HARPER 6/17/2015

- What should be equality? $\Gamma \vdash M \equiv N : A$

- Nat, zero, succ, rec_{Nat}

eg) plus : Nat $\to$ Nat $\to$ Nat

$\lambda x \lambda y\ rec(x ; x ; succ(x'))(y)$

Fact: $\forall m,n \in \mathbb{N}$ plus $\bar{m}\ \bar{n} \equiv \overline{m+n} =$ plus $\bar{n}\ \bar{m}$

pf. by mathematical induction.

$\beta \begin{cases} x : Nat \vdash \text{plus } x \text{ zero} \equiv x \\ x : Nat, y : Nat \vdash \text{plus } x\ succ(y) \equiv succ(\text{plus } x\ y) \end{cases}$

"true by calculation"

$x : Nat \vdash \text{plus } 0\ x \not\equiv x$

$\hookrightarrow$ without unicity (mathematical induction)

Families of Types $\approx$ Predicates aka propositional functions

Logic $\approx$ predicates

FP $\approx$ $n : Nat \vdash Seq(n)$ type

STT - simple type theory

A type

Γ ctx

$\Gamma \vdash M : A$

$\gamma : \Gamma' \to \Gamma$

$x_1 : A'$ $\quad x_1 : A_1 \ldots x_n : A_n$

$x[A]$

$\Gamma' \vdash \gamma^* M : A$

DTT - dependent type theory

context / (closed) type

Γ ctx

$\Gamma \vdash A$ type

$\Gamma \vdash M : A$

family

element

$(\gamma : \Gamma' \to \Gamma)$

calculational equivalence

$\Gamma \equiv \Gamma'$

$\Gamma \vdash A \equiv A'$ type

$\Gamma \vdash M \equiv M' : A$

$(\gamma \equiv \gamma' : \Gamma' \to \Gamma)$

Functionality

$$\frac{x : A \vdash M_x : B_x \qquad P \equiv P' : A}{[P/x]M \equiv [P'/x]M : \begin{cases}[P/x]B \\ [P'/x]B\end{cases}}$$

Functionality

$$\frac{x : A \vdash B_x \text{ type} \qquad M \equiv M' : A}{[M/x]B \equiv [M'/x]B \text{ type}}$$

eg) $Seq(\text{plus } M \text{ zero}) \equiv Seq\ M$ type

Respect for Type Equivalence:

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash A \equiv A'}{\Gamma \vdash M : A'} \qquad \frac{\Gamma \vdash M \equiv M' : A \quad \Gamma \vdash A \equiv A'}{\Gamma \vdash M \equiv M' : A'}$$

* Something being of type A is a matter of syntax, not a theorem. * Should be consequence of rules.

Which families do we have?

- $n : Nat \vdash Seq(n)$ type
- Π, Σ
- all STT

$IF(7, \text{"7"})(x) : \underbrace{IF(Nat, String)(x)}_{\text{family}}$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma, x : Bool \vdash IF(A,B)(x) \text{ type}}$$

Replace families/elements with universes:

$$\Gamma \vdash A : \mathcal{U} \qquad \Gamma \vdash A \equiv A' : \mathcal{U}$$

so can replace w/ next line:

$$\Gamma \vdash M : A \qquad \Gamma \vdash M \equiv M' : A$$

But can't have nested within self.

Infinite hierarchy of universes $\mathcal{U}_0 : \mathcal{U}_1 : \mathcal{U}_2 \ldots$
cumulative "$\mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \mathcal{U}_2 \ldots$" ie $\dfrac{\Gamma \vdash M : \mathcal{U}_i}{\Gamma \vdash M : \mathcal{U}_{i+1}}$

$$\frac{}{\Gamma \vdash Nat : \mathcal{U}_0} \qquad \frac{}{\Gamma \vdash 0, 1 : \mathcal{U}_0}$$

$$\frac{\Gamma \vdash A : \mathcal{U}_i \quad \Gamma \vdash B : \mathcal{U}_i}{\Gamma \vdash A \overset{\times}{\to} B : \mathcal{U}_i} \qquad \frac{}{\Gamma \vdash \mathcal{U}_i : \mathcal{U}_{i+1}} \text{ (hierarchy)}$$

(inconsistent: $\mathcal{U} : \mathcal{U}$).

$$\frac{}{\Gamma, x : Nat \vdash Seq(x) : \mathcal{U}_0} \qquad \frac{\Gamma \vdash A : \mathcal{U}_i \quad \Gamma, x : A \vdash B : \mathcal{U}_i}{\Gamma \vdash \Pi x : A . B : \mathcal{U}_i}$$

