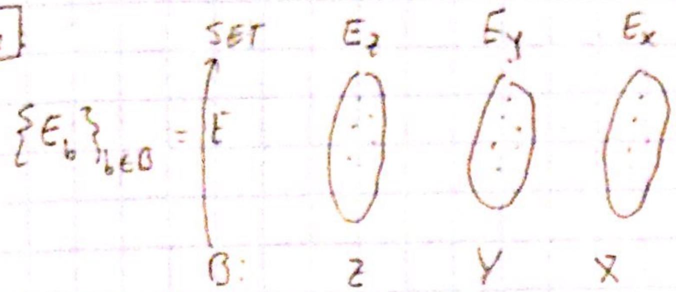
MOREHOUSE 6/19/2015

DEPENDENCE AND FIRST ORDER LOGIC

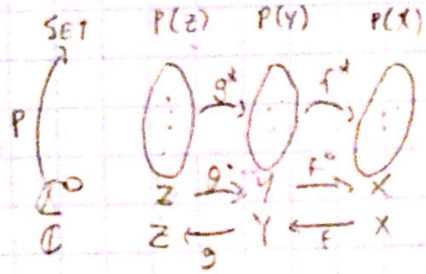
$\boxed{\text{Dependence}}$



BCC: The category of small bicartesian closed categories and functors respecting bicartesian closed structure.

⇓ generalize ⇓

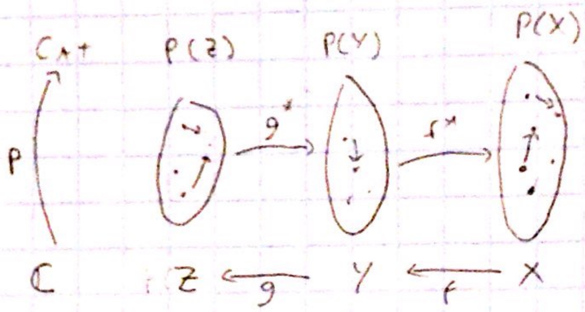
Pre sheaf
A contravariant SET-value functor:
$P : \mathbb{C}^o \to SET$



⇓ generalize: ⇓

we are interested in BCC categories.

Indexed Category
$P : \mathbb{C}^o \to CAT =$



- $\mathbb{C}$ is base category
- $X : \mathbb{C}$, $P(\mathbb{C})$ is fiber over X
- for $f :: \mathbb{C}$, f^* is a reindexing functor

Language of Types

- Atomic Types $\Upsilon = (X, Y, Z \ldots)$
  $[\![X]\!] : \mathbb{C}$ (with finite products)
- Typing contexts (lists of distinct typed variables)
  $[\![x_1 : X_1, \ldots x_n : Y_n]\!] := [\![Y_1]\!] \times \ldots \times [\![Y_n]\!]$
- Context Weakening: $\hat{x} := \Phi, x : X \to \Phi$
  $[\![\hat{x}]\!] := \pi_0 : [\![\Phi]\!] \times [\![X]\!] \to [\![\Phi]\!]$
- Function symbols - an arity indexed collection
  $F \in \mathcal{F}(\underbrace{Y_1, \ldots Y_n}_{\text{argument type}} ; \underbrace{X}_{\text{return type}})$

This is a morphism in index category, where objects in index category are typing category

→ $[\![f]\!] := [\![Y_1]\!] \times \ldots [\![Y_n]\!] \to [\![X]\!]$

$\boxed{\text{Terms in Context}}$

→ $[\![\Phi \mid t : X]\!] := [\![\Phi]\!] \to [\![X]\!]$

- Lifted Variable for $x_i : X_i \in \Phi$
  $[\![\Phi \mid x_i : X_i]\!] := [\![\Phi]\!] \xrightarrow{\pi_i} [\![X_i]\!]$

- Applied function symbol for $F \in \mathcal{F}(Y_1, \ldots Y_n ; X)$ and terms $\Phi \mid t_i : Y_i$
  $[\![\Phi \mid f(\vec{t}) : X]\!] := [\![\Phi]\!] \xrightarrow{\langle [\![t_1]\!], \ldots [\![t_n]\!] \rangle} [\![Y_1]\!] \times \ldots \times [\![Y_n]\!] \xrightarrow{[\![f]\!]} [\![X]\!]$

already exists

Context Extension for $\Phi \mid t : X$ and $y : Y \notin \Phi$
$[\![\Phi, y : Y \mid t : X]\!] := [\![\Phi]\!] \times [\![Y]\!] \xrightarrow{[\![\hat{y}]\!]} [\![\Phi]\!] \xrightarrow{[\![t]\!]} [\![X]\!]$

Term Substitution for terms $\Phi, y : Y \mid t : X$ and $\Phi \mid s : Y$
$[\![\Phi \mid t[y \mapsto s] : X]\!] := [\![\Phi]\!] \underbrace{\xrightarrow{\langle id, [\![s]\!] \rangle}}_{\text{single substitution}} [\![\Phi]\!] \times [\![Y]\!] \xrightarrow{[\![t]\!]} [\![X]\!]$

