

1D Potts model

N atoms, each with p states, 1d chain

$$H = -J \sum_{i=1}^N \sum_{i_n=1}^p \delta_{i_n, i_{n+1}}$$

$$i_{N+1} = i_1$$

a) Free energy for $p=2$ ($N \rightarrow \infty$)

b) Free energy for general p .

c) Internal energy $\langle H \rangle$ at low and high T limits.

$$T_{ij} = \begin{pmatrix} e^{\beta J} & 1 & \dots & 1 \\ 1 & e^{\beta J} & & \\ & & \ddots & \\ 1 & & & \end{pmatrix}$$

$$\begin{aligned} Z &= \text{tr}(e^{-\beta H}) = \sum_{\sigma_i=1}^p \langle \sigma_i | e^{-\beta H} | \sigma_i \rangle \\ &= \sum_{\sigma_1} \sum_{\sigma_2} \sum_{\sigma_3} \dots \langle \sigma_1 | P_{12} | \sigma_2 \rangle \langle \sigma_2 | P_{23} | \sigma_3 \rangle \dots \\ &= \text{tr}((P_{ij})^N) \approx \text{tr} \begin{pmatrix} E_{\max} & & \\ & 0 & \\ & & \ddots \end{pmatrix}^N \approx E_{\max}^N \end{aligned}$$

$$p=2 \rightarrow \begin{vmatrix} e^{\beta J} - \lambda & 1 \\ 1 & e^{\beta J} - \lambda \end{vmatrix} = 0 \rightarrow (e^{\beta J} - \lambda)^2 - 1 = 0$$

$$\lambda = e^{\beta J} \pm 1$$

choose largest eigenvalue

$$F = -kTN \ln \lambda_{\max} = -kTN \ln(e^{\beta J} + 1) = -kTN \ln \left[e^{\frac{\beta J}{2}} + e^{-\frac{\beta J}{2}} \right] e^{\frac{\beta J}{2}}$$

b) for general vector $\begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix}$

$$\hat{T}_{ij} \begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix} = \lambda \begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix}$$

$$= -kTN \ln(2 \cosh \frac{\beta J}{2}) - \frac{N}{2} J$$

similar to Ising model

$$\begin{array}{c} -J \\ -0 \end{array} \rightarrow \begin{array}{c} +\frac{J}{2} \\ -\frac{J}{2} \end{array} + \frac{J}{2}$$

$$\forall a_i: \left(e^{\beta J} a_i + 1 \cdot \sum_{j \neq i} a_j = \lambda a_i \right)$$

$$e^{\beta J} a_i + \sum_{j \neq i} a_j - a_i = \lambda a_i$$

$$a_i (\lambda + 1 - e^{\beta J}) = \sum_{j \neq i} a_j = \text{const (doesn't depend on } i)$$

$$\sum_{i=1}^p a_i = p \cdot \text{const} \quad \left(\sum_{i=1}^p 1 = p \right)$$

$$\left(\sum_{i=1}^p a_i \right) (\lambda + 1 - e^{\beta J}) = p \left(\sum_{i=1}^p a_i \right)$$

$$\sum a_i = 0 \quad \downarrow \quad \forall a_i = 0 \quad \text{or} \quad (\lambda + 1 - e^{\beta J}) = p \neq 0 \quad \text{or} \quad \lambda + 1 - e^{\beta J} = 0$$

$$\boxed{\lambda_{\max} = e^{\beta J} + p - 1}$$

$$\lambda_{\text{other}} = e^{\beta J} - 1 < \lambda_{\max}$$

degenerate eigvals

$$F = -kT N \ln(e^{\beta J} + p - 1)$$

$$\begin{aligned} c) E &= -\frac{\partial \ln Z}{\partial \beta} = \frac{\partial}{\partial \beta} (\beta F) = F + \beta \frac{\partial F}{\partial \beta} \\ &= -N \frac{e^{\beta J}}{e^{\beta J} + p - 1} = \begin{cases} -\frac{J}{p} N & T \rightarrow \infty, \beta \rightarrow 0 \\ -JN & T \rightarrow 0, \beta \rightarrow \infty \end{cases} \end{aligned}$$

At $T=0$ all atoms in the same state as there is no entropy gain.

At $T \rightarrow \infty$ entropy wins & we have atoms equally distributed so that prob. that n atoms is of same spin is $\frac{1}{p}$