of A. If J is δ-stable, set $\ell = 1$ and let k denote the subfield of central (τ, δ)-constants in A. If J is not δ-stable, let ℓ be the number of τ^t-orbits contained in $\mathcal{X}$, and let k denote the subfield of central (τ^t, δ^t)-constants in a τ^t-simple factor of A. Then there is a continuous closed surjection $\phi \colon \mathcal{Y} \to$* spec$k^\ell[z]$, *where z is an indeterminate. If $I \in$* spec$k^\ell[z]$ *and I is not one of the maximal ideals of the form*

$$\{(f_1, \ldots, f_\ell) \in k^\ell[z] = (k[z])^\ell \mid f_j \in (z)\}$$

for some $j \in \{1, \ldots, \ell\}$, then $\phi^{-1}(I)$ is a singleton. If $\phi^{-1}(I)$ is not a singleton, then it consists of a single τ-orbit of spec$S$ *of cardinality n.*

Proof: (I) follows from (5.3) and (5.4), while (II.1) follows from (8.5.II).

(II.2) First note by (5.12) that $\mathcal{Y}$ is equal to the set of prime ideals of S which lie over at least one prime ideal in $\mathcal{X}$. Parts (i–iii) follow from (10.8) and (9.8). To prove (iv), first note by (10.1.2) that we may assume without loss of generality that R is (τ, δ)-simple artinian. If R is simple then the desired conclusion follows from (10.1.5) and (10.8iv). If R is semisimple but not simple, then the conclusion follows from (10.1.3). Finally, if R is not semisimple then the conclusion follows from (10.1.6). □

We conclude with a characterization of the prime ideals in S under the hypothesis (p, t). However, our results here are less complete than in the previous two cases.

Theorem 11.3. *Assume that condition (p, t) of (2.7) holds, and set $N = \{1, t, tp, tp^2, \ldots\}$.*

(I) *Each prime ideal P of S lies directly over at least one prime ideal of R, and if P lies directly over two distinct prime ideals then they are contained within a single finite τ-orbit of* spec$R$.

(II.1) *Let Q be a prime ideal of R whose τ-orbit is infinite, let A denote the Goldie quotient ring of R/Q, and denote images in A with an overbar. There exists a prime ideal of S lying directly over Q if and only if there exist $c \in A$ and $m \in N$ such that $\overline{\delta^m(r)} = \tau^m(r)c - cr$ for all $r \in R$. Moreover, at most one prime ideal of S lies directly over Q.*

(II.2) *Let n be a positive integer, let $\mathcal{X}$ be a τ-orbit in* spec$R$ *of cardinality n, and let $\mathcal{Y}$ denote the set of prime ideals in S lying directly over at least one ideal in $\mathcal{X}$. Set $J = \bigcap \mathcal{X}$, let A denote the Goldie quotient ring of R/J, and let I denote the maximal (τ, δ)-ideal contained in J.*

(i) *The set $\mathcal{Y}$ is nonempty if and only if J is nilpotent modulo I.*

(ii) *Suppose that no positive power of τ induces an inner automorphism of A.*

(a) *The set $\mathcal{Y}$ is nonempty if and only if J is δ^m-stable for some $m \in N$.*

(b) *If $\mathcal{Y}$ is nonempty but there exists no $m \in N$ such that δ^m maps J to itself and further induces an inner τ^m-derivation of A, then $\mathcal{Y}$ is equal to a τ-orbit of* spec$S$ *of cardinality no greater than n.*

(c) *If $\mathcal{Y}$ is nonempty and there exists $m \in N$ such that δ^m maps J to itself and further induces an inner τ^m-derivation of A, then*

$\mathcal{Y}$ equals the union of two τ-orbits in spec$S$*, each of cardinality no greater than n.*

(iii) *Suppose that $\mathcal{Y}$ is infinite. Then for some positive integer $\ell \leq n$ there is a continuous closed surjection $\phi\colon \mathcal{Y} \to \operatorname{spec} k^{\ell}[z]$, where z is an indeterminate and k is a central subfield of A. If $I \in \operatorname{spec} k^{\ell}[z]$ and I is not one of the maximal ideals of the form*

$$\{(f_1, \dots, f_\ell) \in k^{\ell}[z] = (k[z])^{\ell} \mid f_j \in (z)\}$$

for some $j \in \{1, \dots, \ell\}$, then $\phi^{-1}(I)$ is a singleton. If $\phi^{-1}(I)$ is not a singleton, then it consists of a single τ-orbit of spec$S$ *of cardinality n.*

Proof: (I) follows from (5.3) and (5.4), while (II.1) follows from (8.5.II).

(II.2) Part (i) follows from (9.6), recalling from (2.2ii) and (2.3i) that the prime radical of I is the intersection of a finite τ-orbit of prime ideals of R. Part (ii) follows from (9.8) and (10.9), while part (iii) follows from the argument used for (11.2.II.2iv). $\square$

Remark 11.4. The two preceding results may also be viewed from the following perspective. First, impose equivalence relations $\sim$ on spec$R$ and spec$S$ by declaring two prime ideals P_α, P_β to be equivalent if and only there exist non-negative integers m, n such that $\tau^m(P_\alpha) = P_\beta$ and $\tau^n(P_\beta) = P_\alpha$. (In other words, each prime ideal is equivalent to itself, and two distinct prime ideals are equivalent exactly when they are contained within a single finite τ-orbit.) Denote by spec$R/{\sim}$ and spec$S/{\sim}$ the resulting quotients. Second, if P is a prime ideal of S then let $\Phi(P)$ denote the set of prime ideals Q of R such that P lies directly over Q. From (5.3) and (5.4) it follows that the correspondence $P \to \Phi(P)$ induces single valued functions spec$S \to$ spec$R/{\sim}$ and spec$S/{\sim} \to$ spec$R/{\sim}$. Parts (II.2) of (11.2) and (11.3) are essentially descriptions of the fibers of these functions. (A direct analog of this point of view is featured in [**32**].)

12. Irreducible finite dimensional representations of q-skew extensions.

We now specialize the results of the previous section to finite dimensional representations, a setting in which the Noether-Skolem Theorem allows us to simplify the discussion considerably. To begin, let R be a noetherian algebra over an algebraically closed field k, and let $S = R[y;\tau,\delta]$ where τ is a k-linear automorphism of R and δ is a k-linear τ-derivation of R. Further suppose that (τ,δ) is a q-skew derivation for some nonzero $q \in k$. Observe that one of the cases $(*,0)$, $(0,t)$, or (p,t) of (2.7) must now hold; hence, the results presented here are essentially corollaries to the previous section. Classifying the irreducible finite dimensional representations of S is tantamount to classifying the co-finite-dimensional maximal ideals of S, which is what we actually do here.

For a k-algebra A let $\max_f A$ denote the set of co-finite-dimensional (over k) maximal ideals of A, and if α is an automorphism of A let $\max_f^\alpha A$ denote the set of co-finite-dimensional α-prime ideals of A (that is, the set of co-finite-dimensional maximal α-ideals).

Theorem 12.1. *Assume either that q is not a root of unity, or that k has characteristic zero and $q = 1$.*

(I) *Let P be a co-finite-dimensional maximal ideal of S, and let $I = P \cap R$. Then either $I \in \max_f R$ and I is not τ-stable, or $I \in \max_f^\tau R$ and I is δ-stable.*

(II.1) *Suppose that Q is a co-finite-dimensional maximal ideal of R which is not τ-stable.*

(i) *If P is a prime ideal of S contracting to Q then $P \in \max_f S$.*

(ii) *There exists a co-finite-dimensional maximal ideal of S contracting to Q if and only if there exists $c \in R$ such that $\delta(r) - \tau(r)c + cr \in Q$ for all $r \in R$.*

(iii) *If there exists $c \in R$ as in* (ii), *then $Q + (y+c)S = Q + S(y+c)$ is the unique co-finite-dimensional maximal ideal of S contracting to Q.*

(II.2) *Suppose that J is a co-finite-dimensional τ-prime ideal of R which is δ-stable, and let z be an indeterminate.*

(i) *If P is a prime ideal of S contracting to J, then either $P = JS$ or $P \in \max_f S$.*

(ii) *If J is prime, then the set of co-finite-dimensional maximal ideals of S contracting to J is homeomorphic to* $\max k[z]$.

(iii) *If J is not prime, then the set of co-finite-dimensional maximal ideals of S contracting to J is homeomorphic to* $\max k[z, z^{-1}]$.

Proof: (I) This follows from (10.4).

(II.1) Suppose there exists a prime ideal P of S contracting to Q. By (5.9ii), P is not equal to $\operatorname{ann}(S/QS)_S$, and therefore $P \in \max_f S$ by (5.7i). Parts (ii) and (iii) then follow from (11.1.II.1) and (8.5.I.3).

(II.2) We may assume that $J = 0$. From (3.14) it follows that δ and some positive power of τ are inner on R. Parts (i-iii) now follow from (2.3iii) together with the observation that the subfield of central τ-constants in R/J is just $k.1$. $\square$

Remark 12.2. The example given in (8.7) shows that in the situation of (12.1), a co-finite-dimensional maximal ideal of R which is not τ-stable need not be the contraction of any prime ideal of S.

Theorem 12.3. *Assume that k has characteristic zero and that q is a primitive t^{th} root of unity for some integer $t > 1$.*

(I) *Each co-finite-dimensional maximal ideal P of S lies directly over at least one maximal ideal in $\max_f R$, and if P lies directly over two distinct maximal ideals in $\max_f R$ then they are contained in a single finite τ-orbit of $\max_f R$.*

(II.1) *Suppose that Q is a co-finite-dimensional maximal ideal of R whose τ-orbit is infinite.*

(i) *If P is a prime ideal of S lying directly over Q then $P \in \max_f S$.*

(ii) *There exists a co-finite-dimensional maximal ideal of S lying directly over Q if and only if there exist $c \in R$ and $m \in \{1, t\}$ such that $\delta^m(r) - \tau^m(r)c + cr \in Q$ for all $r \in R$.*

(iii) *If there exist $c \in R$ and $m \in \{1, t\}$ as in (ii), with m minimal, then the ideal $\operatorname{ann}[S/(QS + (y^m + c)S)]_S$ is the unique co-finite-dimensional maximal ideal of S lying directly over Q.*

(II.2) *Let n be a positive integer, let $\mathcal{X}$ be a τ-orbit in* $\max_f R$ *of cardinality n, and let $\mathcal{Y}$ denote the set of ideals in* $\max_f S$ *lying directly over at least one ideal in $\mathcal{X}$. Set $J = \bigcap \mathcal{X}$.*

(i) *If J is not δ^t-stable, then $\mathcal{Y}$ is empty.*

(ii) *Suppose that J is δ^t-stable. If J is δ-stable, set $\ell = 1$, while if J is not δ-stable, let ℓ be the number of τ^t-orbits contained in $\mathcal{X}$. Then there is a continuous closed surjection $\phi\colon \mathcal{Y} \to \max k^\ell[z]$, where z is an indeterminate. If $I \in \max k^\ell[z]$ and I is not one of the maximal ideals of the form*

$$\{(f_1, \dots, f_\ell) \in k^\ell[z] = (k[z])^\ell \mid f_j \in (z)\}$$

for some $j \in \{1, \dots, \ell\}$, then $\phi^{-1}(I)$ is a singleton. If $\phi^{-1}(I)$ is not a singleton, then it consists of a single τ-orbit of $\max_f S$ *of cardinality n.*

Proof: The theorem follows from arguments analogous to those used in proving (11.2) and (12.1). $\square$

We leave to the reader the corresponding specialization of (11.3) to co-finite-dimensional maximal ideals.

