J Syst Sci Syst Eng (Dec 2007) 16(4): 385-413
DOI: 10.1007/s11518-007-5058-2
ISSN: 1004-3756 (Paper) 1861-9576 (Online)
CN11-2983/N

A SURVEY OF STACKELBERG DIFFERENTIAL GAME MODELS IN SUPPLY AND MARKETING CHANNELS

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Abstract

Stackelberg differential game models have been used to study sequential decision making in noncooperative games in diverse fields. In this paper, we survey recent applications of Stackelberg differential game models to the supply chain management and marketing channels literatures. A common feature of these applications is the specification of the game structure: a decentralized channel composed of a manufacturer and independent retailers, and a sequential decision procedure with demand and supply dynamics and coordination issues. In supply chain management, Stackelberg differential games have been used to investigate inventory issues, wholesale and retail pricing strategies, and outsourcing in dynamic environments. The underlying demand typically has growth dynamics or seasonal variation. In marketing, Stackelberg differential games have been used to model cooperative advertising programs, store brand and national brand advertising strategies, shelf space allocation, and pricing and advertising decisions. The demand dynamics are usually extensions of the classical advertising capital models or sales-advertising response models. We begin by explaining the Stackelberg differential game solution methodology and then provide a description of the models and results reported in the literature.

Keywords: Stackelberg differential games, supply chain management, marketing channels, open-loop equilibria, feedback policies, channel coordination, optimal control

1. Introduction

The study of differential games was initiated by Isaacs (1965) with applications to warfare and pursuit-evasion problems. Since then, differential game models have been used extensively to study problems of conflict arising in such diverse disciplines as operations management, marketing, and economics. Particularly related to our review are applications of differential games in supply chain management and marketing channels. A number of books and papers have surveyed applications in these areas (e.g., Jørgensen 1982, Feichtinger et al. 1994, Dockner et al. 2000, Erickson 1992, 1995, 1997). However, all of these surveyed applications consist of duopoly and oligopoly situations in which the decisions are made simultaneously, and the objective is to obtain Nash equilibrium solutions.

More recently, differential games have been formulated to study the hierarchical or sequential decision making situations that exist in supply chains and marketing channels, and for which a reasonable solution concept is that of Stackelberg equilibrium (Stackelberg 1952). Stackelberg differential game (SDG) models have been used to study conflicts and coordination issues associated with inventory and production policies, outsourcing, capacity and shelf space allocation decisions, dynamic competitive advertising strategies, and pricing for new products. It is these studies that we shall survey in this paper.

It should be noted at the outset that the literatures on supply chain and marketing channels management are closely related, since both deal with physical delivery of the product from suppliers to end users through intermediaries and in the most efficient manner. However, the supply chain literature has been more concerned with production quantity and inventory decisions, and demand uncertainty, whereas marketing papers have looked at pricing and promotions decisions, customer heterogeneity, and brand positioning. Competition and channel coordination are important issues in both. Since these topics are rapidly converging, we will therefore provide a combined overview in this paper.

In supply chain management, Stackelberg differential games have been applied to the study of topics such as inventory and production issues, wholesale and retail pricing strategies, and outsourcing in dynamic environments. The underlying demand may have growth dynamics or seasonal variation. In the marketing area, we discuss applications such as cooperative advertising programs, shelf space allocation, and price and advertising decisions. The demand dynamics are usually advertising capital models (e.g., Nerlove and Arrow 1962) or sales-advertising response models (e.g., Vidale-Wolfe 1957 and Sethi 1983). In the former, advertising is considered as an investment in the stock of goodwill and, in the latter, sales-advertising response is specified as a direct relation between the rate of change in sales and advertising.

The fact that the papers that have applied SDG models to supply and marketing channels problems are relatively few in number and recent, suggests that the literature is still at a formative stage. Although, over the last two decades, supply chain management as a whole has garnered much research attention, most studies of strategic interactions between the channel members are based on the static newsvendor framework. By static, we mean that although these models may have two stages, one where the leader moves and the second where the follower moves, thereby modeling sequential decision making in the channel, they overlook strategic issues that arise when the firms interact with each other repeatedly over time and make decisions in a dynamic fashion. We therefore hope that this survey, by focusing on dynamic interactions between the channel members, generates further interest in this emerging area of research.

We shall limit the scope of our survey to applications using differential games, i.e., where the strategies are in continuous time with a finite or infinite horizon, and where the Stackelberg equilibrium is the solution concept. Thus, we exclude from its purview dynamic games played in discrete time. We also do not review applications in economics, for which we refer the interested readers to Bagchi (1984).

This paper is organized as follows. In Section 2, we introduce the concepts and methodology of Stackelberg differential games. In Section 3, we review SDG models in the area of supply chain management. In Section 4, we discuss their applications to marketing channels. Section 5 concludes the paper. Table 1 summarizes the notations used in the paper and Table 2 summarizes the descriptions of the models surveyed in the paper.

Table 1 Notations

*	Optimal/ equilibrium levels	$\hat{c}_i$	Unit advertising cost
i	Denotes the i^{th} player	f	Production function, function defining $\dot{x}$
$A, \hat{A}$	Advertising level	h_i	Unit inventory/ backlog cost
$B, \hat{B}$	Local advertising	h_i^+	Unit inventory holding cost
C_i	Cost function of production or of advertising	h_i^-	Unit backlog cost
D	Demand function, Revenue rate	m_i	Margin
G, G_i	Goodwill	p, p_i	Retail price
$H_i, \overline{H}_i$	Current-value Hamiltonians	q	Manufacturer's share of revenue
I_i	Inventory level	r, r_1, r_2	Effectiveness of advertising
J_i	Objective function	t_1^d, t_2^d	Time paremeters describing periodic demand
K	Infrastructure captital	t_s, t_f	Start and end fo promotional period
L_i	Labor force	u	Advertising expenditure
M	Market size	u^i	Control variable
N	Number of firms	u^{2^0}	Optimal response of player 2
Q_i	Production rate, Processing rate	w	Wholesale price
$\overline{Q}_i$	Capacity limit	x	State variable, Sales rate
S	Shelf space	α	External market influence
S_i	Salvage value, Unit salvage value	$\alpha_1, \alpha_2, \alpha_3$	Demand parameters
T	Planning horizon	β	Internal market influence
T_2	Optimal response function of player 2	δ	Decay rate
V_i	Value function	$\theta, \hat{\theta}$	Manufacturer's share of the retailer's advertising cost
X	Cumulative sales	λ_i	Adjoint variable, Shadow price
a	Market potential	π_i	Instantaneous profit rate
a_i, b_i, d, e_i	Problem parameters	ρ, ρ_i	Discount rate
a_l, a_s, b_l, b_s	Advertising effectiveness	ϕ, ψ	Adjoint variable, Shadow price
b	Price sensitivity	ω_i	Coefficient of incentive strategy
c_i	Unit production cost, Unit advertising cost	Ω_i	Feasible set of controls

Table 2 Summary of model description

Paper	Dynamics	L's decisions	F's decisions	Solution[1]
Breton, Jarrar, and Zaccour (2006)	Lanchester type	Ad. effort	Ad. effort	FSE
Chintagunta and Jain (1992)	NA dynamics	Ad. effort	Ad. effort	OLNE
Desai (1992)	Seasonal	Production rate, Price	Price	OLSE
Desai (1996)	Seasonal	Production rate, Price	Price	OLNE, OLSE
Eliashberg and Steinberg (1987)	Seasonal	Production rate, Price	Price	OLSE
Gutierrez and He (2007)	Bass type	Price	Price	OLSE
He, Prasad, and Sethi (2007)	Sethi 1983	Participation rate, Price	Ad. Effort, Price	FSE
He and Sethi (2008)	Bass type	Price	Price	OLSE
Jain and Chintagunta (2002)	NA dynamics	Ad. Effort, Price	Ad. Effort, Price	OLNE, FNE
JSZ (2000)[2]	NA dynamics	Participation rate, Ad. Effort	Ad. effort	FNE, FSE
JSZ (2001)[2]	NA dynamics	Ad. Effort, Price	Ad. Effort, Price	FNE, FSE
JTZ (2003)[3]	NA dynamics	Participation rate, Ad. Effort	Ad. Effort, Price	FSE
JTZ (2006)[3]	NA dynamics	Ad. effort	Ad. effort	FSE
Karray and Zaccour (2005)	NA dynamics	Ad. Effort, (coop), Price	Ad. Effort, Price	FSE
Kogan and Tapiero (2006)	General	Price	Price	OLNE, OLSE
Kogan and Tapiero (2007a)	General	Price, Production rate	Price	OLSE
Kogan and Tapiero (2007b)	General	Processing rate	Production rate	OLSE
Kogan and Tapiero (2007c)	General	Price	Order quantity	OLSE
Kogan and Tapiero (2007d)	General	Price	Order quantity	OLSE
Kogan and Tapiero (2007e)	General	Price	Price	OLSE
Martin-Herran and Taboubi (2005)	NA dynamics	Ad. Effort, Incentive	Shelf-space	FNE, FSE

[1]The symbol OLNE = Open-loop Nash equilibrium, FNE = Feedback Nash equilibrium, OLSE = Open-loop Stackelberg equilibrium, and FSE = Feedback Stackelberg equilibrium.
[2]JSZ is the abbreviation for Jørgensen, Sigue, and Zacccour.
[3]JTZ is the abbreviation for Jørgensen, Taboubi, and Zaccour.

