6/19/2015 MOREHOUSE

start on notes pg 45

~~Recall~~

A coproduct of objects A and B is a cospan on A and B ($A \xrightarrow{p_0} P \xleftarrow{p_1} B$) so that for any cospan on A and B ($A \xrightarrow{x_0} X \xleftarrow{x_1} B$) $\exists! \, t: X \leftarrow P$ $t \circ p_i = x_i$

$$\begin{tikzcd} & X & \\ A \arrow[ur, "x_0"] \arrow[r, "p_0"'] & P \arrow[u, "t"] & B \arrow[ul, "x_1"'] \arrow[l, "p_1"] \end{tikzcd}$$

A and B are the ~~factors~~ cases

p_0 and p_1 are the ~~projections~~ insertions (ι_0, ι_1)

t is the copair of x_0, x_1 ($[x_0, x_1]$)

P is written "A+B"

Lemmas

- Identity expansion
- Uniqueness: Coproducts are unique up to an insertion preserving iso
- Coproduct functor:

  $- + - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

Interpreting Disjunction

- $[\![A \vee B]\!] := [\![A]\!] + [\![B]\!]$
- intro rules:

  $\dfrac{A}{A \vee B} \vee I_1 \quad \dfrac{B}{A \vee B} \vee I_2$ as $[\![\vee I_0]\!] := \iota_0$, $[\![\vee I_1]\!] := \iota_1$

- elim rule:

  $$\dfrac{A \vee B \quad \begin{matrix} \overline{A} \\ \mathcal{D}_0 \\ C \end{matrix} \quad \begin{matrix} \overline{B} \\ \mathcal{D}_1 \\ C \end{matrix}}{C} \vee E$$

  $[\![\vee E]\!] := [\![-]\!]$ not sufficiently general

$$\dfrac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}$$

or

$$\dfrac{\Gamma\ A \vee B \quad \begin{matrix} \overline{\Gamma, A} \\ \mathcal{D}_0 \\ C \end{matrix} \quad \begin{matrix} \overline{\Gamma, B} \\ \mathcal{D}_1 \\ C \end{matrix}}{C}$$

obs: $X \times 0 \cong 0$

An exponential of objects A and B is an object E and arrow $\varepsilon: E \times A \to B$ so that for any object X and arrow $f: X \times A \to B$, $\exists! \, \lambda(f): X \to E$.

$X \xrightarrow{\lambda(f)} E$ $\qquad \lambda(f) \times A \cdot \varepsilon = f$

$$\begin{tikzcd} X \times A \arrow[r, "f"] \arrow[dr, "\lambda(f) \times id(A)"'] & B \\ & E \times A \arrow[u, "\varepsilon"'] \end{tikzcd}$$

dist: $X \times (A + B) \cong (X \times A) + (X \times B)$

$$\begin{tikzcd} & X \times (A+B) \arrow[d, "dist", "\cong"'] & \\ X \times A \arrow[r, "\iota_0"] \arrow[dr, "f"'] & (X \times A) + (X \times B) \arrow[d, "{[f,g]}"] & X \times B \arrow[l, "\iota_1"'] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

ε is evaluation map

~~$\lambda(f)$ is the "curry" of f~~

$E \xrightarrow{\lambda(\varepsilon)} E$

$$\begin{tikzcd} E \times A \arrow[r, "\varepsilon"] \arrow[dr, "\lambda(\varepsilon) \times A"'] & B \\ & E \times A \arrow[u, "\varepsilon"'] \end{tikzcd}$$

so $\lambda(\varepsilon) = id(E)$ [identity expansion]

Lemma. (Uniqueness)

Exponentials are unique up to an evaluation preserving iso, so write "$A \supset B$" or "B^A"

* Ex. why $i \cdot \lambda(f) = \lambda(i \times A \cdot f)$?

$$Y \xrightarrow{i} X \xrightarrow{\lambda(f)} A \supset B$$

$$\begin{tikzcd}
Y \times A \arrow[r, "i \times A"] \arrow[drr, bend right, "(i \cdot \lambda(f)) \times A"'] & X \times A \arrow[r, "f"] \arrow[dr, "\lambda(f) \times A"] \arrow[d, phantom, "②"] & B \arrow[d, phantom, "①"] \\
& {} & (A \supset B) \times A \arrow[u]
\end{tikzcd}$$

So prove triangles ① and ② commute. ① is definition.

lemma

Exponential functor

$A \supset -$

$\mathbb{C} \to \mathbb{C}$

$B \mapsto A \supset B$

$$\begin{tikzcd}
A \supset B \arrow[rr, dashed] & & A \supset C \\
(A \supset B) \times A \arrow[r, "\varepsilon"] & B \arrow[r, "g"] & C
\end{tikzcd}$$

type checks.

$g : B \to C \mapsto A \supset g := \lambda(\varepsilon \cdot g)$

Interpreting Implication

$[\![A \supset B]\!] := [\![A]\!] \supset [\![B]\!]$

- intro rule:

$$\dfrac{\begin{array}{c}\overline{A} \\ \mathcal{D} \\ B\end{array}}{A \supset B} \supset I \quad \text{or} \quad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \supset I \quad \text{or} \quad \dfrac{\begin{array}{c}\overline{\Gamma, A} \\ \mathcal{D} \\ \Gamma \quad B\end{array}}{A \supset B} \supset I$$

$[\![\supset I]\!] := \lambda([\![-]\!])$

- elim rule

$$\dfrac{A \supset B \quad A}{B} \supset E \qquad [\![\supset E]\!] := \varepsilon$$

