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This paper deals with imperfect preventive maintenance (PM) optimisation problem. The system to be maintained is typically a production system assumed to be continuously monitored and subject to stochastic degradation. To assess such degradation, the proposed maintenance model takes into account both corrective maintenance (CM) and PM. The system undergoes PM whenever its reliability reaches an appropriate value, while CM is performed at system failure. After a given number of maintenance actions, the system is preventively replaced by a new one. Both CM as well as PM are considered imperfect, i.e. they bring the system to an operating state which lies between two extreme states, namely the *as bad as old* state and *as good as new* state. The imperfect effect of CM and PM is modelled on the basis of the hybrid hazard rate model. The objective of the proposed PM optimisation model consists on finding the optimal reliability threshold together with the optimal number of PM actions to maximise the average availability of the system. A mathematical model is then proposed. To solve this problem an algorithm is provided. A numerical example is presented to illustrate the proposed maintenance optimisation model.

**Keywords:** reliability; imperfect preventive maintenance; optimisation; hazard rate model



## 1. Introduction

Systems of production of good and services are the vast majority of most industrial capital. Production strategies have been widely investigated with the main focus is to ensure low production costs. To be competitive, companies should develop a cost-efficient strategies allowing to maximise their profit, on the one hand, and to exploit rationally their production systems, on the other hand. Production systems are indeed subject to random deterioration processes with respect to both age and usage. Such a deterioration impacts not only the production system itself but also the product quality. Therefore, to ensure a rational exploitation of production systems and to keep high product quality, maintenance activities are usually performed as solutions to asses the degradation of the production system. Maintenance activities are performed whenever the system fails or as preventive maintenance (PM). PM activities are planned to reduce the risk of failure occurrence and, either to minimise the total cost induced by both production system operations and maintenance, or to maximise the availability of the production systems.

The growing importance of the maintenance of production systems has lead to an increasing interest in the development and implementation of maintenance optimisation models for stochastic degrading production systems. Different researchers have produced many interesting and significant results for a huge variation of maintenance optimisation models.

The framework of PM for reparable systems was initiated by Barlow and Hunter in their seminal paper (Barlow and Hunter 1960) (see also the book by Barlow and Proschan 1996). Since, a large and growing variety of mathematical models appeared in the literature for the design of optimal maintenance policies for reparable systems. For a survey, the reader may refer, for example, to Cho and Parlar (1991), Dekker (1996) and Jardine and Tsang (2006), and the references therein. Further generalisations of Barlow and Hunter work are proposed recently by Nakagawa and his coauthors (see the recent reference Nakagawa and Mizutani 2009). In Nakagawa and Mizutani (2009), extended the well-known results of periodic replacement with minimal repair policy as well as that of block replacement policy from infinite into finite time horizon setting. For more details about PM policies made within finite time horizon, one may refer to the book of Nakagawa (2008). Lugtigheid and his coauthors gave also an interesting review of a number of maintenance models (Lugtighei, Jardine, and Jiang 2007).

Dealing with complex reparable systems, for example, aircraft, pipeline, nuclear power plant, to name a few, imperfect PM operations are needed and usually adopted. Such PM operations when performed bring the system to operate between the two extremes operating states, namely the *as good as new* state and the *as bad as old* state. As already pointed out in the