2. Basics of the Stackelberg Differential Game

A differential game has the following structure: 1) The state of the dynamic system at any time t is characterized by a set of variables called the state variables. Typical state variables are market share, sales, inventory, goodwill, etc. 2) There are decision variables called controls that are chosen by the game players: For example, order quantity, pricing, advertising, and shelf-space decisions etc. 3) The evolution of the state variables over time is described by a set of differential equations involving both state and control variables. 4) Each player has an objective function (e.g., the present value of the profit stream over time) that it seeks to maximize by its choice of decisions.

There are several different concepts of equilibria in differential games. In this paper, we are concerned with SDG models. Like the static Stackelberg game, due originally to Stackelberg (1952), an SDG is hierarchical (or sequential) in nature. We consider an SDG involving two players labeled as 1 (the leader) and 2 (the follower) making decisions over a finite horizon T. We shall also remark on what needs to be done when $T=\infty$. Let $x(t)\in R^n$ denote the vector of state variables, $u^1\in\Omega_1\subset R^{m_1}$ denote the control vector of the leader, and $u^2\in\Omega_2\subset R^{m_2}$ denote the control vector of the follower. The sequence of the play is as follows. The leader announces its control path $u^1(\cdot)$. Then, the follower decides its optimal control path $u^2(\cdot)$ in response to the announced $u^1(\cdot)$. A Stackelberg equilibrium is obtained when the leader solves its problem taking into account the optimal response of the follower.

Further specification is required in terms of the information structure of the players in order to precisely define a Stackelberg equilibrium. In this section, we shall consider two different information structures. The first one, called the *open-loop* information structure, assumes that the players must formulate their decisions at time t only with the knowledge of the initial condition of the state at time zero. The second one, called the *feedback* information structure, assumes that the players use their knowledge of the current state at time t in order to formulate their decisions at time t. We give the procedure for obtaining an open-loop solution in Section 2.1. The feedback solution procedure will be described in Section 2.2.

2.1 Open-Loop Solution

The solution procedure is that of backward induction. That is, like the leader, we first anticipate the follower's best response to any announced policy of the leader. The anticipation is derived from solving the optimization problem of the follower given the leader's policy. We then substitute the follower's response function into the leader's problem and solve for the leader's optimal policy. This policy of the leader, together with the retailer's best response to that policy, constitutes a Stackelberg equilibrium of the SDG. For the open-loop solution, given an announced decision $u^1(\cdot)=(u^1(t),0\le t\le T)$ by the leader, the follower's optimal control problem is

$$\max_{u^2(\cdot)}[J_2(t,x_0,u^2(\cdot);u^1(\cdot)) = \int_0^T e^{-\rho_2 t}\pi_2(t,x(t),u^1(t),u^2(t))dt + e^{-\rho_2 T}S_2(x(T))]$$

subject to the state equation

$$\dot{x}(t)=f(t,x(t),u^1(t),u^2(t)),\ x(0)=x_0, \quad (1)$$

where $\rho_2>0$ is the follower's discount rate, π_2 is its instantaneous profit rate function, $S_2(x(T))$ is the salvage value, and x_0 is the initial state. Recall that the open-loop information structure for both players means that the controls $u^1(t)$ and $u^2(t)$ at time t depend only on the initial state x_0. We assume that $f(t,\cdot,u^1,u^2)$, $\pi_2(t,\cdot,u^1,u^2)$, and $S_2(\cdot)$ are continuously differentiable on $R^n, \forall t \in [0,T]$.

Let $u^{2^0}(\cdot)$ denote the follower's optimal response to the announced control $u^1(\cdot)$ by the leader. The optimal response of the follower must be made at time zero and will, in general, depend on the entire announced control path $u^1(\cdot)$ by the leader and the initial state x_0. Clearly, given $u^1(\cdot)$, which can be expressed as a function of x_0 and t, the follower's problem reduces to a standard optimal control problem. This problem is solved by using the maximum principle (see e.g., Sethi and Thompson 2000), which requires introducing an adjoint variable or a shadow price that decouples the optimization problem over the interval $[0,T]$ into a sequence of static optimization problems, one for each instant $t \in [0,T]$. Note also that the shadow price at time t denotes the per unit value of a change in the state $x(t)$ at time t, and the change $\dot{x}(t)$ in the state at time t requires only the time t control $u^1(t)$ from the leader's announced control trajectory $u^1(\cdot)$. Nevertheless, we shall see later that the follower's optimal response $u^{2^0}(t)$ at any given time t will depend on the entire $u^1(\cdot)$ via the solution of the two-boundary problem resulting from the application of the maximum principle. Let us now proceed to solve the follower's problem given $u^1(\cdot)$.

The follower's current-value Hamiltonian is

$$\begin{aligned} &H_2(t,x,\lambda,u^1,u^2) \\ &\quad = \pi_2(t,x,u^1,u^2) + \lambda' f(t,x,u^1,u^2), \end{aligned} \tag{2}$$

where $\lambda \in R^n$ is the (column) vector of the shadow prices associated with the state variable x, the (row) vector λ' denotes its transpose, and it satisfies the adjoint equation

$$\begin{aligned} &\dot{\lambda}' = \rho\lambda' - \frac{\partial}{\partial x} H_2(t,x,\lambda,u^1,u^2), \\ &\lambda'(T) = \frac{\partial}{\partial x} S_2(x(T)). \end{aligned} \tag{3}$$

Here we have suppressed the argument t as is standard in the control theory literature, and we will do this whenever convenient and when there arises no confusion in doing so. Note that the gradient $\partial H_2 / \partial x$ is a row vector by convention.

The necessary condition for the follower's optimal response denoted by u^{2^0} is that it maximizes H_2 with respect to $u^2 \in \Omega_2$, i.e.,

$$u^{2^0}(t) = \arg\max_{u^2 \in \Omega_2} H_2(t,\lambda(t),x(t),u^1(t),u^2). \tag{4}$$

Clearly, it is a function that depends on t, x, λ, and u^1. A usual procedure is to obtain an optimal response function $T_2(t,x,\lambda,u^1)$ so that $u^{2^0}(t) = T_2(t,x(t),\lambda(t),u^1(t))$. In the case when $u^{2^0}(t)$ is an interior solution, T_2 satisfies the first-order condition

$$\frac{\partial}{\partial u^2} H_2(t,x,\lambda,u^1,T_2) = 0. \tag{5}$$

If the Hamiltonian H_2 defined in (2) is jointly concave in the variables x and u^2 for any given u^1, then the condition (5) is also sufficient for the optimality of the response T_2 for the given u^1.

It is now possible to complete our discussion of the dependence of the follower's optimal response T_2 on $u^1(\cdot)$ and x_0. Note that the time t response of the follower is given by

$$u^{2^0}(t) = T_2(t,x(t),\lambda(t),u^1(t)). \tag{6}$$

But the resulting two-point boundary value problem given by (1), (2) with $T_2(t,x,\lambda,u^1)$ in place of u^2 makes it clear that the shadow price $\lambda(t)$ at time t depends on the entire $u^1(\cdot)$ and x_0. Thus, we have the dependence of $u^{2^0}(t)$, and therefore of $u^{2^0}(\cdot)$, on $u^1(\cdot)$ and x_0.

If we can explicitly solve for the optimal response $T_2(t,x,\lambda,u^1)$, then we can specify the leader's problem to be

$$\begin{aligned}\max_{u^1(\cdot)} [J_1(t,x,u^1(\cdot)) &= \int_0^T e^{-\rho_1 t}\pi_1(t,x,u^1,T_2(t,x,\lambda,u^1))dt \\ &+e^{-\rho_1 T}S_1(x(T))],\end{aligned} \tag{7}$$

$$\dot{x} = f(t,x,u^1,T_2(t,x,\lambda,u^1)),\ x(0)=x_0, \tag{8}$$

$$\begin{aligned}&\dot{\lambda}' = \rho_2\lambda' - \frac{\partial}{\partial x}H_2(t,x,\lambda,u^1,T_2(t,x,\lambda,u^1)),\\ &\lambda'(T) = \frac{\partial}{\partial x}S_2(x(T)),\end{aligned} \tag{9}$$

where $\rho_1 > 0$ is the leader's discount rate and the differential equations in (8) and (9) are obtained by substituting the follower's best response $T_2(t,x,\lambda,u^1)$ for u^2 in the state equation (1) and the adjoint equation (3), respectively. We formulate the leader's Hamiltonian

$$\begin{aligned}&\bar{H}_1(t,x,\lambda,\phi,\psi,u^1)\\ &=\pi_1(t,x,u^1,T_2(t,x,\lambda,u^1))\\ &\quad+\phi' f(t,x,u^1,T_2(t,x,\lambda,u^1))\\ &\quad+\psi'[\rho\lambda-(\frac{\partial}{\partial x}H_2(t,x,\lambda,u^1,T_2(t,x,\lambda,u^1)))'],\end{aligned} \tag{10}$$

where $\phi \in R^n$ and $\psi \in R^n$ are the vectors of the shadow prices associated with x and λ, respectively, and they satisfy the adjoint equations

$$\begin{aligned}\dot{\phi}' &= \rho_1\phi' - \frac{\partial}{\partial x}\bar{H}_1(t,x,\lambda,\phi,\psi,u^1,T_2(t,x,\lambda,u^1))\\ &= \rho_1\phi' - \frac{\partial}{\partial x}\pi_1(t,x,u^1,T_2(t,x,\lambda,u^1))\\ &\quad -\phi'\frac{\partial}{\partial x}f(t,x,u^1,T_2(t,x,\lambda,u^1))\\ &\quad +\psi'\frac{\partial^2}{\partial x^2}H_2(t,x,\lambda,u^1,T_2(t,x,\lambda,u^1)),\\ &\phi'(T) = \frac{\partial}{\partial x}S_1(x(T)) - \frac{\partial^2}{\partial x^2}S_2(x(T))\psi(T),\end{aligned} \tag{11}$$

$$\begin{aligned}\dot{\psi}' &= \rho_1\psi' - \frac{\partial}{\partial \lambda}\bar{H}_1(t,x,\lambda,\phi,\psi,u^1,T_2(t,x,\lambda,u^1))\\ &= \rho_1\psi' - \frac{\partial}{\partial \lambda}\pi_1(t,x,u^1,T_2(t,x,\lambda,u^1))\\ &\quad -\phi'\frac{\partial}{\partial \lambda}f(t,x,u^1,T_2(t,x,u^1,\lambda))\\ &\quad +\psi'\frac{\partial^2}{\partial x^2}H_2(t,x,\lambda,u^1,T_2(t,x,\lambda,u^1)),\\ &\psi(0)=0.\end{aligned} \tag{12}$$

