(Hw 2009 8.1)

$\boxed{C8}$ We work in the Grand Canonical Ensemble:

Single site: $\mathcal{Z}_1 = 1 + e^{\beta\mu} \cdot \frac{a^3}{\lambda^3} = 1 + \zeta \frac{a^3}{\lambda^3}$

M sites: $\mathcal{Z}_M = \left(1 + \zeta \frac{a^3}{\lambda^3}\right)^M$

(Also from $\mathcal{Z} = \sum_{N=0}^{M} \zeta^N \left(\frac{a^3}{\lambda^3}\right)^N \cdot \frac{M!}{N!(M-N)!} = \left(1+\zeta\frac{a^3}{\lambda^3}\right)^M$)

a) $N = \zeta \frac{\partial \ln \mathcal{Z}}{\partial \zeta} = \zeta M \frac{\partial \ln\left(1+\zeta\frac{a^3}{\lambda^3}\right)}{\partial \zeta} = \zeta M \frac{a^3/\lambda^3}{1+\zeta\frac{a^3}{\lambda^3}}$

$n = \frac{N}{Ma^3} = \frac{\zeta/\lambda^3}{1+\zeta a^3/\lambda^3}$

b) $\ln \mathcal{Z} = \beta P V = M \ln\left(1+\zeta\frac{a^3}{\lambda^3}\right)$

$na^3 = \frac{1}{\frac{1}{\zeta}\frac{\lambda^3}{a^3}+1} \rightarrow \frac{1}{\zeta}\frac{\lambda^3}{a^3} = \frac{1}{na^3} - 1$

$$\rightarrow \boxed{\zeta\frac{a^3}{\lambda^3} = \frac{na^3}{1-na^3}}$$

$Pa^3 = kT \ln\left(1+\zeta\frac{a^3}{\lambda^3}\right) = kT \ln\left(1+\frac{na^3}{1-na^3}\right)$

$$\boxed{Pa^3 = kT \ln\left(\frac{1}{1-na^3}\right)}$$

as $n \to 0$: $kT\ln\left(\frac{1}{1-na^3}\right) = -kT\ln\left(1-na^3\right) \approx kT\, na^3$

$P \to nkT$ as ideal gas.

As $na^3 \to 1$ we get a singularity, ie. the pressure diverges since we simply cannot add any more particles.

c) For a phase transition we require an attraction (nearest-neighbor or otherwise) which will lead to a 1^{st} order transition