6/16/2015 MOREHOUSE

BEHAVIORAL REASONING

Functor:

for cat $\mathbb{C}$ and $\mathbb{D}$, object map $F_0 : \mathbb{C}_0 \to \mathbb{D}_0$

arrow map: $F_1 : \mathbb{C}_1 \to \mathbb{D}_1$

such that:

- $f : \mathbb{C}(A \to B) \longmapsto F_1(f) : \mathbb{D}(F_0(A) \to F_0(B))$

$A \xrightarrow{id} A \longmapsto F_0(a) \xrightarrow{id} F_0(a)$

$A \xrightarrow{f} B \xrightarrow{g} C \longmapsto F_0(A) \xrightarrow{F_1(f)} F_0(A) \xrightarrow{F_1(g)} F_0(C)$

Composition structure of functors

- identity functor

$id(\mathbb{C})$

$$\begin{tikzcd} \mathbb{C}: A \arrow[d, "f"'] & & A \arrow[d, "f"] \\ B \arrow[rr, phantom, "\xrightarrow{id}"] & & B \end{tikzcd}$$

- Functor compositions

$F \cdot G$

$$\begin{tikzcd} \mathbb{C} \quad A \arrow[d, "f"'] & \mathbb{D} \quad C \arrow[d, "g"'] & \mathbb{E} \quad E \arrow[d, "h"'] \\ B \arrow[r, phantom, "\longmapsto"] & D \arrow[r, phantom, "\longmapsto"] & F \end{tikzcd}$$

Size Matters

- A small collection := a set
- $\mathbb{C}$ a small collection if $\mathbb{C}_1$ is small.
- Category of small categories "CAT"
- CAT does not contain CAT as object.
  b/c discrete categories are sets, and all sets are too large to be a set.
- $\mathbb{C}$ is locally small if all homs are small. ie, $\forall A, B$, $\mathbb{C}(A \to B)$ is a set.

$\boxed{\text{*Unless specified, all categories are locally small*}}$

If we fix $X : \mathbb{C}$, define function $A : \mathbb{C} \longmapsto \mathbb{C}(X \to A)$

This extends to a functor:

$$\frac{\mathbb{C}(X \to \_)}{\mathbb{C} \longmapsto \text{set}}$$

$A \mapsto \mathbb{C}(X \to A)$

$f : A \to B \longmapsto \mathbb{C}(X \to f) := \_ \cdot f : \mathbb{C}(X \to A) \to \mathbb{C}(X \to B)$

Respect for composition structure:

check: $\mathbb{C}(X \to id(A)) = id(\mathbb{C}(X \to A))$

$\mathbb{C}(X \to f \cdot g) = \mathbb{C}(X \to f) \cdot \mathbb{C}(X \to g)$

In SET,

A "product" $A \times B := \{ (a, b) \mid a \in A, b \in B \}$

In small categories, the "product category" $\mathbb{C} \times \mathbb{D}$ has:

- objects $(\mathbb{C} \times \mathbb{D})_0 = \mathbb{C}_0 \times \mathbb{D}_0$
- arrows $(\mathbb{C} \times \mathbb{D})_1 = \mathbb{C}_1 \times \mathbb{D}_1$
  with $\delta^i((f, p)) = (\delta^i(f), \delta^i(p))$
- identity: $id(A, X) := (id(A), id(X))$
- composition: $(f, p) \cdot (g, q) = (f \cdot g, p \cdot q)$

For category $\mathbb{C}$, form the <u>opposite category</u> $\mathbb{C}^{o}$
- objects: $\mathbb{C}^{o}_{o} := \mathbb{C}_{o}$
- arrows: $\mathbb{C}^{o}(A \to B) := \mathbb{C}(B \to A)$
- identity: $id(A) :: \mathbb{C}^{o} = id(A) :: \mathbb{C}$
- composition: $f \cdot g : \mathbb{C}^{o} := g \cdot f :: \mathbb{C}$