In (10), we are using the notation $\bar{H}_1$ for the Hamiltonian to recognize that it uses the optimal response function of the follower for u^2 in its definition. In (11) and (12), we have used the envelope theorem (see, e.g., Derzko et al. 1984) in taking the derivative $\partial\bar{H}_1/\partial x$.

Finally, it is important to remark that unlike in Nash differential game, the adjoint equation and the transversality condition $\phi'(T)$ in (11) have a second-order term each on account of the facts that the adjoint equation (9) is a state equation in the leader's problem and that it has a state-dependent terminal condition, also in (9), arising from the sequential game structure of SDG. The necessary optimality condition for the leader's optimal control $u^{1*}(t)$ is that it maximizes $\overline{H}_1$ with respect to $u^1 \in \Omega_1$, i.e.,

$$u^{1*}(t) = \arg\max_{u^1 \in \Omega_1} \overline{H}_1(t, x(t), \lambda(t), \phi(t), \psi(t), u^1). \tag{13}$$

In the case when $u^1(t)$ is an interior solution, this is equivalent to the condition

$$\frac{\partial}{\partial u^1} \overline{H}_1(t, x(t), \lambda(t), \phi(t), \psi(t), u^{1*}(t)) = 0. \tag{14}$$

Also if $\overline{H}_1$ is jointly concave in the variables x, λ, and u^1, then (14) is also sufficient for optimality of $u^{1*}(t)$. Once we have obtained $u^{1*}(t)$ from (14), we can substitute it into T_2 to obtain the optimal control $u^{2*}(t)$ of the follower.

In many cases, it may not be possible to have an explicit expression for the response function T_2. Then, the alternate method would be to impose (5) as an equality constraint on the leader's optimization problem (7)-(9). It is easy to redo the above procedure in these cases and derive the following result. Note that this will require a Lagrange multiplier function $\mu(\cdot) \in R^n$ associated with the constraint (5).

Theorem 1 *For the SDG under consideration in this subsection, assume that*

i) $f(t, \cdot, u^1, u^2)$ *is twice continuously differentiable on* R^n, $\forall t \in [0, T]$.

ii) $\pi^2(t, \cdot, u^1, u^2)$ *and* $S_2(\cdot)$ *are twice continuously differentiable on* R^n, $\forall t \in [0, T]$.

iii) $\pi^1(t, \cdot, u^1, u^2)$ *and* $S_1(\cdot)$ *are twice continuously differentiable on* R^n, $\forall t \in [0, T]$.

iv) $H_2(t, x, u^1, \cdot)$ *is continuously differentiable and strictly convex on* Ω_2.

v) $f(t, x, \cdot, u^2)$, $\pi_i(t, x, \cdot, u^2)$, $i = 1, 2$, *are continuously differentiable on* Ω_1.

If $u^{i*}(t), 0 \le t \le T, i = 1, 2$, *provide an open-loop Stackelberg solution with* 1 *as the leader and* 2 *as the follower,* $u^{1*}(t)$ *is interior to the set* Ω_i, *and* $\{x^*(t), 0 \le t \le T\}$ *denotes the corresponding state trajectory, then there exist continuously differentiable functions* $\lambda(\cdot)$, $\phi(\cdot)$, $\psi(\cdot)$ *:* $[0, T] \to R^n$ *and a continuous function* $\mu(\cdot) : [0, T] \to R^n$, *such that the following relations are satisfied:*

$$\dot{x}^* = f(t, u^{1*}, u^{2*}),\ x^*(0) = x_0,$$
$$\dot{\lambda}' = \rho_2 \lambda' - \frac{\partial}{\partial x} H_2(t, x^*, \lambda, u^{1*}, u^{2*}),$$
$$\lambda'(T) = \frac{\partial}{\partial x} S_2(x^*(T)),$$
$$\dot{\phi}' = \rho_1 \phi' - \frac{\partial}{\partial x} H_1(t, x^*, \lambda, \phi, \psi, \mu, u^{1*}, u^{2*}),$$
$$\phi'(T) = \frac{\partial}{\partial x} S_1(x^*(T)) - \frac{\partial^2}{\partial x^2} S_2(x^*(T)) \psi(T),$$
$$\dot{\psi}' = \rho_1 \psi' - \frac{\partial}{\partial \lambda_2} H_1(t, x^*, \lambda, \phi, \psi, \mu, u^{1*}, u^{2*}),$$
$$\psi(0) = 0,$$
$$\frac{\partial}{\partial u^1} H_1(t, x^*, \lambda, \phi, \psi, \mu, u^{1*}, u^{2*}) = 0,$$
$$\frac{\partial}{\partial u^2} H_1(t, x^*, \lambda, \phi, \psi, \mu, u^{1*}, u^{2*}) = 0,$$
$$\frac{\partial}{\partial u^2} H_2(t, x^*, \lambda, u^{1*}, u^{2*}) = 0,$$

where H_2 *is defined by (2) and* H_1 *is defined by*

$$\begin{aligned} H_1(t, x, \lambda, \phi, \psi, \mu, u^1, u^2) &= \pi_1(t, x, u^1, u^2(t, x, \lambda, u^1)) \\ &\quad + \phi' f(t, x, u^1, u^2(t, x, \lambda, u^1)) \\ &\quad + \psi'[\rho\lambda - (\frac{\partial}{\partial x} H_2(t, x, \lambda, u^1, u^2))'] \\ &\quad + \frac{\partial}{\partial u^2} H_2(t, x, \lambda, u^1, u^2)\mu. \end{aligned}$$

We mention briefly that in infinite horizon problems, i.e., when $T = \infty$, it is usually assumed that the salvage values $S_1 = S_2 = 0$. Then the practice is to replace the terminal conditions on λ, ϕ and ψ by $\lim_{T\to\infty} e^{-\rho_2 T}\lambda(T) = 0$, $\lim_{T\to\infty} e^{-\rho_1 T}\phi(T) = 0$, and $\lim_{T\to\infty} e^{-\rho_1 T}\psi(T) = 0$, respectively. We should note that these conditions are not necessary, but are sufficient when coupled with appropriate concavity conditions.

It is known that in general open-loop Stackelberg equilibria are time inconsistent. This means that given an opportunity to revise its strategy at any future time after the initial time, the leader would benefit by choosing another strategy than the one it chose at the initial time. Thus, an open-loop Stackelberg equilibrium only makes sense if the leader can credibly pre-commit at time zero its strategy for the entire duration of the game. In many management settings such commitment is not observed in practice. Nevertheless, there is a considerable literature dealing with open-loop Stackelberg equilibria, on account of their mathematical tractability.

2.2 Feedback Solution

In the preceding section, we have described an open-loop Stackelberg solution concept to solve the SDG. We shall now develop the procedure to obtain a feedback Stackelberg solution. While open-loop solutions are said to be static in the sense that decisions can be derived at the initial time, without regard to the state variable evolution beyond that time, feedback equilibrium strategies at any time t are functions of the values of the state variables at that time. They are perfect state-space equilibria because the necessary optimality conditions are required to hold for all values of the state variables, and not just the values that lie on the optimal state-space trajectories. Therefore, the solutions obtained continue to remain optimal at each instant of the time after the game has begun. Thus, they are known as *subgame* perfect because they do not depend on the initial conditions.

Since differential games evolve in continuous time, it is worthwhile to elaborate on the physical interpretation of their sequential move order. A feedback Stackelberg solution to an SDG can be conceptualized as the limit of the feedback Stackelberg solutions of a sequence of discrete-time dynamic games, each one obtained by time discretization of the original differential game, with $(k+1)^{st}$ game in the sequence corresponding to a finer discretization than the k^{th} one. Each of these games can be termed as a "sampled-state" game, as the only time discretization is in the information set comprising the state vector. Furthermore, as explained in Basar and Olsder (1999), the feedback Stackelberg solution of each state-sampled game can be obtained by solving a sequence of approximately defined open-loop Stackelberg games. Consequently, the limiting solution involves solutions of a sequence of open-loop Stackelberg games, each one defined on an infinitesimally small sub-interval, which means that we now must obtain Stackelberg solutions based on the incremental profit at each time *t*.

