

# Universal Constructions and Interpretation of Intuitionistic Propositional Logic

Defn

A terminal object is an object  $T$  such that for any object  $X$ :

$$\exists ! x: X \rightarrow T$$

$\circ(X)$

Socrates Dictum

Know Thyself  
("Pole thyself")

i.e., set  $X := T$ , so  $T \xrightarrow{\circ(T)} T$   
 $\text{Id}$

Lemma:

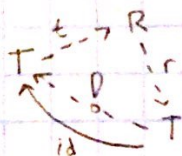
Identity expansion:  $\circ(T) = \text{id}(T)$

Lemma Uniqueness

Terminals are unique up to a unique iso.

PF

If  $T$  and  $R$  are terminals.

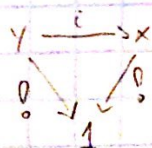


Since unique,  $t \cdot r = \circ = \text{id}(T)$   
 and  $r \cdot t = \text{id}(R)$  }  $t$  is iso.

and the only, since  $R$  been terminal means only one morphism from  $T$ , thus only one iso.

So we write "1" for a terminal.

Ex. pre-composing with a gang  
 why  $i \cdot \circ(X) = \circ(Y)$ ?



Terminals in

- SET are singletons
- CAT are singletons
- PREORD are ? \*
- a pre-order category are ? \*

## Interpret Truth

- Truth  $\llbracket T \rrbracket := 1$

- intro rule:

$$\frac{\Gamma \vdash A}{\Gamma \vdash T} \text{Tr} \Leftrightarrow \frac{\Gamma \vdash T}{\Gamma \vdash T} \text{Tr} \Leftrightarrow \frac{\Gamma}{T} \text{Tr}$$

$$\llbracket \text{Tr} \rrbracket := \circ(\llbracket \Gamma \rrbracket): \llbracket \Gamma \rrbracket \rightarrow \llbracket T \rrbracket$$



Defn/

An initial object is an object  $S$  such that for any object  $X$ :

$$\exists \underset{\circlearrowleft}{0} s: S \rightarrow X$$

$$S \xrightarrow{\circlearrowleft 0(x)} X$$

Refer to arbitrary initial object as  $0$ .

- Id expansion:  $\circlearrowleft(0) = id(0)$
- Uniqueness: Initial objects are unique up to unique iso.
- Composition with a cobary

$$X \xrightarrow{i} Y$$

$$\circlearrowleft(x) \cdot i = \circlearrowleft(y)$$

Initials are:

- Set are empty sets
- Cat are empty categories.

Interpreting Falshood

- Falshood  $\llbracket \perp \rrbracket = 0$
- elim rule:

$$\frac{\perp}{A} \vdash \quad \llbracket \perp \rrbracket = \circlearrowleft(\llbracket A \rrbracket) : \llbracket \perp \rrbracket \rightarrow \llbracket A \rrbracket$$

Defn

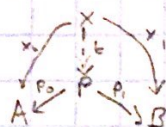
A product of objects  $A$  and  $B$  is a span on  $A$  and  $B$ ,

$$i.e.: A \xleftarrow{p_0} P \xrightarrow{p_1} B$$

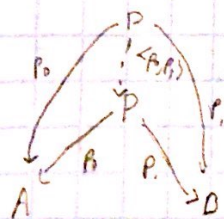
so that for any span on  $A$  and  $B$  ( $A \xleftarrow{x_0} X \xrightarrow{x_1} B$ )

$$\exists ! t: X \rightarrow P, t \cdot p_i = x_i \quad i.e.:$$

$$t = \langle x_0, x_1 \rangle$$



Vocabulary:  $A$  and  $B$  are Factor  
 $p_0$  and  $p_1$  are projections  
 $t$  is the tuple of  $x_0, x_1$



Lemma: Identity expansion for products is  $id(P) = \langle p_0, p_1 \rangle$

Lemma:

Uniqueness: Products are unique up to a projection-preserving iso.

$$P \xleftarrow{p_0} P \xrightarrow{p_1} B$$

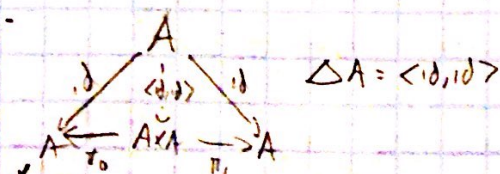
$$s \cdot p_i = p_i \quad [P \text{ is product}]$$

$$= p_i \quad [Q \text{ is product}]$$

$$\Rightarrow s \cdot t = id(P)$$

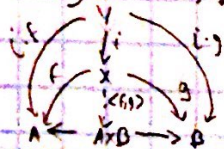
$$t \cdot s = id(Q)$$

So  $s$  is an iso, and this only one respecting projections.



