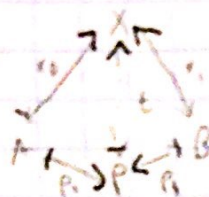


Defn.

A ^{co}product of objects A and B is a ^{co}span on A and B $(A \xrightarrow{p_0} P \xleftarrow{p_1} B)$
 so that for any ^{co}span on A and B $(A \xrightarrow{r_0} X \xleftarrow{r_1} B)$
 $\exists ! t: X \rightarrow P$ $(\circ p_i \circ r_i = \text{id})$



A and B are the factors cases
 p_0 and p_1 are the ^{insertions} ("i.i.")
 t is the ^{couple} of r_0, r_1 $([r_0, r_1])$
 P is written " $A+B$ "

Lemmas

- Identity expansion
- Uniqueness: Coproducts are unique up to an insertion preserving iso
- Coproduct functor:
 $- + -: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

Interpreting Disjunction

$[A \vee B] := [A] + [B]$

- intro rules:

$$\frac{A}{A \vee B} \vee I_1 \quad \frac{B}{A \vee B} \vee I_2 \quad \text{as } [x_0] := t_0, [x_1] := t_1$$

- elim rule: $\frac{\Gamma \vdash A \quad \Gamma \vdash B \quad \frac{C}{C} \vee E}{\Gamma \vdash C} \vee E$

$[ve] := [t_0, t_1]$
 not sufficient
 general

obs: $X \times 0 \cong 0$

An exponential of objects A and B is an object E and arrow $e: E \times A \rightarrow B$ so that for any object X and arrow $f: X \times A \rightarrow B$, $\exists ! \lambda(f): X \rightarrow E$.

$$X \times A \xrightarrow{f} B \quad \lambda(f) \times A \cdot e = f$$

E is evaluation map
 $\lambda(f)$ is the "curry" of f
 $E \xrightarrow{\lambda(e)} E$

$$E \times A \xrightarrow{e} B \quad \text{so } \lambda(e) = \text{id}(E) \quad [\text{identity expansion}]$$

Lemma. (Uniqueness)

Exponentials are unique up to an evaluation preserving iso, so write " $A \Rightarrow B$ " or " B^A "

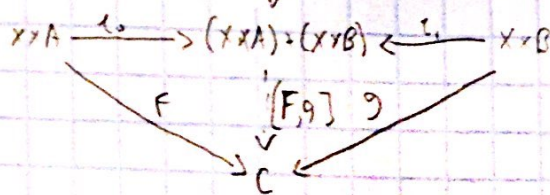
$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}$$

dist: $X \vee (A+B) \cong (X \vee A) + (X \vee B)$

$X \vee (A+B)$

dist \vdash



* Ex. why $i \cdot \lambda(f) = \lambda(i \times A \cdot f)$?

$$Y \xrightarrow{i} X \xrightarrow{\lambda(f)} A \triangleright B$$

$$Y \times A \xrightarrow{i \times A} X \times A \xrightarrow{f} B$$

$$(i \cdot \lambda(f)) \times A \xrightarrow{\quad} (A \triangleright B) \times A$$

So prove triangles ① and ② commute.

① is definition.

Lemma

Exponential functor

$$A \triangleright -$$

$$\mathbb{C} \rightarrow \mathbb{C}$$

$$B \mapsto A \triangleright B$$

$$g: B \rightarrow C \mapsto A \triangleright g := \lambda(\varepsilon \cdot g)$$

$$\begin{array}{c} A \triangleright B \xrightarrow{\quad} A \triangleright C \\ (A \triangleright B) \times A \xrightarrow{\varepsilon} B \xrightarrow{g} C \end{array}$$

→ type checks.

Interpreting Implication

$$\llbracket A \triangleright B \rrbracket := \llbracket A \rrbracket \triangleright \llbracket B \rrbracket$$

- intro rule:

$$\frac{\frac{\frac{\Gamma \vdash A}{\vdash B} \triangleright I}{\Gamma \vdash A \triangleright B} \text{ or } \frac{\Gamma \vdash A \vdash B}{\Gamma \vdash A \triangleright B} \triangleright I \text{ or } \frac{\frac{\Gamma \vdash A}{\vdash B} \triangleright I}{\Gamma \vdash A \triangleright B} \triangleright I$$

$$\llbracket \triangleright I \rrbracket := \lambda(\llbracket - \rrbracket)$$

- elim rule

$$\frac{A \triangleright B \quad A \triangleright E}{B} \triangleright E \quad \llbracket \triangleright E \rrbracket := \varepsilon$$

Categories with:

