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On typing knowledge graphs

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Abstract. ...

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1 Introduction

A knowledge graph (abbr. KG) includes a data and knowledge representation language in the form of a graph. It is the dictionaries, such as RDF-Schema [18], which attaches meanings to the edges of a graph, turning a graph into a language for the representation of data and knowledge. In comparison to the data structures used in programming languages [15, 8], the conceptual schema of a KG is complex in the sense that any data structure can be represented as a graph. Let us depict some of the features of the data and knowledge representation language of a KG.

First, a KG includes taxonomies of classes and properties [2]. Second, typing of ground identifiers, as well as typing of the predicates with the domain and range types are stored in a KG. Further, similarly to the *roles* [3] of a knowledge base, the predicates of a KG are treated as objects that are included in a taxonomy of predicates. For each predicate we have the definition of one or more triple types stored in a KG. Finally, the predicates are inherited through the taxonomy of classes, and, from the other perspective, the domain and range types of the predicates are inherited through the hierarchy of predicates.

Furthermore, the data and KR language of a KG, which is based on RDF and RDF-Schema [17, 18], does not include variables as in the case for the expressions of a programming language. All information needed for typing a ground triple is in a KG.

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2 Preliminaries

2.1 Knowledge graph

This section defines a knowledge graph as a RDF graph [17] using RDF-Schema [18] for the representation of the structural part of a knowledge base.

Let I be a set of URI-s, B be a set of blanks and L be a set of literals. Let us also define sets $S = I \cup B$, $P = I$, and $O = I \cup B \cup L$. A *RDF triple* is a triple $(s, p, o) \in S \times P \times O$. A *RDF graph* $g \subseteq S \times P \times O$ is a set of triples. Set of all graphs will be denoted as G .

The complete set of triples of a RDF graph is in the text denoted as Δ . We use Δ when we want to refer to the original set of KG triples and not the data model of a KG presented in the following Section.

2.2 A data model of a KG

To abstract away the details of the RDF data model we unify the representation of knowledge graph by separating solely between the identifiers and triples. The identifiers include the set of individual identifiers and the set of class identifiers. The triples include the set of ground triples and the set of triple types.

However, since typing of a KG is based on the separation between the values and the types, we use the following classification of identifiers and triples. The set of values \mathcal{V} includes the individual identifiers \mathcal{V}_i and the individual (ground) triples \mathcal{V}_t . To be able to refer to the specific subsets of \mathcal{V}_i in the rules, we also introduce the set $\mathcal{V}_l \subseteq \mathcal{V}_i$, which denotes a set of literal values, and the set $\mathcal{V}_p \subseteq \mathcal{V}_i$ that refers to the set of predicates from a KG. Finally, a set $\mathcal{T}_p \in \mathcal{T}_i$ stands for all predicates of a KG that now have the role of types. Note that predicates are treated both as values and as types.

The set of all valid types \mathcal{T} of a KG comprises the class identifiers \mathcal{T}_i , i.e. types of individual identifiers \mathcal{V}_i , and the triple types \mathcal{T}_t , i.e. the types of individual triples \mathcal{V}_t .

The set of class identifiers \mathcal{T}_i related by subclass relation is a poset forming a taxonomy of classes. Similarly also predicates, now in the role of types, are ordered in a poset. The meaning of class identifiers is established by their interpretations. The interpretation of a class identifier c comprises the instances of a given class c as well as the instances of all c 's sub-classes.

The individual triples include solely the individual identifiers from \mathcal{V}_i in places of S and O , and predicates \mathcal{V}_p in the place of P component of a triple. The types of individual triples are product types $S * p * O$ where $S, O \in \mathcal{T}_i$ and $p \in \mathcal{T}_p$. The product types are in our data model of a KG written as a triple (S, p, O) . The triple types are ordered by a subtype relation to form a poset. The subtype relation among the triple types is defined on the basis of the subtype relation among the classes and predicates. A more detailed formal definition of a KG is given in [21].

2.3 Typing rule language

We do not use standard typing rule language [15, 8] that includes a context Γ where the types of the variables are stored. We use a meta-language that is rooted in first-order logic (abbr., FOL) but similar to the one used by Pierce in [15] for the representation of records and subtyping.

The rules are composed of a set of premises and a conclusion. The premises can be expressions stating the set membership of an object, typing and subtyping judgements, or expressions in the FOL including the previous two forms. The expressions of FOL

can express complex premises such as the requirements for the LUB and GLB triple types. The conclusion part of the rule is a typing or subtyping judgment.

The symbols used in a rule are grounded by stating their membership in the sets of identifiers (\mathcal{V} , \mathcal{V}_i and \mathcal{V}_t) and triples (\mathcal{T} , \mathcal{T}_i and \mathcal{T}_t) defined in Section 2.1. When we write $O \in S$ then we mean the *existence* of O in a set S . We use universal quantification $\forall O \in S, p(O)$ when we state that some property $p(O)$ holds for all objects O from S . The premises of the rule are treated from the left to the right. The quantification of the symbols binds the symbols up to the last premise unless defined differently by the parentheses.

Similar to [6], we differ between two interpretations of rules. First, the *generator* view of rules is the forward interpretation where rules infer the types from the types derived by premises. Second, the *type-checking* view of the rules is the backward interpretation. Given the expression and its type, the rules are applied backwards to verify that the expression has that type.

3 Typing identifiers

In this section we present the basic typing of ground identifiers \mathcal{V}_i . The rules for typing ground identifiers are further used for typing ground triples \mathcal{V}_t in Section 5. Typing of literals \mathcal{V}_l is described in Section 3.1. The rules for deriving the stored types of ground identifiers are given in Section 3.2. Finally, the sub-typing relation \preceq is defined for the class identifiers \mathcal{T}_i and the types of ground identifiers are presented in Section 3.3.

3.1 Typing literals

Literal values $\mathcal{V}_l \subseteq \mathcal{V}_i$ are the instances of literal types $\mathcal{T}_l \subseteq \mathcal{T}_i$. The literal types are in RDF provided by the RDF-Schema dictionary [18]. RDF-Schema defines a list of literal types, such as `xsd:integer`, `xsd:string`, or `xsd:boolean`.

The literals are composed of literal values and literal types. For example, the literal `"literal"^^xsd:string` includes the literal value `"365"` and a type `xsd:integer`. Typing of literals is defined by the following rule.

$$\frac{L \in \mathcal{V}_l \quad T \in \mathcal{T}_l \quad \text{"L"}^{\wedge\wedge} T \in \Delta}{L : T} \quad (1)$$

The rule states that a literal value L is of a type T if a literal `"L"^^T` is an element of a KG Δ . A literal type T is referencing a type from the RDF-Schema dictionary.

3.2 Ground typing and subtyping of identifiers

The typing expression $V :_1 T$ is a *ground typing* relation $:_1$ that links a value $V \in \mathcal{V}$ to a ground type $T \in \mathcal{T}$. The ground typing relation is a one-step typing relation based on typing stored in a KG.