$[\![[y \mapsto s]]\!] := \langle id, [\![s]\!] \rangle$

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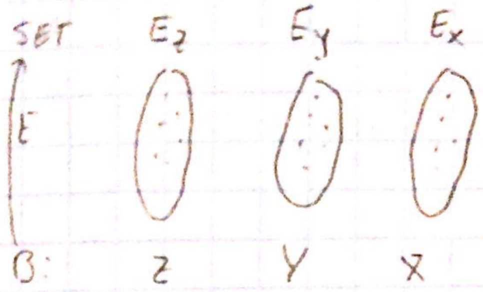
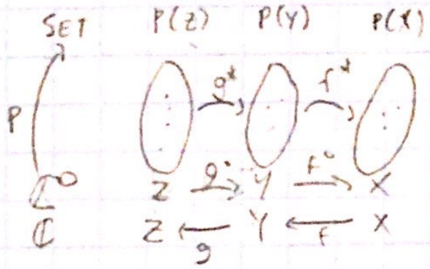
Dependence

$\{E_b\}_{b \in B} =$



⇓ generalize ⇓



presheaf
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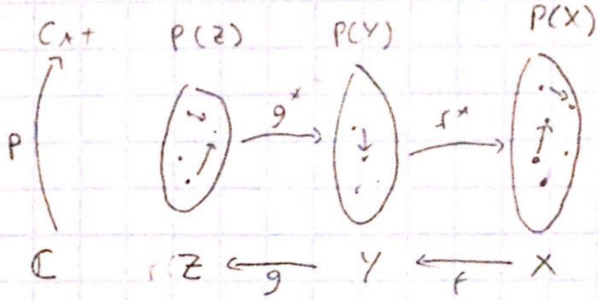
⇓ generalize: ⇓



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single substitution.
$[\![[y \mapsto s]]\!] := \langle id, [\![s]\!] \rangle$

- Applied function symbol for $f \in \mathcal{F}(Y_1, \ldots Y_n; X)$ and terms $\Phi \mid t_i : Y_i$
  $[\![\Phi \mid f(\vec{t}) : X]\!] := [\![\Phi]\!] \xrightarrow{\langle [\![t_1]\!], \ldots [\![t_n]\!] \rangle} [\![Y_1]\!] \times \ldots \times [\![Y_n]\!] \xrightarrow{[\![f]\!]} [\![X]\!]$

already exists

Relation Symbols - arity indexed collection

$R \in \mathcal{R}(X_1, \dots X_n)$

predicate := atomic proposition

If $\mathbb{C}$ is a cartesian category interpreting the term language and $P : \mathbb{C}^{o} \to BCC$:

$[\![R]\!] : P([\![X_1]\!] \times \dots \times [\![X_n]\!])$ Eg. Prime $\in \mathcal{R}(\mathbb{N})$

$P(\mathbb{N})$: x — prime

$\mathbb{C}$: $[\![\mathbb{N}]\!]$

$$\boxed{\begin{array}{l}\text{Propositions in Context:}\\ \underbrace{\Phi}_{\text{typing context}} \,|\, \underbrace{A}_{\text{proposition}}\ \underbrace{\text{PROP}}_{\text{judgement}} \\ [\![\Phi | A \text{ prop}]\!] : P([\![\Phi]\!])\end{array}}$$

Applied Relation Symbol for $R \in \mathcal{R}(Y_1, \dots Y_n)$ and terms $\Phi | t_i : Y_i$:

$[\![\Phi | R(\vec{t}) \text{ prop}]\!] :=$

$$\begin{tikzcd}
{[\![R]\!]} \arrow[rr, mapsto] & & {[\![R(\vec{t})]\!]} \\
{[\![Y_1]\!] \times \dots \times [\![Y_n]\!]} & & {[\![\Phi]\!]} \arrow[ll, "{\langle [\![t_1]\!], \dots [\![t_n]\!] \rangle}"]
\end{tikzcd}$$

$\mathbb{C}$: types and typing contexts are objects

arrows are terms

Context Extension for $\Phi | A$ prop and $y : Y \notin \Phi$

$[\![\Phi, y:Y | A \text{ prop}]\!] :=$

$$\begin{tikzcd}
{[\![A]\!]} \arrow[rr, mapsto] & & {[\![A]\!]} \\
{[\![\Phi]\!]} & & {[\![\Phi]\!] \times [\![Y]\!]} \arrow[ll, "{[\![\hat{y}]\!]}"]
\end{tikzcd}$$

Term Substitution for Prop $\Phi, y:Y | A$ prop and term $\Phi | t : Y$

$[\![\Phi | A[y \mapsto t]]\!] =$

$$\begin{tikzcd}
{[\![A]\!]} \arrow[rr, mapsto] & & {[\![A[y \mapsto t]]\!]} \\
{[\![\Phi]\!] \times [\![Y]\!]} & & {[\![\Phi]\!]} \arrow[ll, "{\langle id, [\![t]\!] \rangle}", "{[\![[y \mapsto t]]\!]}"']
\end{tikzcd}$$