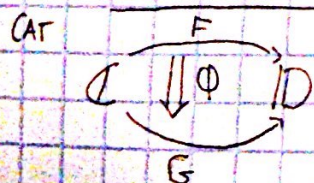
- (\mathbb{I}, \times) are cartesian (finite products and terminal object)
- $(\mathbb{I}, \times, \triangleright)$ are cartesian closed
- $(\mathbb{O}, +)$ are co-cartesian
- $(\mathbb{O}, !, +, \times)$ are bicartesian
- $(\mathbb{O}, !, +, \times, \triangleright)$ are bicartesian closed

for parallel functors $F, G: \mathbb{C} \rightarrow \mathbb{D}$, a natural transformation ϕ from F to G :

- for $A: \mathbb{C}$, an arrow $\phi(A): \mathbb{D}(F(A) \rightarrow G(A))$ (the component)

- for $F: \mathbb{C}(A \rightarrow B)$ a naturality square:

$$\begin{array}{ccc} \mathbb{C} & & \mathbb{D} \\ A \xrightarrow{F} B & & F(A) \xrightarrow{F(f)} F(B) \\ \downarrow \phi(A) & & \downarrow \phi(B) \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$



in notes, this behavioral characterization of natural transformation is explored.

ADJUNCTIONS - Almost Inverses

anti-parallel functors F and G form an adjunction (" $F \dashv G$ ") if for any $A: \mathcal{C}, B: \mathcal{D}$ there is a natural bijection of hom-sets:

- bijection: $\mathcal{C}(A \rightarrow G(B)) \xrightarrow{\sim} \mathcal{D}(F(A) \rightarrow B)$

- naturality: $\mathcal{C}: A' \xrightarrow{a} A \xrightarrow{f=g^b} G(B) \xrightarrow{G(g)} G(C) \xrightarrow{G(g)} G(C)$
 $\mathcal{D}: F(A') \xrightarrow{F(a)} F(A) \xrightarrow{f^a} B \xrightarrow{b} B'$

$F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ form $F \dashv G$ if:

Universal Property of unit:

$\exists \eta: \text{id}(\mathcal{C}) \rightarrow F \cdot G$

$\forall f: \mathcal{C}(A \rightarrow G(B))$

$\exists! g: \mathcal{D}(F(A) \rightarrow B)$

$\eta(A) \cdot G(g) = f$

$\mathcal{C}: A \xrightarrow{f} G(B)$
 $\mathcal{D}: F(A) \xrightarrow{g} B$
 $(G \circ F)(A) \xrightarrow{G(f)} G(B)$
 $\eta(A) \uparrow$

Universal Property of counit

$\exists \epsilon: G \cdot F \rightarrow \text{id}(\mathcal{D})$

$\forall g: \mathcal{D}(F(A) \rightarrow B)$

$\exists! f: \mathcal{C}(A \rightarrow G(B))$

$F(f) \cdot \epsilon(B) = g$

$\mathcal{C}: A \xrightarrow{f} G(B)$
 $\mathcal{D}: F(A) \xrightarrow{g} B$
 $F(f) \downarrow$
 $\epsilon(B) \uparrow$
 $(F \circ G)(B)$

* Ex. Explain why: Hint: probe $\eta(A)$ and $\epsilon(B)$ with themselves

$\mathcal{C} \xrightarrow{\lambda A} \mathcal{C}$
 $\mathcal{C} \xrightarrow{\epsilon} \mathcal{C}$

Product functor and Implication functor are adjoint.

$\mathcal{C}: X \xrightarrow{\lambda(F)} A \triangleright B$
 $\mathcal{C}: X \times A \xrightarrow{f} B$

$\mathcal{C}: X \times A \xrightarrow{f} B$
 $\mathcal{C}: X \times A \xrightarrow{f} B$

$\mathcal{C}: X \times A \xrightarrow{f} B$
 $\mathcal{C}: X \times A \xrightarrow{f} B$

$\Gamma \vdash A$
 $\Gamma \vdash B$
 $\Gamma \vdash A \triangleright B$
 $\Gamma \vdash A$
 $\Gamma \vdash B$

justified by diagram commutativity

long way around.

short path.

Unintended: $\text{id}(A \triangleright B) = \lambda(\epsilon)$

$A \triangleright B = \frac{\frac{\epsilon}{A \triangleright B} \quad \frac{A \triangleright B \quad A}{B} \triangleright E}{A \triangleright B} \triangleright I$

Elaborate: $\lambda \cdot \lambda F = \lambda(\lambda \cdot A \cdot F)$
 $\Gamma \vdash A$
 $\Gamma \vdash B$
 $\Gamma \vdash A \triangleright B$
 $\Gamma \vdash B$
 $\Gamma \vdash A \triangleright B$

For negative connectives: λ :

- Interpreted as right adjoint functors

$\llbracket \lambda I \rrbracket = _b$ $\llbracket \lambda \Rightarrow_r \rrbracket := \beta_R$

$\llbracket \lambda E \rrbracket := \epsilon$

$\llbracket \lambda \Rightarrow_e \rrbracket := \text{identity expansion}$

For positive connectives: λ

"left adjoint" functors

end page 68.

read diagram