In this section we deal with the identifiers $I \in \mathcal{V}_i$ which are the values of type $T \in \mathcal{T}_i$. The ground types are directly linked to the ground identifiers via a stored

typing relation in the form $(I, rdf : type, T) \in \Delta$. The expression $I :_1 T$ states that a class identifier T is a ground type of an individual identifier I .

The subtyping expression $T_1 \preceq_1 T_2$ defines a subtype relationship between the types T_1 and T_2 . In the case we deal with the identifiers, the relation $T_1 \preceq_1 T_2$ denotes the subclass relations since $T_1, T_2 \in \mathcal{T}_i$. The relation \preceq_1 is a one-step relation that is stored in Δ . Note that a unique notation for ground typing allows us to address differently the *stored* and the *derived* types of ta KG.

The rule for the one-step typing relation $:_1$ is defined using the predicate `rdf:type`.

$$\frac{I \in \mathcal{V}_i \quad T \in \mathcal{T}_i \quad (I, rdf:type, T) \in \Delta}{I :_1 T} \quad (2)$$

The individual entity I can have more than one stored types. By using Rule 2 as a generator, it synthesizes all types $T_j^{j \in [1, n]}$ such that $I :_1 T_j^{j \in [1, n]}$.

If we want to obtain all valid ground types of I , the rule can be used either in some other rule that employs it as a generator, or we can update above rule to generate a \wedge -type including all the types of I as presented in Section 4.

The one-step subtyping relationship \preceq_1 is defined on classes by using the RDF predicate `rdfs:subClassOf` as follows.

$$\frac{C_1, C_2 \in \mathcal{T}_i \quad (C_1, rdfs:subClassOf, C_2) \in \Delta}{C_1 \preceq_1 C_2} \quad (3)$$

The rule for the definition of the one-step subtyping relationship \preceq_1 is based on the predicate `rdfs:subPropertyOf`.

$$\frac{P_1, P_2 \in \mathcal{T}_p \quad (P_1, rdfs:subPropertyOf, P_2) \in \Delta}{P_1 \preceq_1 P_2} \quad (4)$$

The predicates have in the above rule the role of types in the sense that they represent the names of the binary relations.

3.3 Typing and subtyping identifiers

The one-step relationship \preceq_1 is extended with the reflectivity, transitivity and antisymmetry to obtain the subtyping relationship \preceq . The relation \preceq forms a partial ordering of class identifiers. The ground typing relation $:_1$ is then extended with the *rule of subsumption* presented as Rule 11 to obtain a typing relation $:$.

First, the one-step relationship \preceq_1 is generalized to the relationship \preceq defined over class identifiers \mathcal{T}_i .

$$\frac{I_1, I_2 \in \mathcal{T}_i \quad I_1 \preceq_1 I_2}{I_1 \preceq I_2} \quad (5)$$

Next, the subtyping relationship \preceq is reflexive.

$$\frac{I \in \mathcal{T}_i}{I = I} \quad (6)$$

The subtype relationship is also transitive.

$$\frac{I_1, I_2, I_3 \in \mathcal{T}_i \quad I_1 \preceq I_2 \quad I_2 \preceq I_3}{I_1 \preceq I_3} \quad (7)$$

Finally, the subtype relationship is asymmetric which is expressed using the following rule.

$$\frac{I_1, I_2 \in \mathcal{T}_i \quad I_1 \preceq I_2 \quad I_2 \preceq I_1}{I_1 = I_2} \quad (8)$$

As a consequence of the rules 6-8 the relation \preceq is a poset.

Knowledge graphs include a special class \top that represents the root class of the ontology. In RDF ontologies \top is usually represented by the predicate owl:Thing [9]. The following rule specifies that all class identifiers are more specific than \top .

$$\frac{\forall S \in \mathcal{T}_i}{S \preceq \top} \quad (9)$$

The one-step typing relation $:_1$ is now extended to the typing relation $:$ that takes into account the subtyping relation \preceq . The following rule states that a stored type is a type.

$$\frac{I \in \mathcal{V}_i \quad C \in \mathcal{T}_i \quad I :_1 C}{I : C} \quad (10)$$

The link between the typing relation and subtype relation is provided by adding a typing rule called *rule of subsumption* [15].

$$\frac{I \in \mathcal{T}_i \quad S, T \in \mathcal{T}_i \quad I : S \quad S \preceq T}{I : T} \quad (11)$$

4 Intersection and union types

– *CHECK: upper and lower bounds in the following rules.*

The meaning of the \wedge and \vee types can be seen through their interpretations. The instances of the intersection type $T_1 \wedge T_2$ are objects belonging to both T_1 and T_2 . The type $T_1 \wedge T_2$ is the greatest lower bound of the types T_1 and T_2 . In general, $\wedge[T_i^{i \in [1, n]}]$ is the greatest lower bound (abbr. GLB) of types $T_i^{i \in [1, n]}$ [13, 14]. The instances of the type $\wedge[T_i^{i \in [1, n]}]$ form a maximal set of objects that belong to all types T_i .

The rules for the \wedge and \vee types presented in this section are general—they apply for the identifier types \mathcal{I}_c and triple types \mathcal{T}_t . The set of types $\tau = \mathcal{I}_c \cup \mathcal{T}_t$ is used to ground the types in the rules.

The interpretation of a type $\wedge[T_i^{i \in [1, n]}]$ is included in interpretation of every particular type T_i . Hence, the type $\wedge[T_i^{i \in [1, n]}]$ is the greatest lower bound of types $T_i^{i \in [1, n]}$. This is stated by the following rule.

$$\frac{T_i^{i \in [1, n]} \in \tau}{\wedge[T_i^{i \in [1, n]}] \preceq T_i^{i \in [1, n]}} \quad (12)$$

Further, the following rule states that if the type S is a subtype of every type $T_i^{i \in [1, n]}$ then S is a subtype of $\bigwedge[T_i^{i \in [1, n]}]$.

$$\frac{S \in \tau \quad T_i^{i \in [1, n]} \in \tau \quad S \preceq T_i^{i \in [1, n]}}{S \preceq \bigwedge[T_i^{i \in [1, n]}]} \quad (13)$$

The opposite to the above rule, the following rule states the necessary conditions that must be met so that a type T is a supertype of a \bigwedge -type $\bigwedge[S_i^{i \in [1, n]}]$.

$$\frac{T \in \tau \quad S_i^{i \in [1, n]} \in \tau \quad S \in \{S_i^{i \in [1, n]}\} \quad S \preceq T}{\bigwedge[S_i^{i \in [1, n]}] \preceq T} \quad (14)$$

The intersection and union types are dual. This can be seen also from the duality of the rules for the \bigwedge and \bigvee types.

