3 Indices and Standard Form

3.1 Index Notation

Here we revise the use of index notation. You will already be familiar with the notation for squares and cubes

$$\boxed{\begin{aligned} a^2 &= a \times a \\ a^3 &= a \times a \times a \end{aligned}}\ , \text{ and}$$

this is generalised by defining:

$$\boxed{a^n = \underbrace{a \times a \times \ldots \times a}_{n \text{ of these}}}$$

Example 1

Calculate the value of:

(a) 5^2 (b) 2^5 (c) 3^3 (d) 10^4

Solution

(a) $5^2 = 5 \times 5$
$= 25$

(b) $2^5 = 2 \times 2 \times 2 \times 2 \times 2$
$= 32$

(c) $3^3 = 3 \times 3 \times 3$
$= 27$

(d) $10^4 = 10 \times 10 \times 10 \times 10$
$= 10\ 000$

Example 2

Copy each of the following statements and fill in the missing number or numbers:

(a) $2^{\square} = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

(b) $9 = 3^{\square}$

(c) $1000 = 10^{\square}$

(d) $5^3 = \square \times \square \times \square$

Solution

(a) $2^7 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

(b) $9 = 3 \times 3 = 3^2$

(c) $1000 = 10 \times 10 \times 10 = 10^3$

(d) $5^3 = 5 \times 5 \times 5$

Example 3

(a) Determine 2^5.

(b) Determine 2^3.

(c) Determine $2^5 \div 2^3$.

(d) Express your answer to (c) in index notation.

Solution

(a) $2^5 = 32$

(b) $2^3 = 8$

(c) $2^5 \div 2^3 = 32 \div 8$
$\quad\quad\quad\;\;= 4$

(d) $4 = 2^2$

Exercises

1. Calculate:

(a) 2^3	(b) 10^2	(c) 3^2
(d) 10^3	(e) 9^2	(f) 3^3
(g) 2^4	(h) 3^4	(i) 7^2

2. Copy each of the following statements and fill in the missing numbers:

(a) $10 \times 10 \times 10 \times 10 \times 10 = 10^{\square}$

(b) $3 \times 3 \times 3 \times 3 = 3^{\square}$

(c) $7 \times 7 \times 7 \times 7 \times 7 = 7^{\square}$

(d) $8 \times 8 \times 8 \times 8 \times 8 = 8^{\square}$

(e) $5 \times 5 = 5^{\square}$

(f) $19 \times 19 \times 19 \times 19 = 19^{\square}$

(g) $6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 = 6^{\square}$

(h) $11 \times 11 \times 11 \times 11 \times 11 \times 11 = 11^{\square}$

3. Copy each of the following statements and fill in the missing numbers:

(a) $8 = 2^{\square}$

(b) $81 = 3^{\square}$

(c) $100 = 10^{\square}$

(d) $81 = 9^{\square}$

(e) $125 = 5^{\square}$

(f) $1\,000\,000 = 10^{\square}$

(g) $216 = 6^{\square}$

(h) $625 = 5^{\square}$

4. Is 10^2 bigger than 2^{10} ?

5. Is 3^4 bigger than 4^3 ?

6. Is 5^2 bigger than 2^5 ?

7. Copy each of the following statements and fill in the missing numbers:

(a) $49 = \square^{2}$

(b) $64 = \square^{3}$

(c) $64 = \square^{6}$

(d) $64 = \square^{2}$

(e) $100\,000 = \square^{5}$

(f) $243 = \square^{5}$

8. Calculate:

(a) $2^2 + 2^3$

(b) $2^2 \times 2^3$

(c) $3^2 + 2^2$

(d) $3^2 \times 2^2$

(e) $2^3 \times 10^3$

(f) $10^3 + 2^5$

9. Calculate:

(a) $(3+2)^4$ (b) $(3-2)^4$

(c) $(7-4)^3$ (d) $(7+4)^3$

10. Writing your answers in index form, calculate:

(a) $10^2 \times 10^3$ (b) $2^3 \times 2^7$

(c) $3^4 \div 3^2$ (d) $2^5 \div 2^2$

(e) $10^6 \div 10^2$ (f) $5^4 \div 5^2$

11. (a) Without using a calculator, write down the values of *k* and *m*.

$$64 = 8^2 = 4^k = 2^m$$

(b) Complete the following:

$$2^{15} = 32\,768$$

$$2^{14} = \boxed{}$$

(KS3/99Ma/Tier 5-7/P1)

3.2 Laws of Indices

There are three rules that should be used when working with indices:

When m and n are positive integers,

1. $a^m \times a^n = a^{m+n}$

2. $a^m \div a^n = a^{m-n}$ or $\dfrac{a^m}{a^n} = a^{m-n}$ $(m \geq n)$

3. $\left(a^m\right)^n = a^{m \times n}$

These three results are logical consequences of the definition of a^n, but really need a formal proof. You can 'verify' them with particular examples as below, but this is not a proof:

$$\begin{aligned} 2^7 \times 2^3 &= (2\times2\times2\times2\times2\times2\times2)\times(2\times2\times2) \\ &= 2\times2\times2\times2\times2\times2\times2\times2\times2\times2 \\ &= 2^{10} \qquad \text{(here } m=7,\ n=3 \text{ and } m+n=10\text{)} \end{aligned}$$

or,

$$2^7 \div 2^3 = \frac{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2}$$

$$= 2 \times 2 \times 2 \times 2$$

$$= 2^4 \qquad \text{(again } m = 7,\ n = 3 \text{ and } m - n = 4\text{)}$$

Also, $$\left(2^7\right)^3 = 2^7 \times 2^7 \times 2^7$$

$$= 2^{21} \quad \text{(using rule 1)} \qquad \text{(again } m = 7,\ n = 3 \text{ and } m \times n = 21\text{)}$$

The proof of the first rule is given below:

Proof

$$a^m \times a^n = \underbrace{a \times a \times \ldots \times a}_{m \text{ of these}} \times \underbrace{a \times a \times \ldots \times a}_{n \text{ of these}}$$

$$= \underbrace{a \times a \times \ldots \times a \times a \times a \ldots \times a}_{(m + n) \text{ of these}}$$

$$= a^{m + n}$$

The second and third rules can be shown to be true for all positive integers m and n in a similar way.

