B23

a) Grand Canonical ensemble:

$$Q(r) = \sum_{N=0}^{\infty} e^{\beta\mu N} \cdot Z_N(r)$$

$$Z_N(r) = \frac{\overset{\text{(spin)}}{2^N}}{N!}\left(\int d^3p\, e^{-\beta\frac{p^2}{2m}} \int dA \cdot e^{+\frac{\beta G M m}{r}} \cdot \Delta h\right)^N$$

define $V = A \cdot \Delta h$

$$= \frac{2^N}{N!}\left(\frac{V}{\lambda^3} e^{\frac{\beta G M m}{r}}\right)^N$$

$$Q(r) = \sum_{N=0}^{\infty} \frac{1}{N!}\left(\frac{2V}{\lambda^3} \cdot e^{\beta\left(\mu + \frac{GMm}{r}\right)}\right)^N$$

$$= \exp\left(\frac{2V}{\lambda^3} \cdot e^{\beta\left(\mu + \frac{GMm}{r}\right)}\right)$$

$$\Omega(r) = -kT \ln Q(r) = -2kT \frac{V}{\lambda^3} \cdot e^{\beta\left(\mu + \frac{GMm}{r}\right)}$$

$$N(r) = -\frac{\partial \Omega}{\partial \mu} = \frac{2V}{\lambda^3} e^{\beta\mu + \beta\frac{GMm}{r}}$$

$$\boxed{n(r) = \frac{2}{\lambda^3} e^{\beta\mu + \frac{\beta G M m}{r}}}$$

$\boxed{B23}$ Neutron star, $U = -\frac{GMm}{r}$

a) $n(r) = \frac{2}{V} \sum_p e^{-\beta\left(\frac{p^2}{2m} - \mu - \frac{mMG}{r}\right)}$ (2 spins)

$= \frac{2}{\lambda^3} \cdot e^{\beta\mu + \beta\frac{mMG}{r}}$

$n(r_0) = \frac{2}{\lambda^3} \cdot e^{\beta\mu + \frac{\beta mMG}{r_0}}$ $\mu(r) = \mu = \text{const}$

$n(r) = n(r_0) \cdot e^{\beta GMm\left(\frac{1}{r} - \frac{1}{r_0}\right)}$

$n(r \to \infty) > 0$ ie no confinement.

b) $n(r) = \frac{2}{V} \sum_p \frac{1}{e^{\beta\left(\frac{p^2}{2m} - \mu - \frac{GMm}{r}\right)} + 1}$

$= \frac{2}{\lambda^3} f_{3/2}(\underbrace{e^{\beta\mu + \beta\frac{GMm}{r}}}_{\text{"}z^*\text{"}})$

at $T \to 0$ $f_{3/2}(z^*) = \frac{4}{3\sqrt{\pi}}\left[(\ln z^*)^{3/2} + \frac{\pi^2}{8}(\ln z^*)^{-\frac{1}{2}} + \dots\right]$
(pathria 8.1.30)

$\rightarrow n(r)\Big|_{T\to 0} = 2\left(\frac{2\pi m}{\beta \hbar^2}\right)^{3/2} \cdot \frac{4}{3\sqrt{\pi}} \beta^{3/2}\left(\mu + \frac{GMm}{r}\right)^{3/2}$

$= \frac{8\pi}{3h^3} \cdot (2m)^{3/2}\left(\mu + \frac{GMm}{r}\right)^{3/2}$

$n(r_0) > 0 \rightarrow -\frac{GMm}{r_0} < \mu$, for $\mu < 0$ we have

$n(r) > 0 \rightarrow -\frac{GMm}{r} < \mu \rightarrow \boxed{r_0 < r < \frac{GMm}{|\mu|}}$

c) for $T \neq 0$ the expansion needs $T << \mu + \frac{GMm}{r}$

at $r \to \infty$ $\to$ $T << \mu$ $\to$ $\mu >> 0$

$n(\mu + \frac{GMm}{r})\Big|_{r \to \infty} \to n(\mu)$ always positive

for $\mu > 0$, ie. no confinement.

for $\mu < 0$ again $r < \frac{GMm}{|\mu|} \to n(r) \approx r^{-3/2}$

No confinement!

For $e^{\left(\frac{\mu + \frac{GMm}{r}}{kT}\right)} << 1$

ie for $|\mu| >> kT$, $\mu < 0$

ie. for $e^{-\frac{|\mu|}{kT}} << 1$ we have

$f_{3/2}\left(e^{\left(\frac{\mu + \frac{GMm}{r}}{kT}\right)}\right) \approx e^{\frac{\mu + \frac{GMm}{r}}{kT}}$

giving us the Classical sol.