Universal Constructions and
Interpretation of Intuitionistic Propositional Logic

Defn

A terminal object is an object T such that for any object X:

$$\exists! \underbrace{x}_{!(X)} : X \dashrightarrow T$$

Socrates Dictum

Know Thyself
("Probe thyself")

ie, set $X := T$, so
$$\begin{tikzcd} T \arrow[r, dashed, bend left, "!"] \arrow[r, bend right, "id"'] & T \end{tikzcd}$$

lemma:

Identity expansion: $!(T) = id(T)$

lemma Uniqueness

Terminals are unique up to a unique iso.

Pf

If T and R are terminals.

$$\begin{tikzcd} & R \arrow[d, dashed, "r"] \\ T \arrow[ur, dashed, "t"] & T \arrow[l, dashed, "!"] \arrow[l, bend left, "id"] \end{tikzcd}$$

Since unique, $t \cdot r = ! = id(T)$ and $r \cdot t = id(R)$ } t is iso, and the only, since R being terminal means only one morphism from T, thus only one iso.

So we write "1" for a terminal.

Ex. pre-composing with a bang

why $i \cdot !(X) = !(Y)$?

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr, "!"'] & & X \arrow[dl, "!"] \\ & 1 & \end{tikzcd}$$

Terminals in
- SET are singletons
- CAT are singletons
- PREORD are ? *
- a pre-order category are ? *

Interpret Truth

- Truth $[\![\top]\!] := 1$
- intro rule:

$$\frac{}{\top}\top I \iff \frac{}{\Gamma \vdash \top}\top I \iff \frac{\Gamma}{\top}\top I$$

$$[\![\top I]\!] := !([\![\Gamma]\!]) : [\![\Gamma]\!] \to [\![\top]\!]$$

Defn/

An <u>initial object</u> is an object S such that for any object X:

$\exists! \underbrace{s}_{\text{¡}(X)} : S \to X \qquad S \xrightarrow{\text{¡}(X)} X$

Refer to arbitrary initial object as 0.

- Id expansion: $\text{¡}(0) = id(0)$
- Uniqueness: Initial objects are unique up to unique iso.
- Composition with a cobang

$$\begin{tikzcd} X \arrow[r, "i"] & Y \\ 0 \arrow[u, dashed, "\text{¡}(X)"] \arrow[ur, dashed, "\text{¡}(Y)"'] & \end{tikzcd} \qquad \text{¡}(X) \cdot i = \text{¡}(Y)$$

Initials in:
- SET are empty sets
- CAT are empty categories.

<u>Interpreting Falsehood</u>
- Falsehood $[\![\bot]\!] = 0$
- elim rule:

$$\frac{\bot}{A}\,\bot E \qquad [\![\bot E]\!] = \text{¡}([\![A]\!]) : [\![\bot]\!] \to [\![A]\!]$$

Defn

A <u>product</u> of objects A and B is a <u>span</u> on A and B,
i.e. $A \xleftarrow{p_0} P \xrightarrow{p_1} B$
So that for any span on A and B $(A \xleftarrow{x_0} X \xrightarrow{x_1} B)$
$\exists! \, t : X \to P, \; t \cdot p_i = x_i$ i.e.
$t = \langle x_0, x_1 \rangle$

$$\begin{tikzcd} & X \arrow[dl, bend right, "x_0"'] \arrow[dr, bend left, "x_1"] \arrow[d, dashed, "t"] & \\ A & P \arrow[l, "p_0"] \arrow[r, "p_1"'] & B \end{tikzcd}$$

Vocabulary: A and B are <u>Factors</u>
p_0 and p_1 are <u>projections</u>
t is the <u>tuple</u> of x_0, x_1

$$\begin{tikzcd} & P \arrow[dl, bend right, "p_0"'] \arrow[dr, bend left, "p_1"] \arrow[d, dashed, "\langle p_0, p_1 \rangle"] & \\ A & P \arrow[l, "p_0"] \arrow[r, "p_1"'] & B \end{tikzcd}$$

Lemma: Identity expansion for products is $id(P) = \langle p_0, p_1 \rangle$

Lemma:
Uniqueness: Products are unique up to a projection-preserving iso.

Pf.
$$\begin{tikzcd} & P \arrow[dl, "p_0"'] \arrow[dr, "p_1"] \arrow[d, dashed, "s"] & \\ A & Q \arrow[l, "q_0"] \arrow[r, "q_1"'] & B \\ & P \arrow[ul, "p_0"] \arrow[ur, "p_1"'] \arrow[u, dashed, "t"] & \end{tikzcd}$$

$s \cdot t \cdot p_i$
$= s \cdot q_i$ [P is product]
$= p_i$ [Q is product]
so $s \cdot t = id(P)$
$t \cdot s = id(Q)$
So s is an iso, and the only one respecting projections.

$$\begin{tikzcd} & A \arrow[dl, "id"'] \arrow[dr, "id"] \arrow[d, dashed, "\langle id, id \rangle"] & \\ A & A \times A \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & A \end{tikzcd} \qquad \Delta A = \langle id, id \rangle$$

So write $A \times B$ for product, and π_0, π_1 for projections

Ex. Why $i \cdot \langle f, g \rangle = \langle i \cdot f, i \cdot g \rangle$?