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review by Pham and Wang (1996), and also demonstrated by the huge existing literature, imperfect PM is a major concern in the field of engineering asset management and has gained much attention of researchers in reliability and maintenance theory. To model imperfect PM, Pham and Wang (1996) listed methods and techniques among them the so-called age reduction model initially due to the work by Malik (1979). In Malik (1979), the age reduction model is defined on the basis of the concept of system *virtual age* according to which a system becomes *younger* whenever it undergoes a PM. Roughly speaking, after an imperfect PM, the age $t$ of the system is reduced to $\alpha \times t$ where $\alpha$ is the age reduction coefficient ($0 \leq \alpha \leq 1$). Accordingly, the system becomes *as good as new* if its age is reset to zero ($\alpha = 0$) while it becomes *as bad as old* if the age reduction coefficient $\alpha = 1$. In the later case, the imperfect PM corresponds to the well-known minimal repair which consists to restore the system to the state immediately preceding the failure. In the work by Nguyen and Murthy (1981), the authors investigated the problem of optimal PM policies for repairable systems. They assumed that after a repair the systems age becomes zero while the system failure rate increases by the increasing of the number of repairs carried out. Nakagawa (1988) (see also Nakagawa 2008; Nakagawa and Mizutani 2009) introduced the concept of hazard rate increased factor to model imperfect PM. According to this model, the imperfect PM could be seen as a replacement of the existing system with a new but less reliable one. In other words, after an imperfect PM, the initial hazard rate $h(t)$ of the system is modified to be $\beta \times h(t)$ where $\beta$ is such that $\beta \geq 1$ and called the *adjustment factor*. Following this modelling approach, the hazard rate function increases with the number of imperfect PM carried out. On the basis of these imperfect PM models, Lin, Zuo, and Yam (2000) proposed a hybrid model derived by combining age reduction as well as hazard rate increased effects. The hybrid hazard rate model not only reduces the effective age to a certain value, but it further changes the slope of the hazard rate function. If the hazard rate function of the system was $h(T)$ just before performing a PM at time $T$, then it becomes $\beta \times h(\alpha T + t)$ right after the PM, where $\beta \geq 1$, $0 \leq \alpha \leq 1$ and $t \geq 0$. When $\beta = 1$, the hybrid model reduces to age reduction model, while it reduces to hazard rate adjustment model when $\alpha = 0$. This hybrid model is used as a modelling approach for imperfect PM in the work by Zhou, Xi, and Lee (2007). In another work proposed by Lin, Zuo, and Yam (2001), the problem of PM analysis and optimisation is studied for repairable systems with two categories of failure modes, namely the maintainable and non-maintainable failure modes. Only the system failure rate corresponding to the maintainable failure mode is altered whenever PM actions are performed. A similar work to that of Lin, Zuo, and Yam (2001) is proposed by El-Ferik and Ben-Daya (2006) to develop an age-based hybrid model for imperfect PM for repairable systems with the two categories of failure modes initially introduced in Lin, Zuo, and Yam (2001). In El-Ferik and Ben-Daya (2006), an analysis is conducted to determine the existence and uniqueness of optimal imperfect PM policy.

Exploiting the hazard rate model as a modelling approach for the imperfect PM of deteriorating reparable systems is recognised as a very useful and practical method in maintenance engineering. Indeed, several preventive maintenance model based on hazard rate function have been developed and successfully applied to complex industrial systems from different area such as energy production, transportation and distribution (Zhao et al. 2006; Sun, Ma, and Morris 2009; Xia et al. 2012), and mining as well as food-processing industries and utility industry (Jardine and Tsang 2006).

The present paper proposes a PM optimisation approach for a system which is assumed to be continuously monitored and degrades stochastically. The system undergoes PM whenever its reliability reaches a given threshold level $R_{th}$. In the case where the system fails before reaching the threshold $R_{th}$, a corrective maintenance (CM) is then carried out. After a number $N$ of maintenance cycles (a maintenance cycle ends either by a CM or a PM which occurs first), the system is replaced by a new one. Both preventive and CM actions are made identically on the basis of the hybrid hazard model. As pointed out by Wang (2000), the choice of reliability threshold $R_{th}$ and number $N$ of maintenance cycles to be experienced before replacing the system by a new one, will obviously have an economic impact on the performance of the maintenance policy. Indeed, on the one hand, a low value of the threshold $R_{th}$ implies a long and an uninterrupted use of the system but with, however, an increased risk of failures. On the other hand, if the reliability threshold is set to a high value, the number of system failures is then reduced but with an increasing preventive maintenance cost and unavailability. Therefore, the maintenance optimisation problem to be solved consists on finding the joint optimal reliability threshold $R_{th}^*$ together with the optimal number $N^*$ of maintenance cycles to maximise the average availability of the system.

Although the present work is based on the hybrid hazard rate model initially introduced by Lin, Zuo, and Yam (2000), the maintenance policy it dealt with is different from that used in Lin, Zuo, and Yam (2000). In fact, in (Lin, Zuo, and Yam 2000), the system under study undergoes, on the one hand, PM whenever its corresponding failure rate reaches a given value, and, on the other hand, minimal repair at failure (refer to model 2 in Lin, Zuo, and Yam 2000). The system is completely renewed after the $(N-1)$th PM. The resulting maintenance cost optimisation is clearly different from the one we propose. Also, the present work differs from that of El-Ferik and Ben-Daya (2006) where the system to be maintained is considered as operating under a maintenance policy known as of type I in the work by Nguyen and Murthy (1981). According to this maintenance policy, the system undergoes the $k$th PM either at failure or at a given system age from the last PM, which occurs first. At the end of the $(N-1)$th PM, the system is renewed. The objective consists then to determine the optimal ages and the optimal number of PM before restoring the system. Unlike the PM model we propose, reliability threshold is not addressed in the work by El-Ferik and Ben-Daya (2006) where decision variables are the dates and the number of PM. In our work, however, reliability threshold as well as the number of PM are the decision variables whereas dates at which PM are performed are derived from the reliability threshold, i.e. PM dates are evaluated as function of the reliability threshold. The work by Liao, Pan, and Xi (2010) is quite similar to that of Lin, Zuo, and Yam (2000). In both works, the same maintenance policy is used and the objective functions are similar except that Liao, Pan, and Xi (2010) consider, reliability threshold instead of hazard rate threshold, and additional operational and breakdown costs are also taken into account. The objective of maintenance policies adopted in the above-mentioned comparable works (Lin, Zuo, and Yam 2000; El-Ferik and Ben-Daya 2006; Liao, Pan, and Xi 2010) consists to minimise the expected total cost per unit of time in infinite time horizon. The present work, however, deals with availability maximisation. Since, it can be considered as a parallel development to the existing works.