Let $V_i(t,x)$ denote the feedback Stackelberg profits-to-go of player i in current-value term at time *t*, at state x. For any given policy $u^1(t,x)$ by the leader, the follower's HJB equation is

$$\rho_2 V_2 - \frac{\partial V_2(t,x)}{\partial t} = \max_{u_2 \in \Omega_2} \frac{\partial V_2(t,x)}{\partial x} f(t,x,u^1(t,x),u^2) + \pi_2(t,x,u^1(t,x),u^2), \quad (15)$$

with $V_2(T,x) = S_2(x)$. The follower's instantaneous reaction function is given by

$$T_2(t,x,u^1,\frac{\partial V_2}{\partial x}) = \arg\max_{u^2 \in \Omega_2} [\frac{\partial V_2}{\partial x} f(t,x,u^1,u^2) + \pi_2(t,x,u^1,u^2)]. \quad (16)$$

Given this, the leader's HJB equation can be written as

$$\rho_1 V_1 - \frac{\partial V_1(t,x)}{\partial t} = \max_{u^1 \in \Omega_1} [\pi_1(t,x,u^1,T^2(t,x,u^1,\frac{\partial V_2}{\partial x})) + \frac{\partial V_1(t,x)}{\partial x} f(t,x,u^1,T_2(t,x,u^1,\frac{\partial V_2}{\partial x}))], \quad (17)$$

with $V_1(T,x) = S_1(x)$. Maximizing the right-hand side of (17) yields the optimal feedback pricing for the leader, i.e.,

$$u^{1*}(t,x) = \arg\max_{u^1 \in \Omega_1} [\pi_1(t,x,u^1,T^2(t,x,u^1,\frac{\partial V_2}{\partial x})) + \frac{\partial V_1}{\partial x} f(t,x,u^1,T_2(t,x,u^1,\frac{\partial V_2}{\partial x}))]. \quad (18)$$

Then, the optimal control policy for the follower is obtained by substituting $u^{1*}(t,x)$ for u^1 in (16), i.e.,

$$u^{2*}(t,x) = T_2(t,x,u^{1*}(t,x),\frac{\partial V_2}{\partial x}). \quad (19)$$

Substituting (18) and (19) into (15) and (17), we can rewrite the HJB equation for the game as the system

$$\rho_1 V_1 - \frac{\partial V_1(t,x)}{\partial t} = \pi_1(t,x,u^{1*}(t,x),u^{2*}(t,x)) + \frac{\partial V_1(t,x)}{\partial x} f(t,x,u^{1*}(t,x),u^{2*}(t,x)),$$

$$\rho_2 V_2 - \frac{\partial V_2(t,x)}{\partial t} = \pi_2(t,x,u^{1*}(t,x),u^{2*}(t,x)) + \frac{\partial V_2(t,x)}{\partial x} f(t,x,u^{1*}(t,x),u^{2*}(t,x)),$$

$V_i(T,x) = S_i(x), i = 1,2.$

This system can be solved to obtain $V_1(t,x)$ and $V_2(t,x)$, and the feedback Stackelberg strategies $u^{1*}(t,x)$ and $u^{2*}(t,x)$.

3. Supply Chain Management Applications

In this section we shall review applications in the supply chain management area. The leader is the supplier or manufacturer who decides on variables such as the wholesale price and/or production rate, whereas the retailer is the follower and its decision variables can be, for example, retail price and shelf-space allocation.

3.1 Eliashberg and Steinberg (1987): Pricing and Production with Constant Wholesale Price

The paper by Eliashberg and Steinberg (1987) considers a decentralized assembly system composed of a manufacturer and a distributor. The distributor processes the product and its demand has seasonal fluctuations. As an aside, Pekelman (1974) used a general time-varying demand function $D(t) = a(t) - b(t)p(t)$, where $a(t)$ is the market potential, $b(t)$ is the coefficient of price sensitivity, and $p(t)$ is the distributor's price. In Eliashberg and Steinberg

(1987), it is assumed that $b(t)$ is constant and $a(t)$ has the form $a(t) = -\alpha_1 t^2 + \alpha_2 t + \alpha_3$, $0 \le t \le T$, where α_1, α_2 and α_3 are positive parameters and $T = \alpha_2/\alpha_1$.

The manufacturer and the distributor play an SDG with the manufacturer as the leader and the retailer as the follower. The distributor determines its processing, pricing, and inventory policies. The manufacturer decides its production rate and a constant wholesale price. The distributor's problem is

$$\max_{I_R(\cdot),p(\cdot)} \{J_R = \int_0^T [p(t)D(t) - wQ_R(t) - C_R(Q_R(t)) - h_R I_R(t)]dt\},$$

$$\dot{I}_R(t) = Q_R(t) - D(t),$$

$$I_R(0) = I_R(T) = 0,\ I_R(t) \ge 0,$$

where $Q_R(t)$ is its processing rate, $C_R(Q_R)$ is its processing cost function, h_R is its inventory holding cost per unit, and $I_R(t)$ is its inventory level. Similar to Pekelman (1974), a linear holding cost function is used. It is assumed that the processing cost function is increasing and strictly convex.

The paper shows that the distributor follows a two-part processing strategy. During the first part of the processing schedule, it processes at a constant increasing rate. This policy builds up inventory initially and then draws down inventory until it reaches zero at a time t_R^*. During the second part, which begins at the stockless point t_R^*, it processes at precisely the market demand rate. Pricing also follows a two-part strategy. The price charged by the distributor is first increasing at a decreasing rate and then decreasing at an increasing rate. The inventory builds up for a while and then reaches zero. From then on, the distributor processes just enough to meet demand.

An intuitive interpretation is as follows. The distributor, facing a seasonal demand that increases and then decreases, can smooth out processing operations by carrying inventory initially due to the assumption of convex processing cost. In other words, if it did not carry any inventory throughout the entire horizon, it could incur higher costs due to the convexity of the processing cost function.

Turning to the manufacturer's problem, let Q_M, I_M, h_M, and $C_M(Q_M)$ denote its production rate, inventory level, inventory holding cost per unit, and the production cost function, respectively. The manufacturer's problem is given by

$$\max_{Q_M(\cdot),w} \{J_M = \int_0^T [(w - c_M)Q_R(w,t) - C_M(Q_M(t)) - h_M I_M(t)]dt\},$$

$$\dot{I}_M(t) = Q_M(t) - Q_R(w,t),$$

$$I_M(0) = I_M(T) = 0,$$

$$Q_M(t) \ge Q_R(t) \ge 0, I_M(t) \ge 0,$$

where c_M is its cost per unit transferred to the distributor and $Q_R(w,t)$ is the best response of the distributor at time t given w. It is assumed that $w > c_M$ and that the production cost function is quadratic.

The manufacturer's policies are characterized as following a two-part production policy. During the first part, it produces at a constantly increasing rate. During the second part, which begins at the manufacturer's stockless point t_M^*, it produces at exactly the distributor's processing rate. In general, if the manufacturer's inventory holding cost per unit is sufficiently low and the distributor's processing efficiency and inventory holding cost per unit are high, then the manufacturer can also smooth out its operations.

3.2 Desai (1992): Pricing and Production with Retailer Carrying No Inventory

Desai (1992) analyzes the production and pricing decisions in a channel composed of a manufacturer that produces the goods and sells them through a retailer who serves the final market. This paper differs from Eliashberg and Steinberg (1987) in three ways. First, it allows the manufacturer to change the wholesale price over time. Second, the retailer is not allowed to carry inventory. Third, a quadratic holding cost function is assumed. The retailer faces a price-dependent seasonal demand. As in Eliashberg and Steinberg (1987), Desai (1992) uses a time varying demand function $D(t) = a(t) - bp(t)$, where $a(t) = \alpha_1 + \alpha_2 \sin(\pi t/T)$ and T is the duration of the season. The retailer (the follower) decides on the pricing policy, and its problem is given by

$$\max_{p(\cdot)} [J_R = \int_0^T [p(t) - w(t)]D(t)dt],$$

where $p(t)$ is the retail price and $w(t)$ is the wholesale price charged by the manufacturer. Note that Eliashberg and Steinberg assumed the wholesale price to be constant. Here, the manufacturer (the leader) decides on the wholesale price $w(t)$ and its production rate $Q_M(t)$. Its problem is

$$\max_{w(\cdot), Q_M(\cdot)} [J_M = \int_0^T [w(t)D(t) - c_M Q_M^2(t) - h_M I_M^2(t)]dt + S_M I_M(T)],$$

where h_M is the inventory holding cost, c_M is the per unit production cost, $I_M(t)$ is the inventory level at time t, and S_M is the per unit salvage value. The inventory dynamics is

$$\dot{I}_M(t) = \frac{dI_M}{dt} = Q_M(t) - D(t), \quad I_M(0) = I_{M0},$$

where $Q_M(t) \geq 0$ and $D(t) \geq 0$.

The result is that once the production rate becomes positive, it does not become zero again, which implies production smoothing. However, none of the gains of production smoothing are passed on to the retailer. The optimal production rate and the inventory policy are a linear combination of the nominal demand rate, the peak demand factor, the salvage value, and the initial inventory. In the scenario where the retailing operation does not require an effort, the pricing policies of the manufacturer and the retailer and the production policy of the manufacturer have the synergistic effect that an increase in the manufacturer's price or production rate or the retailer's price leads to an increase in the rate of change of inventory. However, in the scenario where the retailing operation does benefit from the effort, the retailer's pricing policy may not necessarily be synergistic with the other policies.

