

Functorfor cat \mathcal{C} and \mathcal{D} , object map $F_0: \mathcal{C}_0 \rightarrow \mathcal{D}_0$ arrow map: $F_1: \mathcal{C}_1 \rightarrow \mathcal{D}_1$

such that:

$$F: \mathcal{C}(A \rightarrow B) \longrightarrow \mathcal{D}(F_0(A) \rightarrow F_0(B))$$

$$A \xrightarrow{id} A \mapsto F_0(A) \xrightarrow{id} F_0(A)$$

$$A \xrightarrow{f} B \xrightarrow{g} C \mapsto F_0(A) \xrightarrow{F_1(f)} F_0(B) \xrightarrow{F_1(g)} F_0(C)$$

Composition structure of functors

- identity functor

$$id(\mathcal{C})$$

$$\begin{array}{ccc} \mathcal{C}: A & \xrightarrow{id} & A \\ f \downarrow & & \downarrow f \\ B & & B \end{array}$$

- functors composition

$$F \circ G$$

$$\begin{array}{ccccc} \mathcal{C} & A & & \mathcal{D} & C & & \mathcal{E} & E \\ f \downarrow & & \xrightarrow{\quad} & g \downarrow & & \xrightarrow{\quad} & h \downarrow & \\ B & & & D & & & F & \end{array}$$

Size Matters

• A small collection := a Set

• \mathcal{C} a small collection if \mathcal{C}_0 is small.

• Category of small categories "CAT"

• CAT does not contain CAT as object.

b/c discrete categories are sets, and all sets are too large to be a set.

• \mathcal{C} is locally small if all homs are small. i.e., $\forall A, B, \mathcal{C}(A \rightarrow B)$ is a set.(unless specified, all categories are locally small)If we fix $x: \mathcal{C}$, define function $A: \mathcal{C} \mapsto \mathcal{C}(x \rightarrow A)$

This extends to a functor:

$$\mathcal{C}(x \rightarrow -)$$

$$\mathcal{C} \mapsto \text{set}$$

$$A \mapsto \mathcal{C}(x \rightarrow A)$$

$$f: A \rightarrow B \mapsto \mathcal{C}(x \rightarrow f) := - \circ f: \mathcal{C}(x \rightarrow A) \rightarrow \mathcal{C}(x \rightarrow B)$$

Respect for composition structure:

$$\text{check: } \mathcal{C}(x \rightarrow id(A)) = id(\mathcal{C}(x \rightarrow A))$$

$$\mathcal{C}(x \rightarrow f \circ g) = \mathcal{C}(x \rightarrow f) \cdot \mathcal{C}(x \rightarrow g)$$

In SET,

$$A \text{ "product" } A \times B := \{ (a, b) \mid a \in A, b \in B \}$$

In small categories, the "product category" $\mathcal{C} \times \mathcal{D}$ has:- objects $(\mathcal{C} \times \mathcal{D})_0 = \mathcal{C}_0 \times \mathcal{D}_0$ - arrows $(\mathcal{C} \times \mathcal{D})_1 = \mathcal{C}_1 \times \mathcal{D}_1$

$$\text{with } \delta^i((f, p)) = (\delta^i(f), \delta^i(p))$$

- identity: $id(A, x) := (id(A), id(x))$ - composition: $(f, p) \cdot (g, q) = (f \circ g, p \circ q)$

For category \mathcal{C} , form the opposite category \mathcal{C}^o

- objects: $\mathcal{C}^o := \mathcal{C}$

- arrows: $\mathcal{C}^o(A \rightarrow B) := \mathcal{C}(B \rightarrow A)$

- identity: $\text{id}(A) :: \mathcal{C}^o = \text{id}(A) :: \mathcal{C}$

- composition: $f \cdot g :: \mathcal{C}^o := g \cdot f :: \mathcal{C}$

Dual constructions
Dual Theorems

Essential Sameness

in SET: Bijection, i.e. $X \cong Y$ if $\exists f: X \rightarrow Y$ that is injective & surjective

in Category: p is injective if $\forall f, g: W \rightarrow X, \forall w \in W. (p \circ f)(w) = (p \circ g)(w)$
implies $f(w) = g(w)$

by function extensionality if:

$$\forall f, g: W \rightarrow X. f \circ p = g \circ p \Rightarrow f = g$$

Behavioral Injectivity: Monomorphism — monic / Monic is an adjective

ie a pre^{in} -cancelable arrow

$$\text{ie on } m :: \mathcal{C} \text{ s.t. } \forall f, g: \mathcal{C} \quad f \cdot m = g \cdot m \Rightarrow f = g$$