The rest of this paper is organised as follows. The next section gives a brief review of the hybrid hazard rate model as a modeling technique of imperfect PM. On the basis of results of Section 2, the PM problem is introduced and its corresponding mathematical model is proposed in Section 3. To solve this optimisation problem an algorithm is also proposed. In Section 4, a numerical example is provided to illustrate the proposed PM approach. The results obtained from the numerical example are discussed. Conclusion and future works are drawn in Section 5.



## 2. The hybrid hazard rate model for imperfect PM

In this section, the hybrid hazard rate model as modeling framework of imperfect maintenance is presented. For more details about this model, the reader may refer to the work by Lin, Zuo, and Yam (2000). Let us consider a system whose lifetime is randomly distributed and for which the corresponding hazard rate function is denoted by $h_1(t)$. This later is defined by the conditional probability of the first failure of the system at the next time given that it is presently working. Based on this definition, three observations are made (Lin, Zuo, and Yam 2000). The first states that the definition of the hazard rate assumes that the system is presently operating. The second observation that can be made is that the health condition of the system is assumed to be completely described by its hazard rate function. The third observation states that the hazard rate reflects the overall operating history of the system. Such a history includes, in addition to operating conditions, failures and repairs as well as PM actions previously performed on the system.

To assess system degradation, assume that imperfect PM are carried out at the end of a given time intervals. The length of the $k$th PM cycle is denoted by $T_k$ $(k = 1, 2, \ldots)$. According to the hybrid hazard rate model (Lin, Zuo, and Yam 2000), the hazard rate function $h_{k+1}(t)$ of the system after the $k$th PM is defined recursively as :

$$h_{k+1}(t) = \beta_k h_k(t + \alpha_k T_k) \quad t \in [0, T_{k+1}[, \; k = 1, 2, \ldots \tag{1}$$

where $\alpha_k$ and $\beta_k$ states, respectively, for the age reduction coefficient and the hazard rate increase coefficient (adjustment factor) such that $0 \leq \alpha_1 < \alpha_2 < \cdots \leq 1$ and $1 \leq \beta_1 < \beta_2 < \cdots$. For $t \in [0, T_1[$, $h_1(t)$ is the hazard rate of the system which is initially assumed to be a new one. From the above equation, the hazard rate function $h_k(t)$, for $k = 1, 2, \ldots,$ and $t \in [0, T_k[$, can be written in the following form:

$$h_k(t) = B_k h_1 \left(A_k + t\right), \tag{2}$$

where $A_k = \sum_{i=1}^{k-1} \alpha_i T_i$ and $B_k = \prod_{i=1}^{k-1} \beta_i$ such that $A_1 = 0$ and $B_1 = 1$.

By exploiting the recursion rule given by Equation (1), the system reliability $R_k(t)$ for the $k$th PM cycle is given according to the theoretical relationship between system reliability and hazard rate function:

$$R_k(t) = \exp\left(-\int_0^t h_k(x)dx\right), \tag{3}$$

which can equivalently be written as:

$$R_k(t) = \exp\left(-B_k \int_{A_k}^{t+A_k} h_1(x)dx\right). \tag{4}$$

If PM times $T_k$ $(k = 1, \ldots)$ correspond to instants where the system reliability reaches the threshold level $R_{th}$, it follows that:

$$R_{th} = R_k(T_k) \tag{5}$$

$$= \exp\left(-\int_0^{T_k} h_k(t)dt\right) \tag{6}$$

$$= \exp\left(-B_k \int_{A_k}^{T_k+A_k} h_1(t)dt\right). \tag{7}$$

If, furthermore, we denote by $H_1(t) = \int_0^t h_1(x)dx$ the cumulative hazard rate, then the above equations are equivalent to:

$$R_{th} = \exp\left(B_k(\mathcal{H}_1(A_k) - \mathcal{H}_1(A_k + T_k))\right), \tag{8}$$

