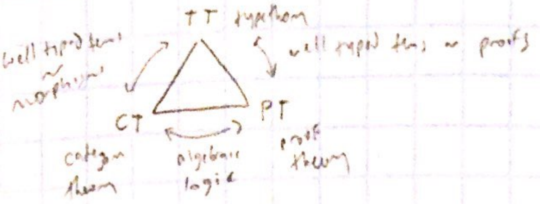
OVERVIEW HARPER 6/15/2015

Computational Trinitarianism



Things to keep an eye on:

1. ABSTRACTION - "behavioral characterization". ex. $\mathbb{N}$ is characterized by universal property of being the least set containing zero and closed under successions.

2. STRICTURES, NOT AFFORDANCES - "less is more". ie, conservative extensions. when you add more things, you invalidate all results for existing language.

"Expressive power stems from restriction of language"

⟶ Constructivity - Axiomatic freedom. You can introduce axioms that allow restricted version of feature.

3. PROOF RELEVANCE. Proofs are mathematical objects,

4. IDENTITY. "no entity without identity". What is equality?

Truth vs. Proof

1. Truth tables - denotational view of mathematics
2. Euclid - operational view of mathematics

Boolean Algebra

A complemented distributed lattice

1 pre-order. $\dfrac{}{x \le x}$ $\dfrac{x \le y \quad y \le z}{x \le z}$, $x \equiv y$ iff $x \le y$ and $y \le x$

2. Finite meets and joins. (ie, lattice)

Meets $\begin{cases} \text{greatest elt: } \dfrac{}{x \le 1} \\ \dfrac{z \le x,\ z \le y}{z \le x \wedge y} \quad \dfrac{}{x \wedge y \le x} \quad \dfrac{}{x \wedge y \le y} \end{cases}$

joins $\begin{cases} \text{least elt: } \dfrac{}{0 \le x} \\ \dfrac{}{x \le x \vee y} \quad \dfrac{}{y \le x \vee y} \quad \dfrac{x \le z \quad y \le z}{x \vee y \le z} \end{cases}$

3. Complement. $\overline{x} \vee x \ge 1$
$0 \ge \overline{x} \wedge x$

$\boxed{4.\ \text{Exponential.}\quad y^x \triangleq \overline{x} \vee y}$ not part of defn, just obser.

5. Distributive

$x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z)$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Heyting Algebra

A lattice with exponentials

1. pre-order
2. finite meets and joins
3. Exponentials.

$$y^x \wedge x \leq y$$

$$\frac{z \wedge x \leq y}{z \leq y^x}$$

BA = HA w/ complement

closed world — open world

* Ex. every HA is distributive

very useful
Yoneda lemma (prove!)
$x \leq y$ iff $\forall z, z \leq x$ implies $z \leq y$

* Ex. 1. $x \leq x \vee y$ (weakening)
$x \wedge y \leq x$
2. $x \leq x \wedge x$ (contraction)
3. $x \wedge y \leq y \wedge x$ (exchange).

In HA, $\sim x \triangleq 0^x$. so $\sim x \wedge x \leq 0$, but not true that $1 \leq \sim x \vee x$

What does this have to do with logic?

A prop, A true.

A_1 true, ..., A_n true $\vdash$ A true

$A_1 \wedge \ldots \wedge A_n \leq A$

What about proofs?

- Add evidence for an ordering

$x \leq y \rightsquigarrow f : x \leq y$, aka $f : x \to y$

$$\frac{}{id : x \to x} \qquad \frac{g : x \to y \quad f : y \to z}{f \circ g : x \to z}$$

meets

$$\begin{tikzcd} & z \arrow[d, dashed, "\langle f, g \rangle"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & x \times y \arrow[dl, "fst"'] \arrow[dr, "snd"] & \\ x & & y \end{tikzcd}$$

Identity

1. when is $f = g : x \to y$
2. $\Gamma \vdash M = N : A$

Lindenbaum Algebra

$[B] = \{ \beta \mid B \equiv A \}$

$[A] \wedge [B] = [A \wedge B]$

$[A] \leq [B]$ iff $A \vdash B$

Theorem

$\Gamma \vdash A$ iff $\forall$ HA H $[\Gamma]_H \leq [A]_H$.

$\rightarrow$ soundness

$\leftarrow$ completeness

What are introduction and elimination rules for proof derivation?

$$\begin{array}{c} \mathcal{D} \\ A \\ \vdots \end{array} \quad \vdash \quad \begin{array}{c} \mathcal{D} \\ A \end{array}$$

Logical frameworks is study of this.