3.3 Desai (1996): Pricing and Production with Further Processing by the Retailer

This paper extends Desai (1992) by requiring the retailer to process the goods received from the manufacturer before they can be sold in the final market. The manufacturer makes the production and pricing decisions while the retailer decides on the processing rate and pricing policies. The paper considers optimal policies under three types of contracts: contracts under which the manufacturer charges a constant price throughout a season, contracts under which the retailer processes at a constant rate throughout the season, and contracts under which the manufacturer and retailer cooperate to make decisions jointly. It is shown that the type of contract does not significantly impact the retailer's price. However, the type of contract has an impact on the manufacturer's price and the production rate as well as the retailer's processing rate. If the demand is not highly seasonal, a constant processing rate contract will lead to higher production and processing rates, and a lower manufacturer's price compared to a constant manufacturer's price contract.

3.4 Gutierrez and He (2007): Life-Cycle Channel Coordination

Gutierrez and He (2007) study the intertemporal channel coordination issues in an innovative durable product (IDP) supply chain composed of a manufacturer who produces the IDP and a retailer who serves the final market.

The demand in all the above papers has seasonal variations which are exogenously determined and independent of the retail price. Therefore, there is no interdependence between the demand in different periods. In contrast, the demand in this paper evolves according to a Bass (1969) type diffusion process. Specifically, the demand is affected by the external and internal market influences (i.e., word-of-mouth) as well as the retail price. The word-of-mouth creates an interdependence between the current and future demand.

The manufacturer is the leader and announces the wholesale price trajectory. The retailer follows by deciding on the retail price trajectory. While the manufacturer is far-sighted, i.e., it maximizes its life-cycle profits, two possibilities for the retailer are considered: (1) a far-sighted strategy of maximizing the life-cycle profit, and (2) a myopic strategy of maximizing the instantaneous profit rate at any time t. They address the following research questions: Does the manufacturer prefer the retailer to be far-sighted or myopic? Does the retailer prefer to be far-sighted or myopic? They derive open-loop pricing strategies for both players.

Far-sighted Retailer. When the retailer is far-sighted, for a given wholesale price path $w(\cdot)$, it chooses in response a retail price path $p(\cdot)$ by solving the problem

$$\max_{p(\cdot)} \int_0^T \{[p(t) - w(t) - c_R]\dot{X}(t)\}dt \tag{20}$$

$$\begin{aligned} &\dot{X}(t) = (M - X(t))(\alpha + \beta X(t))(1 - bp(t)), \\ &X(0) = X_0, \end{aligned} \tag{21}$$

where c_R is the per unit selling cost including any opportunity cost, α and β are internal and external influence parameters, b is the price sensitivity parameter, and X_0 is the initial number of adopters. The manufacturer's problem is

$$\max_{w(\cdot)} \int_0^T [w(t) - c_M]\dot{X}(t)dt, \tag{22}$$

$$\begin{aligned} &\dot{X}(t) = \frac{1}{2}F(X(t))\{1 - b[w(t) + c_R - \lambda_R(t)]\}, \\ &X(0) = X_0, \end{aligned} \tag{23}$$

$$\begin{aligned} \dot{\lambda}_R(t) = -\frac{1}{4b}&[-\alpha + \beta M - 2\beta X(t)] \bullet \\ &\{1 - b[w(t) + c_R - \lambda_R(t)]\}^2, \ \lambda_R(T) = 0, \end{aligned} \tag{24}$$

where c_M is the per unit production cost and λ_R is the retailer's shadow price. Note that X and λ_R are the manufacturer's two state variables, and their evolution incorporates the retailer's best response.

Myopic Retailer. For each $t \in [0,T]$, the retailer selects its response $p(t)$ in order to maximize its instantaneous profit rate at time t subject to the state equation (21). Accordingly, we remove the retailer's shadow price λ_R from the retailer's best response and then obtain the manufacturer's optimization problem as (22) and (23) with $\lambda_R(t)$ removed from (23).

The paper finds that the manufacturer does not always find it more profitable for the retailer to be far-sighted, and may sometimes be better off if the retailer is myopic. On the other hand both the manufacturer and the retailer are better off if the retailer is far-sighted when the final market is insufficiently penetrated. However, if the market is close to saturation such as at the end of the planning horizon, the manufacturer will shift its preference and be better off with a myopic retailer, while the retailer prefers to be far-sighted. However, monitoring the retailer sales volume or retail price becomes an implementation necessity when the manufacturer offers a wholesale price contract assuming the retailer is myopic.

3.5 He and Sethi (2008): Pricing and Slotting Decisions

He and Sethi (2008) extend the work of Gutierrez and He (2007) by considering the impact of shelf space allocation (or another promotional device for that matter) on retail demand. Here it is assumed that retail demand is a concave increasing function of the shelf-space of merchandise displayed on the shelf. This is operationalized by introducing a multiplicative term $\sqrt{S(t)}$ to the right-hand side of (21), where $S(t)$ is the shelf space allocated to the product at time t, and by including a linear cost of shelf space in the retailer's objective function (20). The solution is for the equilibrium wholesale and retail pricing and slotting decisions. However, in connection with the strategic profitability foci of the retailer, myopic or far-sighted, the obtained results are similar to those in Gutierrez and He (2007).

3.6 Kogan and Tapiero (2007a): Inventory Game with Endogenous Demand

This paper considers a supply chain consisting of a manufacturer (leader) and a retailer (follower) facing a time-dependent endogenous demand depending on the price set by the retailer. Furthermore, the retailer has a finite processing capacity, which requires consideration of the effect of inventory. Thus, the retailer must decide on the retail price $p(t)$ as well as the order quantity $Q_R(t)$. The manufacturer, on the other hand, has ample capacity and must decide on only the wholesale price $w(t)$. It is assumed that the game is played over a season of length T which includes a short promotional period $[t_s, t_f]$ such as the Christmas time, during which both the demand potential $a(t)$ and the customer price sensitivity $b(t)$ are high. Specifically, the demand $D(t) = a(t) - b(t)p(t)$, where

$$a(t) = \begin{cases} a_1, \, t < t_s \;\; and \;\; t \geq t_f \\ a_2, \, t_s \leq t < t_f, \end{cases}$$

$$b(t)=\begin{cases} b_1,\ t<t_s\ \ and\ \ t\ge t_f \\ b_2,\ t_s\le t<t_f, \end{cases}$$

with $a_2 > a_1$ and $b_2 > b_1$. The manufacturer is assumed to be restricted to setting a constant wholesale price w_1 in the regular periods and $w_2 \le w_1$ in the promotion period. The manufacturer has ample capacity and produces exactly according to the retailer's order to maximize its profit:

$$\max_{w(\cdot)} \int_0^T [w(t)Q_R(t) - c_M Q_R(t)]dt$$

subject to $w(t) \ge c_M$, where c_M is the per unit production cost.

The retailer's selects its order quantity $Q_R(t)$ and the retail price $p(t)$, $0 \le t < T$, by solving the problem:

$$\max_{Q_R(\cdot),p(\cdot)} \int_0^T \{p(t)[a(t)-b(t)p(t)] \\ -c_R Q_R(t) - w(t)Q_R(t) - h(X(t))\}dt,$$

$$\dot{I}_R(t) = Q_R(t) - [a(t) - b(t)p(t)],$$

$$0 \le Q_R(t) \le \bar{Q};\ \ a(t)-b(t)p(t) \ge 0; p(t) \ge 0,$$

where $\bar{Q}$ is the retailer's maximum processing rate.

The optimal solution to the centralized problem as well as the Stackelberg equilibrium is obtained. The analysis requires the system to begin in a steady state at time 0, go to a transient state in response to promotional decisions, and then revert back to the steady state by the end of the season at time *T*. Thus, the solution is meant to be implemented in a rolling horizon fashion.

Under reasonable conditions on the parameters, formulas for the equilibrium values of the regular and promotional wholesale prices for the manufacturer are derived. It is shown that it is beneficial for the retailer to change pricing and processing policies in response to the reduced promotional wholesale price and the increased customer price sensitivity during the promotion. The change is characterized by instantaneous jump upward in quantities ordered and downward in retailer prices at the instant when the promotion period starts and vice versa just when the promotion ends. In fact, the retailer starts lowering its prices sometime before the promotion starts. This causes a greater demand when the promotion period begins, thereby taking advantage of the reduced wholesale price during the promotion. This is accomplished gradually to strike a trade-off between the surplus/backlog cost and the wholesale price over time. Specifically, any reduction in the wholesale price results first in backlog and then in surplus. An opposite scenario takes place on the side when the promotion periods end.

In the symmetric case when unit backlog and surplus costs are equal, the typical equilibrium as shown in Figure 1 is obtained. As can be seen, due to symmetric costs, the transient solution is symmetric with respect to the midpoint of the promotion phase.

Finally, due to inventory dynamics, the traditional two-part tariff does not coordinate the supply chain as it does in the static case. This is because the manufacturer when setting the promotional wholesale price ignores not only the retailer's profit margin from sales, but also the profit margin from handling inventories. However, in the special case where the manufacturer fixes a wholesale price throughout the season, the retailer's problem becomes identical to the centralized problem and the supply chain is coordinated.