By applying the ln function to and dividing both sides of the above equality by $B_k$, we obtain:

$$\frac{\ln(R_{th})}{B_k} = \mathcal{H}_1(A_k) - \mathcal{H}_1(A_k + T_k), \tag{9}$$

this implies that:

$$\mathcal{H}_1(A_k + T_k) = \mathcal{H}_1(A_k) - \frac{\ln(R_{th})}{B_k}, \tag{10}$$

from which the time instant $T_k$ to perform the $k$th PM can easily be obtained such that:

$$T_k = \mathcal{H}_1^{-1}\left(\mathcal{H}_1(A_k) - \frac{\ln(R_{th})}{B_k}\right) - A_k, \tag{11}$$

where $H_1^{-1}(t)$ states for the inverse function of $H_1(t)$.

Dealing with the particular case where the system lifetime is assumed to be Weibull distributed with the shape and scale parameter denoted, respectively, by $\xi$ and $\eta$, we have:

$$h_1(t) = \left(\frac{\xi}{\eta}\right)\left(\frac{t}{\eta}\right)^{\xi-1}, \quad \text{and} \quad \mathcal{H}_1(t) = \left(\frac{t}{\eta}\right)^{\xi}, \tag{12}$$

and the reliability $R_k(t)$ for the $k$th maintenance cycle can be obtained as:

$$R_k(t) = \exp\left[B_k\left(\left(\frac{A_k}{\eta}\right)^{\xi} - \left(\frac{t + A_k}{\eta}\right)^{\xi}\right)\right]. \tag{13}$$

while time $T_k$ of the $k$th PM given by Equation (11) becomes:

$$T_k = \eta\left(\left(\frac{A_k}{\eta}\right)^{\xi} - \frac{\ln(R_{th})}{B_k}\right)^{\frac{1}{\xi}} - A_k, \tag{14}$$

To illustrate, the hazard rate variation after PM actions is drawn in Figure 1. This figure corresponds to the case where the scale and shape parameters of Weibull lifetime distribution are set to $\eta = 40$ and $\xi = 2.5$, while reliability threshold $R_{th} = 0.9$. The age reduction coefficients $\alpha_k$ as well as the increase hazard rate coefficients (the adjustment factor) $\beta_k$ are evaluated according to the following formula:

$$\alpha_k = \frac{k}{3k+1}; \quad \beta_k = \frac{4k+1}{3k+1}. \tag{15}$$

For a horizon composed of four cycles, PM instants $T_k$ $(k = 1, \ldots, 4)$ are obtained by using Equation (14) and evaluated to $T_1 = 16.26$, $T_2 = 11.04$, $T_3 = 7.30$ and $T_4 = 4.95$. From Figure 1, it is shown that, right after PM, the hazard rate of the system is reduced to a certain value (effect due to the age reduction coefficient) and then increases more quickly than it behave just before performing the PM (effect of the hazard rate increased coefficient). Furthermore, as one can expects, more the system is aging more frequent are PM actions.

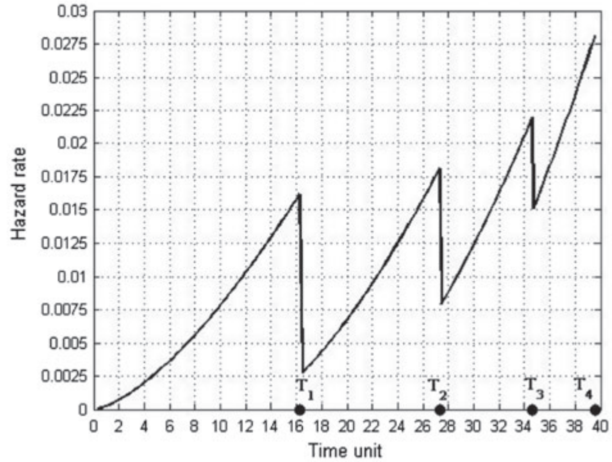


Figure 1. Changes of the system hazard rate due to imperfect PM.