We can see an important result using rule 2:

$$\frac{x^n}{x^n} = x^{n - n} = x^0$$

but $\frac{x^n}{x^n} = 1$, so

$$\boxed{x^0 = 1}$$

This is true for any non-zero value of x, so, for example, $3^0 = 1$, $27^0 = 1$ and $1001^0 = 1$.

Example 1

Fill in the missing numbers in each of the following expressions:

(a) $2^4 \times 2^6 = 2^{\square}$ (b) $3^7 \times 3^9 = 3^{\square}$

(c) $3^6 \div 3^2 = 3^{\square}$ (d) $\left(10^4\right)^3 = 10^{\square}$

Solution

(a) $2^4 \times 2^6 = 2^{4+6}$

$= 2^{10}$

(b) $3^7 \times 3^9 = 3^{7+9}$

$= 3^{16}$

(c) $3^6 \div 3^2 = 3^{6-2}$

$= 3^4$

(d) $\left(10^4\right)^3 = 10^{4\times 3}$

$= 10^{12}$

Example 2

Simplify each of the following expressions so that it is in the form a^n, where n is a number:

(a) $a^6 \times a^7$ (b) $\dfrac{a^4 \times a^2}{a^3}$ (c) $\left(a^4\right)^3$

Solution

(a) $a^6 \times a^7 = a^{6+7}$

$= a^{13}$

(b) $\dfrac{a^4 \times a^2}{a^3} = \dfrac{a^{4+2}}{a^3}$

$= \dfrac{a^6}{a^3}$

$= a^{6-3}$

$= a^3$

(c) $\left(a^4\right)^3 = a^{4\times 3}$

$= a^{12}$

Exercises

1. Copy each of the following statements and fill in the missing numbers:

(a) $2^3 \times 2^7 = 2^{\square}$ (b) $3^6 \times 3^5 = 3^{\square}$

(c) $3^7 \div 3^4 = 3^{\square}$ (d) $8^3 \times 8^4 = 8^{\square}$

(e) $\left(3^2\right)^5 = 3^{\square}$ (f) $\left(2^3\right)^6 = 2^{\square}$

(g) $\dfrac{3^6}{3^2} = 3^{\square}$ (h) $\dfrac{4^7}{4^2} = 4^{\square}$

2. Copy each of the following statements and fill in the missing numbers:

(a) $a^3 \times a^2 = a^{\square}$ (b) $b^7 \div b^2 = b^{\square}$

(c) $\left(b^2\right)^5 = b^{\square}$ (d) $b^6 \times b^4 = b^{\square}$

(e) $\left(z^3\right)^9 = z^{\square}$ (f) $\dfrac{q^{16}}{q^7} = q^{\square}$

3. Explain why $9^4 = 3^8$.

4. Calculate:

(a) $3^0 + 4^0$ (b) $6^0 \times 7^0$

(c) $8^0 - 3^0$ (d) $6^0 + 2^0 - 4^0$

5. Copy each of the following statements and fill in the missing numbers:

(a) $3^6 \times 3^{\square} = 3^{17}$ (b) $4^6 \times 4^{\square} = 4^{11}$

(c) $\dfrac{a^6}{a^{\square}} = a^4$ (d) $\left(z^{\square}\right)^6 = z^{18}$

(e) $\left(a^{19}\right)^{\square} = a^{95}$ (f) $p^{16} \div p^{\square} = p^7$

(g) $\left(p^{\square}\right)^8 = p^{40}$ (h) $q^{13} \div q^{\square} = q$

6. Calculate:

(a) $\dfrac{2^3}{2^2} + 3^0$ (b) $\dfrac{3^4}{3^3} - 3^0$

(c) $\dfrac{5^4}{5^2} + \dfrac{6^2}{6}$ (d) $\dfrac{7^7}{7^5} - \dfrac{5^9}{5^7}$

(e) $\dfrac{10^8}{10^5} - \dfrac{5^6}{5^3}$ (f) $\dfrac{4^{17}}{4^{14}} - \dfrac{4^{13}}{4^{11}}$

7. Fill in the missing numbers in each of the following expressions:

(a) $8^2 = 2^{\square}$ (b) $81^3 = 9^{\square} = 3^{\square}$

(c) $25^6 = 5^{\square}$ (d) $4^7 = 2^{\square}$

(e) $125^4 = 5^{\square}$ (f) $1000^6 = 10^{\square}$

(g) $81 = \square^4$ (h) $256 = \square^4 = \square^8$

8. Fill in the missing numbers in each of the following expressions:

(a) $8 \times 4 = 2^{\square} \times 2^{\square}$
$= 2^{\square}$

(b) $25 \times 625 = 5^{\square} \times 5^{\square}$
$= 5^{\square}$

(c) $\dfrac{243}{9} = \dfrac{3^{\square}}{3^{\square}}$
$= 3^{\square}$

(d) $\dfrac{128}{16} = \dfrac{2^{\square}}{2^{\square}}$
$= 2^{\square}$

9. Is each of the following statements true or false?

(a) $3^2 \times 2^2 = 6^4$ (b) $5^4 \times 2^3 = 10^7$

(c) $\dfrac{6^8}{2^8} = 3^8$ (d) $\dfrac{10^8}{5^6} = 2^2$

10. Copy and complete each expression:

(a) $\left(2^6 \times 2^3\right)^4 = \left(2^{\square}\right)^4 = 2^{\square}$

(b) $\left(\dfrac{3^6}{3^2}\right)^5 = \left(3^{\square}\right)^5 = 3^{\square}$

(c) $\left(\dfrac{2^3 \times 2^4}{2^7}\right)^4 = \left(2^{\square}\right)^4 = 2^{\square}$

(d) $\left(\dfrac{3^2 \times 9}{3^3}\right)^4 = \left(3^{\square}\right)^4 = 3^{\square}$