13. Quantized Weyl algebras.

In this and the next section we narrow our focus to two of the examples of q-skew extensions which originally motivated our investigation. In this part, we provide a concise description of the prime spectra of q-Weyl algebras $A_1(T,q) = T\{x,y\}/\langle yx - qxy - 1\rangle$ when T is a noetherian algebra over a field k and $q \in k$ is a primitive t^{th} root of unity for some $t > 1$. (The case when q is not a root of unity is elementary and was discussed in (2.9).) In particular, we show that $\operatorname{spec}A_1(T,q)$ is a disjoint union of two subsets homeomorphic to

$$\operatorname{spec}T[x,x^{-1}] \qquad \text{and} \qquad \operatorname{spec}T[x^t,y^t,(1-(1-q)^t x^t y^t)^{-1}].$$

13.1. (i) Choose a field k, a noetherian k-algebra T, and an element $q \in k$ which is a primitive t^{th} root of unity for some integer $t > 1$. Since the case $q = 1$ is well understood, we shall also assume that $q \neq 1$. Thus if $\operatorname{char}k = p > 0$, the integer t cannot be a power of p. Note that if $p \mid t$, then q is also a primitive $(t/p)^{\text{th}}$ root of unity. Hence, we may factor out whatever powers of p divide t. Therefore there is no loss of generality in assuming that t is invertible in k. In particular, either case $(0,t)$ or case (p,t) of (2.7) will be in effect.

Throughout the section, let S denote the q-Weyl algebra $A_1(T,q)$. As in (2.8), $S = R[y;\tau,\delta]$ where $R = T[x]$, while τ is the automorphism of R which fixes T pointwise but sends x to qx, and δ is the τ-derivation of R which vanishes on T but sends x to 1. Recall from (2.8) that $\delta\tau = q\tau\delta$.

(ii) Extend τ to an automorphism of S sending y to $q^{-1}y$ as in (2.4ii). In order to briefly describe the types of τ-orbits in $\operatorname{spec}S$ which may occur, let P be a prime ideal of S, and let $u = yx - xy = (q-1)xy + 1$. It is easy to verify that $us = \tau(s)u$ for each $s \in S$. Letting $\widetilde{S}$ denote the localization of S at powers of the normal element u, and extending τ to $\widetilde{S}$, we further obtain the equality $\tau(s) = usu^{-1}$ for each $s \in \widetilde{S}$. Therefore, P is τ-stable when $u \notin P$, since in this case P is the contraction of a prime ideal of $\widetilde{S}$ (e.g., [**16**, 9.22]). Now assume that P contains u. As shown in [**13**, 8.2], the inclusion $R \hookrightarrow T[x,x^{-1}]$ extends uniquely to an isomorphism from S/uS onto $T[x,x^{-1}]$ which sends y to $(1-q)^{-1}x^{-1}$. Consequently, $P = uS + QS$ for some prime ideal Q of R which does not contain x, and the τ-orbit of P corresponds exactly to the τ-orbit of Q.

(iii) Note that τ^t equals the identity on both R and S. It follows from [**13**, 8.1] that $\delta(x^t) = 0$, whence x^t commutes with y, and so x^t is central in S. On the other hand, it follows from (3.8i,iii) that δ^t is a derivation on R such that $y^t r = r y^t + \delta^t(r)$ for all $r \in R$. Since δ^t vanishes on $T \cup \{x\}$, we must have $\delta^t = 0$, and thus y^t is central in S.

Now set $U = T[x, y^t]$ and $C = T[x^t, y^t]$, and note that τ restricts to automorphisms of both U and C. Since U is a centralizing extension of C, each prime ideal of U contracts to a prime ideal of C. Also, since C is easily checked to be the fixed ring of U with respect to τ, and since the order of τ is invertible in U, it follows from [**35**, 3.4, 4.2] that contraction to C produces a homeomorphism from $\mathrm{spec}^\tau U$ onto $\mathrm{spec} C$.

We also observe that S is a centralizing extension of C and hence that each prime ideal of S contracts to a prime ideal of C. Therefore, because τ fixes C pointwise, it follows that each τ-prime ideal of S contracts to a prime ideal of C.

(iv) Recall from (3.8iii) that S is a free right and left U-module with $\{1, y, \ldots, y^{t-1}\}$ serving as a basis for both sides. We note that S is also free as a right or left C-module. Extend δ to a τ-derivation of U as in (3.8iv), and set $S^* = U[\theta; \tau, \delta]$. Then $\theta^t - y^t$ is a normal element of S^*, and we may identify S with $S^*/(\theta^t - y^t)S^*$. In fact, it follows as in (iii) that θ^t is central in S^*, and thus $\theta^t - y^t$ is central.

Lemma 13.2. *Let P' be a prime ideal of $U = T[x, y^t]$. Then there exists exactly one prime ideal of S lying directly over P'.*

Proof: Let $S^* = U[\theta; \tau, \delta]$ and $S = S^*/(\theta^t - y^t)S^*$ as in (13.1iv). It suffices to show that there is a unique prime ideal of S^* which contains the central element $w = \theta^t - y^t$ and lies directly over P'. Let A denote the Goldie quotient ring of U/P', and let V denote the A-S^*-bimodule $A \otimes_U S^*$.

Since w is central in S^*, it follows that

$$Vw = A.(1 \otimes w).S^* = \sum_{i=0}^{\infty} A.(1 \otimes w).\theta^i,$$

and so Vw is a proper nonzero A-S^*-sub-bimodule of V. Choose a maximal proper A-S^*-sub-bimodule B of V containing Vw, and let $P = \mathrm{ann}(V/B)_{S^*}$. It follows from (5.7ii) that P is a prime ideal of S^* lying directly over P', and it is clear that $w \in P$.

Let P_1 be any prime ideal of S^* which contains w and lies directly over P'. Since $w \in P_1$ while

$$\mathrm{ann} V_{S^*} = \mathrm{ann}(S^*/P'S^*)_{S^*} \subseteq P'S^*,$$

we see that $P_1 \neq \mathrm{ann} V_{S^*}$. It now follows from (5.7i) that there exists a maximal proper A-S^*-sub-bimodule $B_1 \subseteq V$ such that $P_1 = \mathrm{ann}(V/B_1)_{S^*}$. To show that $P_1 = P$, it will be enough to show that $B_1 = B$. Note that $Vw \subseteq VP_1 \subseteq B_1$. Therefore it now suffices to show that B is the unique maximal proper A-S^*-sub-bimodule of V containing Vw.

Assume that $x \notin P'$. Since x is a central element of U satisfying $\tau^i(x) = q^i x$ for all i, it follows that there does not exist an integer i in $\{1, \ldots, t-1\}$ such that τ^i maps P' to itself and induces an inner automorphism of A. Thus in this case (7.3i) implies that there is a unique maximal proper A-S^*-sub-bimodule of V containing Vw.

Finally, assume that $x \in P'$. Then the images of both x and $\tau(x)$ in A are zero, and so there does not exist an element $c \in A$ such that $\overline{\delta(x)} = \tau(x)c - cx$. Therefore in this case it follows from (8.3ii) that Vw is a maximal proper A-S^*-sub-bimodule of V. $\square$

Proposition 13.3. (i) *If P' is a prime ideal of $U = T[x, y^t]$ then the set of prime ideals of S lying over P' is contained within a single τ-orbit of* spec$S$.

(ii) *Let J be a prime ideal of $C = T[x^t, y^t]$. Then $JS = SJ$ and the set of prime ideals of S minimal over JS coincides with a τ-orbit of* spec$S$. *Moreover, the prime ideals of S minimal over JS are precisely the prime ideals of S that contract to J.*

Proof: (i) By (13.1iv) and (5.13i), any prime ideal of S that lies over P' is contained in the τ-orbit of a prime ideal that lies directly over P'. Since by (13.2) there is only one prime ideal of S that lies directly over P', part (i) follows.

(ii) Since all elements of C commute with x and y, it is immediate that $JS = SJ$. Obviously any prime ideal of S that contracts to J must contain JS. By incomparability (e.g., [**34**, 13.8.14iii]), it follows that all prime ideals of S that contract to J are minimal over JS. Now let P be any prime ideal of S minimal over JS. Observe that since S is a free right or left C-module, S/JS is a free right or left (C/J)-module, and so S/JS is torsionfree as a right or left (C/J)-module. Since the regular elements of S/JS are also regular modulo the minimal prime ideal P/JS (e.g., [**16**, 10.8, 6.5]), it follows that S/P is torsionfree as a right or left (C/J)-module, and consequently $P \cap C = J$. Therefore the prime ideals of S minimal over JS are precisely the prime ideals of S that contract to J.