The choice of reliability threshold $R_{th}$ as well as the number $N$ of PM and replacement actions to be performed economically impact the performance of the maintenance policy. Indeed, on the one hand, a low value of the reliability threshold $R_{th}$ implies a long and an uninterrupted use of the system but with, however, an increased risk of failures. On the other hand, if the reliability threshold is set to a high value, the number of system failures is then reduced but with an increasing number of PM actions that induce an important maintenance cost as well as system unavailability. The maintenance optimisation problem addressed in the present paper consists then on finding a trade off between the reliability threshold and the number of PM actions to maximise the average system availability.

## 3. Maintenance optimisation model

According to the PM strategy adopted in the present paper, the system may experience $N$ maintenance cycles. Within each cycle, the system is maintained either when it fails or its reliability reaches the threshold $R_{th}$. Corrective maintenance is carried out whenever the system fails while PM is performed in the case where the system survives until the required reliability threshold is reached. Expected durations of corrective and PM are, respectively, denoted by $T_c$ and $T_p$. At the end of the $N$th cycle, the system is replaced by a new one. The expected duration of system replacement is denoted by $T_r$. The optimal maintenance policy consist then on finding optimal values of both the number $N$ of cycles and the critical system reliability threshold $R_{th}$ that maximises the average system availability. The average system availability is then a function of the both decision variables $N$ and $R_{th}$, and hereafter denoted by $\mathcal{A}_v(R_{th}, N)$ and defined to be such that:

$$\mathcal{A}_v(R_{th}, N) = \frac{E(UpTime)}{E(UpTime) + E(DownTime)}, \tag{16}$$

where $E(UpTime)$ and $E(DownTime)$ corresponds to the expected system up time and down time, respectively. To determine these times, it is worth noticing that the maintenance policy adopted induces a renewal process. A renewal cycle is the time between two consecutive replacements and composed of the times where the system alternates between the up state and down state. These expected values are given as:

$$E(UpTime) = \sum_{k=1}^{N} UpTime(k), \text{ and} \tag{17}$$

$$E(DownTime) = \sum_{k=1}^{N} DownTime(k), \tag{18}$$

where $UpTime(k)$ and $DownTime(k)$, for $k \in \{1, \ldots, N\}$ represents the average time, during the $k$th maintenance cycle, where the system is in its up state and down state, respectively. Accordingly, the average availability is written as:

$$\mathcal{A}_v(N,R_{th}) = \frac{\sum\limits_{k=1}^{N} UpTime(k)}{\sum\limits_{k=1}^{N}(UpTime(k)+DownTime(k))}. \tag{19}$$

The following proposition gives $UpTime(k)$ and $DownTime(k)$.

Proposition 3.1 *During the kth maintenance cycle, $DownTime(k)$ and $UpTime(k)$ are evaluated such that:*

$$DownTime(k) = \begin{cases} T_c(1-R_{th})+T_pR_{th} & \text{if } k=1,\dots,N-1, \\ T_r & \text{if } k=N. \end{cases} \tag{20}$$

$$UpTime(k) = \int\limits_0^{T_k} R_k(t)dt. \tag{21}$$

*Proof* Let us denote by the random variable $X_k$ the residual lifetime of the system before the $k$th maintenance (preventive or corrective) is performed. Let $F_k(t)$ be the cumulative distribution function of $X_k$ and $R_k(t) = 1 - F_k(t)$ be its reliability function. Since the system undergoes maintenance whenever its reliability reaches the threshold value $R_{th}$ or fails before. It follows that probabilities to perform preventive and corrective maintenance are, respectively, given by $P(X_k > T_k) = R_k(T_k)$, which is equal to $R_{th}$, and $P(X_k \leqslant T_k) = F_k(T_k)$, for $k = 1,\dots,N-1$. Dealing with the $N$th cycle, the system is replaced either at failure or at the time where the reliability threshold is reached. This induces a duration $T_r$ with probability 1. On the other hand, the system remains in its operating state for a random time given by the random variable $\min(X_k, T_k)$ which also corresponds to the up time. It follows that the expected system operating time is given by:

$$E\left(\min(X_k,T_k)\right) = \int\limits_0^{T_k} R_k(t)dt,$$

this ends the proof. □

From the results of the above proposition, it follows that the average system availability $\mathcal{A}(N,R_{th})$:

$$\mathcal{A}_v(N,R_{th}) = \frac{\sum\limits_{k=1}^{N}\left(\int\limits_0^{T_k} R_k(t)dt\right)}{(N-1)\left(T_c(1-R_{th})+T_pR_{th}\right)+T_r+\sum\limits_{k=1}^{N}\left(\int\limits_0^{T_k} R_k(t)dt\right)}, \tag{22}$$

By maximising the system availability $\mathcal{A}_v(N,R_{th})$, the optimal maintenance policy can be determined according to the optimal values obtained for both the reliability threshold and the number of maintenance cycles.