(e) $\left(\dfrac{6^2 \times 6^8}{6^3}\right)^4 = \left(6^{\square}\right)^4 = 6^{\square}$

(f) $\left(\dfrac{7^8}{7^2 \times 7^3}\right)^5 = \left(7^{\square}\right)^5 = 7^{\square}$

3.3 Negative Indices

Using negative indices produces fractions. In this section we practice working with negative indices. From our work in the last section, we see that

$$a^2 \div a^3 = a^{2-3} = a^{-1}$$

but we know that

$$a^2 \div a^3 = \frac{a \times a}{a \times a \times a} = \frac{1}{a}, \text{ a fraction.}$$

So clearly,

$$\boxed{a^{-1} = \frac{1}{a}}$$

In same way,

$$\boxed{\begin{aligned} a^{-2} &= \frac{1}{a^2} \\ &= \frac{1}{a \times a} \\ a^{-3} &= \frac{1}{a^3} \\ &= \frac{1}{a \times a \times a} \end{aligned}}$$

and, in general,

$$\boxed{a^{-n} = \frac{1}{a^n}}$$

for positive integer values of n. The three rules at the start of section 3.2 can now be used for any integers m and n, not just for positive values.

Example 1

Calculate, leaving your answers as fractions:

(a) 3^{-2} (b) $2^{-1} - 4^{-1}$ (c) 5^{-3}

Solution

(a) $3^{-2} = \dfrac{1}{3^2}$

$= \dfrac{1}{9}$

(b) $2^{-1} - 4^{-1} = \dfrac{1}{2} - \dfrac{1}{4}$

$= \dfrac{1}{4}$

(c) $5^{-3} = \dfrac{1}{5^3}$

$= \dfrac{1}{125}$

Example 2

Simplify:

(a) $\dfrac{6^7}{6^9}$ (b) $6^4 \times 6^{-3}$ (c) $\left(10^2\right)^{-3}$

Solution

(a) $\dfrac{6^7}{6^9} = 6^{7-9}$

$= 6^{-2} = \dfrac{1}{6^2} = \dfrac{1}{36}$

(b) $6^4 \times 6^{-3} = 6^{4+(-3)}$

$= 6^{4-3} = 6^1 = 6$

(c) $\left(10^{2}\right)^{-3} = 10^{-6}$

$$= \frac{1}{10^{6}}$$

$$= \frac{1}{1000\,000}$$

Exercises

1. Write the following numbers as fractions *without using any indices*:

(a) 4^{-1} (b) 2^{-3} (c) 10^{-3}

(d) 7^{-2} (e) 4^{-3} (f) 6^{-2}

2. Copy the following expressions and fill in the missing numbers:

(a) $\frac{1}{49} = \frac{1}{7^{\square}} = 7^{\square}$ (b) $\frac{1}{100} = \frac{1}{10^{\square}} = 10^{\square}$

(c) $\frac{1}{81} = \frac{1}{9^{\square}} = 9^{\square}$ (d) $\frac{1}{16} = \frac{1}{2^{\square}} = 2^{\square}$

(e) $\frac{1}{10\,000\,000} = \frac{1}{10^{\square}} = 10^{\square}$ (f) $\frac{1}{1024} = \frac{1}{2^{\square}} = 2^{\square}$

3. Calculate:

(a) $4^{-1} + 3^{-1}$ (b) $6^{-1} + 2^{-1}$

(c) $5^{-1} - 10^{-1}$ (d) $10^{-2} - 10^{-3}$

(e) $4^{-1} - 10^{-1}$ (f) $6^{-1} + 7^{-1}$

4. Simplify the following expressions giving your answers in the form of a number to a power:

(a) $4^{7} \times 4^{-6}$ (b) $5^{7} \times 5^{-3}$

(c) $\frac{7^{4}}{7^{-6}}$ (d) $\left(3^{2}\right)^{-4}$

(e) $\left(6^{-2}\right)^{-3}$ (f) $8^{4} \times 8^{-9}$

(g) $\frac{7^{2}}{7^{-2}}$ (h) $\frac{8^{9}}{8^{-9}}$

5. Copy each of the following expressions and fill in the missing numbers;

(a) $\dfrac{1}{9} = 3^{\square}$

(b) $\dfrac{1}{100} = 10^{\square}$

(c) $\dfrac{1}{125} = 5^{\square}$

(d) $\dfrac{5}{5^4} = 5^{\square}$

(e) $\dfrac{6^2}{6^3} = 6^{\square}$

(f) $\dfrac{2^2}{2^{10}} = 2^{\square}$

6. Simplify the following expressions:

(a) $\dfrac{x^8}{x^3}$

(b) $\dfrac{x^7}{x^9}$

(c) $\dfrac{x^4}{x^8}$

(d) $\left(x^6\right)^{-4}$

(e) $\left(\dfrac{1}{x^2}\right)^4$

(f) $\left(x^{-8}\right)^3$

7. Copy and complete the following statements:

(a) $0.1 = 10^{\square}$

(b) $0.25 = 2^{\square}$

(c) $0.0001 = 10^{\square}$

(d) $0.2 = 5^{\square}$

(e) $0.001 = 10^{\square}$

(f) $0.02 = 50^{\square}$

8. Copy the following expressions and fill in the missing numbers:

(a) $\dfrac{x^4}{x^{\square}} = x^2$

(b) $x^6 \times x^{\square} = x^2$

(c) $x^9 \times x^{\square} = x^2$

(d) $\dfrac{x^7}{x^{\square}} = x^{-2}$

(e) $\dfrac{x^3}{x^{\square}} = x^4$

(f) $\left(x^3\right)^{\square} = x^{-6}$

9. Copy the following expressions and fill in the missing numbers:

(a) $\frac{1}{8} = 2^{\square}$ (b) $\frac{1}{25} = 5^{\square}$

(c) $\frac{1}{81} = 9^{\square}$ (d) $\frac{1}{10\,000} = 10^{\square}$

10. If $a = b^3$ and $b = \frac{1}{c^2}$, express a as a power of c, without having any fractions in your final answer.

3.4 Standard Form

Standard form is a convenient way of writing very large or very small numbers. It is used on a scientific calculator when a number is too large or too small to be displayed on the screen.