The instances from the union type $T_1 \vee T_2$ are either the instances of T_1 or T_2 , or the instances of both types. Therefore, $\bigvee[T_i^{i \in [1, n]}]$ is the least upper bound of types $T_i^{i \in [1, n]}$ [13].

$$\frac{T_i^{i \in [1, n]} \in \tau}{T_i^{i \in [1, n]} \preceq \bigvee[T_i^{i \in [1, n]}]} \quad (15)$$

Finally, the following rule defines the necessary conditions to be met for a type T to be a supertype of a type $\bigvee[S_i^{i \in [1, n]}]$.

$$\frac{T \in \tau \quad S_i^{i \in [1, n]} \in \tau \quad S_i^{i \in [1, n]} \preceq T}{\bigvee[S_i^{i \in [1, n]}] \preceq T} \quad (16)$$

Again, the opposite rule that defines the premises that must hold so that S is a subtype of a type $\bigvee[T_i^{i \in [1, n]}]$.

$$\frac{T \in \tau \quad S_i^{i \in [1, n]} \in \tau \quad S \in \{S_i^{i \in [1, n]}\} \quad T \preceq S}{T \preceq \bigvee[S_i^{i \in [1, n]}]} \quad (17)$$

Note that besides checking the subtype relation between a type treated as a whole and some logical type, Rules 12-17 can be used to check the subtyping among arbitrary logical types.

4.1 The join and meet types

The \bigvee and \bigwedge types are logical types defined through the sets of instances. Given two types T and S we have a least upper bound $S \vee T$, and a greatest lower bound $S \wedge T$ types where $S \vee T$ denotes a minimal set of objects that are of type S or T (or both), and $S \wedge T$ denotes a maximal set of objects that are of type S and T .

A KG includes a stored poset of classes and triple types that represent types of the individual objects and ground triples. The poset can be used to compute a join $S \sqcup T$ and a meet $S \sqcap T$. Usual definition of the join and meet operators is by using a least

upper bound and a greatest lower bound if they exist [15], respectively. However, in a KG we are also interested in the upper bound and lower bound types [5]. Let us present an example.

Example 1. Let $P = (U, \preceq)$ be a partially ordered set P such that $U = \{a, b, c, d, e\}$ and the relation $\preceq = \{a \preceq c, a \preceq d, b \preceq c, b \preceq d, c \preceq e, d \preceq e\}$. The upper bounds of $S = \{a, b\}$ are the elements c and d . Since there is no lower upper bounds, the upper bounds $\{c, d\}$ are minimal upper bounds. The least upper bound of S is e .

In the case that we remove the element e from P then P does not have a least upper bound but it still has two minimal upper bounds c and d . \square

The least upper bound (abbr. LUB) is by definition one element. It has to be related to all upper bounds via the relationship \preceq . On the other hand, the most interesting upper and lower bounds are minimal upper bounds (abbr. MUB) and maximal lower bounds (abbr. MLB) [11]. They are lower than the least upper bound and higher than the greatest lower bound, respectively. They represent more detailed information about the parameter set of types S than the LUB type of S .

The join $J = \sqcup[T_i^{i \in [1, n]}]$ is a set of MUB types $J_j^{j \in [1, m]} \in J$ such that J_j is an upper bound with $T_i^{i \in [1, n]} \preceq J_j$, and there is no such L where $T_i^{i \in [1, n]} \preceq L$ without also having $J_j \preceq L$. Since we have a top type \top defined in a KG, the join of arbitrary two types always exists.

The meet of types $T_i^{i \in [1, n]}$, $M = \cap[T_i^{i \in [1, n]}]$, is a set of the maximal lower bound types $M_j^{j \in [1, m]} \in M$ such that M_j is lower bound with $M_j \preceq T_i^{i \in [1, n]}$, and all other lower bounds U with $U \preceq T_i^{i \in [1, n]}$ entail $U \preceq M_j$. Note that the meet of the set of types from a KG does not always exist.

The join type is related to the \vee -type. Given a set of types $\{T_i^{i \in [1, n]}\}$, the join $J = \sqcup[T_i^{i \in [1, n]}]$ is a set of types $J_j^{j \in [1, m]} \in J$ that are the minimal upper bounds such that $T_i \preceq J_j$ for $i \in [1, n]$. On the other hand, Rule 15 for the \vee -types states $T_i \preceq \vee[T_i^{i \in [1, n]}]$. However, the join type and \vee -type differ in the interpretation.

$$\llbracket \vee[T_i^{i \in [1, n]}] \rrbracket_\Delta = \bigcup_{i \in [1..n]} \llbracket T_i \rrbracket_\Delta \subseteq \bigcup_{j \in [1, m]} \llbracket J_j \rrbracket_\Delta = \llbracket \sqcup[T_i^{i \in [1, n]}] \rrbracket_\Delta$$

While the interpretation of the type $\vee[T_i^{i \in [1, n]}]$ includes precisely the instances of all T_i , the interpretation of the type $\sqcup[T_i^{i \in [1, n]}]$ contains the instances of minimal upper bound types. The interpretation of $\sqcup[T_i^{i \in [1, n]}]$ can include interpretations of classes that are not among $T_i^{i \in [1, n]}$.

A meet type of $T_i^{i \in [1, n]}$ may not exist in a poset of types from a KG. In general, the meet types $M = \cap[T_i^{i \in [1, n]}]$ exist in a class ontology if the types $T_i^{i \in [1, n]}$ are *bounded below* [15] which means that there exists a type L such that $L \preceq T_i$ for all i . The meet types are not frequent on the lower levels of a class ontology from a KG.

As in the case of the \vee -type and the join type, the semantics of the \wedge -type is similar to the semantics of the meet type. A \wedge -type is a type that implements logical view of the greatest lower bound type. Differently, the meet types are based on the concrete

poset of KG types and represent concrete types though their interpretation is contained in the interpretation of a \wedge -type. The type $\wedge[T_i^{i \in [1, n]}]$ denotes the intersection $\bigcap \llbracket T_i \rrbracket_\Delta$ while the interpretation of a meet type $M_j \in \cap[T_i^{i \in [1, n]}]$ includes the interpretation of the meet types from M . Note that the instances of the meet types are in the intersection of the instances of types $T_i^{i \in [1, n]}$. The set $\bigcap \llbracket T_i \rrbracket_\Delta$ can also include objects that are not instances of any meet type from M . Hence,

$$\llbracket \wedge[T_i^{i \in [1, n]}] \rrbracket_\Delta = \bigcap_{i \in [1..n]} \llbracket T_i \rrbracket_\Delta \supseteq \bigcap_{j \in [1, m]} \llbracket M_j \rrbracket_\Delta = \llbracket \cap[T_i^{i \in [1, n]}] \rrbracket_\Delta.$$

In type-checking the ground triples, the join types are used in the procedure for checking the types derived bottom-up against the stored schema of a KG as presented in Section 5.2. The join as well as meet types are useful in the procedure for type-checking basic graph patterns [20]. The \vee and \wedge -types are logical types that can be simplified in the typing positions of a graph pattern by using typing rules, and can be approximated by using join and meet types to obtain a more precise concrete type of a graph pattern variable.

