Comparison of Direct Methods for NMPC Applied to a Thrust Vector Drone

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Introduction

1. What is a Thrust Vector Drone and Why they are useful for Rocket Design

Thrust Vector Drone Equations of Motion

1. What goes in this section

1. *Equations*:

$$\dot{\vec{x}} = \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \\ q_x \\ q_y \\ q_z \\ q_w \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\dot{\vec{u}} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ T_{avg} \\ T_{diff} \end{bmatrix}$$

$$\dot{\vec{p}} = \vec{v}$$

$$\dot{\vec{v}} = \frac{1}{m} R(\vec{q}) \vec{F}_b + \vec{g}$$

$$\dot{\vec{q}} = \frac{1}{2} Q(\vec{\omega}) \vec{q}$$

$$\dot{\vec{\omega}} = I^{-1} \Big(\vec{M}_b - \vec{\omega} \times (I\,\vec{\omega}) \Big)$$

$$\dot{x} = v_x$$

$$\dot{y} = v_y$$

$$\dot{z} = v_z$$

$$\vec{F_b} = T \begin{bmatrix} \sin\theta_2 \\ -\sin\theta_1\cos\theta_2 \\ \cos\theta_1\cos\theta_2 \end{bmatrix}$$

$$\vec{l} = \begin{bmatrix} 0 \\ 0 \\ -l \end{bmatrix}$$

$$\vec{M_z} = \begin{bmatrix} 0 \\ 0 \\ M_z \end{bmatrix}$$

$$\vec{M_b} = \vec{l} \times \vec{F_b} + \vec{M_z} = \begin{bmatrix} -lT\sin\theta_1\cos\theta_2 \\ -lT\sin\theta_2 \\ M_z \end{bmatrix}$$

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

$$R(\vec{q}) = \begin{bmatrix} 1-2(q_y^2+q_z^2) & 2(q_xq_y - q_wq_z) & 2(q_xq_z + q_wq_y) \\ 2(q_xq_y + q_wq_z) & 1-2(q_x^2+q_z^2) & 2(q_yq_z - q_wq_x) \\ 2(q_xq_z - q_wq_y) & 2(q_yq_z + q_wq_x) & 1-2(q_x^2+q_y^2) \end{bmatrix}$$

$$Q(\vec{\omega}) = \begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix}$$

$$\dot{x} = v_x$$

$$\dot{y} = v_y$$

$$\dot{z} = v_z$$

$$\dot{v_x} = \frac{\sin\theta_2 * T(1-2(q_y^2+q_z^2)) - \sin\theta_1\cos\theta_2 * 2T(q_xq_y - q_wq_z) + \cos\theta_1\cos\theta_2 * 2T(q_xq_z + q_wq_y)}{m}$$

$$\dot{v_y} = \frac{\sin\theta_2 * 2T(q_xq_y + q_wq_z) - \sin\theta_1\cos\theta_2 * T(1-2(q_x^2+q_z^2)) + \cos\theta_1\cos\theta_2 * 2T(q_yq_z - q_wq_x)}{m} + g$$

$$\dot{v_z} = \frac{\sin\theta_2 * 2T(q_xq_z - q_wq_y - \sin\theta_1\cos\theta_2 * 2T(q_yq_z + q_wq_x) + \cos\theta_1\cos\theta_2 * T(1-2(q_x^2+q_y^2))}{m}$$

$$\dot{q_x} = \frac{\omega_z q_y - \omega_y q_z + \omega_x q_w}{2}$$

$$\dot{q_y} = \frac{-\omega_z q_x + \omega_x q_z + \omega_y q_w}{2}$$

$$\dot{q_z} = \frac{\omega_y q_x - \omega_x q_y + \omega_z q_w}{2}$$

$$\dot{q_w} = \frac{-\omega_x q_x - \omega_y q_y - \omega_z q_z}{2}$$

$$\dot{\omega_x} = \frac{-lT\sin\theta_1\cos\theta_2 - \omega_y\omega_z I_{zz} + \omega_y\omega_z I_{yy}}{I_{xx}}$$

$$\dot{\omega_y} = \frac{-lT\sin\theta_2 - \omega_x\omega_z I_{xx} + \omega_x\omega_z I_{zz}}{I_{yy}}$$

$$\dot{\omega_z} = \frac{M_z - \omega_x\omega_y I_{yy} + \omega_x\omega_y I_{xx}}{I_{zz}}$$

2. *Quaternions:* Why we chose to use them.