Before using standard form, we revise multiplying and dividing by powers of 10.

Example 1

Calculate:

(a) 3×10^4 (b) 3.27×10^3

(c) $3 \div 10^2$ (d) $4.32 \div 10^4$

Solution

(a) $3 \times 10^4 = 3 \times 10\,000$

$= 30\,000$

(b) $3.27 \times 10^3 = 3.27 \times 1000$

$= 3270$

(c) $3 \div 10^2 = \frac{3}{100}$

$= 0.03$

(d) $4.32 \div 10^4 = \dfrac{4.32}{10\,000}$

$= \dfrac{432}{1\,000\,000}$

$= 0.000432$

These examples lead to the approach used for standard form, which is a reversal of the approach used in Example 1.

> In *standard form*, numbers are written as
>
> $$a \times 10^n$$
>
> where $1 \le a < 10$ and n is an integer.

Example 2

Write the following numbers in standard form:

(a) 5720 (b) 7.4

(c) 473 000 (d) 6 000 000

(e) 0.09 (f) 0.000621

Solution

(a) $5720 = 5.72 \times 1000$

$= 5.72 \times 10^3$

(b) $7.4 = 7.4 \times 1$

$= 7.4 \times 10^0$

(c) $473\,000 = 4.73 \times 100\,000$

$= 4.73 \times 10^5$

(d) $6\,000\,000 = 6 \times 1\,000\,000$

$= 6 \times 10^6$

(e) $0.09 = \dfrac{9}{100}$

$= 9 \div 10^2$

$= 9 \times 10^{-2}$

(f) $0.000621 = \dfrac{6.21}{10\,000}$

$= \dfrac{6.21}{10^{4}}$

$= 6.21 \times 10^{-4}$

Example 3

Calculate:

(a) $\left(3 \times 10^{6}\right) \times \left(4 \times 10^{3}\right)$

(b) $\left(6 \times 10^{7}\right) \div \left(5 \times 10^{-2}\right)$

(c) $\left(3 \times 10^{4}\right) + \left(2 \times 10^{5}\right)$

Solution

(a) $\left(3 \times 10^{6}\right) \times \left(4 \times 10^{3}\right) = (3 \times 4) \times \left(10^{6} \times 10^{3}\right)$

$= 12 \times 10^{9}$

$= 1.2 \times 10^{1} \times 10^{9}$

$= 1.2 \times 10^{10}$

(b) $\left(6 \times 10^{7}\right) \div \left(5 \times 10^{-2}\right) = (6 \div 5) \times \left(10^{7} \div 10^{-2}\right)$

$= 1.2 \times 10^{9}$

(c) $\left(3 \times 10^{4}\right) + \left(2 \times 10^{5}\right) = 30\,000 + 200\,000$

$= 230\,000$

$= 2.3 \times 10^{5}$

Note on Using Calculators

Your calculator will have a key EE or EXP for entering numbers in standard form.

For example, for 3.2×10^{7}, press

3 · 2 EXP 7

which will appear on your display like this:

```
3.2   07
```

Some calculators also display the ' $\times 10$ ' part of the number, but not all do. You need to find out what your calculator displays. Remember, you must always write the ' $\times 10$ ' part when you are asked to give an answer in standard form.

Exercises

1. Calculate:

(a)	6.21×1000	(b)	8×10^{3}	(c)	4.2×10^{2}
(d)	$3 \div 1000$	(e)	$6 \div 10^{2}$	(f)	$3.2 \div 10^{3}$
(g)	6×10^{-3}	(h)	9.2×10^{-1}	(i)	3.6×10^{-2}

2. Write each of the following numbers in standard form:

(a)	200	(b)	8000
(c)	9 000 000	(d)	62 000
(e)	840 000	(f)	12 000 000 000
(g)	61 800 000 000	(h)	3 240 000

3. Convert each of the following numbers from standard form to the normal decimal notation:

(a)	3×10^{4}	(b)	3.6×10^{4}	(c)	8.2×10^{3}
(d)	3.1×10^{2}	(e)	1.6×10^{4}	(f)	1.72×10^{5}
(g)	6.83×10^{4}	(h)	1.25×10^{6}	(i)	9.17×10^{3}

4. Write each of the following numbers in standard form:

(a)	0.0004	(b)	0.008
(c)	0.142	(d)	0.0032
(e)	0.00199	(f)	0.000000062
(g)	0.0000097	(h)	0.00000000000021

5. Convert the following numbers from standard form to the normal decimal format:

(a) 6×10^{-2} (b) 7×10^{-1} (c) 1.8×10^{-3}

(d) 4×10^{-3} (e) 6.2×10^{-3} (f) 9.81×10^{-4}

(g) 6.67×10^{-1} (h) 3.86×10^{-5} (i) 9.27×10^{-7}

6. Without using a calculator, determine:

(a) $(4 \times 10^{4}) \times (2 \times 10^{5})$ (b) $(2 \times 10^{6}) \times (3 \times 10^{5})$

(c) $(6 \times 10^{4}) \times (8 \times 10^{-9})$ (d) $(3 \times 10^{-8}) \times (7 \times 10^{-4})$

(e) $(6.1 \times 10^{6}) \times (2 \times 10^{-5})$ (f) $(3.2 \times 10^{-5}) \times (4 \times 10^{-9})$

7. Without using a calculator, determine:

(a) $(9 \times 10^{7}) \div (3 \times 10^{4})$ (b) $(8 \times 10^{5}) \div (2 \times 10^{-2})$

(c) $(6 \times 10^{-2}) \div (2 \times 10^{-3})$ (d) $(6 \times 10^{4}) \div (3 \times 10^{-6})$

(e) $(4.8 \times 10^{12}) \div (1.2 \times 10^{3})$ (f) $(3.6 \times 10^{8}) \div (9 \times 10^{3})$

8. Without a calculator, determine the following, giving your answers in both normal and standard form::

(a) $(6 \times 10^{5}) + (3 \times 10^{6})$ (b) $(6 \times 10^{2}) + (9 \times 10^{3})$