4.2 Typing with \wedge and \vee types

- *The use of \wedge and \vee types to describe identifiers.*
 - $v = \mathcal{I}_i \cup \mathcal{T}_i$ and $\tau = \mathcal{I}_c \cup \mathcal{T}_t$
 - General rules are defined to work with identifiers and triples.
 - Hence typing rules can be used to type idents and triples.
- *For $V \in v$ gather ground types of identifiers with \wedge -type as $V :_\downarrow \wedge[T_i^{i \in [1, n]}]$.*
 - $V \in v \quad \forall T_i \in \tau, t :_\downarrow T_i.$
 - The ground type of V is a type $T_g = \wedge[T_i^{i \in [1, n]}]$.
 - The following rule gathers all ground types of $V \in v$.

$$\frac{V \in v \quad T_i^{i \in [1, n]} \in \tau \quad V :_\downarrow T_i^{i \in [1, n]}}{V :_\downarrow \wedge[T_i^{i \in [1, n]}]} \quad (18)$$

Let's have a look at \wedge type composed of V 's ground types $T_i^{i \in [1, n]}$ in the case $V \in \mathcal{I}_i$. In Yago [9], often V has a set of very specific classes C_s but also some general classes C_g . The general classes C_g are close to the classes used in the stored triple types. If stored typing of V is correct, then C_g includes the classes that are supertypes of classes from C_s .

The ground type $\wedge[T_i^{i \in [1, n]}]$ can include pairs of types $T_i \preceq T_k$ with $i \neq k$. Depending on the further use, we can either compute the minimal or the maximal elements from the poset $\{T_i^{i \in [1, n]}\}$ with respect to \preceq [5]. The super-types of the minimal elements of $\{T_i^{i \in [1, n]}\}$ include all valid types of V . Hence, we use the set of minimal elements from $\{T_i^{i \in [1, n]}\}$ as the starting point to explore the relations between the ground triple types and the user-defined triple types of a triple t including V as a component.

The operator MIN is defined on a poset of types $(\{T_i^{i \in [1, n]}\}, \preceq)$. Given a set of types $\{T_i^{i \in [1, n]}\}$ the MIN operator retains types $S_j^{j \in [1, m]} \in \{T_i^{i \in [1, n]}\}$ such that $\#T_k^{k \in [1, n]}(T_k \prec S_j)$. All pairs of types $S_k, S_l \in \{S_j^{j \in [1, m]}\}$ with $k \neq l$ are *incomparable*, i.e., $S_1 \not\preceq S_2 \equiv S_1 \not\leq S_2 \wedge S_1 \not\geq S_2$. The logical rule for the operation MIN is as follows.

$$\frac{V \in v \quad V :_{\downarrow} \wedge [T_i^{i \in [1, n]}] \quad S \in \{T_k^{k \in [1, n]}\} \quad \forall i \in [1, n], S \preceq T_i \vee S \not\preceq T_i}{V :_{\downarrow} S} \quad (19)$$

The rule says that S is a minimal type of a ground type $\wedge [T_i^{i \in [1, n]}]$. S is minimal since all other T_i types are either more general or equal (\succeq), or not related to S . The rule generates all MIN types of $\wedge [T_i^{i \in [1, n]}]$.

The following rule is used for gathering all MIN types of $\wedge [T_i^{i \in [1, n]}]$. The result is a conjunction of minimal types $\wedge [S_1 \dots S_m]$.

$$\frac{V \in v \quad \forall i \in [1, m], V :_{\downarrow} S_i}{V :_{\downarrow} \wedge [S_j^{j \in [1, m]}]} \quad (20)$$

The rule for filtering $\wedge [T_i^{i \in [1, n]}]$ of all $T_i \succeq T_j$ where $i \neq j$ by using algorithmic typing is defined as follows. The algorithm implementing the operation MIN is presented in Section 6.1.

$$\frac{V \in v \quad \vdash V :_{\downarrow} \wedge [T_i^{i \in [1, n]}] \quad \vdash \{S_j^{j \in [1, m]}\} = \downarrow [T_i^{i \in [1, n]}]}{\vdash V :_{\downarrow} \wedge [S_j^{j \in [1, m]}]} \quad (21)$$

Now we have a minimal ground type of a value V in the form of a conjunction of minimal types S_i of V . The types that are important from the perspective of typing computer languages defined on values from a KG are the user-defined types of triples. The definition of a user-defined triple type requires the knowledge about the meaning of the binary relationship defined by a predicate—this is reflected in the selection of the domain and range of the predicate.

The first step in verifying the relations between the ground and user-defined types of a value V is the computation of a join type from a ground type of V . The join type of a ground type should be a subtype of the user-defined types that describe the value V . If the join type is not a subtype of the user-defined type then there is an error in stored types of a value V .

Let us now present the typing rules that, given $V \in v$, determine the join of $\{T_i^{i \in [1, n]}\}$ as $V : \sqcup [T_i^{i \in [1, n]}]$. Recall from Section 4.1 that we defined the operation join as the minimal upper bounds of a set $\{T_i^{i \in [1, n]}\}$. Let $P = (\tau, \preceq)$ and $\{T_i^{i \in [1, n]}\} \subseteq P$. A join $\sqcup [T_i^{i \in [1, n]}]$ is a set of minimal upper bounds $\{S_j^{j \in [1, m]}\}$ that are related to all types $T_i^{i \in [1, n]}$ via \preceq , and are minimal.

Rules 22-23 present the logical definition of the join type. The following Rule 22 determines one join type but can be used to generate all join types.

$$\frac{V \in v, V :_{\downarrow} \wedge [T_i^{i \in [1, n]}] \quad S \in \tau, T_i^{i \in [1, n]} \preceq S \quad \forall P \in \tau, (T_i^{i \in [1, n]} \preceq P \wedge S \preceq P) \vee P \not\preceq S}{V :_{\sqcup} S} \quad (22)$$

The individual join types derived by the above rule are gathered into one \wedge type of join types by using the following rule.

$$\frac{V \in v \quad S_i^{i \in [1, m]} \in \tau \quad \forall i \in [1, m], V :_{\sqcup} S_i}{V :_{\sqcup} \wedge [S_i^{i \in [1, m]}]} \quad (23)$$

The following Rule 24 derives the complete join type in one step. The operator $\text{join } \sqcup [T_i^{i \in [1, n]}]$ returns as a result a set of minimal upper bound types of $T_i^{i \in [1, n]}$. Since all the join types are valid types of V , we can group them into one \wedge type.

$$\frac{V \in v \quad \vdash V :_{\downarrow} \wedge [T_i^{i \in [1, n]}] \quad \vdash \{S_j^{j \in [1, m]}\} = \sqcup [T_i^{i \in [1, n]}]}{\vdash V :_{\sqcup} \wedge [S_j^{j \in [1, m]}]} \quad (24)$$

The join types are used as the starting point for searching the correct user-defined type. In the case the predicate p has two different definitions in two different contexts, then the paths from join types to the domain and range classes determines the definition of the triple type. The details are presented in Section 5.3.