Knowing the initial hazard rate function $h_1(t)$ of the system, solutions $T_k$ $(k=1,\dots)$ of Equation (11) are derived as functions of the reliability level $R_{th}$. Substituting these solutions into the PM optimisation model (Equation 22) this later becomes also a function of $R_{th}$. Computing the first derivative $\frac{\partial \mathcal{A}_v(N,R_{th})}{\partial R_{th}}$ and setting it equal to the null value, leads to the following equation:

$$\mathcal{A}_v(N,R_{th}) = \frac{\sum\limits_{k=1}^{N}\left(\frac{1}{B_kh_1(A_k+T_k)}\right)}{(N-1)(T_c-T_p)+\sum\limits_{k=1}^{N}\left(\frac{1}{B_kh_1(A_k+T_k)}\right)}. \tag{23}$$

The value of the optimal reliability threshold $R^*_{th}$ as a solution of Equation (23) is then obtained as a function of the number $N$ of maintenance cycles. The optimal value $N^*$ of $N$ is obtained by maximising the right-hand side of Equation (23):

$$\frac{\sum\limits_{k=1}^{N}\left(\frac{1}{B_kh_1(A_k+T_k)}\right)}{(N-1)(T_c-T_p)+\sum\limits_{k=1}^{N}\left(\frac{1}{B_kh_1(A_k+T_k)}\right)}, \tag{24}$$

while time instants where PM actions should be carried out are derived straightforward from Equation (11). Accordingly, the following algorithm is proposed to the computation of both optimal reliability threshold as well as the optimal PM and replacement schedule.

**Algorithm 1:**

**Step 1**: Compute solutions $T_k$ of Equation (11). Theses solutions are functions of reliability threshold $R_{th}$,
**Step 2**: Use solutions in Step 1 to solve Equation (23) with respect to $R_{th}$,
**Step 3**: Based on Steps 1 & 2, choose $N$ to maximise the function given by Equation (24),
**Step 4**: From results of Steps 1 & 2 together with the optimal number $N$ from Step 3, compute $T_k$ $(k = 1, \ldots, N)$,

In the following section, a numerical example is provided to illustrate the proposed PM optimisation model. The above algorithm is implemented and the overall results obtained in the following experiments are discussed.



## 4. Numerical example

In this section a numerical example is conducted for a system whose lifetime follows a Weibull distribution with shape and scale parameters are, respectively, denoted by $\xi$ and $\eta$. These two parameters as well as average duration times $T_c$, $T_p$ and $T_r$ are assumed to be known. Usually, such time parameters are obtained from maintenance engineers or estimated from reliability historical data. To conduct our example, we only need to know the ratios $\delta_{cp} = \frac{T_c}{T_p}$ and $\delta_{rp} = \frac{T_r}{T_p}$. In the present numerical example, the following parameter values are arbitrary and considered only for illustration purpose. The ratio $\delta_{cp} = 2$ while the ratio $\delta_{rp}$ is made variable such that $\delta_{rp} \in \{10, 50, 100, 500\}$. Parameters $\xi$ and $\eta$ of the system lifetime Weibull distribution are set to $\xi = 3.85$ and $\eta = 350$ (given in time unit). Age reduction coefficients $\alpha_k$ as well as hazard rate increasing coefficients $\beta_k$ are obtained from the following formula:

$$\alpha_k = \frac{k}{3k+2}, \text{ and } \beta_k = \frac{2k+3}{k+2}, \quad (k = 1, \ldots). \tag{25}$$

First, let us assume that the system should experience its mission with a required reliability greater than $R_{th} = 0.31$, i.e. the system should undergoe PM whenever its reliability reaches the value 0.31. For the value of the ratio $\delta_{rp}$ fixed to $\delta_{rp} = 100$, Figure 2 gives the average availability vs. the number $N$ of PM and replacement. From this figure, the optimal number $N^*$ of PM (including the system replacement) is found to be $N^* = 6$. Thus, the system experiences 5 PM actions and then replaced by a new one. According to this maintenance policy, the system reliability vs. time is drawn in Figure 3. From this figure, the first PM action is performed at time $T_1 = 367.02$ (unit of time), from that time the system undergoes the second PM at $T_2 = 248.30$ and so on. The system is replaced at $T_6 = 22, 27$ unit of time after the 5th PM action. The system average availability induced by such a maintenance policy is $\mathcal{A}_v(N^*, R_{th}) = 86.29\%$ (see Figure 2).