(c) $6 \times 10^{5} - 1 \times 10^{4}$ (d) $8 \times 10^{-2} + 9 \times 10^{-3}$

(e) $6 \times 10^{-4} + 8 \times 10^{-3}$ (f) $6 \times 10^{-4} - 3 \times 10^{-5}$

9. Use a calculator to determine:

(a) $(3.4 \times 10^{6}) \times (2.1 \times 10^{4})$ (b) $(6 \times 10^{21}) \times (8.2 \times 10^{-11})$

(c) $(3.6 \times 10^{5}) \times (4.5 \times 10^{7})$ (d) $(8.2 \times 10^{11}) \div (4 \times 10^{-8})$

(e) $(1.92 \times 10^{6}) \times (3.2 \times 10^{-11})$ (f) $(6.2 \times 10^{14})^{3}$

10. The radius of the earth is 6.4×10^{6} m. Giving your answers in standard form, correct to 3 significant figures, calculate the circumference of the earth in:

(a) m (b) cm (c) mm (d) km

11. Sir Isaac Newton (1642-1727) was a mathematician, physicist and astronomer.

In his work on the gravitational force between two bodies he found that he needed to multiply their masses together.

(a) Work out the value of the mass of the Earth multiplied by the mass of the Moon. Give your answer in standard form.

Mass of Earth	=	5.98×10^{24} kg
Mass of Moon	=	7.35×10^{22} kg

Newton also found that he needed to work out the square of the distance between the two bodies.

(b) Work out the square of the distance between the Earth and the Moon. Give your answer in standard form.

Distance between Earth and Moon	=	3.89×10^{5} km

Newton's formula to calculate the gravitational force (F) between two bodies is $F = \dfrac{G m_1 m_2}{R^2}$ where

G is the gravitational constant, m_1 and m_2 are the masses of the two bodies, and R is the distance between them.

(c) Work out the gravitational force (F) between the Sun and the Earth using the formula $F = \dfrac{G m_1 m_2}{R^2}$ with information in the box below.

Give your answer in standard form.

$m_1 m_2$	=	1.19×10^{55} kg^2
R^2	=	2.25×10^{16} km^2
G	=	6.67×10^{-20}

(KS3/95/Ma/Levels 6-8/P1)

12. (a) Which of these statements is true?

(i) 4×10^3 is a larger number than 4^3.

(ii) 4×10^3 is the same size as 4^3.

(iii) 4×10^3 is a smaller number than 4^3.

Explain your answer.

(b) One of the numbers below has the same value as 3.6×10^4. Write down the number.

36^3 $\quad 36^4$ $\quad (3.6 \times 10)^4$ $\quad 0.36 \times 10^3$ $\quad 0.36 \times 10^5$

(c) One of the numbers below has the same value as 2.5×10^{-3}. Write down the number.

25×10^{-4} $\quad 2.5 \times 10^3$ $\quad -2.5 \times 10^3$ $\quad 0.00025$ $\quad 2500$

(d) $\left(2 \times 10^2\right) \times \left(2 \times 10^2\right)$ can be written more simply as 4×10^4.

Write the following values as simply as possible:

(i) $\left(3 \times 10^2\right) \times \left(2 \times 10^{-2}\right)$

(ii) $\dfrac{6 \times 10^8}{2 \times 10^4}$

(KS3/98/Ma/Tier 6-8/P1)

3.5 Fractional Indices

Indices that are fractions are used to represent square roots, cube roots and other roots of numbers.

$$a^{\frac{1}{2}} = \sqrt{a} \qquad \text{for example,} \qquad 9^{\frac{1}{2}} = 3$$

$$a^{\frac{1}{3}} = \sqrt[3]{a} \qquad \text{for example,} \qquad 8^{\frac{1}{3}} = 2$$

$$a^{\frac{1}{4}} = \sqrt[4]{a} \qquad \text{for example,} \qquad 625^{\frac{1}{4}} = 5$$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Example 1

Calculate:

(a) $81^{\frac{1}{2}}$ (b) $1000^{\frac{1}{3}}$ (c) $4^{-\frac{1}{2}}$

Solution

(a) $81^{\frac{1}{2}} = \sqrt{81}$

$= 9$

(b) $1000^{\frac{1}{3}} = \sqrt[3]{1000}$

$= 10$

(c) $4^{-\frac{1}{2}} = \dfrac{1}{4^{\frac{1}{2}}}$

$= \dfrac{1}{\sqrt{4}}$

$= \dfrac{1}{2}$

Exercises

1. Calculate:

(a) $49^{\frac{1}{2}}$ (b) $64^{\frac{1}{2}}$ (c) $16^{\frac{1}{2}}$

(d) $81^{-\frac{1}{2}}$ (e) $100^{-\frac{1}{2}}$ (f) $25^{-\frac{1}{2}}$

(g) $9^{\frac{1}{2}}$ (h) $36^{-\frac{1}{2}}$ (i) $144^{\frac{1}{2}}$

2. Calculate:

(a) $8^{\frac{1}{3}}$ (b) $8^{-\frac{1}{3}}$ (c) $125^{\frac{1}{3}}$

(d) $64^{-\frac{1}{3}}$ (e) $216^{\frac{1}{3}}$ (f) $1000000^{-\frac{1}{3}}$

3. Calculate:

(a) $32^{\frac{1}{5}}$ (b) $64^{-\frac{1}{2}}$ (c) $10000^{\frac{1}{4}}$

(d) $81^{-\frac{1}{4}}$ (e) $625^{\frac{1}{4}}$ (f) $100000^{-\frac{1}{5}}$

4. Calculate:

(a) $\left(\dfrac{4\times 8}{2}\right)^{\frac{1}{2}}$ (b) $\left(\dfrac{9\times 27}{3}\right)^{\frac{1}{4}}$

(c) $\left(\dfrac{125\times 5}{25}\right)^{\frac{1}{2}}$ (d) $\left(\dfrac{625}{5}\right)^{-\frac{1}{3}}$