Since JS is a τ-ideal of S, the set of prime ideals of S minimal over JS is a finite union of τ-orbits. Thus it only remains to show that any two prime ideals P_α, P_β of S minimal over JS are in the same τ-orbit of spec$S$. Choose prime ideals P'_α and P'_β of U such that P_α lies over P'_α and P_β lies over P'_β. Note that $J = P_\alpha \cap C \subseteq P'_\alpha \cap C$, and similarly $J \subseteq P'_\beta \cap C$. Set $I_\alpha = \bigcap_i \tau^i(P'_\alpha)$ and $I_\beta = \bigcap_i \tau^i(P'_\beta)$, and note that J is contained in both $I_\alpha \cap C$ and $I_\beta \cap C$. It follows from (13.1iv) and (5.12i) that there exists a positive integer m such that $I_\alpha^m \subseteq P_\alpha$ and $I_\beta^m \subseteq P_\beta$. Then $(I_\alpha \cap C)^m \subseteq P_\alpha \cap C = J$, whence $I_\alpha \cap C = J$, and similarly $I_\beta \cap C = J$. It now follows from (13.1iii) that $I_\alpha = I_\beta$, and so P'_α and P'_β are in the same τ-orbit of spec$U$. Therefore we conclude from (i) that P_α and P_β must be in the same τ-orbit of spec$S$, as desired. □

Theorem 13.4. *Let $S = A_1(T, q)$ where T is a noetherian algebra over a field k and q is a primitive t^{th} root of unity in k for some integer $t > 1$ which is invertible in k. If $C = T[x^t, y^t]$, then the rule $P \mapsto P \cap C$ defines a homeomorphism from* spec$^\tau S$ *onto* spec$C$.

Proof: As noted in (13.1iii), each τ-prime ideal of S contracts to a prime ideal of C. We thus obtain a mapping $\phi\colon \text{spec}^\tau S \to \text{spec}C$ given by the rule $\phi(P) = P \cap C$. It is clear that ϕ is continuous, and it follows from (13.3ii) that ϕ is a bijection. Finally, since the extension $C \hookrightarrow S$ satisfies Going Up (e.g., [**34**, 13.8.14ii]), we conclude that ϕ is a closed map, and the theorem is proved. □

13.5. Let $u = yx - xy = (q-1)xy + 1$ as in (13.1ii), and observe that u^t is a central element of S. In particular, u^t is contained within the center of $A_1(k, q)$, which by [**3**, 2.1] is equal to $k[x^t, y^t]$. This last assertion allows us to deduce that

$$u^t \;=\; \sum_{i=0}^{t} \tbinom{t}{i}(q-1)^i(xy)^i \;=\; (q-1)^t q^{t(t-1)/2} x^t y^t + 1.$$

(This result can also be obtained from a direct calculation similar to [**13**, 7.4].) Note that if t is odd, then $q^{t(t-1)/2} = 1$, while if t is even, then $q^{t(t-1)/2} =$

$q^{-t/2} = -1$. Hence, we can simplify the expression above for u^t as follows: $u^t = 1 - (1-q)^t x^t y^t$.

Theorem 13.6. *Let $S = A_1(T, q)$ where T is a noetherian algebra over a field k and q is a primitive t^{th} root of unity in k for some integer $t > 1$ which is invertible in k. Set $u = yx - xy$.*

(i) *There is a homeomorphism from the set $\{P \in \operatorname{spec} S \mid u \in P\}$ onto the set $\{Q \in \operatorname{spec} T[x] \mid x \notin Q\}$ given by the rule $P \mapsto P \cap T[x]$.*

(ii) *There is a homeomorphism from the set $\{P \in \operatorname{spec} S \mid u \notin P\}$ onto the set $\{I \in \operatorname{spec} T[x^t, y^t] \mid 1-(1-q)^t x^t y^t \notin I\}$ given by the rule $P \mapsto P \cap T[x^t, y^t]$.*

Proof: (i) This follows from (13.1ii).

(ii) By (13.1ii), u is a normal element of S and any prime ideal of S that does not contain u is τ-stable. Since $\tau(u) = u$, we thus see that the set of prime ideals of S not containing u coincides with the set of τ-prime ideals not containing u. Further, since u is normal a prime ideal of S contains u if and only if it contains u^t. Finally, we have $u^t = 1 - (1-q)^t x^t y^t$ by (13.5). Therefore the homeomorphism given in (13.4) restricts to a homeomorphism from $\{P \in \operatorname{spec} S \mid u \notin P\}$ onto $\{I \in \operatorname{spec} T[x^t, y^t] \mid u^t \notin I\}$, and part (ii) is proved. □

14. Prime factors of coordinate rings of quantum matrices.

Turning to the second of our motivating examples, we calculate Goldie ranks of prime factors of the algebras $\mathcal{O}_q(M_2(k))$, known as *coordinate rings of quantum* 2×2 *matrices*, where q is a nonzero element of a base field k. (A description of $\mathcal{O}_q(M_2(k))$ in terms of generators and relations is presented in (14.6).) We prove that all prime factors of $\mathcal{O}_q(M_2(k))$ have rank 1 (i.e., are completely prime) when q is not a root of unity. When q is a primitive t^{th} root of unity we prove that the prime factors of $\mathcal{O}_q(M_2(k))$ have rank at most t, and when t is even we prove that the prime factors have rank no greater than $t/2$. These conclusions do not depend on the choice of k.

We require the full extent of the analysis developed thus far to obtain the above results, and we are able to use our machinery because $\mathcal{O}_q(M_2(k))$ can be written as a q^4-skew polynomial ring (see (14.6)). In fact, it follows easily from the description given in [**38**] that for any integer $n \geq 2$, the algebra $\mathcal{O}_q(M_n(k))$ can be presented as an n^2-fold iterated q^4-skew extension of k. In [**15**], we develop an iterated version of some of our machinery to obtain the conclusion that all prime ideals of $\mathcal{O}_q(M_n(k))$ are completely prime when q is not a root of unity (also see [**14**]).

We first consider a class of examples containing $\mathcal{O}_q(M_2(k))$, the twisted q-Weyl algebras $A_1(T, \tau, q)$, and certain q-skew extensions included among the algebras studied by Jordan in [**25**]. The construction also covers those algebras termed "uninteresting" in [**6**] (so labelled because there is a finite bound on the dimensions of their finite dimensional irreducible representations).

14.1. (i) Let T be a noetherian algebra over a field k, let τ be a k-algebra automorphism of T, and set $R = T[x; \tau^{-1}]$. Choose nonzero elements $\alpha, \beta \in k$ and a central element $d \in T$ such that $\tau(d) = \alpha^{-1} d$, and extend τ to the automorphism of R such that $\tau(x) = \beta x$. As seen in (2.8), there is a unique k-linear τ-derivation δ on R such that $\delta = 0$ on T while $\delta(x) = d$. Let $S = R[y; \tau, \delta]$. Then S may be defined as the k-algebra generated by $T \cup \{x, y\}$ with the relations

$$\begin{cases} yx &= \beta xy + d \\ ax &= x\tau(a) \qquad \text{for all } a \in T \\ ya &= \tau(a)y \qquad \text{for all } a \in T. \end{cases}$$

Since the τ-derivations $\tau^{-1}\delta\tau$ and δ both vanish on T and since $\tau^{-1}\delta\tau(x) = \alpha\beta d = \alpha\beta\delta(x)$, we see that $\tau^{-1}\delta\tau = \alpha\beta\delta$. Thus (τ, δ) is a q-skew derivation, where $q = \alpha\beta$.

(ii) Note that d is a normal element of S, that dT is a τ-stable ideal of T, and that $S/dS \cong (T/dT)[x; \tau^{-1}][y; \tau]$. Furthermore, because the multiplicative set $\mathcal{C} = \{\lambda d^n \mid 0 \neq \lambda \in k \text{ and } n \in \mathbf{N}\}$ is a τ-stable Ore set in both T and R, the algebras T, R, S can all be localized with respect to $\mathcal{C}$, by [**13**, 1.4]. Moreover, since these localizations amount to adjoining a multiplicative inverse for d, we shall denote them by $T[d^{-1}]$, $R[d^{-1}]$, $S[d^{-1}]$. It follows from [**13**, 1.4] that (τ, δ) extends uniquely to a skew derivation on $R[d^{-1}]$ and that $S[d^{-1}] \cong R[d^{-1}][y; \tau, \delta] \cong T[d^{-1}][x; \tau^{-1}][y; \tau, \delta]$.

All prime ideals of S that contain d arise, of course, from prime ideals of S/dS. On the other hand, if P is a prime ideal of S not containing d, then by its normality d is regular modulo P, and in this case P arises from a prime ideal of $S[d^{-1}]$. To state these two observations more precisely: the natural k-algebra maps $S \to S/dS$ and $S \to S[d^{-1}]$ induce homeomorphisms from $\operatorname{spec}(S/dS)$ and $\operatorname{spec} S[d^{-1}]$ onto subsets of $\operatorname{spec} S$, and $\operatorname{spec} S$ is the disjoint union of these two subsets. Thus to understand $\operatorname{spec} S$ in general, it suffices to understand the cases in which either $d = 0$ or d is invertible.

14.2. We shall make repeated use of the following elementary facts about skew polynomial rings of the form $B = A[x; \sigma]$, where A is a noetherian ring and σ is an automorphism of A. These facts can be found, for instance, in [**34**, Chapter 10].

(i) If σ is an inner automorphism of A, then $B = A[w]$ for some central indeterminate w.

(ii) If I is a σ-prime ideal of A, then $IB = BI$ is a prime ideal of B, and $B/IB \cong (A/I)[x; \sigma]$.

(iii) If P is a prime ideal of B not containing x, then $P \cap A$ is a σ-prime ideal of A.

(iv) The prime ideals of B that contain x are precisely the ideals of the form $I + xB$ where I is a prime ideal of A.

(v) If A is σ-simple and no nonzero power of σ is inner on A, then all nonzero prime ideals of B contain x.

Lemma 14.3. *Let R be as in (14.1i). Assume that $q \neq 1$ and that q is not a root of unity. If Q is a prime ideal of R such that $d, x \notin Q$ and $Q \neq (Q \cap T)R$, then the τ-orbit of Q is infinite.*

Proof: Since $x \notin Q$, the contraction $Q \cap T$ is a τ-prime ideal of T and $R/(Q \cap T)R \cong (T/(Q \cap T))[x; \tau^{-1}]$, by (14.2ii,iii). Therefore, there is no loss of generality in assuming that $Q \cap T = 0$. Hence T is now a τ-prime ring, τ extends to an automorphism of the Goldie quotient ring A of T, and Q extends to a nonzero prime ideal $\widetilde{Q}$ of the localization $A[x; \tau^{-1}]$. Since it suffices to show that the τ-orbit of $\widetilde{Q}$ is not finite, we may assume without loss of generality that $T = A$. That is, we may assume that T is artinian and τ-simple.