The present work seek, however, to ensure a balance between the reliability threshold and the number of PM actions to minimise the expected total maintenance and replacement cost per unit of time in the infinite time horizon. Indeed, for different values of the reliability threshold $R_{th}$ together with the ratio $\delta_{rp}$ fixed to $\delta_{rp} = 100$, Figure 4 gives the optimal

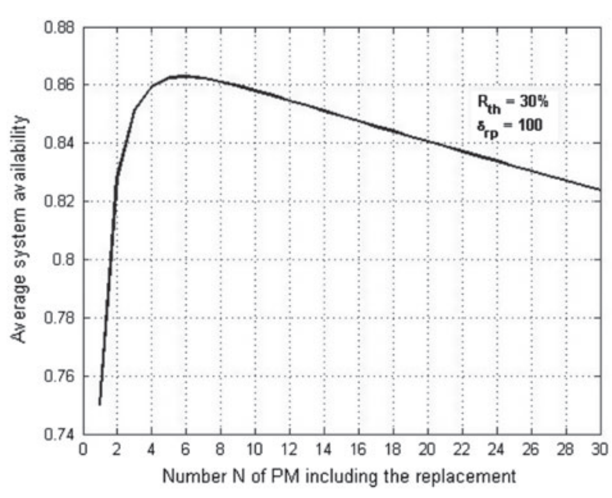


Figure 2. Average availability vs. the number of maintenance cycles: case of $R_{th} = 0.31$ and $\delta_{rp} = 100$.

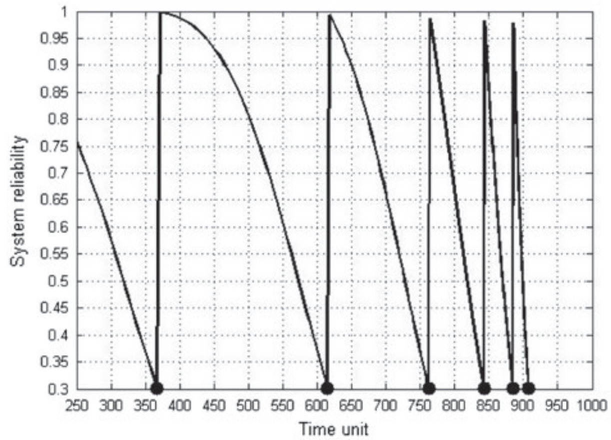


Figure 3. Changes of system reliability under the CBM where $\delta_{rp} = 100$ and $R_{th} = 0.31$.

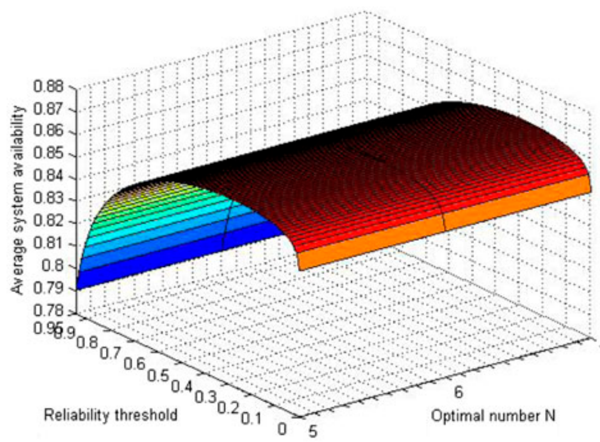


Figure 4. Average availability and the optimal number of maintenance cycles vs. reliability threshold.

number $N^*$ of PM including the system replacement and the induced average availability. In this figure, it is shown that higher is the reliability threshold more important is the number of PM. This implies that, as one can expect, to keep higher the system reliability more frequent preventive maintenance actions need to be carried out. Consequently, the system becomes less and less available. The same consequence is also obtained whenever the reliability threshold is set to a low value. Indeed, a low value of the threshold implies a long and an uninterrupted use of the system but with, however, an increased risk of failures which increase the number of corrective maintenance and by the way decrease the system availability. Therefore, the maintenance optimisation problem to be solved consists on finding the joint optimal reliability threshold together with the optimal number of maintenance cycles to maximise the average availability of the system.