5. Is each of the following statements *true* or *false*:

(a) $16^{\frac{1}{2}} = 8$ (b) $16^{\frac{1}{4}} = 2$

(c) $81^{\frac{1}{3}} = 9$ (d) $\left(\dfrac{1}{100}\right)^{-\frac{1}{2}} = 10$

6. Simplify:

(a) $\left(x^{9}\right)^{\frac{1}{3}}$ (b) $\left(a^{10}\right)^{-\frac{1}{2}}$

(c) $\dfrac{a}{a^{\frac{1}{2}}}$ (d) $\dfrac{a^{\frac{1}{2}}}{a}$

7. Simplify:

(a) $\dfrac{x^{\frac{3}{2}}}{x}$ (b) $\dfrac{x}{x^{\frac{3}{2}}}$

(c) $\dfrac{a^{\frac{1}{3}}}{a}$ (d) $\dfrac{a^{\frac{1}{3}}}{a^{\frac{1}{2}}}$

8. Calculate:

(a) $4^{-\frac{1}{2}} + 4^{\frac{1}{2}}$ (b) $\left(9^{0} + 9^{\frac{1}{2}}\right)^{\frac{1}{2}}$

(c) $\left(256^{\frac{1}{2}}\right)^{\frac{1}{2}}$ (d) $\left(9 - 9^{0}\right)^{\frac{1}{3}}$

Levelling-Up Basic Mathematics

Logarithms

Robin Horan

The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic competence in the use of logarithms.

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Table of Contents

1. Logarithms (Introduction)

Let a and N be positive real numbers and let $N = a^n$. Then n is called the *logarithm of* N to the base a. We write this as

$$n = \log_a N.$$

Examples 1

(a) Since $16 = 2^4$, then $4 = \log_2 16$.

(b) Since $81 = 3^4$, then $4 = \log_3 81$.

(c) Since $3 = \sqrt{9} = 9^{\frac{1}{2}}$, then $1/2 = \log_9 3$.

(d) Since $3^{-1} = 1/3$, then $-1 = \log_3(1/3)$.

Exercise

Use the definition of logarithm given on the previous page to determine the value of x in each of the following.

1. $x = \log_3 27$
2. $x = \log_5 125$
3. $x = \log_2(1/4)$
4. $2 = \log_x(16)$
5. $3 = \log_2 x$

2. Rules of Logarithms

Let a, M, N be positive real numbers and k be any number. Then the following important rules apply to logarithms.

$$\begin{array}{lrcl}
1. & \log_a MN & = & \log_a M + \log_a N \\
2. & \log_a \left(\frac{M}{N}\right) & = & \log_a M - \log_a N \\
3. & \log_a \left(m^k\right) & = & k \log_a M \\
4. & \log_a a & = & 1 \\
5. & \log_a 1 & = & 0
\end{array}$$

3. Logarithm of a Product

1. ⟵ **Proof that** $\log_a MN = \log_a M + \log_a N$.

Examples 2

(a) $\log_6 4 + \log_6 9 = \log_6(4 \times 9) = \log_6 36.$

If $x = \log_6 36$, then $6^x = 36 = 6^2$.

Thus $\log_6 4 + \log_6 9 = 2$.

(b) $\log_5 20 + \log_4 \left(\frac{1}{4}\right) = \log_5 \left(20 \times \frac{1}{4}\right).$

Now $20 \times \frac{1}{4} = 5$ so $\log_5 20 + \log_4 \left(\frac{1}{4}\right) = \log_5 5 = 1.$

Quiz. To which of the following numbers does the expression $\log_3 15 + \log_3 0 \cdot 6$ simplify?

(a) 4 (b) 3 (c) 2 (d) 1

4. Logarithm of a Quotient

1. ⟵ **Proof that** $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$.

Examples 3

(a) $\log_2 40 - \log_2 5 = \log_2\left(\frac{40}{5}\right) = \log_2 8$.

If $x = \log_2 8$ then $2^x = 8 = 2^3$, so $x = 3$.

(b) If $\log_3 5 = 1.465$ then we can find $\log_3 0 \cdot 6$.

Since $3/5 = 0 \cdot 6$, then $\log_3 0 \cdot 6 = \log_3\left(\frac{3}{5}\right) = \log_3 3 - \log_3 5$.

Now $\log_3 3 = 1$, so that $\log_3 0 \cdot 6 = 1 - 1 \cdot 465 = -0 \cdot 465$

Quiz. To which of the following numbers does the expression $\log_2 12 - \log_2\left(\frac{3}{4}\right)$ simplify?

(a) 0 (b) 1 (c) 2 (d) 4

5. Logarithm of a Power

1. $\longleftarrow$ **Proof that** $\log_a\left(m^k\right) = k\log_a M$

Examples 4

(a) Find $\log_{10}(1/10000)$. We have $10000 = 10^4$, so $1/10000 = 1/10^4 = 10^{-4}$.

Thus $\log_{10}(1/10000) = \log_{10}\left(10^{-4}\right) = -4\log_{10} 10 = -4$, where we have used rule 4 to write $\log_{10} 10 = 1$.

(b) Find $\log_{36} 6$. We have $6 = \sqrt{36} = 36^{\frac{1}{2}}$.

Thus $\log_{36} 6 = \log_{36}\left(36^{\frac{1}{2}}\right) = \frac{1}{2}\log_{36} 36 = \frac{1}{2}$.

Quiz. If $\log_3 5 = 1 \cdot 465$, which of the following numbers is $\log_3 0 \cdot 04$?

(a) -2.930 (b) -1.465 (c) -3.465 (d) 2.930

6. Use of the Rules of Logarithms

In this section we look at some applications of the rules of logarithms.