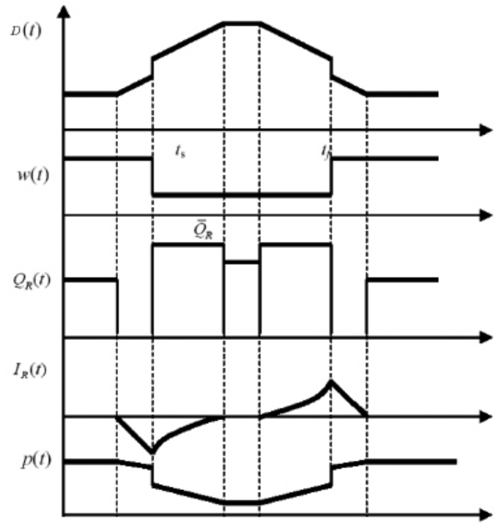

Figure 1 Optimal policies with promotion

3.7 Kogan and Tapiero (2007b): Inventory Game with Exogenous Demand

This game differs from Kogan and Tapiero (2007a) in two ways. First, the demand is no longer price-dependent, and so pricing is not an issue. This simplifies the problem. Second, both the manufacturer and the retailer have limited capacities in contrast to the previous game in which only the retailer has a limited capacity. The objective of each member is to decide on the production/processing rates that minimize inventory costs over an infinite planning horizon. This paper assumes that the retailer is the leader who decides on the order quantity. Let h_R^+, h_M^+, h_R^-, h_M^- be the retailer's inventory holding cost, the manufacturer's inventory holding cost, the retailer's backlog cost, and the manufacturer's shortage cost per unit, respectively. Let $I_R^+ = \max\{I_R, 0\}$ and $I_R^- = \max\{-I_R, 0\}$. The retailer's objective is to find a production rate Q_R in order to minimize its all inventory related costs. Let $h_R(I_R(t)) = h_R^+ I_R^+(t) + h_R^- I_R^-(t)$ denote its total inventory cost at time t. Its problem is given by

$$\min_{Q_R(\cdot)} \int_{t_s}^{t_f} h_R(I_R(t))dt,$$

$$\dot{I}_R(t) = Q_R(t) - D(t),\ 0 \le Q_R(t) \le \bar{Q}_R,$$

where I_R is the retailer's inventory level, Q_R is the retailer's processing rate, $\bar{Q}_R$ is the maximum processing rate, and $D(t)$ is a step-wise and periodic customer demand rate for the product given as

$$D(t) = \begin{cases} d > \bar{Q}_R, t_1^d + jT < t \le t_2^d + JT, j = 1, 2..., \\ d < \bar{Q}_M, t_2^d + (j-1)T < t \le t_1^d + JT, j = 1, 2..., \end{cases}$$

where $\bar{Q}_M$ is the manufacturer's production limit. The manufacturer's problem is

$$\min_{Q_M(\cdot)} \int_{t_s}^{t_f} h_M(I_M(t))dt = \min_{Q_M} \int_{t_s}^{t_f} h_M(I_M(t))dt$$

$$\dot{I}_M(t) = Q_M(t) - Q_R(t),\ 0 \le Q_M(t) \le \bar{Q}_M,$$

where $h_M(I_M(t)) = h_M^+ I_M^+(t) + h_M^- I_M^-(t)$ is the manufacturer's inventory cost at time t.

For the centralized channel, if the retailer's backlog cost is lower than the manufacturer's, i.e., $h_R^- < h_M^-$, and the retailer's holding cost is higher than the manufacturer's, i.e., $h_R^+ > h_M^+$, ($h_M^+ \ne h_R^-$, $h_M^- \ne h_R^+$), then there is no backlog on the manufacturer's side. For the decentralized channel in which the manufacturer is the leader, the retailer processes products at the maximum rate when it has an inventory surplus; when there is a shortage, the retailer reduces the processing rate to equal the maximum production rate of the manufacturer. This complicates the ordering decisions which require multiple switching points induced by coordinating inventory decisions and capacity limitation. While the demand is not price-dependent, it varies with time. Thus, the optimal control problems faced by each of the players is a standard dynamic inventory problem as discussed, e.g., by Bensoussan et al. (1974).

Since production control appears linearly in these problems, the optimal production policy can have three possible regimes: maximum production, (singular) production on demand, and zero production. Sequencing of these regimes depends on the time-varying nature of the demand and inventory cost parameters.

As for coordination, the authors show that if the retailer pays the manufacturer for its inventory related cost, then the centralized solution is attained. They also show that a two-part tariff contract can be obtained to coordinate the supply chain.

3.8 Kogan and Tapiero (2007c): Production Balancing with Subcontracting

This paper considers a supply chain consisting of one manufacturer (follower) and one supplier (leader). The supplier has ample capacity and so the inventory dynamics is not an issue. The manufacturer, on the other hand, has a limited capacity, and its decisions depend on the available inventory. The product demand rate at time t is $a(t)\tilde{D}$, where $\tilde{D}$ is a random variable and $a(t)$ is known as the demand shape parameter. The realization of $\tilde{D}$ is observed only at the end of a short selling season, such as in the case of fashion goods, and as a result, the manufacturer can only place an advance order to obtain an initial end-product inventory, which is then used to balance production over time with the limited in-house capacity.

Thus, the problem is a dynamic version of the newsvendor problem which incorporates production control. The supplier's problem is to set a constant wholesale price to maximize its profit from the advance order from the manufacturer. The manufacturer must decide on the advance order as well as the production rate over time in order to minimize its total expected cost of production, inventory/backlog, and advance order. The authors show that it is possible to transform this problem into a deterministic optimal control problem that can be solved to obtain the manufacturer's best response to the supplier's announced wholesale price.

The paper assumes that the unit in-house production cost is greater than the supplier's cost , and shows that if the supplier makes profit (i.e., has a positive margin), then the manufacturer produces more in-house and buys less than in the centralized solution. This is due to the double marginalization not unlike in the static newsvendor problem. Furthermore, if the manufacturer is myopic, it also orders less than the centralized solution even though it does not produce in-house, since it does not take into account the inventory dynamics.

While the optimal production rate over time would depend on the nature of the demand, it is clear that the optimal production policy will have intervals of zero production, maximum production, and a singular level of production. The authors also solve a numerical example and obtain the equilibrium wholesale price, the manufacturer's advance order quantity, and the production rate over time.

3.9 Kogan and Tapiero (2007d): Outsourcing Game

This paper considers a supply chain consisting of one producer and multiple suppliers, all with limited production capacities. The suppliers are the leaders and set their wholesale prices over time to maximize profits. Following this, the producer decides on its in-house production rate and supplements this by ordering from a selection of suppliers over time in order to meet a random demand at the end of a planning period T. The producer incurs a penalty for any unmet demand. Unlike in Kogan and Tapiero (2007c), no assumption regarding the cost of in-house production and the supplier's production cost is made here. The producer's goal is to minimize its expected cost.

As in Kogan and Tapiero (2007c), it is possible to transform the producer's problem into a deterministic optimal control problem. Because the producer's problem is linear in its decisions, the production rate has one of three regimes as in the previous model, and its ordering rate from any chosen supplier will also have similar three regimes.

The authors show that the greater the wholesale price of a supplier, the longer the producer waits before ordering from that supplier. This is because the producer has an advantage over that supplier, up to and until a breakeven point in time for outsourcing to this supplier is reached. As in Kogan and Tapiero (2007c), here also when a supplier sets its wholesale price strictly above its cost over the entire horizon, then the outsourcing order quantity is less than that in the centralized solution.

Again, the supply chain is coordinated if the suppliers set their wholesale prices equal to their unit costs and get lump-sum transfers from the producer.

3.10 Kogan and Tapiero (2007e): Pricing Game

This paper concerns an SDG between a supplier (the leader) who chooses a wholesale price for a product and a retailer (the follower) who responds with a retail product price. The authors incorporate learning on the part of the supplier whose production cost declines as more units are produced. The demand for the product is assumed to be decreasing in the retail price. The authors show that myopic pricing is optimal for the retailer. Under a certain profitability condition, the retail price is greater than in the centralized solution because of double marginalization. However, the gap is larger than that in the static pricing game. This is because the higher retail price implies less cost reduction learning in the dynamic setting in comparison to that in the centralized solution, whereas in the static framework no learning is involved in the centralized solution and in the Stackelberg solution.

3.11 Kogan and Tapiero (2007f): Co-Investment in Supply Chains

This paper considers a supply chain with N firms. Let $K(t)$ denote the current level of the supply chain infrastructure capital, $L = (L_1, ..., L_N)$ denote a vector of the labor force, $I = (I_1, ..., I_N)$ denote a vector of investment policy, and $Q_j = f(K, L_j)$ denote an aggregate production function of the j^{th} firm, where $\frac{\partial f}{\partial K} \geq 0$, $\frac{\partial f}{\partial L_j} > 0$ for $L \neq 0$, $\frac{\partial f(K,0)}{\partial L_j} = 0$, and $\frac{\partial^2 f}{\partial L_j^2} < 0$. The process of capital accumulation is given by

$$\dot{K}(t) = -\delta K(t) + \sum_{j=1}^{N} I_j(t), \quad K(0) = K_0,$$
$$f(K(t), L_j(t)) \geq I_j(t) \geq 0, j = 1, ..., N.$$

The j th firm's objective is to maximize its discounted total profit, i.e.,

$$\max_{L_j(\cdot), I_j(\cdot)} \{ \int_0^\infty e^{-\rho_j t} [p_j(t) f(K(t), L_j(t)) - c_j(t) L_j(t) - C_I((1-\theta) I_j(t))] dt \}, \quad j = 1, ..., N,$$

where p_j is the price, c_j is the unit labor cost, $C_I(\cdot)$ is the investment cost function, and θ is the portion of the cost that is subsidized.