– *Example from a KG.*

– *Let's have a look at \sqcup types of I 's ground types $S_1 \dots S_n$ from a KG.*

- *In many cases I has a single type S_1 which is the same as the join type S .*
- *The super-classes of the join type S have to be part of stored triple types*
- *to be selected as the types of subject or object.*
- *Often join type S is close to the classes that are components of stored triple types.*

– *Can the situation with two super-predicates of a predicate p lead to two different definitions of p ?*

5 Typing triples

– *We would like to type of a triple $t \in \mathcal{T}_i$.*

– *There are two basic aspects of a triple type.*

- *1. $t : T_g$ is computed bottom-up: from the stored types of triple components.*
- *2. $t : T_u$ can be computed from the user-defined types of properties.*
- *The relation $T_g \preceq T_u$ must hold if the typing of KG is correct.*
- *If the predicate p of type T_u is defined in multiple contexts, some of disjunctively*
- *linked components of T_u may not be related to T_g .*
- *The filtering of T_u is done by Rule 36.*

– *About the types that are computed bottom-up.*

- *Ground type of a triple is computed first by extending $:_{\downarrow}$ to triples.*
- *A triple can have multiple ground types $T_g = (\wedge [S_i^{i \in [1, k]}], p, \wedge [O_j^{j \in [1, m]}])$.*

- Next, the join type $\sqcup[T_g]$ of ground type T_g is derived.
- The type $\sqcup[T_g]$ is used as a stepping stone to determine the final type of t .
 - To be used for type-checking graph patterns.
- Stored triple types are user-defined types.
 - Stored types for a predicate p are defined via the predicates rdfs:domain and rdfs:range .
 - From the top of the ontology, the stored type $:\uparrow$ is determined based on p .
 - However, given p we can have a triple type (T_s, p, T_o) such that T_s or T_o are defined for some $p' \succeq p$.
 - Special case: p' has two domains T_s^1 and T_s^2 —type is then $(T_s^1 \vee T_s^2, p, _) \equiv (T_s^1, p, _) \vee (T_s^2, p, _)$.
 - $\mathcal{T}_\uparrow = \{(T_s, p, T_o) \mid p \in \mathcal{I}_p \wedge (p, \text{rdfs:domain}, T_s) \in \Delta \wedge (p, \text{rdfs:range}, T_o) \in \Delta\}$
 - Derived types of the stored types are computed using Rule 11.
 - Derived types of \mathcal{T}_\uparrow include the complete top of the ontology
- Stored triple types for a given predicate p are computed as MIN of valid stored types for p .
 - The MIN types of types obtained using $:\uparrow$ are the smallest triple types
 - including MIN classes as components.
 - Stored type have to be minimal to have minimal interpretation (e.g., type of a triple pattern).
 - Finally, the type $:\downarrow$ of t is determined by summing alternative $:\downarrow$ types.
 - Note there can be more than one \downarrow -type.
 - This happens when triple types include property that is defined in two different contexts.
- Interactions between the \wedge/\vee types of triple components and triples must be added.
 - Analogy between the types of functions in LC and types of triples.
 - Show rules relating \wedge/\vee types and triple types. Example.
 - E.g., $(S_1 \wedge S_2) * p * R = S_1 * p * R \wedge S_2 * p * R$.
 - Are all rules covered?
- Predicates should be treated in the same way as the classes.
 - They can have a rich hierarchy.
 - Note: Discussion on special role of predicates and their relations to classes?
 - Mention Cyc as the practical KB with rich hierarchy of predicates.

5.1 Triple types

A type of a triple $(s, p, o) \in v$ is a triple $(D, p, R) \in \tau$ such that $s : D$ and $o : R$ holds. The types D and R are type expressions that represent either a class or a \wedge -type composed of classes.

The \wedge -types reflect the semantics of RDF-Schema [18] which permits the definition of multiple domains and ranges of the predicate p . Consequently, each predicate p has exactly one triple type of the form

$$(\wedge[S_i^{i \in [1, n]}], p, \wedge[O_j^{j \in [1, m]}]).$$

Example 2. If p has two domains D_1 and D_2 , and a single range type R then the type corresponding to p is $(D_1 \wedge D_2, p, R)$. \square

The interpretation of a triple type (D, p, R) is defined as follows.

$$\llbracket (D, p, R) \rrbracket_{\Delta} = \{(s, p, o) \mid s \in \llbracket D \rrbracket_{\Delta} \wedge o \in \llbracket R \rrbracket_{\Delta}\}$$

The subtype relationship among the triples is defined on the basis of the subtype relationship among the classes and the \wedge -types defined on classes, and predicates. The following rule defines the relationship \preceq between two triple types.

$$\frac{T_1 \in \tau, T_1 = (D_1, p_1, R_1) \quad T_2 \in \tau, T_2 = (D_2, p_2, R_2) \quad D_1 \preceq D_2 \quad p_1 \preceq p_2 \quad R_1 \preceq R_2}{T_1 \preceq T_2} \quad (25)$$

It is obvious from the definition of the subtyping relationship among the triple types that $\llbracket T_1 \rrbracket_{\Delta} \subseteq \llbracket T_2 \rrbracket_{\Delta}$. Note that Rule 25 handles the \wedge -types of the subject and object through Rules 12-14.

5.2 Ground types of a triple

The ground types of a triple t are either a stored ground type, a minimal ground type, or a join ground type. The stored ground type includes types that are stored in a KG. The minimal ground type then consists of solely the minimal types from the stored ground types. Finally, the join ground type is the conjunction of minimal upper bound types [11].

A ground type of an individual identifier I is a class C related to I by one-step type relationship $:\downarrow$, as presented by Rule 2. In terms of the concepts of a knowledge graph, C and I are related by the relationship `rdf:type`.

A ground type of a triple $t = (I_s, p, I_o)$ is a product type $T_s * p * T_o$ that includes the ground types of t 's components I_s and I_o , and the property p which now has the role of a type. A ground type of a triple is defined by the following rule.

$$\frac{t \in v, t = (I_s, p, I_o) \quad T_s, T_o \in \tau \quad I_s : \downarrow T_s \quad I_o : \downarrow T_o \quad p : \downarrow \text{rdf:Property}}{t : \downarrow T_s * p * T_o} \quad (26)$$

The type T_s is either a class identifier or a \wedge -type composed of a conjunction of class identifiers. The predicates are treated differently to the subject and object components of triples. The predicates have the role of classes while they are instances of `rdf:Property`.

The minimal ground type of a triple t can be obtained by using the minimal ground types of the triple components.

$$\frac{t \in v, t = (I_s, p, I_o) \quad T_s, T_o \in \tau \quad I_s : \downarrow T_s \quad I_o : \downarrow T_o \quad p : \downarrow \text{rdf:Property}}{t : \downarrow T_s * p * T_o} \quad (27)$$

The component types T_s and T_o of the minimal ground type $T_s * p * T_o$ can represent \wedge -types. The following rule transforms a triple type including \wedge -types into a \wedge -type of

simple triple types composed of a class identifiers in place of S and O components. The rule is expressed in a general form by using the typing relation ":" which can be replaced by any labeled typing relation.

$$\frac{t \in v \quad t : \wedge[S_i^{i \in [1,n]} * p * \wedge[R_j^{j \in [1,m]}]}{t : \wedge_{i \in [1,n], j \in [1,m]}[S_i * p * T_j]} \quad (28)$$

The type in the conclusion of the rule is constructed by the Cartesian product of the sets of types belonging to types of S and O components. Since each of types S_i and R_j are valid for the components S and O , respectively, then also the triple types from the conclusion are valid.

