D29 (HW 2009 11.1) Quantum Oscillator

$$\ddot{x} + \gamma \dot{x} + \Omega^2 x = A(t)$$

$$\varphi_A(\omega) = \frac{\hbar\omega\gamma}{M} \coth \frac{\hbar\omega}{2k_B T}$$

a) $$(-\omega^2 - i\omega\gamma + \Omega^2) x_\omega = A_\omega$$

$$|x_\omega|^2 = \frac{|A_\omega|^2}{(\omega^2-\Omega^2)^2 + \omega^2\gamma^2} \qquad \left(|v(\omega)|^2 = \omega^2 |x(\omega)|^2\right)$$

Wiener - Kinchin: $$\frac{|v(\omega)|^2}{|A(\omega)|^2} = \frac{\varphi_v(\omega)}{\varphi_A(\omega)}$$

$$\rightarrow \boxed{\varphi_v(\omega) = \frac{\omega^2}{(\omega^2-\Omega^2)^2 + \omega^2\gamma^2} \cdot \frac{\hbar\omega\gamma}{M} \coth\left(\frac{\hbar\omega}{2kT}\right)}$$

b) $$\lim_{\gamma \to 0} \frac{\gamma\omega}{(\omega^2-\Omega^2)^2 + \omega^2\gamma^2} = \pi\delta(\omega^2-\Omega^2)$$

$$\left(\pi\delta(x) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{x^2+\varepsilon^2}\right), \quad \delta(x^2-\alpha^2) = \frac{1}{2|\alpha|}\left(\delta(x+\alpha) + \delta(x-\alpha)\right)$$

$$= \frac{\pi}{2\Omega}\left(\delta(\omega-\Omega) + \delta(\omega+\Omega)\right)$$

$$\varphi_v(\omega)\Big|_{\gamma \to 0} = \frac{\pi\hbar\omega^2}{2M\Omega}\left[\delta(\omega-\Omega) + \delta(\omega+\Omega)\right] \coth\frac{\hbar\omega}{2kT}$$

$$\langle v^2(t) \rangle = \int \varphi_v(\omega) \cdot \frac{d\omega}{2\pi} = \frac{\hbar\Omega}{2M} \coth\frac{\hbar\Omega}{2kT}$$

$$\frac{1}{2} M \langle v^2 \rangle = \frac{\hbar\Omega}{4} \cdot \coth\frac{\hbar\Omega}{2kT} = \frac{1}{2}\hbar\Omega\left[\frac{1}{2} + \frac{1}{e^{\beta\hbar\Omega}-1}\right]$$

but $\hat{\mathcal{H}} = (a^{+}a + \frac{1}{2}) \cdot \hbar\Omega$

$$\langle a^{+}a \rangle = \frac{1}{e^{\beta\hbar\Omega} - 1}$$

$$\rightarrow E = \langle \hat{\mathcal{H}} \rangle = \hbar\Omega \left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\Omega} - 1} \right)$$

but kinetic energy $\frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} E$
for harmonic oscillator.