Since $Q \neq 0$ and $x \notin Q$, the ring $T[x, x^{-1}; \tau^{-1}]$ is not simple. As in (2.3iii), the center of R is equal to $K[z]$, where K is the field of central τ-constants in T, and $z = ux^\ell$ for some positive integer ℓ and some unit $u \in T$ such that $\tau^{-1}(u) = u$ and $\tau^{-\ell}(a) = u^{-1}au$ for all $a \in T$. Note that $\tau(z) = \beta^\ell z$, and that $\alpha^\ell d = \tau^{-\ell}(d) = u^{-1}du = d$. Since $d \neq 0$, it follows that $\alpha^\ell = 1$. From (2.3iii) we see that $Q = fR$, where f is the unique monic polynomial of minimal degree (say n) in $Q \cap K[z]$. Further, f is irreducible in $K[z]$. Since $x \notin Q$ and x is normal in R, we have $z \notin Q$. Thus the constant term of f must be nonzero.

If $\tau^m(Q) = Q$ for some positive integer m, then $\tau^m(f) \in Q$, and it follows from the minimality of n that $\tau^m(f) = \beta^{\ell mn} f$. Since the constant term of f is nonzero, we conclude that $\beta^{\ell mn} = 1$. But then $q^{\ell mn} = 1$, contradicting our hypotheses. Therefore the τ-orbit of Q must be infinite. □

Proposition 14.4. *Let S be as in (14.1i). Assume that $q \neq 1$ and that q is not a root of unity.*

(I) *If P is a prime ideal of S which does not contain d, then either $P \cap R = KR$ for some τ-prime ideal K of T or else $P \cap R$ is a prime ideal of R such that $d, x \notin P \cap R$ and $P \cap R \neq (P \cap T)R$.*

(II.1) *Suppose that Q is a prime ideal of R such that $d, x \notin Q$ and $Q \neq (Q \cap T)R$. Then there exists a unique prime ideal P in S contracting to Q, and $\operatorname{rank}(S/P) = \operatorname{rank}(R/Q)$.*

(II.2) *Suppose that K is a τ-prime ideal of T, let A denote the Goldie quotient ring of R/KR, and extend (τ, δ) to A as in (2.3i). The natural map $R \to R/KR \hookrightarrow A$ extends uniquely to an algebra homomorphism $\varphi\colon S \to A[\theta; \tau, \delta]$ sending y to θ.*

(i) *The prime ideals of S contracting to KR are precisely the inverse images of the prime ideals of $A[y;\tau,\delta]$ under the map φ.*

(ii) *The coset $x+KR$ is invertible in A, and $A[y;\tau,\delta]=A[z;\tau]$ where $z=y-(1-q)^{-1}d(x+KR)^{-1}$. In particular, $zA[y;\tau,\delta]$ is a nonzero prime ideal of $A[y;\tau,\delta]$.*

(iii) *There exist more than two prime ideals in $A[y;\tau,\delta]$ if and only if for some integers m, n with $n>0$ there exists a unit u in the Goldie quotient ring of T/K such that $\tau(u)=\beta^n u$ and $ua=\tau^m(a)u$ for all $a\in T/K$.*

Proof: (I) By (10.4), $P\cap R$ is either a δ-stable τ-prime ideal of R or a prime ideal which is not τ-stable.

Assume first that $P\cap R$ is a δ-stable τ-prime ideal. Then $P\cap R=\bigcap_i \tau^i(Q)$ for some prime ideal Q of R whose τ-orbit is finite. From the assumption $d\notin P$ it follows that $d\notin Q$. Since $P\cap R$ is δ-stable and $\delta(x)\notin P$, we must have $x\notin P\cap R$, and consequently $x\notin Q$. Thus $K=Q\cap T$ is a τ-prime ideal of T by (14.2iii), and $Q=KR$ by (14.3). Now $\tau(Q)=Q$, and therefore $P\cap R=Q=KR$.

Now assume that $P\cap R=Q$ is a prime ideal of R which is not τ-stable. Let A be the Goldie quotient ring of R/Q, and denote images in A with an overbar. According to (11.1.II.1i), there exists $c\in A$ such that $\overline{\delta(r)}=\tau(r)c-cr$ for all $r\in R$. In particular, $\overline{d}=\beta xc-cx$. Since $d\notin Q$, it follows that $x\notin Q$. Then $Q\cap T$ is a τ-prime ideal of T by (14.2iii), and since $(Q\cap T)R$ is τ-stable, we conclude that $Q\neq(Q\cap T)R$.

(II.1) By (14.3), the τ-orbit of Q is infinite. Let A be the Goldie quotient ring of R/Q, and note that the coset $\overline{x}=x+Q$ is a nonzero normal element of A, whence $\overline{x}$ is invertible in A. Set $c=(q-1)^{-1}\overline{d}(\overline{x})^{-1}\in A$, and observe that $\tau(a)c-ca=0=\overline{\delta(a)}$ for all $a\in T$, while also

$$\tau(x)c-cx \;=\; \beta(q-1)^{-1}\overline{x}\,\overline{d}(\overline{x})^{-1}-(q-1)^{-1}\overline{d} \;=\; (q-1)^{-1}(\beta\alpha-1)\overline{d} \;=\; \overline{\delta(x)}.$$

It follows that the τ-derivations $r\mapsto\tau(r)c-cr$ and $r\mapsto\overline{\delta(r)}$ from R to the R-R-bimodule A agree. Therefore, the desired conclusions follow from (8.5.I).

(II.2) The existence and uniqueness of φ are immediate from the fact that $\theta(r+KR)=(\tau(r)+KR)\theta+(\delta(r)+KR)$ for all $r\in R$. Since KR is a (τ,δ)-ideal of R, there is no loss of generality in assuming that $K=0$ by (2.1vi). In particular, R is prime by (14.2ii). To begin, part (i) is clear since A is simple.

Now set $c = (1-q)^{-1}dx^{-1}$ in A. As in the proof of (II.1), the inner τ-derivation $r \mapsto \delta_c(r) = cr - \tau(r)c$ agrees with δ on R, and so $\delta = \delta_c$ on A. Therefore part (ii) follows from (2.1iv).

It remains to prove part (iii). Let F denote the Goldie quotient ring of T. If there exist $m, n \in \mathbf{Z}$ and $u \in F$ with the given properties, set $v = ux^{m-n}$. Then v is a unit in A, and we observe that $vb = \tau^n(b)v$ for all $b \in T \cup \{x\}$. It follows that $\tau^n(a) = vav^{-1}$ for all $a \in A$, and so by (2.3iii) there are infinitely many prime ideals in $A[z;\tau] = A[y;\tau,\delta]$.

Conversely, if $A[z;\tau]$ has more than two prime ideals, then by (2.3iii) there exist a positive integer n and a unit $v \in A$ such that $vb = \tau^n(b)v$ for all $b \in A$. Now consider the skew-Laurent series ring $F((x;\tau^{-1}))$, as defined for example in [**16**, 1ZA.a]. There is a canonical embedding of A into $F((x;\tau^{-1}))$, which allows us to write $v = \sum_{i=h}^{\infty} v_i x^i$ for some $h \in \mathbf{Z}$ and some $v_i \in F$. Choose an index j such that $v_j \neq 0$. Since

$$\sum_{i=h}^{\infty} v_i \tau^{-i}(c) x^i \;=\; vc \;=\; \tau^n(c)v \;=\; \sum_{i=h}^{\infty} \tau^n(c) v_i x^i$$

for all $c \in F$, we obtain $v_j a = \tau^{n+j}(a)v_j$ for all $a \in F$. Moreover, from the equation $vx = \tau^n(x)v$ it follows that $\tau(v_j) = \beta^n v_j$. Now $v_j F = F v_j$ is a nonzero τ-ideal of F, and since F is τ-simple, we conclude that v_j is invertible in F. Therefore the requirements of (iii) are satisfied with $m = n + j$ and $u = v_j$. $\square$

14.5. The following example illustrates the use of (14.4). Let $S = A_1(k[z], \tau, q)$, where k is a field of characteristic zero, τ is the k-algebra automorphism of $k[z]$ such that $\tau(z) = z + 1$, and q is a nonzero element of k which is neither 1 nor a root of unity. Then S is an algebra of the type constructed in (14.1), with $T = k[z]$ and $d = 1$, while $\alpha = 1$ and $\beta = q$.

Since T is τ-simple and no nonzero power of τ is inner on T, it follows from (14.2iv,v) that the only nonzero prime ideals in the algebra $R = T[x;\tau^{-1}]$ are the ideals $I + xR$ for prime ideals I of T. Consequently, it follows from (14.4.I) that all prime ideals of S contract to 0 in R. Thus spec$S$ is homeomorphic to spec$A[y;\tau,\delta]$, where A is the Goldie quotient ring of R. It is easily checked that there do not exist a positive integer n and a nonzero element $u \in k(z)$ such that $\tau(u) = \beta^n u$. Hence, it follows from (14.4.II.2iii) that $A[y;\tau,\delta]$ has only two prime ideals.

Therefore, S has precisely two prime ideals in this example.

We note for any nonzero prime ideal I of T that the prime ideal $I + xR$ in R has infinite τ-orbit but that there are no prime ideals of S lying over $I + xR$. This provides alternate examples to (8.7).

14.6. Let k be a field and q a nonzero element of k. We can describe $\mathcal{O}_q(M_2(k))$ as the k-algebra with generators x, u, v, y and relations

$$\begin{aligned} ux &= q^{-2}xu, & vx &= q^{-2}xv, & uv &= vu, \\ yu &= q^{-2}uy, & yv &= q^{-2}vy, & xy - yx &= (q^2 - q^{-2})uv. \end{aligned}$$

Further, it can be shown that the monomials $u^g v^h x^i y^j$, for nonnegative integers g, h, i, j, are linearly independent over k (see [**38**]).

The k-subalgebra of $\mathcal{O}_q(M_2(k))$ generated by u and v is just the commutative polynomial ring $T = k[u, v]$. The k-subalgebra of $\mathcal{O}_q(M_2(k))$ generated by u, v, x is the skew polynomial ring $R = T[x; \tau^{-1}]$, where τ is the k-algebra automorphism of T such that $\tau(u) = q^{-2}u$ and $\tau(v) = q^{-2}v$. Finally, $\mathcal{O}_q(M_2(k))$ itself can be expressed as the skew polynomial ring $R[y; \tau, \delta]$, where τ has been extended to an automorphism of R mapping x to itself and δ is the k-linear τ-derivation on R such that $\delta(u) = \delta(v) = 0$ while $\delta(x) = (q^{-2} - q^2)uv$. Thus $\mathcal{O}_q(M_2(k))$ is an algebra of the type constructed in (14.1), with $d = (q^{-2} - q^2)uv$ and $\beta = 1$. Since $\tau(d) = q^{-4}d$, it follows from (14.1) that (τ, δ) is a q^4-skew derivation on R.