The above decomposition of a triple type into a set of triple types is useful when we check the ground types against the user-defined types to select the valid user-defined type of a triple. This is detailed in Section 5.3.

Finally, the join of a set of ground types is a set of minimal upper bound types. Similarly to the previous two rules, the join is defined on the basis of joins of triple type components S and O .

$$\frac{t \in v, t = (I_s, p, I_o) \quad T_s, T_o \in \tau \quad I_s :_{\sqcup} T_s \quad I_o :_{\sqcup} T_o \quad p :_{\downarrow} \text{rdf:Property}}{t :_{\sqcup} T_s * p * T_o} \quad (29)$$

The join type $T_s * p * T_o$ includes in the components S and O the \wedge -types comprising one or multiple MUB classes. When we convert this type into a conjunction of single MUB triple types then each MUB triple type stands for all ground triple types of t .

We can easily see that each particular triple type is a MUB type since it includes MUB types in its components. Since the MUB types of the components are incomparable by \preceq then also the MUB triple types obtained by Rule 29 are incomparable.

– All rules defined for the ground triple types rely on the computation of MIN and MUB types of its components.

– This is because the stored types as well as the stored poset relation \preceq is defined on the class ontology.

5.3 Stored triple types

The stored triple types are types of triples defined by RDF-Schema [18] vocabulary. Each predicate has the domain and range defined either directly, when domain and range are defined for the predicate p , or indirectly, if the domain and range is defined for p 's super-predicates and inherited to the predicate p .

When a KG is restricted by using RDF-Schema we can specify one or more domain and range types. The semantics of RDF-Schema [19] interprets multiple domains and ranges with \wedge -type. If p has two domains T_1 and T_2 then p can link subjects I that are of type T_1 and T_2 . The domain type of p is then $T_1 \wedge T_2$, or, in terms of OWL [12], $\text{owl:intersectionOf}(T_1 \ T_2)$. The RDF-Schema does not allow the definition of the domain of a predicate with the \vee -type. Consequently, each predicate can have only one meaning.

Let us now inspect the same problem from the perspective of predicate modelling. In the case one predicate stands for one sense of a predicate, a polysemic predicate is specialized to sub-predicates modelling different senses. In the case that predicates of a KG have many senses, this representation can lead to predicate explosion. However, the reasoning is simpler if all predicates have unique meaning. The use of one sense for one predicate is detailed in Section 5.3.1.

The second way of modelling predicate sense is definition of the same predicate in multiple contexts. The context can be defined in many ways. Examples of contexts in KGs are RDF named graphs [4], microtheories of Cyc [16], and spaces in Scone [7]. The corresponding reasoners of KGs can define and use contexts. A predicate can be defined differently in different contexts but within one context we can only have one meaning of a predicate.

The context (sense) of a predicate can also be defined by using triple types including a given predicate. In other words, a sense of a predicate depends on the context implemented by a triple type. The definition of one predicate in multiple contexts is studied in 5.3.2.

5.3.1 KGs with RDF-Schema. Let us first define the rules for finding a stored type of a triple in the case RDF-Schema semantics is used in a KG. We first determine all stored triple types for a given triple $t = (s, p, o)$. A stored type is constructed by selecting triple types that are composed of a predicate p and the domain and range types linked to predicates $p' \succeq p$. The domain and range types can be defined for the predicate p and/or inherited from the predicates $p' \succ p$.

$$\frac{t \in v, t = (s, p, o) \quad p_1, p_2 \in \tau, p \preceq p_1 \preceq p_2 \quad \begin{array}{l} T_i^{i \in [1, n]} \in \tau, (p_1, \text{domain}, T_i) \in \Delta \quad S_j^{j \in [1, m]} \in \tau, (p_2, \text{range}, S_j) \in \Delta \end{array}}{t :_{\uparrow} \wedge [T_i^{i \in [1, n]}] * p * \wedge [S_j^{j \in [1, m]}]} \quad (30)$$

The above rule generates the types of the domain and range of a predicate p . The valid domain or range types of the predicate p can be any of types defined as domain or range for some $p' \succeq p$. Note that domain and range types can be defined for different predicates p_1 and p_2 "above" p ($p_1, p_2 \succeq p$) which have to be comparable, i.e., $p_1 \sim p_2$. This condition restricts the domain and range to be defined on the same path from p to some maximal element m of the predicate p poset. In the second part of the premise the domain and range types are gathered for the selected predicates p_1 and p_2 , respectively.

The above Rule 30 generates all valid stored types of a triple t . From the set of all valid stored types of t we select the subset including only the minimal types. The following rule is a logical judgment for a minimal stored triple type of a t .

$$\frac{t \in \mathcal{T}_i \quad T \in \tau, t :_{\uparrow} T \quad S_i^{i \in [1, n]} \in \tau, t :_{\uparrow} S_i \quad T \preceq S_i \vee T \not\preceq S_i}{t :_{\downarrow} T} \quad (31)$$

The first part of premise says that t is a ground triple and there exists $T \in \tau$ which is a type of t . The second part of the premise requires that T is the minimal type of all types S of t . In other words, there is no type $S_i^{i \in [1, n]}$ of t that is a subtype of T . Hence,

T is the minimal type of the stored triple types of t . Note that if the stored typing of a KG (using RDF.Schema) is correct then the condition $T \sim S_i$ is always *true*.

In the case that our KG is restricted to RDF-Schema data model then one predicate must represent exactly one meaning. Therefore, Rule 31 generates exactly one minimal type. Furthermore, under the restrictions of RDF-Schema we can not define a predicate p with two meanings. If we would specify two different domains or ranges of p then reasoner would treat the domain and range types as \wedge -types. Each instance of the domain (range) type have to be an instance of all specified types of the domain (range).

5.3.2 KGs with the contextual representation. The collective findings of the research in the area of Cognitive science [10] show that natural language is inherently contextual, and the context is essential in the human representation of knowledge and reasoning.

While the current trend is to enforce exactly one meaning of a predicates in KGs, the contextual representation and reasoning allows the definition of multiple senses of a predicate. There are many motivations for adopting contextual representation and reasoning in a KG. First, with the evolvement of KGs there are many examples where a KG is represented in the modular way splitting the dataset into parts that correspond to the contexts. The meaning of a predicate can be different in different contexts while the reasoner is able to disambiguate among the different meanings of a predicate. The practical examples of KGs using contexts include the named graphs in DBpedia [1], Wiki Data [22] and Yago [9]. Another example is Cyc [16] that uses microtheories to represent different contexts. Similarly to Cyc, Scone [7] is a KR system that can define spaces (contexts) and uses contextual reasoning.

