a. Understand the concept of a complex number.

NOTE:

- A complex number is a number consisting of **a real part** and **an imaginary part**.
- A complex number form is **a + bi**, where **a** is a real part and **bi** is an imaginary part.
  For example: 15 + 2i (15 is a real part and 2i is an imaginary part)
- Properties of complex number:

$$i = \sqrt{-1} \text{ and } i^2 = -1$$
$$(-1)^{even\ power} = 1$$
$$(-1)^{odd\ power} = -1$$

Example 1:

Express the following in terms of i:

a. $\sqrt{-25}$

$= \sqrt{-1(25)}$

$= \sqrt{-1}\sqrt{25}$

$= 5i$ (answer)

b. $-\sqrt{-11}$

$= -\sqrt{-1(11)}$

$= -\sqrt{-1}\sqrt{11}$

$= -\sqrt{11}i$ (answer)

Example 2:

Simplify i^{10}.

$i^{10} = i^{2(5)}$

$= (-1)^5$

$= -1$ (answer)

Example 3:

Solve $-7i^3 + 2i^8$.

$= -7i^2 i + 2i^{2(4)}$

$= -7(-1)i + 2(-1)^{(4)}$

$= 7i + 2(1)$

$= 7i + 2$ (answer)

EXERCISE Express each of the following in terms of i:		
1. $\sqrt{-16}$ $Ans: 4i$	2. $\sqrt{-7}$ $Ans: \sqrt{7}i$	3. $-\sqrt{-4}$ $Ans: -2i$
4. $\sqrt{-10}$ $Ans: \sqrt{10}i$	5. $\sqrt{-49}$ $Ans: 7i$	6. $-\sqrt{-13}$ $Ans: \sqrt{-13}i$
EXERCISE Solve each of the following:		
7. $-1-8i-4-i$ $Ans: -5-9i$	8. i^{17} $Ans: i$	9. $(5i)^3$ $Ans: -125i$
10. $-6(-2+9i)$ $Ans: 12-54i$	11. $2-i^{12}$ $Ans: 1$	12. $-5i8i$ $Ans: 40$
13. i^6 $Ans: -1$	14. i^{11} $Ans: -i$	15. $-i^3+3i^{20}-6i^{43}$ $Ans: 3+7i$

b. Calculate the operation of complex number:

i. Addition
ii. Subtraction
iii. Multiplication
iv. Division
v. Conjugate

Note and Example : ADDITION

- Complex numbers are **added** by adding the real and imaginary parts of the summands:

$$(a + b\,i) + (c + d\,i) = \underbrace{(a + c)}_{\text{Real part}} + \underbrace{(b + d)}_{\text{Imaginary part}}\,i$$

- Example:
  $(2 + 3\,i) + (-4 + 5\,i) = (2 + (-4)) + (3 + 5)\,i$

  $= -2 + 8\,i$ (answer)

Note and Example : SUBTRACTION

- Subtraction of complex number is defined by:

$$(a + b\,i) - (c + d\,i) = (a - c) + (b - d)\,i$$

- Example:
  $(2 + 3\,i) - (-4 + 5\,i) = (2 - (-4)) + (3 - 5)\,i$

  $= 6 - 2\,i$ (answer)

Note and Example : MULTIPLICATION

- The multiplication of two complex numbers $a + b\,i$ and $c + d\,i$ is defined as follows:
  $(a + b\,i)(c + d\,i) = (ac - bd) + (ad + bc)i$

- Multiplying complex numbers works like multiplying two binomials by using the FOIL method. Recall that FOIL means to multiply the **F**irst terms, **O**utside terms, **I**nside terms and **L**ast terms.

- However you do not need to memorize the above definition as the multiplication can be carried out using properties similar to those of the real numbers and the added property $i^2 = -1$.

- Example:

$$(2+5\text{i})\,(6+3\text{i}) = 12\boxed{+\,6i+30i}+15\boxed{i^2}$$

$$= 12 + 36i + 15(-1)$$

$$= 12 + 36i - 15$$

$$= -3 + 36i$$ (answer)