Dual constructions
Dual Theorems

Essential Sameness
in SET: Bijection, ie $X \cong Y$ if $\exists p : X \to Y$ that is injective & surjective

in Category: p is injective if $\forall f, g : W \to X$, $\forall w \in W.$ $(p \circ f)(w) = (p \circ g)(w)$
implies $f(w) = g(w)$

by function extensionality if:
$\forall f, g : W \to X.\ f \cdot p = g \cdot p \supset f = g$

Behavioral Injectivity: Monomorphism — pneumonic / Monic is an adjective
ie a post-cancelable arrow
ie an $m :: \mathbb{C}$ s.t. $\forall f, g : \mathbb{C}$ $f \cdot m = g \cdot m \supset f = g$

lemma:
i. identity arrows are monic
ii. compositions of monics are monic
iii. if $m \cdot n$ is monic then m is monic

Pf i. $f \cdot id = g \cdot id \Rightarrow$ [unit law of composition] $f = g$.
ii. $f \cdot m \cdot n = g \cdot m \cdot n \Rightarrow$ [n monic] $f \cdot m = g\,m \Rightarrow$ [m monic] $f = g$
iii. $f \cdot m = g \cdot m \Rightarrow$ [whiskering] $f \cdot m \cdot n = g \cdot m \cdot n \Rightarrow$ [assumption] $f = g$

Dual property of being monic: Epimorphism
ie a pre-cancelable arrow
ie. an $e :: \mathbb{C}$ s.t. $\forall f, g : \mathbb{C}$ $e \cdot f = e \cdot g \supset f = g$

- In SET monic = injective, epic = surjective
- Consider Mon, $(\mathbb{N}, +, 0) \hookrightarrow (\mathbb{Z}, +, 0)$
is monic epic but not surjective
so monic + epic is not sufficient for "essentially the same"

Def Split monic - a post-invertible arrow
ie. $s :: \mathbb{C}$ s.t. $\exists r :: \mathbb{C}$ $s \cdot r = id$
Split epic - a pre-invertible arrow
ie $s :: \mathbb{C}$ s.t. $\exists r :: \mathbb{C}$ $r \cdot s = id$
lemma. split monics are monics
let s be split monic, with $s \cdot r = id$
$f \cdot s = g \cdot s \Rightarrow$ [whisker] $f \cdot s \cdot r = g \cdot s \cdot r \Rightarrow$ [assumption] $f \cdot id = g \cdot id \ldots$
lemma. Functors preserve split monics (and split epics).

Axiom of Choice

Given a family of nonempty sets, there is a function choosing an element of each

$$\begin{tikzcd} E \arrow[d, "f"'] & P(E) \hookrightarrow Set \\ B & B \arrow[u, "f^*"'] \end{tikzcd}$$

- $\forall b \in B.\ f^*(b)$ nonempty $\iff$ f is a surjection
- Axiom of choice $\iff$ every epi is split

If $s \cdot r = id$ then s is a <u>section</u> for r
and r is a <u>retraction</u> for s

- So being a split monic means having a retraction
  being a split epic means having a section

lemma:

If an arrow f has a section s and a retraction r then $s = r$

pf

$$\begin{tikzcd} B \arrow[r, "s"] \arrow[rrr, bend left=50, "s"] \arrow[rr, bend right, "id"'] \arrow[rrr, bend right=60, "r"'] & A \arrow[r, "f"] \arrow[rr, bend left, "id"] & B \arrow[r, "r"] & A \end{tikzcd}$$

$\square$

- If f has section and retraction g then g has retraction and section f.
- So $f \cdot g = id$ and $g \cdot f = id$

def Arrow $f : A \to B$ is iso if there is arrow $g : B \to A$ called an inverse of f
so that $f \cdot g = id(A)$ and $g \cdot f = id(B)$

* $\boxed{\text{Ex. inverses are unique}}$

Notation: $A \cong B$ for there exists iso between A and B

Safe Temporal

$$\overline{A @ t+1}$$

$$\frac{B @ t+1 \quad \bigcirc A @ t}{\bigcirc B @ t}\ \delta I \qquad \frac{\bigcirc A @ t}{A @ t+1}$$