The second motivation for using contextual representation of KGs is the problem of a predicate with mutiple senses represented with multiple sub-predicates. In a query—whether expressed in natural language, logic or as a database query—it is difficult to disambiguate the correct sub-predicate for the particular query. The alternative is that a user must explicitly select a correct sense of a predicate (i.e., a sub-predicate) in the query.

Finally, a predicate can be compared to a mathematical function since it represents a binary relation. In mathematics, a function is not represented by its name only, but with a function type including, besides the function name, also the types of its domain and range. Types of function can disambiguate among the different functions with the same name but different domain and range types. Similarly, the types can be effectively used to disambiguate the meaning of the predicate in a KG.

To be able to study the behavior of KG predicates with multiple senses in the presence of triple types, we propose a minimal KR schema language that includes the triple types stored in a KG as triples. In the case there are more than one triple types including a predicate p then these are treated as alternatives. For example, if a KG includes triples (c_1, p, c_2) and (c_3, p, c_4) , where $c_i \in \mathcal{I}_c$, then the type of ground triples including p is $c_1 * p * c_2 \vee c_3 * p * c_4$.

Further, the proposed KR language can use \wedge and \vee -types in subject and object components of triple types. The \wedge and \vee -types are often implemented in KGs in the

form of OWL type constructors `owl:intersectionOf` and `owl:unionOf`¹. The use of \vee -type in place of the domain or range type is redundant since it can be expressed with multiple triple types. Hence, the *minimal* KR schema language includes solely the triple types of the form

$$\wedge[c_i^{i \in [1, n]}] * p * \wedge[c_j^{j \in [1, m]}].$$

Let us now present the rules which derive the stored types of a ground triple t in the case our minimal KR schema language is used for the definition of the KG schema. First, the following Rule 32 generates all valid alternatives of stored triple types given a triple t . Note that D_i and R_i are the types of domains and ranges of p that can stand for a \wedge -type.

$$\frac{t \in v, t = (s, p, o) \quad T_i^{i \in [1, n]} \in \tau, T_i = (D_i, p, R_i) \quad T_i^{i \in [1, n]} \in \Delta}{t : \uparrow \vee [T_i^{i \in [1, n]}]} \quad (32)$$

A stored triple type $\vee [T_i^{i \in [1, n]}]$ of t is a disjunction of all valid stored triple types of t . Similarly to Rules 19-20, which are defined to find minimal types of a set of classes, the following Rules 33-34 generate the set of minimal stored triple types of a set including all valid stored types of t .

$$\frac{t \in v \quad t : \uparrow \vee [T_i^{i \in [1, n]}] \quad S \in \{T_k^{k \in [1, n]}\} \quad \forall i \in [1, n], S \preceq T_i \vee S \not\preceq T_i}{V : \Downarrow S} \quad (33)$$

The rule says that S is a minimal stored triple type of a stored triple type $\vee [T_i^{i \in [1, n]}]$. S is minimal since all other types T_i are either more general or equal (\succeq), or not related to S . The following rule gathers all minimal stored triple types of $\vee [T_i^{i \in [1, n]}]$.

$$\frac{t \in v \quad \forall i \in [1, m], t : \Downarrow S_i}{t : \Downarrow \vee [S_j^{j \in [1, m]}]} \quad (34)$$

The above rule is identical to the Rule 20 except that in this settings it handles triples and not identifiers. The result is a disjunction of minimal stored triple types $S_j^{j \in [1, m]}$.

The following Rule 35 is an algorithmic typing rule for deriving a minimal type of a type $\vee [T_i^{i \in [1, n]}]$ by computing a set of minimal stored triple types $\{S_j^{j \in [1, m]}\}$ from a poset of all valide stored triple types $(T_i^{i \in [1, n]}, \preceq)$. The algorithm implementing the operation minimum is presented in Section 6.1.

$$\frac{t \in v \quad \vdash t : \uparrow \vee [T_i^{i \in [1, n]}] \quad \vdash \{S_j^{j \in [1, m]}\} = \Downarrow [T_i^{i \in [1, n]}]}{\vdash t : \Downarrow \vee [S_j^{j \in [1, m]}]} \quad (35)$$

¹ The OWL union and intersection type constructors are employed mostly in domain specific KGs like biomedical and genomic ontologies. In these scientific fields the knowledge base includes large ontologies where new classes can be defined as logical combinations of existing classes.

5.4 Typing a triple

- Why using \wedge and \sqcap types for typing a triple t ?
 - We would like to check typing of a triple $t \in \mathcal{T}_i$.
 - We compute first the ground type $T_g = \wedge[T_i^{i \in [1, n]}]$ and a stored type T_s of t .
 - The ground type T_g is computed from the ground types of t 's components.
 - The subtype relation should hold $T_g \preceq T_s$.
- Two ways of defining semantics.
 - 1) enumeration style: stored types are enumerated as alternatives (\vee).
 - 2) packed together: alternative types are packed in one \vee type.
 - One advantage of (1) is that individual glb types can be processed further individually.
 - Advantage of (2) is the higher-level semantics without going in implementation.
- Stored types have to be related to all join ground types to represent the correct type of a triple.
- It seems it would be easier to check the pairs one-by-one using (1) in algorithms.
- In case of using complete types in the phases, types would further have to be processed by \wedge, \vee rules.

The type of a triple $t = (s, p, o)$ is computed by first deriving the base type T and the top type S of t . Then, we check if S is reachable from T through the sub-class and sub-property hierarchies, i.e., $T \preceq S$.

$$\frac{t \in \mathcal{T}_i \quad T \in \mathcal{T}_t, t : \downarrow T \quad S \in \mathcal{T}_t, t : \uparrow S \quad T \preceq S}{t : S} \quad (36)$$

- How to compute $T \preceq S$? Refer to position where we have a description.
- Order the possible derivations, gatherings (groupings) ... of types.
- Possible diagnoses.
- Components not related to a top type of a triple?
- Components related to sub-types of a top type?
- Above pertain to all components.

6 Implementation of type-checking

6.1 Computing MIN type

- We have a tuple $t = (s, p, o)$.
 - The task is to compute MIN type of t 's stored types.
 - The algorithm is using the predicates $p' \preceq p$ to obtain all domains and ranges of p .
- FUNCTION `typeGlbStored` (p : Property, cnt : Integer, d_p, r_p : Set): Type
- BEGIN
- if $d_p = \{\}$ then $d_p = \{c_s \mid (p, rdfs:domain, c_s) \in G\}$

```

– if  $r_p = \{\}$  then  $r_p = \{c_o \mid (p, \text{rdfs:range}, c_o) \in G\}$ 
– if  $d_p \neq \{\} \wedge r_p \neq \{\}$  then
– RETURN  $(\vee d_p, p, \vee r_p)$ 
–  $ts = \{\}$ 
– for  $p' \ ((p, \text{rdfs:SubPropertyOf}, p') \in G)$ 
– BEGIN
–  $T_{p'} = \text{typeGlbStored}(p', cnt + 1, d_p, r_p)$ 
–  $ts = ts \cup \{T_{p'}\}$ 
– END
– RETURN  $\vee ts$ 
– END

```

- Start with a set $\{p\}$ and close the set by using `rdfs:superPropertyOf` marking them with the “distance” from p .
 - Gather all domain and range types of $p' \preceq p$ in d_p and r_p , respectively, marking each domain and range class with the distance of the related p' .
 - Generate types $\vee(s_t, p, o_t)$ where $s_t \in d_p$, $o_t \in r_p$ and both s_t and o_t are marked with the minimal values.
 - Note that there can be more than one element s_t (and o_t) marked with a minimal value in d_p (and r_p). Hence Cartesian product of the selected domains and ranges are used to generate types (s_t, p, o_t) .