Lemma 14.7. *Let R, T, τ be as in* (14.6). *Assume that $q \neq 1$ and that q is not a root of unity.*

(i) *The prime ideals of T with finite τ-orbit are precisely the homogeneous prime ideals. In particular, all finite τ-orbits in* spec$T$ *are singletons.*

(ii) *If Q is a prime ideal of R and $I = Q \cap T$, then I is a prime ideal of T. Moreover, if u and v are not both contained in I, then either $Q = I + xR$ or $Q = IR$.*

(iii) *If J is a τ-prime ideal of R, then J is a prime ideal of R and $J \cap T$ is a homogeneous prime ideal of T. Consequently, all finite τ-orbits in* spec$R$ *are singletons.*

(iv) *All prime ideals of R are completely prime.*

Proof: (i) Consider a prime ideal I of T such that $\tau^m(I) = I$ for some positive integer m. Given $f \in I$, write $f = f_0 + f_1 + \ldots + f_n$ where each f_i is a homogeneous polynomial in T with total degree i. For $j = 0, 1, \ldots,$ we have

$$f_0 + q^{-2jm} f_1 + q^{-4jm} f_2 + \ldots + q^{-2njm} f_n \;=\; \tau^{jm}(f) \;\in\; I.$$

Since $1, q^{-2}, q^{-4}, \ldots$ are all distinct, it follows that $f_0, \ldots, f_n \in I$. Thus I is a homogeneous ideal.

Since homogeneous ideals of T are τ-stable, part (i) follows.

(ii) If $x \in Q$, then $Q = I + xR$ and I is a prime ideal of T. Now suppose that $x \notin I$. Then I is a τ-prime ideal of T by (14.2iii), and so I is a prime ideal of T by (i). Moreover, if u and v are not both in I, no positive power of τ is inner on the quotient field of T/I. In this case, it follows from (2.3iii) that $Q = IR$.

(iii) Since J is τ-prime, $J = \bigcap_i \tau^i(Q)$ for some prime ideal Q of R whose τ-orbit is finite. Then $I = Q \cap T$ is a prime ideal of T by (ii), and I has finite τ-orbit. It now follows from (i) that I is homogeneous. In particular, I is τ-stable, and hence $J \cap T = I$. Thus $J \cap T$ is a homogeneous prime ideal.

If u and v are not both in I, then by (ii) either $Q = I + xR$ or $Q = IR$. In either case, $\tau(Q) = Q$ and so $J = Q$. On the other hand, if $u, v \in I$ then $Q = uR + vR + H$ for some prime ideal H of $k[x]$. Then $\tau(Q) = Q$, and again $J = Q$. Therefore J is prime.

The last statement of (iii) follows immediately.

(iv) This follows from (ii) together with the fact that $R/(uR + vR)$ is a commutative ring. □

Lemma 14.8. *Let $S = \mathcal{O}_q(M_2(k))$, and assume that q is neither 1 nor a root of unity. Let x, u, v, y be as in* (14.6), *and let U be the subalgebra of S generated by x, u, y.*

(i) *Let P be a prime ideal of S containing u. If $I = P \cap U$, then I is a prime ideal of U. Moreover, if x and y are not both in I, then either $P = I + vS$ or $P = IS$.*

(ii) *All prime ideals of S/uS are completely prime.*

Proof: Observe that u is a normal element of S, and that S/uS can be described as the k-algebra with generators $\overline{x}$, $\overline{y}$, $\overline{v}$ and relations

$$\overline{v}\,\overline{x} \;=\; q^{-2}\overline{x}\,\overline{v}, \qquad \overline{v}\,\overline{y} \;=\; q^{2}\overline{y}\,\overline{v}, \qquad \overline{x}\,\overline{y} \;=\; \overline{y}\,\overline{x}.$$

Thus $\overline{U} = (U + uS)/uS$ is just a commutative polynomial ring $k[\overline{x}, \overline{y}]$, and $S/uS = \overline{U}[\overline{v}; \sigma]$ where σ is the k-algebra automorphism of $\overline{U}$ such that $\sigma(\overline{x}) = q^{-2}\overline{x}$ and $\sigma(\overline{y}) = q^2\overline{y}$.

Now view $\overline{U}$ as a $\mathbf{Z}$-graded k-algebra in which $\overline{y}$ has graded degree 1 while $\overline{x}$ has graded degree -1. Then for any homogeneous element $f_i \in \overline{U}$ with graded degree i, we have $\sigma(f_i) = q^{2i} f_i$. As in the proof of (14.7i), it follows that the prime ideals of $\overline{U}$ with finite σ-orbit are all σ-stable. Hence, (i) and (ii) follow from the arguments used to prove (14.7ii,iv). $\square$

Theorem 14.9. *If $S = \mathcal{O}_q(M_2(k))$ where $q \neq 1$ and q is not a root of unity, then all prime ideals of S are completely prime.*

Proof: We shall use the description of S given in (14.6). By (14.8), all prime ideals of S containing u are completely prime, and similarly all prime ideals of S containing v are completely prime.

Now consider a prime ideal P of S such that $u, v \notin P$, and set $J = P \cap R$. By (10.4), J is either a δ-stable τ-prime ideal of R or a prime ideal which is not τ-stable. In either case, it follows from (14.7iii,iv) that J is a completely prime ideal of R. If J is not τ-stable, then by (14.7iii) the τ-orbit of J is infinite. In this case, it follows from (8.5.I) that $\operatorname{rank}(S/P) = \operatorname{rank}(R/J) = 1$, whence P is completely prime.

At this point, we may assume that J is δ-stable and τ-stable. By (14.7iii), the contraction $I = J \cap T$ is a homogeneous prime ideal of T. Since $u, v \notin I$, it follows from (14.7ii) that either $J = I + xR$ or $J = IR$. On the other hand, the element $d = (q^{-2} - q^2)uv$ is not in P, and so (14.4.I) shows that $x \notin P$. Thus we must have $J = IR$. In view of (14.4.II.2i), we see that it now suffices to show that all prime ideals in the algebra $A[y; \tau, \delta]$ are completely prime, where A is the Goldie quotient ring of R/J.

By (14.4.II.2ii), the coset $x + J$ is invertible in A, and δ is an inner τ-derivation on A. Moreover, since $(x + J)^{-1}(a + J)(x + J) = \tau(a + J)$ for $a \in \{u,\ v,\ x\}$, we see that τ is inner on A. Therefore $A[y; \tau, \delta] = A[w]$ for some central indeterminate w, by (2.1iv) and (14.2i). Consequently, $A[y; \tau, \delta]$ is isomorphic to a localization of the algebra $R^* = T^*[x, x^{-1}; \tau^{-1}]$, where $T^* = (T/I)[d^{-1}][w]$ and τ has been extended from $(T/I)[d^{-1}]$ to an automorphism of T^* fixing w. It thus suffices to show that all prime ideals of R^* are completely prime.

Now T^* is a homomorphic image of the algebra $T' = k[u, u^{-1}, v, v^{-1}, w]$, and τ lifts to a k-algebra automorphism of T' in the obvious fashion. As in the proof of (14.7i), any prime ideal of T' with finite τ-orbit is homogeneous with respect to u and v, and thus is τ-stable. Hence, all τ-prime ideals of T' are prime, and so all τ-prime ideals of T^* are prime.

If Q^* is any prime ideal of R^*, it follows from (14.2iii) that the contraction $I^* = Q^* \cap T^*$ is a τ-prime ideal of T^*, and so I^* is a prime ideal of T^*. Since T^* is commutative, no nonzero power of τ can be inner on the quotient field F^* of T^*/I^*, and so $F^*[x, x^{-1}; \tau^{-1}]$ is simple. Thus $Q^* = I^*T^*$, and so R^*/Q^* is isomorphic to the domain $(T^*/I^*)[x, x^{-1}; \tau^{-1}]$. Therefore Q^* is a completely prime ideal of R^*, as desired. □

14.10. We now turn to the case when q is a root of unity. Here we shall need the observation that if U is a noetherian ring generated by n^2 elements as a module over its center, for some positive integer n, then all prime factors of U have rank at most n. (This assertion follows since each Goldie quotient ring of a prime factor of U will be a vector space of dimension at most n^2 over a central subfield.)

14.11. Let $S = \mathcal{O}_q(M_2(k))$ with the notation of (14.6), and suppose that q is a primitive t^{th} root of unity for some odd integer $t > 1$. Then also q^2 and q^4 are primitive t^{th} roots of unity. It follows immediately that u^t and v^t are central in S, and that τ^t is the identity on R. From [**13**, 1.1], we see that

$$\delta(x^t) = \sum_{i=0}^{t-1} x^i dx^{t-1-i} = \sum_{i=0}^{t-1} q^{4i} dx^{t-1} = 0,$$

and hence x^t is central in S. By (3.8i,iii), δ^t is a derivation on R, and $y^t r = \tau^t(r)y^t + \delta^t(r)$ for all $r \in R$. Since δ^t vanishes on u, v, x, it follows that $\delta^t = 0$ and that y^t is central in S.

Now S is generated as a module over its center by t^4 elements, namely the monomials $u^g v^h x^i y^j$ with $0 \le g, h, i, j \le t-1$. It therefore follows from (14.10) that all prime factor rings of S have rank at most t^2. With a more detailed analysis, we shall reduce the upper bound on these ranks to t. Moreover, since S does have prime factors with rank t (see (14.12)), this bound is sharp.

The following lemma allows us to deal globally with those prime ideals of S which contain x^t but do not contain uv, instead of repeatedly appealing to

(3.9) and (3.12). The method of proof is that used in (3.9iii) and [**13**, 7.5]. (We note that [**13**, 7.5] cannot be applied directly, since the image of $d^{-1}x$ in $S[d^{-1}]/x^tS[d^{-1}]$ is not central.)

Lemma 14.12. *Let $S = \mathcal{O}_q(M_2(k))$, and suppose that q is a primitive t^{th} root of unity for some odd integer $t > 1$. If u, v, x, y, d are as in* (14.6), *then*

$$S[d^{-1}]/x^tS[d^{-1}] \cong M_t(k[u, u^{-1}, v, v^{-1}, y^t]).$$

Proof: Set $R' = R[d^{-1}]/x^tR[d^{-1}]$ and $S' = S[d^{-1}]/x^tS[d^{-1}]$, identify S' with $R'[y; \tau, \delta]$, and let z denote the coset $d^{-1}x + x^tR[d^{-1}]$ in R'. Then $\tau(z) = q^4z$ and $\delta(z) = 1$, while $z^t = 0$. By [**13**, 1.1], $\delta(z^m) = \sum_{i=0}^{m-1} q^{4i}z^{m-1}$ for all positive integers m, from which we see that $\alpha = \delta^{t-1}(z^{t-1})$ is a nonzero element of k. Set $w = \alpha^{-1}z^{t-1}$, so that $\delta^{t-1}(w) = 1$ and $wz = 0$.