The authors derive the Nash strategy as well as the Stackelberg strategy for the supply chain where firms are centralized and controlled by a "supply chain manager". Their results show that the Stackelberg strategy applied to consecutive subsets of firms will result in an equilibrium identical to that obtained in case of a Nash supply chain. The implication is that it does not matter who the leader is and who the followers are.

4. Marketing Channel Applications

Research in marketing has applied differential game models to study dynamic advertising strategies in competitive markets. These papers mainly focus on the horizontal interactions such as the advertising competition between brands, and accordingly seek Nash equilibria (Deal 1979, Deal et al. 1979, Teng and Thompson 1983, Erickson 1992, Chintagunta and Jain 1992, Fruchter and Kalish 1997).

Recently, a few SDG models have appeared that study the vertical interaction in marketing channels. In terms of their dynamics, these can be categorized into two groups, advertising capital models and sales-advertising response models. Advertising capital models consider advertising as an investment in the stock of goodwill $G(t)$ as in the model of Nerlove and Arrow (1962) (NA, thereafter). The advertising capital is affected by the current and past advertising expenditures by a firm on the demand for its products. It changes over time according to

$$\dot{G}(t) = A(t) - \delta G(t), \; G(0) = G_0 \geq 0, \qquad (25)$$

where $A(t)$ is the current advertising investment (in dollars) and δ is a constant positive decay rate.

Sales-advertising response models characterize a direct relation between the rate of change in sales and advertising. The Vidale-Wolfe (1957) (VW, hereafter) advertising model is given by

$$\dot{x}(t) = \frac{dx(t)}{dt} = rA(t)(1-x(t)) - \delta x(t),\ \ x(0) = x_0,$$

where x is the market share, A is the advertising rate, and r is the effectiveness of advertising. The Sethi (1983) model is a variation of the VW model, and it is given by

$$\dot{x}(t) = rA(t)\sqrt{1-x(t)} - \delta x(t),\ \ x(0) = x_0. \tag{26}$$

Sethi (1983) also gave a stochastic extension of his model. Little (1979) discusses some of the desirable features of the advertising models that have appeared in the literature.

4.1 Jørgensen, Sigue, and Zaccour (2000): Dynamic Cooperative Advertising

This paper studies a channel in which the manufacturer and retailer can make advertising expenditures that have both long-term and short-term impacts on the retail demand. Specifically, the long-term advertising affects the carry-over effect of advertising, while the short term advertising influences current retail sales only.

The manufacturer controls its rate of long-term advertising effort $A(t)$ and short-term advertising effort $\hat{A}(t)$. The retailer controls its long-term advertising rate $B(t)$ and short-term advertising rate $\hat{B}(t)$. An extended NA model describes the dynamics of the goodwill as

$$\dot{G}(t) = a_l A(t) + b_l B(t) - \delta G(t),\ \ G(0) = G_0, \tag{27}$$

where a_l and b_l are positive parameters that capture the effectiveness of the long-term advertising of the manufacturer and the retailer, respectively. At any instant of time, the demand D is given by

$$D(\hat{A}, \hat{B}, G) = (a_S \hat{A} + b_S \hat{B})\sqrt{G},$$

where a_S and b_S are parameters that capture the effectiveness of the manufacturer's and retailer's short-term advertising, respectively.

Suppose the manufacturer and the retailer can enter into a cooperative advertising program in which the manufacturer pays a certain share of the retailer's advertising expenditure. The manufacturer is the Stackelberg game leader in designing the program. It announces its advertising strategies and support rates for the retailer's long-term and short-term advertising efforts.

The manufacturer's problem is

$$\max_{A(\cdot),\hat{A}(\cdot),\theta,\hat{\theta}} [J_M = \int_0 e^{-\rho t}\{m_M D - \frac{1}{2}c_M A^2(t) - \frac{1}{2}\hat{c}_M \hat{A}^2(t) - \frac{1}{2}\theta(t)c_R B^2(t) - \frac{1}{2}\hat{\theta}(t)\hat{c}_R(t)\hat{B}^2(t)\}dt]$$

and the retailer's is

$$\max_{B(\cdot),\hat{B}(\cdot)} [J_R = \int_0 e^{-\rho t}\{m_R D - \frac{1}{2}(1-\theta(t))c_R B^2(t) - \frac{1}{2}(1-\hat{\theta}(t))c_R(t)\hat{B}^2(t)\}dt],$$

where m_R and m_M are the manufacturer's and the retailer's margins, respectively, $\theta(t)$ and $\hat{\theta}(t)$ are the percentages that the manufacturer pays of the retailer's long-term and short-term advertising costs, respectively.

The results show that both the manufacturer and the retailer prefer full support to either of long-term or short-term support alone, which in turn is preferred to no support at all.

4.2 Jørgensen, Sigue, and Zaccour (2001) : Impact of Leadership on Channel Efficiency

This paper studies the effects of strategic interactions in both pricing and advertising in a channel consisting of a manufacturer and a retailer. It considers three scenarios: the manufacturer and retailer simultaneously choose their margins and advertising rates; sequential with the retailer as the leader; and sequential with the manufacturer as the leader. The manufacturer controls its margin $m_M(t)$ and the rate of advertising $A(t)$. The retailer controls its margin $m_R(t)$ and the advertising rate $B(t)$. The demand rate $D(t)$ follows a NA type of dynamics, and is given by

$$D(t) = B(t)(a - bp(t))\sqrt{G(t)},$$

where $p(t)$ is the retail price, a and b are positive parameters, and $G(t)$ is the stock of brand goodwill given by (27). It is assumed that the retailer is myopic, meaning that it is only concerned with the short-term effects of its pricing and advertising decisions. The manufacturer is concerned with its brand image. The manufacturer's total profit is

$$J_M = \int_0^\infty e^{-\rho t}[m_M(t)B(a - bp(t))\sqrt{G(t)} - \frac{1}{2}c_M A^2(t)]dt$$

and the retailer's is

$$J_R = \int_0^\infty e^{-\rho t}[m_R(t)B(t)(a - bp(t))\sqrt{G(t)} - \frac{1}{2}c_R B^2(t)]dt$$

The authors show that the manufacturer's leadership and the retailer's leadership in a channel are not symmetric as in pure pricing games. The manufacturer's leadership improves channel efficiency and is desirable in terms of consumer welfare, but the retailer's leadership is not desirable for channel efficiency and for consumer welfare.

4.3 Karray and Zaccour (2005): Advertising for National and Store Brands

This paper considers a marketing channel composed of a national manufacturer and a retailer who sells the manufacturer's national product (labeled as 1) and may also introduce a private label (denoted as 2) at a lower price than the manufacturer's brand. The paper finds that while the retailer is better off by introducing the private label, the manufacturer is worse off. Furthermore, it investigates whether a cooperative advertising program could help the manufacturer to alleviate the negative impact of the private label.

The manufacturer decides on the national advertising $A(t)$. The retailer controls the promotion efforts $B_1(t)$ for the national brand and efforts $B_2(t)$ for the store brand. The retailer's effort has an immediate impact on sales, but they do not affect the dynamics of the goodwill of the national brand. The goodwill $G(t)$ of the national brand evolves according to the NA dynamics given by equation (25). The demand functions D_1 for the national brand and D_2 for the store brand are as follows:

$$D_1(B_1, B_2, G) = aB_1 - b_1 B_2 + e_1 G - dG^2,$$

$$D_2(B_1, B_2, G) = aB_2 - b_2 B_1 - e_2 G,$$

where a, b_1, b_2, e_1, e_2, and d are positive parameters.

Karray and Zaccour consider three scenarios:

1) Game N: The retailer carries only the national brand (labeled as 1) and no cooperative advertising program. It is shown that the retailer promotes the national brand at a positive constant rate and the advertising strategy is decreasing in the goodwill.

2) Game S: The retailer offers both a store brand and the national brand and there is no cooperative advertising program. It is shown that the retailer will promote the national brand if the marginal revenue from doing so exceeds the marginal loss on the store brand.

3) Game C: The retailer offers both brands and the manufacturer proposes to the retailer a cooperative advertising program. Here the retailer will always promote the store brand. The retailer will also promote the national brand, but only under certain conditions specified in the paper.

The conclusion is that the introduction of store brand always hurts the manufacturer. The cooperative advertising program is profit Pareto-improving for both players.

4.4 Jørgensen, Taboubi, and Zaccour (2003): Retail Promotions with Negative Brand Image Effects

In this paper, the manufacturer advertises in the national media to build up the image for its brand. The retailer locally promotes the brand (by such means as local store displays and advertising in local newspapers) to increase sales revenue, but these local promotional efforts are assumed to be harmful to the brand image. This paper analyzes two firms in a cooperative program in which the manufacturer supports the retailer's promotional efforts by paying part of the cost incurred by the retailer when promoting the brand. The two firms play an SDG where the manufacturer is the leader. The paper addresses the question of whether the cooperative promotion program is profitable and whether the retailer's decision on being a myopic or far-sighted will affect the implementation of a cooperative program.