6.2 Computing join type

- A ground type of a triple $t = (s, p, o)$ is a type (s_t, p, o_t) such that $(s, \text{rdf:type}, s_t)$ and $(o, \text{rdf:type}, o_t)$.
 - The sets of types s_t and o_t are stored in the sets g_s and g_o , respectively.
 - The ground type of t is then $T_t = (\wedge g_s, p, \wedge g_o)$.
 - The lub of a type T_t is computed as follows.
 - For each $s_t \in g_s$ ($o_t \in g_o$) compute a closure of a set $\{s_t\}$ ($\{o_t\}$) with respect to the relationship `rdfs:superClassOf` obtaining the sets c_s and c_o .
 - Each step of the closure newly obtained classes are marked with the number of steps if new in the set, and the maximum of both numbers of steps otherwise.
 - The maximum guarantees the monotonicity: $s_t \prec_{\downarrow} s'_t \Rightarrow m(s_t) > m(s'_t)$.
 - Proof: Assume $m(s_t) \leq m(s'_t)$. Since $s_t \prec_{\downarrow} s'_t$ then $m(s'_t) + 1$ is maximum of s_t . Contradiction.
- The lub of T_t is computed by intersecting all sets c_s (c_o) obtained for each s_t (o_t), yielding a lub type of g_s (g_o).
 - The intersection of sets is computed step-by-step. Initially, intersection is the first set c_s (c_o) of some s_t (o_t).
 - In each step, a new set c_s (c_o) for another s_t (o_t) is intersected with previous result.
 - The classes that are in the result (intersection) are merged by selecting the maximum number of closure steps for each class in the intersection.

- The reason for this is to obtain the number of steps within which all ground classes reach a given lub class.
- Finally, the lub classes are selected from the resulted intersection of all c_s (c_o).
- The class with the smallest number of steps is taken. Then, it is deleted from the set together with all its super-classes.
- If set is not empty the previous step is repeated.

6.3 Relating LUB ground and MIN stored types

7 Related work

- Comparing typing relation in an OO model with a KG [15].
- The values from KGs have similar structure to the values of object-oriented models.
- However, the predicates of a KG are more expressive than the data members of the classes.
- Similarly, the record types form a lattice under subtype relationship with least upper bound and greatest lower bound based on sets of record attributes.
- Include the differences between Pierce’s (classical) sub-typing view of stored sub-class relationships among classes and the approach taken in this paper
- Pierce treats classes as generators of objects that inherit methods and data members from its super-class.
- The methods are inherited by copying the definitions in each subclass and then explicitly calling the method in the superclass.
- List the differences: classes are identifiers, there is a sub-class relationship included in a sub-typing relationship.

8 Conclusions

References

1. S. Auer, C. Bizer, G. Kobilarov, J. Lehmann, R. Cyganiak, and Z. Ives. Dbpedia: A nucleus for a web of open data. In *The Semantic Web*, volume 4825 of *Lecture Notes in Computer Science*, pages 722–735. Springer, 2007.
2. F. Baader, D. Calvanese, D. McGuinness, D. Nardi, and P. Patel-Schneider. *Description Logic Handbook*. Cambridge University Press, 2002.
3. R. J. Brachman and J. G. Schmolze. An overview of the kl-one knowledge representation system. *Cognitive Science*, 9:171–216, 1985.
4. J. J. Carroll, C. Bizer, P. Hayes, and P. Stickler. Named graphs, provenance and trust. In *Proceedings of the 14th International Conference on World Wide Web (WWW '05)*, pages 613–622. ACM, 2005.
5. B. A. Davey and H. A. Priestley. *Introduction to lattices and order*. Cambridge University Press, 2nd edition, 2002.

6. J. Dunfield and N. Krishnaswami. Bidirectional typing. *ACM Comput. Surv.*, 54(5), May 2021.
7. S. E. Fahlman. Using scone’s multiple-context mechanism to emulate human-like reasoning. In *Advances in Cognitive Systems, Papers from the 2011 AAAI Fall Symposium, Arlington, Virginia, USA, November 4-6, 2011*, volume FS-11-01 of *AAAI Technical Report*. AAAI, 2011.
8. J. R. Hindley. *Basic simple type theory*. Cambridge University Press, USA, 1997.
9. J. Hoffart, F. M. Suchanek, K. Berberich, and G. Weikum. Yago2: A spatially and temporally enhanced knowledge base from wikipedia. *Artificial Intelligence*, 194(0):28 – 61, 2013. Artificial Intelligence, Wikipedia and Semi-Structured Resources.
10. D. L. Hollister, A. J. Gonzalez, and J. Hollister. Contextual reasoning in human cognition and its implications for artificial intelligence systems. In *Modeling and Using Context (CONTEXT 2017)*, volume 10257 of *Lecture Notes in Computer Science*, pages 599–608. Springer, 2017.
11. S. B. Knudstorp. The modal logic of minimal upper bounds, 2024.
12. Owl 2 web ontology language. <http://www.w3.org/TR/owl2-overview/>, 2012.
13. B. C. Pierce. Programming with intersection types, union types, and polymorphism, 1991.
14. B. C. Pierce. Intersection types and bounded polymorphism, 1996.
15. B. C. Pierce. *Types and Programming Languages*. MIT Press, 1 edition, Feb. 2002.
16. D. Ramachandran, P. Reagan, and K. Goolsbey. First-orderized researchcyc: Expressivity and efficiency in a common-sense ontology. In *AAAI Reports*. AAAI, 2005.
17. Resource description framework (rdf). <http://www.w3.org/RDF/>, 2004.
18. Rdf schema. <http://www.w3.org/TR/rdf-schema/>, 2004.
19. Rdf schema. <https://www.w3.org/TR/rdf-mt/>, 2004.
20. I. Savnik. Type-checking graph patterns. Technical Report Technical Report (In preparation), FAMNIT, University of Primorska, 2025.
21. I. Savnik, K. Nitta, R. Skrekovski, and N. Augsten. Type-based computation of knowledge graph statistics. *Annals of Mathematics and Artificial Intelligence*, 2025.
22. D. Vrandečić and M. Krötzsch. Wikidata: A free collaborative knowledgebase. *Communications of the ACM*, 57(10):78–85, 2014.