Note and Example : DIVISION

- A division problem is usually given in **fraction** form. To solve a division problem, we will need to know the **conjugate** of the denominator.
- The conjugate of the complex number z = a + bi is defined as $\tilde{z}$ = a – bi.
- $\tilde{z}$ also can be denoted as $\bar{z}$ or z^*.
- It is simply the same numbers of the complex number but with a different sign between them. For example, the conjugate of $2+5i$ is $2-5i$.
- $\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di}$
- Example:

$$\frac{2}{2+5i} = \frac{2}{2+5i} \times \frac{2-5i}{2-5i}$$

$$= \frac{4-10i}{4-25i^2}$$

$$= \frac{4-10i}{4-25(-1)}$$

$$= \frac{4-10i}{29}$$

$$= \frac{4}{29} - \frac{10i}{29}$$ (answer)

EXERCISE Solve:		
1. $(-7+9i)+(2-10i)$ $Ans: -5-i$	2. $(5+2i)+(3-2i)$ $Ans: 8$	3. $(-2+4i)+(3-9i)$ $Ans: 1-5i$

4. $(8 + 5i) - (2 + 3i)$ $Ans: 6 + 2i$	5. $(-2+4i)-(3+9i)$ $Ans: -5 - 5i$	6. $(5-2i)-2(3+i)$ $Ans: -1 - 4i$
7. $(1 + 2i)(6 + 3i)$ $Ans: 15i$	8. $(3 + 4i)(3 - 4i)$ $Ans: 25$	9. $(4 + i)\ (4 + i)$ $Ans: 15 + 8i$
10. $\frac{6}{1-i}$ $Ans: 3 + 3i$	11. $\frac{4+3i}{2-3i}$ $\frac{-1+18i}{13}$	12. $\frac{1-4i}{2+4i}$ $Ans: \frac{-7-6i}{10}$

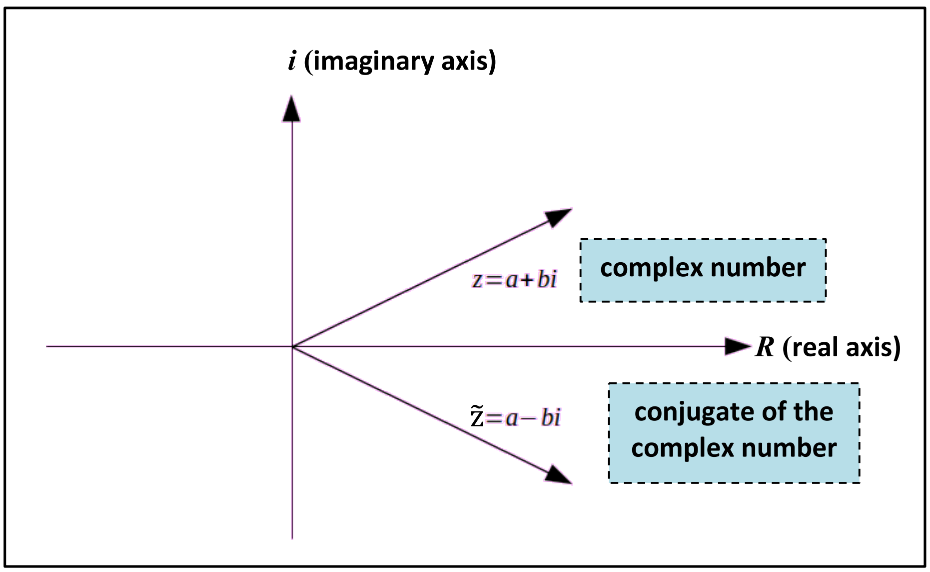
c. Understand graphical representation of a complex number through Argand Diagram:

i. Argand Diagram.

ii. Modulus and argument.

Note

- **Argand diagram** used to plot a complex number. On the horizontal axis are the real numbers and on the vertical axis are the imaginary numbers.
- Example of an argand diagram:



- **Modulus** of a complex numbers, $|z| = R = \sqrt{a^2 + b^2}$
- **Argument** of a complex numbers, $arg\ (z) = \tan^{-1}\left(\frac{b}{a}\right)$
- Quadrant position and formula to find $arg\ (z)$ or θ:

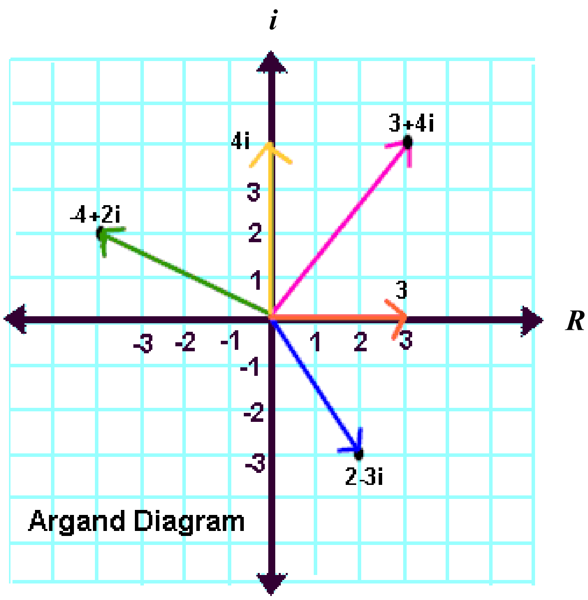
2nd Quadrant $arg(z) = 180° - \alpha$	**1st Quadrant** $arg(z) = \alpha$
3rd Quadrant $arg(z) = 180° + \alpha$	**4th Quadrant** $arg(z) = 360° - \alpha$

Example 1:

Draw an argand diagram for the following complex numbers:

a) 3 + 4i b) 2 - 3i c) -4 + 2i

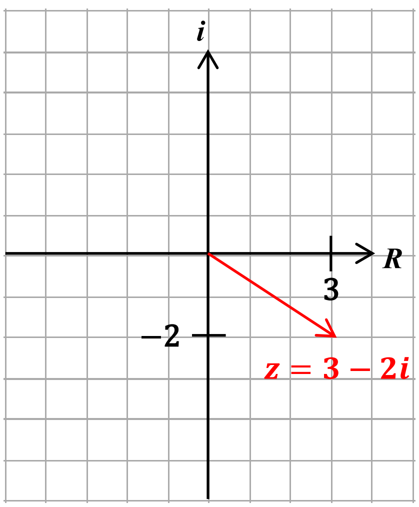
d) 3 (which is really 3 + 0i) e) 4i (which is really 0 + 4i)



(answer)

Example 2:

Find the modulus and the argument for $z = 3 - 2i$.



modulus, $|\boldsymbol{z}| = \sqrt{(3)^2 + (-2)^2}$

$= \sqrt{13}$ (answer)

argument, $\boldsymbol{arg}\ (\boldsymbol{z}) = tan^{-1}\left(\frac{-2}{3}\right)$

$\alpha = -33.69°$

$\theta = \boxed{360° - 33.69°}$

$\theta = 326.31°$ (answer)

- Note that θ = **360°−** for the **4th quadrant**.
- Ignore the '−ve' sign when you substitute α into formula.

EXERCISE Draw an Argand Diagram to represent the following complex number:		
1. $2 + 5i$	2. $4 - 5i$	3. $-3 + 6i$
4. $-5 - 2i$	5. $-6i$	6. -3

EXERCISE

Find the modulus and the argument for:

7. $z = -4 + i$	8. $z = 3 + 2i$	9. $z = -3 - 2i$
$\lvert z\rvert = \sqrt{17}$ $\theta = 165.964°$	$\lvert z\rvert = \sqrt{13}$ $\theta = 33.69°$	$\lvert z\rvert = \sqrt{13}$ $\theta = 213.69°$

10. $z = 3 + 4i$ $\lvert z \rvert = 5$ $\theta = 53.13°$	11. $z = -1 - i$ $\lvert z \rvert = \sqrt{2}$ $\theta = 225°$	12. $z = 2 + 7i$ $\lvert z \rvert = \sqrt{53}$ $\theta = 74.055°$

d. Understand Complex Number in other form:

i. Complex number in Polar form and Exponential form.

ii. Multiplication and division of complex number in polar form.

Note:

- Form of complex number:

Cartesian Form	**Polar Form**	**Exponential Form**
$Z = a + bi$	$Z = \lvert z \rvert \angle \theta$ (θ should be in degree)	$Z = Re^{\theta i}$ (θ should be in radian)

- **Multiplication** and **division** of complex number in polar form:

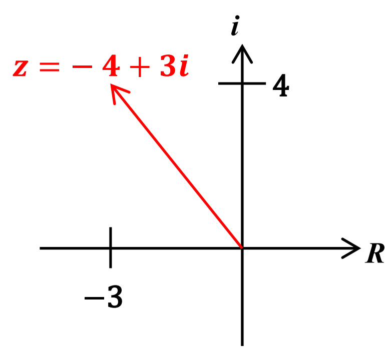
Multiplication	**Division**
Modulus1 × Modulus2 ∠ arg1 + arg2	Modulus1 ÷ Modulus2 ∠ arg1 − arg2

Example 1:

Change the complex number Z = – 3 + 4i to Polar and Exponential form.

$$|z| = R = \sqrt{(-3)^2 + (4)^2}$$
$$= \sqrt{25}$$
$$= 5$$

$$arg\ z = tan^{-1}\left(\frac{4}{-3}\right)$$
$$\alpha = -53.13°$$
$$\theta = 180° - 53.13°$$
$$\theta = 126.87°$$



REMEMBER !