Examples 5

(a) $\log_4 1 = 0.$

(b) $\log_{10} 10 = 1.$

(c) $\log_{10} 125 + \log_{10} 8 = \log_{10}(125 \times 8) = \log_{10} 1000$
$= \log_{10}\left(10^3\right) = 3\log_{10} 10 = 3.$

(d) $2\log_{10} 5 + \log_{10} 4 = \log_{10}\left(5^2\right) + \log_{10} 4 = \log_{10}(25 \times 4)$
$= \log_{10} 100 = \log_{10}\left(10^2\right) = 2\log_{10} 10 = 2.$

(e) $3\log_a 4 + \log_a(1/4) - 4\log_a 2 = \log_a\left(4^3\right) + \log_a(1/4) - \log_a\left(2^4\right)$
$= \log_a\left(4^3 \times \frac{1}{4}\right) - \log_a\left(2^4\right) = \log_a\left(4^2\right) - \log_a\left(2^4\right)$
$= \log_a 16 - \log_a 16 = 0.$

Exercise

Use the rules of logarithms to simplify each of the following.

1. $3\log_3 2 - \log_3 4 + \log_3\left(\frac{1}{2}\right)$

2. $3\log_{10} 5 + 5\log_{10} 2 - \log_{10} 4$

3. $2\log_a 6 - (\log_a 4 + 2\log_a 3)$

4. $5\log_3 6 - (2\log_3 4 + \log_3 18)$

5. $3\log_4(\sqrt{3}) - \frac{1}{2}\log_4 3 + 3\log_4 2 - \log_4 6$

7. Quiz on Logarithms

In each of the following, find x.

Begin Quiz

1. $\log_x 1024 = 2$

(a) 2^3 (b) 2^4 (c) 2^2 (d) 2^5

2. $x = (\log_a \sqrt{27} - \log_a \sqrt{8} - \log_a \sqrt{125})/(\log_a 6 - \log_a 20)$

(a) 1 (b) 3 (c) 3/2 (d) -2/3

3. $\log_c(10 + x) - \log_c x = \log_c 5)$

(a) 2.5 (b) 4.5 (c) 5.5 (d) 7.5

End Quiz

Score: Correct

8. Change of Bases

There is one other rule for logarithms which is extremely useful in practice. This relates logarithms in one base to logarithms in a different base. Most calculators will have, as standard, a facility for finding logarithms to the base 10 and also for logarithms to base e (natural logarithms). What happens if a logarithm to a different base, for example 2, is required? The following is the rule that is needed.

$$\boxed{\log_a c = \log_a b \times \log_b c}$$

1. ⟵ **Proof of the above rule**

The most frequently used form of the rule is obtained by rearranging the rule on the previous page. We have

$$\log_a c = \log_a b \times \log_b c \quad \text{so} \quad \log_b c = \frac{\log_a c}{\log_a b}\,.$$

Examples 6

(a) Using a calculator we find that $\log_{10} 3 = 0 \cdot 47712$ and $\log_{10} 7 = 0 \cdot 84510$. Using the above rule,

$$\log_3 7 = \frac{\log_{10} 7}{\log_{10} 3} = \frac{0 \cdot 84510}{0 \cdot 47712} = 1 \cdot 77124\,.$$

(b) We can do the same calculation using instead logs to base e. Using a calculator, $\log_e 3 = 1 \cdot 09861$ and $\log_e 7 = 1 \cdot 94591$. Thus

$$\log_3 7 = \frac{\ln 7}{\ln 3} = \frac{1 \cdot 94591}{1 \cdot 09861} = 1 \cdot 77125\,.$$

The calculations have all been done to five decimal places, which explains the slight difference in answers.

(c) Given only that $\log_{10} 5 = 0 \cdot 69897$ we can still find $\log_2 5$, as follows. First we have $2 = 10/5$ so

$$\begin{aligned}
\log_{10} 2 &= \log_{10}\left(\frac{10}{5}\right) \\
&= \log_{10} 10 - \log_{10} 5 \\
&= 1 - 0 \cdot 69897 \\
&= 0 \cdot 30103\,.
\end{aligned}$$

Then

$$\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2} = \frac{0 \cdot 69897}{0 \cdot 30103} = 2 \cdot 32193\,.$$

Solutions to Quizzes

Solution to Quiz:

Using rule 1 we have

$$\log_3 15 + \log_3 0 \cdot 6 = \log_3(15 \times 0 \cdot 6) = \log_3 9$$

But $9 = 3^2$ so

$$\log_3 15 + \log_3 0 \cdot 6 = \log_3 3^2 = 2.$$

End Quiz

Solution to Quiz:

Using rule 2 we have

$$\log_2 12 - \log_2 \left(\frac{3}{4}\right) = \log_2 \left(12 \div \frac{3}{4}\right)$$

Now we have $12 \div \frac{3}{4} = 12 \times \frac{4}{3} = \frac{12\times 4}{3} = 16$.

Thus $\log_2 12 - \log_2 \left(\frac{3}{4}\right) = \log_2 16 = \log_2 2^4$.

If $x = \log_2 2^4$, then $2^x = 2^4$, so $x = 4$. End Quiz

Solution to Quiz:

Note that

$$0 \cdot 04 = 4/100 = 1/25 = 1/5^2 = 5^{-2}.$$

Thus

$$\log_3 0 \cdot 04 = \log_3 \left(5^{-2}\right) = -2 \log_3 5.$$

Since $\log_3 5 = 1 \cdot 465$, we have

$$\log_3 0 \cdot 05 = -2 \times 1 \cdot 465 = -2.930.$$

End Quiz

Solutions to Problems

Problem 1.

Since

$$x = \log_3 27$$

then, by the definition of a logarithm, we have

$$3^x = 27.$$

But $27 = 3^3$, so we have

$$3^x = 27 = 3^3,$$

giving

$$x = 3.$$

□

Problem 2.

Since $x = \log_{25} 5$ then, by the definition of a logarithm,

$$25^x = 5.$$

Now

$$5 = \sqrt{25} = 25^{\frac{1}{2}},$$

so that

$$25^x = 5 = 25^{\frac{1}{2}},$$

From this we see that $x = 1/2$. $\square$

Problem 3.

Since $x = \log_2(1/4)$, then, by the definition of a logarithm,

$$2^x = 1/4 = 1/(2^2) = 2^{-2}.$$

Thus $x = -2$. $\square$

Problem 4.

Since $2 = \log_x(16)$ then, by the definition of logarithm,

$$x^2 = 16 = 4^2.$$

Thus

$$x = 4.$$

$\square$

Problem 5.

