6/19/2015 HARPER

Identification Type

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathrm{refl}_A(M) : M =_A M}\ \mathrm{I}$$

$$\frac{\Gamma, x, y : A,\ z : x =_A y \vdash C_{x,y,z} : \mathcal{U} \qquad \Gamma, x : A \vdash c : [x, x, \mathrm{refl}_A(x)/x, y, z]C \qquad \Gamma \vdash \alpha : M =_A N}{\Gamma \vdash J[x,y,z.C](x.c)(\alpha) : [M, N, \alpha]C}$$

special case:

$$\frac{\Gamma, x : A, y : A \vdash C_{x,y} : \mathcal{U} \qquad \Gamma, x : A \vdash c : [x,x/x,y]C \qquad \Gamma \vdash \alpha : M =_A N}{\Gamma \vdash J'[x,y.C](x.c)(\alpha) : [M,N/x,y]C}$$

$J'[x,y.C](x.c)(\mathrm{refl}_A M) \equiv [M/x]c$

$\boxed{\text{transport}}$

$$\frac{\Gamma, z : A \vdash B : \mathcal{U} \qquad \Gamma \vdash \alpha : M =_A N}{\Gamma \vdash \mathrm{tr}[z.B](\alpha) : [M/z]B \to [N/z]B}$$

$\mathrm{tr}[z.B](\mathrm{refl}_A M)(P) \equiv P$

Derived Notion "Path Over" aka "Correlation"

$\boxed{\text{identity} = \text{path}}$

suppose $z : A \vdash B : \mathcal{U}$ $\quad \alpha : M =_A N$

$$\begin{array}{ccc} P & =^{z.B}_{\alpha} & Q \\ \vdots & & \vdots \\ [M/z]B & & [N/z]B \end{array} \quad \triangleq \ \mathrm{tr}[z.B](\alpha)(P) =_{[N/z]B} Q$$

Identification is an equivalence relation

I. $\mathrm{refl}_A(M) : M =_A M$

II. $\dfrac{\alpha : M =_A N}{\alpha^{-1} : N =_A M}$ is derivable $\qquad$ Computational Equation: $(\mathrm{refl}\,M)^{-1} \equiv \mathrm{refl}\,M$

Pf. identification induction on α:

$\alpha^{-1} \triangleq J'[x,y.\ y =_A x](x.\mathrm{refl}_A x)(\alpha) : [M,N/x,y](y =_A x)$

$: N =_A M.$

III. $\dfrac{\alpha : M =_A N \qquad \beta : N =_A P}{\alpha \cdot \beta : M =_A P}$ is derivable.

Pf.#1 ~~Pf. $\alpha \cdot \beta \triangleq J[x,y$~~

$x, y, z : A,\ p : \mathrm{Id}_A(x,y),\ q : \mathrm{Id}_A(y,z) \vdash p \cdot q : \mathrm{Id}_A(x,z)$

$x, y : A,\ p : \mathrm{Id}_A(x,y) \vdash \prod_{z:A} \underbrace{\mathrm{Id}(y,z) \to \mathrm{Id}(x,z)}.$

$\lambda z\,[\lambda u : \mathrm{Id}(\ ,z).\ J'[x,y.\mathrm{Id}(y,z) \to \mathrm{Id}(x,z)](x.\lambda x.x)(p) : \ ]$

Computationally: $\mathrm{refl}_A(M) \cdot q \equiv q$. But not $p \cdot \mathrm{refl}(N) \equiv p$.

Pf#2. Exercise. example 4 in proof, 5.1. holds

Functionality (relatable to functoriality)

$f : A \to B,\ x : A,\ y : A,\ p : x =_A y \vdash ap_f(p) : f(x) =_B f(y)$ is derivable

$ap_f(p) \triangleq J'[u,v.\ f(u) =_B f(v)](u.\ refl_B(f(u)))(p) : f(u) =_B g(u)$

for Π:

$f : \Pi_{x:A}.B,\ x,y : A,\ p : x =_A y \vdash \boxed{apd_f(p)} : f(x) \overset{x.B}{=}_p g(x)$.

*Exercise

Identification is an ∞-groupoid.