3. *Thrust Model:*

$$T = aT_{avg}^2 + bT_{avg} + c$$

$$M_z = dI_{zz}T_{diff}$$

$$T_1 = T_{avg} + T_{diff}/2$$

$$T_2 = T_{avg} - T_{diff}/2$$

NMPC

1. what goes in this section

Model predictive control (MPC) is a control algorithm that solves a linear optimization problem at each time step. The set of instance constraints can be used to enforce the equations of motion and restrict the values of control inputs or state variables. For example, MPC constraints can be used to eliminate solutions that violate minimum or maximum control values or require control values to change faster than mechanically possible. This gives MPC algorithms more control over the types of solutions produced than linear quadratic regulators (LQR) or proportional integral derivative (PID) controllers. Unlike LQR and PID methods which are designed to find an optimal control input for a given moment in time, an MPC solution considers a series of control inputs over a finite time horizon. This allows MPC solutions to create a *plan* of actions that can anticipate and adjust to future issues. For example, the optimal speed for a car at a given instance might change if the algorithm knew that it was entering a curve in the road in the next few time steps. The ability to anticipate future conditions allows MPC to construct more robust solutions. The control plan is rebuilt at each timestep allowing the method to react to disturbances and noise. Nonlinear model predictive control (NMPC) is a variant of MPC that allows nonlinear constraints.

The biggest advantage of using NMPC to control our thrust vector drone is the allowance of nonlinear constraints. The 6DOF equations of motion that describe our dynamics are strongly nonlinear. An LQR or MPC method requires a linear model. The 6DOF equations can be linearized around the fixed point in which the drone is vertically balanced, however, the linearized equations quickly become inaccurate if the drone moves away from the fixed point. Working directly with the nonlinear equations allows more accurate predictions of the drone state. Like MPC, NMPC allows us to enforce the mechanical limitations of our thrust motors and gimbal servos.

A prime drawback of MPC and NMPC algorithms over LQR and PID is their higher computational cost. To use NMPC to control a thrust vector drone, we need to solve an NMPC instance in less than 0.02 seconds on a Raspberry Pi 5. Given an NMPC problem instance, there are a variety of known ways to formulate the instance as nonlinear programming problem. Here we give our NMPC problem instance and present three different ways to formulate this instance as a nonlinear programming problem. We then compare the quality of solutions produced and their relative computational efficiency.

To use NMPC the problem instance needs to be parameterized over the control trajectory, the differential equations for the system dynamics need to be integrated forward, and then the resulting problem can be optimized over control parameters. There are different ways to parameterize the problem and integrate it. These choices will affect the size of the problem and the speed at which it can be solved. Also the accuracy of the solution.

Terms we should know *single shooting*, *multiple shooting*, *direct collocation methods*, *pseudo spectral methods* What are these and relative advantages / disadvantages.

$$
\begin{gathered}
\min_{u(t)} \int_{t_0}^{t_f} \ell(x,u,t)\,dt \;+\; \phi(x_f) \\
\ell(x,u,t) = (x(t)-x_{ref})^T Q (x(t)-x_{ref}) + (u(t)-u_{ref})^T R (u(t)-u_{ref}) \\
\phi(x_f) = (x(t_f)-x_{ref})^T Q_f (x(t_f)-x_{ref}) \\
-\theta_{max} \le \theta_1 \le \theta_{max} \\
-\theta_{max} \le \theta_2 \le \theta_{max} \\
-\dot{\theta}_{max} \le \dot{\theta}_1 \le \dot{\theta}_{max} \\
-\dot{\theta}_{max} \le \dot{\theta}_2 \le \dot{\theta}_{max} \\
T_{min} \le T_{avg} + T_{diff}/2 \le T_{max} \\
T_{min} \le T_{avg} - T_{diff}/2 \le T_{max} \\
T_{diff_{min}} \le T_{diff} \le T_{diff_{max}} \\
0 \le z
\end{gathered}
$$

1 Direct Methods

Orthagonal Collocation using do-mpc

Multiple Shooting (with Runge–Kutta discretization)

Need to talk about time period $[t_i, t_f]$ and how we divide it into sizes of Δt. For multiple shooting, we divide our time horizon into N evenly spaced nodes $\{t_0, t_1, \ldots, t_N\}$. At each node we make a copy of our state x_k and control u_k variables. Within each time step $[t_k, t_{k+1}]$ we forward integrate our differential equations f using Runge-Kutta to create a constraint that enforces the change of state between time steps.

$$
\begin{aligned}
k_1 &= f(x_k, u_k, t_k), \\
k_2 &= f\big(x_k + \tfrac{\Delta t}{2} k_1,\, u_k,\, t_k + \tfrac{\Delta t}{2}\big)\,, \\
k_3 &= f\big(x_k + \tfrac{\Delta t}{2} k_2,\, u_k,\, t_k + \tfrac{\Delta t}{2}\big)\,, \\
k_4 &= f(x_k + \Delta t\, k_3,\, u_k,\, t_k + \Delta t)\,, \\
x_{k+1} &= x_k + \tfrac{\Delta t}{6}\Big(k_1 + 2k_2 + 2k_3 + k_4\Big).
\end{aligned} \tag{1}
$$

Where f here needs a reference to our differential equations function from earlier. Need to talk about initial conditions, objective function,

$$
x_0 = X_0
$$

$$
\sum_{k=0}^{N-1} \Big[(x_k - x_r)^T Q (x_k - x_r) + (u_k - u_r)^T R (u_k - u_r)\Big] + (x_N - x_r)^T Q (x_N - x_r) \tag{2}
$$