Observe that $\delta^i(w) \in zR'$ for all $i < t-1$, whence $y^{t-1}w \equiv 1 \pmod{zS'}$, and so $wy^{t-1}w = w$. Thus $e = wy^{t-1}$ is an idempotent in S' such that $eS' = wS'$. Since $z^iS'/z^{i+1}S' \cong wS'$ for $i = 0, \dots, t-1$, it follows that $S'_{S'} \cong (S'/zS')^t$. Therefore $S' \cong M_t(E)$, where E is the endomorphism ring of $(S'/zS')_{S'}$.

Now $E \cong \mathbf{I}/zS'$, where $\mathbf{I}$ denotes the idealizer of zS' in S'. Clearly u, u^{-1}, v, v^{-1}, y^t are all in $\mathbf{I}$. For any positive integer h not divisible by t, we have

$$y^hz = q^{4h}zy^h + \binom{h}{1}_q y^{h-1},$$

with $\binom{h}{1}_q$ invertible in k by (2.6iv). It thus follows that any element of $\mathbf{I}$ with y-degree h must have its leading coefficient in zR'. Consequently, we conclude that $\mathbf{I}$ equals the subalgebra of S' generated by $\{u, u^{-1}, v, v^{-1}, y^t\} \cup zS'$, and therefore $E \cong k[u, u^{-1}, v, v^{-1}, y^t]$. □

Lemma 14.13. *Let $S = \mathcal{O}_q(M_2(k))$, and suppose that q is a primitive t^{th} root of unity for some odd integer $t > 1$. Let u, v, and R be as in* (14.6). *Then each of the algebras $R[u^{-1}]$, $(S/uS)[x^{-1}]$, and $(S/uS)[y^{-1}]$ can be generated by t^2 elements as a module over its center.*

Proof: The algebra $C = k[u^{-1}v, u^t, x^t]$ is contained in the center of $R[u^{-1}]$, and $R[u^{-1}]$ is generated as a C-module by the monomials u^ix^j with $0 \le i, j \le t-1$. Similarly, $(S/uS)[x^{-1}]$ can be generated by t^2 elements as a module over the central subalgebra $k\{\overline{xy}, \overline{x}^t, (\overline{x})^{-t}, \overline{v}^t\}$, and $(S/uS)[y^{-1}]$ can be generated by t^2 elements as a module over the central subalgebra $k\{\overline{xy}, \overline{y}^t, (\overline{y})^{-t}, \overline{v}^t\}$. $\square$

Theorem 14.14. *If $S = \mathcal{O}_q(M_2(k))$ where q is a primitive t^{th} root of unity for some odd integer $t > 1$, then all prime factor rings of S have rank at most t.*

Proof: Let P be a prime ideal of S. First, it follows from (14.12) that if $uv \notin P$ while $x^t \in P$, then $\operatorname{rank}(S/P) = t$. Second, if $u \in P$ but x and y are not both contained in P, then $\operatorname{rank}(S/P) \leq t$ by (14.13) and (14.10). On the other hand, if P contains u, x, y, then S/P is commutative and so $\operatorname{rank}(S/P) = 1$. Therefore $\operatorname{rank}(S/P) \leq t$ if $u \in P$, and similarly if $v \in P$.

It remains to consider the case that $u, v, x^t \notin P$. Then P lies directly over some prime ideal Q of R, and by (5.12i) the prime ideals of R minimal over $P \cap R$ all lie in the τ-orbit of Q. Since P does not contain any power of u, v, or x^t, it follows that $u, v, x^t \notin Q$. In particular, $x \notin Q$, and so the contraction $I = Q \cap T$ is a τ-prime ideal of T by (14.2iii).

Let J be the maximal (τ, δ)-ideal of R contained within Q. By (9.6i), Q is minimal over J. Since the prime radical of R/J is τ-prime (2.3i), it follows that the prime ideals of R minimal over J are precisely the prime ideals in the τ-orbit of Q. If A denotes the artinian classical quotient ring of R/J, then some prime factor of $A[y; \tau, \delta]$ is isomorphic to a localization of S/P. Hence, it now suffices to show that all prime factors of $A[y; \tau, \delta]$ have rank at most t.

Now x is a normal element of R which is not in Q and hence is not in any of the prime ideals minimal over J. Since R has an artinian classical quotient ring, it follows that x is regular modulo J, and thus that the coset $x + J$ is invertible in A. We then observe that $(x + J)^{-1} a (x + J) = \tau(a)$ for $a \in \{u + J,\ v + J,\ x + J\}$, and consequently that τ is inner on A. Further, it follows from calculations similar to those in the proof of (14.4.II.1) that δ is an inner τ-derivation on A. Therefore $A[y; \tau, \delta] = A[w]$ for some central indeterminate w, by (2.1iv) and (14.2i). Hence, $A[y; \tau, \delta]$ contains a nilpotent ideal N such that $A[y; \tau, \delta]/N$ is isomorphic to a finite direct product of copies of $F[w]$ where F denotes the Goldie quotient ring of R/Q. It thus suffices to show that all prime factors of $F[w]$ have rank at most t.

Since $u \notin Q$, the induced ideal $Q[u^{-1}]$ is a prime ideal of $R[u^{-1}]$, and F is isomorphic to the Goldie quotient ring of $R[u^{-1}]/Q[u^{-1}]$. Consequently, it follows from (14.13) that F can be generated by t^2 elements as a module over its center, and hence the same is true for $F[w]$. Therefore by (14.10) all prime factors of $F[w]$ have rank at most t, as desired. □

Theorem 14.15. *If $S = \mathcal{O}_q(M_2(k))$ where q is a primitive t^{th} root of unity for some even integer $t > 2$, then all prime factor rings of S have rank at most $t/2$.*

Proof: Since q^2 is a primitive $(t/2)^{\text{th}}$ root of unity, we can proceed as in (14.11)–(14.14), with t replaced by either $t/2$ or $t/4$ as appropriate. In particular, the correct analog of (14.12) when t is divisible by 4 is

$$S[d^{-1}]/x^{t/4}S[d^{-1}] \;\cong\; M_{t/4}(k[u,u^{-1},v,v^{-1},y^{t/4}]).$$

We leave the details to the reader. □

15. Chains of prime ideals in iterated Ore extensions.

15.1. Let R be a noetherian ring, let τ be an automorphism of R, and let $S = R[y; \tau, \delta]$. Suppose that $P_0 \subset P_1 \subseteq P_2$ are prime ideals of S which all lie over a particular prime ideal of R. It follows from (5.11) that if τ extends to an automorphism of S then $P_1 = P_2$. (For the case when τ or δ is trivial, the assertion that $P_1 = P_2$ is well known, at least under the hypothesis that $P_0 \cap R = P_1 \cap R$; see for example [**34**, 10.6.9ii, 10.6.4ii, 14.2.9].) We conclude our study by proving that the above conclusion remains valid when the assumption concerning τ is dropped, and we further generalize the result to certain iterated Ore extensions. The generalization depends on extending the localization-theoretic results of Section 4.

15.2. (i) Let $R = R_0 \subset R_1 \subset \cdots \subset R_n = T$ be a sequence of embedded noetherian rings such that for $1 \le i \le n$ it holds that $R_i = R_{i-1}[y_i; \tau_i, \delta_i]$ for an automorphism τ_i of R_{i-1}, a τ_i-derivation δ_i of R_{i-1}, and an element $y_i \in R_i$. Further assume that $\tau_i(R) = R$ and $\delta_i(R) \subseteq R$ for $1 \le i \le n$. Let $\mathcal{G}$ denote the group of automorphisms of R generated by $\tau_1, \ldots, \tau_n$, and $\mathcal{G}^+$ the subsemigroup of $\mathcal{G}$ generated by the τ_i.

(ii) The proof of the following fact is left to the reader: Let B denote an R-R-bimodule subfactor of T as above, and let $b \in B$. Then $R.b.R$ is noetherian as both a right and a left R-module.

15.3. Let $R = R_0 \subset \cdots \subset R_n = T$ and $\mathcal{G}^+$ be as in (15.2i). Let P be a prime ideal of T, and let C be the Goldie quotient ring of T/P. Choose $C_0, \ldots, C_m$ and $Q_1, \ldots, Q_m$ for T as in (4.1).

Lemma. *Fix $1 \le i \le m$. There exists an $\alpha \in \mathcal{G}^+$ such that $\alpha(Q_i) = Q_m$.*

Proof: Adapt the proof of (4.4) as follows. Let A be the Goldie quotient ring of R/Q_m. From (4.1) and (15.2ii) we see that there exists an R-R-bimodule subfactor B of T/Q_mT such that the left annihilator of B is Q_m, the right annihilator of B is Q_i, and such that ${}_A(A \otimes_R B)_R$ is a simple A-R-bimodule.

