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Algorithms for Big Data

Spring Semester 2022

Exercise Set 6

Definition 1 (Hadamard matrix) *We define* $H_1 = \begin{bmatrix}1\end{bmatrix}$ *and* $H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$. *We write* $F = \frac{1}{\sqrt{n}} H_n$, *dropping* n *from the index (and assuming* n *is a power of two).*

Exercise 1:
Show that $\|Fx\|_2 = \|x\|_2$ for any $x \in \mathbb{R}^n$.

Exercise 2:
Show that $FF = I$.

Exercise 3:
Show algorithm that given $x \in \mathbb{R}^n$ computes Fx in time $\mathcal{O}(n \log n)$.

Definition 2 (Fourier matrix) *Let* $\omega = 1^{1/n} = \cos(2\pi/n) + i\sin(2\pi/n)$. *We define* W *as a* $n \times n$ *matrix such that*

$$W = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \dots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \dots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix}.$$

Exercise 4:
Show that $\|Wx\|_2 = \|x\|_2$ for any $x \in \mathbb{C}^n$.

Exercise 5:
Show that $W\overline{W} = I$.

Exercise 6:
Show algorithm that given $x \in \mathbb{C}^n$ computes Wx in time $\mathcal{O}(n \log n)$.