Group has unit : refl, inv : symmetry, mult : transitivity } equivalence

"Group is groupoid with only one type"

Groupoid Laws

$unit_L : refl(M) \cdot \alpha =_{M =_A N} \alpha$

$unit_R : \alpha =_{M =_A N} \alpha \cdot refl(N)$

$unit_L$
$$\begin{tikzcd} M \arrow[r, "refl"] \arrow[rr, bend right, "\alpha"'] & M \arrow[r, "\alpha"] & N \end{tikzcd}$$

or

$inv_L : \alpha^{-1}\alpha =_{N =_A N} refl(N)$

$inv_R : \alpha\alpha^{-1} =_{M =_A M} refl(M)$

$unit_R$
$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[rr, bend right, "\alpha"'] & N \arrow[r, "refl"] & N \end{tikzcd}$$

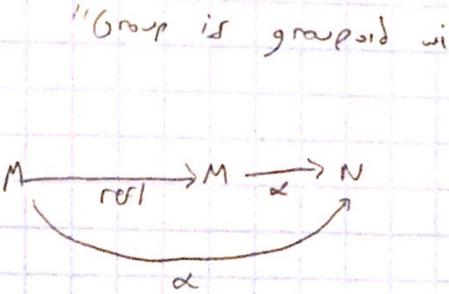
inv_L
$$\begin{tikzcd} N \arrow[r, "\alpha^{-1}"] \arrow[rr, bend right, "refl"'] & M \arrow[r, "\alpha"] & N \end{tikzcd}$$

$assoc : \alpha \cdot (\beta\gamma) =_{M = P} (\alpha\beta)\gamma$

*

$\boxed{\text{Exercise: all by identification induction}}$

assoc



Structure of Identifications

$$\boxed{Id_{A \times B}(M,N) \simeq Id_A(fst\ M, fst\ N) \times Id_B(snd\ M, snd\ N)}$$ *

Exercise: show this (via identification induction).

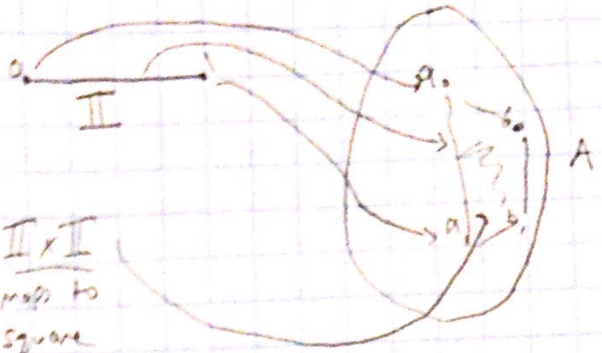
$Id_{A \to B}(f,g) \xrightarrow{happly} \prod_{x:A} f(x) =_B g(x) \simeq \prod_{x,y:A} x =_A y \to f(x) =_B g(y)$

*ex not equivalence. *ex

Univalence $Id_U(A,B) \simeq \overbrace{A \simeq B}^{\text{almost true}}$ ← want this type to have at most one element up to higher identification.

$\Sigma f : A \to B.\ IsEquiv(f)$

Interval



$\mathbb{I} \times \mathbb{I}$ maps to square

$$\left.\begin{array}{c} \overline{\quad 0 : \mathbb{I} \quad} \\[1em] \overline{\quad 1 : \mathbb{I} \quad} \\[1em] \overline{\quad seg : 0 =_{\mathbb{I}} 1 \quad} \end{array}\right\} I$$

$$\frac{\Gamma, z : \mathbb{I} \vdash C_z : \mathcal{U} \quad \Gamma \vdash c_0 : [0/z]C_z \quad \Gamma \vdash c_1 : [1/z]C_z \quad \Gamma \vdash s : c_0 =^{z.C}_{seg} c_1}{\Gamma, z : \mathbb{I} \vdash rec_{\mathbb{I}}[z.C](\qquad)(z) : C}$$

Fun Fact

OTT + Id + $\mathbb{I}$ has function extensionality (funext).

1 $\underbrace{\mathbb{I} \to A}_{\text{"geometric lines in A"}} \simeq \underbrace{\Sigma_{x,y:A}.\, x =_A y}_{\text{"free path space"}}$

"algebraic lines in A"

aka "identifications"

2. $\mathbb{I} \to (A \to B) \cong (\mathbb{I} \times A) \to B$

$\cong (A \times \mathbb{I}) \to B \longleftarrow$ "homotopy"

fun. ext. $\cong A \to (\mathbb{I} \to B)$

"paths between functions are same as paths between ranges."

Higher Inductive Definition $\approx$ Proof relevant Quotient.

Problem:

Is there a computational interpretation of $\underbrace{\text{HiTT}}$ higher type theory.

if only identification is refl, then J reduces

but then what about $J[\ ](x.C)(seg) \equiv ?$