Chebyshev Pseudospectral Method

Maintains accuracy with far fewer nodes. Disadvantages: A more complex NLP. doesn't do well with discontinuous controls.

$$\tau_j = \cos\left(\frac{j\pi}{N}\right), \qquad j = 0, 1, ..., N$$

$$D_{N_{00}} = \frac{2N^2+1}{6}$$

$$D_{N_{jj}} = \frac{-x_j}{2(1-x_j^2)}$$

$$D_{N_{ij}} = \frac{c_i}{c_j}\frac{(-1)^{i+j}}{x_i - x_j}$$

$$D_{N_{NN}} = -\frac{2N^2+1}{6}$$

$$c_{i/j} = \begin{cases} 2 & i/j = 0 \text{ or } N, \\ 1 & \text{otherwise.} \end{cases}$$

$$D_6 = \begin{bmatrix} \frac{73}{6} & -(8+4\sqrt{3}) & 4 & -2 & \frac{4}{3} & 4\sqrt{3}-8 & \frac{1}{2} \\ 2+\sqrt{3} & -\sqrt{3} & -(1+\sqrt{3}) & \frac{2}{\sqrt{3}} & 1-\sqrt{3} & \frac{1}{\sqrt{3}} & \frac{1-\sqrt{3}}{1+\sqrt{3}} \\ -1 & 1+\sqrt{3} & -\frac{1}{3} & -2 & 1 & 1-\sqrt{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{2}{\sqrt{3}} & 2 & 0 & -2 & \frac{2}{\sqrt{3}} & -\frac{1}{2} \\ -\frac{1}{3} & \sqrt{3}-1 & -1 & 2 & \frac{1}{3} & -(1+\sqrt{3}) & 1 \\ \frac{1-\sqrt{3}}{1+\sqrt{3}} & \sqrt{3} & \sqrt{3}-1 & -\frac{2}{\sqrt{3}} & 1+\sqrt{3} & \sqrt{3} & -(2+\sqrt{3}) \\ -\frac{1}{2} & 8-4\sqrt{3} & -\frac{4}{3} & 2 & -4 & 8+4\sqrt{3} & -\frac{73}{6} \end{bmatrix}$$

$$\dot{x}(\tau_i) \;\approx\; \sum_{j=0}^{N} D_{ij}\, x(\tau_j), \qquad i = 0, \ldots, N.$$

$$t(\tau) = \frac{t_f - t_0}{2}\,\tau + \frac{t_f + t_0}{2}, \qquad \frac{dt}{d\tau} = \frac{t_f - t_0}{2}.$$

$$\sum_{j=0}^{N} D_{ij}\, x_j = \frac{t_f - t_0}{2}\, f(x_i, u_i), \qquad i = 1, \ldots, N-1.$$

Using Clenshaw-Curtis quadrature.

$$w_{\mathrm{j_{even}}} = \begin{cases} \dfrac{2}{N}\left(1 - \displaystyle\sum_{n=1}^{\frac{N}{2}-1} \frac{2}{4n^2-1}\cos\!\left(\frac{2nj\pi}{N}\right) - \frac{(-1)^j}{N^2-1}\right), & 1 \le j \le N-1, \\ \dfrac{1}{N^2-1}, & j = 0 \text{ or } j = N \quad (\text{since } N \text{ even}). \end{cases}$$

$$w_{\mathrm{j_{odd}}} = \begin{cases} \dfrac{2}{N}\left(1 - \displaystyle\sum_{n=1}^{\frac{N}{2}-1} \frac{2}{4n^2-1}\cos\!\left(\frac{2nj\pi}{N}\right)\right), & 1 \le j \le N-1, \\ \dfrac{1}{N^2}, & j = 0 \text{ or } j = N. \end{cases}$$

$$J = \frac{t_f - t_0}{2}\int_{t_0}^{t_f} \ell(x(t), u(t), t)\, dt \;+\; \phi(x(t_f))$$

$$J \approx \frac{t_f - t_0}{2} \sum_{j=0}^{N} w_j \, \ell(x_j, u_j, t_j) \; + \; \phi(x_N), \qquad t_j = \frac{t_f - t_0}{2} \, \tau_j + \frac{t_f + t_0}{2}.$$