Let V denote the A-R-bimodule $A \otimes_R T = A \otimes_R (T/Q_mT)$, and note that ${}_A(A \otimes_R B)_R$ is an A-R-bimodule factor of V. Denote the canonical image $1 \otimes t$ in V of an element $t \in T$ by $\overline{t}$. Note that the set $\{\overline{y}_1^{i_1} \cdots \overline{y}_n^{i_n} \mid i_1, \ldots, i_n \ge 0\}$ forms a basis for ${}_AV$. Also note, for each $r \in R$, that

$$\overline{y}_1^{i_1} \cdots \overline{y}_n^{i_n} r \;=\; \alpha(r)\overline{y}_1^{i_1} \cdots \overline{y}_n^{i_n} + p,$$

where $\alpha \in \mathcal{G}^+$, and where if p is written in terms of the above basis for ${}_AV$, then the total degree of p is less than $i_1 + \cdots + i_n$. Hence, there is a filtration $0 = V_0 \subset V_1 \subset \cdots$ of A-R-sub-bimodules of V such that $V = \bigcup_j V_j$, and such that for $j \geq 1$ it holds that ${}_A(V_j/V_{j-1})_R \cong {}_A(A^{\alpha_j})_R$ for some $\alpha_j \in \mathcal{G}^+$. Recall from (4.3) that each ${}_A(A^{\alpha_j})_R$ is simple. Hence, there exists a non-negative integer ℓ such that ${}_A(A \otimes_R B)_R \cong {}_A(A^{\alpha_\ell})_R$, and consequently, $\alpha_\ell^{-1}(Q_m) = Q_i$. □

Proposition 15.4. *Let $R = R_0 \subset \cdots \subset R_n = T$ and $\mathcal{G}^+$ be as in (15.2i). Let $S = R_j$, for some fixed $j \in \{0, \ldots, n\}$. Let P be a prime ideal of T, let $F = S/(P \cap S)$, and let $E = R/(P \cap R)$. Let $\mathcal{C}$ denote the set of regular elements of E.*

(i) *$\mathcal{C}$ is an Ore set of E, and $E\mathcal{C}^{-1} = \mathcal{C}^{-1}E$ is the artinian quotient ring of E.*

(ii) *$\mathcal{C}$ is an Ore set of regular elements of F. In particular, F naturally embeds into its quotient ring $F\mathcal{C}^{-1} = \mathcal{C}^{-1}F$.*

Proof: Let $C_0, \ldots, C_m = C$ and $Q_1, \ldots, Q_m$ be as in (4.1). It follows from (15.3) that the rings R/Q_i all have the same classical Krull dimension, say d. The proof now follows from (4.5) by an argument identical to that for (4.6). □

Lemma 15.5. *Let S be a noetherian ring containing a noetherian subring R, and suppose that there exists an element $y \in S$ such that $yR + R = Ry + R$ and*

$$S \;=\; R + yR + y^2R + \cdots \;=\; R + Ry + Ry^2 + \cdots .$$

Then $\mathrm{rKdim}(S) \leq \mathrm{rKdim}(R) + 1$.

Proof: Since there is nothing to prove in case $y \in R$, assume that $y \notin R$. By [**34**, 1.2.10, 6.5.6], the associated graded ring $\mathrm{gr}S$ with respect to the filtration

$$0 \subseteq R \subseteq R + yR \subseteq R + yR + y^2R \subseteq \cdots$$

is a noetherian ring such that $\mathrm{rKdim}(S) \leq \mathrm{rKdim}(\mathrm{gr}S)$. Consequently, after replacing S by $\mathrm{gr}S$, we may assume that $Ry = yR$. Next, using [**34**, 6.3.8iii], choose a prime ideal P of S such that $\mathrm{rKdim}(S) = \mathrm{rKdim}(S/P)$. Note that

$$\mathrm{rKdim}(R) \;\geq\; \mathrm{rKdim}(R/(P \cap R)).$$

Next, replace S with S/P and R with $R/(P \cap R)$. Note that the equality $Ry = yR$ still holds, so y is a normal element of S. Since S is prime, it follows that y is regular. It is now straightforward to check that there exists an automorphism τ of S such that $\tau(s)y = ys$ for all $s \in S$. Note that $\tau(R) = R$, and hence S is a homomorphic image of $R[x;\tau]$ for some indeterminate x. The lemma now follows from [**34**, 6.5.4i]. $\square$

Lemma 15.6. *Let $R = R_0 \subset \cdots \subset R_n = T$ be as in* (15.2i), *let P be a prime ideal of T, and let $\mathcal{C}$ denote the set of regular elements of $R/(P \cap R)$. Then $\mathcal{C}$ is an Ore set in T/P and* $\operatorname{rKdim}([T/P]\mathcal{C}^{-1}) \leq n$.

Proof: By (15.4), $\mathcal{C}$ is an Ore set of regular elements of $R_j/(P \cap R_j)$, for $0 \leq j \leq n$; in particular, $\mathcal{C}$ is Ore in T/P. Moreover, $[R/(P \cap R)]\mathcal{C}^{-1}$ is artinian. Hence, it suffices to show that

$$\operatorname{rKdim}([R_{j+1}/(P \cap R_{j+1})]\mathcal{C}^{-1}) \;\leq\; \operatorname{rKdim}([R_j/(P \cap R_j)]\mathcal{C}^{-1}) + 1$$

for $0 \leq j < n$.

Fix j as above, and retain the notation of (15.4). Let $G = R_{j+1}/(P \cap R_{j+1})$, and let x denote the canonical image in G of y_{j+1}. Let $K = F\mathcal{C}^{-1}$ and $L = G\mathcal{C}^{-1}$, and identify E, F, and G with their images in L. It is straightforward to verify that

$$L \;=\; K + Kx + Kx^2 + \cdots \;=\; K + xK + x^2K + \cdots,$$

and that $xK + K = Kx + K$. The lemma now follows from (15.5). $\square$

Theorem 15.7. *Let $R = R_0 \subset \cdots \subset R_n = T$ be a sequence of noetherian rings where each R_i is equal to $R_{i-1}[y_i;\tau_i,\delta_i]$ for some element $y_i \in R_i$, some automorphism τ_i of R_{i-1} such that $\tau_i(R) = R$, and some τ_i-derivation δ_i of R_{i-1} such that $\delta_i(R) \subseteq R$. Let Q be a prime ideal of R. Suppose that $P_0, \ldots, P_n$ are prime ideals of T such that $P_0 \subset \cdots \subset P_n$, and such that $P_0, \ldots, P_n$ all lie over Q. If P is a prime ideal of T properly containing P_n, then P cannot also lie over Q.*

Proof: Let $\mathcal{C} = \mathcal{C}_R(P_0 \cap R)$, that is, the set of elements of R regular modulo $P_0 \cap R$. It follows, for example, from [**16**, 10.8, 6.5], that $\mathcal{C} \subseteq \mathcal{C}_R(Q)$. Suppose that $P \cap R \subseteq Q$; then P is disjoint from $\mathcal{C}$. Since $\mathcal{C}$ is Ore in T/P_0 by (15.4), the chain $P_0 \subset \cdots \subset P_n \subset P$ of prime ideals of T extends to a chain

$$0 = I_0 \subset \cdots \subset I_n \subset I_{n+1}$$

of prime ideals of $[T/P_0]\mathcal{C}^{-1}$; see for example [**16**, 9.22]. However, by (15.6), $\operatorname{rKdim}([T/P_0]\mathcal{C}^{-1}) \leq n$. Since the classical Krull dimension is less than or equal to the right Krull dimension of any noetherian ring, we have arrived at a contradiction. The theorem follows. $\square$

REFERENCES

1. F. W. Anderson and K. R. Fuller, Rings and Categories of Modules (Springer-Verlag, New York, 1974).

2. G. E. Andrews, The Theory of Partitions, Encyclopedia of Math and its Applic. (G. C. Rota, Ed.), Vol. 2 (Addison-Wesley, Reading, 1976).

3. M. Awami, M. Van den Bergh, and F. Van Oystaeyen, Note on derivations of graded rings and classification of differential polynomial rings, Bull. Soc. Math. Belg., Sér. A, 40 (1988), 175–183.

4. E. J. Behr, Enveloping algebras of Lie superalgebras, Pacific J. Math., 130 (1987), 9–25.

5. A. D. Bell, When are all prime ideals in an Ore extension Goldie?, Comm. Alg., 13 (1985), 1743–1762.

6. A. D. Bell and S. P. Smith, Some 3-dimensional skew polynomial rings, preprint, University of Wisconsin, Milwaukee, 1990.

7. W. Borho, On the Joseph-Small additivity principle for Goldie ranks, Comp. Math., 47 (1982), 3–29.

8. G. Cauchon and J. C. Robson, Endomorphisms, derivations, and polynomial rings, J. Alg., 53 (1978), 227–238.

9. J. Cigler, Operatormethoden für q-Identitäten, Monatshefte für Math., 88 (1979), 87–105.

10. E. Cisneros, M. Ferrero, and M. I. Conzález, Prime ideals of skew polynomial rings and skew Laurent polynomial rings, Math. J. Okayama Univ., 32 (1990), 61–72.

11. J. Dixmier, Enveloping Algebras (North-Holland, Amsterdam, 1977).

12. A. W. Goldie and G. O. Michler, Ore extensions and polycyclic group rings, J. London Math. Soc. (2), 9 (1974), 337–345.

13. K. R. Goodearl, Prime ideals in skew polynomial rings and quantized Weyl algebras, J. Alg., 150 (1992), 324–377.

14. ________, Uniform ranks of prime factor rings of skew polynomial rings, i Ring Theory, Proc. Ohio State – Denison University Conf., 1992 (S. K. Jain and S. T. Rizvi, Eds.), World Scientific, to appear.

15. K. R. Goodearl and E. S. Letzter, Prime factor algebras of the coordinate ring of quantum matrices, Proc. Amer. Math. Soc., to appear.

16. K. R. Goodearl and R. B. Warfield, Jr., An Introduction to Noncommutative Noetherian Rings, London Math. Soc. Student Texts 16 (Cambridge Univ. Press, Cambridge, 1989).

17. T. Hayashi, Q-Analogues of Clifford and Weyl algebras – spinor and oscillator representations of quantum enveloping algebras, Commun. Math. Phys., 127 (1990), 129–144.

18. I. N. Herstein, Noncommutative Rings, Carus Mathematical Monographs 15 (Math. Assoc. of America, Washington, 1968).

19. R. S. Irving, Prime ideals of Ore extensions over commutative rings. II, J. Alg., 58 (1979), 399–423.

20. N. Jacobson, The Theory of Rings, Math. Surveys 1 (Amer. Math. Soc., Providence, 1943).

21. A. V. Jategaonkar, Localization in Noetherian Rings, London Math. Soc. Lecture Notes 98 (Cambridge University Press, Cambridge, 1986).

22. D. A. Jordan, Ore extensions and Jacobson rings, Ph. D. Dissertation (1975), University of Leeds.

23. ________, Noetherian Ore extensions and Jacobson rings, J. London Math. Soc. (2), 10 (1975), 281–291.

24. ________, Simple skew Laurent polynomial rings, Comm. Alg., 12 (1984), 135–137.

25. ________, Iterated skew polynomial rings and quantum groups, J. Alg., to appear.

26. A. A. M. Kamal, Some remarks on Ore extension rings, to appear.

27. E. E. Kirkman and L. W. Small, q-Analogs of harmonic oscillators and related rings, Israel J. Math., to appear.

28. T. Y. Lam, K. H. Leung, A. Leroy, and J. Matczuk, Invariant and semi-invariant polynomials in skew polynomial rings, Israel Math. Conference Proc., 1 (1989), 247–261.

29. A. Leroy and J. Matczuk, Prime ideals of Ore extensions, Comm. Alg., 19 (1991), 1893–1907.

30. E. S. Letzter, Primitive ideals in finite extensions of noetherian rings, J. London Math. Soc. (2), 39 (1989), 427–435.

