HARPER 6/16/2015

Eg. Every HA is distributive

$x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$

1. eq-ive to $y \vee z \leq ((x \wedge y) \vee (x \wedge z))^x$

$y \leq ((x \wedge y) \vee (x \wedge z))^x \qquad z \leq ((x \wedge y) \vee (x \wedge z))^x$

$x \wedge y \leq ((x \wedge y) \vee (x \wedge z)) \qquad z \wedge x \leq (x \wedge y) \vee (x \wedge z)$

2. Suppose $(x \wedge y) \vee (x \wedge z) \leq u$. want to show $x \wedge (y \vee z) \leq u$

sufficient to show $y \vee z \leq u^x$

So $y \leq u^x$ and $z \leq u^x$

$x \wedge y \leq u \qquad x \wedge z \leq u$

$x \wedge y \leq (x \wedge y) \vee (x \wedge z) \leq u$, Same for z.

3. Use the completeness theorem for intuitionistic propositional logic.

$A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$

or

$x : A \times (B + C) \vdash M : (A \times B) + (A \times C)$

4. Use categories.

Given: Intuitionistic Propositional Logic / Simple Type Theory

$$\frac{}{\Gamma \vdash 1 \text{ type}}\ 1\text{-F}$$

$$\frac{}{\Gamma \vdash \langle\rangle : 1}\ 1\text{-I}$$

(no 1-E)

β: none

η: $\Gamma \vdash \langle\rangle \equiv M : 1$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{A \times B \text{ type}}\ \times F$$

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N \rangle : A \times B}\ \times I$$

$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash fst(M) : A \qquad snd(M) : B}\ \times E$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \to B \text{ type}}\ \to F$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}\ \to I$$

$$\frac{\Gamma \vdash M : A \to B \quad , \ N : A}{\Gamma \vdash M(N) : B}\ \to E$$

β: $\Gamma \vdash (\lambda x.M)N \equiv [N/x]M : B$

η: $\Gamma \vdash \lambda x.M(x) \equiv M : A \to B$

$$\frac{}{\Gamma \vdash 0 \text{ type}}\ 0F$$

β: none

no 0I

η: $\Gamma, z:0 \vdash R \equiv abort\ z$

$$\frac{\Gamma \vdash M : 0}{\Gamma \vdash abort\ M : A}\ 0E$$

β: $\Gamma \vdash fst(\langle M, N \rangle) \equiv M : A$

$\quad snd(\langle M, N \rangle) \equiv N : A$

η: $\Gamma \vdash \langle fst\ M, snd\ M \rangle \equiv M : A \times B$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A + B \text{ type}}\ +F \qquad \frac{\Gamma \vdash M : A}{\Gamma \vdash inl\ M : A + B}\ +I_1 \qquad \frac{\Gamma \vdash N : B}{\Gamma \vdash inr\ N : A + B}\ +I_2$$

$$\frac{\Gamma \vdash M : A + B \quad \Gamma, x : A \vdash N : C \quad \Gamma, y : B \vdash P : C}{\Gamma \vdash case\ M(x.N, y.P) : C}\ +E$$

β: $case(x.M, y.N)(inl\ P) = [P/x]M$

$\quad case(x.M, y.N)(inr\ Q) = [Q/y]N$

$$\eta: \frac{\Gamma \vdash R(inl\ P) \equiv [P/x]M \qquad \Gamma \vdash R(inr\ Q) \equiv [Q/y]N}{\Gamma, z : A + B \vdash R_z = case(x.M, y.N)}$$

"Data" Types

eg) Nat

$$\frac{}{\Gamma \vdash zero : Nat} \; Nat\text{-}I_1$$

$$\frac{}{\Gamma, x{:}Nat \vdash succ(x) : Nat} \; Nat\text{-}I_2$$

$$\frac{\Gamma \vdash M : C \qquad \Gamma, x{:}C \vdash N : C}{\Gamma, z{:}Nat \vdash iter_{Nat}(M, x.N)(z) : C} \; Nat\text{-}E$$

(β) rules

where $iter(M, x.N)(zero) \equiv M$

$iter(M, x.N)(succ\,x) = [iter(M,x.N)(x)/x]N$

extensional characterization $\left\{ \begin{array}{l} \underline{\text{Fact}}: \text{ for all } m \in \mathbb{N} \\ p2(\bar{m}) = succ(succ(\bar{m})) \end{array} \right.$

where $p_2 = iter(succ(succ(zero)), x.succ(x))$.

$$(\eta) \; \frac{\text{for all } m \in \mathbb{N} \; [m/z] R \equiv iter(\_)(m) : C}{\Gamma, z{:}Nat \vdash R \equiv iter(M, x.N)(z) : C}$$

not legal
This is a problem!

Analytic vs. Synthetic Judgement

$\Gamma \vdash M \equiv N : A$ — ANALYTIC
self evident
requires no evidence

SYNTHETIC
requires evidence

Families of Types

sets $\{X_i\}_{i \in I}$ "$X : I \to Set$"

Motivation: Props-as-Types

Type ~ Prop
Family ~ Predicate
"typical function" ~ "propositional function"

First Order Logic

$x : \tau \vdash \phi \text{ prop}$

eg. $x, y : \mathbb{Z} \vdash x = y \text{ prop}$