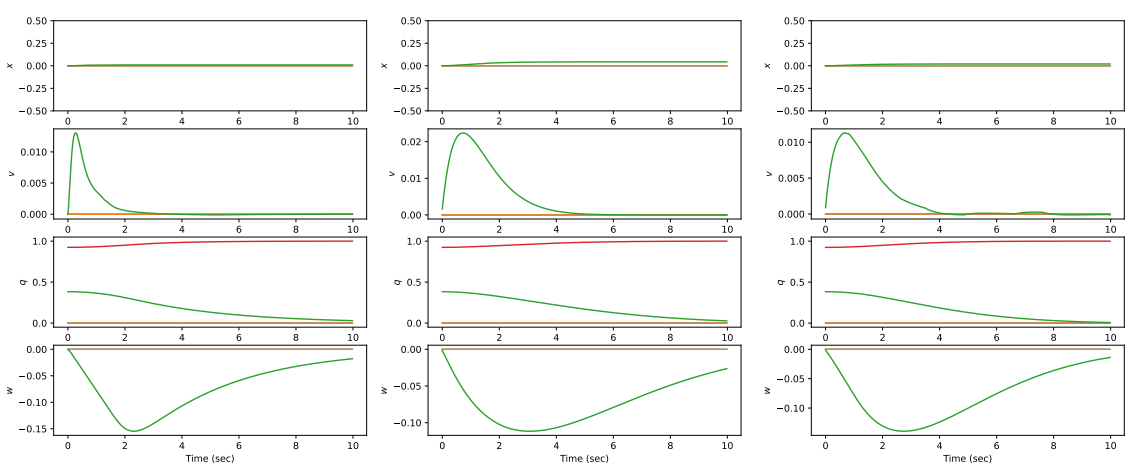
Comparison of Methods

To compare our different formulations for the NLP problem, we ran a series of simulations each with the drone beginning in a unique and perturbed starting state. In all cases, the drone goal state was to be vertically balanced and motionless at the point $(x, y, z) = (0, 0, 0)$ in three dimensional space. For each simulation, we computed the average time required to solve the NLP problem at each iteration. The quality of the solutions produced by each method was assessed by comparing the state and control graphs through time. All experiements were run on a Macbook Pro 3.49GHz Apple M2 chip with 8GB RAM. Code was implemented in Python 3.13 using CasADI 3.7.1.

NLP Solver

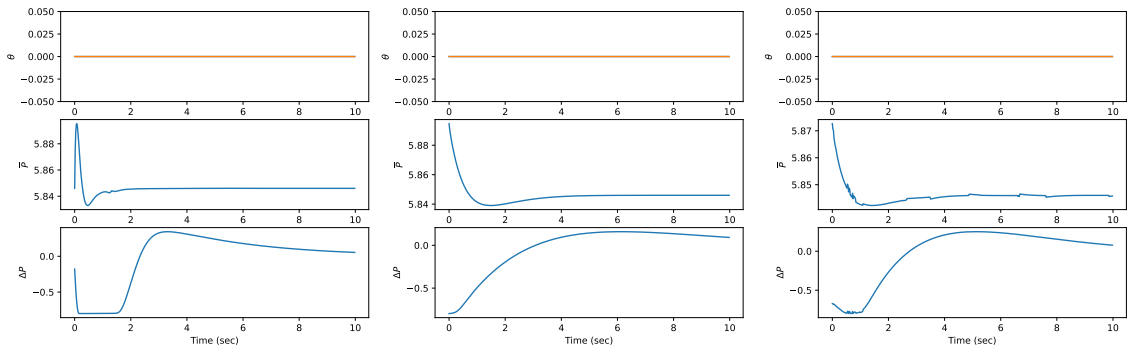
To solve our non-linear programming (NLP) problems we used the `ipopt` solver that comes installed with CasADi 3.7.1. This solver requires an additional subroutine to solve sparse matrix systems. We experimented with the `mumps` solver that comes with CasADi, and also tried using the `ma27` and `ma57` solvers from HSL (Harwell Subroutine Library) [HSL13]. All solvers were compiled with Apple clang version 17.0.0. The HSL `ma27` solver produced the most efficient solutions overall. All solvers and experiments were run with the same solver settings.

```
ipopt_settings = {
    'ipopt.max_iter': 100,
    'ipopt.tol': 1e-3,
    'ipopt.acceptable_tol': 3e-2,
    'ipopt.linear_solver': 'ma27',
}
```



(a) orthagonal collocation (b) multiple shooter (c) Chebyshev pseudospectral

Figure 1: State data on a simulation requiring a 45° rotation about the z axis.



(a) orthagonal collocation (b) multiple shooter (c) Chebyshev pseudospectral

Figure 2: Control data on a simulation requiring a 45° rotation about the z axis.

