2D XY Model - Low Temperature Limit

The XY model is defined as follows:

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

1) Find the correlation function $C_{\vec{r}0} = \langle \cos(\theta_{\vec{r}} - \theta_0) \rangle$
2) Does long-range order exist in the system? Explain.
3) At higher T's, there exist vortex excitations with energy $E = \ln\left(\frac{a}{R}\right)$. Find the temperature these vortices proliferate.

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1) At $T \to 0$ minimum energy dictates ($\forall_{ij} : \theta_i = \theta_j$)
therefore at low T $\cos(\theta_i - \theta_j) \approx 1 - \frac{1}{2}(\theta_i - \theta_j)^2$
Translational invariance suggests we move to Fourier space, where

$$\theta_i = \frac{1}{\sqrt{N}} \sum_{\vec{k}} \tilde{\theta}_k e^{-i\vec{k}\cdot\vec{r}_i} \qquad \tilde{\theta}_k = \frac{1}{W} \sum_{\vec{r}_i} \theta_i e^{i\vec{k}\cdot\vec{r}_i}$$

$$\vec{k}_i = \frac{2\pi}{\sqrt{N}a}(n_x, n_y)$$

[we discard $k=0$ modes as they are rotations of all spins]

Orthogonality gives $\sum_{\vec{r}_i} e^{i\vec{k}\cdot\vec{r}_i} = N\delta_{\vec{k},0}$

$$\sum_i [\theta_i - \theta_{i+a\hat{x}}]^2 = \frac{1}{N} \sum_i \left[\sum_{\vec{k}} \tilde{\theta}_k e^{-i\vec{k}\cdot\vec{r}_i} - \sum_{\vec{k}} \tilde{\theta}_k e^{-i\vec{k}\cdot\vec{r}_i} e^{-ik_x a} \right]^2$$

$$= \frac{1}{N} \sum_i \left[\sum_{\vec{k}} \tilde{\theta}_k e^{-i\vec{k}\cdot\vec{r}_i} (1 - e^{-ik_x a}) \right]^2$$

$$= \frac{1}{N} \sum_{\vec{k}_1, \vec{k}_2} \sum_i \tilde{\theta}_{k_1} e^{-i\vec{k}_1\cdot\vec{r}_i} (1 - e^{-ik_{1x} a}) \tilde{\theta}_{k_2} e^{-i\vec{k}_2\cdot\vec{r}_i} (1 - e^{-ik_{2x} a})$$

$$= \frac{1}{N} \sum_{\bar{k}_1, \bar{k}_2} \tilde{\theta}_{\bar{k}_1} \tilde{\theta}_{\bar{k}_2} \left(1 - e^{-ik_{1x}a} - e^{-ik_{2x}a} + e^{-i(k_1+k_2)_x a}\right) \underbrace{\sum_{\bar{r}_i} e^{-i(\bar{k}_1+\bar{k}_2)\bar{r}_i}}_{= N\delta_{\bar{k}_1, -\bar{k}_2}}$$

$$= \sum_{\bar{k}} \tilde{\theta}_{\bar{k}} \tilde{\theta}_{-\bar{k}} \left(2 - e^{-ik_x a} - e^{+ik_x a}\right) = \sum_{\bar{k}} \tilde{\theta}_{\bar{k}} \tilde{\theta}_{-\bar{k}} \left(2 - 2\cos(k_x a)\right)$$

using $|k| \ll 1$ because $\lambda \sim \frac{1}{k} \gg 1$
(low energy excitations) $\cos(k_x a) \approx 1 - \frac{1}{2}(k_x a)^2$

$$\approx \sum_{\bar{k}} \tilde{\theta}_{\bar{k}} \tilde{\theta}_{-\bar{k}}\, k^2 a^2$$

Giving us: $\mathcal{H} = \frac{+J}{2} \sum_{k} (ka)^2 \tilde{\theta}_k \tilde{\theta}_{-k}$ but

$\theta_i \in \mathbb{R} \Leftrightarrow \tilde{\theta}_{-k} = \theta_k^*$ giving

$$\mathcal{H} = \frac{+J}{2} \sum_{k} (ka)^2 |\theta_k|^2 \quad \text{or} \quad J \sum_{k>0} (ka)^2 |\theta_k|^2$$

Partition function now factorizes:

$$Z = \int_{-\infty}^{\infty} \prod_{k} d\tilde{\theta}_k \, e^{-\frac{\beta J}{2}(ka)^2 |\tilde{\theta}_k|^2} = \prod_{k} \int_{-\infty}^{\infty} d\theta \, e^{-\frac{\beta J}{2}(ka)^2 \theta^2} =$$

$$= \prod_{k} \sqrt{\frac{2\pi}{\beta J (ka)^2}} = \prod_{k>0} \frac{2\pi}{\beta J (ka)^2}$$

