PFENNING 6/19/2015

Linear Logic and Concurrency

Linear hypothetical judgment

$\underbrace{A_1, \ldots A_n}_{\text{resources}} \vdash C$

$$\frac{\Delta, A \vdash B}{\Delta \vdash A \multimap B}\multimap R \qquad \frac{\Delta' \vdash A \quad \Delta'', B \vdash C}{\Delta', \Delta'', A \multimap B \vdash C}\multimap L$$

$A \oplus B$ - internal choice

$A \otimes B$ - simultaneous conjunction

$A \multimap B$ - linear implication

$A \& B$ - alternative conjunction or external choice

$$\frac{\Delta' \vdash A \quad \Delta'' \vdash B}{\Delta', \Delta'' \vdash A \otimes B}\otimes R \qquad \frac{\Delta, A, B \vdash C}{\Delta, A \otimes B \vdash C}\otimes L \qquad \frac{}{A \vdash A} ID_A$$

$$\frac{\Delta \vdash A \quad \Delta', A \vdash C}{\Delta, \Delta' \vdash C} cut_A$$

ND		SEQ
Local soundness reduction	$\cong$	Cut reduction
Local completeness expansion	$\cong$	identity expansion
global "soundness" "normalization"	$\cong$	cut elimination
global "completeness"	$\cong$	(full) identity expansion

$$\frac{}{\cdot \vdash 1} 1R \qquad \frac{\Delta \vdash C}{\Delta, 1 \vdash C} 1L$$

Identity Expansion with $A \otimes B$.

$$\frac{}{A \otimes B \vdash A \otimes B} ID_{A \otimes B} \quad \underset{E}{\Longrightarrow} \quad \dfrac{\dfrac{\dfrac{}{A \vdash A} ID_A \quad \dfrac{}{B \vdash B} ID_B}{A, B \vdash A \otimes B}}{A \otimes B \vdash A \otimes B} \otimes L$$

Identity Expansion of 1

$$\frac{}{1 \vdash 1} ID_1 \quad \underset{E}{\Longrightarrow} \quad \dfrac{\dfrac{}{\cdot \vdash 1} 1R}{1 \vdash 1} 1L$$

$$\frac{\Delta \vdash A}{\Delta \vdash A \oplus B} \oplus R_1 \qquad \frac{\Delta \vdash B}{\Delta \vdash A \oplus B} \oplus R_2 \qquad \frac{\Delta, A \vdash C \quad \Delta, B \vdash C}{\Delta, A \oplus B \vdash C} \oplus L$$

We want

$A \otimes 1 \cong A$

$A \oplus 0 \cong A$

$A \& \top \cong A$

no 0R. $\qquad \frac{}{\Delta, 0 \vdash C} 0L$

$$\frac{\Delta, A \vdash C}{\Delta, A \& B \vdash C} \& L_1 \qquad \frac{\Delta, B \vdash C}{\Delta, A \& B \vdash C} \& L_2 \qquad \frac{\Delta \vdash A \quad \Delta \vdash B}{\Delta \vdash A \& B} \& R \qquad \frac{}{\Delta \vdash \top} \top R \quad \text{no } \top L$$

Eg: $\$1 \otimes \$1 \multimap T\&C$

dollars in sequence and malfunction: $\$1 \multimap (\$1 \multimap (T \& C) \otimes 1)$

gives money back if it malfunctions: $\$1 \multimap (\$1 \multimap (T \& C) \oplus (\$1 \otimes \$1))$

in ND: (for review)

$A_1 \ldots A_n \overset{ND}{\vdash} A$

correspondence:

$x_1 : A_1, \ldots x_n : A_n \vdash M : A$

local reduction drives computation:

$(\lambda x.M) N \Longrightarrow_R [N/x]M$

$\boxed{\text{Concurrent Computation}}$

$\underbrace{c_1 : A_1, \ldots c_n : A_n}_{\text{uses channels } c_1 \ldots c_n}^{\Delta} \vdash \overset{\text{process}}{P} :: \underbrace{(c : A)}_{\text{provides } c}$ session type

$$\frac{\Delta \vdash P :: (c : A) \quad \Delta', c : A \vdash Q :: (d : D)}{\Delta, \Delta' \vdash (c \leftarrow \text{spawn } P ; Q) :: (d : D)} Cut$$

$$\frac{\Delta, d : A \vdash Q :: (c : B)}{\Delta \vdash (d \leftarrow \text{recv } c ; Q) :: (c : A \multimap B)} \multimap R$$

$$\frac{\Delta' \vdash P :: (d : A) \quad \Delta'', c : B \vdash Q :: (e : E)}{\Delta', \Delta'', c : A \multimap B \vdash \text{send } c (d \leftarrow P) ; Q :: (e : E)} \multimap L$$

$$\frac{}{d:A \vdash [c \leftarrow d] :: (c:A)}\ \mathrm{ID}_A$$

"forwarding"

$$\frac{\Delta \vdash P :: (c:A) \qquad \Delta \vdash Q :: (c:B)}{\Delta \vdash \text{case}(\text{recv } c,\ \text{inl} \Rightarrow P,\ \text{inr} \Rightarrow Q) :: (c : A \& B)}\ \&R$$

$$\frac{\Delta, c:A \vdash R :: (d:D)}{\Delta, c:A \& B \vdash \text{send inl};\ R :: (d:D)}\ \&L_1$$

for $\&L_2$, sends inr.

$$\frac{}{\cdot \vdash \text{send } c \langle\rangle}\ 1R$$

$$\frac{\Delta \vdash Q :: (d:D)}{\Delta, c:1 \vdash \underbrace{\text{wait } c}_{\langle\rangle \leftarrow \text{recv } c};\ Q :: (d:D)}\ 1L$$

asymmetry of interpretation of $\otimes$.

$c : A \multimap B$	recv $d:A$ along c, continue as B
$c : A \otimes B$	send $d:A$ along c, continue as B
$c : 1$	terminate
$c : A \& B$	recv inl or inr, continue as A or B
$c : A \oplus B$	send inl or inr, continue as A or B

Implementation of this called SILL

Cut elimination means that everything eventually terminates.