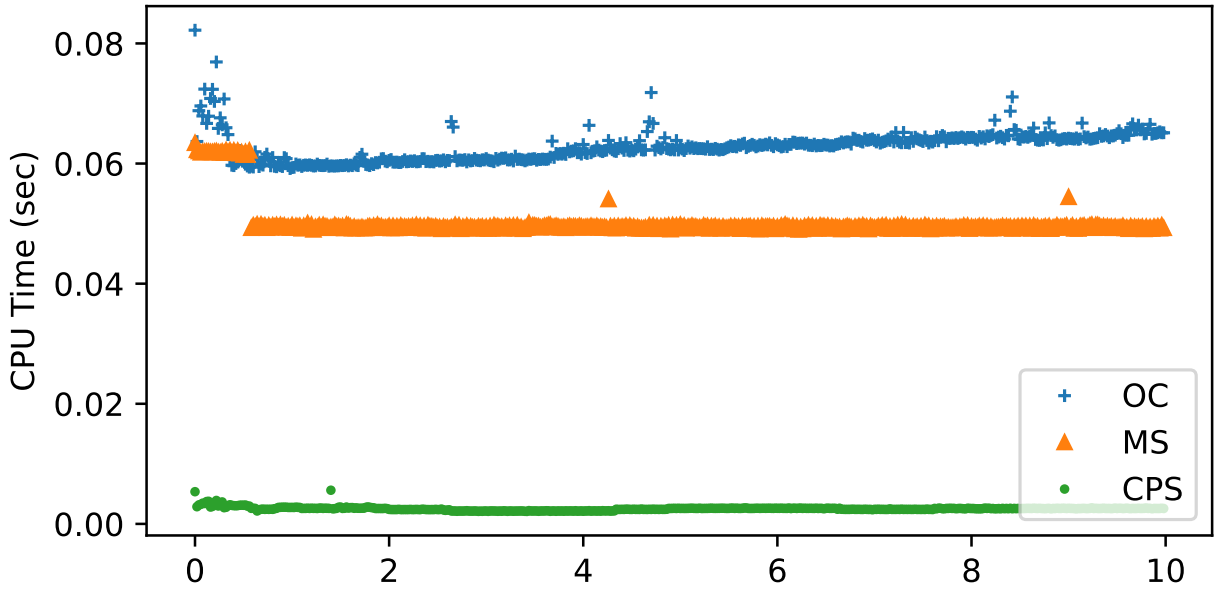
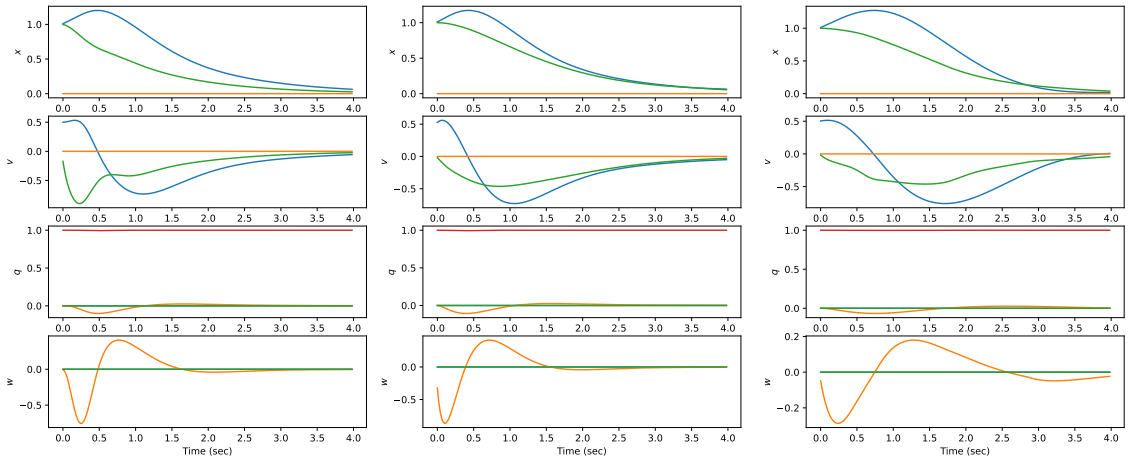
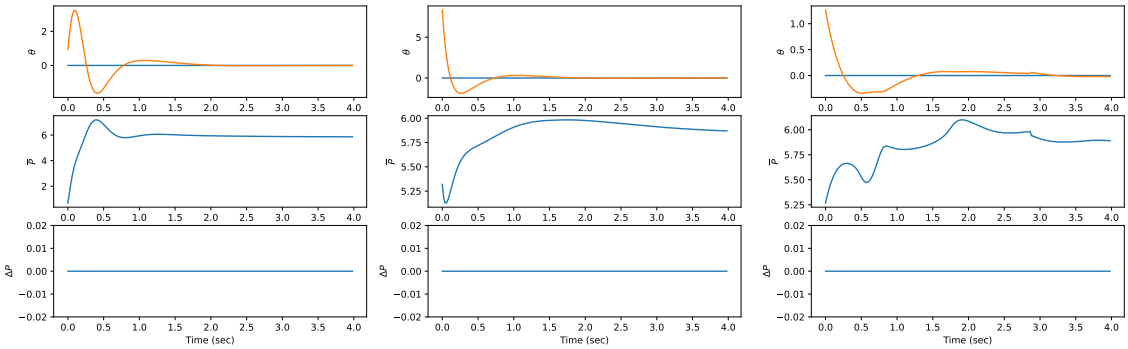


Figure 3: CPU time for simulation requiring a 45° rotation about the z axis.