$\Sigma x : A . B : \mathcal{U}_i$

$$\frac{\Gamma, x : A \vdash M_x : B_x}{\Gamma \vdash \lambda x . M : \Pi x : A . B} \Pi\text{-}I \qquad \frac{\Gamma \vdash M : \Pi x : A . B \quad \Gamma \vdash N : A}{\Gamma \vdash M(N) : [N/x] B}$$

β $\Gamma \vdash (\lambda x . M)(N) \equiv [N/x] M : [N/x] B$

eg: $\underbrace{\lambda X . \lambda x . x : (\Pi X : \mathcal{U}_i . X \to X) : \mathcal{U}_{i+1}}_{\text{predicativity}}$

η $\Gamma \vdash \lambda x . M x \equiv M : \Pi x : A . B$

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : [M/x] B_x}{\Gamma \vdash \langle M, N \rangle : \Sigma x : A . B} \Sigma I \qquad \frac{\Gamma \vdash M : \Sigma x : A . B}{\Gamma \vdash fst\, M : A}$$

$\Gamma \vdash snd\, M : [fst\, M/x] B$

In Props as types:
$\Pi \sim \forall$
$\Sigma \sim \exists$
↳ constructive existential

β: $fst \langle M, N \rangle \equiv M$, $snd \langle M, N \rangle \equiv N$

not constructive

Fact *

$\exists a, b$ irrational ($\notin \mathbb{Q}$) s.t. $a^b \in \mathbb{Q}$

Ex. what does this prove?

Pf consider $a = b = \sqrt{2}$
either $\sqrt{2}^{\sqrt{2}}$ is in $\mathbb{Q}$ or not:
if rational, done. else, $a = \sqrt{2}^{\sqrt{2}}$, $b = \sqrt{2}$.
$\sqrt{2}^{\sqrt{2}^{\sqrt{2}}} = \sqrt{2}^2 = 2 \in \mathbb{Q}$. $\square$

$\lambda x : Nat . \_\_ : Vec\, x$

(Int, Vec x)

(Int, Vec N)

(1, [0])

~~(1, [0, 0])~~

If, revisited.

STT

$$\dfrac{\Gamma \vdash M : C \qquad \Gamma \vdash N : C}{\Gamma, z : \mathrm{Bool} \vdash \mathrm{if}(M, N)(z) : C}$$

DTT

$$\dfrac{\Gamma, z : \mathrm{Bool} \vdash C_z : \mathcal{U} \qquad \Gamma \vdash M : [tt/z]C \qquad \Gamma \vdash N : [ff/z]C}{\Gamma, z : \mathrm{Bool} \vdash \mathrm{if}[z.C](M, N)(z) : C_z}$$

↳ motive

principle: "proof by induction"

"GADTs"

STT

$$\dfrac{\Gamma \vdash M : C \qquad \Gamma, x : \mathrm{Nat}, y : C \vdash N : C}{\Gamma, z : \mathrm{Nat} \vdash \mathrm{rec}_{\mathrm{Nat}}(M, x, y.N) : C}$$

DTT

$$\dfrac{\Gamma, z : \mathrm{Nat} \vdash C_z : \mathcal{U} \qquad \Gamma \vdash M : [\mathrm{zero}/z]C \qquad \Gamma, x : \mathrm{Nat}, y : [x/z]C \vdash N : [\mathrm{succ}\, x/z]C}{\Gamma, z : \mathrm{Nat} \vdash \mathrm{rec}_{\mathrm{Nat}}(M; x, y.N)(z) : C_z}$$

> * Ex. choose $z : \mathrm{Nat} \vdash \mathrm{Seq}\, z$,
> write program w/ type
> $z : \mathrm{Nat} \vdash \_\_\_\_ : \mathrm{Seq}\, z$
> that does!

Idea: C_z can compute in interesting ways.

ex. $EQ : \mathrm{Nat} \to \mathrm{Nat} \to \mathcal{U}$

s.t. $EQ(\bar{m})(\bar{m}) \equiv 1$

$EQ(\bar{m})(\bar{n}) \equiv 0 \quad (m \neq n)$

recursively:

$EQ(\bar{0})(\bar{0}) \equiv 1$

$EQ(\bar{0})(\mathrm{succ}(\_)) \equiv 0$

$EQ(\mathrm{succ}(\_))(\bar{0}) \equiv 0$

$EQ(\mathrm{succ}(x))(\mathrm{succ}(y)) \equiv EQ(x)(y)$

> * Exercise: write out with $\mathrm{rec}_{\mathrm{Nat}}$

> * Exercise: $x : \mathrm{Nat} \vdash \_\_\_\_$ ← fill in
> s.t. $EQ(x)(x)$

Suppose $f : \mathrm{Nat} \to \mathrm{Nat}$

and $\_\_ : EQ(M, N)$

Is it the case that

$\_\_ : EQ(fM, fN)$?