2nd Quadrant
$arg(z) = 180° - \alpha$

- Polar Form:

$Z = |z| \angle \theta$

$Z = 5 \angle\ 126.87°$ (answer)

- Exponential Form:

$Z = Re^{\theta i}$

$$\theta\ in\ radian = \frac{126.87° \times \pi}{180}$$

$= 2.214$ rad

$Z = 5e^{2.214i}$ (answer)

Example 2:

Express $Z = \sqrt{45}\angle\ 13°$ to Exponential form.

$|z| = \sqrt{45}$

$\arg(z)\,, \theta = 13°$

- Exponential Form:

$Z = Re^{\theta i}$

$\theta\ in\ radian = \dfrac{13° \times \pi}{180}$

$= 0.227$ rad

$Z = \sqrt{45}e^{0.227i}$ (answer)

Example 3:

Given $Z_1 = 6\angle 25°$ and $Z_2 = 17\angle 44°$. Find the value of:

a) Z_1Z_2

$= 6\angle 25°. 17\angle 44°$
$= 6 \times 17\angle 25° + 44°$
$= 102\angle 69°$ (answer)

b) $\dfrac{Z_1}{Z_2}$

$= \dfrac{6\angle 25°}{17\angle 44°}$

$= \dfrac{6}{17}\angle 25° - 44°$

$= \dfrac{6}{17}\angle - 19°$ (answer)

c) $\dfrac{2Z_1}{Z_1Z_2}$

$= \dfrac{2(6\angle 25°)}{6\angle 25°.17\angle 44°}$

$= \dfrac{12\angle 50°}{102\angle 69°}$

$= \dfrac{2}{17}\angle 50° - 69°$

$= \dfrac{2}{17}\angle - 19°$ (answer)

EXERCISE

Express each of the following complex number in polar and exponential form:

1. $z = -1 + 5i$ PF: $\sqrt{26} \angle 101.31°$ EF: $\sqrt{26}e^{1.768i}$	2. $z = -1 - 5i$ PF: $\sqrt{26} \angle 258.69°$ EF: $\sqrt{26}e^{4.515i}$	3. $z = 3 - 3i$ PF: $\sqrt{18} \angle 315°$ EF: $\sqrt{18}e^{5.498i}$
4. $z = -6 + 4i$ PF: $\sqrt{52} \angle 146.31°$ EF: $\sqrt{52}e^{2.554i}$	5. $z = -1 - i$ PF: $\sqrt{2} \angle 225°$ EF: $\sqrt{2}e^{3.927i}$	6. $z = 2 + 9i$ PF: $\sqrt{85} \angle 77.471°$ EF: $\sqrt{85}e^{1.352i}$

EXERCISE Find the value of Z_1Z_2 if:		
7. $Z_1 = 3\angle 15°$, $Z_2 = 7\angle 14°$ $21\angle 29°$	8. $Z_1 = 10\angle 115°$, $Z_2 = 2\angle 140°$ $20\angle 255°$	9. $Z_1 = 3\angle 65°$, $Z_2 = 15\angle 41°$ $45\angle 106°$

EXERCISE Find the value of $\frac{Z_1}{Z_2}$ if:		
10. $Z_1 = 3\angle 15°$, $Z_2 = 7\angle 14°$ $\frac{3}{7}\angle 1°$	11. $Z_1 = 10\angle 115°$, $Z_2 = 2\angle 140°$ $5\angle -25°$	12. $Z_1 = 3\angle 65°$, $Z_2 = 15\angle 41°$ $\frac{1}{5}\angle 24°$

EXERCISE Find the value of $\frac{3Z_2}{Z_1Z_2}$ if:		
13. $Z_1 = 3\angle 15°$, $Z_2 = 7\angle 14°$ $1\angle 13°$	14. $Z_1 = 10\angle 115°$, $Z_2 = 2\angle 140°$ $\frac{3}{10}\angle 165°$	15. $Z_1 = 3\angle 65°$, $Z_2 = 15\angle 41°$ $1\angle 17°$