31. ________, Finite correspondence of spectra in noetherian ring extensions, Proc. Amer. Math. Soc., 116 (1992), 645–652.

32. ________, Prime and primitive ideals in enveloping algebras of solvable Lie superalgebras, Contemp. Math., 130 (1992), 237–255.

33. D. R. Malm, Simplicity of partial and Schmidt differential operator rings, Pacific J. Math., 132 (1988), 85–112.

34. J. C. McConnell and J. C. Robson, Noncommutative Noetherian Rings (Wiley-Interscience, New York, 1987).

35. S. Montgomery, Prime ideals in fixed rings, Comm. Alg., 9 (1981), 423–449.

36. S. Montgomery and S. P. Smith, Skew derivations and $U_q(sl(2))$, Israel J. Math., 72 (1990), 158–166.

37. H. Morikawa, On ζ_n-Weyl algebra $W_r(\zeta_n, Z)$, Nagoya Math. J., 113 (1989), 153–159.

38. S. P. Smith, Quantum groups: an introduction and survey for ring theorists, in Noncommutative Rings (S. Montgomery and L. Small, Eds.), MSRI Publ. 24 (Springer-Verlag, New York, 1992), 131–178.

39. R. P. Stanley, Enumerative Combinatorics, Vol. I (Wadsworth & Brooks/Cole, Monterey, 1986).

40. M. G. Voskoglou, Prime ideals of skew polynomial rings, Riv. Mat. Univ. Parma, 15 (1989), 17–25.

41. R. B. Warfield, Jr., Prime Ideals in Ring Extensions, J. London Math. Soc. (2), 28 (1983), 453–460.

42. E. Wexler-Kreindler, Propriétés de transfert des extensions d'Ore, in Séminaire d'Algèbre Paul Dubreil 1976-77, Lecture Notes in Math. 641 (Springer-Verlag, Berlin, 1978), pp. 235–251.

Department of Mathematics, University of Utah, Salt Lake City, Utah, 84112

Current address of first author: Department of Mathematics, University of California, Santa Barbara, CA 93106

Current address of second author: Department of Mathematics, Texas A&M University, College Station, TX 77843

Editorial Information

To be published in the *Memoirs*, a paper must be correct, new, nontrivial, and significant. Further, it must be well written and of interest to a substantial number of mathematicians. Piecemeal results, such as an inconclusive step toward an unproved major theorem or a minor variation on a known result, are in general not acceptable for publication. *Transactions* Editors shall solicit and encourage publication of worthy papers. Papers appearing in *Memoirs* are generally longer than those appearing in *Transactions* with which it shares an editorial committee.

As of March 4, 1994, the backlog for this journal was approximately 7 volumes. This estimate is the result of dividing the number of manuscripts for this journal in the Providence office that have not yet gone to the printer on the above date by the average number of monographs per volume over the previous twelve months, reduced by the number of issues published in four months (the time necessary for preparing an issue for the printer). (There are 6 volumes per year, each containing at least 4 numbers.)

A Copyright Transfer Agreement is required before a paper will be published in this journal. By submitting a paper to this journal, authors certify that the manuscript has not been submitted to nor is it under consideration for publication by another journal, conference proceedings, or similar publication.

Information for Authors and Editors

Memoirs are printed by photo-offset from camera copy fully prepared by the author. This means that the finished book will look exactly like the copy submitted.

The paper must contain a *descriptive title* and an *abstract* that summarizes the article in language suitable for workers in the general field (algebra, analysis, etc.). The *descriptive title* should be short, but informative; useless or vague phrases such as "some remarks about" or "concerning" should be avoided. The *abstract* should be at least one complete sentence, and at most 300 words. Included with the footnotes to the paper, there should be the 1991 *Mathematics Subject Classification* representing the primary and secondary subjects of the article. This may be followed by a list of *key words and phrases* describing the subject matter of the article and taken from it. A list of the numbers may be found in the annual index of *Mathematical Reviews*, published with the December issue starting in 1990, as well as from the electronic service e-MATH **[telnet e-MATH.ams.org** (or **telnet 130.44.1.100**). Login and password are **e-math]**. For journal abbreviations used in bibliographies, see the list of serials in the latest *Mathematical Reviews* annual index. When the manuscript is submitted, authors should supply the editor with electronic addresses if available. These will be printed after the postal address at the end of each article.

Electronically prepared manuscripts. The AMS encourages submission of electronically prepared manuscripts in $\mathcal{AMS}$-TEX or $\mathcal{AMS}$-LATEX because properly prepared electronic manuscripts save the author proofreading time and move more quickly through the production process. To this end, the Society has prepared "preprint" style files, specifically the amsppt style of $\mathcal{AMS}$-TEX and the amsart style of $\mathcal{AMS}$-LATEX, which will simplify the work of authors and of the production staff. Those authors who make use of these style files from the beginning of the writing process will further reduce their own effort. Electronically submitted manuscripts prepared in plain TEX or LATEX do not mesh properly with the AMS production systems and cannot, therefore, realize the same kind of expedited processing. Users of plain TEX should have little difficulty learning AMS-TEX, and LATEX users will find that AMS-LATEX is the same as LATEX with additional commands to simplify the typesetting of mathematics.

Guidelines for Preparing Electronic Manuscripts provides additional assistance and is available for use with either AMS-TEX or AMS-LATEX. Authors with FTP access may obtain *Guidelines* from the Society's Internet node `e-MATH.ams.org` (130.44.1.100). For those without FTP access *Guidelines* can be obtained free of charge from the e-mail address `guide-elec@math.ams.org` (Internet) or from the Customer Services Department, American Mathematical Society, P.O. Box 6248, Providence, RI 02940-6248. When requesting *Guidelines*, please specify which version you want.

At the time of submission, authors should indicate if the paper has been prepared using AMS-TEX or AMS-LATEX. The *Manual for Authors of Mathematical Papers* should be consulted for symbols and style conventions. The *Manual* may be obtained free of charge from the e-mail address `cust-serv@math.ams.org` or from the Customer Services Department, American Mathematical Society, P.O. Box 6248, Providence, RI 02940-6248. The Providence office should be supplied with a manuscript that corresponds to the electronic file being submitted.

Electronic manuscripts should be sent to the Providence office immediately after the paper has been accepted for publication. They can be sent via e-mail to `pub-submit@math.ams.org` (Internet) or on diskettes to the Publications Department, American Mathematical Society, P. O. Box 6248, Providence, RI 02940-6248. When submitting electronic manuscripts please be sure to include a message indicating in which publication the paper has been accepted.

Two copies of the paper should be sent directly to the appropriate Editor and the author should keep one copy. The *Guide for Authors of Memoirs* gives detailed information on preparing papers for *Memoirs* and may be obtained free of charge from the Editorial Department, American Mathematical Society, P. O. Box 6248, Providence, RI 02940-6248. For papers not prepared electronically, model paper may also be obtained free of charge from the Editorial Department.

Any inquiries concerning a paper that has been accepted for publication should be sent directly to the Editorial Department, American Mathematical Society, P. O. Box 6248, Providence, RI 02940-6248.

Editors

This journal is designed particularly for long research papers (and groups of cognate papers) in pure and applied mathematics. Papers intended for publication in the *Memoirs* should be addressed to one of the following editors:

Ordinary differential equations, partial differential equations, and applied mathematics to JOHN MALLET-PARET, Division of Applied Mathematics, Brown University, Providence, RI 02912-9000; e-mail: am438000@brownvm.brown.edu.

Harmonic analysis, representation theory, and Lie theory to ROBERT J. STANTON, Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, OH 43210-1174; electronic mail: stanton@function.mps.ohio-state.edu.

Ergodic theory, dynamical systems, and abstract analysis to DANIEL J. RUDOLPH, Department of Mathematics, University of Maryland, College Park, MD 20742; e-mail: djr@math.umd.edu.

Real and harmonic analysis to DAVID JERISON, Department of Mathematics, MIT, Rm 2–180, Cambridge, MA 02139; e-mail: jerison@math.mit.edu.

Algebra and algebraic geometry to EFIM ZELMANOV, Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53706-1388; e-mail: zelmanov@math.wisc.edu

Algebraic topology and differential topology to MARK MAHOWALD, Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208-2730; e-mail: mark@math.nwu.edu.

Global analysis and differential geometry to ROBERT L. BRYANT, Department of Mathematics, Duke University, Durham, NC 27706-7706; e-mail: bryant@math.duke.edu.

Probability and statistics to RICHARD DURRETT, Department of Mathematics, Cornell University, White Hall, Ithaca, NY 14853-7901; e-mail: rtd@cornella.cit.cornell.edu.

Combinatorics and Lie theory to PHILIP J. HANLON, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1003; e-mail: phil.hanlon@math.lsa.umich.edu.

Logic and universal algebra to GREGORY L. CHERLIN, Department of Mathematics, Rutgers University, Hill Center, Busch Campus, New Brunswick, NJ 08903; e-mail: cherlin@math.rutgers.edu.

Algebraic number theory, analytic number theory, and automorphic forms to WEN-CHING WINNIE LI, Department of Mathematics, Pennsylvania State University, University Park, PA 16802-6401; e-mail: wli@math.psu.edu.

Complex analysis and nonlinear partial differential equations to SUN-YUNG A. CHANG, Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90024-1555; e-mail: chang@math.ucla.edu.

All other communications to the editors should be addressed to the Managing Editor, PETER SHALEN, Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60680; e-mail: U10123@uicvm.uic.edu.

Recent Titles in This Series

(Continued from the front of this publication)

(See the AMS catalog for earlier titles)

Prime Ideals in Skew and q-Skew Polynomial Rings

K. R. Goodearl and E. S. Letzter

There has been continued interest in skew polynomial rings and related constructions since Ore's initial studies in the 1930s. New examples not covered by previous analyses have arisen in the current study of quantum groups. The aim of this work is to introduce and develop new techniques for understanding the prime ideals in skew polynomial rings $S = R[y; \tau, \delta]$, for automorphisms τ and τ-derivations δ of a noetherian coefficient ring R. Goodearl and Letzter give particular emphasis to the use of recently developed techniques from the theory of noncommutative noetherian rings. When R is an algebra over a field k on which τ and δ act trivially, a complete description of the prime ideals of S is given under the additional assumption that $\tau^{-1}\delta\tau = q\delta$ for some nonzero $q \in k$. This last hypothesis is an abstraction of behavior found in many quantum algebras, including q-Weyl algebras and coordinate rings of quantum matrices, and specific examples along these lines are considered in detail.

ISBN 0-8218-2583-6

9 780821 825839