Quantifiers

$$\begin{tikzcd}
BCC & P([\![\Phi]\!]) \arrow[rr, bend left, phantom, "\perp"] \arrow[rr, "{[\![\hat{x}]\!]^*}"] & & P([\![\Phi, x:X]\!]) \arrow[ll, bend right, "{[\![\exists x:X. _]\!]}"'] \arrow[ll, bend left, "{[\![\forall x:X. _]\!]}"] \\
\mathbb{C} \arrow[u, "P"] & {[\![\Phi]\!]} & & {[\![\Phi, x:X]\!]} \arrow[ll, "{[\![\hat{x}]\!]}"]
\end{tikzcd}$$

Universal Quantification

$P(\Phi)$: $\Gamma \xrightarrow{\vartheta^{\flat}} \forall x:X.A$

$$\begin{tikzcd}
P(\Phi, x:X): & \hat{x}^*\Gamma \arrow[r, "\vartheta"] \arrow[dr, "\hat{x}^*(\vartheta^{\flat})"'] & A \\
& & \hat{x}^*(\forall x:X.A) \arrow[u, "\varepsilon(A)"'] \arrow[ul, phantom, "\Uparrow \beta_R"]
\end{tikzcd}$$

Universal Quantification continued

Intro rule: $\_^{\flat}$

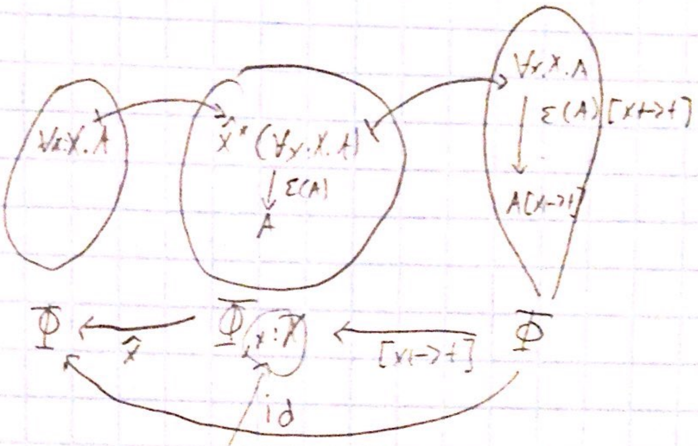
$$\begin{array}{c} \hat{x}^* \Gamma \\ \mathcal{D} \\ A \end{array} \xmapsto{\ \_^{\flat}\ } \dfrac{\Gamma \quad \begin{array}{c}\overline{\hat{x}^*\Gamma} \\ \mathcal{D} \\ A\end{array}}{\forall x{:}X.\,A}\,\forall I$$

$\to P(\Phi, x X)$

$P(\Phi)$

Elim rule $\varepsilon(-)$

$$A \mapsto \dfrac{\hat{x}^*(\forall x X A)}{A}$$



compare with:

$$\dfrac{\forall x{:}X.\,A \quad t{:}X}{A[x\mapsto t]}$$

which is composition of two steps. this is interesting as it'll let us do proof with intermediate.

← useful for proof search

allows us to use context variable or metavariable. can choose when to re-index by substitution.

$$\dfrac{\hat{x}^*\Gamma \quad \begin{array}{c}\overline{\hat{x}^*\Gamma} \\ \mathcal{D} \\ A\end{array}}{\hat{x}^*(\forall x{:}X.\,A)}\,\hat{x}^*(\forall I) \atop \overline{\quad A \quad}\,\forall E \quad \overset{\Longrightarrow}{\beta_R} \quad \begin{array}{c}\hat{x}^*\Gamma \\ \mathcal{D} \\ A\end{array}$$

Note: never have to choose representative term. Normally thats present in Universal quantifier reduction.

$$id(\forall x{:}X.A) = (\varepsilon\_)^{\flat}$$

$$\forall x{:}X{:}A = \dfrac{\forall x{:}X.A \quad \dfrac{\hat{x}^*(\forall x{:}X.A)}{A}\,\forall E}{\forall x{:}X.\,A}\,\forall I$$

1 2 1 1 3 1 2 3 ... hmm.
n=1 n=2 n=1 n=2

one on if off. all others off if on.

If enter room and off, elect yourself counter, turn on, set n=1.
If enter room and on, stop counting, turn off. only do this once until become counter.

1 2 1 3 1
n=1 n=2 n=3
~~on~~ off on off done

1 1 2 1 2 3 2
n=1 n=1 n=2 n=3
on off ~~off~~ off on off done

1 2 3 2 3 2 3
n=1 n=1 n=2 n=3
on off on off on off done

1 2 1 2 3 1
n=1 n=2 n=2 done.

~~1 2 1 2 1~~ fail.
n=1 n=2 n=3