Propositions as Types

Curry '35
Howard '69

$$\frac{M:A \qquad N:B}{\langle M,N\rangle : A\wedge B}\wedge I$$

where $M:A$ means M is a proof (term) of A.

Prop	Type
$A\wedge B$	$A\times B$
$A\supset B$	$A\to B$
$A\vee B$	$A+B$
$\bot$	0
$\top$	1

$$\frac{M:A\wedge B}{\pi_1 M : A}\wedge E_1 \qquad \frac{M:A\wedge B}{\pi_2 M : B}\wedge E_2$$

$$\frac{\begin{array}{c}\overline{x:A}^{\,x}\\ \vdots \\ M:B\end{array}}{(\lambda x.M): A\supset B}\supset I^x \qquad \frac{M:A\supset B \qquad N:A}{M\,N : B}$$

eg:

$$\frac{\dfrac{\begin{array}{c}\overline{x:A}^{\,x}\\ \mathcal{D}\\ M:B\end{array}}{\lambda x.M : A\supset B}\supset I^x \qquad N:A}{(\lambda x.M)\,N : B}\supset E \quad \underset{R}{\Longrightarrow} \quad \begin{array}{c}\mathcal{E}\\ N:A\\ {[N/x]}\mathcal{D}\\ {[N/x]}M:B\end{array}$$

So $(\lambda x.M)\,N \Rightarrow [N/x]\,M$ ([β] reduction)

$$\begin{array}{c}\mathcal{D}\\ M:A\supset B\end{array} \quad \underset{E}{\Longrightarrow} \quad \frac{\dfrac{\begin{array}{c}\mathcal{D}\\ M:A\supset B\end{array} \qquad \overline{x:A}^{\,x}}{Mx:B}}{(\lambda x.Mx): A\supset B}$$

So $M : A\supset B \underset{E}{\Longrightarrow} \lambda x.\,Mx$ ([η] expansion)

Prop	$\longleftrightarrow$	Type
Proof	$\longleftrightarrow$	λ-term
Intro Rule	$\longleftrightarrow$	Constructors
Elim Rule	$\longleftrightarrow$	Destructor
Reduction	$\longleftrightarrow$	Computation Step
Expansion	$\longleftrightarrow$	Extensional Equality

$$\frac{M:A}{\mathrm{inl}\,M : A\vee B}\vee I_1 \qquad \frac{M:B}{\mathrm{inr}\,M : A\vee B}\vee I_2$$

$$\frac{M:A\vee B \qquad \begin{array}{c}\overline{x:A}^{\,x}\\ N:C\end{array} \qquad \begin{array}{c}\overline{y:B}^{\,y}\\ P:C\end{array}}{\mathrm{case}(M,\ x.N,\ y.P) : C}\vee E^{x,y}$$

$\mathrm{case}(\mathrm{inl}\,M,\ x.N,\ y.P) \underset{R}{\Longrightarrow} [M/x]\,N$

$\mathrm{case}(\mathrm{inr}\,M,\ x.N,\ y.P) \Longrightarrow [M/y]\,P$

$$\begin{array}{c}\mathcal{D}\\ A\vee B\end{array} \underset{E}{\Longrightarrow} \frac{\begin{array}{c}\mathcal{D}\\ A\vee B\end{array} \qquad \dfrac{\overline{A}^{\,x}}{A\vee B}\vee I_1 \qquad \dfrac{\overline{B}^{\,y}}{A\vee B}\vee I_2}{A\vee B}\vee E$$

$M : \underset{E}{\Longrightarrow} \mathrm{case}(M,\ x.\mathrm{inl}\,x,\ y.\mathrm{inr}\,y)$

$\bot$ no introduction rule

$$\frac{M:\bot \ \mathit{true}}{\mathrm{abort}\,M : C \ \mathit{true}}\bot E$$

like a zero arity disjunction.

$$\overline{\ \top\ }\,\top I$$ - like zero arity conjunction.

$\top$ no elimination rules

$$A\supset(B\supset C) \vdash (A\wedge B)\supset C$$

$$\frac{\dfrac{\dfrac{f : A\supset(B\supset C) \qquad \dfrac{\overline{p:A\wedge B}^{\,p}}{\pi_1 p : A}\wedge E_1}{f(\pi_1 p) : B\supset C}\supset E \qquad \dfrac{\overline{p:A\wedge B}^{\,p}}{\pi_2 p : B}\wedge E_2}{f(\pi_1 p)(\pi_2 p) : C}\supset E}{\lambda p.\,f(\pi_1 p)(\pi_2 p) : (A\wedge B)\supset C}\supset I^p$$

or:

isomorphism

$f : A\supset(B\supset C) \vdash \lambda p.(f(\pi_1 p)(\pi_2 p)) : (A\wedge B)\supset C$

$g : (A\wedge B)\supset C \vdash \lambda x.\,\lambda y.\,g(\langle x,y\rangle) : A\supset(B\supset C)$

$\qquad\qquad :A \quad :B \quad :C$

$f \stackrel{?}{=} \lambda x.\lambda y.(\lambda p.(f(\pi_1 p)(\pi_2 p))\langle x,y\rangle)$

$\underset{R}{\Longrightarrow} \lambda x\,\lambda y\, f(\pi_1\langle x,y\rangle)(\pi_2\langle x,y\rangle)$

$\underset{R}{\Longrightarrow} \lambda x\,\lambda y\, f(x)(y)$

$f \underset{E}{\Longrightarrow} \lambda x.\,f x \underset{E}{\Longrightarrow} \lambda x(\lambda y.\,f x)y)$

Ex. check the other direction

Ex. is $A\supset(B\supset C)$ and $(A\supset B)\supset(A\supset C)$ isomorphism