Since $3 = \log_2 x$, by the definition of logarithm, we must have

$$2^3 = x.$$

Thus $x = 8$. $\square$

Problem 1.

Let $m = \log_a M$ and $n = \log_a N$, so, by definition, $M = a^m$ and $N = a^n$. Then

$$MN = a^m \times a^n = a^{m+n},$$

where we have used the appropriate rule for exponents. From this, using the definition of a logarithm, we have

$$m + n = \log_a(MN).$$

But $m+n = \log_a M + \log_a N$, and the above equation may be written

$$\log_a M + \log_a N = \log_a(MN),$$

which is what we wanted to prove. □

Problem 1.

As before, let $m = \log_a M$ and $n = \log_a N$. Then $M = a^m$ and $N = a^n$. Now we have

$$\frac{M}{N} = \frac{a^m}{a^n} = a^{m-n},$$

where we have used the appropriate rule for indices. By the definition of a logarithm, we have

$$m - n = \log_a \left(\frac{M}{N}\right).$$

From this we are able to deduce that

$$\log_a M - \log_a N = m - n = \log_a \left(\frac{M}{N}\right).$$

□

Problem 1.

Let $m = \log_a M$, so $M = a^m$. Then

$$M^k = (a^m)^k = a^{mk} = a^{km},$$

where we have used the appropriate rule for indices. From this we have, by the definition of a logarithm,

$$km = \log_a \left(M^k\right).$$

But $m = \log_a M$, so the last equation can be written

$$k \log_a M = km = \log_a \left(M^k\right),$$

which is the result we wanted. $\square$

Problem 1. First of all, by rule 3, we have $3\log_3 2 = \log_3(2^3) = \log_3 8$. Thus the expression becomes

$$\log_3 8 - \log_3 4 + \log_3\left(\frac{1}{2}\right) = \left[\log_3 8 + \log_3\left(\frac{1}{2}\right)\right] - \log_3 4\,.$$

Using rule 1, the first expression in the [ ] brackets becomes

$$\log_3\left(8 \times \frac{1}{2}\right) = \log_3 4.$$

The expression then simplifies to

$$\log_3 4 - \log_3 4 = 0.$$

$\square$

Problem 2.

First we use rule 3:

$$3\log_{10}5 = \log_{10}\left(5^3\right)$$

and

$$5\log_{10}2 = \log_{10}\left(2^5\right).$$

Thus

$$3\log_{10}5 + 5\log_{10}2 = \log_{10}\left(5^3\right) + \log\left(2^5\right) = \log_{10}\left(5^3\times 2^5\right),$$

where we have used rule 1 to obtain the right hand side. Thus

$$3\log_{10}5 + 5\log_{10}2 - \log_{10}4 = \log_{10}\left(5^3\times 2^5\right) - \log_{10}4$$

and, using rule 2, this simplifies to

$$\log_{10}\left(\frac{5^3\times 2^5}{4}\right) = \log_{10}\left(10^3\right) = 3\log_{10}10 = 3.$$

$\square$

Problem 3.

Dealing first with the expression in brackets, we have

$$\log_a 4 + 2\log_a 3 = \log_a 4 + \log_a \left(3^2\right) = \log_a \left(4 \times 3^2\right),$$

where we have used, in succession, rules 3 and 2. Now

$$2\log_a 6 = \log_a \left(6^2\right)$$

so that, finally, we have

$$\begin{aligned} 2\log_a 6 - (\log_a 4 + 2\log_a 3) &= \log_a \left(6^2\right) - \log_a \left(4 \times 3^2\right) \\ &= \log_a \left(\frac{6^2}{4 \times 3^2}\right) \\ &= \log_a 1 \\ &= 0. \end{aligned}$$

$\square$

Problem 4.

Dealing first with the expression in brackets we have

$$2\log_3 4 + \log_3 18 = \log_3\left(4^2\right) + \log_3 18 = \log_3\left(4^2 \times 18\right),$$

where we have used rule 3 first, and then rule 1. Now, using rule 3 on the first term, followed by rule 2, we obtain

$$\begin{aligned}
5\log_3 6 - (2\log_3 4 + \log_3 18) &= \log_3\left(6^5\right) - \log_3\left(4^2 \times 18\right) \\
&= \log_3\left(\frac{6^5}{4^2 \times 18}\right) \\
&= \log_3\left(\frac{2^5 \times 3^5}{4^2 \times 2 \times 9}\right) \\
&= \log_3\left(3^3\right) \\
&= 3\log_3 3 = 3,
\end{aligned}$$

since $\log_3 3 = 1$.

$\square$

Problem 5.

The first thing we note is that $\sqrt{3}$ can be written as $3^{\frac{1}{2}}$. We first simplify some of the terms. They are

$$3\log_4\sqrt{3} = 3\log_4\left(3^{\frac{1}{2}}\right) = \frac{3}{2}\log_4 3,$$

$$\log_4 6 = \log_4(2\times 3) = \log_4 2 + \log_4 3.$$

Putting all of this together:

$$\begin{aligned} & 3\log_4(\sqrt{3}) - \tfrac{1}{2}\log_4 3 + 3\log_4 2 - \log_4 6 \\ = \quad & \frac{3}{2}\log_4 3 - \frac{1}{2}\log_4 3 + 3\log_4 2 - (\log_4 2 + \log_4 3) \\ = \quad & \left(\frac{3}{2} - \frac{1}{2} - 1\right)\log_4 3 + (3-1)\log_4 2 \\ = \quad & 2\log_4 2 = \log_4\left(2^2\right) = \log_4 4 = 1. \end{aligned}$$

$\square$

Problem 1.

Let $x = \log_a b$ and $y = \log_b c$. Then, by the definition of logarithms,

$$a^x = b \quad \text{and} \quad b^y = c.$$

This means that

$$c = b^y = (a^x)^y = a^{xy},$$

with the last equality following from the laws of indices. Since $c = a^{xy}$, by the definition of logarithms this means that

$$\log_a c = xy = \log_a b \times \log_b c.$$

$\square$