By varying simultaneously the reliability threshold $R_{th}$ and the number $N$ of maintenance cycles, Table 1 gives the optimal PM and replacement schedules for the different values assigned to the ratio $\delta_{rp}$. For each value of $\delta_{rp}$, Table 2 shows the system average availability $\mathcal{A}_v(N^*, R^*_{th})$ as well as the optimal reliability threshold $R^*_{th}$ and optimal number $N^*$ of maintenance cycles. From Table 2, it is found that the number of PM is increasing with respect to the ratio $\delta_{rp}$. Accordingly, when the ratio $\delta_{rp}$ becomes important, the proposed PM policy suggests to delay as far as possible the system replacement by performing more and more PM actions. Table 1 gives also time intervals for scheduled PM and replacement. From this table, it can also be seen that such time intervals decrease by the increase of maintenance cycles number. This is mainly due to both the system degradation and the imperfect effect of PM actions carried out on the system. For example, in the case where the ratio $\delta_{rp} = 50$, results from Table 1 suggest to perform 4 PM after which the system is replaced by a new one. The first PM of the system is performed after $T_1 = 370.22$ time unit. From that time and after $T_2 = 250.46$ time unit the system undergoes the second PM and so on. The system is then replaced by a new one 42.8 time unit following the fourth PM. With respect to this maintenance policy, the system reliability vs. time is depicted in Figure 5. From this figure, the optimal value corresponding to the reliability threshold as a control limit parameter is found to be equal to 28.9%. This value is also obtained from Table 2 for $\delta_{rp} = 50$.

Table 1. Optimal PM and replacement planing obtained by varying the ratio $\delta_{rp}$.

| $\delta_{rp}$ | 10 | 50 | 100 | 500 |
|---|---|---|---|---|
| | 363.88 | 370.22 | 373.46 | 376.76 |
| | 246.17 | 250.46 | 252.65 | 254.89 |
| | 145.64 | 148.17 | 149.47 | 150.79 |
| | | 80.91 | 81.62 | 82.35 |
| | | 42.79 | 43.17 | 43.55 |
| | | | 22.69 | 22.87 |
| | | | | 12.00 |
| | | | | 6.29 |

Table 2. Average availability corresponding to joint optimal number $N^*$ of maintenance cycles and reliability threshold $R^*_{th}$.

| $\delta_{rp}$ | 10 | 50 | 100 | 500 |
|---|---|---|---|---|
| $N^*$ | 3 | 5 | 6 | 8 |
| $R^*_{th}(\%)$ | 31.3 | 28.9 | 27.7 | 26.5 |
| $\mathcal{A}_v(N^*, R^*_{th})(\%)$ | 97.79 | 92.18 | 86.3 | 57.56 |

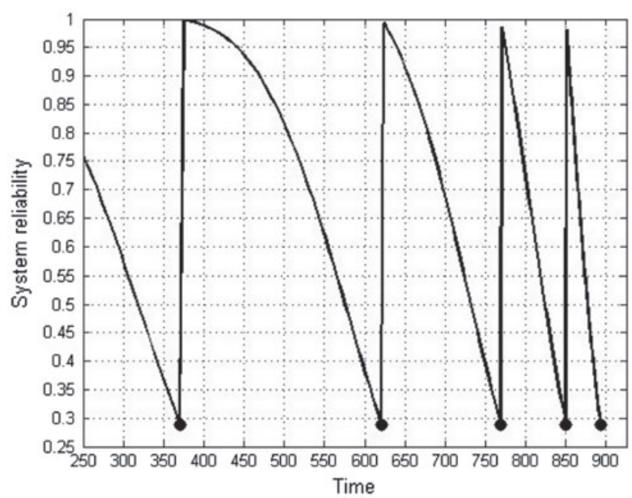


Figure 5. Changes of system reliability under the optimal CBM: case of $\delta_{rp} = 50$.



## 5. Conclusion

This paper proposed a imperfect PM optimisation model. The system considered is assumed to be continuously monitored and subject to stochastic degradations. Imperfect PM actions are undertaken on the basis of hybrid hazard rate model and the condition to perform a PM corresponds to a system reliability threshold. The hybrid hazard model offers the advantage to combine the effect of two coefficients, namely the age reduction coefficient and the hazard rate increase coefficient. Furthermore, it allows to represent the system degradation and the imperfect effect of PM. A mathematical model is then proposed and numerically solved. The optimal maintenance policy allows to derive the critical reliability threshold together with the number of maintenance cycles that maximise the average availability of the system.

We are currently working on an extension of the present work to deal with multicomponent systems. An important issue also consist to deal with the problem of integrated optimisation of such maintenance strategy while considering production demand and resources requirements. It is also important to investigate how coefficients of the hybrid hazard rate model can be obtained and to study the impact of their variability on the maintenance optimisation model.

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