(a) orthagonal collocation (b) multiple shooter (c) Chebyshev pseudospectral

Figure 4: State data on a simulation with initial conditions: $x = 1$, $z = 1$, $v_x = 0.5$



(a) orthagonal collocation (b) multiple shooter (c) Chebyshev pseudospectral

Figure 5: Control data on a simulation with initial conditions: $x = 1$, $z = 1$, $v_x = 0.5$

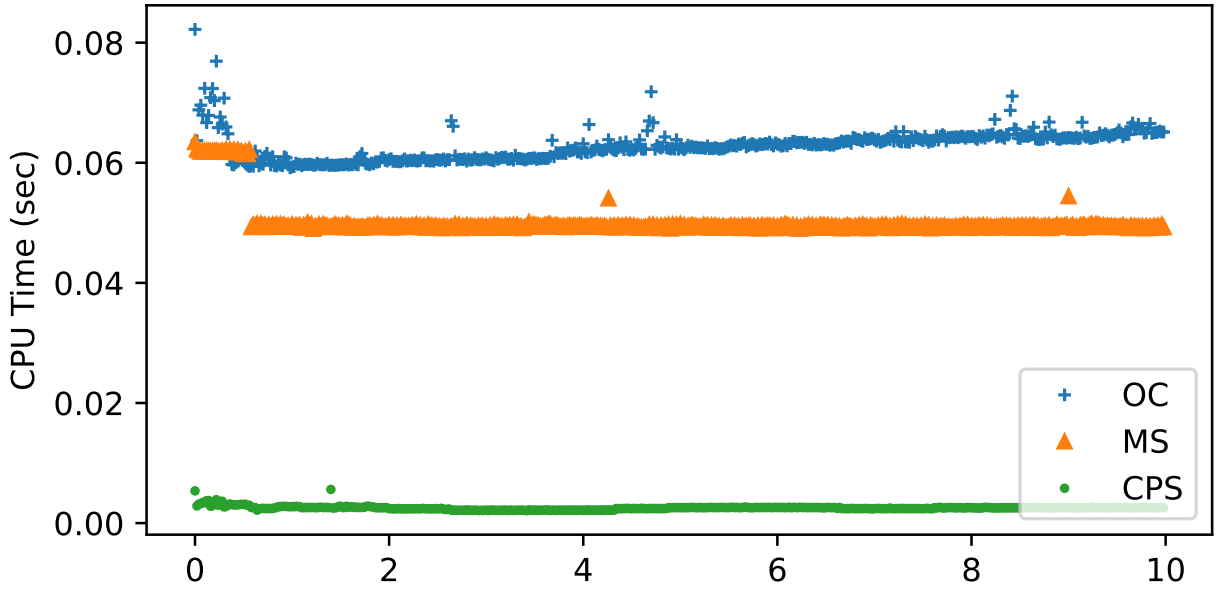
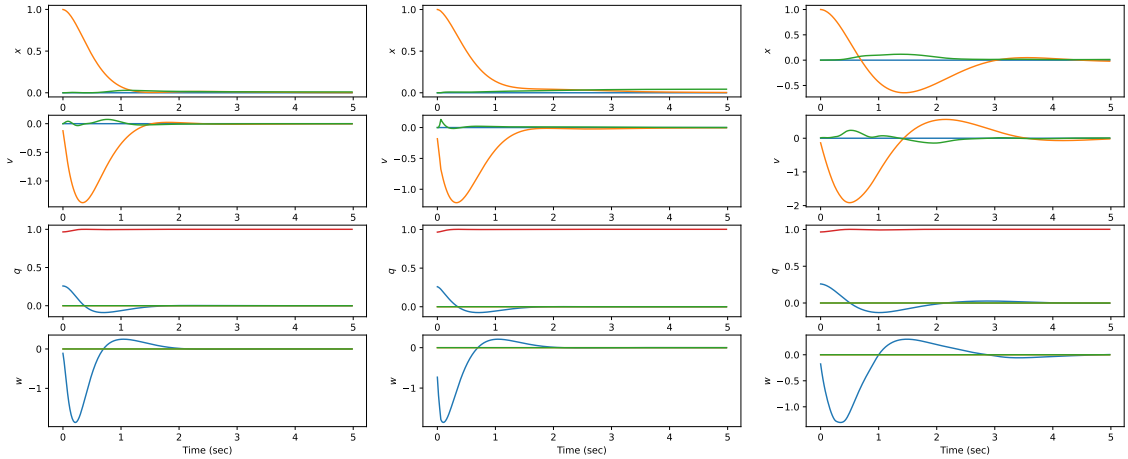
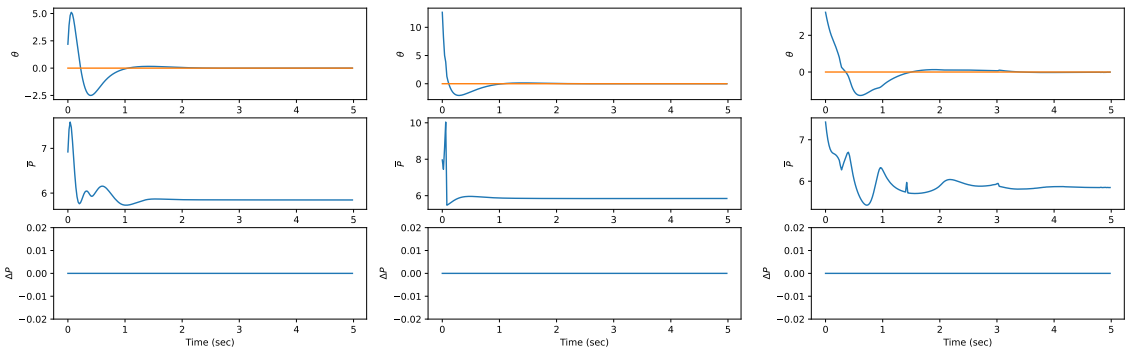


Figure 6: CPU time for a simulation with initial conditions: $x = 1$, $z = 1$, $v_x = 0.5$



(a) orthagonal collocation (b) multiple shooter (c) Chebyshev pseudospectral

Figure 7: State data on a simulation with initial conditions: $y = 1{,}15°$ rotation about x axis.