Now the corr. function:

$$\langle \cos(\theta_{\bar r} - \theta_{\bar 0}) \rangle = \langle e^{i(\theta_{\bar r} - \theta_0)} \rangle$$ (imaginary part $= 0$)

$$= \left\langle e^{\frac{i}{\sqrt N}\sum_k \tilde\theta_k (e^{i\bar k\cdot\bar r} - 1)} \right\rangle = \frac{\int \prod_k \left(d\theta_k\, e^{i\mu_k \theta_k} e^{-\frac{\beta J}{2}(ka)^2|\theta_k|^2} \right)}{\int \prod_k \left(d\tilde\theta_k\, e^{-\frac{\beta J}{2}(ka)^2|\theta_k|^2} \right)}$$

with $\mu_k = \frac{1}{\sqrt N}(e^{i\bar k\cdot \bar r} - 1)$

$\theta_k = X + iY$, $\mu_k = \mu_1 + i\mu_2$, $\tilde\theta_{-k} = \theta_k^*$, $\mu_{-k} = \mu_k^*$

$$\int_0^\infty e^{i(\mu_k\theta_k + \mu_{-k}\theta_{-k})} e^{-\beta J (ka)^2 |\theta_k|^2}\, d\theta_k\, d\theta_k$$ (integrate over half plane)

$$\int_0^\infty e^{2i(\mu_1 X - \mu_2 Y)} e^{-\beta J (ka)^2 (X^2 + Y^2)}\, dX dY$$

$$= \left(\int_0^\infty dX \cdot e^{2i\mu_1 X - \beta J (ka)^2 X^2} dX \right) \left(e^{-2i\mu_2 Y} e^{-\beta J (ka)^2 Y^2} dY \right)$$

$$e^{-\frac{\mu_1^2}{\beta J (ka)^2}} \int_0^\infty e^{-\beta J (ka)^2 \left(X - \frac{i\mu_1}{J(ka)^2}\right)^2} dX \cdot e^{-\frac{\mu_2^2}{\beta J (ka)^2}} \int$$

cancel with denominator in partition function

$$= e^{-\frac{(\mu_1^2 + \mu_2^2)}{\beta J (ka)^2}} \rightarrow g(r) = e^{-\frac{1}{2\beta J}\sum_k \frac{|\mu_k|^2}{(ka)^2}} = e^{-\frac{1}{2\beta J}\underbrace{\sum_k \frac{2 - 2\cos(\bar k \cdot \bar r)}{(ka)^2}}_{= f(\bar r)}}$$

How to sum $f(\bar{r})$?

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(\bar{r}) = \sum_{\bar{k}} 2\cos(\bar{k}\cdot\bar{r}) = 2N\delta_{\bar{r},0}$$

This is the Poisson equation for a charge in 2d, giving

$$f(\bar{r}) = \frac{1}{\pi}\ln r$$

Therefore $g(\bar{r}) = g(|\bar{r}|) \propto e^{\frac{-T\ln r}{2\pi J}} = r^{\frac{-T}{2\pi J}}$

No long range order in the system! This is quasi-long range.

(as $J \to \infty$ we have $g(r) \to 1$)

Mermin-Wagner (1966):

"Continuous symmetries cannot be spontaneously broken at finite T in systems with sufficiently short-range interactions in dimensions $d \leq 2$"

Vortices in the XY-model

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \approx E_0 + \frac{J}{2} \sum_{\langle ij \rangle} (\theta_i - \theta_j)^2$$

$$\approx E_0 + \frac{J}{2} \int d\bar{r} (\nabla \theta)^2 \qquad (E_0 = 2JN \text{ for completely alligned rotors})$$

Extrema of $\mathcal{H}$: $\quad \frac{\delta \mathcal{H}}{\delta \theta(\bar{r})} = 0 \implies \nabla^2 \theta(\bar{r}) = 0$

Two options: $\theta(\bar{r}) = \text{const}$ (normal G.S.)

$\oint (\bar{\nabla}\theta) \cdot d\bar{l} = 2\pi n \leftarrow$ enclosing a vortex

$$\oint (\bar{\nabla}\theta) \cdot d\bar{l} = |\bar{\nabla}\theta| \cdot 2\pi r = 2\pi n \rightarrow \quad |\nabla \theta| = \frac{n}{r}$$

$$E_{vor} = \frac{J}{2} \int d\bar{r} \cdot (\bar{\nabla}\theta)^2 = \frac{1}{2} \cdot 2\pi \cdot J n^2 \int_a^R r\,dr \cdot \frac{1}{r^2} = n^2 \pi J \ln\left(\frac{R}{a}\right)$$

energy goes as n^2, so we take $n = 1$

$$E_{vor} = \pi J \ln\left(\frac{R}{a}\right)$$

vortex size: a, therefore $\left(\frac{R}{a}\right)^2$ possible vortex configurations.

$$S_{vort} = k_B \ln\left(\left(\frac{R}{a}\right)^2\right) = 2k_B \ln\left(\frac{R}{a}\right)$$

$$F_{vort} = E_v - TS_v = \ln\left(\frac{R}{a}\right)(\pi J - 2T) \rightarrow$$

$F < 0 \Leftrightarrow$ vortex generation favorable $\quad \boxed{T_c = \frac{\pi}{2} J}$

and for $g(r) \sim r^{-\frac{T}{2\pi J}} \sim r^{-\frac{T}{4T_c}} \Big|_{T_c} \sim r^{-\frac{1}{4}} \sim r^{-\eta} \Leftrightarrow \left[\eta(T_c) = \frac{1}{4}\right]$

where does this argument fail?