Ex. why  $i: \langle F, G \rangle = \langle i \cdot F, i \cdot G \rangle$ ?

So write  $A \times B$  for product, and  $\pi_0, \pi_1$  for projections





## Defn

Product of arrows

for  $f: X \rightarrow A$  and  $g: Y \rightarrow B$

$$f \times g := \langle \pi_0 \circ f, \pi_1 \circ g \rangle$$

$$\begin{array}{ccccc} X & \xleftarrow{\pi_0} & X \times Y & \xrightarrow{\pi_1} & Y \\ f \downarrow & & \downarrow (f \times g) & & \downarrow g \\ A & \xleftarrow{\pi_0} & A \times B & \xrightarrow{\pi_1} & B \end{array}$$

Lemma / Product Functor

If  $\mathcal{C}$  has products for objects, there is a functor

$- \times - : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$   
name of product of categories  
alleged functor

$$\begin{array}{ccc} & \Gamma & \\ \downarrow \sigma_0 & \downarrow & \downarrow \sigma_1 \\ A & \xleftarrow{\pi_0} & A \times B & \xrightarrow{\pi_1} & B \end{array}$$

## Interpreting Conjunction

-  $\llbracket A \wedge B \rrbracket := \llbracket A \rrbracket \times \llbracket B \rrbracket$

- intro rule:

$$\frac{A \quad B \quad \Lambda E \Leftrightarrow \quad \Gamma \vdash A \quad \Gamma \vdash B \quad \Lambda I}{\Gamma \vdash A \wedge B} \quad \Lambda E \Leftrightarrow \quad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \quad \Lambda I$$

$$\llbracket \Lambda I \rrbracket := \langle \llbracket - \rrbracket^I, \llbracket - \rrbracket^I \rangle$$

- elim rules:

$$\frac{A \wedge B \quad \Lambda E_1}{B} \quad \frac{A \wedge B \quad \Lambda E_2}{A}$$

$$\llbracket \Lambda E_i \rrbracket := \pi_i$$

## Unbiased Products

- nullary product  
iso to terminal object.

$$\begin{array}{c} X \\ \downarrow \\ 1 \end{array}$$

- unary product

$$\begin{array}{c} X \\ \downarrow \pi \\ A \end{array}$$

let  $p := A$ ,  $\pi := \text{id}(A)$ ,  $\langle x \rangle := x$

contraction:  $\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \quad \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \times \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket$   
exchange:  $\frac{\Gamma, A \vdash B}{\Gamma, B, A \vdash C} \quad \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \times \llbracket B \rrbracket \xrightarrow{\text{id} \times \sigma} \llbracket \Gamma \rrbracket \times \llbracket B \rrbracket \times \llbracket A \rrbracket \rightarrow \llbracket C \rrbracket$

Unbiased Contexts:

if  $\Gamma := A_0, \dots, A_n$  then

$$\llbracket \Gamma \rrbracket := \llbracket A_0 \rrbracket \times \dots \times \llbracket A_n \rrbracket$$

Structural Contexts

$\llbracket \Gamma \vdash A \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \xrightarrow{\text{id} \times 1} \llbracket \Gamma \rrbracket \times 1$   
weakening:  $\frac{\Gamma \vdash B}{\Gamma, A \vdash B} \quad \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \xrightarrow{\pi_0} \llbracket \Gamma \rrbracket \rightarrow \llbracket B \rrbracket$

- ternary:  $A \times B \times C \cong (A \times B) \times C \cong A \times (B \times C)$

lemma: Products are associative (up to canonical isomorphism)

lemma: Products are unital (up to canonical isomorphism)

$$A \times 1 \cong A \cong 1 \times A$$

lemma: Products are symmetric (up to canonical isomorphism)

$$A \times B \cong B \times A \leftarrow \text{called } \sigma$$

## Interpreting Propositional Contexts:

- Empty context "0" " $\emptyset$ "  $\llbracket \emptyset \rrbracket := 1$

- Context extension  $\llbracket \Gamma, A \rrbracket := \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket$

\* Ex.  $\Gamma, A \vdash A$  interpret } do like above rule)  
 $\frac{\Gamma \vdash A \quad \Gamma, A \vdash B}{\Gamma \vdash B}$  cut