6/18/2015 PFENNING

CUT ELIMINATION; LINEAR LOGIC

$$\begin{array}{c} A_1\downarrow \ldots A_n\downarrow \\ \vdots \\ C\uparrow \end{array} = \Gamma\downarrow \vdash A\uparrow$$ (two ways of writing Natural Deduction)

$\boxed{\Gamma\downarrow \vdash A\uparrow \text{ ND} \qquad \Gamma \Rightarrow A \text{ SEQ}}$

Theorem.

(i) If $\Gamma\downarrow \overset{\mathcal{D}}{\vdash} A\uparrow$ then $\Gamma \Rightarrow A$.

(ii) If $\Gamma\downarrow \vdash A\downarrow$ and $\Gamma, A \Rightarrow C$ then $\Gamma \Rightarrow C$

Case $\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma\downarrow \vdash A_1\uparrow} \quad \overset{\mathcal{D}_2}{\Gamma\downarrow \vdash A_2\uparrow}}{\Gamma\downarrow \vdash A_1 \wedge A_2\uparrow}\wedge I$

$\Gamma \Rightarrow A_1$ by IH(i) on $\mathcal{D}_1$

$\Gamma \Rightarrow A_2$ by IH(i) on $\mathcal{D}_2$

$\Gamma \Rightarrow A_1 \wedge A_2$ by $\wedge R$

similar for other introduction/right rules

Case $\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Gamma\downarrow \vdash A\downarrow}}{\Gamma\downarrow \vdash A\uparrow}\uparrow\downarrow$

$\Gamma, A \Rightarrow A$ by id rule

$\Gamma \Rightarrow A$ by IH $\mathcal{D}'$

Case $\mathcal{D} = \overline{\Gamma\downarrow, A\downarrow \vdash A\downarrow}$

$\Gamma, A, A \Rightarrow C$ assumption

$\Gamma, A \Rightarrow C$ by contraction.

Case $\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma\downarrow \vdash A \wedge B\downarrow}}{\Gamma\downarrow \vdash A\downarrow}\wedge E_1$

$\Gamma, A \Rightarrow C$ assumption

$\Gamma, A \wedge B, A \Rightarrow C$ weakening

$\Gamma, A \wedge B \Rightarrow C$ by $\wedge L_1$

$\Gamma \Rightarrow C$ by IH on $\mathcal{D}_1$

Other cases are similar

A true iff $A\uparrow$ if we add $\dfrac{A\uparrow}{A\downarrow}$, what happens in Sequent Calculus:

Case: $\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Gamma\downarrow \vdash A\uparrow}}{\Gamma\downarrow \vdash A\downarrow}$

$\Gamma, A \Rightarrow C$ assumption

$\Gamma \Rightarrow A$ by IH ($\mathcal{D}'$)

$\Gamma \Rightarrow C$

$$\frac{\Gamma \Rightarrow A \quad \Gamma, A \Rightarrow C}{\Gamma \Rightarrow C}\text{Cut}$$

Theorem. Admissibility of Cut

If $\Gamma \overset{\mathcal{D}}{\Rightarrow} A$ and $\Gamma, A \overset{\mathcal{E}}{\Rightarrow} C$ then $\Gamma \overset{\mathcal{F}}{\Rightarrow} C$

Case $\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma \Rightarrow A_1} \quad \overset{\mathcal{D}_2}{\Gamma \vdash A_2}}{\Gamma \Rightarrow A_1 \wedge A_2}$ $\quad \dfrac{\Gamma, A_1 \wedge A_2, A_1 \Rightarrow C}{\Gamma, A_1 \wedge A_2 \Rightarrow C} = \mathcal{E}$

$$\Gamma, A_1 \Rightarrow C \quad \text{by "IH" } A_1 \wedge A_2, \mathcal{D}^{+A_1}, \overset{\mathcal{F}}{\mathcal{E}}$$

$$\Gamma \Rightarrow C \quad \text{by "IH" on } A_1, D_1, \mathcal{F}$$

Overall strategy is induction on first A, second on $\underset{\leq}{\mathcal{D}}$ or $\underset{=}{\mathcal{E}}$

or $\underset{=}{\mathcal{D}}$ $\underset{\leq}{\mathcal{E}}$

Linear Logic $\boxed{\text{GIRARD '87}}$ - adds $!A$ to indicate you can re-use it.
("of course")

$$\overbrace{A_1, \ldots, A_n}^{\Delta} \vdash C$$

$\underbrace{A_1, \ldots, A_n}_{\text{resources}}$

resources
= must use them exactly once in proof.

$$\frac{}{A \vdash A}\,\text{id} \qquad \frac{\Delta, A, B \vdash C}{\Delta, A \otimes B \vdash C}\,\otimes L \qquad \frac{\Delta \vdash A \quad \Delta' \vdash B}{\Delta, \Delta' \vdash A \otimes B}\,\otimes R$$

$$\frac{\Delta, A \vdash B}{\Delta \vdash A \multimap B}\,\multimap R \qquad \frac{\Delta \vdash A \quad \Delta', B \vdash C}{\Delta, \Delta', A \multimap B \vdash C}\,\multimap L$$

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Delta \vdash A} \quad \overset{\mathcal{E}}{\Delta' \vdash B}}{\Delta, \Delta' \vdash A \otimes B}\,\otimes R \qquad \dfrac{\overset{\mathcal{F}}{\Delta'', A, B \vdash C}}{\Delta'', A \otimes B \vdash C}\,\otimes L}{\Delta, \Delta', \Delta'' \vdash C}\,\text{Cut } A \otimes B$$

reduction

$$\dfrac{\Delta' \vdash B \qquad \dfrac{\overset{\mathcal{D}}{\Delta \vdash A} \quad \overset{\mathcal{F}}{\Delta'', A, B \vdash C}}{\Delta, \Delta'', B \vdash C}\,\text{Cut } A}{\Delta, \Delta', \Delta'' \vdash C}\,\text{Cut } B$$