$$\begin{tikzcd} & Y \arrow[ddl, bend right, "i \cdot f"'] \arrow[ddr, bend left, "i \cdot g"] \arrow[d, "i"] & \\ & X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, "\langle f, g \rangle"] & \\ A & A \times B \arrow[l] \arrow[r] & B \end{tikzcd}$$

Defn

Product of arrows

for $f : X \to A$ and $g : Y \to B$

$f \times g := \langle \pi_0 \cdot f, \pi_1 \cdot g \rangle$

$$\begin{tikzcd} X \arrow[d, "f"'] & X \times Y \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] \arrow[d, dashed, "f \times g"] & Y \arrow[d, "g"] \\ A & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B \end{tikzcd}$$

Lemma / Product Functor

If $\mathbb{C}$ has product for objects, there is a functor

$$\underbrace{- \times -}_{\text{name of alleged functor}} : \underbrace{\mathbb{C} \times \mathbb{C}}_{\text{product of categories}} \to \mathbb{C} \quad : \text{CAT}$$

$$\begin{tikzcd} & \Gamma \arrow[dl, bend right, "\mathcal{D}_0"'] \arrow[d, dashed] \arrow[dr, bend left, "\mathcal{D}_1"] & \\ A & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B \end{tikzcd}$$

Interpreting Conjunction

- $[\![A \wedge B]\!] := [\![A]\!] \times [\![B]\!]$
- intro rule:

$$\frac{A \quad B}{A \wedge B} \wedge I \iff \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge I \iff \Gamma \; \frac{\overset{\overline{\Gamma}}{\mathcal{D}_0} \atop A \quad \overset{\overline{\Gamma}}{\mathcal{D}_1} \atop B}{A \wedge B} \wedge I$$

ie ↗

$$[\![\wedge I]\!] := \langle [\![\overset{\mathcal{D}_0}{-}]\!], [\![\overset{\mathcal{D}_1}{-}]\!] \rangle$$

- elim rules:

$$\frac{A \wedge B}{B} \wedge E_1 \qquad \frac{A \wedge B}{A} \wedge E_0 \qquad [\![\wedge E_i]\!] := \pi_i$$

Unbiased Products

$$\begin{tikzcd} & X \arrow[dl, bend right, "x_0"'] \arrow[d, dashed, "\langle x_1 \ldots x_n \rangle"] \arrow[dr, bend left, "x_n"] & \\ A_0 & P \arrow[l, "\pi_0"] \arrow[r, "\pi_n"'] & A_n \end{tikzcd}$$

- nullary product
  iso to terminal object.

$$\begin{tikzcd} X \arrow[d, dashed, "\langle \rangle"] \\ P \end{tikzcd} \quad \langle \rangle \cong 1$$

- unary product

$$\begin{tikzcd} & X \arrow[dl] \arrow[d, dashed, "\langle x \rangle"] \\ A & P \arrow[l, "\pi"] \end{tikzcd}$$

let $P := A$, $\pi := id(A)$, $\langle x \rangle := x$

- ternary: $A \times B \times C \cong (A \times B) \times C \cong A \times (B \times C)$

contraction: $\dfrac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}$ $\quad [\![\Gamma]\!] \times [\![A]\!] \xrightarrow{id \times \Delta} [\![\Gamma]\!] \times [\![A]\!] \times [\![A]\!] \to [\![B]\!]$

exchange: $\dfrac{\Gamma, B, A \vdash C}{\Gamma, A, B \vdash C}$ $\quad [\![\Gamma]\!] \times [\![A]\!] \times [\![B]\!] \xrightarrow{id \times \sigma} [\![\Gamma]\!] \times [\![B]\!] \times [\![A]\!] \to [\![C]\!]$

Unbiased Contexts:

if $\Gamma := A_0, \ldots A_n$ then

$[\![\Gamma]\!] := [\![A_0]\!] \times \ldots \times [\![A_n]\!]$

Structural Contexts

$[\![\Gamma \vdash A]\!] : [\![\Gamma]\!] \to [\![A]\!]$ $\quad \xrightarrow{id \times !} [\![\Gamma]\!] \times 1 \cong$

weakening: $\dfrac{\Gamma \vdash B}{\Gamma, A \vdash B}$ $\quad [\![\Gamma]\!] \times [\![A]\!] \xrightarrow{\pi_0} [\![\Gamma]\!] \to [\![B]\!]$

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lemma: Products are associative (up to canonical isomorphism)

lemma: Products are Unital (up to canonical isomorphism)
ie. $A \times 1 \cong A \cong 1 \times A$

lemma: Products are symmetric (up to canonical isomorphism)
ie. $A \times B \cong B \times A$ ← called σ

Interpreting Propositional Contexts:

- Empty context "$\cdot$" "$\emptyset$" $\quad [\![\emptyset]\!] := 1$
- Context extension $[\![\Gamma, A]\!] := [\![\Gamma]\!] \times [\![A]\!]$

Ex. $\overline{\Gamma, A \vdash A}$ init $\quad$ interpret } do like above rules

$$\frac{\Gamma \vdash A \quad \Gamma, A \vdash B}{\Gamma \vdash B} \text{cut}$$