(a) orthagonal collocation (b) multiple shooter (c) Chebyshev pseudospectral

Figure 8: Control data on a simulation with initial conditions: $y = 1{,}15°$ rotation about x axis.

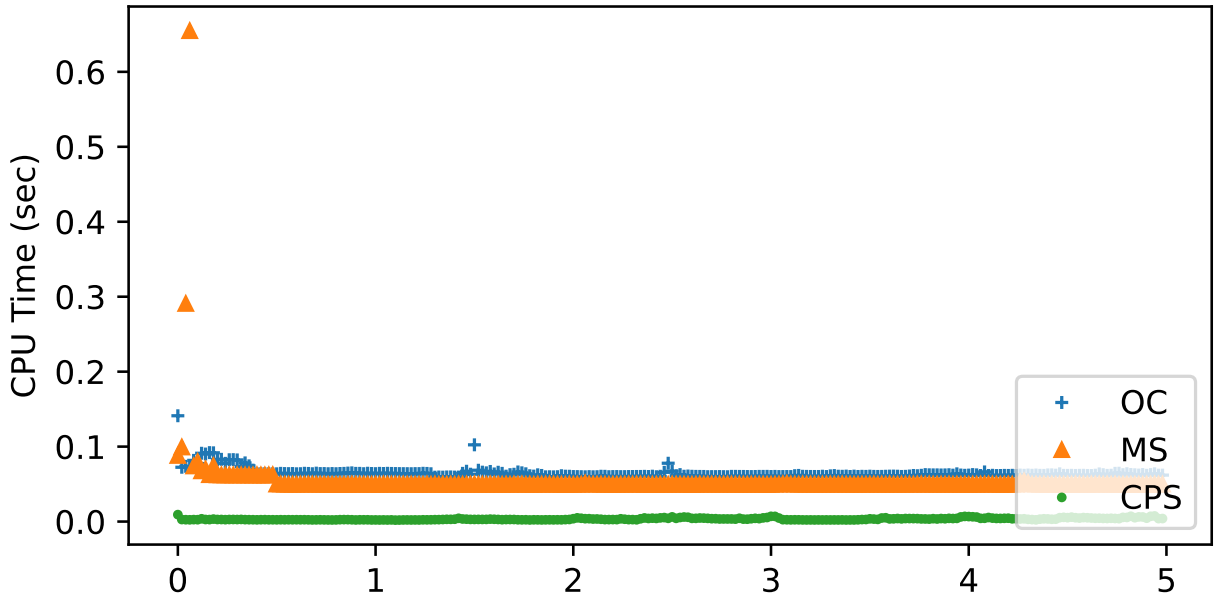
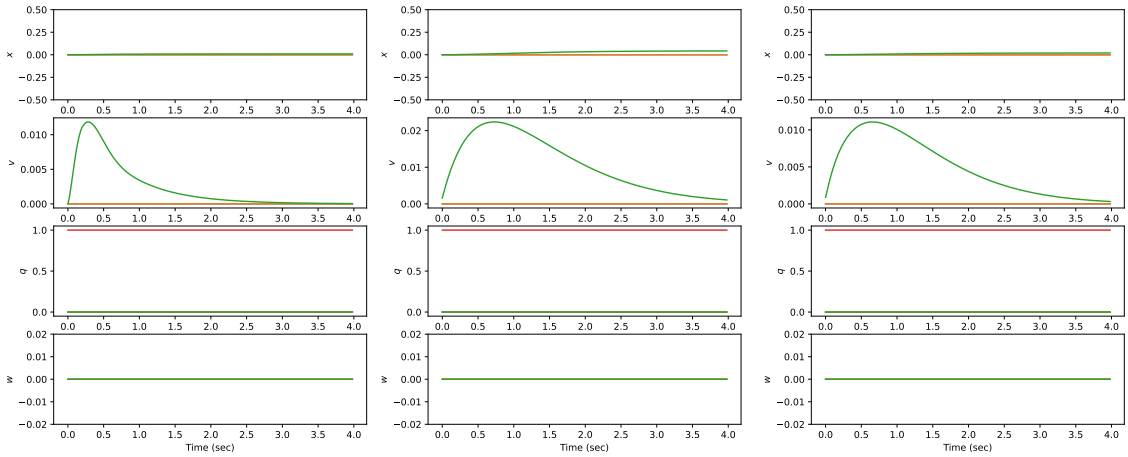
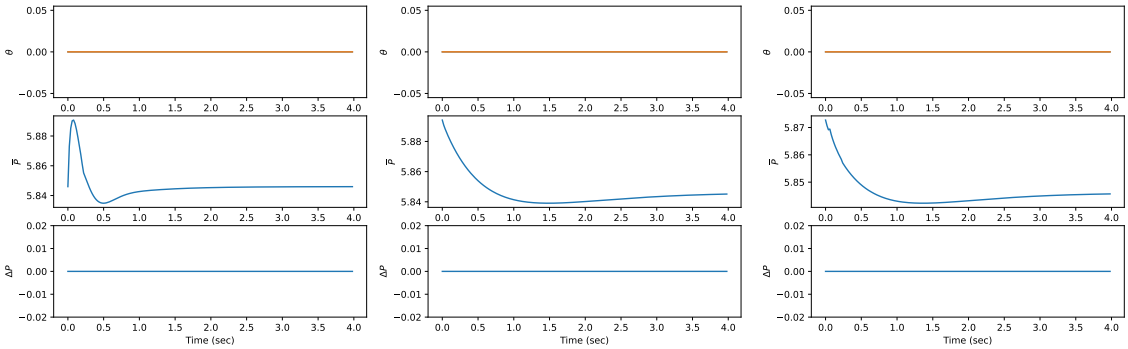