Let $A(t)$, $B(t)$, and $G(t)$ denote the manufacturer's advertising rate, the retailer's promotional rate, and the brand image, respectively. The dynamics of $G(t)$ is described by the differential equation

$$\dot{G}(t) = aA(t) - bB(t) - \delta G(t),\ G(0) = G_0 > 0,$$

where a and b are positive parameters measuring the impact of the manufacturer's advertising and retailer's promotion, respectively, on the brand image. The margin on the product is $m(B(t), G(t)) = dB(t) + eG(t)$, where d and e are positive parameters that represent the effects of promotion and brand image on the current sales revenue. With this formulation of the demand and the revenue, the retailer faces a trade-off between the sales volume and the negative impact of the local advertising on the brand image.

The manufacturer and the retailer incur quadratic advertising and promotional costs $C(A(t)) = c_A A^2/2$ and $C(B(t)) = c_B B^2/2$, respectively. The manufacturer's objective functional is

$$J_M = \int_0^\infty \{e^{-\rho t} q[dB(t) + eG(t)] - \frac{c_A A(t)^2}{2} - \frac{c_B \theta(t) B(t)^2}{2}\} dt$$

and the retailer's is

$$J_R = \int_0^\infty \{e^{-\rho t}(1-q)[dB(t)+eG(t)] - \frac{c_B(1-\theta(t))B(t)^2}{2}\}dt,$$

where q is the manufacturer's fraction of the margin and $\theta(t)$ is the fraction the manufacturer contributes to the retailer's promotion cost.

It is shown that a cooperative program is implementable if the initial value of the brand image G_0 is sufficiently small, and if the initial brand image is "intermediate" but promotion is not "too damaging" (i.e., b is small) to the brand image.

4.5 Martin-Herran and Taboubi (2005): Shelf-space Allocation

This paper considers a supply chain with one retailer and two manufacturers. The retailer has limited shelf-space and must decide on the allocation of the shelf-space to the two products. Let $S_1(t)$ denote the fraction of the shelf-space allocated to product i, $i \in \{1,2\}$, then $S_2(t) = 1 - S_1(t)$. At time t, each manufacturer i decides on its advertising strategy $A_i(t)$ and a shelf-space dependent display allowance $W_i(t) = \omega_i(t)S_i(t)$, where $\omega_i(t)$ is the coefficient of the incentive strategy. The retail demand is a function of the shelf-space and goodwill given by

$$D_i(t) = S_i(t)[a_i G_i(t) - \frac{1}{2}b_i S_i(t)],$$

where G_i is the goodwill for brand i evolving according to the NA dynamics. This model implies that the shelf-space has a diminishing marginal effect on the sales.

The paper assumes that the retailer and two manufacturers play an SDG game with the retailer as the follower, while the two manufacturers play a Nash game, i.e., they announce simultaneously their advertising and incentive strategies ω_1 and ω_2.

The authors show that manufacturers can affect the retailer's shelf-space allocation decisions through the use of incentive strategies (push) and/or advertising investment (pull). Depending on the system parameters, the manufacturer should choose between incentive strategies and/or advertising.

4.6 Jørgensen, Taboubi, and Zaccour (2006): Incentives for Retail Promotions

This paper considers a channel composed of a manufacturer selling a particular brand to the retailer. The manufacturer invests in national advertising, which improves the image of its brand, while the retailer makes local promotions for the brand. The manufacturer and the retailer play an SDG with the manufacturer as the leader. The dynamics of the goodwill stock follows the classical NA dynamics given in (25). Jørgensen et al. (2006) assume that the manufacturer and retailer apportion a fixed share of the total revenue. They consider two scenarios: joint maximization and individual maximization. They show that the manufacturer advertises more in the joint maximization scenario than under individual maximization. This result does not depend on whether or not the manufacturer supports the retailer's promotion in the individual maximization case.

4.7 Breton, Jarrar, and Zaccour (2006): Feedback Stackelberg Equilibrium with a Lanchester-Type Model and Empirical Application

This paper studies dynamic equilibrium advertising strategies in a duopoly. A Lanchester-type model provides the market share dynamics for the two competitors in the duopoly. Let $A_i(t)$ denote the advertising expenditure of player $i \in \{1,2\}$ at any instant of time $t \in [0,\infty)$ and let $x(t)$ denote the market share of Player 1 (the leader) at time t. The market share of Player 2 (the follower) is thus $(1-x(t))$. The market share evolves as follows

$$\dot{x}(t) = r_1\sqrt{A_1(t)}(1-x(t)) - r_2\sqrt{A_2(t)}x(t),\ x(0) = x_0, \tag{28}$$

where the positive constant $r_i, i \in \{1,2\}$, denotes the advertising effectiveness of player i and x_0 is the initial market share of Player 1.

In this paper, the authors use a discrete-time version of (28). They assume the following sequence of the game. At stage k, the leader observes the state of the system and chooses an optimal advertising level $A_1(k)$; the follower does not play at this stage. At the next stage, the follower observes the state of the system and chooses an optimal advertising level $A_2(k+1)$; the leader does not play at this state. This procedure leads to a feedback Stackelberg equilibrium (FSE).

The authors empirically test the discrete-time model specification by using a dataset of Coke and Pepsi advertising expenditures. They compare the fit of the FSE against the method in Jarrar et al. (2004) for the feedback Nash equilibrium (FNE). Note that in the continuous-time game, the FNE coincides with the FSE, and Rubio (2006), in a more general setting, also observes this coincidence. However, in the discrete-time version, the solutions need not coincide. The authors find that the FSE fits the actual cola industry advertising expenditures better than the FNE, which suggests that the advertising decision making in this industry follows a sequential rather than simultaneous pattern. However, they find that there is no significant difference in being a leader or a follower.

4.8 He, Prasad, and Sethi (2007): Cooperative Advertising

In a decentralized channel, while the retailer often incurs the full cost of advertising, it only captures a portion of the benefits. This creates an incentive for the retailer to under-advertise. A few papers, such as Berger (1972), Bergen and John (1997) and Huang, Li and Mahajan (2001), have investigated the use of cooperative advertising programs to improve channel performance. All of these papers derive the results in a static setup. In contrast, He, Prasad, and Sethi investigate cooperative advertising in a dynamic supply chain. The manufacturer announces a participation rate, i.e., it will provide a proportion of the retailer's advertising expenditure. In addition, it also announces the wholesale price. In response, the retailer obtains its optimal advertising and retail pricing policies. The paper models this problem as an SDG and provides feedback solutions to the optimal advertising and pricing policies for the manufacturer and the retailer. The market potential dynamics is given by the Sethi model (26). The retailer solves the following problem:

$$V_R(x) = \max_{p(\cdot),A(\cdot)} \int_0^\infty e^{-\rho t}[(p(t) \\ -w(x(t)))D(p(t))x(t) \\ -(1-\theta(x(t)))A^2(t)]dt,$$

where $w(x)$ and $\theta(x)$ are the announced feedback policies of the manufacturer and $p(t)$ and $A(t)$ denote the retailer's retail price and advertising effort at time t.

The manufacturer solves the following problem:

$$V_M(x) = \max_{w(\cdot),\theta(\cdot)} \int_0^\infty e^{-\rho t}[(w(t)-c)\cdot \\ D(p(x(t) \mid w(t),\theta(t)))x(t) \\ -\theta(t)A^2(x(t) \mid w(t),\theta(t))]dt,$$

where $p(x(t) \mid w,\theta)$ and $A(x(t) \mid w,\theta)$ are the optimal responses of the retailer to the announced w and θ.

The authors also solve the model for a vertically integrated channel. Comparing its results to those for the decentralized channel, they find that the decentralized channel has higher than optimal prices and lower than optimal advertising. Furthermore, they derive conditions under which it is optimal for the manufacturer not to indulge in cooperative advertising.

Whereas wholesale price by itself cannot correct for these problems, it is shown that a combination of wholesale price and co-op advertising allows the channel to achieve the coordination outcome. Thus for the manufacturer, decision making that jointly optimizes co-op advertising and price is advantageous.

The authors also solve the stochastic extension of the problem in which the dynamics is an Ito equation version of (28) also developed in Sethi (1983).

5. Conclusions

The areas of supply chain management and marketing channels have attracted a great deal of attention over the last two decades. While most of the models in these areas are based on static settings such as the newsvendor model in operations management, and, therefore, are limited to examine the one-shot interactions between the channel members, the insights under the assumption of the static setting do not always carry over to dynamic situations. In practice, the channel members often interact with each other frequently. It is thus natural to explore how their decisions evolve over time. For such situations, the differential game modeling approaches discussed in this survey can be very useful.

A number of reasons have limited the applications of the Stackelberg differential games. We can see that a majority of the papers in this survey use open-loop equilibria because of their mathematical tractability. Even then, numerical analysis is often used to get insights into the impact of the key parameters on the issues under examination. A major drawback of the open-loop solution is that they are in general not time consistent. On the other hand, the feedback Stackelberg equilibria are hard to obtain.

We conclude by pointing out future research avenues. One interesting avenue is to incorporate uncertainty into the sales-advertising response dynamics. As we can see that all of the models are deterministic, even though there are many situations that can be affected by random factors. For these situations, it would be of interest to model them as stochastic differential games. An attempt in this direction is already made by He et al. (2007) as noted at the end of Section 4.8. Another possible avenue is to collect data to perform empirical research that could estimate the parameters, as well as optimal feedback policies.

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