Categories with:
- $(1, \times)$ are cartesian. (finite products and terminal objects)
- $(1, \times, \supset)$ are cartesian closed
- $(0, +)$ are cocartesian
- $(0, 1, +, \times)$ are bicartesian
- $(0, 1, +, \times, \supset)$ are bicartesian closed

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for parallel functors $F, G : \mathbb{C} \to \mathbb{D}$, a <u>natural transformation</u> φ from F to G:
- for $A : \mathbb{C}$, an arrow $\varphi(A) : \mathbb{D}(F(A) \to G(A))$ (the <u>component</u>)
- for $f : \mathbb{C}(A \to B)$ a <u>naturality square</u>:

$\mathbb{C}$: $A \xrightarrow{f} B$

D:

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\varphi(A)"'] & F(B) \arrow[d, "\varphi(B)"] \\
G(A) \arrow[r, "G(f)"'] & G(B)
\end{tikzcd}$$

CAT

$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \varphi"] & \mathbb{D}
\end{tikzcd}$$

$\leftarrow$ in notes, this behavioral characterization of natural transformation is explored.

ADJUNCTIONS - Almost Inverses

$\mathbb{C} \to \mathbb{D}$ $\mathbb{D} \to \mathbb{C}$

anti-parallel functors F and G form an adjuction ("$F \dashv G$") if for any $A : \mathbb{C}$, $B : \mathbb{D}$ there is a natural bijection of hom-sets:

{F is left adjoint, G is right adjoint}

three equivalent definitions

- bijection: $\dfrac{\mathbb{C}(A \to G(B))}{\mathbb{D}(F(A) \to B)}$ $\_^{\#}$, $\_^{b}$

- naturality: $\mathbb{C}: A' \xrightarrow{a} A \xrightarrow{f = g^b} G(B) \xrightarrow{G(b)} G(B')$
$\mathbb{D}: F(A') \xrightarrow{F(a)} F(A) \xrightarrow{g = f^{\#}} B \xrightarrow{b} B'$

$F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{C}$ form $F \dashv G$ if:

unit of adjunction

Universal Property of unit:

$\exists \eta : id(\mathbb{C}) \to F \cdot G.$
$\forall f : \mathbb{C}(A \to G(B)).$
$\exists! g : \mathbb{D}(F(A) \to B).$
$\eta(A) \cdot G(g) = f$

$$\begin{array}{ll} & (G \circ F)(A) \\ & \eta(A) \uparrow \quad \searrow G(g) \\ \mathbb{C}: & A \xrightarrow{f} G(B) \\ \hline \mathbb{D}: & F(A) \xrightarrow[g]{} B \end{array}$$

Universal Property of count

$\exists \varepsilon : G \cdot F \to id(\mathbb{D}).$
$\forall g : \mathbb{D}(F(A) \to B).$
$\exists! f : \mathbb{C}(A \to G(B)).$
$F(f) \cdot \varepsilon(B) = g$

$$\begin{array}{ll} \mathbb{C}: & A \xrightarrow{f} G(B) \\ \hline \mathbb{D} & F(A) \xrightarrow[g]{} B \\ & F(f) \searrow \quad \nearrow \varepsilon(B) \\ & (F \circ G)(B) \end{array}$$

β_R

Ex. Explain why:
- $(\eta(A))^{\#} = id(F(A))$
- $(\varepsilon(B))^{b} = id(G(B))$

Hint: probe $\eta(A)$ and $\varepsilon(B)$ with them selves

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "- \times A"] \arrow[r, phantom, "\perp"] & \mathbb{C} \arrow[l, bend left, "A \supset -"] \end{tikzcd}$$

Product functor and Implication functor are adjoint.

$$\begin{array}{l} \mathbb{C}: X \xrightarrow{\lambda(f)} A \supset B \\ \hline \mathbb{C}: X \times A \xrightarrow{f} B \\ \lambda(f) \times id(A) \searrow \quad \uparrow \varepsilon(B) \\ \qquad (A \supset B) \times A \end{array}$$

long way around

short path

$$\dfrac{\dfrac{\begin{matrix}\Gamma \quad \lambda \\ \mathcal{D} \\ B\end{matrix}}{A \supset B}\supset I \qquad A}{B}\supset E \quad \overset{\text{justified by diagram commuting}}{\Longrightarrow_R} \quad \begin{matrix}\Gamma \quad A \\ \mathcal{D} \\ B\end{matrix}$$

Uninterpreted $id(A \supset B) = \lambda(\varepsilon)$

$$\begin{matrix}\mathcal{E} \\ A \supset B\end{matrix} = \dfrac{\dfrac{\begin{matrix}\mathcal{E} \\ A \supset B\end{matrix} \quad A}{B}\supset E}{A \supset B}\supset I$$

Elaborate: $\iota \cdot \lambda f = \lambda(\iota \times A \cdot f)$

$$\dfrac{\begin{matrix}\Gamma \\ \mathcal{E} \\ C\end{matrix} \quad \begin{matrix}\overline{C\ A} \\ \mathcal{D} \\ B\end{matrix}}{A \supset B}\supset I = \dfrac{\begin{matrix}\Gamma \\ \mathcal{E} \\ C \quad A \\ \mathcal{D} \\ B\end{matrix}}{A \supset B}\supset I$$

For negative connectives χ:
- Interpreted as right adjoint functors
- $[\![\chi I]\!] := \_^{b}$
- $[\![\chi E]\!] := \varepsilon$
- $[\![\chi \Rightarrow_R]\!] := \beta_R$
- $[\![\chi \Rightarrow_E]\!] :=$ identity expansion

For positive connectives χ:
"left adjoint" functors

end page 68.

Read differently