6/15/2015 MOREHOUSE

Category Basics

"by assuming less can apply to more situations"

- Logical objects as categorical objects
- Proofs as arrows

A <u>Category</u> $\mathbb{C}$ has the following:

$\boxed{\text{data}}$

- An object collection '$\mathbb{C}_0$'
  "$A : \mathbb{C}$" $:= A \in \mathbb{C}_0$
- A morphism collection '$\mathbb{C}_1$'
  "$f :: \mathbb{C}$" $:= f \in \mathbb{C}_1$
  '$\mathbb{C}(A \to B)$' $:= \{ f :: \mathbb{C} \mid \partial^-(f) = A \text{ and } \partial^+(f) = B \}$
- Boundary functions - from arrows to objects.
  + Domain "∂^-"
  + Codomain "∂^+"
- Identity arrow functions from objects to arrows
  "id" s.t. $id(A) \in \mathbb{C}(A \to A)$
- Arrow composition
  if $f : A \to B$ and $g : B \to C$
  $f \cdot g : A \to C$

$\boxed{\text{laws}}$

- left unit law. for $f : A \to B$, $id(A) \cdot f = f$.
- right unit law. for $f : A \to B$, $f \cdot id(B) = f$.
- associative law. for $f : A \to B$, $g : B \to C$, $h : C \to D$, $(f \cdot g) \cdot h = f \cdot (g \cdot h)$

$\boxed{\text{terms}}$ - just to shorten things:

- Arrows f and g are <u>composable</u> if $\partial^+(f) = \partial^-(g)$
- Arrows f and g are <u>parallel</u> if $\partial^i(f) = \partial^i(g)$
- " " " <u>anti-parallel</u> if composable both ways
- An <u>endomorphism</u> is a self-composing arrow
- A <u>path</u> is a "serially composable" list of arrows
- Notation: $g \circ f = f \cdot g$.
- Notation: use "A" instead of $id(A)$ when obvious.

<u>Diagrams</u>

$A \xrightarrow{f} B \xrightarrow{g} C$

<u>Commutative Diagram</u>

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "f \cdot g"'] & & C \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & C \arrow[r, "h"] & D \end{tikzcd}$$

"whiskering"

$f \cdot g_1 = f \cdot g_2$, $g_1 \cdot h = g_2 \cdot h$

"pasting" (transitivity of equality, by combining diagrams)

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ A \arrow[ur, "id"] \arrow[rr, "f"] \arrow[dr, "f"'] & & B \\ & B \arrow[ur, "id"'] & \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[rr, "f \cdot g"] \arrow[d, "f"'] & & C \arrow[d, "h"] \\ B \arrow[urr, "g"] \arrow[rr, "g \cdot h"'] & & D \end{tikzcd}$$

Example Categories

- the empty category "$\mathbb{0}$"

- $\star \circlearrowleft^{id}$ A singleton category "$\mathbb{1}$"

- $\bullet\circlearrowleft \ \bullet\circlearrowleft \ \bullet\circlearrowleft$ Discrete categories. Any set can be thought of as a discrete category

- For any pre-order set $(P, \leq)$ we have a preorder category $\mathbb{P}$.

  Objects: $\mathbb{P}_0 := P$

  Arrows: $\mathbb{P}(x \to y) := \begin{cases} \{\text{"}x \leq y\text{"}\} & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases}$

  Identities: $id(x) := \text{"}x \leq x\text{"}$

  Composition: $\text{"}x \leq y\text{"} \cdot \text{"}y \leq z\text{"} = \text{"}x \leq z\text{"}$

  A pre-order category that is not discrete:

  $\bullet \longrightarrow \bullet$ is the Interval category "$\mathbb{I}$"

  e.g. $(\mathbb{N}, +, 0)$

- A monoid $(M, *, \varepsilon)$ can be regarded as a monoid category $\mathbb{M}$, with:

  Objects: $\mathbb{M}_0 := \{\star\}$

  Arrows: $\mathbb{M}(\star \to \star) := M$

  Identity: $id(\star) = \varepsilon$

  Composition: $x \cdot y := x * y$

  A simple monoid category that is not discrete.

  $\star \circlearrowleft^{s}$. so need $s \cdot s$, $s \cdot s \cdot s$ ... if all distinct, this is $(\mathbb{N}, +, 0)$.

Notation: $[\![x]\!] \leftarrow$ interpretation of

Categories of propositions and proofs

Objects: interpret propositions (& propositional contexts)

$[\![A]\!] : \mathbb{C} \quad [\![\Gamma]\!] : \mathbb{C}$

Arrows: interpret derivations

$$\left[\!\!\left[\begin{array}{c}\Gamma \\ \mathcal{D} \\ A\end{array} \right]\!\!\right] : [\![\Gamma]\!] \to [\![A]\!]$$

Identities: identity derivations

Composition: Later, with more machinery

Functor morphism of categories

for categories $\mathbb{C}$ and $\mathbb{D}$, a functor F from $\mathbb{C}$ to $\mathbb{D}$ has:
- function on objects $F_0 : \mathbb{C}_0 \to \mathbb{D}_0$
- function on arrows $F_1 : \mathbb{C}_1 \to \mathbb{D}_1$

s.t. $F : \mathbb{C}(A \to B) \longmapsto F_1(f) : \mathbb{D}(F_0(A) \to F_0(B))$

and $A \xrightarrow{id} A \longmapsto F(A) \xrightarrow{id} F(A)$

and

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "f \cdot g"] & & C \end{tikzcd} \longmapsto \begin{tikzcd} & F(B) \arrow[dr, "F(g)"] & \\ F(A) \arrow[ur, "F(f)"] \arrow[rr, "F(f \cdot g)"] & & F(C) \end{tikzcd}$$

- Structured sets as categories ↑ what we talked about above

- Categories **of** structured sets
  - if No structure:

    SET - the category of sets and functions. Objects: Sets, Arrows: functions. Id: $\lambda x.x$ Comp: function composition
  - PREORD - the category of preordered sets.

    Objects: preordered sets, Arrows: monotone maps. Id: $\lambda x.x$. Comp: function composition
  - MON - category of monoids

    Objects: monoids. Arrows: monoid homomorphism