Lemma:

i. identity arrows are monic

ii. compositions of monics are monic

iii. if $m \cdot n$ is monic then m is monic

pf i. $f \cdot \text{id} = g \cdot \text{id} \Rightarrow [\text{unit law of composition}] f = g.$

ii. $f \cdot m \cdot n = g \cdot m \cdot n \Rightarrow [n \text{ monic}] f \cdot m = g \cdot m \Rightarrow [m \text{ monic}] f = g$

iii. $f \cdot m = g \cdot m \Rightarrow [\text{whisker}] f \cdot m \cdot n = g \cdot m \cdot n \Rightarrow [\text{assumption}] f = g$

Dual property of being monic: Epimorphism

ie a pre^{out} -cancelable arrow

$$\text{ie. on } e :: \mathcal{C} \text{ s.t. } \forall f, g: \mathcal{C} \quad e \cdot f = e \cdot g \Rightarrow f = g$$

- In SET monic = injective, epic = surjective

- Consider Mon, $(\mathbb{N}, +, 0) \hookrightarrow (\mathbb{Z}, +, 0)$

is pre^{in} but not surjective

So monic + epic is not sufficient for "essentially the same"

Def Split monic - a pre^{in} -invertible arrow

$$\text{ie. } s :: \mathcal{C} \text{ s.t. } \exists r :: \mathcal{C} \quad s \cdot r = \text{id}$$

Split epic - a pre^{out} -invertible arrow

$$\text{ie. } s :: \mathcal{C} \text{ s.t. } \exists r :: \mathcal{C} \quad r \cdot s = \text{id}$$

Lemma. Split monics are monic,

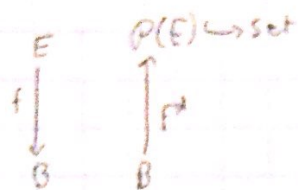
let s be split monic, with $s \cdot r = \text{id}$

$$f \cdot s = g \cdot s \Rightarrow [\text{whisker}] f \cdot g \cdot r = g \cdot s \cdot r \Rightarrow [\text{assumption}] f \cdot \text{id} = g \cdot \text{id} \dots$$

Lemma. Functors preserve split monics (and split epics).

Axiom of Choice

Given a family of nonempty sets, there is a function choosing an element of each



• $\forall b \in B, f^*(b) \text{ nonempty} \iff f \text{ is a surjection}$

• Axiom of choice \iff every epi is split

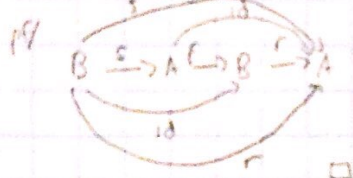
If $S \cdot r = \text{id}$ then S is a section for r
and r is a retraction for S

- So being a split mono means having a retraction

being a split epi means having a section

Lemma:

If an arrow f has a section s and a retraction r then $s = r$



- If f has section and retraction g then g has retraction and section f .

- So $f \cdot g = \text{id}$ and $g \cdot f = \text{id}$

def Arrow $f: A \rightarrow B$ is iso if there is arrow $g: B \rightarrow A$ called an inverse of f
so that $f \cdot g = \text{id}(A)$ and $g \cdot f = \text{id}(B)$

* Ex. inverses are unique

Notation: $A \cong B$ for there exists iso between A and B

Safe Term!

$A @ t+1$

$$\frac{B @ t+1 \quad OA @ t}{OB @ t} \text{OI}$$

$$\frac{OA @ t}{A @ t+1}$$