Figure 9: CPU time for simulation with initial conditions: $y = 1{,}15°$ rotation about x axis.



(a) orthagonal collocation (b) multiple shooter (c) Chebyshev pseudospectral

Figure 10: State data on a simulation with initial state equal to goal state.



(a) orthagonal collocation (b) multiple shooter (c) Chebyshev pseudospectral

Figure 11: Control data on a simulation with initial state equal to goal state.

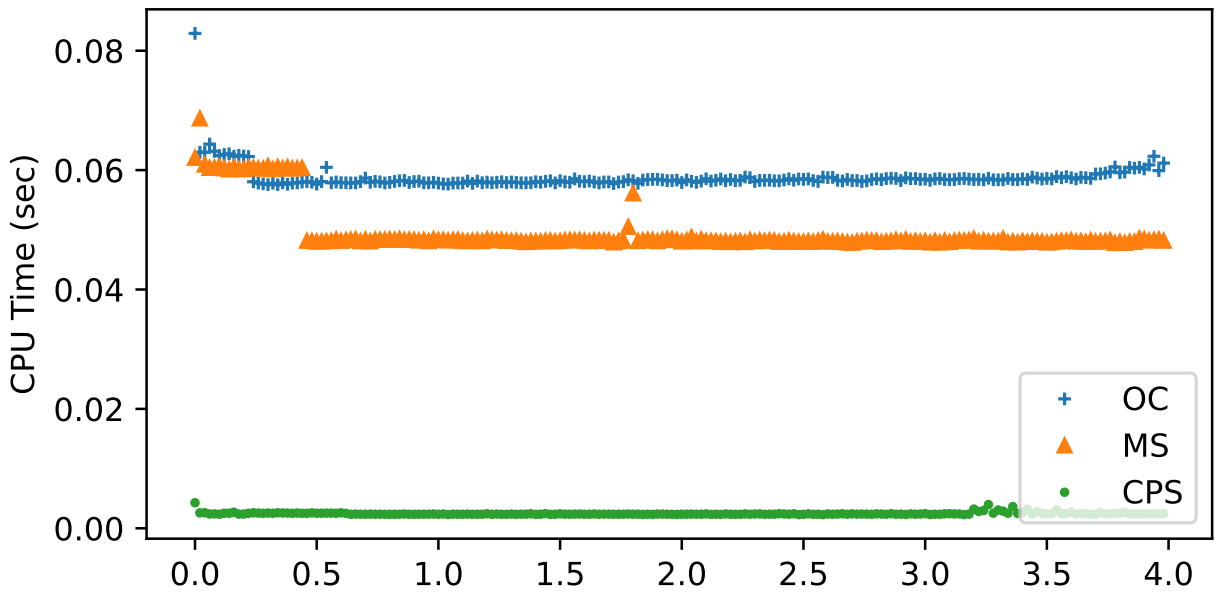


Figure 12: CPU time for simulation with initial state equal to goal state.

initial state	OC	MS	CPS
45° rotation about z axis	0.063	0.056	0.002
$x = 1$, $z = 1$, $v_x = 0.5$	0.059	0.046	0.003
$y = 1$,15° rotation about x axis	0.063	0.06	0.004
initial state equal to goal state	0.062	0.056	0.002

Table 1: Comparison of average CPU time in seconds for solution of NLP for a series of simulations. Methods include orthagonal collocation (OC), multiple shooter (MS), and Chebyshev pseudospectral method (CPS).

Our goal is to run our control algorithm at 50Hz on the raspberry Pi 5. The CPU speed of the Pi is 2.4GHz while our experiments were run on a machine with a CPU speed of 3.49GHz. To meet our goal of 50Hz on the Pi we need a solution time on the experiment machine of less than 0.014 seconds.

Conclusions

References

[HSL13] HSL. A collection of fortran codes for large scale scientific computation., 2013.